Correction to E. Zeidler, Quantum Field Theory, Vol. I.

Please, replace Section 7.24.7 on page 449 by the following new version. The author would like to thank Professor William Faris from the University of Arizona (U.S.A.) for drawing his attention to this error in the original version. The original wrong formulation of Theorem 7.39 claims more than one needs in physics. In terms of physics, the appearance of the solution φ_0 in the corrected theorem corresponds to the fact that the LSZ reduction formula used by physicists is only valid on mass-shell.

7.24.7 The Magic LSZ Reduction Formula for Scattering Functions

Let us use the extended quantum action functional $Z(J,\varphi)$ in order to define the scattering functional

$$\boxed{\mathsf{S}(\varphi) := \frac{Z(0,\varphi)}{Z(0,0)}.}$$

The functional derivative

$$\mathsf{S}_n(x_1,\ldots,x_n) := \frac{\delta^n \mathsf{S}(\varphi)}{\delta\varphi(x_1)\cdots\delta\varphi(x_n)}\Big|_{\varphi=0} \qquad (7.130)$$

is called the n-point scattering function. The physical meaning of this function will be discussed in Sect. 14.2.5. For example, if $\kappa = 0$, then $\mathsf{S}(\varphi) = 1$. In terms of physics, the triviality of $\mathsf{S}(\varphi)$ tells us that there is no proper scattering if the interaction vanishes. By Taylor expansion,

$$\mathsf{S}(\varphi) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{x_1,\ldots,x_n \in \mathcal{M}} \mathsf{S}_n(x_1,\ldots,x_n)\varphi(x_1)\cdots\varphi(x_n)(\Delta^4 x)^n.$$

This is regarded as a formal power series expansion. In perturbation theory, we consider terms up to order κ^m for fixed $m = 1, 2, \ldots$, where κ denotes the coupling constant. Define the modified n-point scattering function

$$\hat{\mathsf{S}}_n(x_1,\ldots,x_n) := \frac{1}{(\mathrm{i}\hbar)^n} \left\{ \prod_{k=1}^{n} (D_{x_k} + \mathrm{i}\varepsilon I) C_n \right\} (x_1,\ldots,x_n),$$

which depends on the correlation function C_n. Here, the symbol D_{x_k} stands for the linear operator D from the equation of motion (7.122) which acts on the kth variable of the correlation function C_n.

Theorem 7.39. *Choose the function $\varphi_0 : \mathcal{M} \to \mathbb{R}$ such that $D\varphi_0 = 0$. For all $x_1,\ldots,x_n \in \mathcal{M}$ and $n = 1, 2, \ldots$, we have the following LSZ reduction formulas: The sum*

$$\sum_{x_1,\ldots,x_n\in\mathcal{M}} \mathsf{S}_n(x_1,\ldots,x_n)\varphi_0(x_1)\cdots\varphi_0(x_n)(\Delta^4 x)^n$$

is equal to the sum

$$\sum_{x_1,\ldots,x_n\in\mathcal{M}} \hat{\mathsf{S}}_n(x_1,\ldots,x_n)\varphi_0(x_1)\cdots\varphi_0(x_n)(\Delta^4 x)^n + O(\varepsilon) \quad \text{as } \varepsilon \to +0.$$

Proof. The proof is an elementary consequence of the chain rule for partial derivatives and the relation

$$R_\varepsilon(D+\mathrm{i}\varepsilon)^{-1} = -I$$

together with $(D+\varepsilon I)\varphi_0 = \varepsilon\varphi_0$. To simplify notation, we set $D_\varepsilon := D+\varepsilon I$.

Step 1: To understand the simple idea of the proof, let us start with the case where $N=1$. Then

$$Z_{\text{free}}(J,\varphi) := \mathrm{e}^{\mathrm{i}R_\varepsilon J^2/2\hbar}\,\mathrm{e}^{\mathrm{i}\varphi J/\hbar}, \qquad \varphi, J \in \mathbb{R}.$$

Here, D_ε and R_ε are nonzero real numbers with $(-D_\varepsilon)R_\varepsilon = 1$. For the indices $k=0,1,2,\ldots,$ let us also introduce the differential operators

$$\frac{\delta}{\delta J} := \frac{1}{\Delta^4 x}\,\frac{\partial}{\partial J}, \qquad A := \frac{\delta^k}{\delta J^k}, \qquad \frac{\delta}{\delta\varphi} := \frac{1}{\Delta^4 x}\,\frac{\partial}{\partial\varphi}.$$

Let $n=1,2,\ldots$ We claim that

$$\frac{\delta^n A Z_{\text{free}}}{\delta\varphi^n}(0,0)\varphi_0^n(\Delta^4 x)^n = (-D_\varepsilon)^n\frac{\delta^n A Z_{\text{free}}}{\delta J^n}(0,0)\varphi_0^n(\Delta^4 x)^n + O(\varepsilon)$$

as $\varepsilon \to 0$. This is equivalent to

$$\boxed{\frac{\partial^n A Z_{\text{free}}}{\partial\varphi^n}(0,0)\varphi_0^n = (-D_\varepsilon)^n\frac{\partial^n A Z_{\text{free}}}{\partial J^n}(0,0)\varphi_0^n + O(\varepsilon)} \tag{0.1}$$

as $\varepsilon \to 0$, which is the key relation of our proof.

(i) First choose $n=1$. To simplify notation, set $\alpha := \frac{\mathrm{i}}{\hbar}$. Obviously,

$$\frac{\partial Z_{\text{free}}(J,\varphi)}{\partial\varphi} = \alpha J Z_{\text{free}}(J,\varphi).$$

Furthermore,

$$\frac{\partial Z_{\text{free}}(J,\varphi)}{\partial J} = \alpha(R_\varepsilon J + \varphi)Z_{\text{free}}(J,\varphi).$$

Setting $\varphi = 0$,

$$\frac{\partial Z_{\text{free}}}{\partial J}(J,0) = \alpha R_\varepsilon J Z_{\text{free}}(J,0).$$

Noting that $-D_\varepsilon R_\varepsilon = 1$, we get

$$-D_\varepsilon \frac{\partial Z_{\text{free}}}{\partial J}(J,0) = \alpha J Z_{\text{free}}(J,0).$$

Hence

$$-D_\varepsilon \frac{\partial Z_{\text{free}}}{\partial J}(J,0) = \frac{\partial Z_{\text{free}}}{\partial \varphi}(J,0).$$

Applying the differential operator A to this, we obtain

$$-D_\varepsilon \frac{\partial A Z_{\text{free}}}{\partial J}(J,0) = \frac{\partial A Z_{\text{free}}}{\partial \varphi}(J,0).$$

Setting $J = 0$, we get the claim (0.1) for $n = 1$.

(ii) Now choose $n = 2$. This step will show the typical feature of our proof. Differentiation with respect to φ yields

$$\frac{\partial^2 Z_{\text{free}}}{\partial \varphi^2}(J,0) = \alpha^2 J^2 Z_{\text{free}}(J,0). \tag{0.2}$$

Observe that

$$\frac{\partial^k \mathrm{e}^{\mathrm{i}\varphi J/\hbar}}{\partial J^k}|_{\varphi=0} = 0, \qquad k = 1, 2, \ldots$$

Therefore, setting $\mathcal{Z}_{\text{free}}(J) := Z_{\text{free}}(J,0)$, we get

$$\frac{\partial^2 Z_{\text{free}}}{\partial J^2}(J,0) = \frac{d^2 \mathcal{Z}_{\text{free}}(J)}{dJ^2}.$$

Hence

$$\frac{\partial^2 Z_{\text{free}}}{\partial J^2}(J,0) = \alpha^2 R_\varepsilon^2 J^2 Z_{\text{free}}(J,0) + \alpha R_\varepsilon Z_{\text{free}}(J,0).$$

Multiplying this by the factor D_ε^2 and noting that $D_\varepsilon R_\varepsilon = -1$ (and hence $D_\varepsilon^2 R_\varepsilon = -D_\varepsilon$), we get

$$D_\varepsilon^2 \frac{\partial^2 Z_{\text{free}}}{\partial J^2}(J,0) = \alpha^2 J^2 Z_{\text{free}}(J,0) - \alpha Z_{\text{free}}(J,0) D_\varepsilon.$$

This yields

$$\frac{\partial^2 Z_{\text{free}}}{\partial \varphi^2}(J,0) = D_\varepsilon^2 \frac{\partial^2 Z_{\text{free}}}{\partial J^2}(J,0) + r$$

with the remainder $r := \alpha Z_{\text{free}}(J,0) D_\varepsilon$. Since $D_\varepsilon \varphi_0 = 0$, we get

$$r\varphi_0^2 = 0.$$

This implies

$$\frac{\partial^2 Z_{\text{free}}}{\partial \varphi^2}(J,0)\varphi_0^2 = D_\varepsilon^2 \frac{\partial^2 Z_{\text{free}}}{\partial J^2}(J,0)\varphi_0^2.$$

Applying the differential operator A to this, we get

$$\frac{\partial^2 AZ_{\text{free}}}{\partial\varphi^2}(J,0)\varphi_0^2 = D_\varepsilon^2 \frac{\partial^2 AZ_{\text{free}}}{\partial J^2}(J,0)\varphi_0^2.$$

Finally, setting $J = 0$, we obtain the claim (0.1) for $n = 2$.

(iii) For $n = 3, 4, \ldots$, the proof of (0.1) proceeds analogously.

Step 2: Let $N = 2, 3, \ldots$ In contrast to Step 1, now we have to use partial derivatives with respect to the real variables $J(1), \ldots, J(N), \varphi(1), \ldots, \varphi(N)$. Let $D_\varepsilon(x, y)$ and $\mathcal{R}_\varepsilon(x, y)\Delta^4 x\Delta^4 y$ denote the entries of the matrix D_ε and R_ε, respectively. Here, $R_\varepsilon = -D_\varepsilon^{-1}$. Then

$$\langle J|R_\varepsilon J\rangle = \sum_{x,y\in\mathcal{M}} J(x)\mathcal{R}_\varepsilon(x, y)J(y)\Delta^4 x\Delta^4 y,$$

and $\sum_{y\in\mathcal{M}} D_\varepsilon(x, y)\mathcal{R}_\varepsilon(y, z)\Delta^4 y = -\delta_{x,z}$. The same argument as in Step 1 tells us that the functional derivative

$$\frac{\delta^n \mathsf{S}}{\delta\varphi(x_1)\cdots\delta\varphi(x_n)}(0)$$

is equal to

$$(-1)^n \sum_{y_1,\ldots,y_n\in\mathcal{M}} D_\varepsilon(x_1, y_1)\cdots D_\varepsilon(x_n, y_n)\frac{\delta^n Z}{\delta J(y_1)\cdots\delta J(y_n)}(0, 0) + r.$$

Since $D_\varepsilon\varphi_0 = \varepsilon\varphi_0$ and hence $\sum_{y\in\mathcal{M}} D_\varepsilon(x, y)\varphi_0(y) = \varepsilon\varphi_0(x) = O(\varepsilon)$, the remainder r has the crucial property that

$$\sum_{x_1,\ldots,x_n\in\mathcal{M}} r\varphi_0(x_1)\cdots\varphi_0(x_n) = O(\varepsilon) \qquad \text{as } \varepsilon\to 0.$$

Replacing the functional derivatives by S_n and C_n, we obtain that the n-point scattering function $\mathsf{S}_n(x_1, \ldots, x_n)$ is equal to

$$\frac{1}{(\mathrm{i}\hbar)^n} \sum_{y_1,\ldots,y_n\in\mathcal{M}} D(x_1, y_1)\cdots D(x_n, y_n)C_n(y_1, \ldots, y_n) + r.$$

Multiplying this by $\varphi_0(x_1)\cdots\varphi_0(x_n)(\Delta^4 x)^n$ and summing over the variables $x_1, \ldots, x_n \in \mathcal{M}$, we obtain the desired LSZ reduction formula. □

Correction to E. Zeidler, Quantum Field Theory, Vol. I.

Please, replace Section 14.2.5 on page 785 by the following new version.

14.2.5 The Magic LSZ Reduction Formula for the S-Matrix

Scattering functions. Parallel to the generating functional $Z(J)$ for the correlation functions, let us define the so-called scattering functional

$$\boxed{\mathsf{S}(\varphi) := \frac{Z(0,\varphi)}{Z(0,0)}} \qquad (0.3)$$

by using the extended quantum action functional $Z = Z(J,\varphi)$ from Sect. 14.2.3. The scattering functional is normalized by $\mathsf{S}(0) = 1$. Furthermore, for $n = 1, 2, \ldots$, let us define the scattering functions by setting

$$\mathsf{S}_n(x_1, \ldots, x_n) := \frac{\delta^n \mathsf{S}(\varphi)}{\delta\varphi(x_1)\cdots\delta\varphi(x_n)}\Big|_{\varphi=0}.$$

The scattering functions $\mathsf{S}_n(x_1, \ldots, x_n)$ are symmetric with respect to the variables $x_1, \ldots, x_n$. Equivalently, we get

$$\mathsf{S}(\varphi) := 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathbb{M}^{4n}} \mathsf{S}_{\mathsf{n}}(x_1, \ldots, x_n)\varphi(x_1)\cdots\varphi(x_n) d^4x_1 \cdots d^4x_n.$$

This means that the scattering functional S is the generating functional for the family of scattering functions $\mathsf{S}_1, \mathsf{S}_2, \ldots$.

The modified scattering functions. Let $n = 1, 2, \ldots$ Motivated by Theorem 7.39 (see the correction above), we introduce the modified scattering functions by setting

$$\boxed{\hat{\mathsf{S}}_n(x_1, \ldots, x_n) := \frac{1}{\mathrm{i}^n}\left\{\prod_{k=1}^{n}(D_{x_k} + \mathrm{i}\varepsilon)\right\} C_n(x_1, \ldots, x_n).} \qquad (0.4)$$

Here, we define

$$D_{x_k} := -\Box_{x_k} - m_0^2.$$

Note that the wave operator $\Box_{x_k}$ acts on the variable x_k of the full correlation function C_n.

***S*-matrix elements.** Let us briefly discuss the physical meaning of the modified scattering functions. By (14.14), the function

$$u_{\mathbf{p}}(\mathbf{x}, t) := \frac{\mathrm{e}^{\mathrm{i}(\mathbf{px} - E_{\mathbf{p}}t)}}{\sqrt{\mathcal{V}}}$$

is a solution of the free Klein–Gordon equation $(\Box + m_0^2)\varphi = 0$. Here, we set

$$E_{\mathbf{p}} := \sqrt{m_0^2 + \mathbf{p}^2}.$$

The normalization factor $\mathcal{V} > 0$ is chosen in such a way that

$$\int_{\mathcal{C}} |u_{\mathbf{p}}(\mathbf{x}, t))|^2 \, d^3\mathbf{x} = \int_{\mathcal{C}} d^3\mathbf{x} = 1$$

where $\mathcal{C}$ is an arbitrary bounded open subset of $\mathbb{R}^3$. In terms of physics, the function $u_{\mathbf{p}}$ describes a homogenous stream of free mesons of particle number density $1/\mathcal{V}$. In other words, there is precisely one particle in the box $\mathcal{C}$ of volume $\mathcal{V}$. Each particle has the momentum vector $\mathbf{p}$ and the energy $E_{\mathbf{p}}$. Now consider a scattering process in the box $\mathcal{C}$ with

- n incoming particle streams of free mesons at remote past, $t = -\infty$, described by the functions
$$u_{\mathbf{p}_1}, \ldots, u_{\mathbf{p}_n}$$
of momentum vectors $\mathbf{p}_1, \ldots, \mathbf{p}_n$, respectively.
- After scattering, there appear m outgoing particle streams of free mesons at remote future, $t = +\infty$, described by the functions
$$u_{\mathbf{p}_{n+1}}, \ldots, u_{\mathbf{p}_{n+m}}$$
of momentum vectors $\mathbf{p}_{n+1}, \ldots, \mathbf{p}_{n+m}$, respectively.

Set $v_{\mathbf{p}_k} := u_{\mathbf{p}_k}, k = 1, \ldots, n$ and $v_{\mathbf{p}_l} := u^\dagger_{\mathbf{p}_l}, l = n+1, \ldots, n+m$. We define

$$\boxed{\mathsf{S}_{\mathbf{p}_1,\ldots,\mathbf{p}_{n+m}} := \int_{\mathbb{M}^{4(n+m)}} \hat{\mathsf{S}}_{n+m}(x_1, \ldots, x_{n+m}) \prod_{k=1}^{n+m} v_{\mathbf{p}_k}(x_k) d^4x_k.} \qquad (0.5)$$

This is called the S-matrix element of the scattering process. The real number

$$|\mathsf{S}_{\mathbf{p}_1,\ldots\mathbf{p}_{2n}}|^2$$

is the so-called transition probability of the scattering process. This transition probability can be used in order to compute the cross section of the scattering process. This is precisely the quantity which can be measured in scattering experiments. Many applications to concrete physical processes will be considered in the following volumes.

The LSZ axiom. We postulate that the elements of the S-matrix are obtained by the so-called LSZ reduction formulas (0.5). This is the so-called LSZ axiom.

Renormalization. The LSZ reduction formula (0.5) tells us how to reduce the S-matrix elements to the correlation functions. By Sect. 14.2.4, we have to pass to the renormalized correlation functions. Therefore, physicists replace the correlation functions in formula (0.4) by the renormalized correlation functions.

> *This way, physicists get the renormalized scattering functions, and hence the renormalized S-matrix elements.*

Remark. Note the following. Physicists use different (formal) approaches for computing the S-matrix. It turns out that all of these approaches yield the same S-matrix elements, and hence we obtain the same cross sections for scattering processes in (renormalized) perturbation theory. This will be thoroughly studied in the following volumes together with additional physical motivations.

Correction to Volume I, page 263, line 7 from below.

By electronic accident, some sentences are missing.

(begin correction)
(or ambient isotopic) iff there exists an orientation-preserving diffeomorphism

$$F : \mathbb{R}^3 \to \mathbb{R}^3$$

which maps the set $f(\mathbb{S}^1)$ onto $g(\mathbb{S}^1)$. The knot is called an unknot iff it is equivalent to the unit circle.

A link is a finite collection of pairwise disjoint knots. Note that knots are special links. Graphically, links are represented by projections onto a fixed plane with crossings marked as over and under (Fig. 5.23). There exists a complete classification of all knots having at most 16 crossing points. This list comprehends 1, 701, 936 knots.[1] Two links are called equivalent iff they consist of equivalent knots.

> *The main task of knot theory is to decide whether two knots or links are equivalent.*

For example, one wants to know whether a knot is trivial, that is, it is equivalent to the unknot. To this end, one introduces link invariants which possess the characteristic property that equivalent links have the same link invariant. A crucial link invariant is the Jones polynomial ...
(end of correction)

[1] See the introduction to knot theory by Adams (1994) and the article by Hoste et al. (1998).