Eberhard Zeidler

Quantum Field Theory
I: Basics in Mathematics and Physics

A Bridge between Mathematicians and Physicists

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TO THE MEMORY OF JÜRGEN MOSER
(1928–1999)

Preface

Daß ich erkenne, was die Welt im Innersten zusammenhält.[1]
Faust

Concepts without intuition are empty, intuition without concepts is blind.
Immanuel Kant (1724–1804)

The greatest mathematicians like Archimedes, Newton, and Gauss have always been able to combine theory and applications into one.
Felix Klein (1849–1925)

The present comprehensive introduction to the mathematical and physical aspects of quantum field theory consists of the following six volumes:

Volume I: Basics in Mathematics and Physics
Volume II: Quantum Electrodynamics
Volume III: Gauge Theory
Volume IV: Quantum Mathematics
Volume V: The Physics of the Standard Model
Volume VI: Quantum Gravity and String Theory.

Since ancient times, both physicists and mathematicians have tried to understand the forces acting in nature. Nowadays we know that there exist four fundamental forces in nature:

- Newton's gravitational force,
- Maxwell's electromagnetic force,
- the strong force between elementary particles, and
- the weak force between elementary particles (e.g., the force responsible for the radioactive decay of atoms).

In the 20th century, physicists established two basic models, namely,

- the Standard Model in cosmology based on Einstein's theory of general relativity, and
- the Standard Model in elementary particle physics based on gauge theory.

[1] So that I may perceive whatever holds the world together in its inmost folds. The alchemist Georg Faust (1480–1540) is the protagonist of Goethe's drama *Faust* written in 1808.

One of the greatest challenges of the human intellect is the discovery of a unified theory for the four fundamental forces in nature based on first principles in physics and rigorous mathematics. For many years, I have been fascinated by this challenge. When talking about this challenge to colleagues, I have noticed that many of my colleagues in mathematics complain about the fact that it is difficult to understand the thinking of physicists and to follow the pragmatic, but frequently non-rigorous arguments used by physicists. On the other hand, my colleagues in physics complain about the abstract level of the modern mathematical literature and the lack of explicitly formulated connections to physics. This has motivated me to write the present book and the volumes to follow.

It is my intention to build a bridge between mathematicians and physicists.

The main ideas of this treatise are described in the Prologue to this book. The six volumes address a broad audience of readers, including both undergraduate students and graduate students as well as experienced scientists who want to become familiar with the mathematical and physical aspects of the fascinating field of quantum field theory. In some sense, we will start from scratch:

- For students of mathematics, I would like to show that detailed knowledge of the physical background helps to motivate the mathematical subjects and to discover interesting interrelationships between quite different mathematical questions.
- For students of physics, I would like to introduce fairly advanced mathematics which is beyond the usual curriculum in physics.

For historical reasons, there exists a gap between the language of mathematicians and the language of physicists. I want to bridge this gap.[2] I will try to minimize the preliminaries such that undergraduate students after two years of studies should be able to understand the main body of the text. In writing this monograph, it was my goal to follow the advise given by the poet Johann Wolfgang von Goethe (1749–1832):

Textbooks should be attractive by showing the beauty of the subject.

Ariadne's thread. In the author's opinion, the most important prelude to learning a new subject is strong motivation. Experience shows that highly motivated students are willing to take great effort to learn sophisticated subjects.

I would like to put the beginning of Ariadne's thread into the hands of the reader.

[2] On November 7th 1940, there was a famous accident in the U.S.A. which was recorded on film. The Tacoma Narrows Bridge broke down because of unexpected nonlinear resonance effects. I hope that my bridge between mathematicians and physicists is not of Tacoma type.

Remember the following myth. On the Greek island of Crete in ancient times, there lived the monster Minotaur, half human and half bull, in a labyrinth. Every nine years, seven virgins and seven young men had to be sacrificed to the Minotaur. Ariadne, the daughter of King Minos of Crete and Pasiphaë fell in love with one of the seven young men – the Athenian Prince Theseus. To save his life, Ariadne gave Theseus a thread of yarn, and he fixed the beginning of the thread at the entrance of the labyrinth. After a hard fight, Theseus killed the Minotaur, and he escaped from the labyrinth by the help of Ariadne's thread.[3] For hard scientific work, it is nice to have a kind of Ariadne's thread at hand. The six volumes cover a fairly broad spectrum of mathematics and physics. In particular, in the present first volume the reader gets information about

- the physics of the Standard Model of particle physics and
- the magic formulas in quantum field theory,

and we touch the following mathematical subjects:

- finite-dimensional Hilbert spaces and a rigorous approach to the basic ideas of quantum field theory,
- elements of functional differentiation and functional integration,
- elements of probability theory,
- calculus of variations and the principle of critical action,
- harmonic analysis and the Fourier transform, the Laplace transform, and the Mellin transform,
- Green's functions, partial differential equations, and distributions (generalized functions),
- Green's functions, the Fourier method, and functional integrals (path integrals),
- the Lebesgue integral, general measure integrals, and Hilbert spaces,
- elements of functional analysis and perturbation theory,
- the Dirichlet principle as a paradigm for the modern Hilbert space approach to partial differential equations,
- spectral theory and rigorous Dirac calculus,
- analyticity,
- calculus for Grassmann variables,
- many-particle systems and number theory,
- Lie groups and Lie algebras,
- basic ideas of differential and algebraic topology (homology, cohomology, and homotopy; topological quantum numbers and quantum states).

We want to show the reader that many mathematical methods used in quantum field theory can be traced back to classical mathematical problems. In

[3] Unfortunately, Theseus was not grateful to Ariadne. He deserted her on the Island of Naxos, and she became the bride of Dionysus. Richard Strauss composed the opera *Ariadne on Naxos* in 1912.

particular, we will thoroughly study the relation of the procedure of renormalization in physics to the following classical mathematical topics:

- singular perturbations, resonances, and bifurcation in oscillating systems (renormalization in a nutshell on page 625),
- the regularization of divergent infinite series, divergent infinite products, and divergent integrals,
- divergent integrals and distributions (Hadamard's finite part of divergent integrals),
- the passage from a finite number of degrees of freedom to an infinite number of degrees of freedom and the method of counterterms in complex analysis (the Weierstrass theorem and the Mittag–Leffler theorem),
- analytic continuation and the zeta function in number theory,
- Poincaré's asymptotic series and the Ritt theorem in complex analysis,
- the renormalization group and Lie's theory of dynamical systems (one-parameter Lie groups),
- rigorous theory of finite-dimensional functional integrals (path integrals).

The following volumes will provide the reader with important additional material. A summary can be found in the Prologue on pages 11 through 15.

Additional material on the Internet. The interested reader may find additional material on my homepage:

Internet: `www.mis.mpg.de/ezeidler/`

This concerns a carefully structured panorama of important literature in mathematics, physics, history of the sciences and philosophy, along with a comprehensive bibliography. One may also find a comprehensive list of mathematicians, physicists, and philosophers (from ancient until present time) mentioned in the six volumes. My homepage also allows links to the leading centers in elementary particle physics: CERN (Geneva, Switzerland), DESY (Hamburg, Germany), FERMILAB (Batavia, Illinois, U.S.A.), KEK (Tsukuba, Japan), and SLAC (Stanford University, California, U.S.A.). One may also find links to the following Max Planck Institutes in Germany: Astronomy (Heidelberg), Astrophysics (Garching), Complex Systems in Physics (Dresden), Albert Einstein Institute for Gravitational Physics (Golm), Mathematics (Bonn), Nuclear Physics (Heidelberg), Werner Heisenberg Institute for Physics (Munich), and Plasmaphysics (Garching).

Apology. The author apologizes for his imperfect English style. In the preface to his monograph *The Classical Groups*, Princeton University Press, 1946, Hermann Weyl writes the following:

> The gods have imposed upon my writing the yoke of a foreign tongue that was not sung at my cradle.
>
> "Was das heissen will, weiss jeder,
> Der im Traum pferdlos geritten ist,"[4]

[4] Everyone who has dreamt of riding free, without the need of a horse, will know what I mean.

> I am tempted to say with the Swiss poet Gottfried Keller (1819–1890). Nobody is more aware than myself of the attendant loss in vigor, ease and lucidity of expression.

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I hope that the reader of this book enjoys getting a feel for the unity of mathematics and physics by discovering interrelations between apparently completely different subjects.

Leipzig, Fall 2005 *Eberhard Zeidler*