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**A nonlocal anisotropic model
for phase transitions
Part II: Asymptotic behaviour
of rescaled energies**

by

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A Nonlocal Anisotropic Model for Phase Transitions
Part II: Asymptotic Behaviour of Rescaled Energies

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Abstract: we study the asymptotic behaviour as $\varepsilon \to 0$, of the nonlocal models for phase transition described by the scaled free energy

$$F_\varepsilon(u) := \frac{1}{4\varepsilon} \int_{\Omega \times \Omega} J_\varepsilon(x' - x)\big(u(x') - u(x)\big)^2 dx'dx \; + \frac{1}{\varepsilon} \int_\Omega W\big(u(x)\big)\, dx \;,$$

where u is a scalar density function, W is a double-well potential which vanishes at ± 1, J is a non-negative interaction potential and $J_\varepsilon(h) := \varepsilon^{-N} J(h/\varepsilon)$. We prove that the functionals F_ε converge in a variational sense to the anisotropic surface energy

$$F(u) := \int_{Su} \sigma(\nu_u) \;,$$

where u is allowed to take the values ± 1 only, ν_u is the normal to the interface Su between the phases $\{u = +1\}$ and $\{u = -1\}$, and σ is the surface tension. This paper concludes the analisys started in [AB].

Keywords: phase transitions, singular perturbations, Γ-convergence, nonlocal integral functionals.

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1. Introduction and statement of the result

In this paper we study the asymptotic behaviour as $\varepsilon \to 0$ of the functionals F_ε defined by

$$F_\varepsilon(u, \Omega) := \frac{1}{4\varepsilon} \int_{\Omega \times \Omega} J_\varepsilon(x' - x)\big(u(x') - u(x)\big)^2 dx'dx \; + \frac{1}{\varepsilon} \int_\Omega W\big(u(x)\big)\, dx \;, \qquad (1.1)$$

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and of their minimizers subject to suitable constraints. Here u is a real function on the open set $\Omega \subset \mathbb{R}^N$, ε is a positive scaling parameter, $J : \mathbb{R}^N \to [0, +\infty]$ is an even function and $J_\varepsilon(y) := \varepsilon^{-N} J(y/\varepsilon)$, $W : \mathbb{R} \to [0, +\infty)$ is a continuous function which vanishes at ± 1 only (see paragraph 1.2 for precise assumptions).

The function u can be interpreted as a macroscopic density of a scalar intrinsic quantity which describes the configurations of a given system; the second integral in (1.1) forces a minimizing configuration u to take values close to the "pure" states $+1$ and -1 (phase separation), while the first integral represents an interaction energy which penalizes the spatial inhomogeneity of the configuration (surface tension). A relevant example is given in equilibrium Statistical Mechanics by the continuum limit of Ising spin systems on lattices; in that setting u represents a macroscopic magnetization density and J is a ferromagnetic Kac potential (cf. [ABCP] and references therein).

If the parameter ε is small, when we minimize F_ε subject to the volume constraint $\int_\Omega u = c$ with $|c| < \text{vol}(\Omega)$, then the second term in (1.1) prevails; roughly speaking a minimizer u_ε takes values close to -1 or $+1$, and the transition between the two phases occurs in a thin layer with thickness of order ε.

It is therefore natural to consider the asymptotic behaviour of this model as ε tends to 0; accordingly we expect that the minimizers u_ε converge (possibly passing to a subsequence) to a limit function u which takes values ± 1 only. More precisely our main result states the following (see Theorem 1.4 and remarks below): as $\varepsilon \to 0$ the functionals F_ε converge, in the sense of Γ-convergence in $L^1(\Omega)$, to a limit energy F which is finite only when $u = \pm 1$ a.e., and in that case is given by

$$F(u) := \int_{Su} \sigma(\nu_u) \, d\mathscr{H}^{N-1} \, , \tag{1.2}$$

where Su is the interface between the phases $\{u = +1\}$ and $\{u = -1\}$, ν_u if the normal field to Su and σ is a suitable strictly positive even function on $\mathbb{R}^N$; $\mathscr{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdoff measure.

The definition of Γ-convergence immediately implies that the minimizers u_ε of F_ε converge in $L^1(\Omega)$ to minimizers of F. This means that when $\varepsilon \to 0$ the model associated with the energy F_ε converge to the classical van der Waals model for phase separation associated with the (anisotropic) surface tension σ.

A first result of this type was proved in the isotropic case (that is, when J is radially symmetric) in [ABCP], although with a different method and for a particular choice of W only. In the isotropic case $F(u)$ reduces to the measure of the interface Su multiplied by a positive factor σ, which is obtained by taking a suitable (unscaled) one-dimensional functional $\overline{F}(v, \mathbb{R})$ of type (1.1) and computing its infimum over all $v : \mathbb{R} \to [-1, 1]$ which tend to $+1$ at $+\infty$ and to -1 at $-\infty$.

We show that in the general anisotropic case the value of the function σ at some unit vector e is obtained by solving a certain minimum problem on an unbounded N-dimensional stripe; this is called the optimal profile problem associated with the direction e (see paragraph 1.3). In [AB] it was proved that for every e such a problem admits at least one solution $u : \mathbb{R}^N \to [-1, 1]$ which is constant with respect to all directions orthogonal to e.

We emphasize that the proof of the Γ-convergence theorem does not depend on this existence result (cf. Remark 1.10). Indeed our result can also be extended to the multi-phase case, that is, when u takes values in $\mathbb{R}^m$ and W is a function on $\mathbb{R}^m$ which vanishes at finitely many affinely independent points (see paragraph 1.12), while so far there are no existence results for the corresponding optimal profile problem (cf. [AB], section 4b).

Finally we underline that the assumption $J \geq 0$, which in the statistical model is addressed as the *ferromagnetic assumption*, is crucial in all our proofs, while the other assumptions on J and W are close to be optimal (see paragraphs 1.2 and 1.13).

For a fixed positive ε our model closely recall the Cahn-Hilliard model for phase separation (see [CH]), which is described by the energy functional

$$I_\varepsilon(u) := \frac{\varepsilon}{2} \int_\Omega |\nabla u|^2 + \frac{1}{\varepsilon} \int_\Omega W(u) \ . \tag{1.3}$$

Indeed F_ε can be obtained from I_ε by replacing the term $\big|\nabla u(x)\big|$ in the first integral in (1.3) with the average of the finite differences $\frac{1}{\varepsilon}\big|u(x + \varepsilon h) - u(x)\big|$ with respect to the measure distribution $J(h)\,dh$.

The Γ-convergence of the functionals I_ε to a limit (isotropic) energy of the form (1.2) was proved by L. Modica and S. Mortola (see [MM], [Mo]) and then extended to more general anisotropic functionals in [Bou], [OS]. It is important to recall that the term $\int |\nabla u|^2$ in (1.3) was derived in [CH] as a first order approximation of a more general and complicated quadratic form. Indeed our result shows that the asymptotic behaviour of these functionals is largely independent of the choice of this quadratic form. A first result in this direction was already obtained in [ABS]; the one-dimensional functionals considered there, were defined as in (1.1) with $J(h) = 1/h^2$ (see also paragraph 1.13 below).

We recall here that the evolution model associated with the energy F_ε is described, after a suitable time scaling, by the nonlocal parabolic equation

$$u_t = \varepsilon^{-2}\big(J_\varepsilon * u - u - f(u)\big) \ , \tag{1.4}$$

where f is the derivative of W and we assumed $\|J\|_1 = 1$; the analog for the energy I_ε is the scaled Allen-Cahn equation $u_t = \triangle u - \varepsilon^{-2}f(u)$. The asymptotic behaviour of the solutions of (1.4) has been widely studied in the isotropic case (see for instance [DOPT1-3], [KS1-2]) and leads to a motion by mean curvature in the sense of viscosity solutions; the generalization to the anisotropic case has been given in [KS3]. Analogous results have been proved for the solutions of the scaled Allen-Cahn equation (see for instance [BK], [DS], [ESS], [Ilm]).

An interesting mathematical feature of the functional F_ε is that they are not local; this means that given disjoint sets A and A', the energy $F_\varepsilon(u, A \cup A')$ stored in $A \cup A'$ is strictly larger than the sum of $F_\varepsilon(u, A)$ and $F_\varepsilon(u, A')$. More precisely we have

$$F_\varepsilon(u, A \cup A') = F_\varepsilon(u, A) + F_\varepsilon(u, A') + 2\Lambda_\varepsilon(u, A, A') \ , \tag{1.5}$$

where the *locality defect* $\Lambda_\varepsilon(u, A, A')$ is defined as

$$\Lambda_\varepsilon(u, A, A') := \frac{1}{4\varepsilon} \int_{A \times A'} J_\varepsilon(x' - x)\big(u(x') - u(x)\big)^2 dx'dx \tag{1.6}$$

for every $A, A' \subset \mathbb{R}^N$ and every $u : A \cup A' \to \mathbb{R}$. The meaning of $\Lambda_\varepsilon(u, A, A')$ is quite clear, since it represents the scaled interaction energy between A and A'. In this paper we always assume a proper decay of J at infinity, namely (1.8), in order to guarantee that $\Lambda_\varepsilon(u, A, A')$ vanishes as $\varepsilon \to 0$ whenever the sets A and A' are distant.

Before passing to precise statements, we need to fix some general notation.

In the following Ω is a bounded open subset of $\mathbb{R}^N$, and it is called *regular* when has Lipschitz boundary (for $N = 1$, when it is a finite union of distant open intervals). Unless differently stated all sets and functions are assumed to be Borel measurable.

Every set in $\mathbb{R}^N$ is usually endowed with the Lebesgue measure $\mathscr{L}_N$, and we simply write $\int_B f(x)\,dx$ for the integrals over B and $|B|$ for $\mathscr{L}_N(B)$, while we never omit an explicit mention

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of the measure when it differs from $\mathscr{L}_N$. As usual $\mathscr{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure.

1.1. BV functions and sets of finite perimeter

For every open set Ω in $\mathbb{R}^N$, $BV(\Omega)$ denotes the space of all functions $u : \Omega \to \mathbb{R}$ *with bounded variation*, that is, the functions $u \in L^1(\Omega)$ whose distributional derivative Du is represented by a bounded $\mathbb{R}^N$-valued measure on Ω. We denote by $BV(\Omega, \pm 1)$ the class of all $u \in BV(\Omega)$ which take values ± 1 only. For every function u on Ω, Su is the set of all essential singularities, that is, the points of Ω where u has no approximate limit; if $u \in BV(\Omega)$ the set Su is *rectifiable*, and this means that it can be covered up to an $\mathscr{H}^{N-1}$-negligible subset by countably many hypersurfaces of class C^1.

The essential boundary of a set $E \subset \mathbb{R}^N$ is the set $\partial_* E$ of all points in Ω where E has neither density 1 nor density 0. A set $E \subset \Omega$ has finite perimeter in Ω if its characteristic function 1_E belongs to $BV(\Omega)$, or equivalently, if $\mathscr{H}^{N-1}(\partial_* E \cap \Omega)$ is finite; in this case $\partial_* E$ is rectifiable, and we may endow it with a measure theoretic normal ν_E (defined up to $\mathscr{H}^{N-1}$-negligible subsets) so that the measure derivative $D1_E$ is represented as

$$D1_E(B) = \int_{\partial_* E \cap B} \nu_E \, d\mathscr{H}^{N-1} \quad \text{for every } B \subset \Omega.$$

A function $u : \Omega \to \pm 1$ belongs to $BV(\Omega, \pm 1)$ if and only if $\{u = +1\}$ (or $\{u = -1\}$ as well) has finite perimeter in Ω. In this case Su agrees with the intersection of the essential boundary of $\{u = +1\}$ with Ω, and the previous formula becomes

$$Du(B) := 2 \int_{Su \cap B} \nu_u \, d\mathscr{H}^{N-1} \quad \text{for every } B \subset \Omega, \tag{1.7}$$

where ν_u is a suitable normal field to Su. We claim that Su is the interface between the phases $\{u = +1\}$ and $\{u = -1\}$ in the sense that it contains every point where both sets have density different than 0. For further results and details about BV functions and finite perimeter sets we refer the reader to [EG], chapter 5.

1.2. Hypotheses on J and W

Unless differently stated, the interaction potential J and the double-well potential W which appear in (1.1) satisfy the following assumptions:

(i) $J : \mathbb{R}^N \to [0, +\infty)$ is an even function (i.e., $J(h) = J(-h)$) in $L^1(\mathbb{R}^N)$ and satisfies

$$\int_{\mathbb{R}^N} J(h) \, |h| \, dh < \infty \ . \tag{1.8}$$

(ii) $W : \mathbb{R} \to [0, +\infty)$ is a continuous function which vanishes at ± 1 only and has at least linear growth at infinity (cf. the proof of Lemma 1.11).

1.3. The optimal profile problem and the definition of σ

We first define the auxiliary unscaled functional $\mathscr{F}$ by

$$\mathscr{F}(u, A) := \frac{1}{4} \int_{x \in A, \, h \in \mathbb{R}^N} J(h) \left(u(x+h) - u(x) \right)^2 dx \, dh \ + \int_{x \in A} W \left(u(x) \right) dx \tag{1.9}$$

for every set $A \subset \mathbb{R}^N$ and every $u : \mathbb{R}^N \to \mathbb{R}$. Hence $\mathscr{F}(u, A) = F_1(u, A) + \Lambda_1(u, A, \mathbb{R}^N \setminus A)$.

Figure 1: the sets C, Q_C and T_C.

A function $u : \mathbb{R}^N \to \mathbb{R}$ is called C-periodic if $u(x + re_i) = u(x)$ for every x and every $i = 1, \ldots, N-1$, where r is the side length of C and $e_1, \ldots, e_{N-1}$ are its axes. We denote by $X(C)$ the class of all functions $u : \mathbb{R}^N \to [-1, 1]$ which are C-periodic and satisfy

$$\lim_{x_e \to +\infty} u(x) = +1 \quad \text{and} \quad \lim_{x_e \to -\infty} u(x) = -1 \ , \tag{1.10}$$

and finally we set

$$\sigma(e) := \inf \left\{ |C|^{-1} \mathscr{F}(u, T_C) : C \in \mathscr{C}_e, \ u \in X(C) \right\} \ . \tag{1.11}$$

The minimum problem (1.11) is called the *optimal profile problem* associated with the direction e, and a solution is called an *optimal profile for transition in direction e*. In [AB] we proved that the minimum in (1.11) is attained, and there exists at least one minimizer u which depends only on the variable x_e, and more precisely $u(x) = \gamma(x_e)$ where $\gamma : \mathbb{R} \to [-1, 1]$ is the optimal profile associated with a certain one-dimensional functional F^e.

For the rest of this section Ω is a fixed regular open subset of $\mathbb{R}^N$.

Theorem 1.4. *Under the previous assumptions the following three statements hold:*

(i) Compactness: *let be given sequences (ε_n) and $(u_n) \subset L^1(\Omega)$ such that $\varepsilon_n \to 0$, and $F_{\varepsilon_n}(u_n, \Omega)$ is uniformly bounded; then the sequence (u_n) is relatively compact in $L^1(\Omega)$ and each of its cluster points belongs to $BV(\Omega, \pm 1)$.*

(ii) Lower bound inequality: *for every $u \in BV(\Omega, \pm 1)$ and every sequence (u_ε) such that $u_\varepsilon \to u$ in $L^1(\Omega)$, we have*
$$\liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon, \Omega) \geq F(u) \ ;$$

(iii) Upper bound inequality: *for every $u \in BV(\Omega, \pm 1)$ there exists a sequence (u_ε) such that $u_\varepsilon \to u$ in $L^1(\Omega)$ and*
$$\limsup_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon, \Omega) \leq F(u) \ .$$

Remark 1.5. Statements (ii) and (iii) of Theorem 1.4 can be rephrased by saying that the functionals $F_\varepsilon(\cdot, \Omega)$, or in short F_ε, Γ-converge in the space $L^1(\Omega)$ to the functional F given by (1.2) for all functions $u \in BV(\Omega, \pm 1)$ and extended to $+\infty$ in $L^1(\Omega) \setminus BV(\Omega, \pm 1)$.

For the general theory of Γ-convergence we refer the reader to [DM]; for the applications of Γ-convergence to phase transition problems we refer to the early paper of Modica [Mo], and to [Al] for a review of some of these results and related mathematical issues.

Remark 1.6. As every Γ-limit is lower semicontinuous, we infer from the previous remark that the functional F given in (1.2) is weakly* lower semicontinuous and coercive on $BV(\Omega, \pm 1)$.

The coercivity of F implies that the infimum of $\sigma(e)$ over all unit vectors $e \in \mathbb{R}^N$ is strictly positive, while the semicontinuity implies that the 1-homogeneous extension of the function σ to $\mathbb{R}^N$, namely the function $x \mapsto |x|\, \sigma(x/|x|)$, is convex (see for instance [AmB], Theorem 3.1). Notice that it is not immediate to recover this convexity result directly from the definition of σ in (1.11).

Remark 1.7. Statement (iii) of Theorem 1.4 can be refined by choosing the approximating sequence (u_ε) so that $\int_\Omega u_\varepsilon = \int_\Omega u$ for every ε (we will not prove this refinement of statement (iii); in fact one has to slightly modify the construction of the approximating sequence (u_ε) given in Theorem 5.2). This way we can fit with a prescribed volume constraint: given $c \in (-|\Omega|, |\Omega|)$, then the functionals F_ε Γ-converge to F also on the class Y_c of all $u \in L^1(\Omega)$ which satisfy the volume constraint $\int_\Omega u = c$.

A sequence (v_ε) in Y_c is called a *minimizing sequence* if v_ε minimizes $F_\varepsilon(\cdot, \Omega)$ in Y_c for every $\varepsilon > 0$, and is called a *quasi-minimizing sequence* if $F_\varepsilon(v_\varepsilon, \Omega) = \inf\{F_\varepsilon(u, \Omega) : u \in Y_c\} + o(1)$. Using the semicontinuity result given in [AB], Theorem 4.7, and the truncation argument given in Lemma 1.11 below, we can prove that a minimizer of $F_\varepsilon(\cdot, \Omega)$ in Y_c exists provided that W is of class C^2 and $\ddot{W}(t) \geq -d_\varepsilon$ for every $t \in [-1, 1]$, where d_ε is defined by

$$d_\varepsilon := \operatorname*{ess\,inf}_{x \in \Omega} \frac{1}{2} \int_\Omega J_\varepsilon(x' - x)\, dx' \ .$$

Notice that d_ε tends to $\frac{1}{4}\|J\|_1$ as $\varepsilon \to 0$.

Now by a well-known property of Γ-convergence and by statement (i) of Theorem 1.4 we infer the following:

Corollary 1.8. *Let (v_ε) be a minimizing or a quasi-minimizing sequence for F_ε on Y_c. Then (v_ε) is relatively compact in $L^1(\Omega)$, and every cluster point v minimizes F among all functions $u \in BV(\Omega, \pm 1)$ which satisfy $\int_\Omega u = c$. Equivalently, the set $E := \{v = 1\}$ solves the minimum problem*

$$\min\left\{ \int_{\partial_* E} \sigma(\nu_E)\, d\mathcal{H}^{N-1} : \ E \text{ has finite perimeter in } \Omega \text{ and } |E| = \tfrac{1}{2}(c + |\Omega|) \right\} \ .$$

1.9. Outline of the proof of Theorem 1.4 for N = 1

In order to explain the idea of the proof of Theorem 1.4 and the connection with the optimal profile problem, now we briefly sketch the proof of statement (ii) and (iii) for the one-dimensional case (the proof of statement (i) being slightly more delicate).

In this case σ becomes the infimum of $F_1(\cdot, \mathbb{R})$ over the class X of all $u : \mathbb{R} \to [-1, 1]$ which converge to $+1$ at $+\infty$ and to -1 at $-\infty$ (cf. (1.11)). We assume for simplicity that Ω is the interval $(-1, 1)$, and that $u(x) = -1$ for $x < 0$, $u(x) = +1$ for $x \geq 0$. Then $Su = \{0\}$, and $\sigma \mathcal{H}^0(Su) = \sigma$; a standard localization argument can be used to prove the result in the general case (cf. [Al], section 3a).

We first remark that the functionals F_ε satisfy the following rescaling property: given $\varepsilon > 0$ and $u : \mathbb{R} \to \mathbb{R}$ we set $u^\varepsilon(x) := u(\varepsilon x)$, and then a direct computation gives

$$F_\varepsilon(u, \mathbb{R}) = F_1(u^\varepsilon, \mathbb{R}) \ . \tag{1.12}$$

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Let us consider now the lower bound inequality. First we reduce to a sequence (u_ε) which converges to u in $L^1(\Omega)$ and satisfies $|u_\varepsilon| \leq 1$; then we extend each u_ε to the rest of $\mathbb{R}$ by setting $u_\varepsilon(x) := -1$ for $x \leq -1$, $u_\varepsilon(x) := 1$ for $x \geq 1$. The key point of the proof is to show that

$$F_\varepsilon(u_\varepsilon, \Omega) \simeq F_\varepsilon(u_\varepsilon, \mathbb{R}) \quad \text{as } \varepsilon \to 0. \tag{1.13}$$

By identity (1.5), (1.13) can be written in term of the locality defect Λ_ε, and more precisely it reduces to $\Lambda_\varepsilon(u_\varepsilon, \Omega, \mathbb{R} \setminus \Omega) = o(1)$; notice that in general this equality may be false, but using the decay estimates for the locality defect given in section 2 we can prove that it is true if we replace Ω with another interval, which may be chosen arbitrarily close to Ω.

By (1.12) and the definition of σ we get $F_\varepsilon(u_\varepsilon, \mathbb{R}) = F_1(u_\varepsilon^\varepsilon, \mathbb{R}) \geq \sigma$, and then (1.13) yields

$$\liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon, \Omega) \geq \sigma .$$

The proof of the upper bound inequality is even more simple: we take an optimal profile γ (i.e., a solution of the minimum problem which defines σ) and we set $u_\varepsilon(x) := \gamma(x/\varepsilon)$ for every $\varepsilon > 0$. Then $u_\varepsilon(x)$ converge to $u(x)$ for every $x \neq 0$, and (1.12) yields

$$F_\varepsilon(u_\varepsilon, \Omega) \leq F_\varepsilon(u_\varepsilon, \mathbb{R}) = F_1(\gamma, \mathbb{R}) = \sigma .$$

Remark 1.10. It is clear from this brief sketch that the shape of the optimal profile plays no rôle in the proof of Theorem 1.4, nor it does the fact that the minimum in (1.11) is attained: in case no optimal profiles were available, it would suffices to replace γ with functions in X which "almost" minimize $F_1(\cdot, \mathbb{R})$. This could be indeed the case when one considers the vectorial version of this problem (see paragraph 1.12)

Nevertheless the existence of the optimal profile has a deeper meaning than it appears above. Indeed if (v_ε) is a sequence of minimizers of F_ε which converges to some $v \in BV(\Omega, \pm 1)$, then we would expect that if we blow-up the functions v_ε at some fixed singular point $\bar{x}$ of v by taking the functions $\gamma_\varepsilon(x) := v_\varepsilon(\varepsilon(x - \bar{x}))$, then γ_ε "resembles" more and more an optimal profile. In other words we expect the optimal profiles to be the asymptotic shapes of the minimizers v_ε about the discontinuity points of v. Yet a precise statement in this direction is far beyond the purposes of this paper.

Concerning proofs, statements (i), (ii), and (iii) of Theorem 1.4 will be proved in sections 3, 4, and 5 respectively, while section 2 is devoted to the asymptotic estimates for the locality defect Λ_ε. In section 6 we study the relation between the Dirichlet integral $\int_\Omega |\nabla u|^2$ and the interaction energy

$$G_\varepsilon(u) := \int_{\Omega \times \Omega} J_\varepsilon(x' - x) \left(\frac{u(x') - u(x)}{\varepsilon} \right)^2 dx' dx .$$

Warning: throughout the proof of Theorem 1.4, that is, in all sections from 2 to 5, we will always restrict ourselves to functions which take values in $[-1, 1]$. We are allowed to do this in view of the following lemma:

Lemma 1.11. *For every function $u : \Omega \to \mathbb{R}$, let Tu denote the truncated function $Tu(x) := (u(x) \wedge 1) \vee -1$. Then $F_\varepsilon(u, \Omega) \geq F_\varepsilon(Tu, \Omega)$ for every $\varepsilon > 0$, and for every sequence (u_ε) such that $F_\varepsilon(u_\varepsilon, \Omega)$ is bounded in ε there holds $\|u_\varepsilon - Tu_\varepsilon\|_1 \to 0$ as $\varepsilon \to 0$.*

Proof. The inequality $F_\varepsilon(u, \Omega) \geq F_\varepsilon(Tu, \Omega)$ is immediate. Let now be given a sequence (u_ε) such that $F_\varepsilon(u_\varepsilon, \Omega) \leq C$ for every ε.

Since W is strictly positive and continuous out of ± 1, and has growth at least linear at infinity (see paragraph 1.2), for every $\delta > 0$ we may find $a > 0$, $M > 0$ and $b > 0$ so that

$W(t) \geq a$ when $1 + \delta \leq |t| \leq M$ and $W(t) \geq b|t|$ when $M \leq |t|$. Then we define A_ε and B_ε as the sets of all $x \in \Omega$ where $u_\varepsilon(x)$ satisfies respectively $1 + \delta \leq |u_\varepsilon(x)| \leq M$ and $M \leq |u_\varepsilon(x)|$. Hence

$$\|u_\varepsilon - T u_\varepsilon\|_1 \leq \delta|\Omega| + M|A_\varepsilon| + \int_{B_\varepsilon} |u_\varepsilon| \leq \delta|\Omega| + \left(\frac{M}{a} + \frac{1}{b}\right) \int_\Omega W(u_\varepsilon) \ .$$

Since $\int_\Omega W(u_\varepsilon) \leq C\varepsilon$, passing to the limit as $\varepsilon \to 0$ we obtain that $\limsup \|u_\varepsilon - T u_\varepsilon\|_1 \leq \delta|\Omega|$, and since δ can be taken arbitrarily small the proof is complete. $\qquad\square$

We conclude this section with a short overlook of the possible generalizations of Theorem 1.4 and some open problems.

1.12. The multi-phase model

In order to describe a multi-phase system one may postulate a free energy of the form (1.1) where u is a *vector* density function on a domain of $\mathbb{R}^N$ taking values in $\mathbb{R}^m$, $W : \mathbb{R}^m \to [0, \infty)$ is a continuous function which vanishes at $k+1$ affinely independent wells $\{\alpha_0, \dots, \alpha_k\}$ (and therefore $k \leq m$), and J is the usual interaction potential.

Theorem 1.4 holds provided we make the following modifications: $BV(\Omega, \pm 1)$ is replaced by the class $BV(\Omega, \{\alpha_i\})$ of all functions $u \in BV(\Omega, \mathbb{R}^m)$ which takes values in $\{\alpha_0, \dots, \alpha_k\}$ only, and the functional F is now defined by

$$F(u) := \sum_{i<j} \int_{S_{ij}} \sigma_{ij}(\nu_{ij}) \, d\mathscr{H}^{N-1} \ , \tag{1.14}$$

where S_{ij} is the interface which separates the phases $\{u = \alpha_i\}$ and $\{u = \alpha_j\}$, and precisely $S_{ij} := \partial_*\{u = \alpha_i\} \cap \partial_*\{u = \alpha_j\} \cap \Omega$ (recall that both phases have finite perimeter in Ω), and ν_{ij} is the measure theoretic normal to S_{ij}. For every unit vector e the value $\sigma_{ij}(e)$ is defined by the following version of the optimal profile problem:

$$\sigma_{ij}(e) := \inf \left\{ |C|^{-1} \mathscr{F}(u, T_C) : \ C \in \mathscr{C}_e, \ u \in X^{ij}(C) \right\} \ , \tag{1.15}$$

here we follow the notation of paragraph 1.3 and $X^{ij}(C)$ is the class of all functions $u : \mathbb{R}^N \to \mathbb{R}^m$ which are C-periodic and satisfy the boundary condition

$$\lim_{x_e \to +\infty} u(x) = \alpha_j \quad \text{and} \quad \lim_{x_e \to -\infty} u(x) = \alpha_i \ .$$

This vectorial generalization of Theorem 1.4 can be proved by adapting the proof for the scalar case given below, and using a suitable approximation result for the functions in $BV(\Omega, \{\alpha_i\})$ (cf. the approach in [Ba] for the vectorial version of the Modica-Mortola theorem).

Notice that in this case it is not known whether the optimal profile problem (1.15) admits a solution or not (cf. [AB], section 4b).

1.13. The optimal assumptions on J

As already remarked, in the current approach the ferromagnetic condition $J \geq 0$ cannot be removed; more precisely it plays an essential rôle in the proof of statement (i) of Theorem 1.4, and in particular in the first step of the proof of Theorem 3.1 (on the other hand the proofs of statements (ii) and (iii) do not require the non-negativity of J).

Yet it would be quite interesting to understand the asymptotic behaviour of the functionals F_ε when J is allowed to take also negative values. As far as we know in that case it may well happen that different scalings should be considered for the functional F_ε, and that the Γ-limit has a completely different form.

About the growth assumptions on J, we can replace the hypotheses in paragraph 1.3, namely $J \in L^1(\mathbb{R}^N)$ and (1.8), with the following more general ones (cf. [AB], section 4c): J is even, non-negative, and satisfies

$$\int_{\mathbb{R}^N} J(h)\left(|h| \wedge |h|^2\right) dh < +\infty .$$ (1.16)

We remark that the proof of Theorem 1.4 needs no modifications at all if J does not belong to $L^1(\mathbb{R}^N)$ but still verifies (1.8), while some additional cares have to be taken in the fully general case, and more precisely in the proof of stament (iii) (see in particular the third step in the proof of Theorem 5.2 and the decay of the locality defect in Lemma 2.7). Indeed statements (i) and (ii) can always be recovered from the usual version of Theorem 1.4 by approximating J with an increasing sequence of potentials which satisfy the assumptions in paragraph 1.2.

Finally we notice that if (1.16) does not hold, then the value of $\sigma(e)$ as given by the optimal profile problem (1.11) is always equal to $+\infty$ (cf. [AB], Theorem 4.6). This probably means that a different scaling should be considered in the definition of the functionals F_ε. For instance, if $N = 1$ and $J(h) = 1/h^2$ the "right" scaling is given by

$$\varepsilon \int_{\Omega \times \Omega} \left|\frac{u(x') - u(x)}{x' - x}\right|^2 dx' dx \ + e^{1/\varepsilon} \int_\Omega W\big(u(x)\big)\, dx \ ,$$

or equivalently by multiplying the functionals F_ε defined in (1.1) by an infinitesimal factor of order $|\log \varepsilon|^{-1}$. In this case we obtain again a Γ-limit of the form (1.2) (see for instance [ABS]). However no general result is available when J does not verifies (1.16).

2. Decay estimates for the locality defect.

In this section we study the asymptotic behaviour as ε tends to 0 of the locality defect Λ_ε. Roughly speaking, the goal is to show that the limit of $\Lambda_\varepsilon(u_\varepsilon, A, A')$ is determined only by the asymptotic behaviour of the sequence u_ε close to the intersection of the boundaries of A and A'. The main result of this section is Theorem 2.8.

We first need to fix some additional notation. We define the auxiliary potential $\hat{J}$ by

$$\hat{J}(h) := \int_0^1 J\left(\frac{h}{t}\right) \left|\frac{h}{t}\right| \frac{dt}{t^N} \qquad \text{for every } h \in \mathbb{R}^N.$$ (2.1)

It follows immediately from the definition that $\hat{J}$ is even, non-negative, and satisfies

$$\|\hat{J}\|_1 = \int_{\mathbb{R}^N} J(h)\, |h|\, dh .$$ (2.2)

Definition 2.1. Throughout this section Σ always denotes a subset of a Lipschitz hypersurface in $\mathbb{R}^N$, and is endowed with the Hausdorff measure $\mathscr{H}^{N-1}$; we often omit any explicit mention to this measure.

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Let A be a set of positive measure in $\mathbb{R}^N$, and take a sequence (u_n) of functions from A into $[-1, 1]$, and a sequence (ε_n) of positive real numbers which tends to 0. We say that the ε_n-*traces of* u_n (relative to A) *converge on* Σ *to* $v : \Sigma \to [-1, 1]$ when

$$\lim_{n \to \infty} \int_{y \in \Sigma} \left[\int_{\{h \,:\, y + \varepsilon_n h \in A\}} \hat{J}(h) \left| u_n(y + \varepsilon_n h) - v(y) \right| dh \right] dy = 0. \tag{2.3}$$

Remark 2.2. Notice that we make no assumption on the relative position of A and Σ; in particular they may be even distant. Notice moreover that the notion of "convergence of the ε_n-traces" is introduced without defining what the ε_n-trace of a function is, and in fact there is no such notion. This is due to the fact that for functions in the domain of F_ε, the trace on an $(N-1)$-dimensional manifold cannot be defined (while it is defined for functions in the domain of the Γ-limit, that is, for BV functions).

In view of the definition of the locality defect, it would make more sense to replace the term $\left| u_n(y + \varepsilon_n h) - v(y) \right|$ in (2.3) with its square. But since we restrict ourselves to functions which take values in $[-1, 1]$, the limit in (2.3) is independent of the exponent of $\left| u_n(y + \varepsilon_n h) - v(y) \right|$, and we chose 1 because it simplifies many of the following proofs.

Remark 2.3. We can define the upper $\hat{J}$-density of A at the point $x \in \mathbb{R}^N$ as the upper limit

$$\limsup_{\varepsilon \to 0} \int_{\{h \,:\, x + \varepsilon h \in A\}} \hat{J}(h) \, dh \;,$$

and the lower $\hat{J}$-density as the corresponding lower limit; notice that such densities are local, that is, they do not depend on the behaviour of A out of any open neighborhood of x.

The function v which satisfies (2.3) is uniquely determined for ($\mathscr{H}^{N-1}$-) almost every point of Σ where A has positive J-upper density.

If (2.3) holds for some set A, then it is verified by every A' included in A. Moreover if Σ has finite measure then (2.3) is also verified by every A' such that $A' \setminus A$ has upper $\hat{J}$-density 0 at almost every point of Σ. In particular if are given sets A and A' such that the symmetric difference $A \triangle A'$ has upper $\hat{J}$-density 0 at almost every point of Σ, then A satisfies (2.3) if and only A' does.

Remark 2.4. Condition (2.3) is not easy to verify. If Σ has finite measure then (2.3) holds when

$$\lim_{n \to \infty} u_n(y + \varepsilon_n h) = v(y) \quad \text{for a.e. } y \in \Sigma \text{ and a.e. } h \in A. \tag{2.4}$$

Condition (2.4) holds for instance when u_n converge locally uniformly on some open neighborhood of Σ to a function which, at every point of Σ, is continuous and agrees with v.

Assume now that the functions u_n converge to u in $L^1(A)$. Unfortunately this is not enough to deduce that the ε_n-traces of u_n converge to u on every Lipschitz hypersurface $\Sigma \subset \mathbb{R}^N$, yet this holds for "most" Σ. More precisely, we have the following proposition:

Proposition 2.5. *Take A, (ε_n) and (u_n) as in Definition 2.1; let $g : A \to \mathbb{R}$ be a Lipschitz function, and denote by Σ^t the t-level set of g for every $t \in \mathbb{R}$. If $u_n \to u$ in $L^1(A)$ then, possibly passing to a subsequence, the ε_n-traces of u_n (relative to A) converge to u on Σ^t for a.e. $t \in \mathbb{R}$.*

(Since g admits a Lipschitz extension to $\mathbb{R}^N$, Σ^t is a subset of an oriented closed Lipschitz hypersurface in $\mathbb{R}^N$ for almost every $t \in \mathbb{R}$.)

Proof. To simplify the notation we write $\varepsilon, u_\varepsilon$ instead of ε_n, u_n, we assume that g is 1-Lipschitz and $A = \mathbb{R}^N$ (the general case follows in the same way). For every $\varepsilon > 0$, $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$ we set

$$\Phi_\varepsilon(x) := \int_{\mathbb{R}^N} \hat{J}(h) \left| u_\varepsilon(x + \varepsilon h) - u(x) \right| dh \qquad \text{and} \qquad g_\varepsilon(t) := \int_{\Sigma^t} \Phi_\varepsilon(x)\, dx \ . \tag{2.5}$$

By the coarea formula for Lipschitz functions (see [EG], section 3.3) we get

$$\begin{aligned}
\int_{\mathbb{R}} g_\varepsilon(t)\, dt &= \int_{\mathbb{R}^N} \Phi_\varepsilon(x) \left| \nabla g(x) \right| dx \le \int_{\mathbb{R}^N} \Phi_\varepsilon(x)\, dx \\
&= \int_{\mathbb{R}^N \times \mathbb{R}^N} \hat{J}(h) \left| u_\varepsilon(x + \varepsilon h) - u(x) \right| dx\, dh \\
&\le \int_{\mathbb{R}^N \times \mathbb{R}^N} \hat{J}(h) \left[\left| u_\varepsilon(x + \varepsilon h) - u(x + \varepsilon h) \right| + \left| u(x + \varepsilon h) - u(x) \right| \right] dx\, dh \\
&\le \int_{\mathbb{R}^N} \hat{J}(h) \left[\|u_\varepsilon - u\|_1 + \|\tau_{\varepsilon h} u - u\|_1 \right] dh \ ,
\end{aligned} \tag{2.6}$$

where $\tau_{\varepsilon h} u(x) := u(x + \varepsilon h)$.

Now $\|u_\varepsilon - u\|_1$ tends to 0 by assumption and $\|\tau_{\varepsilon h} u - u\|_1$ tends to 0 as $\varepsilon \to 0$ for every h, and since $\hat{J}$ is summable (cf. (2.2)) we can apply the dominated convergence theorem to the integrals in line (2.6), and we get

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}} g_\varepsilon(t)\, dt = 0 \ .$$

Hence the functions g_ε converge to 0 in $L^1(\mathbb{R})$, and passing to a subsequence we may assume that they also converge pointwise to 0 for a.e. $t \in \mathbb{R}$. Since $g_\varepsilon(t)$ is equal to the double integral in (2.3) (with v replaced by u), the proof is complete. $\qquad\square$

Definition 2.6. Let be given $A, A' \subset \mathbb{R}^N$. We say that the set Σ *divides A and A'* when for every $x \in A$, $x' \in A'$ the segment $[x, x']$ intersects Σ. We say that Σ *strongly divides A and A'* when Σ is the (Lipschitz) boundary of some open set Ω such that $A \subset \overline{\Omega}$ and $A' \subset \mathbb{R}^N \setminus \Omega$.

Now we can state and prove the first decay estimate for the locality defect. Let be given disjoint sets A and A' in $\mathbb{R}^N$ which are divided by Σ, then take positive numbers $\varepsilon_n \to 0$ and functions $u_n : A \cup A' \to [-1, 1]$ and $v, v' : \Sigma \to [-1, 1]$.

Lemma 2.7. *Under the above stated hypotheses, if the ε_n-traces of u_n relative to A and A' converge on Σ to v and v' respectively, then*

$$\limsup_{n \to \infty} \Lambda_{\varepsilon_n}(u_n, A, A') \le \frac{1}{2} \|\hat{J}\|_1 \int_\Sigma \left| v(y) - v'(y) \right| dy \ . \tag{2.7}$$

Proof. To simplify the notation we write ε, u_ε and Λ_ε instead of ε_n, u_n, Λ_{ε_n}. By the definition of Λ_ε, and recalling that $|u_\varepsilon| \le 1$, we obtain

$$\Lambda_\varepsilon(u_\varepsilon, A, A') \le \frac{1}{2\varepsilon} \int_{\mathbb{R}^N} J(h) \underbrace{\left[\int_{A_{\varepsilon h}} \left| u_\varepsilon(x + \varepsilon h) - u_\varepsilon(x) \right| dx \right]}_{I_\varepsilon(h)} dh \ , \tag{2.8}$$

Figure 2: the set $A_{\varepsilon h}$ for given $\varepsilon > 0$ and $h \in \mathbb{R}^N$.

Since the Jacobian determinant of the map which takes $(y, t) \in \Sigma \times [0, 1]$ into $y - t\varepsilon h$ does not exceed $\varepsilon |h|$, by applying the change of variable $x = y - t\varepsilon h$ we get

$$I_\varepsilon(h) \leq \varepsilon |h| \int_\Sigma \Big[\int_{S_{hy}} \big| u_\varepsilon(y + (1 - t)\varepsilon h) - u_\varepsilon(y - t\varepsilon h) \big| \, dt \Big] dy \ ,$$

where S_{hy} is the set of all $t \in [0, 1]$ such that $y - t\varepsilon h \in A$ and $y + (1 - t)\varepsilon h \in A'$. Hence (2.8) yields

$$\Lambda_\varepsilon(u_\varepsilon, A, A') \leq \frac{1}{2} \int\limits_{h \in \mathbb{R}^N, \, y \in \Sigma} J(h) \, |h| \Big[\int\limits_{S_{hy}} \big| u_\varepsilon(y + (1 - t)\varepsilon h) - u_\varepsilon(y - t\varepsilon h) \big| \, dt \Big] dy \, dh \ . \qquad (2.9)$$

Now by the triangle inequality we can estimate $\big| u_\varepsilon(y + (1 - t)\varepsilon h) - u_\varepsilon(y - t\varepsilon h) \big|$ by the sum of the following three terms:

$$\big| v(y) - v'(y) \big| + \big| u_\varepsilon(y - t\varepsilon h) - v'(y) \big| + \big| u_\varepsilon(y + (1 - t)\varepsilon h) - v(y) \big| \ .$$

Accordingly we estimate the double integral at the right hand side of (2.9) by the sum of the corresponding double integrals I_ε^1, I_ε^2 and I_ε^3, that is,

$$\Lambda_\varepsilon(u_\varepsilon, A, A') \leq I_\varepsilon^1 + I_\varepsilon^2 + I_\varepsilon^3 \ . \qquad (2.10)$$

We recall now that $|S_{hy}| \leq 1$ for every h and every y, and then

$$I_\varepsilon^1 := \frac{1}{2} \int\limits_{h \in \mathbb{R}^N, \, y \in \Sigma} J(h) \, |h| \Big[\int\limits_{S_{hy}} \big| v(y) - v'(y) \big| \, dt \Big] dy \, dh$$

$$\leq \frac{1}{2} \Big[\int_{\mathbb{R}^N} J(h) \, |h| \, dh \Big] \Big[\int_\Sigma \big| v(y) - v'(y) \big| \, dy \Big] \ . \qquad (2.11)$$

Since the first integral in line (2.11) is equal to $\|\hat{J}\|_1$ (see (2.2)), inequality (2.7) will follow from (2.10) once we have proved that I_ε^2 and I_ε^3 vanish as $\varepsilon \to 0$. Let us consider I_ε^2:

$$I_\varepsilon^2 := \frac{1}{2} \int\limits_{h \in \mathbb{R}^N, \, y \in \Sigma} J(h) \, |h| \Big[\int_{S_{hy}} \big| u_\varepsilon(y - t\varepsilon h) - v(y) \big| \, dt \Big] dy \, dh$$

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$$\leq^{(1)} \frac{1}{2} \int\limits_{h' \in \mathbb{R}^N,\, y \in \Sigma} \Big[\int_{S_{hy}} J\Big(\frac{h'}{t}\Big) \Big| \frac{h'}{t} \Big| \, |u_\varepsilon(y + \varepsilon h') - v(y)| \, \frac{dt}{t^N} \Big] \, dy \, dh'$$

$$\leq^{(2)} \frac{1}{2} \int\limits_{\Sigma} \Big[\int\limits_{\{h' :\, y + \varepsilon h' \in A\}} \hat{J}(h') |u_\varepsilon(y + \varepsilon h') - v(y)| \, dh' \Big] \, dy \;.$$

Hence I_ε^2 vanishes as $\varepsilon \to 0$ because the ε-traces of u_ε relative to A converge to v on Σ. In a similar way one can prove that I_ε^3 vanishes as $\varepsilon \to 0$. $\qquad\square$

Now we can state the main result of this section. Let be given disjoint sets $A, A' \subset \mathbb{R}^N$, and Σ such that one of the following holds:
(a) the sets A and A' are divided by Σ (cf. Definition 2.6);
(b) the sets A and A' are strongly divided by a Lipschitz boundary S with finite measure and $\Sigma = \partial A \cap \partial A'$;
(c) either A or A' is a bounded set with Lipschitz boundary and $\Sigma = \partial A \cap \partial A'$.

Take then positive numbers $\varepsilon_n \to 0$ and functions $u_n : A \cup A' \to [-1, 1]$.

Theorem 2.8. *Under the above stated hypotheses we have*

$$\limsup_{n \to \infty} \Lambda_{\varepsilon_n}(u_n, A, A') \leq \|\hat{J}\|_1 \mathscr{H}^{N-1}(\Sigma) \;. \tag{2.12}$$

Moreover if the ε_n-traces of u_n relative to A and A' converge on Σ respectively to v and v', then

$$\limsup_{n \to \infty} \Lambda_{\varepsilon_n}(u_n, A, A') \leq \frac{1}{2} \|\hat{J}\|_1 \int_\Sigma \big| v(y) - v'(y) \big| \, dy \;. \tag{2.13}$$

Proof. Notice that (2.12) follows by applying (2.13) to the functions $\overline{u}_n$ which are equal to 1 on A and to -1 on A' (with $v = 1$ and $v' = -1$) and then using the obvious inequality $\Lambda_{\varepsilon_n}(u_n, A, A') \leq \Lambda_{\varepsilon_n}(\overline{u}_n, A, A')$.

Let us prove (2.13). When (a) holds it is enough to apply Lemma 2.7, while (c) clearly implies (b). Assume that (b) holds.

First of all we notice that in this case we can always modify the boundary S so that $S \cap \partial A = S \cap \partial A' = \Sigma$. Now we extend v and v' to 0 in $S \setminus \Sigma$, and then the ε_n-traces of u_n relative to A and A' converge on S to v and v' respectively (use Remark 2.3, recalling that both A and A' have upper J-density 0 at every point of $S \setminus \Sigma$). Now it is enough to apply Lemma 2.7 with S instead of Σ. $\qquad\square$

3. Proof of the compactness result

The following theorem implies statement (i) of Theorem 1.4, and shows that the domain of the Γ-limit of the functionals F_ε is included in $BV(\Omega, \pm 1)$.

Theorem 3.1. *Let Ω be a regular open set and let be given sequences (ε_n) and (u_n) such that $\varepsilon_n \to 0$, $u_n : \Omega \to [-1, 1]$, and $F_{\varepsilon_n}(u_n, \Omega)$ is bounded. Then the sequence (u_n) is relatively compact in $L^1(\Omega)$ and each of its cluster points belongs to $BV(\Omega, \pm 1)$.*

---

[1] Apply the change of variable $h = -h'/t$.

[2] Recall (2.1) and that $|S_{hy}| \leq 1$.

Proof. In order to simplify the notation we replace as usual ε_n, u_n, and F_{ε_n} with ε, u_ε, and F_ε.

We need the following inequality, which may be proved by a direct computation: for every non-negative $g \in L^1(\mathbb{R}^N)$ and every $u : \mathbb{R}^N \to \mathbb{R}$ there holds

$$\int\limits_{\mathbb{R}^N \times \mathbb{R}^N} (g * g)(y) \big| u(x+y) - u(x) \big| \, dy \, dx \leq 2\|g\|_1 \int\limits_{\mathbb{R}^N \times \mathbb{R}^N} g(y) \big| u(x+y) - u(x) \big| \, dy \, dx \; . \qquad (3.1)$$

The proof of the theorem is now divided into two steps.

Step 1. We first prove the thesis under the assumption that each u_ε takes values ± 1 only.

We extend each function u_ε to 1 in $\mathbb{R}^N \setminus \Omega$, and then we observe that

$$\int\limits_{\mathbb{R}^N \times \mathbb{R}^N} J_\varepsilon(y) \big| u_\varepsilon(x+y) - u_\varepsilon(x) \big| \, dy \, dx = O(\varepsilon) \; . \qquad (3.2)$$

Indeed the assumption $u_\varepsilon = \pm 1$ implies $\big| u_\varepsilon(x') - u_\varepsilon(x) \big| = \frac{1}{2}\big(u_\varepsilon(x') - u_\varepsilon(x) \big)^2$, and then by the definition of F_ε we obtain

$$\frac{1}{\varepsilon} \int\limits_{\mathbb{R}^N \times \mathbb{R}^N} J_\varepsilon(x' - x) \big| u_\varepsilon(x') - u_\varepsilon(x) \big| \, dx'dx =$$

$$= 2F_\varepsilon(u_\varepsilon, \mathbb{R}^N) = 2F_\varepsilon(u_\varepsilon, \Omega) + 4\Lambda_\varepsilon(u_\varepsilon, \Omega, \mathbb{R}^N \setminus \Omega) \; .$$

We apply inequality (2.12) with $A = \Omega$ and $A' = \mathbb{R}^N \setminus \Omega$ to show that $\Lambda_\varepsilon(u_\varepsilon, \Omega, \mathbb{R}^N \setminus \Omega)$ is uniformly bounded in ε (recall that we are considering only a subsequence ε_n which converges to 0), while $F_\varepsilon(u_\varepsilon, \Omega)$ is uniformly bounded by assumption. Hence (3.2) is proved.

Now we combine inequality (3.1) with $g := J_\varepsilon$ and inequality (3.2), and we obtain

$$\int\limits_{\mathbb{R}^N \times \mathbb{R}^N} (J_\varepsilon * J_\varepsilon)(y) \big| u_\varepsilon(x+y) - u_\varepsilon(x) \big| \, dy \, dx = O(\varepsilon) \; . \qquad (3.3)$$

Since $J * J$ is a non-negative continuous function, we may find a non-negative smooth function φ (not identically 0) with compact support such that

$$\varphi \leq J * J \quad \text{and} \quad |\nabla\varphi| \leq J * J \; . \qquad (3.4)$$

We set $c := \int_{\mathbb{R}^N} \varphi(y) \, dy$ and for every $y \in \mathbb{R}^N$ and every $\varepsilon > 0$ we define

$$\varphi_\varepsilon(y) := \frac{1}{c\varepsilon^N}\varphi(y/\varepsilon) \quad \text{and} \quad w_\varepsilon(y) := \varphi_\varepsilon * u_\varepsilon(y) \; . \qquad (3.5)$$

The functions φ_ε are smooth and non-negative, have integral equal to 1, and converge to the Dirac mass centered at 0 as $\varepsilon \to 0$. We claim that the sequence (w_ε) is asymptotically equivalent to (u_ε) in $L^1(\mathbb{R}^N)$, and that the gradients ∇w_ε are uniformly bounded in $L^1(\mathbb{R}^N)$. Once this claim is proved we could infer that the sequence (w_ε) is relatively compact in $L^1(\Omega)$ and each of its cluster points belongs to $BV(\Omega, \pm 1)$, and the same holds for the sequence (u_ε).

Now it remains to prove the claim. We have

$$\int_{\mathbb{R}^N} |w_\varepsilon - u_\varepsilon| \, dx = \int_{\mathbb{R}^N} \Big| \int_{\mathbb{R}^N} \varphi_\varepsilon(y)\big(u_\varepsilon(x+y) - u_\varepsilon(x)\big) \, dy \Big| \, dx$$

$$\leq \int\limits_{\mathbb{R}^N \times \mathbb{R}^N} \big| \varphi_\varepsilon(y) \big| \big| u_\varepsilon(x+y) - u_\varepsilon(x) \big| \, dy \, dx$$

$$\leq^{(3)} \frac{1}{c} \int\limits_{\mathbb{R}^N \times \mathbb{R}^N} (J_\varepsilon * J_\varepsilon)(y) \big| u_\varepsilon(x+y) - u_\varepsilon(x) \big| \, dy \, dx \; =^{(4)} O(\varepsilon) \; .$$

$^{(3)}$ By (3.4) and (3.5) we obtain $\varphi_\varepsilon \leq \frac{1}{c}J_\varepsilon * J_\varepsilon$.

$^{(4)}$ Apply estimate (3.3).

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Moreover

$$\int_{\mathbb{R}^N} |\nabla w_\varepsilon| \, dx = \int_{\mathbb{R}^N} \Big| \int_{\mathbb{R}^N} \nabla \varphi_\varepsilon(y) \, u_\varepsilon(x+y) \, dy \Big| \, dx$$

$$=^{(5)} \int_{\mathbb{R}^N} \Big| \int_{\mathbb{R}^N} \nabla \varphi_\varepsilon(y) \left(u_\varepsilon(x+y) - u_\varepsilon(x) \right) dy \Big| \, dx$$

$$\leq \int_{\mathbb{R}^N \times \mathbb{R}^N} \big| \nabla \varphi_\varepsilon(y) \big| \big| u_\varepsilon(x+y) - u_\varepsilon(x) \big| \, dy \, dx$$

$$\leq^{(6)} \frac{1}{c\varepsilon} \int_{\mathbb{R}^N \times \mathbb{R}^N} (J_\varepsilon * J_\varepsilon)(y) \big| u_\varepsilon(x+y) - u_\varepsilon(x) \big| \, dy \, dx =^{(7)} O(1) \ ,$$

and the claim is proved.

Step 2. We consider now the general case. For every $s \in \mathbb{R}$ we set

$$T(s) := \begin{cases} -1 & \text{if } s < 0, \\ +1 & \text{if } s \geq 0, \end{cases} \tag{3.6}$$

and then we define

$$v_\varepsilon := T(u_\varepsilon) \ . \tag{3.7}$$

The functions v_ε takes values ± 1 only, and we claim that the sequence (v_ε) is asymptotically equivalent to (u_ε) in $L^1(\Omega)$ and that $F_\varepsilon(v_\varepsilon, \Omega)$ is uniformly bounded. Once proved this claim the thesis will follow from Step 1.

Take δ so that $0 < \delta < 1$, and let K_ε be the set of all $x \in \Omega$ such that $u_\varepsilon(x) \in [-1+\delta, 1-\delta]$. Then $|u_\varepsilon - v_\varepsilon| \leq \delta$ in $\Omega \setminus K_\varepsilon$, and we deduce

$$\int_\Omega |u_\varepsilon - v_\varepsilon| \, dx \leq \delta|\Omega| + \int_{K_\varepsilon} \left(|u_\varepsilon| + |v_\varepsilon| \right) dx \leq \delta|\Omega| + 2|K_\varepsilon| \ . \tag{3.8}$$

Since $\delta > 0$ and W is zero only at ± 1, there exists a positive constant ρ (which depends on δ) such that $W(t) \geq \rho$ for every $t \in [-1+\delta, 1-\delta]$. Hence

$$|K_\varepsilon| \leq \frac{1}{\rho} \int_{K_\varepsilon} W\left(u_\varepsilon(x) \right) dx \leq \frac{\varepsilon}{\rho} F_\varepsilon(u_\varepsilon, \Omega) = \frac{O(\varepsilon)}{\rho}. \tag{3.9}$$

Inequalities (3.8) and (3.9) imply

$$\limsup_{\varepsilon \to 0} \int_\Omega |u_\varepsilon - v_\varepsilon| \, dx \leq \delta|\Omega| \ .$$

As δ is arbitrary, the sequences (u_ε) and (v_ε) are asymptotically equivalent in $L^1(\Omega)$.

It remains to prove that $F_\varepsilon(v_\varepsilon, \Omega)$ is uniformly bounded in ε. Since $\int_\Omega W(v_\varepsilon) \, dy = 0$, we have only to estimate the first integral in the definition of F_ε. Given $s_1, s_2 \in [-1, 1]$ we have that

$$\text{either} \quad |s_1| \leq 1/2 \quad \text{or} \quad \big| T(s_1) - T(s_2) \big| \leq 4|s_1 - s_2| \ .$$

---

[5] Recall that $\int_{\mathbb{R}^N} \nabla \varphi_\varepsilon(y) \, dy = 0$ because φ_ε has compact support.

[6] By (3.4) and (3.5) we obtain $|\nabla \varphi_\varepsilon| \leq \frac{1}{c\varepsilon} J_\varepsilon * J_\varepsilon$.

[7] Apply estimate (3.3).

Hence, if we denote by H_ε the set of all $x \in \Omega$ such that $\big|u_\varepsilon(x)\big| \le 1/2$, we deduce

$$
\begin{aligned}
F_\varepsilon(v_\varepsilon, \Omega) &= \frac{1}{4\varepsilon} \int\limits_{\Omega \times \Omega} J_\varepsilon(x' - x) \left(Tu_\varepsilon(x') - Tu_\varepsilon(x)\right)^2 dx' dx \\
&\le \frac{4}{\varepsilon} \int\limits_{\Omega \times \Omega} J_\varepsilon(x' - x) \left(u_\varepsilon(x') - u_\varepsilon(x)\right)^2 dx' dx + \frac{1}{\varepsilon} \int\limits_{H_\varepsilon \times \Omega} J_\varepsilon(x' - x) \, dx' dx \\
&\le 16 \, F_\varepsilon(u_\varepsilon, \Omega) + \frac{1}{\varepsilon} \|J\|_1 |H_\varepsilon| \; .
\end{aligned}
\tag{3.10}
$$

By the properties of W there exists a positive constant ρ such that $W(t) \ge \rho$ for every t such that $|t| \le 1/2$, and reasoning as in (3.9) we get $|H_\varepsilon| = O(\varepsilon)$; together with (3.10) this proves that $F_\varepsilon(v_\varepsilon, \Omega)$ is uniformly bounded in ε. $\qquad\square$

4. Proof of the lower bound inequality

In this section we prove statement (ii) of Theorem 1.4.

We begin with some notation. For every $\varepsilon > 0$, $A \subset \mathbb{R}^N$ and $u : \mathbb{R}^N \to [-1, 1]$ we define the rescaling of the functional $\mathscr{F}$ given in (1.9) by

$$
\mathscr{F}_\varepsilon(u, A) := \frac{1}{4\varepsilon} \int\limits_{x \in A, \, h \in \mathbb{R}^N} J_\varepsilon(h) \left(u(x + h) - u(x)\right)^2 dx \, dh + \frac{1}{\varepsilon} \int\limits_{x \in A} W\big(u(x)\big) \, dx \; .
\tag{4.1}
$$

Recalling the definitions of F_ε and Λ_ε we obtain:

$$
\mathscr{F}_\varepsilon(u, A) = F_\varepsilon(u, A) + \Lambda_\varepsilon(u, A, \mathbb{R}^N \setminus A) \; .
\tag{4.2}
$$

Let be given now a function u defined on (a subset of) $\mathbb{R}^N$, a point $\bar{x} \in \mathbb{R}^N$ and a positive number r. We define the *blow-up of u centered at $\bar{x}$ with scaling factor r* the function $R_{\bar{x}, r} u$ given by

$$
(R_{\bar{x}, r} u)(x) := u(\bar{x} + rx) \; ;
\tag{4.3}
$$

when $\bar{x} = 0$ we write $R_r u$ instead of $R_{0, r} u$. For every set $A \subset \mathbb{R}^N$ we set, as usual, $\bar{x} + rA := \{\bar{x} + rx : x \in A\}$, and then we easily obtain the following *scaling identities*:

$$
F_\varepsilon(u, \bar{x} + rA) = r^{N-1} F_{\varepsilon/r}(R_{\bar{x}, r} u, A)
\tag{4.4}
$$

$$
\mathscr{F}_\varepsilon(u, \bar{x} + rA) = r^{N-1} \mathscr{F}_{\varepsilon/r}(R_{\bar{x}, r} u, A) \; .
\tag{4.5}
$$

In the proof we also make use of the following well-known results about the blow-up of finite perimeter sets and measures:

4.1. Some blow-up results

Let S be a rectifiable set in $\mathbb{R}^N$ with normal vector field ν; let μ be the restriction of the Hausdorff measure $\mathscr{H}^{N-1}$ to the set S, that is, $\mu := \mathscr{H}^{N-1} \llcorner S$, and let λ be a finite measure on $\mathbb{R}^N$. Then for $\mathscr{H}^{N-1}$-a.e. $\bar{x} \in S$ the density of λ with respect to μ at $\bar{x}$ is given by the following limit:

$$
\frac{d\lambda}{d\mu}(\bar{x}) = \lim_{r \to 0} \frac{\lambda\big(\bar{x} + rQ\big)}{r^{N-1}}
\tag{4.6}
$$

where Q is any unit cube centered at 0 such that $\nu(x)$ is one of its axes.

Let u be a fixed function in $BV(\Omega, \pm 1)$. For every $\bar{x} \in Su$ we denote by $v_{\bar{x}} : \mathbb{R}^N \to \pm 1$ the step function

$$v_{\bar{x}}(x) := \begin{cases} +1 & \text{if } \langle x, \nu_u(\bar{x}) \rangle \geq 0, \\ -1 & \text{if } \langle x, \nu_u(\bar{x}) \rangle < 0. \end{cases} \tag{4.7}$$

Then for $\mathscr{H}^{N-1}$-a.e. $\bar{x} \in Su$, and more precisely for all $\bar{x} \in Su$ such that the density of the measure Du with respect to $|Du|$ exists and is equal to $\nu_u(\bar{x})$, there holds

$$R_{\bar{x},r}u \longrightarrow v_{\bar{x}} \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^N) \text{ as } r \to 0 \tag{4.8}$$

(if u is not defined on the whole of $\mathbb{R}^N$ we take an arbitrary extension).

4.2. Proof of statement (ii) of Theorem 1.4

We can now begin the proof of statement (ii) of Theorem 1.4. We assume therefore that is given a sequence (u_ε) which converges to $u \in BV(\Omega, \pm 1)$ in $L^1(\Omega)$; we have to prove that

$$\liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon, \Omega) \geq \int_{Su} \sigma(\nu_u) \, d\mathscr{H}^{N-1} . \tag{4.9}$$

In the following u_ε and u are fixed. We shall often extract from all positive ε a subsequence (ε_n) which converges to zero; to simplify the notation we shall keep writing ε, F_ε, and u_ε instead of ε_n, F_{ε_n}, u_{ε_n}.

First of all we notice that it is enough to prove inequality (4.9) when the lower limit at the left hand side is finite and then, passing to a subsequence we may assume as well that it is a limit.

Now we follow the approach of [FM]; the main feature of this method consists in the reduction of the lower bound inequality (4.9) to a density estimate (see (4.13)) which has to be verified point by point. What follows, up to equation (4.18), is a straightforward adaptation of this general method (see also [BF], [BFM]).

For every $\varepsilon > 0$ we define the energy density associated with u_ε at the point $x \in \Omega$ as

$$g_\varepsilon(x) := \frac{1}{4\varepsilon} \int_\Omega J_\varepsilon(x' - x) \left(u_\varepsilon(x') - u_\varepsilon(x) \right)^2 dx' + \frac{1}{\varepsilon} W\left(u_\varepsilon(x) \right) , \tag{4.10}$$

and then we consider the corresponding energy distribution

$$\lambda_\varepsilon := g_\varepsilon \cdot \mathscr{L}_N \llcorner \Omega . \tag{4.11}$$

Thus the total variation $\|\lambda_\varepsilon\|$ of the measure λ_ε (on Ω) is equal to $F_\varepsilon(u_\varepsilon, \Omega)$, and since $F_\varepsilon(u_\varepsilon, \Omega)$ is equibounded with respect to ε, possibly passing to a subsequence we can assume that there exists a finite positive measure λ on Ω such that

$$\lambda_\varepsilon \rightharpoonup \lambda \quad \text{weakly* on } \Omega \text{ as } \varepsilon \to 0.$$

Since $F_\varepsilon(u_\varepsilon, \Omega) = \|\lambda_\varepsilon\|$ and $\liminf_{\varepsilon \to 0} \|\lambda_\varepsilon\| \geq \|\lambda\|$, inequality (4.9) is implied by the following:

$$\|\lambda\| \geq \int_{Su \cap \Omega} \sigma(\nu_u) \, d\mathscr{H}^{N-1} . \tag{4.12}$$

In fact, we prove a stronger result: the density of λ with respect to $\mu := \mathscr{H}^{N-1} \llcorner Su$ is greater than or equal to $\sigma(\nu_u)$ at $\mathscr{H}^{N-1}$-a.e. point of Su, that is

$$\frac{d\lambda}{d\mu}(\bar{x}) \geq \sigma\left(\nu_u(\bar{x}) \right) \qquad \text{for } \mathscr{H}^{N-1}\text{-a.e. } \bar{x} \in Su. \tag{4.13}$$

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More precisely, we have the following lemma.

Lemma 4.3. *With the previous notation, inequality (4.13) holds for every $\bar{x} \in Su$ which verifies (4.6) and (4.8).*

Proof. We fix such a point $\bar{x} \in Su$, and we denote by ν the vector $\nu_u(\bar{x})$ and by v the step function $v_{\bar{x}}$ defined in (4.7). Following the notation of paragraph 1.3 we fix an $(N-1)$-dimensional unit cube $C \in \mathscr{C}_\nu$, and we take $Q = Q_C$ and $T = T_C$ accordingly.

As the measures λ_ε weak* converge to λ on Ω as $\varepsilon \to 0$, we have that $\lambda_\varepsilon(A) \to \lambda(A)$ for every set A such that $\lambda(\partial A) = 0$. Since $\lambda(\bar{x} + r(\partial Q)) = 0$ for all positive r up to an exceptional countable set N, we deduce that $\lambda_\varepsilon(\bar{x} + rQ) \to \lambda(\bar{x} + rQ)$ for every positive $r \notin N$. Therefore, recalling (4.6) we write

$$\lim_{\substack{r \to 0 \\ r \notin N}} \left(\lim_{\varepsilon \to 0} \frac{\lambda_\varepsilon(\bar{x} + rQ)}{r^{N-1}} \right) = \lim_{\substack{r \to 0 \\ r \notin N}} \frac{\lambda(\bar{x} + rQ)}{r^{N-1}} = \frac{d\lambda}{d\mu}(\bar{x}) \ . \tag{4.14}$$

Since $u_\varepsilon \to u$ in $L^1(\Omega)$ by assumption and (4.8) holds, we also have

$$\lim_{r \to 0} \left(\lim_{\varepsilon \to 0} R_{\bar{x},r} u_\varepsilon \right) = \lim_{r \to 0} R_{\bar{x},r} u = v \quad \text{in } L^1(Q). \tag{4.15}$$

Therefore by a diagonal argument we may choose sequences (r_n) and (ε_n) so that

$$\lim_{n \to \infty} r_n = \lim_{n \to \infty} (\varepsilon_n / r_n) = 0 \ , \tag{4.16}$$

$$\lim_{n \to \infty} \frac{\lambda_{\varepsilon_n}(\bar{x} + r_n Q)}{r_n^{N-1}} = \frac{d\lambda}{d\mu}(\bar{x}) \ , \tag{4.17}$$

$$\lim_{n \to \infty} R_{\bar{x},r_n} u_{\varepsilon_n} = v \quad \text{in } L^1(Q), \tag{4.18}$$

and then we set $\underline{\varepsilon}_n := \varepsilon_n / r_n$, $v_n := R_{\bar{x},r_n} u_{\varepsilon_n}$. To simplify the notation in the following we write ε, $\underline{\varepsilon}$, r, u_ε and $v_{\underline{\varepsilon}}$ instead of ε_n, $\underline{\varepsilon}_n$, r_n, u_{ε_n} and v_n respectively.

¿From the scaling identity (4.4) and the definition of λ_ε we infer

$$\frac{\lambda_\varepsilon(\bar{x} + rQ)}{r^{N-1}} \geq \frac{F_\varepsilon(u_\varepsilon, \bar{x} + rQ)}{r^{N-1}} = F_{\underline{\varepsilon}}(v_{\underline{\varepsilon}}, Q) \ . \tag{4.19}$$

Keeping in mind (4.17) and (4.19), we can try to prove (4.13) by establishing a precise relation between $F_{\underline{\varepsilon}}(v_{\underline{\varepsilon}}, Q)$ and $\sigma(\nu)$ (see paragraph 1.3).

One possibility is the following: we extend $v_{\underline{\varepsilon}}$ to the strip T by setting $v_{\underline{\varepsilon}} := v$ in $T \setminus Q$, and then we take the C-periodic extension in the rest of $\mathbb{R}^N$. Now, by the scaling identity (4.5) we know that

$$\mathscr{F}_{\underline{\varepsilon}}(v_{\underline{\varepsilon}}, T) \geq \sigma(\nu) \ ,$$

and then it would remain to prove that the difference between $\mathscr{F}_{\underline{\varepsilon}}(v_{\underline{\varepsilon}}, T)$ and $F_{\underline{\varepsilon}}(v_{\underline{\varepsilon}}, Q)$ vanishes as $\underline{\varepsilon} \to 0$; this difference can be written as (cf. (4.20) below)

$$\mathscr{F}_{\underline{\varepsilon}}(v_{\underline{\varepsilon}}, T) - F_{\underline{\varepsilon}}(v_{\underline{\varepsilon}}, Q) = \Lambda_{\underline{\varepsilon}}(v_{\underline{\varepsilon}}, T, \mathbb{R}^N \setminus T) + 2\Lambda_{\underline{\varepsilon}}(v_{\underline{\varepsilon}}, Q, T \setminus Q) \ ,$$

but unfortunately we cannot use Theorem 2.8 to show that it vanishes as $\underline{\varepsilon} \to 0$ because we have no information about the convergence of the $\underline{\varepsilon}$-traces of $v_{\underline{\varepsilon}}$ on the boundaries ∂Q and ∂T.

We overcome this difficulty as follows: as $v_{\underline{\varepsilon}} \to v$ in $L^1(Q)$, Theorem 2.8 shows that for a.e. $t \in (0,1)$ the $\underline{\varepsilon}$-traces of $v_{\underline{\varepsilon}}$ converge to v on the boundary Σ^t of the cube tQ (notice that each Σ^t is the t-level set of the Lipschitz function $g(x) := \operatorname{dist}(x, \partial Q)$).

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We fix for the moment such a t, and we define $\tilde{v}_\varepsilon$ on the stripe tT as

$$\tilde{v}_\varepsilon(x) := \begin{cases} v_\varepsilon(x) & \text{if } x \in tQ, \\ v(x) & \text{if } x \in tT \setminus tQ, \end{cases}$$

and then we take the tC-periodic extension in the rest of $\mathbb{R}^N$. Hence $\tilde{v}_\varepsilon$ belongs to $X(tC)$ (cf. paragraph 1.3), and since $\tilde{v}_\varepsilon = v_\varepsilon$ in tQ

$$\begin{aligned} F_\varepsilon(v_\varepsilon, Q) &\geq F_\varepsilon(v_\varepsilon, tQ) = F_\varepsilon(\tilde{v}_\varepsilon, tQ) = \\ &= F_\varepsilon(\tilde{v}_\varepsilon, tT) - 2\Lambda_\varepsilon(\tilde{v}_\varepsilon, tQ, tT \setminus tQ) \\ &= \mathscr{F}_\varepsilon(\tilde{v}_\varepsilon, tT) - \underbrace{\Lambda_\varepsilon(\tilde{v}_\varepsilon, tT, \mathbb{R}^N \setminus tT)}_{L_\varepsilon^1} - 2\underbrace{\Lambda_\varepsilon(\tilde{v}_\varepsilon, tQ, tT \setminus tQ)}_{L_\varepsilon^2} . \end{aligned} \tag{4.20}$$

Now we claim that both locality defects L_ε^1 and L_ε^2 vanish as $\varepsilon \to 0$; once this is proved we could deduce from the previous formula that

$$\limsup_{\varepsilon \to 0} F_\varepsilon(v_\varepsilon, Q) \geq \limsup_{\varepsilon \to 0} \mathscr{F}_\varepsilon(\tilde{v}_\varepsilon, tT) . \tag{4.21}$$

Let us consider first L_ε^2: the sets tQ and $tT \setminus tQ$ are divided by the boundary Σ^t of tQ, and by the choice of t the ε-trace of $\tilde{v}_\varepsilon$ relative to tQ converge to v on Σ^t (recall that $\tilde{v}_\varepsilon = v_\varepsilon$ on tQ). On the other hand $\tilde{v}_\varepsilon = v$ in $tT \setminus tQ$, and then also the ε-trace relative to $tT \setminus tQ$ converge to v on Σ^t. Hence Theorem 2.8 applies, and L_ε^2 vanishes as $\varepsilon \to 0$.

In a similar way one can prove that also L_ε^1 vanishes as $\varepsilon \to 0$ (it is enough to verify that the ε-trace of $\tilde{v}_\varepsilon$ relative to $\mathbb{R}^N$ converge to v on the boundary of tT).

Eventually we use the scaling identity (4.5) and the definition of $\sigma(\nu)$ to get

$$\mathscr{F}_\varepsilon(\tilde{v}_\varepsilon, tT) = \varepsilon^{N-1}\mathscr{F}\big(R_\varepsilon\tilde{v}_\varepsilon, \tfrac{t}{\varepsilon}T\big) \geq t^{N-1}\sigma(\nu) , \tag{4.22}$$

and putting together (4.17), (4.19), (4.21) and (4.22) we obtain

$$\frac{d\lambda}{d\mu}(\bar{x}) \geq t^{N-1}\sigma(\nu) ;$$

the proof of inequality (4.13) is thus completed by taking t arbitrarily close to 1. $\qquad\square$

5. Proof of the upper bound inequality

Throughout this section Ω is always a regular open set.

Definition 5.1. A *N-dimensional polyhedral set in* $\mathbb{R}^N$ is an open set E whose boundary is a Lipschitz manifold contained in the union of finitely many affine hyperplanes; the *faces* of E are the intersections of the boundary of E with each one of these hyperplanes, and an *edge point* of E is a point which belongs to at least two different faces (that is, a point where ∂E is not smooth). We denote by ν_E the inner normal to ∂E (defined for all points in the boundary which are not edge points).

A *k-dimensional polyhedral set in* $\mathbb{R}^N$ is a polyhedral subset of a k-dimensional affine subspace of $\mathbb{R}^N$. A *polyhedral set in* Ω is the intersection of a polyhedral set in $\mathbb{R}^N$ with Ω.

We say that $u \in BV(\Omega, \pm 1)$ is a *polyhedral function* if there exists an N-dimensional polyhedral set E in $\mathbb{R}^N$ such that ∂E is transversal to $\partial \Omega$ (that is, $\mathscr{H}^{N-1}(\partial E \cap \partial \Omega) = 0$) and $u(x) = 1$ for every $x \in \Omega \cap E$, $u(x) = -1$ for every $x \in \Omega \setminus E$.

Theorem 5.2. *Let $u \in BV(\Omega, \pm 1)$ be a polyhedral function. Then there exists a sequence of functions (u_ε) defined on Ω such that $|u_\varepsilon| \leq 1$ for every ε, u_ε converge to u uniformly on every compact set $K \subset \overline{\Omega} \setminus Su$, and*

$$\limsup_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon, \Omega) \leq \int_{Su} \sigma(\nu_u) \, d\mathscr{H}^{N-1} . \tag{5.1}$$

Proof. Let us fix some notation: E is the polyhedral set associated with u in Definition 5.1; we denote by S the set of all edge points of E which belongs to Ω and by Σ a general face of Su (that is, a face of E). Then S is a finite union of $(N-2)$-dimensional polyhedral sets in Ω, $\partial E = Su$, and we may choose the orientation of Su so that $\nu_E = \nu_u$ (for every point in $Su \setminus S$).

Given open sets A_1, A_2 let $A_1 \sqcup A_2$ we denote the interior of $\overline{A_1} \cup \overline{A_2}$. We define $\mathscr{G}$ as the class of all sets A such that
(i) A is an N-dimensional polyhedral set in Ω, and ∂A and Su are transversal (that is, $\mathscr{H}^{n-1}(Su \cap \partial A) = 0$);
(ii) there exists a sequence of functions (u_ε) defined on $\overline{A}$ such that $|u_\varepsilon| \leq 1$ and

$$u_\varepsilon \to u \quad \text{uniformly on every compact set } K \subset \overline{A} \setminus Su, \tag{5.2}$$

$$\limsup_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon, A) \leq \int_{A \cap Su} \sigma(\nu_u) \, d\mathscr{H}^{N-1} . \tag{5.3}$$

The proof of Theorem 5.2 is achieved by showing that $\Omega \in \mathscr{G}$; this is a consequence of the following three statements:
(a) *if A is an N-dimensional polyhedral set in Ω such that $\mathscr{H}^{N-1}(\overline{A} \cap Su) = 0$, then $A \in \mathscr{G}$;*
(b) *let Σ be a face of Su and let π be the projection map on the affine hyperplane which contains Σ: if A is an N-dimensional polyhedral set in Ω such that $Su \cap \overline{A} = \Sigma$ and $\pi(A) = \Sigma$, then $A \in \mathscr{G}$;*
(c) *if A_1, A_2 belong to $\mathscr{G}$ and are disjoint, then $A_1 \sqcup A_2 \in \mathscr{G}$.*

Step 1: proof of statement (a).

In this case $\mathscr{H}^{N-1}(\partial A \cap Su) = 0$ and $A \cap Su = \varnothing$; then u is constant (-1 or 1) in A, and it is enough to take $u_\varepsilon := u$ for every $\varepsilon > 0$.

Step 2: proof of statement (b).

Property (i) is immediate; let us prove (ii). We denote by e the (constant) inner normal to Σ; therefore Σ lies on some affine hyperplane which is parallel to M; without loss of generality we may assume that Σ lies exactly in M.

Following the notation of paragraph 1.3, for every fixed $\eta > 0$, we can find $C \in \mathscr{C}_e$ and $w \in X(C)$ such that

$$|C|^{-1} \mathscr{F}(w, T_C) \leq \sigma(e) + \eta , \tag{5.4}$$

and then we define

$$u_\varepsilon(x) := w(x/\varepsilon) \quad \text{for every } x \in \mathbb{R}^N . \tag{5.5}$$

Property (5.2) holds because $w(x) \to \pm 1$ as $x_e \to \pm \infty$ (see paragraph 1.3). We claim that

$$\limsup_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon, A) \leq \mathscr{H}^{N-1}(\Sigma) \cdot \big(\sigma(e) + \eta\big) . \tag{5.6}$$

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Without loss of generality, we may assume that C is a unit cube. In order to prove inequality (5.6), for every $\varepsilon > 0$ we cover Σ with the closures of a finite number $h = h(\varepsilon)$ of pairwise disjoint copies of the $(N-1)$-dimensional cube εC, that is, we choose $x_1, \ldots, x_h \in M$ so that

$$\Sigma \subset \bigcup_{i=1}^{h} \left(x_i + \varepsilon \overline{C} \right) .$$

Moreover, since Σ is a polyhedral set in M, the previous covering can be chosen so that

$$h\varepsilon^{N-1} = \mathscr{H}^{N-1}\left(\cup (x_i + \varepsilon \overline{C}) \right) \longrightarrow \mathscr{H}^{N-1}(\Sigma) \quad \text{as } \varepsilon \to 0. \tag{5.7}$$

Notice that, since Σ is the projection of A on M, then $A \subset \cup_i(x_i + \varepsilon \overline{T}_C)$. Then by definition (4.1) we have

$$F_\varepsilon(u_\varepsilon, A) \le F_\varepsilon\left(u_\varepsilon, \cup_i(x_i + \varepsilon T_C) \right) \le \mathscr{F}_\varepsilon\left(u_\varepsilon, \cup_i(x_i + \varepsilon T_C) \right) = \sum_{i=1}^{h} \mathscr{F}_\varepsilon(u_\varepsilon, \varepsilon T_C) , \tag{5.8}$$

where the last equality follows from the fact that $\mathscr{F}_\varepsilon(u_\varepsilon, \cdot)$ is translation invariant and additive. Applying now the scaling identity (4.5) with $\bar{x} = 0$ and $\varepsilon = r$ we get $\mathscr{F}_\varepsilon(u_\varepsilon, \varepsilon T_C) = \varepsilon^{N-1}\mathscr{F}(w, T_C)$, so that by (5.8) and (5.4) we deduce

$$F_\varepsilon(u_\varepsilon, A) \le h\varepsilon^{N-1}\left(\sigma(e) + \eta \right) .$$

Taking into account (5.7) we get (5.6).

Since e agrees with ν_u in $\Sigma = Su \cap A$, (5.3) follows from inequality (5.6) by a simple diagonal argument, and the proof of statement (b) is complete.

Step 3: proof of statement (c).

Given disjoint $A_1, A_2 \in \mathscr{G}$, we set $A := A_1 \sqcup A_2$ and we take sequences (u_ε^1), (u_ε^2) which satisfy property (ii) for A_1 and A_2 respectively. Then we set

$$u_\varepsilon(x) := \begin{cases} u_\varepsilon^1(x) & \text{if } x \in A_1, \\[2mm] u_\varepsilon^2(x) & \text{if } x \in A_2. \end{cases}$$

One can check that properties (i) and (5.2) are satisfied, and that (5.3) reduces to

$$\lim_{\varepsilon \to 0} \Lambda_\varepsilon(u_\varepsilon, A_1, A_2) = 0 .$$

Notice that by (5.2) the ε-traces of u_ε^i relative to A_i converge to u on every Lipschitz hypersurface $\Sigma \subset \overline{A}_i$ such that $\mathscr{H}^{N-1}(\Sigma \cap Su) = 0$ for $i = 1, 2$ (cf. Remark 2.4); in particular this holds true for $\Sigma = \partial A$. Hence the previous identity follows from Theorem 2.8.

Step 4: proof of Theorem 5.2.

It may be verified that Ω may be written as $\Omega = \sqcup A_i$ where the sets A_i are finitely many, pairwise disjoint, and satisfy the hypothesis of statements (a) or (b):

Figure 3: decomposition of Ω as union of $A_0, \ldots, A_9 \in \mathscr{G}$.

Therefore Ω belongs to $\mathscr{G}$ by statement (c), and Theorem 5.2 follows from property (ii). $\square$

In order to complete the proof of Theorem 1.4 we need the following lemma:

Lemma 5.3. *The function σ defined in paragraph 1.3 is upper semicontinuous on the unit sphere of $\mathbb{R}^N$.*

Proof. Fix a unit vector ν in $\mathbb{R}^N$, and for every linear isometry I of $\mathbb{R}^N$ set

$$\hat{\sigma}(I) := \inf \left\{ |C|^{-1} \mathscr{F}(u \circ I, T_C) : C \in \mathscr{C}_\nu, \, u \in X(C) \right\} \tag{5.9}$$

(here we follow the notation of paragraph 1.3). One easily verifies that for every $u \in X(C)$ the map $I \mapsto \mathscr{F}(u \circ I, T_C)$ is continuous on the space $\mathscr{I}$ of all linear isometries of $\mathbb{R}^N$, and therefore $\hat{\sigma}$ is upper semicontinuous on $\mathscr{I}$ because it is defined in (5.9) as an infimum of continuous functions. We deduce the thesis by remarking that $\sigma(e) = \hat{\sigma}(I)$ whenever $e = I\nu$. $\square$

5.4. Proof of statement (iii) of Theorem 1.4

For every $\mathbb{R}^N$-valued Borel measure μ on Ω we set

$$G(\mu) := \int_\Omega \sigma\left(\mu/|\mu|\right) d|\mu| \,, \tag{5.10}$$

where $\mu/|\mu|$ stands for the density of μ with respect to its total variation. Now statement (iii) of Theorem 1.4 reads as follow: for every function $u \in BV\left(\Omega, \pm 1\right)$ there exists a sequence (u_ε) such that $u_\varepsilon \to u$ in $L^1(\Omega)$ and $\limsup F_\varepsilon(u_\varepsilon, \Omega) \le G(Du)$.

By Theorem 5.2 this is true when u is a polyhedral function, and then the general case follows by a simple diagonal argument once we have proved that every function $u \in BV\left(\Omega, \pm 1\right)$ can be approximated (in $L^1(\Omega)$) by a sequence of polyhedral functions (u_n) so that $\limsup G(Du_n) \le G(Du)$.

It is well-known that every $u \in BV\left(\Omega, \pm 1\right)$ can be approximated by polyhedral functions (u_n) in variation, that is, $u_n \to u$ in $L^1(\Omega)$ and $\|Du_n\| \to \|Du\|$ (in fact, when Ω is regular, every set of finite perimeter can be approximated in variation by smooth sets, and then also by polyhedral sets, see for instance [Gi], Theorem 1.24), and then it is enough to prove that G is upper semicontinuous with respect to convergence in variation of measures.

Since σ is a non-negative upper semicontinuous function on the unit sphere of $\mathbb{R}^N$ (Lemma 5.3), then it can be obtained as the limit of an increasing sequence of non-negative continuous

functions σ_n; therefore G is the supremum of the corresponding functionals G_n, and these functionals are continuous with respect to convergence in variation by a well-known result due to Reshetnyak (see for instance the appendix of [LM]). Hence G is upper semicontinuous with respect to convergence in variation, and the proof of statement (iii) of Theorem 1.1 is complete. $\square$

6. Appendix

In this appendix we prove a Γ-convergence result concerning the "gradient part" of the functionals F_ε defined in (1.4). Let $J : \mathbb{R}^N \to \mathbb{R}$ be a non-negative function in $L^1(\mathbb{R}^N)$ (not almost everywhere 0), $L : \mathbb{R} \to \mathbb{R}$ a positive convex function with superlinear growth at infinity, and Ω a bounded open subset of $\mathbb{R}^N$.

For every $y \in \mathbb{R}^N$ we denote by τ_y the translation operator which takes every function u in the function $\tau_y u$ given by $\tau_y u(x) := u(x + y)$ for every x. We set

$$M(z) := \int_{\mathbb{R}^N} L\big(\langle z, y \rangle\big) J(y)\, dy \qquad \forall z \in \mathbb{R}^N . \tag{6.1}$$

Since J is non-negative and L is convex and has superlinear growth at infinity, also M is convex and has superlinear growth at infinity.

Definition 6.1. We define the functionals G_0 and G_ε for every $\varepsilon > 0$ and $u \in L^1(\Omega)$ as

$$G_\varepsilon(u) := \int_{\Omega \times \Omega} J_\varepsilon(x' - x)\, L\left(\frac{u(x') - u(x)}{\varepsilon}\right) dx'dx \tag{6.2}$$

where, as usual, $J_\varepsilon(y) := \varepsilon^{-N} J(y/\varepsilon)$, and

$$G_0(u) := \begin{cases} \displaystyle\int_\Omega M(Du)\, dx & \text{if } u \in W^{1,1}(\Omega), \\[2mm] +\infty & \text{otherwise.} \end{cases} \tag{6.3}$$

Note that the functional G_0 is lower semicontinuous on $L^1(\Omega)$, because M is convex and has superlinear growth at infinity. The following result holds.

Theorem 6.2. *Assume that Ω is convex. Then $G_\varepsilon \leq G_0$ for every $\varepsilon > 0$, and G_0 is the pointwise limit of G_ε. Therefore the functionals G_ε Γ-converge to G_0 in $L^1(\Omega)$.*

Proof. Let us prove that $G_\varepsilon \leq G_0$ for every $\varepsilon > 0$. It is enough to prove this inequality when $u \in W^{1,1}(\Omega)$ (otherwise it is trivial). Since Ω is convex, for almost every couple $(x, x') \in \Omega \times \Omega$ there holds

$$u(x') - u(x) = \int_0^1 \big\langle Du(tx' + (1-t)x), x' - x \big\rangle dt , \tag{6.4}$$

so

$$G_\varepsilon(u) = \int_{\Omega \times \Omega} J_\varepsilon(x' - x)\, L\Big(\int_0^1 \big\langle Du(tx' + (1-t)x), \frac{x' - x}{\varepsilon} \big\rangle dt \Big) dx'dx$$

$$\leq^{(8)} \int_{\substack{x,x' \in \Omega \\ t \in [0,1]}} J_\varepsilon(x' - x)\, L\Big(\big\langle Du(tx' + (1-t)x), \frac{x' - x}{\varepsilon} \big\rangle\Big) dt\, dx'dx$$

_____

$^{(8)}$ Apply Jensen's inequality.

$$\leq^{(9)} \int\limits_{\substack{y\in\Omega,\,h\in\mathbb{R}^N \\ t\in[0,1]}} J(h)\, L\big(\langle D\,u(y),h\rangle\big)\, dt\, dh\, dy = \int_\Omega M\big(D\,u(y)\big)\, dy\ .$$

In order to complete the proof of the theorem, we have to show that for every $u \in L^1(\Omega)$

$$\liminf_{\varepsilon\to 0} G_\varepsilon(u) \geq G_0(u)\ . \tag{6.5}$$

We shall prove inequality (6.5) when J has compact support (the general case may be recovered by approximating J with an increasing sequence of non-negative functions J_n with compact support). We take $r > 0$ so that the support of J is included in $B(0,r)$. If the left term of inequality (6.5) is infinite, there is nothing to prove, and then we can assume that it is finite.

Let A be an open set relatively compact in Ω. For every $\varepsilon > 0$ such that $\text{dist}\,(A, \mathbb{R}^N \setminus \Omega) > \varepsilon r$ and every $u \in L^1(\Omega)$ the following inequality holds:

$$G_\varepsilon(u) \geq \int_{\mathbb{R}^N} \Big[\int_A \varepsilon^{-N} J\Big(\frac{y}{\varepsilon}\Big)\, L\Big(\frac{u(x+y)-u(x)}{\varepsilon}\Big)\, dx \Big]\, dy$$
$$= \int_{\mathbb{R}^N} \Big[\int_A L\Big(\frac{\tau_{\varepsilon h}\, u - u}{\varepsilon}\Big)\, dx \Big]\, J(h)\, dh\ .$$

By applying Fatou's lemma we obtain

$$G_0(u) \geq \liminf_{\varepsilon\to 0} G_\varepsilon(u) \geq \int_{\mathbb{R}^N} \Big[\underbrace{\liminf_{\varepsilon\to 0} \int_A L\Big(\frac{\tau_{\varepsilon h}\, u - u}{\varepsilon}\Big)\, dx}_{P(h)} \Big]\, J(h)\, dh\ . \tag{6.6}$$

Since $G_0(u)$ is finite, $P(h)$ must be finite for almost every h such that $J(h) > 0$. Let be given h so that $P(h)$ is finite, and let (ε_n) be any subsequence converging to 0 so that

$$\int_A L\Big(\frac{\tau_{\varepsilon_n h}\, u - u}{\varepsilon_n}\Big)\, dx \qquad \text{is uniformly bounded with respect to } n.$$

Since L has superlinear growth at infinity by assumption, we obtain that the sequence $(\tau_{(\varepsilon_n h)}u - u)/\varepsilon_n$ is relatively compact in the weak topology of $L^1(A)$, and taking into account that it converges to the partial derivative $\partial u/\partial h$ in the sense of distributions on A [10], we obtain that the partial derivative $\partial u/\partial h$ belongs to $L^1(A)$, the sequence $\big(\tau_{(\varepsilon_n h)}u - u\big)/\varepsilon_n$ converges to $\partial u/\partial h$ weakly in $L^1(A)$, and then well-known semicontinuity theorems yield

$$\liminf_{n\to\infty} \int_A L\Big(\frac{\tau_{(\varepsilon_n h)}u - u}{\varepsilon_n}\Big)\, dx \geq \int_A L\Big(\frac{\partial u}{\partial h}\Big)\, dx\ . \tag{6.7}$$

---

[9] We make the change of variable $(t, x, x') \to (t, h, y)$ where $h := (x' - x)/\varepsilon$ and $y := tx' + (1-t)x$; the corresponding Jacobian determinant is ε^N, and since Ω is convex, y belongs to Ω for every $x, x' \in \Omega$, $t \in [0,1]$.

[10] Let be given a test function $\phi \in \mathcal{D}(A)$: since $\big(\phi - \tau_{-\varepsilon h}\phi\big)/\varepsilon$ converges to $\partial\phi/\partial h$ uniformly on A, we have that

$$\int_A \frac{\tau_{\varepsilon h}u - u}{\varepsilon}\, \phi\, dx = \int_A \frac{\tau_{-\varepsilon h}\phi - \phi}{\varepsilon}\, u\, dx \longrightarrow -\int_A \frac{\partial\phi}{\partial h}\, u\, dx = \Big\langle \frac{\partial u}{\partial h}, \phi \Big\rangle\ .$$

By repeating the previous arguments when h ranges in a basis of $\mathbb{R}^N$, we prove that u belongs to $W^{1,1}(A)$, and since this holds for every A relatively compact in Ω, then u belongs to $W^{1,1}_{\text{loc}}(\Omega)$. Moreover, taking (6.7) into account, (6.6) yields

$$\liminf_{\varepsilon \to 0} G_\varepsilon(u) \geq \int_{\mathbb{R}^N} \left(\int_A L\left(\frac{\partial u}{\partial h}\right) dx \right) J(h)\, dh$$

$$= \int_A \left(\int_{\mathbb{R}^N} J(h)\, L\big(\langle Du, h\rangle\big)\, dh \right) dx = \int_A M(Du)dx\ .$$

Taking the supremum over all A relatively compact in Ω we get

$$\liminf_{\varepsilon \to 0} G_\varepsilon(u) \geq \int_\Omega M(Du)dx\ .$$

Then u belongs to $W^{1,1}(\Omega)$ because M has (super-) linear growth at infinity, and (6.5) is proved.

The fact that the functionals (G_ε) Γ-converges to G_0 is an immediate consequence of the previous results and the $L^1(\Omega)$-lower semicontinuity of G_0 (see [DM], Proposition 5.7). $\qquad\square$

Remark 6.3. The convexity assumption for the domain Ω is needed only in the proof of the inequality $G_0 \geq G_\varepsilon$, in order to have that formula (6.4) makes sense. In fact, it could be replaced by other conditions, e.g., by assuming the existence of an extension operator T which takes each $u \in W^{1,1}(\Omega)$ such that $\int_\Omega M(Du)$ is finite into a function $Tu \in W^{1,1}(\mathbb{R}^N)$ such that $\int_A M(D(Tu))$ is finite for some neighbourhood A of $\overline{\Omega}$ (we do not need that T is either linear or continuous).

Remark 6.4. In the particular case $L(t) := |t|^p$ with $1 < p < \infty$ and J is radially symmetric, M may be easily computed, and we obtain that

$$M(z) = c_p |z|^p \quad \text{with } c_p := \int_{\mathbb{R}^N} J(y)\, \big|\langle y, e\rangle\big|^p dy \tag{6.8}$$

(here e is any unit vector in $\mathbb{R}^N$).

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