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für Mathematik
in den Naturwissenschaften
Leipzig

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by

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Preprint-Nr.: 50 1998

RELAXATION OF SOME MULTI–WELL PROBLEMS

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Abstract. Mathematical models of phase transitions in solids lead to the variational problem, minimize $\int_\Omega W(Du)dx$ where W has a multi-well structure: $W = 0$ on a multi-well set K and $W > 0$ otherwise. We study this problem in two dimensions in the case of equal determinant, i.e., for $K = \mathrm{SO}(2)U_1 \cup \ldots \cup \mathrm{SO}(2)U_k$ or $K = \mathrm{O}(2)U_1 \cup \ldots \cup \mathrm{O}(2)U_k$ for $U_1, \ldots, U_k \in \mathbb{M}^{2\times 2}$ with $\det U_i = \delta$, in three dimensions when the matrices U_i are essentially two-dimensional and also for $K = \mathrm{SO}(3)\hat{U}_1 \cup \ldots \cup \mathrm{SO}(3)\hat{U}_k$ for $U_1, \ldots, U_k \in \mathbb{M}^{3\times 3}$ with $(\operatorname{adj} U_i^T U_i)_{33} = \delta^2$ which arises in the study of thin films. Here $\hat{U}_i$ denotes the (3×2)-matrix formed with the first two columns of U_i. We characterize generalized convex hulls, including the quasiconvex hull, of these sets, prove existence of minimizers and identify conditions for the uniqueness of the minimizing Young measure. Finally, we use the characterization of the quasiconvex hull to propose 'approximate relaxed energies', quasiconvex functions which vanish on the quasiconvex hull of K and grow quadratically away from it.

1. Introduction

Mathematical models for phase transitions in solids lead to the following variational problem (see [BJ1], [BJ3]): Minimize

$$I(u) = \int_\Omega W(Du)dx, \tag{1.1}$$

where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is the deformation of an elastic body which occupies in an ideal unstressed configuration the domain Ω. We assume that the stored energy density W is nonnegative and that the level set $K = \{W = 0\}$ is not empty. The principle of material frame indifference and symmetry properties of the underlying material imply further structure of K. For many materials of interest, K has a multi-well structure,

$$K = \bigcup_{i=1}^{k} \mathrm{SO}(n)U_i.$$

As a consequence, W fails to be quasiconvex and therefore the existence of minimizers cannot be obtained from the direct method in the calculus of variations based on sequential lower semicontinuity of the integral. However, the behaviour of the minimization problem is closely related to quasiconvex hull K^{qc} of the set K: if we minimize I on all Sobolev functions which coincide with the affine mapping $u(x) = Fx$ on $\partial\Omega$, then the infimum of I is zero if and only if F belongs to the quasiconvex hull K^{qc} of K (see e.g. [S]).

Date: November 26, 1998.

1991 *Mathematics Subject Classification.* 49J40,52A30,73B99,73C50,73V25.

Key words and phrases. nonconvex variational problems, generalized convex hulls, existence of minimizers, in–approximation, relaxed energy .

In this paper we characterize generalized convex hulls for multi-well problems in two and three dimensions in the case of equal determinant. More precisely, we prove the following results (see Section 2 for the notation used below):

Theorem 1.1. *Let* $\mathcal{U} = \{U_1, \dots, U_k\} \subset \mathbb{M}^{2\times 2}_{sym}$ *where the matrices* U_i *are positive definite and satisfy* $\det U_i = \delta > 0$.

i) Let $K = \mathrm{SO}(2)U_1 \cup \dots \cup \mathrm{SO}(2)U_k$. *Then* $K^{(2)} = K^{lc} = K^{rc} = K^{qc} = K^{pc}$. *Further, if*

$$\{U_1, \dots, U_n\} = \{U_i \in \mathcal{U} : |U_i\tilde{e}|^2 > \max_{j\neq i} |U_j\tilde{e}|^2 \text{ for some } \tilde{e} \in S^1\},$$

then there exists a set $\mathcal{E}_n = \{e_1, \dots, e_n\} \subset S^1$ *such that any of these hulls is given by*

$$\{F : \det F = \delta, \, |Fe_i|^2 \le \max_{j=1,\dots,n} |U_j e_i|^2, \, i = 1, \dots, n\}.$$

ii) Let $K = \mathrm{O}(2)U_1 \cup \dots \cup \mathrm{O}(2)U_k$. *Then* $K^{(3)} = K^{lc} = K^{rc} = K^{qc} = K^{pc}$ *and any of these hulls is given by*

$$\{F \in \mathbb{M}^{2\times 2} : |\det F| \le \delta, \, |Fe|^2 \le \max_{i=1,\dots,k} |U_i e|^2 \ \forall e \in S^1\}.$$

A similar result holds for the three dimensional case if the wells are essentially two dimensional.

Theorem 1.2. *Let* $\mathcal{U} = \{U_1, \dots, U_k\} \subset \mathbb{M}^{3\times 3}_{sym}$ *where the matrices* U_i *are positive definite and satisfy* $\det U_i = \delta > 0$. *Assume that there exists* $\mu > 0$ *and* $v \in S^2$ *such that* $U_i v = \mu v$ *for* $i = 1, \dots, k$. *Let* $K = \mathrm{SO}(3)U_1 \cup \dots \cup \mathrm{SO}(3)U_k$. *Then* $K^{(2)} = K^{lc} = K^{rc} = K^{qc} = K^{pc}$. *Further, if*

$$\{U_1, \dots, U_n\} = \{U_i \in \mathcal{U} : |U_i\tilde{e}|^2 > \max_{j\neq i} |U_j\tilde{e}|^2 \text{ for some } \tilde{e} \in S^2\},$$

then there exists a set $\mathcal{E}_n = \{e_1, \dots, e_n\} \subset S^2$ *such that any of these hulls is given by*

$$\{F : \det F = \delta, \, F^T F v = \mu^2 v, \, |Fe_i|^2 \le \max_{j=1,\dots,n} |U_j e_i|^2, \, i = 1, \dots, n\}.$$

Applications in the recently developed theory of thin films [LR, BhJ] motivate to consider the following set K:

Theorem 1.3. *Assume that* $U_i \in \mathbb{M}^{3\times 3}_{sym}$, $i = 1, \dots, k$, *are positive definite with* $\mathrm{adj}_{33}\, U_i^2 = \delta^2 > 0$ *and that* $\{e_1, e_2, e_3\}$ *is the standard orthonormal basis in* $\mathbb{R}^3$. *Let* $K = \mathrm{SO}(3)\hat{U}_1 \cup \dots \cup \mathrm{SO}(3)\hat{U}_k$ *where*

$$\mathrm{SO}(3)\hat{U}_i = \{Q\hat{U}_i = (QU_ie_1, QU_ie_2) : Q \in \mathrm{SO}(3)\} \subset \mathbb{M}^{3\times 2}.$$

Then $K^{(3)} = K^{lc} = K^{rc} = K^{qc} = K^{pc}$ *and any of these hulls is given by*

$$\{F \in \mathbb{M}^{3\times 2} : \det(F^T F) \le \delta^2, \, |Fe|^2 \le \max_{i=1,\dots,k} |\hat{U}_i e|^2 \, \forall e \in S^1\}.$$

We use this characterization to propose 'approximate relaxed energies' which may be useful for numerical computations. Minimizing sequences and minimizers of I develop complex oscillatory patterns and this makes numerical computations challenging. Computing with the relaxed energy $I^{\#}$ (which is obtained from I by replacing W with its quasiconvex envelope) is attractive. Many of the numerical difficulties do not arise, the infima coincide, the minimizing sequences of I converge to the minimizers of $I^{\#}$, and recently Ball, Kirchheim and Kristensen [BKK] have shown that under suitable growth hypotheses even the stresses associated with the minimizing sequences of I converge to those associated with the minimizers of $I^{\#}$. Unfortunately, the quasiconvex envelope of W is unknown. However, the practical interest lies in the behavior of the quasiconvex envelope near the set K^{qc}. We use the characterization of this set to propose functions $\bar{W}$ which are quasiconvex, vanish on K^{qc} and grow quadratically away from K^{qc}. In [BD] we adapt the construction to fit measured elastic moduli for various materials.

We illustrate our results with two examples:

i) The two-well problem, which corresponds to an orthorhombic to monoclinic transformation and also arises under suitable assumptions in cubic to tetragonal or orthorhombic transformation, is described in Examples 3.4 and 4.4 (Example 3.4 recovers the results of Ball and James [BJ3]).

ii) The four-well problem which corresponds to a tetragonal to monoclinic transformation and also arises under suitable assumptions in some cubic to monoclinic transformations, is described in Examples 3.7, 4.5 7.3 and 8.3.

Müller and Šverák [MS1], [MS2] recently showed based on Gromov's idea of convex integration that there exist even Lipschitz continuous minimizers of I if F belongs to the interior of the rank-one convex hull of K and if K admits an 'in–approximation' (see Section 6 below for the precise statement); Dacarogna and Marcellini [DM1, DM2] have obtained similar existence results using the Baire's theorem. We show that the sets K^{qc} in Theorems 1.1 and 1.3, but not in Theorem 1.2, admit such in–approximations.

The basic ideas behind the main results are simple, though the details are rather laborious. Two identifications play a crucial role. First, K and consequently the quasiconvex hull K^{qc} is invariant under (multiplication from the left by elements of) SO(2), O(2), and SO(3), respectively. So we can look at the image K_c^{qc} of K^{qc} in the space of 2×2 positive semidefinite symmetric matrices under the map $F \mapsto F^T F$. In other words, we identify the set

$$K_c^{qc} = \{C \in \mathbb{M}^{2\times 2}_{sym} : \det C \geq 0, \sqrt{C} \in K^{qc}\}$$

with K^{qc}. Second, we identify the space $\mathbb{M}^{2\times 2}_{sym}$ of symmetric 2×2 matrices with $\mathbb{R}^3$ using components $\{C_{11}, C_{22}, \sqrt{2}C_{12}\}$. We use the $\sqrt{2}$ in the third component to preserve inner products. Positive semidefinite symmetric matrices correspond to the (affine) half cone

$$\{C : C_{11}C_{22} - C_{12}^2 \geq 0, C_{11} \geq 0, C_{22} \geq 0\}. \tag{1.2}$$

We now give a brief, non-technical discussion of our results. Under the assumptions in Theorem 1.1 *i)* it follows from the minors relation or the weak continuity of the minors that for any $F \in K^{qc}$ and $e \in S^1$, $\det F = \delta$ and $|Fe|^2 \leq \max_{i=1,\dots,k} |U_i e|^2$. Therefore, $K_c^{qc} \subset \mathcal{A}$ where

$$\mathcal{A} = \{C \in \mathbb{M}^{2\times 2}_{sym} : \det C = \delta^2, \langle e, Ce\rangle \leq \max_{i=1,\dots,k} |U_i e|^2 \ \forall\, e \in S^1\}.$$

We now show the converse, $\mathcal{A} \subset K_c^{qc}$. In order to do so, let us look at this set $\mathcal{A}$ in some detail. Clearly the set of all positive definite, symmetric matrices with $\det C = \delta^2$ describes a manifold (hyperboloid), while $\langle e, Ce\rangle = \alpha$ defines a plane in $\mathbb{R}^3$. Thus $\mathcal{A}$ is a subset of this manifold restricted by suitables planes (see Figure 1). Let us elaborate. Figure 2 shows schematically the surface of the hyperboloid.

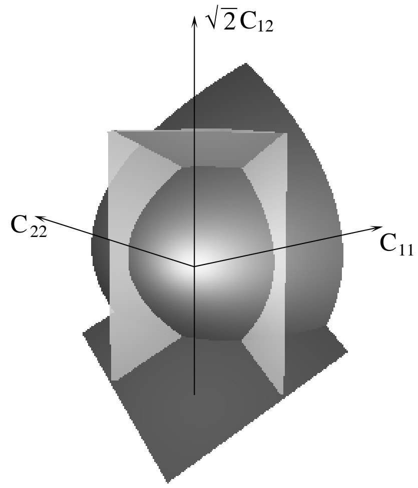

FIGURE 1. *The quasiconvex hull K_c^{qc} for the four-well problem described in Example 3.7 with* SO(2)*-invariant wells.*

For any direction $e \in S^1$ and $\alpha \in \mathbb{R}$, the interesection of the hyperboloid with the plane $\langle e, Ce\rangle = \alpha$ is a (quadratic) curve $\Gamma(e, \alpha)$ which divides the hyperboloid into two parts (see lower left of Figure 2). Start with $\alpha \geq \max_{i=1,\dots,k} |U_i e|^2$ and move the curve (by changing α) till it first touches any of the matrices U_i^2. The set $\mathcal{A}$ is the set that is enclosed by similar curves for all $e \in S^1$. It turns out that if there are k matrices, only k curves are needed to define the boundary of $\mathcal{A}$ (of course this requires a hypothesis that prevents one of the matrices U_i^2 to lie within the set $\mathcal{A}$ defined using the others; otherwise there may be less than k curves). These k curves have the property that they pass through two points U_i^2 and U_j^2. Further, they are rank-one directions in the following sense: we can find $a, n \in \mathbb{R}^2$ such that any C on this curve can be expressed as

$$C = (U_i + ta \otimes n)^T (U_i + ta \otimes n), \quad t \in \mathbb{R}.$$

Therefore, we can obtain any point C on the segment of this curve between U_i^2 and U_j^2 by rank-one lamination and thus $\partial\mathcal{A} \subset K_c^{qc}$. Now pick any point D in the interior of $\mathcal{A}$. There is a rank-one curve passing through D which always lies on the hyperboloid and extends off to infinity in both directions. Therefore, it must intersect $\partial\mathcal{A}$ at two points, and we can obtain D through the lamination of these points. We thus conclude that $\mathcal{A} \subset K_c^{qc}$.

The result and proof of Theorem 1.2 are similar; we use the minors relations to prove one inclusion and lift the constructions above to three dimensions to prove the other.

Let us now turn to part *ii)* of Theorem 1.1 where K consists of k copies of O(2). The fundamental difference between this and the former case can be seen in the special case $k = 1$. While K^{qc} for $K = \mathrm{SO}(2)$ is trivial, i.e. $K^{qc} = \mathrm{SO}(2)$ or $K_c^{qc} = \{I\}$, K^{qc} for $K = \mathrm{O}(2)$ consists of the set of all short maps:

$$K^{qc} = \{F : 0 \leq \lambda_1(F^T F) \leq \lambda_2(F^T F) \leq 1\}$$

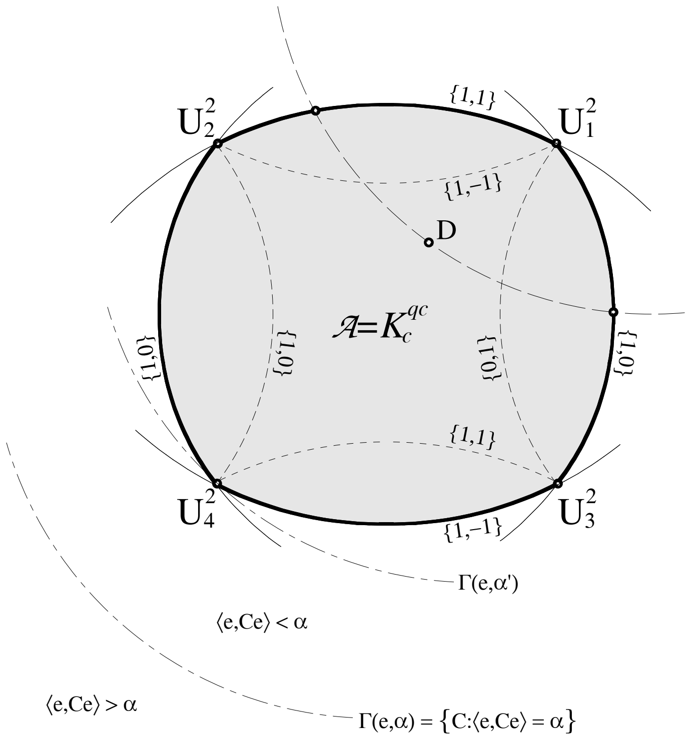

FIGURE 2. *The details of the quasiconvex hull K_c^{qc} for the four-well problem described in Example 3.7 with* SO(2)*-invariant wells.*

where λ_1, λ_2 are the eigenvalues so that

$$K_c^{qc} = \{C : C_{11}C_{22} - C_{12}^2 \geq 0, (C_{11} - 1)(C_{22} - 1) - C_{12}^2 \geq 0\}.$$

The set K_c^{qc} is shown in Figure 3 and is obtained as the intersection of two back-to-back cones given by the two inequalities above, one with apex $C = 0$ and another with apex $C = I$. This is due to the fact that O(2) consists of two copies of SO(2) which have remarkably many rank-one connections: any $Q \in \mathrm{O}(2) \setminus \mathrm{SO}(2)$ is rank-one connected to the identity matrix I. For $k > 1$, the set K_c^{qc} is obtained by combining Figures 1 and 3, i.e., by composing the matrices in $\mathcal{A}$ with short maps. This set is shown in Figure 4, and the boundary consists of $\mathcal{A}$, the cone $\det C = 0$ with apex at $C = 0$, the planes $\langle e, Ce\rangle = \max_{i=1,\dots,k} |U_i e|^2$ and portions of cones with apexes at $U_1^2, \dots U_k^2$. The proof is very similar to that of Theorem 1.1 *i)*: we use the minors relation to find bounds on K_c^{qc} and use lamination to show that these bounds are indeed optimal. Finally note that unlike part *i)* where it is sufficient to use only a finite number of directions e to define the set K^{qc}, in part *ii)* we need all

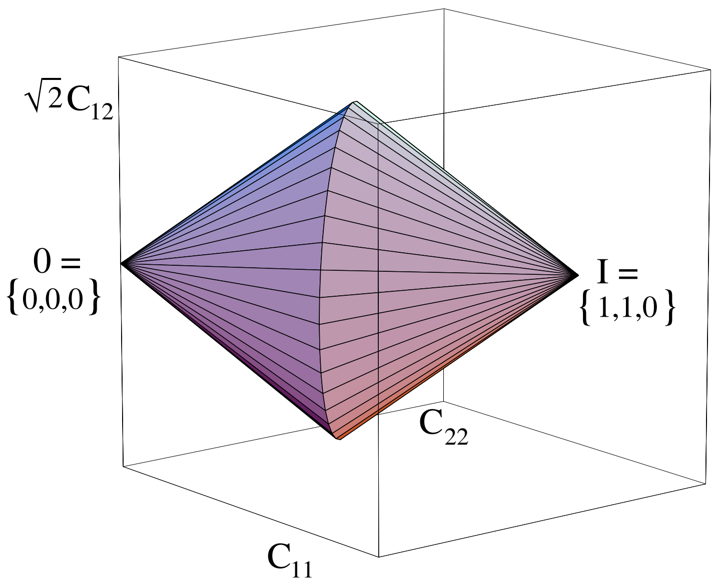

FIGURE 3. *The quasiconvex hull* K_c^{qc} *for* $K = \mathrm{O}(2)$.

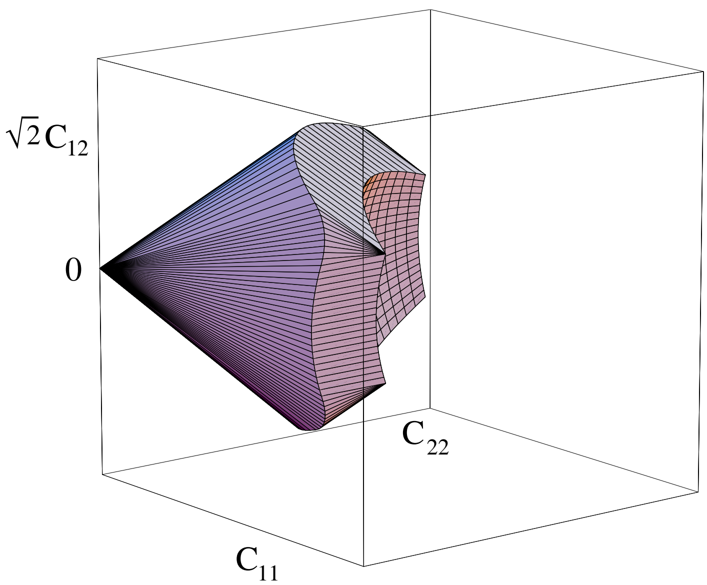

FIGURE 4. *The quasiconvex hull* K_c^{qc} *for the four-well problem described in Example 4.5 with* $\mathrm{O}(2)$*-invariant wells.*

directions $e \in S^1$. A finite subset corresponds to the planar parts of the boundary of K_c^{qc} while the rest defines the cones with apexes at U_i^2.

The set K_c^{qc} when K consists of k copies of $\mathrm{SO}(3)\hat{U}_i$, is described in Theorem 1.3, and is identical to the case when K consists of k copies of $\mathrm{O}(2)$.

The paper is organized as follows. Section 2 collects preliminaries and basic lemmas which are used in the subsequent sections. We suggest that a reader omit it on first reading coming back to it as and when necessary. Theorems 1.1-1.3 will be proven and illustrated with examples in Sections 3-5. Section 6 discusses existence of minimizers while we present in Section 7 uniqueness and non-uniqueness results for microstructures associated with minimizing sequences for the variational problem (1.1). In Section 8 we finally construct approximate relaxed energies.

2. Preliminaries

The generalized convex hulls we are concerned with in this paper are defined as sublevel sets of functions with the corresponding convexity properties. Recall that a function $f : \mathbb{M}^{2\times 2} \to (-\infty, \infty]$ is said to be *convex* if

$$(2.1)\; f(\lambda A + (1-\lambda)B) \le \lambda f(A) + (1-\lambda) f(B) \quad \forall A,\, B \in \mathbb{M}^{2\times 2},\, \lambda \in (0,1);$$

it is said to be *rank-one convex* if (2.1) holds for all A, $B \in \mathbb{M}^{2\times 2}$ with $\operatorname{rank}(A-B) = 1$. Rank-one convexity is a necessary condition for quasiconvexity, the fundamental notion of convexity in the calculus of variations. A function f is *quasiconvex* if

$$\int_{B(0,1)} f(F + D\varphi)dx \ge \int_{B(0,1)} f(F)dx \quad \forall \varphi \in W_0^{1,\infty}(B(0,1);\mathbb{R}^2)$$

(whenever the integral on the left hand side exists). A sufficient condition for quasiconvexity is *polyconvexity*, i.e., there exists a convex function $g : \mathbb{R}^5 \to \mathbb{R}$ such that $f(F) = g(F, \det F)$. We now define for a given compact set $K \subset \mathbb{M}^{2\times 2}$ its *rank-one convex hull* K^{rc} by

$$K^{rc} = \{F \in \mathbb{M}^{2\times 2} : f(F) \le \inf_K f \text{ for all } f : \mathbb{M}^{2\times 2} \to \mathbb{R} \text{ rank-one convex }\}.$$

The *quasiconvex hull* K^{qc}, the *polyconvex hull* K^{pc} and the *convex hull* K^c are defined analogously. Finally we define the *lamination convex hull* K^{lc} in the following way (see [MS1]): Let $K^{(0)} = K$ and define

$$K^{(i+1)} = \big\{\lambda A + (1-\lambda)B : A, B \in K^{(i)}, \operatorname{rank}(A-B) = 1, \lambda \in (0,1)\big\} \cup K^{(i)}.$$

Then

$$K^{lc} = \bigcup_{i=0}^{\infty} K^{(i)}.$$

It follows that

$$K^{lc} \subset K^{rc} \subset K^{qc} \subset K^{pc} \subset K^c, \tag{2.2}$$

(see, e.g., [D], [S]).

We now introduce some notation that we frequently use. Given distinct matrices $U_1, \ldots, U_k \in \mathbb{M}^{2\times 2}_{sym}$, we let

$$\mathcal{U} = \{U_1, \ldots, U_k\}.$$

We note that if $\det U_i = \delta > 0$ for $i = 1, \ldots, k$ then according to the polar decomposition theorem,

$$U_j \notin \bigcup_{i\ne j} \mathrm{SO}(2) U_i$$

for $j = 1, \ldots, k$ so that the SO(2) wells are disjoint. We often use

$$m_{\mathcal{U}}(e) = \max\{|Ue|^2 : U \in \mathcal{U}\}. \tag{2.3}$$

We denote by $e^\perp$ the unique unit vector orthogonal to $e \in S^1$ with $\det(e, e^\perp) = 1$.

We collect in the next two lemmas well-known facts which will be useful throughout the paper.

Lemma 2.1. *Assume that C_1, $C_2 \in \mathbb{M}^{2\times 2}_{sym}$ are positive semidefinite, $C_1 = F_1^T F_1$, $C_2 = F_2^T F_2$. Let $e \in S^1$. Then the following four statements are equivalent:*

i) there exist $Q \in \mathrm{SO}(2)$ and $a \in \mathbb{R}^2$ such that $QF_1 - F_2 = a \otimes e^\perp$;

ii) we have $|F_1 e|^2 = |F_2 e|^2$;

iii) there exists a $v \in \mathbb{R}^2$ such that $C_1 = C_2 + v \otimes e^\perp + e^\perp \otimes v$;

iv) $\det(C_1 - C_2) \le 0$.

Moreover, the vector a in statement i) and the vector v in statement iii) are related by $v = F_2^T a + \frac{1}{2}|a|^2 e^\perp$. Finally, if $\det F_1 = \det F_2$, then $a = \alpha F_2 e$ with $\alpha \in \mathbb{R}$.

Proof: $i) \Rightarrow ii)$: Assume that $QF_1 = F_2 + a \otimes e^\perp$. Then

$$\begin{aligned} C_1 &= (F_2^T + e^\perp \otimes a)(F_2 + a \otimes e^\perp) \\ &= C_2 + F_2^T a \otimes e^\perp + e^\perp \otimes F_2^T a + |a|^2 e^\perp \otimes e^\perp \end{aligned}$$

and *ii)* follows immediately.

$ii) \Rightarrow iii)$: Let $\bar C = C_1 - C_2$. Assume first that $\mathrm{rank}(\bar C) = 1$. Since $\bar C$ is symmetric, there exists a $v \in \mathbb{R}^2$ such that $\bar C = v \otimes v$. By assumption $\langle e, \bar C e\rangle = \langle v, e\rangle^2 = 0$ and thus we obtain *iii)* with $v \,||\, e^\perp$.

Consider now the case $\mathrm{rank}(\bar C) = 2$. Since the eigenvalues λ_i of $\bar C$ satisfy

$$\lambda_1(\bar C) = \min_{v \in S^1} \langle v, \bar C v\rangle < 0 < \max_{v\in S^1} \langle v, \bar C v\rangle = \lambda_2(\bar C),$$

there exist α_1, $\alpha_2 \in \mathbb{R} \setminus \{0\}$ and orthonormal vectors v_1, $v_2 \in \mathbb{R}^2$ such that

$$\begin{aligned} \bar C &= \alpha_2^2 v_2 \otimes v_2 - \alpha_1^2 v_1 \otimes v_1 \\ &= \frac{1}{2}\Big\{(\alpha_2 v_2 - \alpha_1 v_1) \otimes (\alpha_2 v_2 + \alpha_1 v_1) + (\alpha_2 v_2 + \alpha_1 v_1) \otimes (\alpha_2 v_2 - \alpha_1 v_1)\Big\}. \end{aligned}$$

Clearly $\langle e, v_1\rangle \neq 0$ and $\langle e, v_2\rangle \neq 0$ (indeed, if $\langle e, v_1\rangle = 0$, then $v_2 = \gamma e$ with $\gamma \in \{\pm 1\}$ and $e\,\bar C e = \alpha_2^2 \neq 0$, a contradiction). This implies $\langle e, v_2\rangle^2 = \frac{\alpha_1^2}{\alpha_2^2}\langle e, v_1\rangle^2$; since $\{v_1, v_2\}$ is an orthonormal basis of $\mathbb{R}^2$ we infer

$$e = \langle e, v_1\rangle\, v_1 + \langle e, v_2\rangle\, v_2 = \frac{\langle e, v_1\rangle}{\alpha_2}(\alpha_2 v_1 \pm \alpha_1 v_2)$$

and

$$e^\perp = \frac{\langle e, v_1\rangle}{\alpha_2}(\mp \alpha_1 v_1 + \alpha_2 v_2).$$

This proves *iii)* with $v = \alpha_2 v_2 \pm \alpha_1 v_1$.

$iii) \Rightarrow ii)$: This is obvious.

$ii) \Rightarrow i)$: By assumption $|F_1 e|^2 = |F_2 e|^2$ and we may choose $Q \in \mathrm{SO}(2)$ such that $QF_1 e = F_2 e$ or $(QF_1 - F_2)e = 0$. Thus $QF_1 - F_2 = a \otimes n$ is a matrix of rank one and from $(QF_1 - F_2)e = \langle n, e\rangle a = 0$ we deduce that we may choose $n = e^\perp$.

$ii) \Leftrightarrow iv)$: This is also obvious from above (the characterization of $\bar C$).

The relation between a in statement *i)* and v in statement *iii)* follows by direct calculation. Finally, if $\det F_1 = \det F_2$, then *i)* implies $\det F_1 = (\det F_2)(1 + \langle F_2^{-1} a, e^\perp\rangle)$ and thus a must be parallel to $F_2 e$. $\square$

Lemma 2.2. *If* C_1, $C_2 \in \mathbb{M}^{2\times 2}_{sym}$, $C_1 = F_1^T F_1$ *and* $C_2 = F_2^T F_2$ *are positive definite with* $\det C_1 = \det C_2$, *then there exist rotations* $Q_i \in \mathrm{SO}(2)$ *and vectors* a_i, $n_i \in \mathbb{R}^2$, $i = 1, 2$, *such that* n_1 *and* n_2 *are not parallel and* $Q_i F_1 - F_2 = a_i \otimes n_i$. *Moreover,* $|(\lambda Q_1 F_1 + (1-\lambda)F_2)n_2^\perp|^2 < |F_1 n_2^\perp|^2$ *for* $\lambda \in (0,1)$.

Proof: Denote by $\lambda_1 \le \lambda_2$ and $\mu_1 \le \mu_2$ the eigenvalues of C_1 and C_2. Since by hypothesis $\det C_1 = \det C_2$, or $\lambda_1\lambda_2 = \mu_1\mu_2$, we may assume that $\lambda_1 \le \mu_1$ and $\mu_2 \le \lambda_2$, or

$$\min_{w\in S^1} \langle w, C_1 w\rangle \le \min_{w\in S^1} \langle w, C_2 w\rangle \quad \text{and} \quad \max_{w\in S^1} \langle w, C_1 w\rangle \ge \max_{w\in S^1} \langle w, C_2 w\rangle.$$

Therefore, we can deduce from the continuity of the mappings $w \mapsto \langle w, C_1 w\rangle$ and $w \mapsto \langle w, C_2 w\rangle$ that there exists a w_1 such that $\langle w_1, C_1 w_1\rangle = \langle w_1, C_2 w_1\rangle$. By Lemma 2.1 there exist $w_2 \in \mathbb{R}^2$ such that $C_1 = C_2 + w_1^\perp \otimes w_2 + w_2 \otimes w_1^\perp$. The existence of the rank-one connections now follows with $n_1 = w_1^\perp$ and $n_2 = w_2/|w_2|$ from the equivalence $i) \Leftrightarrow iii)$ in Lemma 2.1. The vectors $w_1^\perp$ and w_2 and consequently n_1 and n_2 are not parallel since $\det C_1 = \det(C_2 + \gamma w_1^\perp \otimes w_1^\perp) = (\det C_2)(1 + \gamma\langle w_1^\perp, C_2^{-1} w_1^\perp\rangle) \ne \det C_2$ in view of $\langle w, C_2^{-1} w\rangle \ge \mu_2^{-1} > 0$. The existence of the rank-one connections follows now with $n_1 = w_1^\perp$ and $n_2 = w_2$ from the equivalence $i) \Leftrightarrow iii)$ in Lemma 2.1.

To prove the inequality, note that

$$|F_1 n_2^\perp|^2 = |Q_1 F_1 n_2^\perp|^2 = |F_2 n_2^\perp|^2 + 2\langle n_1, n_2^\perp\rangle\langle F_2 n_2^\perp, a_1\rangle + |a_1|^2 \left(\langle n_1, n_2^\perp\rangle\right)^2.$$

By Lemma 2.1, $|F_1 n_2^\perp|^2 = |F_2 n_2^\perp|^2$ so that

$$2\langle n_1, n_2^\perp\rangle\langle F_2 n_2^\perp, a_1\rangle + |a_1|^2 \left(\langle n_1, n_2^\perp\rangle\right)^2 = 0.$$

Note that $\langle n_1, n_2^\perp\rangle \ne 0$ since n_1 and n_2 are not parallel. Therefore, $\alpha = -\beta < 0$ where $\alpha = 2\langle n_1, n_2^\perp\rangle\langle F_2 n_2^\perp, a_1\rangle$ and $\beta = |a_1|^2\langle n_1, n_2^\perp\rangle$. Finally a calculation shows that

$$|(\lambda Q_1 F_1 + (1-\lambda)F_2)n_2^\perp|^2 < |F_1 n_2^\perp|^2 \iff \lambda\alpha + \lambda^2\beta = \lambda(1-\lambda)\alpha < 0,$$

and we obtain the assertion of the lemma. □

Our characterization of the image of the generalized convex hulls in the three dimensional affine space of symmetric matrices uses the following property of the intersection of the surface $\{C : C \text{ positive definite}, \det C = \delta^2\}$ with the two dimensional hyperplanes $\langle (C_{11}, C_{22}, \sqrt{2}C_{12}), (e_1^2, e_2^2, \sqrt{2}e_1e_2)\rangle = \gamma^2$.

Lemma 2.3. *Assume that* $e \in S^1$ *and* γ, $\delta \in \mathbb{R}$, $\delta > 0$. *Then the set*

$$\Gamma(e;\gamma^2) = \{F^T F : (\det F)^2 = \delta^2,\ |Fe|^2 = \gamma^2\} \subset \mathbb{R}^3$$

is either empty or a smooth one-dimensional manifold which can be parametrized by $t \mapsto F_t^T F_t$ *with* $F_t = F(I + te\otimes e^\perp)$ *for any* $F \in \Gamma(e;\gamma)$.

Proof: Let $E = \{F^T F : \det F \ne 0\} \subset \mathbb{R}^3$ and define $\Phi : E \to \mathbb{R}^2$ by

$$\Phi(X) = \begin{pmatrix} X_{11}X_{22} - X_{12}^2 - \delta^2 \\ e_1^2 X_{11} + e_2^2 X_{22} + 2e_1e_2X_{12} - \gamma^2 \end{pmatrix}.$$

It is easy to see that $\operatorname{rank} D\Phi = 2$ on E and thus $\Phi^{-1}(0)$ is a smooth one-dimensional manifold contained in E. Assume that $\Gamma(e;\gamma) \ne \emptyset$ and let $F_0^T F_0 \in \Gamma(e;\gamma)$. Any $F^T F \in \Gamma(e;\gamma)$ satisfies $|Fe|^2 = |F_0 e|^2$ and thus there exists by Lemma 2.1 an $\alpha \in \mathbb{R}$ such that $F = F_0(I + \alpha e \otimes e^\perp)$. This proves the assertion of the lemma. □

The next lemmas will be important ingredients for the characterization of the boundaries of the generalized convex hulls. Recall that $m_{\mathcal{U}}$ has been defined in (2.3).

Lemma 2.4. *Suppose $F \in \mathbb{M}^{2\times 2}$ satisfies*

i) $|Fe|^2 \leq m_{\mathcal{U}}(e)$ for all $e \in S^1$,
ii) there exist $\tilde{e} \in S^1$ and $i \in \{1,\dots,k\}$ such that $|F\tilde{e}|^2 = |U_i\tilde{e}|^2 > m_{\mathcal{U}\setminus\{U_i\}}(\tilde{e})$.

Then there exists an $\alpha \in \mathbb{R}$ such that $F^TF = U_i^2 - \alpha^2\tilde{e}^\perp \otimes \tilde{e}^\perp$. Moreover, if $\det F = \delta$, then $F = QU_i$ for some $Q \in \mathrm{SO}(2)$.

Proof: In view of Lemma 2.1 there exists a $v \in \mathbb{R}^2$ such that $F^TF = U_i^2 + v \otimes \tilde{e}^\perp + \tilde{e}^\perp \otimes v$. Let $e_\theta = (1+\theta^2)^{-1/2}(\tilde{e} + \theta\tilde{e}^\perp)$. By assumption we may choose $\varepsilon > 0$ small enough such that

$$m_{\mathcal{U}\setminus\{U_i\}}(e_\theta) < |Fe_\theta|^2 \leq |U_ie_\theta|^2 \quad \text{for } |\theta| < \varepsilon.$$

Thus

$$\langle e_\theta, (v \otimes \tilde{e}^\perp + \tilde{e}^\perp \otimes v)e_\theta\rangle = \langle v, e_\theta\rangle\langle\tilde{e}^\perp, e_\theta\rangle = \frac{\theta}{\sqrt{1+\theta^2}}\langle v, e_\theta\rangle \leq 0.$$

We conclude $\sqrt{1+\theta^2}\,\theta\langle v, e_\theta\rangle = \theta\langle\tilde{e}, v\rangle + \theta^2\langle\tilde{e}^\perp, v\rangle \leq 0$ and this implies $v = \gamma\tilde{e}^\perp$ with $\gamma \leq 0$. Thus $F^TF = U_i^2 - \alpha^2\tilde{e}^\perp \otimes \tilde{e}^\perp$. If $\det F = \delta$, then $\delta^2 = \delta^2(1 - \alpha^2|U_i^{-T}\tilde{e}^\perp|^2)$ and therefore $\alpha = 0$. This implies the assertion of the lemma. □

Lemma 2.5. *Suppose there exists $e \in S^1$ and $2 \leq n \leq k$ such that*

$$|U_1e|^2 = \ldots = |U_ne|^2 = m_{\mathcal{U}}(e) > \max\{|U_ie|^2 : i = n+1,\dots,k\}.$$

Set $e_\theta = (1+\theta^2)^{-1/2}(e + \theta e^\perp)$. Then there exist $p, q \in \{1,\dots,n\}$, $p \neq q$ and $\theta_0 > 0$ such that the following three statements hold:

i) $m_{\mathcal{U}}(e_\theta) = |U_pe_\theta|^2 > m_{\mathcal{U}\setminus\{U_p\}}(e_\theta)$ for $-\theta_0 < \theta < 0$;
ii) $m_{\mathcal{U}}(e_\theta) = |U_qe_\theta|^2 > m_{\mathcal{U}\setminus\{U_q\}}(e_\theta)$ for $0 < \theta < \theta_0$;
iii) for all $i \in \{1,\dots,n\}$ we have

$$U_i \in \left(\mathrm{SO}(2)U_p \cup \mathrm{SO}(2)U_q\right)^{(1)}.$$

Proof: It follows from the continuity of the mappings $e \mapsto |U_ie|^2$ that there exists a $\theta_0 > 0$ such that

$$m_{\mathcal{U}}(e_\theta) = \max_{i=1,\dots,n}|U_ie_\theta|^2 > \max_{i=n+1,\dots,k}|U_ie_\theta|^2 \quad \text{for } |\theta| < \theta_0.$$

By Lemma 2.3 (with $F = U_1$) there exist $t_i \in \mathbb{R}$, $i = 1,\dots,n$, such that

$$U_i^2 = U_1^2 + t_i(U_1^2e \otimes e^\perp + e^\perp \otimes U_1^2e) + t_i^2|U_1^2e|^2e^\perp \otimes e^\perp.$$

Relabeling the matrices if necessary, we may assume that $t_1 = 0$ and $t_i > 0$ for $i = 2,\dots,n$. Thus

$$|U_ie_\theta|^2 = |U_1e_\theta|^2 + 2\frac{\theta t_i}{\sqrt{1+\theta^2}}|U_1e|^2 + \frac{\theta^2}{1+\theta^2}(2t_i\langle e^\perp, U_1^2e\rangle + t_i^2|U_1^2e|^2).$$

The conclusions *i)* and *ii)* follow with $t_p = t_1 = 0$ and $t_q = \max_{i=2,\dots,n} t_i$ (where we choose θ_0 sufficiently small).

To prove *iii)* we may assume that $p = 1$, $i = 2$ and $q = 3$ (note that $t_1 = 0 < t_2 < t_3$ by construction of p and q). Let $C(t) = G(t)^TG(t)$ with $G(t) = U_1(I + te \otimes e^\perp)$. Then $C(0) = U_1^TU_1$, $C(t_1) = U_2^TU_2$, and $C(t_2) = U_3^TU_3$. Let V_i be the square root of C_i. By the polar decomposition theorem there exist $Q_i,\, R_i \in \mathrm{SO}(2)$, $i = 1,2,3$ such that $Q_iV_i = G(t_i)$ and $Q_iV_i = R_iU_i$. Let $\lambda = \frac{t_2}{t_3}$. Then

$$(1-\lambda)G(t_1) + \lambda G(t_3) = \frac{t_3 - t_2}{t_3}U_1 + \frac{t_2}{t_3}U_1(I + t_2 e \otimes e^{\perp}) = G(t_2)$$

and therefore $(1-\lambda)R_2^T R_1 U_1 + \lambda R_2^T R_3 U_3 = U_2$. □

Lemma 2.6. *Assume that $U_i \in \mathcal{U}$ and that there exists $\tilde{e} \in S^1$ such that*

$$|U_i\tilde{e}|^2 = m_{\mathcal{U}}(\tilde{e}) > m_{\mathcal{U}\setminus\{U_i\}}(\tilde{e}).$$

Then there exists $U_j \in \mathcal{U}$, $i \neq j$, and $e \in S^1$ such that $|U_ie|^2 = |U_je|^2 = m_{\mathcal{U}}(e)$.

Proof: Suppose the conclusion was wrong. Since $|U_ie|^2 = |U_i(-e)|^2$ we may assume that all vectors $e \in S^1$ are given by $e = e(\varphi) = (\cos\varphi, \sin\varphi)$ with $\varphi \in [0,\pi)$. In particular the map $\varphi \mapsto |U_ie(\varphi)|^2$ is a continuous, periodic map on $[0,\pi]$. By assumption either $|U_ie(\varphi)|^2 > m_{\mathcal{U}\setminus\{U_i\}}(e(\varphi))$ or $|U_ie(\varphi)|^2 < m_{\mathcal{U}\setminus\{U_i\}}(e(\varphi))$. Since the latter case is excluded by assumption, we conclude that the former holds. Choose any $U_k \in \mathcal{U}$, $U_k \neq U_i$. By Lemma 2.2 there exists $t \in \mathbb{R}$, $\bar{e} \in S^1$ and $Q \in \mathrm{SO}(2)$ such that $QU_i - U_k = tU_k\bar{e} \otimes \bar{e}^{\perp}$ and $|U_i\bar{e}|^2 = |U_k\bar{e}|^2 \leq m_{\mathcal{U}}(\bar{e})$. This violates our hypothesis and we deduce that there exists at least one $e \in S^1$ and $U_j \in \mathcal{U}$, $U_j \neq U_i$ such that $|U_ie|^2 = |U_je|^2 = m_{\mathcal{U}}(e)$. □

Lemma 2.7. *Assume that $U_i, U_j \in \mathcal{U}$, $i \neq j$, and that there exists $\varepsilon > 0$ and $e_1 = (\cos\varphi_1, \sin\varphi_1)$ with $\varphi_1 \in [0,\pi)$ such that*

i) $|U_ie_1|^2 = |U_je_1|^2 = m_{\mathcal{U}}(e_1)$,

ii) $m_{\mathcal{U}}(e(\varphi)) = |U_ie(\varphi)|^2 > m_{\mathcal{U}\setminus\{U_i\}}(e(\varphi))$ *for* $\varphi_1 < \varphi < \varphi_1 + \varepsilon$,

iii) $m_{\mathcal{U}}(e(\varphi)) = |U_je(\varphi)|^2 > m_{\mathcal{U}\setminus\{U_j\}}(e(\varphi))$ *for* $\varphi_1 - \varepsilon < \varphi < \varphi_1$.

Then there exists $U_m \in \mathcal{U}$, $m \neq i$, and $e_2 \in S^1$ not parallel to e_1 such that

$$|U_ie_2|^2 = |U_me_2|^2 = m_{\mathcal{U}}(e_2).$$

Proof: Define

$$\tilde{\varphi}_2 = \max\bigl\{\varphi \geq \varphi_1 : |U_ie(\varphi)|^2 = m_{\mathcal{U}}(e(\varphi)) \text{ on } [\varphi_1, \varphi]\bigr\}$$

and let $\varphi_2 = \tilde{\varphi}_2 \mod \pi$. By *ii)*, $\tilde{\varphi}_2 > \varphi_1$ and by *iii)* we conclude $\varphi_2 \neq \varphi_1$. It follows that there exists $\delta > 0$ such that

$$m_{\mathcal{U}}(e(\varphi)) > |U_i(e(\varphi))|^2 \quad \text{for } \varphi_2 < \varphi < \varphi_2 + \delta.$$

The continuity of the mappings $\varphi \mapsto |U_k(e(\varphi))|^2$ implies the assertion of the lemma. □

3. The quasiconvex hull of $\mathrm{SO}(2)U_1 \cup \ldots \cup \mathrm{SO}(2)U_k$

In this section we prove part *i)* in Theorem 1.1 and Theorem 1.2.

We first prove the following version of Theorem 1.1, which uses an infinite number of inequalities to define K^{qc}.

Proposition 3.1. *Assume that $\{U_1, \ldots, U_k\} \subset \mathbb{M}^{2\times 2}_{sym}$ with U_i positive definite and $\det U_i = \delta > 0$. Let $K = \mathrm{SO}(2)U_1 \cup \ldots \cup \mathrm{SO}(2)U_k$. Then $K^{(2)} = K^{lc} = K^{rc} = K^{qc} = K^{pc}$ and any of these hulls is given by*

$$\bigl\{F : \det F = \delta,\, |Fe|^2 \leq \max_{j=1,\ldots,n} |U_je|^2\, \forall e \in S^1\bigr\}.$$

We split the proof of this proposition into a series of lemmas. Let

$$\mathcal{A} = \{F \in \mathbb{M}^{2\times 2} : \det F = \delta,\, |Fe|^2 \le m_{\mathcal{U}}(e)\, \forall e \in S^1\}. \tag{3.1}$$

We will show that $K^{pc} \subset \mathcal{A} \subset K^{(2)}$. This proves the theorem since by (2.2) $K^{lc} \subset K^{pc}$.

Lemma 3.2. *Suppose that the assumptions of Theorem 1.1 hold and that $\mathcal{A}$ is defined by (3.1). Then $K^{pc} \subset \mathcal{A}$.*

Proof: We construct a polyconvex function Φ which vanishes on $\mathcal{A}$ and is positive elsewhere. Let $t_+ = \max\{t, 0\}$ and define for $\nu \in S^1$ the function $g_\nu : \mathbb{M}^{2\times 2} \to \mathbb{R}$ by

$$g_\nu(X) = \big(|X\nu|^2 - m_{\mathcal{U}}(\nu)\big)_+.$$

Clearly g is convex since it is the composition of a convex, nondecreasing function and a convex function.The supremum of convex functions is convex and therefore

$$\Phi(X) = (\det X - \delta)^2 + \sup_{\nu \in S^1} g_\nu(X)$$

is the desired function. $\square$

The reverse inclusion $\mathcal{A} \subset K^{(2)}$ requires some preparation. Let

$$\mathcal{B} = \{F : \det F = \delta,\, |Fe|^2 \le m_{\mathcal{U}}(e)\, \forall\, e \in S^1,\, \exists\, \tilde{e} : |F\tilde{e}|^2 = m_{\mathcal{U}}(\tilde{e})\}. \tag{3.2}$$

As a first step we show in the next lemma that $\mathcal{B} \subset K^{(1)}$. Given U_i, U_j, according to Lemma 2.2 there exists a $Q \in \mathrm{SO}(2)$ and $a, e \in \mathbb{R}^2$ such that $QU_j - U_i = a \otimes e^\perp$. Let

$$\Gamma_{i,j}(e; |U_i e|^2) = \{(U_j + \lambda a \otimes e^\perp)^T (U_j + \lambda a \otimes e^\perp) : \lambda \in [0,1]\} \subset \mathbb{R}^3$$

denote the arc connecting $U_i^T U_i$ and $U_j^T U_j$ on the curve $\Gamma(e; |U_i e|^2)$.

Lemma 3.3. *Assume that $k \ge 2$. Let $F \in \mathcal{B}$ and $C = F^T F$.*

i) There exist $e \in S^1$, $U_p, U_q \in \mathcal{U}$, $p \ne q$, such that $|U_p e|^2 = |U_q e|^2 = m_{\mathcal{U}}(e)$ and $C \in \Gamma_{p,q}(e; m_{\mathcal{U}}(e))$. Moreover, we may choose p and q in such a way that there exist $\tilde{e}_p$, $\tilde{e}_q \in S^1$ such that $m_{\mathcal{U}}(\tilde{e}_p) = |U_p \tilde{e}_p|^2 > m_{\mathcal{U}\setminus\{U_p\}}(\tilde{e}_p)$ and $m_{\mathcal{U}}(\tilde{e}_q) = |U_q \tilde{e}_q|^2 > m_{\mathcal{U}\setminus\{U_q\}}(\tilde{e}_q)$.

ii) We have $\mathcal{B} \subset K^{(1)}$.

Proof: By definition of $\mathcal{B}$ there exists at least one $e \in S^1$ such that $|Fe|^2 = m_{\mathcal{U}}(e)$. If there exists an e such that $m_{\mathcal{U}}(e) = |Fe|^2 = |U_i e|^2 > m_{\mathcal{U}\setminus\{U_i\}}(e)$ for some $i \in 1, \ldots, n$, then it follows from Lemma 2.4 that $F = QU_i$ with $Q \in \mathrm{SO}(2)$ and thus *i)* follows from Lemma 2.6 and Lemma 2.5. Therefore, we may assume (relabeling the matrices if necessary) that there exists $2 \le n \le k$ such that

$$|Fe|^2 = |U_1 e|^2 = \ldots = |U_n e|^2 = m_{\mathcal{U}}(e) > \max\big\{|U_i e|^2 : i = n+1, \ldots, k\big\}.$$

Let p, $q \in \{1, \ldots, n\}$, $p \ne q$ be the indices with the properties stated in Lemma 2.5. By Lemma 2.1, there exist $Q \in \mathrm{SO}(2), \alpha \in \mathbb{R}\backslash\{0\}$ such that $QU_q - U_p = \alpha U_p e \otimes e^\perp$. Note that $\alpha > 0$: Indeed, by expansion and Lemma 2.5 with $e_\theta = (1+\theta^2)^{-1/2}(e + \theta e^\perp)$ we obtain

$$|U_q e_\theta|^2 = |U_p e_\theta|^2 + \frac{2\theta\alpha}{1+\theta^2}|U_p e|^2 + \mathcal{O}(\theta^2) < |U_p e_\theta|^2 \quad \forall \theta \in (-\theta_0, 0)$$

and this proves the asserted inequality. By Lemma 2.3, $F = \tilde{Q}(U_p + s\alpha U_p e \otimes e^\perp)$ for some $\tilde{Q} \in \mathrm{SO}(2), s \in \mathbb{R}$, so that

$$|Fe_\theta|^2 = |U_p e_\theta|^2 + \frac{2\theta\alpha s}{1+\theta^2}|U_p e|^2 + \mathcal{O}(\theta^2).$$

Since $F \in \mathcal{B}$ we have $|Fe_\theta|^2 \le m_{\mathcal{U}}(e_\theta) = |U_p e_\theta|^2$ for $-\theta_0 < \theta < 0$, and we conclude that $s \ge 0$. Similarly, $|Fe_\theta|^2 \le |U_q e_\theta|^2$ for $0 < \theta < \theta_0$ and therefore

$$|U_p e_\theta|^2 + \frac{2\theta\alpha s}{1+\theta^2}|U_p e|^2 + \mathcal{O}(\theta^2) \le |U_p e_\theta|^2 + \frac{2\theta\alpha}{1+\theta^2}|U_p e|^2 + \mathcal{O}(\theta^2)$$

and we conclude that $s \le 1$. This proves *i)*. Finally *ii)* follows from the observation that

$$\big\{F \in \mathbb{M}^{2\times 2} : \det F = \delta,\, F^T F \in \Gamma_{p,q}(e; m_{\mathcal{U}}(e))\big\} \subset \big(\mathrm{SO}(2)U_p \cup \mathrm{SO}(2)U_q\big)^{(1)}$$

using the definition of $\mathcal{B}$. ☐

We are now in a position to prove part *i)* in Proposition 3.1.

Proof of Proposition 3.1: In view of Lemma 3.2 it remains to show that $\mathcal{A} \subset K^{(2)} \subset K^{lc}$. By Lemma 3.3 we have $\mathcal{B} \subset K^{(1)} \subset K^{(2)}$. Assume now that $F \in \mathcal{A}\backslash\mathcal{B}$. Fix any $e \in S^1$ and let $F_t = F(I + te \otimes e^\perp)$ and

$$C(t) = F_t^T F_t = F^T F + t\big(F^T F e \otimes e^\perp + e^\perp \otimes F^T F e\big) + t^2 |Fe|^2 e^\perp \otimes e^\perp.$$

Since $Fe \neq 0$ we conclude $|C(t)|^2 \to \infty$ for $t \to \pm\infty$ and therefore

$$\begin{aligned} t^+ &= \sup\{t > 0 : |F_s e|^2 < m_{\mathcal{U}}(e)\, \forall e \in S^1,\, \forall s \in [0,t]\}, \\ t^- &= \inf\{t < 0 : |F_s e|^2 < m_{\mathcal{U}}(e)\, \forall e \in S^1,\, \forall s \in [0,t]\} \end{aligned}$$

are well-defined and $-\infty < t^- < 0 < t^+ < \infty$. By construction $F^T F$ is contained in the arc connecting C^+ and C^- on the curve $\Gamma(e; |Fe|^2)$. Let $V^\pm$ be the square root of $C^\pm$. Then $F \in (\mathrm{SO}(2)V^+ \cup \mathrm{SO}(2)V^-)^{(1)}$ and since $V^\pm \in \mathcal{B} \subset K^{(1)}$ we conclude $\mathcal{A} \subset K^{(2)}$. This proves the proposition. ☐

The quasiconvex hull of two martensitic wells in two dimensions with equal determinant $\delta > 0$ was first obtained by Ball and James [BJ3]. We recover their result as a special case in Proposition 3.1.

Example 3.4. *(The two-well problem) Assume that* $\det U_1 = \det U_2 = \delta > 0$, $\mathrm{SO}(2)U_1 \neq \mathrm{SO}(2)U_2$, *and let* $K = \mathrm{SO}(2)U_1 \cup \mathrm{SO}(2)U_2$. *Then there exist two vectors* e_1, e_2 *such that*

$$K^{qc} = \big\{F \in \mathbb{M}^{2\times 2} : \det F = \delta,\, |Fe_i|^2 \le \max\{|U_1 e_i|^2,\, |U_2 e_i|^2\},\, i = 1,2\big\}.$$

It is easy to see that for $k = 2$ there exist exactly two rank-one connections between the wells $\mathrm{SO}(2)U_1$ and $\mathrm{SO}(2)U_2$, i.e. there exist $Q_i \in \mathrm{SO}(2)$ and a_i, $e_i \in \mathbb{R}^2$ such that $Q_i U_1 - U_2 = a_i \otimes e_i^\perp$. Let $\mathcal{U} = \{U_1, U_2\}$ and

$$\mathcal{A} = \big\{F \in \mathbb{M}^{2\times 2} : \det F = \delta,\, |Fe_i|^2 \le m_{\mathcal{U}}(e_i),\, i = 1,2\big\}.$$

We have to show that $F \in \mathcal{A}$ implies $|Fe|^2 \le m_{\mathcal{U}}(e)$ for all $e \in S^1$. Assume the contrary. Then there exists an $e \in S^1$ such that $|Fe|^2 > m_{\mathcal{U}}(e)$. Assume first that $|Fe_1|^2 = m_{\mathcal{U}}(e_1)$ (the case that this equality holds for e_2 is similar). There exist $t_0 \in \mathbb{R}$ and $Q_1 \in \mathrm{SO}(2)$ such that $F - Q_1 U_1 = t_0 Q_1 U_1 e_1 \otimes e_1^\perp$. Let $F(t) = Q_1 U_1 + t Q_1 U_1 e_1 \otimes e_1^\perp$. By assumption there exist $t_2 \in \mathbb{R}$ and $Q_2 \in \mathrm{SO}(2)$ such that $F(t_2) = Q_2 U_2$. Since $g(t) = |F(t)e_2|^2 > 0$ is a quadratic function with

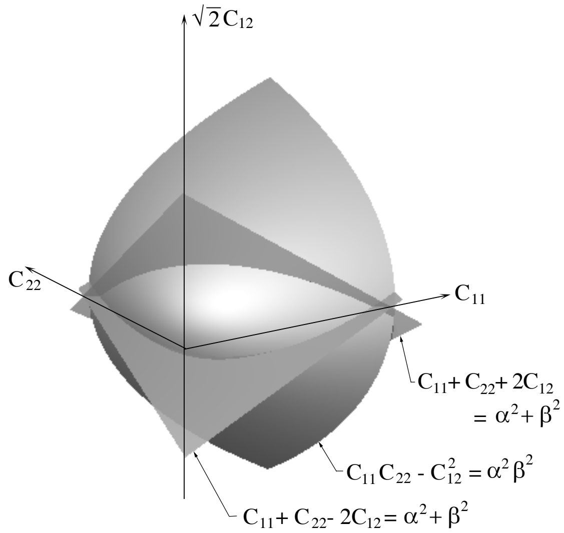

FIGURE 5. *The set* $(\mathrm{SO}(2)U_1 \cup \mathrm{SO}(2)U_2)_c^{qc}$ *for the diagonal matrices* $U_1 = \mathrm{diag}(\alpha,\beta)$ *and* $U_2 = \mathrm{diag}(\beta,\alpha)$.

$g(0) = g(t_2) \geq g(t_0)$ we conclude $t_2 \leq t_0 \leq 0$ or $0 \leq t_0 \leq t_2$. This shows that $F = \lambda Q_1 U_1 + (1-\lambda) Q_2 U_2$ with Q_1, $Q_2 \in \mathrm{SO}(2)$ and $\lambda \in [0,1]$. Thus

$$m_{\mathcal{U}}(e) < |Fe|^2 \leq \lambda |U_1 e|^2 + (1-\lambda)|U_2 e|^2 \leq m_{\mathcal{U}}(e),$$

and we conclude $|Fe|^2 = m_{\mathcal{U}}(e)$, a contradiction. Thus we may assume that $|Fe_i| < m_{\mathcal{U}}(e_i)$ for $i = 1, 2$. Let $F_t = F + tFe \otimes e^{\perp}$. Then $\det F_t = \delta$ and

$$|F_t e_i|^2 = |Fe_i + t\langle e_i, e^{\perp}\rangle Fe|^2.$$

Since e_1 and e_2 are linearly independent, $\langle e_i, e^{\perp}\rangle \neq 0$ for at least one of the two indices and we may choose $s > 0$ such that $|F_s e_1|^2 = m_{\mathcal{U}}(e_1)$ and $|F_s e_2|^2 \leq m_{\mathcal{U}}(e_2)$ (or vice versa). Clearly $F_s \in \mathcal{A}$ and it follows as above that $|F_s e|^2 = |Fe|^2 = m_{\mathcal{U}}(e)$, a contradiction. See Figure 5 for a sketch of the set where U_1 and U_2 are diagonal. Conversely, any set $\tilde{K}$ on the hyperboloid $\{X = F^T F : \det F = \delta\}$ which is bounded by two arcs of the form above can be described by

$$\tilde{K} = \{F^T F : F \in \big(\mathrm{SO}(2)U_1 \cap \mathrm{SO}(2)U_2\big)^{qc}\}.$$

□

We now turn to the proof of part *i*) of Theorem 1.1 which says that the generalized convex hulls are always described by a finite number of vectors as in Example 3.4. The next rather technical lemmas are the main ingredient in the proof. Let

$$\tilde{\mathcal{U}} = \big\{U_i \in \mathcal{U} : |U_i \tilde{e}|^2 > m_{\mathcal{U}\setminus\{U_i\}}(\tilde{e}) \text{ for some } \tilde{e} \in S^1\big\}.$$

Relabeling the matrices if necessary we may assume that

$$\tilde{\mathcal{U}} = \{U_1, \ldots, U_n\}.$$

Lemma 3.5. *Let $\mathcal{U}$ and $\tilde{\mathcal{U}}$ be defined as above. Then*

$$\Big(\bigcup_{i=1}^{k}\mathrm{SO}(2)U_i\Big)^{qc} = \Big(\bigcup_{i=1}^{n}\mathrm{SO}(2)U_i\Big)^{qc}$$

Proof: In view of Proposition 3.1 we only have to show that $|U_ie|^2 \le m_{\tilde{\mathcal{U}}}(e)$ for all $e \in S^1$. It suffices to show this for U_k. Assume that there exists an $e \in S^1$ such that $|U_ke|^2 > m_{\tilde{\mathcal{U}}}(e)$. Relabeling the matrices (if necessary) we may assume that there exists an $\ell \in \{n+1,\dots,k\}$ such that $|U_ie|^2 \ge |U_ke|^2 > m_{\tilde{\mathcal{U}}}(e)$ for $i = \ell,\dots,k$ and $|U_ie|^2 < |U_ke|^2$ for $i = n+1,\dots,\ell+1$. If $k = \ell$ then $|U_ke|^2 > m_{\mathcal{U}\setminus\{U_k\}}$, contradicting $U_k \in \mathcal{U}\setminus\tilde{\mathcal{U}}$. We obtain the same contradiction if there exists an $i \in \{\ell,\dots,k\}$ such that $|U_ie|^2 > \max\{|U_je|^2, j = \ell,\dots,k,\, j \ne i\}$. Thus we may assume (relabeling again the matrices, if necessary) that $U_\ell,\dots,U_k \in \mathcal{U}\setminus\tilde{\mathcal{U}}$ satisfy

$$|U_\ell e|^2 = \ldots = |U_ke|^2 > m_{\{U_1,\dots,U_{l-1}\}}(e)$$

with $\ell < k$. In this situation it follows from Lemma 2.5 *i)* that there exists a $p \in \{\ell,\dots,k\}$ and an $e_\theta \in S^1$ such that

$$|U_pe_\theta|^2 > m_{\mathcal{U}\setminus\{U_p\}}(e_\theta),$$

contradicting the assumption $U_p \in \mathcal{U}\setminus\tilde{\mathcal{U}}$. This proves the assertion of the Lemma. $\square$

Lemma 3.6. *Assume that $n \ge 2$. The set $\tilde{\mathcal{U}}$ has the following properties:*

i) If $e \in S^1$ and $U_i, U_j \in \tilde{\mathcal{U}}$, $i \ne j$, such that $|U_ie|^2 = |U_je|^2 = m_{\mathcal{U}}(e)$, then $|U_\ell e|^2 < m_{\mathcal{U}}(e)$ for all $\ell \in \{1,\dots,n\}\setminus\{i,j\}$.

ii) For all $U_i \in \tilde{\mathcal{U}}$ there exist exactly two matrices $U_{i_1}, U_{i_2} \in \tilde{\mathcal{U}}$, $i \notin \{i_1,i_2\}$ and exactly two non parallel vectors $e_1, e_2 \in S^1$ such that $|U_ie_j|^2 = |U_{i_j}e_j|^2 = m_{\mathcal{U}}(e_j)$ for $j = 1, 2$, and $\Gamma_{i,i_j}(e_j; m_{\mathcal{U}}(e_j)) \subset \{F^TF : F \in \mathcal{B}\}$.

iii) Assume that $\Gamma_\ell(e_\ell; m_{\mathcal{U}}(e_\ell)) = \Gamma_{i_\ell,j_\ell}(e_\ell; m_{\mathcal{U}}(e_\ell))$, $\ell = 1, 2$, are two of the arcs constructed in ii) and let $\mathring{\Gamma}_\ell = \Gamma_\ell\setminus\{U_{i_\ell}^TU_{i_\ell}, U_{j_\ell}^TU_{j_\ell}\}$. Then $\mathring{\Gamma}_1 \cap \mathring{\Gamma}_2 = \emptyset$.

iv) For each $F \in \mathcal{B}$ there exist $U_i, U_j \in \tilde{\mathcal{U}}$, $i \ne j$, and $e \in S^1$ such that $|U_ie|^2 = |U_je|^2 = |Fe|^2 = m_{\mathcal{U}}(e)$ and $F^TF \in \Gamma_{i,j}(e; m_{\mathcal{U}}(e)) \subset \{G^TG : G \in \mathcal{B}\}$.

Proof: We first prove *i)*, *iii)* and *iv)* and then *ii)*.

i) Assume that there are three distinct matrices $U_i, U_j, U_\ell \in \tilde{\mathcal{U}}$ such that $|U_ie|^2 = |U_je|^2 = |U_\ell e|^2 = m_{\mathcal{U}}(e)$. By Lemma 2.5 there exists $\beta \in \{i, j, \ell,\}$ and $Q_\alpha, Q_\gamma \in \mathrm{SO}(2)$, $\lambda \in (0,1)$ such that $U_\beta = \lambda Q_\alpha U_\alpha + (1-\lambda)Q_\gamma U_\gamma$ where $\{\alpha, \beta, \gamma\} = \{i, j, l\}$. Since $U_\beta \in \tilde{\mathcal{U}}$ there exists $\tilde{e} \in S^1$ such that $|U_\beta\tilde{e}|^2 = m_{\mathcal{U}}(\tilde{e}) > m_{\mathcal{U}\setminus\{U_\beta\}}(\tilde{e})$. Then

$$m_{\mathcal{U}}(\tilde{e}) = |U_\beta\tilde{e}|^2 \le \lambda|U_\alpha\tilde{e}|^2 + (1-\lambda)|U_\gamma\tilde{e}|^2 \le m_{\mathcal{U}}(\tilde{e})$$

and therefore $|U_\alpha\tilde{e}|^2 = |U_\gamma\tilde{e}|^2 = m_{\mathcal{U}}(\tilde{e})$. This contradicts the assumption and we conclude $U_\beta \notin \tilde{\mathcal{U}}$.

iii) Assume that $F^TF \in \mathring{\Gamma}_1 \cap \mathring{\Gamma}_2$. By construction there exist $Q_i, Q_j \in \mathrm{SO}(2)$ and $\lambda \in (0,1)$ such that $\lambda Q_iU_{i_2} + (1-\lambda)Q_jU_{j_2} = F$. By assumption

$$m_{\mathcal{U}}(e_1) = |Fe_1|^2 \le \lambda|U_{i_2}e_1|^2 + (1-\lambda)|U_{j_2}e_1|^2 \le m_{\mathcal{U}}(e_1)$$

and thus

$$|U_{i_1}e_1|^2 = |U_{j_1}e_1|^2 = |U_{i_2}e_1|^2 = |U_{j_2}e_1|^2 = m_{\mathcal{U}}(e_1).$$

If $\{i_1, j_1\} \neq \{i_2, j_2\}$, this contradicts *i)* and we obtain the assertion.

Otherwise we conclude by Lemma 2.2 that e_1 and e_2 are not parallel and that there exist a_1, $a_2 \in \mathbb{R}^2$, Q_1, $Q_2 \in \mathrm{SO}(2)$ such that

$$Q_1 U_i - U_j = a_1 \otimes e_1^\perp, \qquad Q_2 U_i - U_j = a_2 \otimes e_2^\perp,$$

where we write U_i and U_j instead of U_{i_ℓ} and U_{j_ℓ}. Let $F_\lambda = U_j + \lambda a_1 \otimes e_1^\perp$. In order to show that the arcs $\Gamma_{i,j}(e_1, m_{\mathcal{U}}(e_1))$ and $\Gamma_{i,j}(e_2, m_{\mathcal{U}}(e_2))$ do not intersect, it suffices to show that $|F_\lambda e_2|^2 < m_{\mathcal{U}}(e_2)$ for $\lambda \in (0,1)$. For $\lambda = 1$ we obtain

$$|Q_1 U_i e_2|^2 = |U_j e_2|^2 + 2\langle e_1^\perp, e_2\rangle\langle U_j e_2, a_1\rangle + \langle e_1^\perp, e_2\rangle^2 |a_1|^2 = 0$$

and thus by assumption

$$2\langle e_1^\perp, e_2\rangle\langle U_j e_2, a_1\rangle + \langle e_1^\perp, e_2\rangle^2 |a_1|^2 = 0.$$

Therefore $\alpha = 2\langle e_1^\perp, e_2\rangle\langle U_j e_2, a_1\rangle < 0$ and $\beta = \langle e_1^\perp, e_2\rangle^2 |a_1|^2 > 0$ (note that $\langle e_1^\perp, e_2\rangle \neq 0$ since e_1 and e_2 are not parallel). Since $|F_\lambda e_2|^2 < m_{\mathcal{U}}(e_2)$ if and only if $\lambda\alpha + \lambda^2\beta = \lambda(1-\lambda)\alpha < 0$, we obtain the assertion.

iv) This follows from Lemma 3.3.

ii) This is easy for $n = 2$ since there are exactly two rank-one connections between the wells. Thus we may assume that $n \geq 3$. Fix U_i. By Lemma 2.6 combined with Lemma 2.5 there exists at least one $e_1 \in S^1$ and $U_j \in \tilde{\mathcal{U}}$, $i \neq j$, such that $|U_i e_1|^2 = |U_j e_1|^2 = m_{\mathcal{U}}(e_1)$. In view of Step 1 we obtain $|U_\ell e_1|^2 < m_{\mathcal{U}}(e_1)$ for $\ell \notin \{i, j\}$ and it follows from Lemma 2.5 that the assumptions *ii)* and *iii)* in Lemma 2.7 are satisfied for some $\varepsilon > 0$. We conclude that there exist at least two linearly independent vectors e_1, $e_2 \in S^1$ such that $|U_i e_1|^2 = |U_j e_1|^2 = m_{\mathcal{U}}(e_1)$ and $|U_i e_2|^2 = |U_\ell e_2|^2 = m_{\mathcal{U}}(e_2)$ with U_j, $U_\ell \in \tilde{\mathcal{U}}$ and $\ell \neq i$.

Assume now that $|U_{i_j} e_j|^2 = |U_i e_j|^2 = m_{\mathcal{U}}(e_j)$ for $j = 1, 2, 3$, where no two of the vectors e_j are parallel and $i \notin \{i_1, i_2, i_3\}$. If $i_1 = i_2 = i_3$, then it is easy to see that $U_i = Q U_{i_1}$ with $Q \in \mathrm{SO}(2)$, violating the general assumptions on $\tilde{\mathcal{U}}$. Thus we may assume that $i_1 \neq i_2$. If $i_1 \neq i_2 = i_3$ then we define $\mathcal{V} = \{U_{i_2}, U_i\}$ and

$$\mathcal{A} = \big\{F \in \mathbb{M}^{2\times 2} : \det F = \delta,\, |F e_j|^2 \leq m_{\mathcal{V}}(e_{i_j}),\, j = 2, 3\big\}.$$

It follows from *i)* that $U_{i_1} \in \mathcal{A}$ and we conclude with the same arguments as in Example 3.4 that

$$U_{i_1} \in \big(\mathrm{SO}(2) U_i \cup \mathrm{SO}(2) U_{i_2}\big)^{qc}.$$

By definition of $\tilde{\mathcal{U}}$ there exists an $\tilde{e} \in S^1$ such that $|U_{i_1}\tilde{e}|^2 = m_{\mathcal{U}}(\tilde{e}) > m_{\mathcal{U}\setminus U_{i_1}}(\tilde{e})$. However, by Example 3.4,

$$m_{\mathcal{U}}(\tilde{e}) = |U_{i_1}\tilde{e}|^2 \leq \max\big\{|U_i\tilde{e}|^2,\, |U_{i_2}\tilde{e}|^2\big\} \leq m_{\mathcal{U}}(\tilde{e}).$$

This is a contradiction. Finally assume that $i_j \neq i_\ell$ for $j \neq \ell$. The curves $\Gamma(e_j; |U_i e_j|^2)$ are the boundary of the regions $|U_i e_j|^2 > m_{\mathcal{U}}(e_j)$ and $|U_i e_j|^2 < m_{\mathcal{U}}(e_j)$ and using the ideas in the proof of *iii)* we see that they intersect only at U_i. Thus each of these regions consists of just one connected component. Consider now the curve $\Gamma(e_1; m_{\mathcal{U}}(e_1))$. Then U_{i_2} and U_{i_3} must lie in the connected component $\{C = F^T F : \det C = \delta^2, |F e_1|^2 < m_{\mathcal{U}}(e_1)\}$. Assume that the angle between the curves $\Gamma(e_1; m_{\mathcal{U}}(e_1))$ and $\Gamma(e_2; m_{\mathcal{U}}(e_2))$ is smaller than the angle between $\Gamma(e_1; m_{\mathcal{U}}(e_1))$ and $\Gamma(e_3; m_{\mathcal{U}}(e_3))$. Since $|U_i e_1|^2 = |U_{i_2} e_2|^2 = m_{\mathcal{U}}(e_2)$ we conclude $|U_{i_1} e_2|^2 < m_{\mathcal{U}}(e_2)$ and thus $|U_{i_3} e_2|^2 > m_{\mathcal{U}}(e_2)$, a contradiction. This proves assertion *ii)* of the lemma. $\square$

With this information at hand we can prove part *i)* in Theorem 1.1.

Proof of part i) of Theorem 1.1: Consider the graph $\mathcal{G} = \mathcal{G}(\mathcal{N}, \mathcal{E})$ where $\mathcal{N} = \tilde{\mathcal{U}}$ is the set of nodes and $\mathcal{E}$ is the set of edges which contains an edge connecting U_i and U_j if and only if there exists an arc $\Gamma_{i,j}(e_{ij}; m(e_{ij}))$ with the properties in Lemma 3.6. Thus there is a one-to-one correspondence of arcs in $\mathcal{B}$ and edges in $\mathcal{E}$ and it follows from Lemma 3.6 that $\mathcal{G}$ is a graph of degree two (i.e. each node is contained in exactly two edges). It is easy to see that $\mathcal{G}$ must consist of disjoint cycles. By Lemma 3.6 the arcs Γ_{ij} corresponding to the edges in the cycles do not intersect and therefore each of these cycles can be interpreted as a closed curve on the hyperboloid $\{\det C = \delta^2\} \subset \mathbb{R}^3$. It is easy to see that the set $\mathcal{A}$ is connected and therefore $\mathcal{G}$ must consist of a single cycle. It follows that $\mathcal{E}$ contains exactly n edges. Let $\mathcal{E}_n$ be the set of normals e_{ij} which define to arcs Γ_{ij} corresponding to the edges in $\mathcal{E}$. By Lemma 3.6, $\mathcal{B}$ is the union of these arcs and therefore K^{qc} is defined by k inequalities. This proves the assertion of the theorem. □

Example 3.7. *(The four-well problem) Assume that $a, b, c > 0$, $a > b$, $ab - c^2 > 0$ and define*

$$U_1 = \begin{pmatrix} a & c \\ c & b \end{pmatrix}, U_2 = \begin{pmatrix} b & c \\ c & a \end{pmatrix}, U_3 = \begin{pmatrix} a & -c \\ -c & b \end{pmatrix}, U_4 = \begin{pmatrix} b & -c \\ -c & a \end{pmatrix}.$$

Let $\mathcal{U} = \{U_1, U_2, U_3, U_4\}$. Then

$$K^{qc} = \big\{F \in \mathbb{M}^{2\times 2} : \det F = \delta, \, |Fe|^2 \le m_{\mathcal{U}}(e) \, \forall e \in \mathcal{E}_4\big\}$$

where

$$\mathcal{E}_4 = \Big\{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\Big\}.$$

See Figures 1 and 2 for a sketch of the set $K^{qc}_c = \{F^T F : F \in K^{qc}\}$ and the rank-one connections defining the boundaries on the manifold $\{\det C = (\det U_1)^2\}$. □

We finally prove Theorem 1.2.

Proof of Theorem 1.2: Let

$$\mathcal{A} = \{F \in \mathbb{M}^{3\times 3} : \det F = \delta, \, F^T F v = \mu^2 v, \, |Fe|^2 \le \max_{i=1,\dots,k} |U_i e|^2 \; \forall e \in S^2\}.$$

We first show that $K^{pc} \subset \mathcal{A}$ by constructing a polyconvex function Φ which vanishes on $\mathcal{A}$ and is positive elsewhere. This generalizes the construction in [BJ4] for the two-well problem. For $\nu \in S^2$ let $g_\nu : \mathbb{M}^{3\times 3} \to \mathbb{R}$ be defined by

$$g_\nu(X) = \big(|X\nu|^2 - m_{\mathcal{U}}(\nu)\big)_+$$

and let

$$\Phi(X) = (\det X - \delta)^2 + \sup_{\nu \in S^2} g_\nu(X) + \big(|Fv|^2 - \mu^2\big)_+ + \big(|\operatorname{cof} Fv|^2 - \frac{\delta^2}{\mu^2}\big)_+.$$

We have to show that $\Phi(X) = 0$ implies $F^T F v = \mu^2 v$. Since $\operatorname{cof} F = (\det F) F^{-T}$ it follows from $\Phi(X) = 0$ that

$$|Fv|^2 \le \mu^2$$

and

$$|\operatorname{cof} Fv|^2 \le \frac{\delta^2}{\mu^2} \Leftrightarrow |F^{-T} v|^2 \le \frac{1}{\mu^2}.$$

Then

$$\begin{aligned}\big|(\frac{1}{\mu}F)(v-\mu^2F^{-1}F^{-T}v)\big| &= \big|\frac{1}{\mu}Fv-\mu F^{-T}v\big|^2\\ &= \frac{1}{\mu^2}|Fv|^2+\mu^2|F^{-T}v|^2-2\le 0,\end{aligned}$$

and since $\det F=\delta>0$ we conclude $v-\mu^2F^{-1}F^{-T}v=0$. This implies the assertion.

We now show that $\mathcal{A}\subset K^{(2)}$. We will reduce the necessary constructions to the two-dimensional situation in Theorem 1.1. Let $F\in\mathcal{A}$. By the polar decomposition theorem we have $F=RU_0$ with $R\in\mathrm{SO}(3)$ and U_0 symmetric and positive definite. Since $F^TFv=U_0^2v=\mu^2v$ we conclude $U_0v=\mu v$. Thus the matrices U_i, $i=0,\dots,k$, have μ as common eigenvalue with corresponding eigenvector v. Choose an orthonormal basis $\{v_1,v_2,v_3=v\}$ and let Q be the rotation with columns v_i. Then

$$Q^TU_iQ=\begin{pmatrix}\hat U_i & 0\\ 0 & \mu\end{pmatrix},$$

with $\hat U_i\in\mathbb{M}^{2\times2}_{sym}$, $\det\hat U_i=\delta/\mu$. Let $\hat{\mathcal{U}}=\{\hat U_1,\dots,\hat U_k\}$. Now define $\hat\pi:\mathbb{R}^3\to\mathbb{R}^2$ by $\hat\pi(u)=(u_1,u_2)$ and $\pi_3:\mathbb{R}^3\to\mathbb{R}$ by $\pi_3(u)=u_3$ for $u\in\mathbb{R}^3$. For $e\in S^2$ we have

$$\begin{aligned}|Fe|^2 &= |RU_0e|^2=|Q^TU_0QQ^Te|^2=|\hat U_0\hat\pi(Q^Te)|^2+\mu^2|\pi_3(Q^Te)|^2\\ &\le m_{\mathcal{U}}(e)=\max_{i=1,\dots,k}|\hat U_i\hat\pi(Q^Te)|^2+\mu^2|\pi_3(Q^Te)|^2.\end{aligned}$$

If we choose e such that $\hat\pi(Q^Te)\in S^1$ then we obtain

$$|\hat U_0e|^2\le m_{\hat{\mathcal{U}}}(e)\quad\forall e\in S^1.$$

It follows from Theorem 1.1 that

$$\hat U_0\in\big(\mathrm{SO}(2)\hat U_1\cup\dots\cup\mathrm{SO}(2)\hat U_k\big)^{(2)}.$$

Since K is invariant under multiplication with elements in $\mathrm{SO}(3)$ from the left and since $\mathrm{rank}(A-B)=1$ if and only if $\mathrm{rank}(Q(A-B)Q^T)=1$ if follows that $F\in K^{(2)}$.

Finally, the reduction from an infinite to a finite number of inequalities in the definition of the hulls follows as in the proof of part *i)* of Theorem 1.1. $\square$

4. The quasiconvex hull of $\mathrm{O}(2)U_1\cup\ldots\cup\mathrm{O}(2)U_k$

In this section we prove part *ii)* of Theorem 1.1. We split the proof into a series of lemmas. As before let $\mathcal{U}=\{U_1,\dots,U_k\}$ and

$$\mathcal{A}=\{F\in\mathbb{M}^{2\times2}:|\det F|\le\delta,\,|Fe|^2\le m_{\mathcal{U}}(e)\,\forall e\in S^1\}. \tag{4.1}$$

We will show that $K^{pc}\subset\mathcal{A}\subset K^{(3)}$. This proves the theorem since by (2.2) $K^{lc}\subset K^{pc}$. We prove first the inclusion $K^{pc}\subset\mathcal{A}$ by constructing a polyconvex function which vanishes exactly on $\mathcal{A}$ and is positive elsewhere.

Lemma 4.1. *Suppose the assumptions of Theorem 1.1 hold and $\mathcal{A}$ is defined by (4.1). Then $K^{pc}\subset\mathcal{A}$.*

Proof: The proof is similar to Lemma 3.2. Let

$$\Phi(X)=(|\det X|-\delta)_++\sup_{\nu\in S^1}g_\nu(X).$$

Since $t \mapsto (|t|-\delta)_+$ is a convex function, Φ is a polyconvex function that vanishes on A and is positive elsewhere. The assertion follows now from the definition of the polyconvex hull. □

The inclusion $\mathcal{A} \subset K^{(3)}$ requires some more work. We prove first two auxiliary results.

Lemma 4.2. *If $F, G \in \mathbb{M}^{2\times 2}$ satisfy*

$$F^T F = G^T G - \alpha G^T e \otimes G^T e$$

for some $\alpha \in [0,1]$ and some $e \in S^1$, then

$$F \in (\mathrm{O}(2)G)^{(1)}.$$

Proof: Since $\alpha \in [0,1]$, there exists $\lambda \in [0,1]$ such that $\alpha = 4\lambda(1-\lambda)$ so that

$$F^T F = G^T G - 4\lambda(1-\lambda) G^T e \otimes G^T e = (G - 2\lambda e \otimes G^T e)^T (G - 2\lambda e \otimes G^T e).$$

Therefore, $|Fv|^2 = |(G - 2\lambda e \otimes G^T e)v|^2$ for all $v \in S^1$ and we conclude that there exists $\tilde{Q} \in \mathrm{O}(2)$ such that

$$F = \tilde{Q}(G - 2\lambda e \otimes G^T e).$$

If we define $Q = I - 2e \otimes e \in \mathrm{O}(2)$, then

$$F = \tilde{Q}(\lambda Q G + (1-\lambda)G) \quad \text{and} \quad QG - G = -2e \otimes G^T e,$$

and this proves the lemma. □

For the statement of the next lemma it is useful to introduce some notation: Let

$$\text{(4.2)} \quad \mathcal{B} = \{F : |\det F| \le \delta,\, |Fe|^2 \le m_{\mathcal{U}}(e)\, \forall\, e \in S_1,\, \exists\, \tilde{e} : |F\tilde{e}|^2 = m_{\mathcal{U}}(\tilde{e})\},$$

$$\text{(4.3)} \quad \mathcal{B}_\alpha = \mathcal{B} \cap \{F : \det F = \alpha\} \quad \text{for} \quad \alpha \in [-\delta, \delta].$$

$$\text{(4.4)} \quad \mathcal{A}_\alpha = \mathcal{A} \cap \{F : \det F = \alpha\} \quad \text{for} \quad \alpha \in [-\delta, \delta].$$

Lemma 4.3. *Assume that $F \in \mathcal{B}$ with $|\det F| < \delta$. Then one of the following alternatives holds:*

i) There exists a unique (up to the sign) $e \in S^1$ such that $|Fe|^2 = |U_i e|^2 > m_{\mathcal{U}\setminus\{U_i\}}(e)$ and $F^T F = U_i^T U_i - \alpha e^\perp \otimes e^\perp$, where $\alpha \in (0, |U_i^{-T} e^\perp|^{-2}]$. Equivalently,

$$F^T F = U_i^T U_i - \tilde{\alpha} U_i^T \tilde{e} \otimes U_i^T \tilde{e} \text{ with } \tilde{e} = U_i^{-T} e^\perp / |U_i^{-T} e^\perp| \in S^1$$

and $\tilde{\alpha} = \alpha |U_i^{-T} e^\perp|^2 \in (0,1]$.

ii) There exists a unique (up to the sign) $e \in S^1$ and a $G \in \mathcal{B}_\delta$ such that (relabeling the matrices if necessary) $|Fe|^2 = |Ge|^2 = |U_1 e|^2 = \cdots = |U_n e|^2 > m_{\mathcal{U}\setminus\{U_1,\dots,U_n\}}(e)$ with $n \ge 2$. Moreover, $F^T F = G^T G - \alpha e^\perp \otimes e^\perp$ with $\alpha \in (0, |G^{-T} e^\perp|^{-2}]$ or, equivalently,

$$F^T F = G^T G - \tilde{\alpha} G^T \tilde{e} \otimes G^T \tilde{e} \quad \text{with} \quad \tilde{e} = G^{-T} e^\perp / |G^{-T} e^\perp| \in S^1$$

and $\tilde{\alpha} = \alpha |G^{-T} e|^2 \in (0,1]$.

Proof: By definition of $\mathcal{B}$ there exists at least one $e \in S^1$ such that $|Fe|^2 = m_{\mathcal{U}}(e)$. Assume first that there exist $e \in S^1$ and $U_i \in \mathcal{U}$ such that $m_{\mathcal{U}}(e) = |Fe|^2 = |U_i e|^2 > \max_{\mathcal{U}\setminus\{U_i\}}$ for some $i \in 1,\dots,k$. It follows from Lemma 2.4 that $F^T F = U_i^2 - \alpha e^\perp \otimes e^\perp$ for some $\alpha \ge 0$. Since $0 \le (\det F)^2 = (\det U_i)^2 (1 - \alpha |U_i^{-T} e^\perp|^2) < \delta^2$ we conclude that $\alpha \in (0, |U_i^{-T} e^\perp|^{-2}]$. The uniqueness of e follows now from $\alpha > 0$ and $|F\tilde{e}|^2 = |U_i \tilde{e}|^2 - \alpha(\tilde{e}, e^\perp)^2$. This proves *i)*.

Assume now that there exists (relabeling the matrices if necessary) $n \in \{2,\dots,k\}$ such that

$$|U_1 e|^2 = \dots = |U_n e|^2 = m_{\mathcal{U}}(e) > m_{\mathcal{U}\setminus\{U_1,\dots,U_n\}}(e).$$

By Lemma 2.5 we find $p,\, q \in \{1,\dots,n\}$, $p \neq q$, such that

a) $m_{\mathcal{U}}(e_\theta) = |U_p e_\theta|^2 > m_{\mathcal{U}\setminus\{U_p\}}(e_\theta)$ for $-\theta_0 < \theta < 0$,
b) $m_{\mathcal{U}}(e_\theta) = |U_q e_\theta|^2 > m_{\mathcal{U}\setminus\{U_q\}}(e_\theta)$ for $\quad 0 < \theta < \theta_0$,

where $e_\theta = \sqrt{1+\theta^2}^{-1}(e + \theta e^\perp)$. According to Lemma 2.1, there exist $a, b \in \mathbb{R}^2$ with

$$F^T F = U_p^2 + a \otimes e^\perp + e^\perp \otimes a, \quad F^T F = U_q^2 + b \otimes e^\perp + e^\perp \otimes b.$$

It follows from *a)* above that

$$|F e_\theta|^2 = |U_p e_\theta|^2 + 2\theta(\langle a, e\rangle + \theta\langle a, e^\perp\rangle) \le m_{\mathcal{U}}(e_\theta) = |U_p e_\theta|^2$$

for all $\theta \in (-\theta_0, 0)$. We conclude $\langle a, e\rangle \ge 0$. Choosing $\theta \in (0,\theta_0)$ and observing *b)* we deduce that $\langle b, e\rangle \le 0$. Therefore there exists $\lambda \in [0,1], t \in \mathbb{R}$ such that

$$\lambda a + (1-\lambda) b = t e^\perp$$

(we allow $t = 0$ if a and b are linearly dependent; in this case we have $\det(F^T F) = \delta^2$). We now define for $\mu \in [0,1]$

$$\begin{aligned} C_{\lambda,\mu} &= \lambda(\mu U_p^2 + (1-\mu) F^T F) + (1-\lambda)(\mu U_q^2 + (1-\mu) F^T F) \\ &= F^T F - 2t\mu e^\perp \otimes e^\perp. \end{aligned}$$

By construction

$$\det C_{\lambda,0} = \det F^T F \le \delta^2,$$

and a simple calculation shows that

$$\begin{aligned} \det C_{\lambda,1} &= \det(\lambda U_p^2 + (1-\lambda) U_q^2) \\ &= \lambda \det U_p^2 + (1-\lambda)\det U_q^2 - \lambda(1-\lambda)\det(U_p^2 - U_q^2) \\ &\ge \delta^2 \end{aligned}$$

since $\det(U_p^2 - U_q^2) \le 0$ according to Lemma 2.1. Therefore, there exists $\tilde\mu$ such that

$$\det C_{\lambda,\tilde\mu} = \delta^2.$$

By construction $\langle e, C_{\lambda,\tilde\mu} e\rangle = m(e)$ and moreover $\langle v, C_{\lambda,\tilde\mu} v\rangle \le m(v)$ for all $v \in S^1$ since $C_{\lambda,\tilde\mu}$ is a convex combination of three matrices which satisfy these inequalities. Therefore, we conclude that there exists $G \in \mathcal{B}_\delta$ such that $G^T G = C_{\lambda,\tilde\mu}$ and

$$F^T F = G^T G + 2t\tilde\mu e^\perp \otimes e^\perp.$$

In particular, $|Ge|^2 = |Fe|^2 = m_{\mathcal{U}}(e)$. Let $\tilde e = G^{-T} e^\perp$. Finally notice that $\det F^T F < \delta^2$ implies that $2t\tilde\mu < 0$ and this proves the uniqueness of e. $\square$

Proof of part ii) in Theorem 1.1: In view of Lemma 4.1 it remains to show that $\mathcal{A} \subset K^{(3)}$. By Lemma 3.3

$$\mathcal{B}_\delta \subset \big(\mathrm{SO}(2) U_1 \cup \dots \cup \mathrm{SO}(2) U_k\big)^{(1)} \subset K^{(1)},$$

and since $\mathcal{B}_{-\delta} = Q\mathcal{B}_\delta$ for any $Q \in \mathrm{O}(2) \setminus \mathrm{SO}(2)$, we conclude that $\mathcal{B}_{\pm\delta} \subset K^{(1)}$. Combining this with Lemma 4.3 and Lemma 4.2, it follows that $\mathcal{B} \subset K^{(2)}$. Now, for any $F \in \mathcal{B}_\alpha$, $\alpha \in [-\delta,\delta] \setminus \{0\}$ we use the arguments in the proof of part *i)* in Theorem 1.1 to construct two rank-one connected matrices on the manifold $\{\det X = \alpha\}$ such that F is contained in the rank-one segment connecting these two matrices. Thus $\mathcal{A} \subset K^{(3)}$. Finally, consider any $F \in \mathcal{A}_0$. Clearly, $F = Q(\beta e \otimes e)$ for some $Q \in \mathrm{O}(2)$, $\beta \in \mathbb{R}$ and $e \in S^1$ and by definition

$$|Fv|^2 = \beta^2 \langle e, v\rangle^2 \leq m_{\mathcal{U}}(v) \quad \forall v \in S^1.$$

By continuity there exists $\gamma^2 \geq \beta^2$ such that $G = \gamma e \otimes e \in \mathcal{B} \subset K^{(2)}$. Therefore, $F^T F = G^T G - \alpha G^T e \otimes G^T e$ with $\alpha = \frac{\gamma^2 - \beta^2}{\gamma^2} \in [0,1]$ and consequently $F \in (\mathrm{O}(2)G)^{(1)} \subset K^{(3)}$ by Lemma 4.2. This implies the assertion of the theorem. ☐

Example 4.4. *(The two-well problem) Assume that* $\mathcal{U} = \{U_1, U_2\}$ *where* $U_1, U_2 \in \mathbb{M}^{2\times 2}$ *with* $\det U_1 = \det U_2 = \delta > 0$ *and that* $\mathrm{O}(2)U_1 \neq O(2)U_2$. *Let* $K = \mathrm{O}(2)U_1 \cup \mathrm{O}(2)U_2$. *Then*

$$K^{qc} = \{F \in \mathbb{M}^{2\times 2} : |\det F| \leq \delta, \, |Fe|^2 \leq m_{\mathcal{U}}(e) \; \forall e \in S^1\}.$$

The set $K_c^{qc} = \{F^T F : F \in K^{qc}\}$ is shown in Figure 6 (which is bounded by the half cone $\{\det C \geq 0\}$ and one sheet of the hyperboloid $\{\det C = \delta^2\}$) and the half spaces

$$\{C \in \mathbb{M}^{2\times 2}_{sym} : \mathrm{tr}(C(e \otimes e)) \leq m_{\mathcal{U}}(e)\}, \quad e \in S^1.$$

The flat parts in the boundary of the set shown in Figure 6 corresponds to the two directions e_i, $i = 1, 2$, with $|U_1 e_i|^2 = |U_2 e_i|^2 = m_{\mathcal{U}}(e_i)$, while the intersection of the half spaces for the other normals generate the two half cones centered at $U_1^T U_1$ and $U_2^T U_2$. In particular, there exists no finite subset of S^1 which describes K^{qc}, in contrary to the case of two $\mathrm{SO}(2)$ invariant wells in Example 3.4 ☐

Example 4.5. *(The four-well problem) Assume that* $a, b, c > 0$, $a > b$, $ab - c^2 > 0$ *and define*

$$U_1 = \begin{pmatrix} a & c \\ c & b \end{pmatrix}, U_2 = \begin{pmatrix} b & c \\ c & a \end{pmatrix}, U_3 = \begin{pmatrix} a & -c \\ -c & b \end{pmatrix}, U_4 = \begin{pmatrix} b & -c \\ -c & a \end{pmatrix}.$$

Let $\mathcal{U} = \{U_1, U_2, U_3, U_4\}$ *and*

$$K = \bigcup_{i=1}^{4} \mathrm{O}(2)U_i.$$

Then

$$K^{qc} = \{F \in \mathbb{M}^{2\times 2} : \det F = \delta, \, |Fe|^2 \leq m_{\mathcal{U}}(e) \; \forall e \in S^1\}.$$

The set $K_c^{qc} = \{F^T F : F \in K^{qc}\}$ is shown in Figure 4. The four flat parts in the boundary correspond now to the four rank-one connections shown in Figure 1 which defined the boundary of the quasiconvex hull of the corresponding set with $\mathrm{SO}(2)$ invariant wells. ☐

5. The quasiconvex hull of $\mathrm{SO}(3)\hat{U}_1 \cup \ldots \cup \mathrm{SO}(3)\hat{U}_k$

In this section we prove Theorem 1.3. We begin with an equivalent description of the set $\mathrm{SO}(3)\hat{F}$. Let $L = \{F \in \mathbb{M}^{3\times 2} : F_{31} = F_{32} = 0\}$ and define $\pi_L : \mathbb{M}^{2\times 2} \to L$ by

$$\pi_L(G) = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \\ 0 & 0 \end{pmatrix}.$$

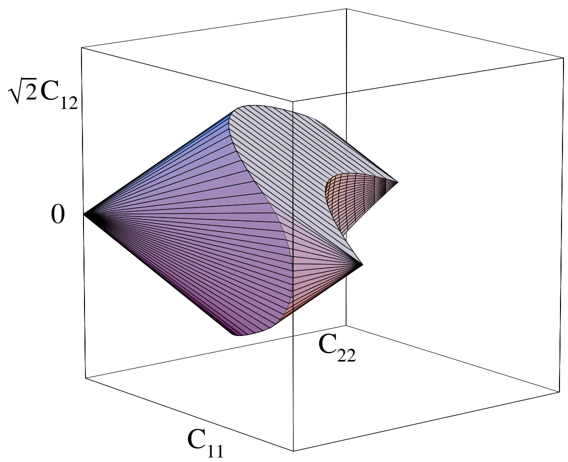

FIGURE 6. *The set* $(\mathrm{O}(2)U_1 \cup \mathrm{O}(2)U_2)_c^{qc}$ *for the diagonal matrices* $U_1 = \mathrm{diag}(a, b)$ *and* $U_2 = \mathrm{diag}(b, a)$. *This is also equal to the set* $(\mathrm{SO}(3)\hat{U}_1 \cup \mathrm{SO}(3)\hat{U}_2)_c^{qc}$ *for the diagonal matrices* $U_1 = \mathrm{diag}(a, b, c)$ *and* $U_2 = \mathrm{diag}(b, a, c)$.

Recall that $\hat{F} = (Fe_1, Fe_2) \in \mathbb{M}^{3\times 2}$ for $F \in \mathbb{M}^{3\times 3}$.

Lemma 5.1. *Let* $F \in \mathbb{M}^{3\times 3}$. *We have* $\mathrm{SO}(3)\hat{F} = \mathrm{SO}(3)\pi_L(G)$ *where* G *is the square root of* $\hat{F}^T\hat{F} \in \mathbb{M}^{2\times 2}$.

Proof: Choose a rotation Q_0 that maps the two-dimensional affine subspace spaned by the first two columns of F to the subspace $\{se_1 + te_2,\, s, t \in \mathbb{R}\}$. Then $Q_0F = \pi_L(G)$ for some matrix $G \in \mathbb{M}^{2\times 2}$. Replacing Q_0 by $(-I + 2e_1 \otimes e_1)Q_0$ or by $(-I + 2e_2 \otimes e_2)Q_0$ if necessary, we may assume that G is positive definite. Finally premultiplying Q_0 by a suitable rotation of the two-dimensional space $\{se_1 + te_2,\, s, t \in \mathbb{R}\}$ we may assume that $Q_0F = \pi_L(G)$ with G positive definite and symmetric. By construction

$$(Q\hat{F})^T(Q\hat{F}) = \hat{F}^T\hat{F} = G^TG = G^2$$

and thus G is the square root of $\hat{F}^T\hat{F}$. The assertion of the lemma follows now easily since $\mathrm{SO}(3)\hat{F} = \mathrm{SO}(3)(Q_0\hat{F})$. □

Let $\hat{\mathcal{U}} = \{\hat{U}_1, \ldots, \hat{U}_k\}$ and define

$$m_{\hat{\mathcal{U}}}(e) = \max\{|\hat{U}_ie|^2 : i = 1, \ldots, k\} \quad \text{for } e \in S^1.$$

Proof of Theorem 1.3: Let

$$\mathcal{A} = \big\{F \in \mathbb{M}^{3\times 2} : \det(F^TF) \le \delta^2,\, |Fe|^2 \le m_{\hat{\mathcal{U}}}(e)\, \forall e \in S^1\big\}.$$

We first show that $K^{pc} \subset \mathcal{A}$. A short calculation shows that for all $F \in \mathbb{M}^{3\times 2}$

$$\det(F^TF) = \mathrm{adj}_{12}^2(F) + \mathrm{adj}_{13}^2(F) + \mathrm{adj}_{23}^2(F),$$

where $\mathrm{adj}_{ij}(F)$ denotes the (2×2)-subdeterminant formed with the i-th and the j-th row of F. Thus

$$h(F) = \big(\det(F^TF) - \delta^2\big)_+$$

is a polyconvex function on $\mathbb{M}^{3\times 2}$. Let

$$g_\nu(F) = \big(|F\nu|^2 - m_{\hat{\mathcal{U}}}(\nu)\big)_+;$$

then

$$\Phi(F) = h(F) + \sup_{\nu \in S^1} g_\nu(F)$$

is a polyconvex function which is zero on $\mathcal{A}$ and positive for all $F \notin \mathcal{A}$. This proves the inclusion $K^{pc} \subset \mathcal{A}$. Thus it remains to show that $\mathcal{A} \subset K^{lc}$.

By Lemma 5.1 we may choose $Q_i \in \mathrm{SO}(3)$ and $G_i \in \mathbb{M}^{2\times 2}_{sym}$ positive definite such that $Q_i\hat{U}_i = \pi_L(G_i)$. Since

$$\mathrm{adj}_{33}(U_i^T U_i) = \mathrm{adj}_{33}\big((Q_iU_i)^T(Q_iU_i)\big) = (\det G_i)^2 = \delta^2$$

we conclude $\det G_i = \delta$ for $i = 1, \dots, k$. Moreover, if we define for $e \in S^1$ the vector $\tilde{e} \in \mathbb{R}^3$ by $\tilde{e} = (e_1, e_2, 0)$, then

$$|\hat{U}_i e|^2 = |U_i\tilde{e}|^2 = |Q_iU_i\tilde{e}|^2 = |G_ie|^2$$

and therefore

$$\max\{|G_ie|^2 : i = 1, \dots, k\} = m_{\hat{\mathcal{U}}}(e). \tag{5.1}$$

Let

$$\tilde{\mathcal{A}} = \Big\{F = Q\pi_L(G) \in \mathbb{M}^{3\times 2} : Q \in \mathrm{SO}(3),\, G \in \big(\mathrm{O}(2)G_1 \cup \ldots \cup \mathrm{O}(2)G_k\big)^{qc}\Big\}.$$

We claim that $\mathcal{A} = \tilde{\mathcal{A}}$. Indeed, let $F = Q\pi_L(G) \in \tilde{\mathcal{A}}$. Then $\det(F^TF) = \det(G)^2 \le \delta^2$ and $|Fe|^2 = |Ge|^2 \le m_{\hat{\mathcal{U}}}(e)$ by (5.1) and Theorem 1.1. Thus $\tilde{\mathcal{A}} \subset \mathcal{A}$. Conversely, let $F \in \mathcal{A}$ and choose $Q \in \mathrm{SO}(3)$ sucht that $Q^TF = \pi_L(G)$. Then $\det(G^TG) = \det(F^TF) \le \delta^2$ and $|Ge|^2 = |Fe|^2 \le m_{\hat{\mathcal{U}}}(e)$. This proves $\mathcal{A} = \tilde{\mathcal{A}}$ and it remains to show that $\tilde{\mathcal{A}} \subset K^{lc}$. For $Q \in \mathrm{SO}(2)$ we define

$$Q^+ = \begin{pmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad Q^- = \big(-I + 2e_2 \otimes e_2\big)Q^+ \in \mathrm{SO}(3).$$

By definition of K,

$$Q^\pm Q_i\hat{U}_i \in K$$

and thus

$$\pi_L(\mathrm{O}(2)G_i) \in K \text{ for } i = 1, \dots, n.$$

Since K is invariant under multiplication by $\mathrm{SO}(3)$ from the left, we conclude $\mathcal{A} = \tilde{\mathcal{A}} \subset K^{lc}$ and the assertion of the theorem follows. □

6. Existence of minimizers

In this section we address the question whether there exist minimizers of the variational problem (1.1). This was an open problem for a long time, but recently fairly general positive results have been obtained in [DM1], [DM2] based on Baire's Theorem and in [MS1], [MS2] based on Gromov's idea of convex integration. Following Gromov [G] and Müller and Šverák [MS1], we define an in–approximation of a given set K in the following way:

Definition 6.1. *Let $K \subset \mathbb{M}^{m\times n}$. A sequence of open sets $V_i \subset \mathbb{M}^{m\times n}$ is called an in–approximation of K if the following three conditions are satisfied:*

i) $V_i \subset V_{i+1}^{lc}$;
ii) *the sets* V_i *are uniformly bounded;*
iii) *if a sequence* $F_i \in V_i$ *converges to* $F \in \mathbb{M}^{m\times n}$ *as* $i \to \infty$, *then* $F \in K$.

In this definition, we replace open sets with relatively open sets, if the set K is a relatively open set with respect to the constraint that one of the minors is fixed (see [MS2]). For example, in case *i)* of Theorem 1.1, the set K and its generalized convex hulls are contained in the smooth manifold $\{\det X = \delta\}$.

We will rely on the following existence result:

Theorem 6.2 ([MS1],[MS2])**.** *Suppose that* $K \subset \mathbb{M}^{m\times n}$ *admits an in–approximation by (relatively) open sets* V_i *in the sense of Definition 6.1. Let* $v \in C^1(\Omega;\mathbb{R}^m)$ *and assume that* $Dv(x) \in V_1$ *for* $x \in \Omega$. *Then there exists a* $u \in W^{1,\infty}(\Omega;\mathbb{R}^m)$ *such that* $u = v$ *on* $\partial\Omega$ *and* $Du \in K$ *a.e.*

In view of this result, it remains to construct in–approximations for the sets under consideration in this paper. For the case of two wells (with equal or different determinant), this has been done in [MS1], [MS2]. We follow their ideas in our multi-well setting.

An important ingredient in the construction of the in–approximation is the characterization of the (relative) interior of the generalized convex hulls. Throughout this section we will assume the following hypotheses:

(H1) $\mathcal{U} = \{U_1, \ldots, U_k\}$, $k \geq 2$, and the matrices $U_i \in \mathbb{M}^{2\times 2}_{sym}$ are positive definite with $\det U_i = \delta > 0$.
(H2) for all $U_i \in \mathcal{U}$ there exists a vector $e \in S^1$ such that $|U_i e|^2 > m_{\mathcal{U}\setminus\{U_i\}}(e)$

(see Lemma 3.5 for a justification of (H2)).

Lemma 6.3. *Assume (H1) and (H2) and let* $K = \mathrm{SO}(2)U_1 \cup \ldots \cup \mathrm{SO}(2)U_k$. *Then the relative interior of* K^{lc} *is given by*

$$\operatorname{rel\,int}(K^{lc}) = \{F \in K^{lc} : |Fe| < m_{\mathcal{U}}(e)\,\forall e \in S^1\}. \tag{6.1}$$

Proof: Let $\mathcal{A}$ denote the right hand side in (6.1) and define

$$\mathcal{B} = \{F \in K^{lc} : \exists\, e \in S^1 : |Fe|^2 = m_{\mathcal{U}}(e)\}.$$

Clearly $K^{lc} = \mathcal{A} \cup \mathcal{B}$. If $F \in \mathcal{A}$, then there exists by compactness of S^1 a $\delta > 0$ such that $|Fe|^2 \leq m_{\mathcal{U}}(e) - \delta$ for all $e \in S^1$. By continuity of the maps $F \mapsto |Fe|^2$ it follows that $F \in \operatorname{rel\,int} K^{lc}$. Conversely, assume that $F \in \mathcal{B}$ with $|Fe|^2 = m_{\mathcal{U}}(e)$, $e \in S^1$. Let $F_t = F(I + te^\perp \otimes e)$. Then $\det F_t = \det F = \delta$ and

$$|F_t e|^2 = |Fe|^2 + 2t\langle Fe, Fe^\perp\rangle + t^2|Fe^\perp|^2.$$

If e is an eigenvector of F^TF, then $F_t \notin K^{lc}$ for all $t \neq 0$ and thus F does not belong to the relative interior of K^{lc}. Otherwise we conclude $F_t \notin K^{lc}$ for $t = s\langle Fe, Fe^\perp\rangle$ with $0 < s < s_0$ and s_0 small enough. This proves the assertion of the lemma. $\square$

Lemma 6.4. *Assume (H1) and (H2). Then there exist for all matrices* $U_i \in \mathcal{U}$ *matrices* $U_i^{(j)} \in \operatorname{rel\,int}(K^{lc})$ *such that* $U_i^{(j)} \to U_i$ *as* $j \to \infty$ *for* $i = 1, \ldots, k$. *Moreover, for each compact set* $E \subset \operatorname{rel\,int}(K^{lc})$ *there exists a* $j_0 \in \mathbb{N}$ *such that*

$$E \subset \Big(\bigcup_{i=1}^k \mathrm{SO}(2)U_i^{(j)}\Big)^{lc}, \qquad j \geq j_0. \tag{6.2}$$

Proof: We first construct for $U_i \in \mathcal{U}$ a sequence of matrices $U_i^{(j)} \in \operatorname{rel\,int}(K^{lc})$ such that $U_i^{(j)} \to U_i$ as $j \to \infty$. By Lemma 3.6 there exist exactly two matrices $U_{i\pm1}$ and vectors $e_{i\pm1} \in S^1$, e_{i-1} not parallel to e_{i+1}, such that

$$|U_i e_{i\pm1}|^2 = |U_{i\pm1}(e_{i\pm1})|^2 = m_{\mathcal{U}}(e_{i\pm1}).$$

Thus there exist $Q_{i\pm1} \in \mathrm{SO}(2)$, $a_{i\pm1} \in \mathbb{R}^2$ such that

$$U_i - Q_{i\pm1}U_{i\pm1} = a_{i\pm1} \otimes e_{i\pm1}^{\perp}.$$

Now let

$$V_{i\pm1}^{\varepsilon} = (1-\varepsilon)U_i + \varepsilon Q_{i\pm1}U_{i\pm1}.$$

By Lemma 6.2 there exists $Q_\varepsilon \in \mathrm{SO}(2)$, $b \in \mathbb{R}^2$, $m \in S^1$ such that

$$Q_\varepsilon V_{i+1}^{\varepsilon} - V_{i-1} = b \otimes m.$$

We claim that

$$W_i^{\lambda,\varepsilon} = \lambda Q_\varepsilon V_{i+1}^{\varepsilon} + (1-\lambda)V_{i-1}^{\varepsilon} \in \operatorname{rel\,int} K^{lc} \qquad \text{for } \varepsilon \in (0,1),\ \lambda \in (0,1).$$

By construction, $W_i^{\lambda,\varepsilon} \in K^{lc}$ and therefore it suffices by Lemma 6.3 to show that

$$|W_i^{\lambda,\varepsilon} e|^2 < m_{\mathcal{U}}(e) \quad \forall e \in S^1.$$

This is immediate in the case $U_{i-1} \neq U_{i+1}$, since

$$m_{\mathcal{U}}(e) = |W_i^{\lambda,\varepsilon}e|^2 \le (1-\varepsilon)|U_i e|^2 + \lambda\varepsilon|U_{i+1}e|^2 + (1-\lambda)\varepsilon|U_{i-1}e|^2 \le m_{\mathcal{U}}(e)$$

implies $|U_i e|^2 = |U_{i\pm1}e|^2 = m_{\mathcal{U}}(e)$, contradicting *i)* in Lemma 3.6.

Assume now that $U_{i-1} = U_{i+1}$. In this case we have by convexity

$$m_{\mathcal{U}}(e) = |W_i^{\lambda,\varepsilon}e|^2 \le \lambda|V_{i+1}^{\varepsilon}e|^2 + (1-\lambda)|V_{i-1}^{\varepsilon}e|^2 \le m_{\mathcal{U}}(e),$$

and we conclude that

$$|V_{i\pm1}^{\varepsilon}e|^2 = m_{\mathcal{U}}(e). \tag{6.3}$$

Consequently, $e = e_{i-1}$ or $e = e_{i+1}$. We may assume that the latter holds. But then by Lemma 2.2 $|V_{i-1}^{\varepsilon}e_{i+1}|^2 < m_{\mathcal{U}}(e_{i+1})$ for $\varepsilon \in (0,1)$, and this contradicts (6.3). Thus $W_i^{\lambda,\varepsilon} \in \operatorname{rel\,int}(K^{lc})$. Now define for example $U_i^{(j)} = W_i^{1/j,1/j}$. Then $U_i^{(j)} \in \operatorname{rel\,int}(K^{lc})$ and $U_i^{(j)} \to U_i$ as $j \to \infty$.

Finally the inclusion (6.2) follows from Lemma 6.3 since by continuity there exists for all $\varepsilon > 0$ a $j_0 > 0$ such that

$$\left|m_{\mathcal{U}}(e) - m_{\{U_1^{(j)},\ldots,U_k^{(j)}\}}(e)\right| < \varepsilon, \quad \forall\, j \ge j_0,\ \forall\, e \in S^1.$$

$\square$

After these preparations we are in a position to prove our first existence result.

Theorem 6.5. *Suppose that* $W : \mathbb{M}^{2\times2} \to \mathbb{R}$, $W \ge 0$, *that* $K = \{W^{-1}(0)\} = \mathrm{SO}(2)U_1 \cup \ldots \cup \mathrm{SO}(2)U_k$, *and that the matrices* U_i *satisfy (H1) and (H2). Assume that* $v \in C^1(\Omega;\mathbb{R}^m)$ *is such that* $\{Dv(x) : x \in \Omega\}$ *is contained in a compact subset of* $\operatorname{rel\,int}(K^{lc})$. *Then there exists a minimizer* u *of the variational problem: Minimize*

$$I(w) = \int_\Omega W(Dw)dx$$

in the class $\{w \in W^{1,\infty}(\Omega;\mathbb{R}^m) : w = v \text{ on } \partial\Omega\}$. *In particular,* $I(u) = 0$.

Proof: In view of Theorem 6.2 it remains to construct an in–approximation of K with relatively open sets V_i such that $\{Dv(x) : x \in \Omega\} \subset V_1$.

The existence of an in-approximation will be a consequence of Lemma 6.3 and Lemma 6.4. Choose $V_1 \subset\subset \operatorname{rel\,int} K^{lc}$ such that $\{Dv(x) : x \in \Omega\} \subset\subset V_1$. Let $\delta_1 > 0$ be given. By Lemma 6.4 we may choose $U_i^{(1)} \in \operatorname{rel\,int} K^{lc}$ such that $|U_i^{(1)} - U_i| < \delta_1$ and

$$\overline{V_1} \subset \Big(\bigcup_{i=1}^{k} \mathrm{SO}(2) U_i^{(1)} \Big)^{lc}.$$

Let $\varepsilon_1 = \operatorname{dist}\big(\{\mathrm{SO}(2)U_1^{(j)} \cup \ldots \cup \mathrm{SO}(2)U_k^{(j)}\}, \partial \operatorname{conv}(K)\big)$ and define

$$V_2 = \big\{F : \det F = \delta, \operatorname{dist}(F, \mathrm{SO}(2)U_1^{(k)} \cup \ldots \cup \mathrm{SO}(2)U_n^{(k)}) < \frac{\varepsilon_1}{2}\big\}.$$

Then $V_1 \subset (V_2)^{lc}$ and $\operatorname{dist}(F, K) < 2\delta_1$ for all $F \in V_2$. Proceeding inductively with δ_1 replaced by $2^{-j}\delta_1$, we obtain an in–approximation of K. This proves the theorem. □

We do not expect similar existence results in three dimensions when the wells are essentially two dimensional, since it is not possible to lift the two dimensional construction in such a manner that they satisfy three dimensional boundary condition.

Remark 6.6. *Let $K = \mathrm{SO}(3)U_1 \cup \ldots \cup \mathrm{SO}(3)U_k$ where $\{U_1, \ldots, U_k\} \subset \mathbb{M}^{3\times 3}_{sym}$ with U_i positive definite, $\det U_i = \delta > 0$ and assume that there exists $\mu > 0$ and $v \in S^2$ such that $U_i v = \mu v$ for $i = 1, \ldots, k$. Assume that Ω is a unit cube with sides parallel to the orthonormal basis $\{e_1, e_2, v\}$. Then, given any $F \in K^{qc} \setminus K$,*

$$I(w) = \int_\Omega W(Dw) dx$$

has no minimizer in the class $\{w \in W^{1,\infty}(\Omega; \mathbb{R}^m) : w = Fx \text{ on } \partial\Omega\}$.

We prove this by contradition. Let y be a minimizer. Notice that $\inf I = 0$ since $F \in K^{qc}$. Therefore, $I(y) = 0$ and consequently, $\nabla y \in K$ a.e. $x \in \Omega$ and hence

$$(\nabla y)^T (\nabla y) v = \mu^2 v \quad \text{and} \quad \det \nabla y = \delta \quad \text{a.e. } x \in \Omega.$$

It follows then, by Theorem 3.1. of Ball and James [BJ2], that y is a plane strain deformation, i.e.,

$$y(x) = Q \begin{pmatrix} y_1(x_1, x_2) \\ y_2(x_1, x_2) \\ \mu x_3 \end{pmatrix}$$

in an orthonormal basis parallel to $\{e_1, e_2, v\}$. Comparing with the boundary condition on the surface $x_3 = 0$, we conclude that $y = Fx$ on Ω. Thus $I(y) = |\Omega| W(F) > 0$, contradicting the assumption that y is a minimizer. □

Now we turn to the case $K = \mathrm{O}(2)U_1 \cup \ldots \cup \mathrm{O}(2)U_k$. We first prove the analogues of Lemma 6.3 and Lemma 6.4 in this situation.

Lemma 6.7. *Suppose that (H1) and (H2) hold and let $K = \mathrm{O}(2)U_1 \cup \ldots \cup \mathrm{O}(2)U_k$. Then the interior of K^{lc} is given by*

$$\operatorname{int} K^{lc} = \{F \in K^{lc} : |\det F| < \delta \text{ and } |Fe|^2 < m_{\mathcal{U}}(e)\ \forall\, e \in S^1\}. \tag{6.4}$$

Proof: Let $\mathcal{A}$ denote the right hand side in (6.4) and define

$$\mathcal{B} = \{F \in K^{lc} : |\det F| = \delta \text{ or } \exists\, e \in S^1 : |Fe|^2 = m_{\mathcal{U}}(e)\}.$$

By continuity, it is easy to see that $\mathcal{A} \subset \operatorname{int} K^{lc}$. Since $K^{lc} = \mathcal{A} \cup \mathcal{B}$ is suffices to show that no point in $\mathcal{B}$ is an interior point of K^{lc}. Assume first that $|\det F| = \delta$. Let $F_\varepsilon = F(I + \varepsilon^2 e \otimes e)$ with $e \in S^1$. Then $|F - F_\varepsilon| = \varepsilon^2 |Fe|$ and $|\det F_\varepsilon| = (1+\varepsilon^2)|\det F|$. Thus $F_\varepsilon \to F$ as $\varepsilon \searrow 0$, but $F_\varepsilon \notin K^{lc}$ for any $\varepsilon > 0$. Therefore F cannot be an interior point of K^{lc}. Assume now that $|\det F| < \delta$ and that there exists an $e \in S^1$ such that $|Fe|^2 = m_{\mathcal{U}}(e)$. It follows from Lemmas 4.2 and 4.3 that there exists a $G \in B_\delta$ with $|Fe|^2 = |Ge|^2$ and $\tilde{Q} \in \mathrm{SO}(2)$, $\tilde{e} \in S^1$, $\lambda > 0$ such that

$$\begin{aligned} F &= \tilde{Q}(G - 2\lambda \tilde{e} \otimes G^T \tilde{e}) \\ &= \tilde{Q}\big(G - \frac{2\lambda}{|G^{-T} e^\perp|^2} G^{-T} e^\perp \otimes e^\perp\big). \end{aligned}$$

Let $F_\varepsilon = F + \varepsilon^2 \tilde{Q} G e \otimes e$. Then $|F - F_\varepsilon| = \varepsilon^2 |Ge|^2$ and $|F_\varepsilon e|^2 = |\tilde{Q} G e + \varepsilon^2 \tilde{Q} G e|^2 = (1+\varepsilon^2) m_{\mathcal{U}}(e)$. Thus $F_\varepsilon \notin K^{lc}$ for $\varepsilon \neq 0$ and hence F is not an interior point of K^{lc}. This proves the lemma. □

Lemma 6.8. *Assume that (H1) and (H2) hold. Then there exist matrices $U_i^{(j)} \in \operatorname{int} K^{lc}$ such that $U_i^{(j)} \to U_i$ as $j \to \infty$, for $i = 1, \ldots, n$. Moreover, for each compact set $E \subset\subset \operatorname{int} K^{lc}$ there exists a $k_0 \in \mathbb{N}$ with*

$$E \subset \Big(\bigcup_{i=1}^{k} \mathrm{O}(2) U_i^{(j)}\Big)^{lc}, \quad j \geq j_0. \tag{6.5}$$

Proof: Let $\tilde{U}_i^{(j)}$ be the sequence of matrices constructed in Lemma 6.4, and let $U_i^{(j)} = \tilde{U}_i^{(j)}(I - \delta_{i,j} e \otimes e)$ with $e \in S^1$. By compactness of S^1 and continuity we may choose $\delta_{i,j} > 0$ such that $|U_i^{(j)} e|^2 < m_{\mathcal{U}}(e)$ for all $e \in S^1$. Then $\det U_i^{(j)} = \det \tilde{U}_i^{(j)}(1 - \delta_{i,j})$ and $U_i^{(j)} \to U_i$ if we choose for example $0 < \delta_{i,j} < \frac{1}{j}$. The inclusion (6.5) follows as in Lemma 6.4. □

Theorem 6.9. *Suppose that $W : \mathbb{M}^{2\times 2} \to \mathbb{R}$, $W \geq 0$, that $K = \{W^{-1}(0)\} = \mathrm{O}(2)U_1 \cup \ldots \cup \mathrm{O}(2)U_k$ and that (H1) and (H2) hold. Assume that $v \in C^1(\Omega; \mathbb{R}^2)$ is such that $\{Dv(x) : x \in \Omega\}$ is contained in a compact subset of $\operatorname{int} K^{lc}$. Then there exists a minimizer u of the variational problem: Minimize*

$$I(w) = \int_\Omega W(Dw)\,dx$$

in the class $\{w \in W^{1,\infty}(\Omega; \mathbb{R}^2) : w = v \text{ on } \partial\Omega\}$. In particular, $I(u) = 0$.

Proof: The proof is analogous to the proof of Theorem 6.5. Choose $V_1 \subset\subset \operatorname{int} K^{lc}$ such that $\{Dv(x) : x \in \Omega\} \subset\subset V_1$. By Lemma 6.8 there exist for $\delta_1 > 0$ matrices $U_i^{(1)} \in \operatorname{int} K^{lc}$ such that $|U_i - U_i^{(1)}| < \delta_1$ and

$$\overline{V_1} \subset \Big(\bigcup_{i=1}^{k} \mathrm{O}(2) U_i^{(1)}\Big)^{lc}.$$

Let $\varepsilon_1 = \operatorname{dist}\big(\mathrm{O}(2)U_1^{(1)} \cup \cdots \cup \mathrm{O}(2)U_k^{(1)}, \partial K^{lc}\big)$ and define

$$V_2 = \{F : \operatorname{dist}(F, \mathrm{O}(2)U_1^{(1)} \cup \cdots \cup \mathrm{O}(2)U_n^{(1)}) < \frac{\varepsilon}{2}\}.$$

Then $V_1 \subset (V_2)^{lc}$ and $\operatorname{dist}(F,K) < 2\delta_1$ for all $F \in V_2$. Proceeding iteratively we obtain the required in–approximation. This proves the theorem. $\square$

Finally we prove an existence result for the SO(3) invariant wells. This requires first a modification of (H1) and (H2). We will assume

(H1′) $\mathcal{U} = \{U_1, \dots, U_k\}$, $k \geq 2$, and the matrices $U_i \in \mathbb{M}^{3\times 3}_{sym}$ are positive definite with $\operatorname{adj}_{33}(U_i^T U_i) = \delta^2 > 0$.

(H2′) Let $Q_i \in \mathrm{SO}(3)$ and $G_i \in \mathbb{M}^{2\times 2}$ be the matrices constructed in Lemma 5.1 with $\pi_L(G_i) = Q_i \hat{U}_i$ and let $\mathcal{G} = \{G_1, \dots, G_k\}$. Then there exists for all $G_i \in \mathcal{G}$ an $e \in S^1$ such that $|G_i e|^2 > m_{\mathcal{G}\setminus\{G_i\}}(e)$.

Lemma 6.10. *Assume that (H1′) and (H2′) hold. Let* $K = \mathrm{SO}(3)\hat{U}_1 \cup \ldots \cup \mathrm{SO}(3)\hat{U}_k$. *Then*

$$\operatorname{int} K^{lc} = \{F \in \mathbb{M}^{3\times 2} : \det(F^T F) < \delta^2,\, |Fe|^2 < m_{\hat{\mathcal{U}}}(e)\,\forall\, e \in S^1\}. \tag{6.6}$$

Proof: Let $\mathcal{A}$ denote the right hand side in (6.6) and define

$$\mathcal{B} = \{F \in K^{lc} : \det(F^T F) = \delta^2 \text{ or } \exists e \in S^1 : |Fe|^2 = m_{\hat{\mathcal{U}}}(e)\}.$$

Then $K^{lc} = \mathcal{A} \cup \mathcal{B}$ and by continuity it is easy to see that $\mathcal{A} \subset \operatorname{int} K^{lc}$. Conversely, assume that $F \in \mathcal{B}$. Since F is not an interior point of K^{lc} if and only if QF is not an interior point for some $Q \in \mathrm{SO}(3)$, we may assume that $F = \pi_L(G)$ with $G \in (\mathrm{O}(2)G_1 \cup \ldots \cup \mathrm{O}(2)G_n)^{lc}$ and $\det(G^T G) = \delta^2$ or $|Ge|^2 = m_{\hat{\mathcal{U}}}(e)$ (see the proof of Lemma 5.1). We conclude as in the proof of Lemma 6.7 that G is not an interior point of $(\mathrm{O}(2)G_1 \cup \ldots \cup \mathrm{O}(2)G_k)^{lc}$ and this implies the assertion of the lemma. $\square$

Lemma 6.11. *Assume that (H1′) and (H2′) hold. Let* $K = \mathrm{SO}(3)\hat{U}_1 \cup \ldots \cup \mathrm{SO}(3)\hat{U}_k$. *Then there exist positive definite matrices* $U_i^{(j)} \in \mathbb{M}^{3\times 3}$ *such that* $\hat{U}_i^{(j)} \in \operatorname{int} K^{lc}$ *and* $U_i^{(j)} \to U_i$ *as* $j \to \infty$. *Moreover, for each compact set* $E \subset\subset \operatorname{int} K^{lc}$ *there exists a* $j_0 \in \mathbb{N}$ *with*

$$E \subset \Big(\bigcup_{i=1}^{k} \mathrm{SO}(3)\hat{U}_i^{(j)}\Big)^{lc}, \quad j \geq j_0.$$

Proof: We may assume that $\hat{U}_i = \pi_L(G_i)$ with G_i as in Lemma 5.1. Let $G_i^{(j)}$ be the sequence of matrices constructed in Lemma 6.8. Then $\pi_L(G_i) \in \operatorname{int} K^{lc}$ and by Lemma 6.10 the matrices

$$U_i^{(j)} = \begin{pmatrix} G_{i,11}^{(k)} & G_{i,12}^{(k)} & U_{i,13} \\ G_{i,21}^{(k)} & G_{i,22}^{(k)} & U_{i,23} \\ 0 & & U_{i,33} \end{pmatrix}$$

have the properties stated in the lemma if we choose j big enough since the set of positive definite matrices is open. $\square$

Theorem 6.12. *Suppose that* $W : \mathbb{M}^{3\times 2} \to \mathbb{R}$, $W \geq 0$, *that* $K = \{W^{-1}(0)\} = \mathrm{SO}(3)\hat{U}_1 \cdots \mathrm{SO}(3)\hat{U}_n$ *and that (H1′) and (H2′) hold. Assume that* $v \in C^1(\Omega;\mathbb{R}^2)$ *is such that* $\{Dv(x) : x \in \Omega\}$ *is contained in a compact subset of* $\operatorname{int} K^{lc}$. *Then there exists a minimizer* u *of the variational problem: Minimize*

$$I(w) = \int_\Omega W(Dw)dx$$

in the class $\{w \in W^{1,\infty}(\Omega;\mathbb{R}^3) : w = v \text{ on } \partial\Omega\}$. *In particular,* $I(u) = 0$.

Proof: This is analogous to the proof of Theorem 6.9. □

7. Uniqueness and non-uniqueness of microstructures

As discussed in the introduction, the direct method in the calculus of variations based on weak lower semicontinuity cannot be applied to obtain existence for the variational problem (1.1). Minimizing sequences typically develop finer and finer oscillations (microstructures) and converge only weakly but not strongly. However, under suitable coercivity and growth assumptions on W (subsequences of) the deformation gradients $\{Du_k\}$ of minimizing sequences generate a gradient Young measure which captures the essential statistics of the oscillations in $\{Du_k\}$ (see for example [T],[B],[KP]). It is a natural question to ask whether the oscillations in the minimizing sequences are unique in the sense that the generated gradient Young measures are unique. In this section we prove that this is only true for some exceptional cases where the measure μ is of the form $\mu = \lambda\delta_A + (1-\lambda)\delta_B$ for A, $B \in K$ with $\operatorname{rank}(A-B) = 1$. For $F \in K^{qc}$ we define

$$\mathcal{M}(F) = \{\mu : \mu \text{ is gradient Young measure with } \operatorname{supp}\mu \subset K, \langle \mu, id\rangle = F\}.$$

In order to prove our non-uniqueness results we will use a special subset of all gradient Young measures, the so-called laminates (see for example [P]). Assume that $F = \lambda A + (1-\lambda)B$ with $\operatorname{rank}(A-B) = 1$ and $\lambda \in (0,1)$. Then $\mu = \lambda\delta_A + (1-\lambda)\delta_B \in \mathcal{M}(F)$. This process of splitting matrices in convex combinations along rank-one lines can be iterated: if $B = \mu C + (1-\mu)D$ with $\operatorname{rank}(C-D) = 1$ and $\mu \in (0,1)$, then $\mu = \lambda\delta_A + (1-\lambda)(\mu\delta_C + (1-\mu)\delta_D) \in \mathcal{M}(F)$. In particular we will use the following result which follows from [BJ3]:

Proposition 7.1. *Assume that U_1 and U_2 are symmetric and positive definite with $\det U_1 = \det U_2 = \delta > 0$. If $F \in (\mathrm{SO}(2)U_1 \cup \mathrm{SO}(2)U_2)^{qc}$ satisfies $|Fe|^2 < m_{\{U_1,U_2\}}(e)$ for all $e \in S^1$, then $\mathcal{M}(F)$ contains at least two laminates supported on three matrices.*

We first consider the case of SO(2) invariant wells. Let $\mathcal{U} = \{U_1, \ldots, U_k\}$ and assume that the hypotheses (H1) and (H2) defined in Section 6 hold. Recall the set $\mathcal{B}$ defined in (3.2). The following proposition shows that the Young measure is unique if and only if F^TF lies on the boundary of K_c^{qc} relative to the hyperboloid $\det C = \delta^2$.

Proposition 7.2. *Let $K = \mathrm{SO}(2)U_1 \cup \ldots \cup \mathrm{SO}(2)U_k$ and $F \in K^{qc}$.*

i) *If $F \in \mathcal{B}$, i.e. if there exist an $e \in S^1$ and U_i, $U_j \in \mathcal{U}$, $i \neq j$, such that $|Fe|^2 = |U_ie|^2 = |U_je|^2 > m_{\mathcal{U}\setminus\{U_i,U_j\}}(e)$, then $\mathcal{M}(F)$ contains a unique element. Indeed, there exist unique Q_i, $Q_j \in \mathrm{SO}(2)$ and $\lambda \in [0,1]$ such that*

$$\mathcal{M}(F) = \{\lambda\delta_{Q_iU_i} + (1-\lambda)\delta_{Q_jU_j}\}.$$

ii) *If $F \notin \mathcal{B}$, i.e. if $|Fe| < m_{\mathcal{U}}(e)$ for all $e \in S^1$, then $\mathcal{M}(F)$ contains more than one element.*

Proof: Assume that $\mu \in \mathcal{M}(F)$ and let $\mu = \lambda_1\mu_1 + \ldots + \lambda_k\mu_k$ where μ_i is a probability measure supported on $\mathrm{SO}(2)U_i$ and $\lambda_i \in [0,1]$ with $\lambda_1 + \ldots + \lambda_k = 1$. By Jensen's inequality

$$|Fe|^2 \leq \sum_{j=1}^{k} \lambda_j \int_{\operatorname{supp}\mu_j} |Ae|^2 d\mu_j(e).$$

The assumptions in i) imply that $\lambda_\ell = 0$ for $\ell \notin \{i,j\}$ and thus $\operatorname{supp}\mu \subset \mathrm{SO}(2)U_i \cup \mathrm{SO}(2)U_j$. Moreover,

$$\int_{\operatorname{supp}\mu} |Ae|^2 d\mu(A) - |Fe|^2 = \int_{\operatorname{supp}\mu} |(F-A)e|^2 d\mu(A) = 0$$

and therefore $\mu = \lambda\delta_{Q_iU_i} + (1-\lambda)\delta_{Q_jU_j}$ where Q_i and Q_j are the uniquely defined rotations with $Q_iU_ie = Q_jU_je = Fe$. Since $U_i \neq U_j$ it follows that λ is uniquely defined and this implies i).

To prove ii) we consider $F(t,v) = F(I + tv \otimes v^\perp)$. Then $\det F(t,v) = \det F$ and there exist $t^+ > 0 > t^-$ such that $F(t^\pm, v) \in \mathcal{B}$. We may assume that $F(t^+,v) \notin K$. By Lemma 3.3 there exist $U_i, U_j \in \mathcal{U}$ such that $F(t^+,v) \in (\mathrm{SO}(2)U_i \cup \mathrm{SO}(2)U_j)^{qc}$. It follows from Example 3.4 that $F(t,v) \in (\mathrm{SO}(2)U_i \cup \mathrm{SO}(2)U_j)^{qc}$ for $t \in (t^+ - \varepsilon, t^+)$, ε small enough, with $|F(t^+,v)|^2 < m_{\{U_i,U_j\}}(e)$ for all $e \in S^1$. Let $F_0 = F(t_0, v)$ with $t_0 \in (t^+ - \varepsilon, t^+)$. Then there exists $\lambda \in [0,1]$ such that $F = \lambda F_0 + (1-\lambda)F(t^-, v)$ and $F_0 - F(t^-,v) = \alpha v \otimes v^\perp$, $\alpha \in \mathbb{R}$. The assertion follows now from Proposition 7.1 since $\mathcal{M}(F_0)$ contains at least two laminates. □

Example 7.3. *(The four-well problem) Consider the four well problem described in Example 3.7. Let*

$$e_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad e_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then $\mathcal{M}(F)$ contains a unique element if and only if $F = QU_1(I + \alpha e_1 \otimes e_1^\perp)$, or $F = QU_2(I + \alpha e_4 \otimes e_4^\perp)$, or $F = QU_4(I + \alpha e_2 \otimes e_2^\perp)$, or $F = QU_3(I + \alpha e_3 \otimes e_3^\perp)$ for some $Q \in \mathrm{SO}(2)$ and for some $\alpha \in \mathbb{R}$. These correspond to the boundary arcs shown in dark in Figure 2. In particular note that F corresponding to some rank-one laminates have more than one element in $\mathcal{M}(F)$, as for example $F = QU_1(I + \alpha e_2 \otimes e_2^\perp)$, or $F = QU_2(I + \alpha e_3 \otimes e_3^\perp)$, or $F = QU_4(I + \alpha e_1 \otimes e_1^\perp)$, or $F = QU_3(I + \alpha e_4 \otimes e_4^\perp)$ which correspond to the dashed arcs shown in Figure 2.

Very similar results hold in three dimensions when the wells are essentially two dimensional. Let U_i satisfy the conditions of Theorem 1.2.

Proposition 7.4. *Let $K = \mathrm{SO}(3)U_1 \cup \ldots \cup \mathrm{SO}(3)U_k$ and $F \in K^{qc}$.*

i) *If $F \in \mathcal{B}$, i.e. if there exist an $e \in S^2$ satisfying $\langle e, v\rangle = 0$ and U_i, $U_j \in \mathcal{U}$, $i \neq j$, such that $|Fe|^2 = |U_ie|^2 = |U_je|^2 > m_{\mathcal{U}\setminus\{U_i,U_j\}}(e)$, then $\mathcal{M}(F)$ contains a unique element. Indeed, there exist unique Q_i, $Q_j \in \mathrm{SO}(3)$ and $\lambda \in [0,1]$ such that*

$$\mathcal{M}(F) = \{\lambda\delta_{Q_iU_i} + (1-\lambda)\delta_{Q_jU_j}\}.$$

ii) *If $F \notin \mathcal{B}$, i.e. if $|Fe| < m_{\mathcal{U}}(e)$ for all $e \in S^2$ satisfying $\langle e, v\rangle = 0$, then $\mathcal{M}(F)$ contains more than one element.*

Proof: The proof follows that of Proposition 7.2 aided by the observation that $F \in K^{qc}$ satisfies $\langle Fe, Fv\rangle = 0$ for all $e \in S^2$ such that $\langle e, v\rangle = 0$. □

We now turn to the $\mathrm{O}(2)$ invariant wells. Let $\mathcal{U} = \{U_1, \ldots, U_k\}$ and assume that the hypotheses (H1) and (H2) defined in Section 6 hold. Recall the set $\mathcal{B}$ defined in (4.2). The following proposition shows that the Young measure is unique if and only if F^TF lies on either the cones with apex U_i^2 or on the intersection of the flat boundary parts with the hyperboloid $\det C = \delta^2$ in K_c^{qc}.

Proposition 7.5. *Let $K = \mathrm{O}(2)U_1 \cup \ldots \cup \mathrm{O}(2)U_k$ and $F \in K^{qc}$.*

i) *If there exists an* $e \in S^1$ *such that* $|Fe|^2 = |U_i e|^2 > m_{\mathcal{U}\setminus\{U_i\}}$, *then* $\mathcal{M}(F)$ *contains a unique element. Indeed, there exist unique* $\lambda \in [0,1]$ *and* $Q^\pm \in \mathrm{O}(2)$ *with* $\det Q^\pm = \pm 1$ *such that*

$$\mathcal{M}(F) = \{\lambda \delta_{Q^+ U_i} + (1-\lambda)\delta_{Q^- U_i}\}.$$

ii) *If there exists an* $e \in S^1$ *such that* $|Fe|^2 = |U_i e|^2 = |U_j e|^2 > m_{\mathcal{U}\setminus\{U_i, U_j\}}(e)$, $i \neq j$, *then there exist unique* $Q_i^\pm, Q_j^\pm \in \mathrm{O}(2)$ *satisfying* $Q_i^\pm U_i e = Q_j^\pm U_j e = Fe, \det Q_i^\pm = \pm 1, \det Q_j^\pm = \pm 1$ *such that*

$$\begin{aligned}\mathcal{M}(F) = \{\mu = {} & \lambda_i^+ \delta_{Q_i^+ U_i} + \lambda_i^- \delta_{Q_i^- U_i} + \lambda_j^+ \delta_{Q_j^+ U_j} + \lambda_j^- \delta_{Q_j^- U_j}, \\ & \lambda_i^\pm, \lambda_j^\pm \in [0,1], \lambda_i^+ + \lambda_i^- + \lambda_j^+ + \lambda_j^- = 1, \\ & (\lambda_i^+ + \lambda_j^+)\delta - (\lambda_i^- + \lambda_j^-)\delta = \det F, \\ & (\lambda_i^+ + \lambda_i^-)\langle U_i e, U_i e^\perp\rangle + (\lambda_j^+ + \lambda_j^-)\langle U_j e, U_j e^\perp\rangle = \langle Fe, Fe^\perp\rangle\}.\end{aligned}$$

Therefore, the set $\mathcal{M}(F)$ *contains a unique element if* $\det F = \pm\delta$ *or if* $\langle U_i e, U_i e^\perp\rangle = \langle Fe, Fe^\perp\rangle$ *or if* $\langle U_j e, U_j e^\perp\rangle = \langle Fe, Fe^\perp\rangle$. *Otherwise* $\mathcal{M}(F)$ *consists of a one-parameter family of measures.*

iii) *If* $|Fe|^2 < m_{\mathcal{U}}(e)$ *for all* $e \in S^1$ *then* $\mathcal{M}(F)$ *contains more than one element.*

Proof: i) It follows as in the proof of i) in Proposition 7.2 that $\operatorname{supp}\mu \subset \mathrm{O}(2)U_i$ and that

$$\int_{\operatorname{supp}\mu} |(F-A)e|^2 d\mu(A) = 0.$$

Since there are exactly two elements $Q^\pm \in \mathrm{O}(2)$ which satisfy $Q^\pm U_i e = Fe$ (one rotation and one reflection) the assertion follows.

ii) We note that there exist exactly four elements $Q_i^\pm, Q_j^\pm \in \mathrm{O}(2)$ which satisfy

$$Q_i^\pm U_i e = Q_j^\pm U_j e = Fe. \tag{7.1}$$

Set

$$\begin{aligned}\tilde{\mathcal{M}} = \{\mu = {} & \lambda_i^+ \delta_{Q_i^+ U_i} + \lambda_i^- \delta_{Q_i^- U_i} + \lambda_j^+ \delta_{Q_j^+ U_j} + \lambda_j^- \delta_{Q_j^- U_j}, \\ & \lambda_i^\pm, \lambda_j^\pm \in [0,1], \lambda_i^+ + \lambda_i^- + \lambda_j^+ + \lambda_j^- = 1, \\ & (\lambda_i^+ + \lambda_j^+)\delta - (\lambda_i^- + \lambda_j^-)\delta = \det F, \\ & (\lambda_i^+ + \lambda_i^-)\langle U_i e, U_i e^\perp\rangle + (\lambda_j^+ + \lambda_j^-)\langle U_j e, U_j e^\perp\rangle = \langle Fe, Fe^\perp\rangle\}.\end{aligned}$$

Now assume that $\mu \in \mathcal{M}(F)$. Then, it follows as in the proof of i) in Proposition 7.2 that

$$\begin{aligned}\mu = {} & \lambda_i^+ \delta_{Q_i^+ U_i} + \lambda_i^- \delta_{Q_i^- U_i} + \lambda_j^+ \delta_{Q_j^+ U_j} + \lambda_j^- \delta_{Q_j^- U_j}, \\ & \lambda_i^\pm, \lambda_j^\pm \in [0,1], \lambda_i^+ + \lambda_i^- + \lambda_j^+ + \lambda_j^- = 1.\end{aligned}$$

Further, the requirement $\langle \mu, id\rangle = F$ implies that

$$\left(\lambda_i^+ Q_i^+ U_i + \lambda_i^- Q_i^- U_i + \lambda_j^+ Q_j^+ U_j + \lambda_j^- Q_j^- U_j\right) e^\perp = Fe^\perp. \tag{7.2}$$

Note that

$$Q_\alpha^+ = \left(I - 2\frac{Q_\alpha^- U_\alpha^{-T} e^\perp \otimes Q_\alpha^- U_\alpha^{-T} e^\perp}{|U_\alpha^{-T} e^\perp|^2}\right) Q_\alpha^-, \quad \alpha = i, j,$$

and hence, we conclude that

$$\begin{aligned}&(\lambda_i^+ + \lambda_i^-)Q_i^- U_i e^\perp + (\lambda_j^+ + \lambda_j^-)Q_j^- U_j e^\perp \\ -&\frac{2\lambda_i^+}{|U_i^{-T}e^\perp|^2} Q_i^- U_i e^\perp - \frac{2\lambda_j^+}{|U_j^{-T}e^\perp|^2} Q_j^- U_j e^\perp = F e^\perp.\end{aligned}$$

We take the inner product of this equation with Fe, recall (7.1) and obtain

$$(\lambda_i^+ + \lambda_i^-)\langle U_i e, U_i e^\perp\rangle + (\lambda_j^+ + \lambda_j^-)\langle U_j e, U_j e^\perp\rangle = \langle Fe, Fe^\perp\rangle.$$

We obtain the final condition,

$$(\lambda_i^+ + \lambda_j^+)\delta - (\lambda_i^- + \lambda_j^-)\delta = \det F,$$

by taking the cross-product ($a \wedge b = a_1 b_2 - a_2 b_1$ for $a, b \in \mathbb{R}^2$) of (7.2) with Fe, recalling (7.1) and noting that for any $A \in \mathbb{M}^{2\times 2}$, $\det A = (Ae) \wedge (Ae^\perp)$. We have proved that $\mu \in \tilde{\mathcal{M}}$ or $\mathcal{M}(F) \subset \tilde{\mathcal{M}}$.

To prove the converse inclusions, let $\mu \in \mathcal{M}$. We note that (7.1) implies that there exist $a, b, c \in \mathbb{R}^2$ such that

$$\begin{aligned}&Q_i^+ U_i - Q_i^- U_i = a \otimes e^\perp, \qquad Q_j^+ U_j - Q_j^- U_j = b \otimes e^\perp, \\ &\left(\frac{\lambda_i^+}{\lambda_i^+ + \lambda_i^-} Q_i^+ U_i + \frac{\lambda_i^-}{\lambda_i^+ + \lambda_i^-} Q_i^- U_i\right) - \\ &\qquad\qquad \left(\frac{\lambda_j^+}{\lambda_j^+ + \lambda_j^-} Q_j^+ U_j + \frac{\lambda_j^-}{\lambda_j^+ + \lambda_j^-} Q_j^- U_j\right) = c \otimes e^\perp.\end{aligned}$$

This implies that μ is a gradient Young measure (in fact as a laminate of rank two). It remains to be shown that $\langle \mu, id\rangle = F$. In view of (7.1), we only have to show (7.2). However, this readily follows from the last two conditions in the definition of $\tilde{\mathcal{M}}$ and the calculations above since for any $u \neq 0, v, w \in \mathbb{R}^2$

$$\langle u, v\rangle = \langle u, w\rangle \text{ and } u \wedge v = u \wedge w \quad \Leftrightarrow v = w.$$

iii) The construction in the proof of Proposition 7.2 implies non-uniqueness for the case $\det F = \delta$ and also for $\det F = -\delta$ (by premultiplying every matrix in the construction by $J = \operatorname{diag}(-1, 1)$). Consider next the case $\det F = 0$. We may assume that $F = \alpha v \otimes v$ with $\alpha > 0$. Since by assumption $|Fe|^2 < m_{\mathcal{U}}(e)$ for all $e \in S^1$ there exists a $\bar{\alpha} > \alpha$ such that $\bar{\alpha} v \otimes v \in \mathcal{B}$. Let $\lambda = \alpha/\bar{\alpha}$. Then $F = (1-\lambda)0 + \lambda\bar{F}$ and since there is more than one laminate with center of mass equal to zero the assertion follows. Consider finally the case $0 < |\det F| < \delta$. We may assume that $0 < \det F = \gamma < \delta$. Choose any $G \in \mathcal{B}_\delta$ of the form $G = \lambda Q U_i + (1-\lambda) U_j$ with $\lambda \in (0,1)$, $Q \in \mathrm{SO}(2)$ and $QU_i - U_j = a \otimes e^\perp$. Let $G_t = G - 2t\tilde{e} \otimes G^T \tilde{e}$ with $\tilde{e} = G^{-T}e^\perp/|G^{-T}e^\perp|$. Since $\det G_t = \det G(1-2t)$ there exists a $\bar{t} \in (0, \frac{1}{2})$ such that $\bar{G} = G_{\bar{t}}$ satisfies $\det \bar{G} = \gamma$. By Lemma 2.2, there exists $R \in \mathrm{SO}(2), a, b \in \mathbb{R}^2$ such that $F = R\bar{G} + a \otimes b$. Let $F_\alpha = R\bar{G} + \alpha a \otimes b$; clearly, there exist $\alpha_0 > 1$ such that $F_{\alpha_0} \in \mathcal{B}_\gamma$ and $F_{\alpha_0} \neq R\bar{G}$. Therefore we can obtain F by laminating $\bar{G}$ and F_{α_0}; the result follows since proof of ii) shows that $\mathcal{M}(\bar{G})$ contains more than one laminate. □

We finally turn to the case of the thin-film wells. Let $\mathcal{U} = \{U_1, \ldots, U_k\}$ and assume that the hypotheses (H1′) and (H2′) defined in Section 6 hold. The result says that the Young measure is unique if and only if $F^T F$ lies on the intersection of the flat boundary regions with the hyperboloid $\det C = \delta^2$ in K_c^{qc}. Notice that unlike the case of the O(2) invariant wells there is no uniqueness in the cones since we can make new constructions which use the third dimension.

Proposition 7.6. *Let* $K = \mathrm{SO}(3)\hat{U}_1 \cup \ldots \cup \mathrm{SO}(3)\hat{U}_k$ *and* $F \in K^{qc}$*. Then* $\mathcal{M}(F)$ *contains a unique element if and only if* $\det(F^T F) = \delta^2$ *and there exists an* $e \in S^1$ *such that* $|Fe|^2 = |\hat{U}_i e|^2 = |\hat{U}_j e|^2 > m_{\hat{\mathcal{U}} \setminus \{\hat{U}_i, \hat{U}_j\}}(e)$, $i \neq j$.

Proof: Consider first the case $\det(F^T F) = \delta^2$ and assume that there exists an $e \in S^1$ such that $|Fe|^2 = |\hat{U}_i e|^2 = |\hat{U}_j e|^2 > m_{\hat{\mathcal{U}} \setminus \{\hat{U}_i, \hat{U}_j\}}(e)$, $i \neq j$. We can adopt the proof of Proposition 7.2 to establish that the Young measure is unique.

Now consider F such that there exists an $e \in S^1$ such that $|Fe|^2 = |\hat{U}_i e|^2 > m_{\hat{\mathcal{U}} \setminus \{\hat{U}_i\}}$. We show that $\mathcal{M}(F)$ contains more than one element. We may assume that $Fe = \hat{U}_i e$ or $F = \hat{U}_i(I - 2\lambda v \otimes v)$ where $u = (e_1, e_2, 0)$, $v = (-e_2, e_1, 0)$ and $\{u, v, w\}$ is an orthonormal basis in $\mathbb{R}^3$. In this basis we have

$$\hat{U}_i = \begin{pmatrix} a & b \\ c & d \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} a & (1-2\lambda)b \\ c & (1-2\lambda)c \\ 0 & 0 \end{pmatrix}.$$

Let

$$R_\theta^\pm = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \mp\sin\theta \\ 0 & \pm\sin\theta & \cos\theta \end{pmatrix}.$$

Note that $R_\theta^\pm \in \mathrm{SO}(3)$ and that

$$R_\theta^+ \hat{U}_i - R_\theta^- \hat{U}_i = (R_\theta^+ - R_\theta^-)\hat{U}_i = 2\sin\theta \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ c & d \end{pmatrix}$$

is rank-one. Therefore, we can laminate $R_\theta^+ \hat{U}_i$ with $R_\theta^- \hat{U}_i$ in the proportion $1/2$ to obtain a Young measure with center of mass

$$\frac{1}{2}(R_\theta^+ + R_\theta^-)U_i = \begin{pmatrix} a & b\cos\theta \\ c & d\cos\theta \\ 0 & 0 \end{pmatrix}$$

which is equal to F for an appropriate choice of θ. For this same F we can follow the proof of Proposition 7.5 i) to obtain a laminate of $Q^+ U_i, Q_- U_i$ where $Q^\pm = \mathrm{diag}(1, \pm 1, 1)$ in the proportion λ. Thus we have constructed two distinct laminates in $\mathcal{M}(F)$.

Finally, for all other cases, we can lift the constructions in the proof of Proposition 7.5 to prove non-uniqueness. □

8. Approximate relaxed energy

The relaxation of the variational problem (1.1) is obtained by replacing W with its lower quasiconvex envelope

$$W^{qc} = \sup\{\Phi : \Phi \leq W, \Phi \text{ quasiconvex}\}.$$

It follows from the invariance of W that

$$W^{qc}(F) = W^{qc}(QF)$$

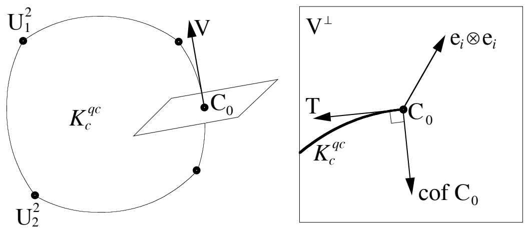

FIGURE 7. *The quasiconvex hull and the space $V^\perp$.*

for all $Q \in \mathrm{SO}(2)$, $\mathrm{O}(2)$ and $\mathrm{SO}(3)$, respectively. Thus there exists $\bar{W}^{qc} : \mathbb{M}^{2\times 2}_{sym} \to \mathbb{R}$ such that

$$W^{qc}(F) = \bar{W}^{qc}(F^TF);$$

the function $\bar{W}^{qc}$ vanishes on K_c^{qc} and grows away from it. We are interested in calculating this function, but this is extremely difficult. However, the practical interest in this function lies near the set K_c^{qc}. Therefore, we construct a function $\bar{W} : \mathbb{M}^{2\times 2}_{sym} \to \mathbb{R}$, which we call the *approximate relaxed energy*, with the following three properties:

(P1) The function $F \mapsto \bar{W}(F^TF)$ is quasiconvex.
(P2) The function $\bar{W}$ vanishes on K_c^{qc} and hence the function $F \mapsto \bar{W}(F^TF)$ vanishes on K^{qc}.
(P3) The function $\bar{W}$ grows quadratically away from K_c^{qc}.

We note that $\bar{W}$ needs to grow quadratically in $C = F^TF$ away from K_c^{qc} in order that the 'linearlized elastic moduli' are positive.

Our approximate relaxed energies are modifications of the functions Φ constructed in Lemmas 3.2, 4.1 and in the proof of Theorem 1.3. Recall that (H1) and (H2) have been defined in Section 6.

Remark 8.1. *Suppose $K = \mathrm{SO}(2)U_1 \cup \ldots \cup \mathrm{SO}(2)U_k$ for $U_i \in \mathbb{M}^{2\times 2}_{sym}$ that satisfy (H1) and (H2), and that $\alpha, \alpha_i > 0$. Then the function*

$$\bar{W}(C) = h(\det C) + \sum_{i=1}^{k} \alpha_i(\langle e_i, Ce_i\rangle - m_{\mathcal{U}}(e_i))_+^2 \tag{8.1}$$

has the properties (P1), (P2) and (P3). Here $\mathcal{E} = \{e_1, \ldots, e_k\}$ is the set of special directions according to Theorem 1.1, $t_+^2 = (\max\{t, 0\})^2$ is the square of the positive part of t and $h : \mathbb{R}^+ \to \mathbb{R}$ is a convex function which satisfies

$$h(\delta^2) = h'(\delta^2) = 0, \quad h''(\delta^2) = \alpha > 0 \quad \textit{and} \quad h(t) \to \infty \textit{ as } t \to 0 \textit{ or } \infty.$$

The convexity of h and $(\cdot)_+^2$ implies that the function $F \mapsto \bar{W}(F^TF)$ is polyconvex and hence (P1) holds; (P2) follows from the characterization of K^{qc} in Theorem 1.1. We now turn to (P3). Recall the identification of symmetric matrices with $\mathbb{R}^3$. In this space, the set K_c^{qc} is a simply connected region in a two-dimensional manifold ($\det C = \delta^2$) whose boundaries are made up of k curves (the intersection of the manifold with the planes $\langle e_i, Ce_i\rangle = m_{\mathcal{U}}(e_i)$). First pick any point C_0 in the interior of K_c^{qc}. It follows from the properties of h that $\bar{W}$ grows quadratically away from C_0 in the direction perpendicular to the manifold at C_0. Now pick any point C_0 on any of the boundary curves. Let V be the tangent (or velocity vector) to the curve at C_0 (see Figure 7) and consider the plane perpendicular to V (Figure 7 right). Since

$$\frac{d}{dt}\det(C_0+tD)\bigg|_{t=0} = \langle\,\mathrm{cof}\,C_0, D\rangle$$

the normal to the manifold at C_0 is in the direction cof C_0. Similarly, the normal to the plane is in the direction $(e_i\otimes e_i)$. Both lie on the plane $V^\perp$ as shown in the figure, and they are not parallel (cof C_0 has rank two while $(e_i\otimes e_i)$ has rank one). Now, h grows quadratically in the directions $\pm\,\mathrm{cof}\,C_0$ while $\alpha_i(\langle e_i, Ce_i\rangle - m_{\mathcal{U}}(e_i))_+^2$ grows quadratically in the direction $(e_i\otimes e_i)$. Consequently, in the plane $V^\perp$, $\bar W$ grows quadratically in every direction away from T which is tangent to K_c^{qc}; in fact, given any $\varepsilon,\theta_0>0$ there exists α_0 such that

$$\bar W(C)\ge \alpha_0|C-C_0|^2\quad \forall C\in V^\perp\quad \text{s.t. } |C-C_0|<\varepsilon, \mathrm{angle}(C-C_0,T)>\theta_0.$$

Note that for given θ_0 and ε the constant α_0 depends only on C_0 and $e_i\otimes e_i$ and this smoothly. Further, the estimate is also true even if C_0 is chosen at the intersection of two curves (i.e., if $C_0=U_i^2$); in fact, such points are obtained as the interesection of two planes $\langle e_{i_1}, Ce_{i_1}\rangle = m_{\mathcal{U}}(e_{i_1})$ and $\langle e_{i_2}, Ce_{i_2}\rangle = m_{\mathcal{U}}(e_{i_2})$ with the manifold and we may use either e_{i_1} or e_{i_2} to establish it. Therefore, given any $\theta_0,\varepsilon>0$ we can choose α_0 independent of the position C_0 on boundary of K_c^{qc} in the above estimate. □

Remark 8.2. *Suppose $K = \mathrm{O}(2)U_1\cup\ldots\cup \mathrm{O}(2)U_k$ for $U_i\in\mathbb{M}^{2\times 2}_{sym}$ that satisfy (H1), (H2) or that $K=\mathrm{SO}(3)\hat U_1\cup\ldots\cup\mathrm{SO}(3)\hat U_k$ for $U_i\in\mathbb{M}^{3\times 3}$ that satisfy (H1′), (H2′). Then the function*

$$\bar W(C) = (\det C-\delta^2)_+^2 + \max_{e\in S^1}(\langle e, Ce\rangle - m_{\mathcal{U}}(e))_+^2 \tag{8.2}$$

has the properties (P1), (P2) and (P3).

This is quite similar to the discussion above. □

Unfortunately, the formula (8.2) above is unsatisfactory since it is not explicit. However, it is possible to make it explicit for specific examples.

Example 8.3. *(The four-well problem) Consider the four well problem described in Example 4.5. Given any $C\in\mathbb{M}^{2\times 2}_{sym}$, let*

$$D=\begin{cases} C-U_1^2 & C_{11}-C_{22}\ge 0\,\&\,C_{12}\ge 0,\\ C-U_2^2 & C_{11}-C_{22}\le 0\,\&\,C_{12}\ge 0,\\ C-U_4^2 & C_{11}-C_{22}\le 0\,\&\,C_{12}\le 0,\\ C-U_3^2 & C_{11}-C_{22}\ge 0\,\&\,C_{12}\le 0.\end{cases}$$

Then $\bar W$ defined in (8.2) can be explicitly written as

$$\begin{aligned}\bar W(C) &= (\det C-\delta^2)_+^2\\ &\quad+\begin{cases} (\lambda(D))_+^2 & \begin{array}{l}(C_{11}-C_{22})(D_{11}-D_{22})\ge 0\\ \textit{and } D_{12}C_{12}\ge 0,\end{array}\\ \max_{e\in\mathcal{E}_4}(\langle e, De\rangle)_+^2 & \textit{otherwise},\end{cases}\end{aligned}\tag{8.3}$$

where

$$\lambda(D)=\left(\frac{D_{11}+D_{22}}{2}\right)+\sqrt{\left(\frac{D_{11}+D_{22}}{2}\right)^2+D_{12}^2}$$

and

$$\mathcal{E}_4 = \Big\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Big\}.$$

We note that

$$\max_{e \in S^1} \left(\langle e, Ce \rangle - m_{\mathcal{U}}(e) \right)_+^2 = \left(\max_{e \in S^1} \left(\langle e, Ce \rangle - m_{\mathcal{U}}(e) \right) \right)_+^2$$

and hence our task is to calculate

$$\max_{e \in S^1} \left(\langle e, Ce \rangle - m_{\mathcal{U}}(e) \right).$$

Let $e = \{\cos\theta, \sin\theta\}$ and

$$f(\theta) = \langle e, Ce \rangle - m_{\mathcal{U}}(e).$$

It is sufficient to look at this function for $\theta \in [0, \pi)$ since our original function is invariant under $e \mapsto -e$. Our task is now to calculate $\max f$ for $\theta \in [0, \pi)$.

But first, we have to calculate $m_{\mathcal{U}}(e) = \max_{i=1,\ldots,4} \langle e, U_i^2 e \rangle$. For any $A \in \mathbb{M}^{2\times 2}_{sym}$,

$$\langle e, Ae \rangle = A_{11} \cos^2\theta + 2A_{12} \cos\theta \sin\theta + A_{22} \sin^2\theta,$$

and it is easy to conclude that

$$m_{\mathcal{U}}(e) = \max_{i=1,\ldots,4} \langle e, U_i^2 e \rangle = \begin{cases} \langle e, U_1^2 e \rangle & \theta \in [0, \frac{\pi}{4}], \\ \langle e, U_2^2 e \rangle & \theta \in [\frac{\pi}{4}, \frac{\pi}{2}], \\ \langle e, U_4^2 e \rangle & \theta \in [\frac{\pi}{2}, \frac{3\pi}{4}], \\ \langle e, U_4^2 e \rangle & \theta \in [\frac{3\pi}{4}, \pi], \end{cases} \tag{8.4}$$

since $a > b, c > 0$ by assumption.

We now claim that $\langle e, Ce \rangle$ and $f(\theta)$ achieve their maximum in the same "quarter interval" $[0, \pi/4]$, $[\pi/4, \pi/2]$, $[\pi/2, 3\pi/4]$ or $[3\pi/4, \pi]$. This is easily verified by contradiction. Let us consider the case $C_{22} \geq C_{11}, C_{12} \geq 0$; then $\langle e, Ce \rangle$ achieves its maximum in $[\pi/4, \pi/2]$. First assume that $f(\theta)$ achieves its maximum for $\theta \in [3\pi/4, \pi]$. Let $\varphi = \theta - \pi/2$ so that $\varphi \in [\pi/4, \pi/2]$. Then a simple calculation using (8.4) shows that

$$f(\varphi) - f(\theta) = (C_{22} - C_{11})(\cos^2\theta - \sin^2\theta) - 4C_{12} \cos\theta \sin\theta \geq 0$$

which contradicts the assumption that f achieves its maximum at θ. Similarly, we can show that $f(\theta)$ cannot achieve its maximum for $\theta \in [0, \pi/4]$ or for $\theta \in [\pi/2, 3\pi/4]$ by checking with $\varphi = \pi/2 - \theta$ and $\varphi = \pi - \theta$ respectively. We can similarly treat the other cases.

Thus, the maximum of f is equal to the maximum of $\langle e, De \rangle$ (for D defined above) for θ restricted to the quarter interval in which $\langle e, Ce \rangle$ achieves its maximum. Now, if the angle corresponding to the eigenvector of the maximal eigenvalue of D lies in this interval, then the maximum of $\langle e, De \rangle$ and that of f is equal to the maximum eigenvalue of D. This is the first possibility in (8.3). If the corresponding angle lies outside this interval, then the maximum of f is equal to the higher of the values of $\langle e, De \rangle$ at the two boundaries of the interval. This is the other possibility of (8.3). □

Acknowledgement

This work was conducted while KB visited the MPI Leipzig in 1997-98 and while GD visited Caltech in 1998. We gratefully acknowledge the partial financial support of the NSF (CMS 9457573) and AFOSR/MURI (F 49602-98-1-0433).

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