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by

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VARIATONAL FORMULATION FOR THE LUBRICATION APPROXIMATION OF THE HELE-SHAW FLOW

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Abstract. It has been recently discovered that both the surface tension driven one-phase Hele-Shaw flow and its lubrication approximation can be understood as (continuous limits of time-discretized) gradient flows of the corresponding surface energy functionals with respect to the Wasserstein metric. Here we complete the connection between the two problems, proving that the time-discretized lubrication approximation is the Γ-limit of suitably rescaled time-discretized Hele-Shaw flows in half space.

1. Introduction

1.1. The Gradient Flow Structure of Hele-Shaw. The surface tension driven one-phase Hele-Shaw problem in $\mathbf{R}^2$ is defined by the following system:

$$\begin{cases} \Delta p = 0 & \text{in } \Omega(t) \\ p = \gamma_2 \kappa & \text{on } \partial\Omega(t) \\ v = -\nabla p \cdot \underline{\nu} & \text{on } \partial\Omega(t). \end{cases} \tag{1.1}$$

Given an initial datum $\Omega_0 \subset \mathbf{R}^2$, (1.1) describes the evolution of the region $\Omega(t)$ occupied by a viscous incompressible fluid in a Hele-Shaw cell [1]. Here, p represents pressure in the fluid; v and $\underline{\nu}$ are the normal velocity field and the outer unit normal field over $\partial\Omega(t)$, respectively; $\gamma_2 > 0$ denotes surface tension and κ stands for the mean curvature of $\partial\Omega(t)$ (positive for a circle). The first two equations

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determine pressure according to Darcy's law, incompressibility of the fluid and Laplace condition; the third one – kinematic condition – closes the system.

It is well-known that

$$\tilde{E}_0(\Omega) = \gamma_2 \mathcal{H}^1(\partial\Omega)$$

is a Liapunov functional for the evolution, but actually the connection between the evolution and the functional is more intimate, as first observed by Almgren [1]. Indeed, for $m > 0$ consider the space

$$\mathcal{M}_m = \left\{ \chi : G \to \{0,1\} \;\middle|\; \textstyle\int_G \chi = m \right\}$$

(here $G = \mathbf{R}^2$); identifying Ω with its characteristic function χ (i.e., $\chi = \chi_\Omega(x)$), we write

$$\tilde{E}_0(\chi) = \gamma_2 \int_{\mathbf{R}^2} d|\nabla \chi|. \tag{1.2}$$

It turns out that, with a suitable choice of the metric tensor, (1.1) can be understood as the gradient flow of $\tilde{E}_0$ on $\mathcal{M}_m$. Let us recall, given a differentiable manifold $\mathcal{M}$ with metric tensor g, that the gradient flow of a differentiable functional E on $(\mathcal{M}, g)$ is given by

$$\begin{aligned} &\rho : [0,\infty) \to \mathcal{M} \\ &g_{\rho(t)}(\partial_t \rho(t), v) = -\langle dE(\rho(t)), v\rangle \quad \forall\, v \in T_{\rho(t)}\mathcal{M}, \quad t \in \mathbf{R}^+. \end{aligned}$$

Without claiming for the moment mathematical rigor, we endow $\mathcal{M}_m$ with a Riemaniann structure: Let $s \to \chi(s)$ be a curve on $\mathcal{M}_m$. Its tangent vector at $s = 0$ is then defined as the normal velocity v of the boundary $\partial\Omega$, whence

$$T_\chi \mathcal{M}_m = \left\{ v : \partial\Omega \to \mathbf{R} \;\middle|\; \textstyle\int_{\partial\Omega} v = 0 \right\}.$$

The constraint $\int_{\partial\Omega} v = 0$ (which is a consequence of mass conservation) allows to identify a tangent vector v – up to additive constant – with the solution p of the elliptic problem

$$\begin{cases} \Delta p = 0 & \text{in } \Omega \\ -\nabla p \cdot \underline{\nu} = v & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

so that

$$T_\chi \mathcal{M}_m \cong \left\{ p : \Omega \to \mathbf{R} \;\middle|\; \Delta p = 0 \text{ in } \Omega \right\}_{/\sim}$$

where $p_1 \sim p_2$ if $p_1 - p_2$ is constant. The Riemaniann structure is completed by the introduction of the metric tensor

$$g_\chi(v_1, v_2) = \int_\Omega \nabla p_1 \cdot \nabla p_2, \quad v_1, v_2 \in T_\chi \mathcal{M}_m \tag{1.4}$$

where p_i are related to v_i through (1.3). It is now easy to see that for any $t > 0$ and any $v \in T_{\chi(t)}\mathcal{M}_m$

$$0 = g_{\chi(t)}(\partial_t \chi(t), v) + \langle d\tilde{E}_0(\chi(t)), v\rangle = \int_{\partial\Omega(t)} (-p(t) + \gamma_2 \kappa(t))v,$$

which together with (1.3) entails (1.1). To give a physical interpretation to the metric tensor g_χ, we observe that

$$g_\chi(v, v) = \inf \left\{ \int_\Omega |\underline{u}|^2, \ \ \operatorname{div} \underline{u} = 0 \ \text{ in } \ \Omega, \ \ \underline{u} \cdot \underline{\nu} = v \ \text{ on } \ \partial\Omega \right\},$$

and that, by Darcy's law, the velocity $\underline{u}$ of the fluid is given by $\underline{u} = -\nabla p$. Therefore $g_\chi(v, v)$ represents the minimal rate of energy dissipation through friction required to generate the infinitesimal perturbation v on the boundary $\partial\Omega$. In conclusion, the Hele-Shaw problem can be viewed as a gradient flow with respect to physically natural quantities: free energy and dissipation of kinetic energy.

1.2. **The Hele-Shaw Flow in Half-Space.** Here we consider the case where $G = H := \mathbf{R} \times \mathbf{R}^+$, and the fluid touches the lateral boundary of the Hele-Shaw cell: $\partial\Omega \cap \partial H \neq \emptyset$. The fluid-glass surface tension γ_1 has to be taken into account, so that the relevant energy functional is

$$\tilde{E}(\chi) = \gamma_2 \int_H d|\nabla \chi| + \gamma_1 \int_{\partial H} \chi^\Gamma, \tag{1.5}$$

where χ^Γ denotes the trace value of χ on ∂H. The set of admissible variations is in this case given by

$$T_\chi \mathcal{M}_m = \left\{ v : \partial\Omega \cap H \to \mathbf{R} \;\middle|\; \textstyle\int_{\partial\Omega \cap H} v = 0 \right\}.$$

A formal computation of the first variation of $\tilde{E}$ yields

$$\langle d\tilde{E}(\chi), v\rangle = \gamma_2 \int_{\partial\Omega\cap H} \kappa v + \sum_{P \in \partial(\partial\Omega \cap \partial H)} \frac{1}{\sin\theta(P)} (\gamma_2 \cos\theta(P) + \gamma_1) v(P),$$

where $\theta(P)$ is the angle which $\partial\Omega$ forms with ∂H at a contact point $P \in \partial(\partial\Omega \cap \partial H)$. At equilibrium, the static contact angle θ is therefore determined by Young's law:

$$\gamma_2 \cos\theta = -\gamma_1. \tag{1.6}$$

In the surface tension driven Hele-Shaw problem, (1.6) is assumed to hold at all stages of evolution (here we are interested in the partial wetting regime $\theta \in (0, \frac{\pi}{2})$, which imposes $0 < -\gamma_1 < \gamma_2$). This corresponds to the idea that the dynamics are governed by two time-scales: a fast one, at triple junctions, which instantaneously enforces (1.6); and a slow one which governs macroscopic motion decreasing surface energy at the fluid-air interface. The kinematic condition

$$\nabla p \cdot \underline{n} = 0 \quad \text{on} \quad \partial\Omega \cap \partial H \tag{1.7}$$

($\underline{n}$ is the outer normal to ∂H) completes the set of equations which define the surface tension driven one-phase Hele-Shaw flow in half space:

$$\begin{cases} \Delta p = 0 & \text{in } \Omega(t) \\ p = \gamma_2 \kappa & \text{on } \partial\Omega(t) \cap H \\ v = -\nabla p \cdot \underline{\nu} & \text{on } \partial\Omega(t) \cap H \\ \nabla p \cdot \underline{n} = 0 & \text{on } \partial\Omega(t) \cap \partial H \\ \cos\theta = -\frac{\gamma_1}{\gamma_2} & \text{on } \partial(\partial\Omega(t) \cap \partial H). \end{cases} \tag{1.8}$$

As before, (1.8) can be understood as the gradient flow of $\tilde{E}$ on $(\mathcal{M}_m, g_\chi)$. We identify any $v \in T_\chi \mathcal{M}_m$ – up to additive constant – with the solution p of the elliptic problem

$$\begin{cases} \Delta p = 0 & \text{in } \Omega \\ \nabla p \cdot \underline{n} = 0 & \text{on } \partial\Omega \cap \partial H \\ -\nabla p \cdot \underline{\nu} = v & \text{on } \partial\Omega \cap H, \end{cases} \tag{1.9}$$

which gives a meaning to the metric tensor g_χ defined by (1.4). Then

$$\begin{aligned} 0 = g_{\chi(t)}(\partial_t \chi(t), v) + \langle d\tilde{E}(\chi(t)), v \rangle \ &= \int_{\partial\Omega(t)} (-p(t) + \gamma_2 \kappa(t)) v \\ &+ \sum_{P \in \partial(\partial\Omega(t) \cap \partial H)} \frac{1}{\sin\theta(t)} (\gamma_2 \cos\theta(t) + \gamma_1) v(P) \end{aligned}$$

for any $v \in T_{\chi(t)} \mathcal{M}_m$ and any $t \in \mathbf{R}^+$, which together with (1.9) entails (1.8).

Remark 1.1. In the gradient flow formulation, the *contact angle condition* "dynamic contact angle equals static contact angle" emerges as a Neumann-type boundary condition, being contained in the differential equation rather than imposed as a constraint on the ambient space. It is a consequence of the choice of the metric tensor – that is, of the dissipation law – and of the energy functional: in this sense it is intrinsically determined by the physics of the problem.

1.3. **The Thin Film Regime.** Assume now that the region Ω filled with fluid is a thin and gently sloping subgraph; that is, Ω is given by

$$\Omega = \Omega_\varepsilon := \left\{(x,y) \in H \;\middle|\; 0 < y < \varepsilon h(x)\right\},$$

where h is a non negative function such that

$$h \in O(1), \quad h' \in O(1),$$

and $\varepsilon \ll 1$ is a small parameter which accounts of the ratio between the typical y-lenghtscale and the typical x-lenghtscale. It is well-known (see e.g. [1], [12] and references therein) that in this regime lubrication theory can be applied, and (1.8) can be approximated by the following evolution problem for the thickness h:

$$\begin{cases} \partial_t h + \partial_x (h \partial_x^3 h) = 0 & \text{in } \{h > 0\} \\ h\partial_x^3 h = 0, \;\; (\partial_x h)^2 = 1 & \text{on } \partial\{h > 0\}. \end{cases} \tag{1.10}$$

This is the *lubrication approximation* of (1.8)*. Let us observe that (1.10) is a free-boundary problem, the unknown free-boundary being given by $\partial\{h > 0\}$. Since the equation is of fourth order, three conditions at $\partial\{h > 0\}$ are expected to be needed for well-posedness: (1.10) requires vanishing height (which defines the free-boundary), vanishing mass flux and prescribed contact angle.

Problem (1.10) also has a gradient flow interpretation. The free energy functional

$$E(h) = \tfrac{1}{2} \int |h'|^2 + \tfrac{1}{2} |\{h > 0\}| \tag{1.11}$$

acts on the ambient space

$$\mathcal{N} = \left\{h : \mathbf{R} \to [0,\infty) \;\middle|\; \textstyle\int h = 1\right\}.$$

We think of the tangent space as

$$T_h \mathcal{N} = \left\{v : \mathbf{R} \to \mathbf{R} \;\middle|\; \textstyle\int v = 0\right\},$$

and identify (up to additive constant) a tangent vector v with the solution π of the elliptic equation

$$v + (h\pi')' = 0. \tag{1.12}$$

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*The differential equation in (1.10) is a particular case of the so called *thin film equation*

$$\partial_t h + \partial_x (h^n \partial_x^3 h) = 0, \quad n \in \mathbf{R}^+;$$

subject to a different boundary condition, namely $\partial_x h = 0$, the thin film equation has been studied recently by many authors; we only quote the pioneering paper [3] and [4], where further references can be found.

The metric tensor g_h is then defined by

$$g_h(v_1, v_2) = \int h\pi_1'\pi_2',$$

where v_i are related to π_i through (1.12). With this choice we obtain

$$\begin{aligned} 0 = g_{h(t)}(\partial_t h(t), v) + \langle dE(h(t)), v\rangle &= \int \pi v - \int h'' v \\ &+ \sum_{P\in\partial\{h>0\}} \frac{v(P)}{2|h'(P)|}(1-|h'(P)|^2), \end{aligned}$$

which entails (1.10). Note that, as for the Hele-Shaw flow in half space, the contact angle condition at the free-boundary is implicitly contained in the differential equation, and not imposed as a constraint on the ambient space.

1.4. **Natural Time Discretizations.** It is possible to give a rigorous mathematical formulation to the gradient flow structures described above by introducing appropriate time discretizations. To this aim we need two premises.

First, any gradient flow on a Riemannian manifold $(\mathcal{M}, g)$ has a natural discretization in time [15, section 4.6]: starting from an initial data $\rho^{(0)} \in \mathcal{M}$, the scheme is given by a sequence $k \in \mathbf{N}$ of variational problems of the form

$$\begin{cases} \rho_\tau^{(k)} \ \text{minimizes} \\ \qquad \frac{1}{2\tau}\mathrm{dist}_g(\rho_\tau^{(k-1)}, \rho)^2 + E(\rho) \\ \text{among all} \ \ \rho \in \mathcal{M}, \end{cases}$$

where k is the time step, τ is its size, and dist_g is the distance induced on $\mathcal{M}$ by g. Note that these discrete problems do not require any differentiable structure, and make sense in a generic metric space $(\mathcal{M}, \mathrm{dist}_g)$.

Secondly, let us consider for a moment the following general setting: A state space

$$\tilde{\mathcal{M}} = \left\{\rho : \mathbf{R}^N \to [0,\infty) \;\middle|\; \textstyle\int_{\mathbf{R}^N} \rho = m\right\},$$

a tangent space

$$T_\rho\tilde{\mathcal{M}} = \left\{v : \mathbf{R}^N \to \mathbf{R} \;\middle|\; \textstyle\int_{\mathbf{R}^N} v = 0\right\},$$

and a metric tensor defined by

$$\tilde{g}_\rho(v_1, v_2) = \int_{\mathbf{R}^N} \rho \nabla p_1 \nabla p_2,$$

with p_i related to v_i via

$$-\mathrm{div}(\rho\nabla p) = v. \tag{1.13}$$

In [15, sections 4.3 and 5.2] it is shown that the distance dist_g induced on $(\tilde{\mathcal{M}}, \tilde{g})$ can be identified with the (L^2-)Wasserstein distance d (cf. section 3). This observation allows to replace dist_g by d in the natural time discretization of gradient flows on $(\tilde{\mathcal{M}}, \tilde{g})$. Let us remark that, when restricted to functions with finite second moments, $\tilde{\mathcal{M}}$ is indeed a metric space with respect to the Wasserstein distance d: therefore the time discretization is based on a well-defined metric structure.

If $N = 1$, then $(\tilde{\mathcal{M}}, \tilde{g})$ coincides with $(\mathcal{N}, g)$. Therefore we introduce the following time discretization of the gradient flow structure for the lubrication approximation (1.10), which only involves the surface energy functional E and the Wasserstein metric:

$$(\mathbf{P}^\tau) \quad \begin{cases} h_\tau^{(k)} \ \text{ minimizes} \\ \qquad\qquad \frac{1}{2\tau} d(h_\tau^{(k-1)}, h)^2 + E(h) \\ \text{among all } \ h \in \mathcal{N}. \end{cases}$$

A long time existence result for (1.10) has been proved in [14] by the second author: the solution is constructed as limit of a suitable interpolation of solutions of $(\mathbf{P}^\tau)$. A-posteriori, this result justifies $(\mathbf{P}^\tau)$ as *time-discretized gradient flow for the lubrication approximation.*

Remark 1.2. It is worth noting that no existence result for (1.10) was previously known. In this case, in other words, establishing a relation between the evolution and the energy functional in terms of a (discretized) gradient flow turns out to be the key for proving existence of solutions for the evolution itself.

Let us now consider the Hele-Shaw flow in $\mathbf{R}^2$. We embed $\mathcal{M}_m$ in $\tilde{\mathcal{M}}$. In view of (1.3) and (1.13), any vector $v \in T_\chi\mathcal{M}_m$ coincides with the vector $\phi_v \in T_\chi\tilde{\mathcal{M}}$ defined by

$$\langle \phi_v, \zeta\rangle = -\int_{\partial\Omega} v\zeta \quad \forall\, \zeta \in C_0^\infty(\mathbf{R}^2),$$

and the metric tensor $\tilde{g}_\chi$ coincides with g_χ on $T_\chi\mathcal{M}_m$:

$$g_\chi(v_1, v_2) = \tilde{g}_\chi(\phi_{v_1}, \phi_{v_2}).$$

Hence, extending $\tilde{E}$ as

$$\tilde{E}_0(\rho) = \begin{cases} \tilde{E}_0(\rho) & \text{if } \rho \in \mathcal{M}_m \\ \infty & \text{else,} \end{cases}$$

Hele-Shaw can be understood as gradient flow of $\tilde{E}_0$ on $(\tilde{\mathcal{M}}, \tilde{g})$. The aforementioned identification allows to replace the induced distance $\mathrm{dist}_{\tilde{g}}$ with the Wasserstein distance d. Since the extended energy compels minimizers to be characteristic sets, one obtains the following scheme for the surface tension driven Hele-Shaw flow:

$$\begin{cases} \chi_\tau^{(k)} \text{ minimizes} \\ \qquad \frac{1}{2\tau} d(\chi_\tau^{(k-1)}, \chi)^2 + \tilde{E}_0(\chi) \\ \text{among all } \chi \in \mathcal{M}_m. \end{cases}$$

In [13, Theorem 1], the second author proves convergence of this scheme to a solution of the original evolution problem: this justifies the scheme itself as *time-discretized gradient flow for Hele-Shaw.* To obtain the natural time discretization for the Hele-Shaw flow in half space, we just have to replace $\tilde{E}_0$ by $\tilde{E}$ (and $G = \mathbf{R}^2$ by $G = H$ in the definition of $\mathcal{M}_m$):

$$\begin{cases} \chi_\tau^{(k)} \text{ minimizes} \\ \qquad \frac{1}{2\tau} d(\chi_\tau^{(k-1)}, \chi)^2 + \tilde{E}(\chi) \\ \text{among all } \chi \in \mathcal{M}_m. \end{cases} \tag{1.14}$$

2. The Main Result

The aim of this paper is to show that lubrication approximation can be understood as a (Γ-)limit of the gradient flow structure for the Hele-Shaw problem in half-space: more precisely, we will prove for fixed $\tau > 0$ that a suitably rescaled sequence of Hele-Shaw discrete evolution problems (1.14) Γ-converges to the discrete evolution problem ($\mathbf{P}^\tau$). In other words, we give here a different way of interpreting and understanding the thin film limit of Hele-Shaw evolution, within the context of gradient flow theory and thus based on physically natural quantities.

2.1. **The Appropriate Scaling.** To mimic the thin film regime, we come back to the rescaling introduced in section 1.3: Given $\varepsilon > 0$, we embed $\mathcal{N} \hookrightarrow \mathcal{M}_\varepsilon$ via

$$\Omega_\varepsilon := \left\{ (x,y) \in H \;\middle|\; 0 < y < \varepsilon h(x) \right\}, \quad h \mapsto \chi_\varepsilon := \chi_{\Omega_\varepsilon}.$$

In view of Young's law (1.6), we have

$$\gamma_1 = -\gamma_2 \cos\theta \sim \gamma_2 \left(\tfrac{\varepsilon^2}{2} - 1\right),$$

so that

$$\tilde{E}(\chi_\varepsilon) = \gamma_2 \int_{\{h>0\}} \sqrt{1 + (\varepsilon h')^2} + \gamma_2 \left(\tfrac{\varepsilon^2}{2} - 1\right) |\{h > 0\}|.$$

Therefore the free energy scales as follows:

$$\tilde{E}(\chi_\varepsilon) \sim \gamma_2 \varepsilon^2 E(h) \quad \text{as} \;\; \varepsilon \to 0. \tag{2.1}$$

With respect to the metric structure, we can confine ourselves to the induced distance: the scaling is then given by the identity

$$\tfrac{1}{\varepsilon} d(\chi_\varepsilon^{(1)}, \chi_\varepsilon^{(2)})^2 = d(\chi_1^{(1)}, \chi_1^{(2)})^2, \tag{2.2}$$

which follows immediately from the definition of Wasserstein metric. In view of (2.1) and (2.2), we define a family of rescaled surface energy functionals $\{E_\varepsilon\}_{\varepsilon>0}$ as

$$E_\varepsilon(\chi) := \tfrac{1}{\varepsilon^2} \left[\int_H d|\nabla\chi| + \left(\tfrac{\varepsilon^2}{2} - 1\right) \int_{\partial H} \chi^\Gamma \right],$$

and consider, for fixed $\tau > 0$, the following rescaled and time-discretized evolutions:

$$(\mathbf{P}_\varepsilon^\tau) \quad \left\{ \begin{array}{l} \chi_\varepsilon^{(k)} \;\; \text{minimizes} \\ \qquad\qquad \tfrac{1}{2\tau\varepsilon} d(\chi_\varepsilon^{(k-1)}, \chi)^2 + E_\varepsilon(\chi) \\ \text{among all} \;\; \chi \in \mathcal{M}_\varepsilon. \end{array} \right.$$

2.2. **The Statement.** We will prove that a subsequence $\{\varepsilon_n\}_{n\in\mathbf{N}}$ of solutions $\{\chi_{\varepsilon_n}^{(k)}\}_{k\in\mathbf{N}}$ of $(\mathbf{P}_\varepsilon^\tau)$ converges to a solution of $(\mathbf{P}^\tau)$, provided the initial data converge. In order to state this result, and in particular to clarify our concept of convergence, let us introduce some notations. We define the sets

$$K_\varepsilon = \left\{ \chi_\varepsilon : H \to \{0,1\} \;\; \mathcal{L}^2\text{−measurable} \;\middle|\; \textstyle\int_H \chi_\varepsilon = \varepsilon, \quad \int_H \tfrac{1}{2}(x^2+y^2)\chi_\varepsilon < \infty \right\},$$

$$K = \left\{ h : \mathbf{R} \to [0,\infty) \;\; \mathcal{L}^1\text{−measurable} \;\middle|\; \textstyle\int h = 1, \quad \int \tfrac{1}{2} x^2 h < \infty \right\}.$$

Given h and a sequence $\{\chi_\varepsilon\}_{\varepsilon\searrow 0}$, we write

$$\chi_\varepsilon \overset{w*}{\rightharpoonup} h \iff \left\{ \begin{array}{l} h \in K, \quad \chi_\varepsilon \in K_\varepsilon \\ \tfrac{1}{\varepsilon} \displaystyle\int_H \zeta(x)\chi_\varepsilon(x,y)\,dx\,dy \to \int \zeta(x)h(x)\,dx \quad \forall\, \zeta \in C_c^0(\mathbf{R}) \end{array} \right.$$

and

$$\chi_\varepsilon \overset{w}{\rightharpoonup} h \iff \begin{cases} h \in K, \quad \chi_\varepsilon \in K_\varepsilon \\ \frac{1}{\varepsilon} \int_H y^2 \chi_\varepsilon(x,y)\,dx\,dy \to 0 \\ \frac{1}{\varepsilon} \int_H \zeta(x)\chi_\varepsilon(x,y)\,dx\,dy \to \int \zeta(x)h(x)\,dx \quad \forall\, \zeta \in C^0_2(\mathbf{R}), \end{cases}$$

where

$$C^0_2(\mathbf{R}^k) = \left\{ \zeta \in C^0(\mathbf{R}^k) \;\Big|\; \sup_{x\in\mathbf{R}^k} \frac{|\zeta(x)|}{1+|x|^2} < \infty \right\}.$$

Note that we use the notation $\overset{w}{\rightharpoonup}$ in a nonstandard way, including in particular the convergence up to the second moments. Now we are ready to state the main result.

Theorem 1. *Let $\tau > 0$, and let $\{\chi_\varepsilon^{(0)}\}_{\varepsilon\searrow 0}$, $h^{(0)}$ be such that*

$$\chi_\varepsilon^{(0)} \overset{w}{\longrightarrow} h^{(0)}.$$

For $\varepsilon > 0$, let $\{\chi_\varepsilon^{(k)}\}_{k\in\mathbf{N}}$ denote a solution of $(\mathbf{P}^\tau_\varepsilon)$. Then for a subsequence

$$\chi_\varepsilon^{(k)} \overset{w}{\longrightarrow} h^{(k)}, \quad E(\chi_\varepsilon^{(k)}) \longrightarrow E(h^{(k)}) \quad \text{as } \varepsilon \searrow 0$$

for any $k \in \mathbf{N}$, where $\{h^{(k)}\}_{k\in\mathbf{N}}$ denotes a solution of $(\mathbf{P}^\tau)$.

The proof is based on the following Proposition of Γ-convergence type.

Proposition 1. *Let $\chi_\varepsilon \overset{w*}{\rightharpoonup} h$, and $\chi_\varepsilon^{(0)} \overset{w*}{\rightharpoonup} h^{(0)}$; then*

(1) $E(h) \le \liminf_{\varepsilon\to 0} E_\varepsilon(\chi_\varepsilon)$;

(2) $d(h^{(0)},h)^2 \le \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon} d(\chi_\varepsilon^{(0)},\chi_\varepsilon)^2$.

For $h \in K$, let

$$\chi_\varepsilon(x,y) = \begin{cases} 1 & 0 < y < \varepsilon h(x) \\ 0 & \text{else;} \end{cases}$$

(3) if $E(h) < \infty$ then $E(h) = \lim_{\varepsilon\to 0} E_\varepsilon(\chi_\varepsilon)$;

(4) if $\int h^3 < \infty$ and $\chi_\varepsilon^{(0)} \overset{w}{\rightharpoonup} h^{(0)}$, then $d(h^{(0)},h)^2 = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} d(\chi_\varepsilon^{(0)},\chi_\varepsilon)^2$.

After introducing the proper definition and some basic properties of the Wasserstein metric in section 3, we provide the proof of Proposition 1 in section 4, and the proof of the main result in section 5.

3. Preliminaries

We list here the properties of the Wasserstein metric which are necessary to our purposes, and give references for more informations on the subject. For $k \in \mathbf{N}$ and $m > 0$, let

$$\mathcal{K} = \mathcal{K}_{k,m} = \{ \ \lambda \text{ non negative Borel measure on } \mathbf{R}^k \text{ such that } \int_{\mathbf{R}^k} d\lambda = m, \ \int_{\mathbf{R}^k} |x|^2 \, d\lambda < \infty \}.$$

Given $\lambda_0, \lambda \in \mathcal{K}$, we introduce (in the language of Monge-Kantorowicz mass transference problem) the space $P(\lambda_0, \lambda)$ of *admissible transference plans*

$$\begin{aligned} P(\lambda_0, \lambda) = \{ \ & p \text{ non negative Borel measure on } \mathbf{R}^k \times \mathbf{R}^k \text{ with} \\ & \text{marginals } \lambda_0, \ \lambda; \ \text{ i.e. such that for all } \zeta \in C_c^0(\mathbf{R}^k) \\ & \int_{\mathbf{R}^{2k}} \zeta(x_0) \, dp(x_0, x) = \int_{\mathbf{R}^k} \zeta(x_0) \, d\lambda_0(x_0) \\ & \int_{\mathbf{R}^{2k}} \zeta(x) \, dp(x_0, x) = \int_{\mathbf{R}^k} \zeta(x) \, d\lambda(x) \qquad \} \end{aligned}$$

The (L^2-)Wasserstein distance $d(\lambda_0, \lambda)$ is defined as

$$d(\lambda_0, \lambda)^2 := \inf_{p \in P(\lambda_0, \lambda)} \int_{\mathbf{R}^{2k}} |x_0 - x|^2 \, dp(x_0, x).$$

Note that $d(\lambda_0, \lambda)$ is finite, since the product measure $\lambda_0 \times \lambda$ belongs to $P(\lambda_0, \lambda)$.

W1 [10] The infimum is attained; that is, there exists $\mu \in P(\lambda_0, \lambda)$ such that

$$d(\lambda_0, \lambda)^2 = \int_{\mathbf{R}^{2k}} |x_0 - x|^2 \, d\mu(x_0, x).$$

Again in the language of mass transference, such a minimum point μ is called *optimal transference plan* (as a matter of fact, the optimal transference plan is unique provided the measures are absolutely continuous with respect to the Lebesgue measure: see [5], [6], [8]). The main feature of the Wasserstein distance, which turns out to be crucial in our analysis, is that it is actually a metric on $\mathcal{K}$, metrizing the weak convergence up to the second moments:

W2 [10], [16], [13] $(\mathcal{K}, d)$ is a metric space, and if $\lambda_n, \lambda \in \mathcal{K}$ are such that

$$\lim_{n \to \infty} \int_{\mathbf{R}^k} \zeta(x) \, d\lambda_n(x) = \int_{\mathbf{R}^k} \zeta(x) \, d\lambda(x) \quad \forall \, \zeta \in C_2^0(\mathbf{R}^k),$$

then $d(\lambda_n, \lambda) \longrightarrow 0$ as $n \to \infty$.

Remark 3.1. In the definition of $(\mathbf{P}_\varepsilon^\tau)$ we have used the symbol d to denote the Wasserstein metric on $\mathcal{K}_{2,\varepsilon}$, while in the definition of $(\mathbf{P}^\tau)$ we have used the same symbol d to denote the Wasserstein metric on $\mathcal{K}_{1,1}$. We shall keep this slight abuse of notation throughout the paper, since the arguments of the metric are sufficient to identify it.

The following simple observation will be frequently used in the sequel. Let $\lambda_0, \lambda \in \mathcal{K}$, $p \in P(\lambda_0, \lambda)$, and assume that $g \in L^1(\mathbf{R}; d\lambda)$; then

$$\int_{\mathbf{R}^{2k}} g(x)\, dp(x,y) = \int_{\mathbf{R}^k} g(x)\, d\lambda(x).$$

Note that, in view of the definition of $\mathcal{K}$, the property holds for any $\mathcal{L}^k$-measurable function $g : \mathbf{R}^k \to \mathbf{R}$ such that $\frac{g(x)}{1+|x|^2} \in L^\infty(\mathbf{R}^k, d\lambda)$.

We conclude the section recalling, for the sake of completeness, some basic properties from measure theory:

Lemma 3.1. *Let $\{\mu_\varepsilon\}_{\varepsilon>0}$ be finite non negative Borel measures on $\mathbf{R}^k$;*

(a) if $\{\mu_\varepsilon\}$ is bounded, then there exist a Borel measure μ and a subsequence (still indexed by ε) such that $\mu_\varepsilon \overset{w*}{\rightharpoonup} \mu$ in the sense of measures, that is

$$\lim_{\varepsilon\to 0} \int_{\mathbf{R}^k} \zeta\, d\mu_\varepsilon = \int_{\mathbf{R}^k} \zeta\, d\mu \quad \forall\, \zeta \in C_c^0(\mathbf{R}^k).$$

Assume that $\mu_\varepsilon \overset{w*}{\rightharpoonup} \mu$ for some Borel measure μ, and let $\eta \in C^0(\mathbf{R}^k; \mathbf{R}^+)$;

(b) if $\{\eta\mu_\varepsilon\}_{\varepsilon>0}$, $\eta\mu$ are finite and $\lim_{\varepsilon\to 0} \int_{\mathbf{R}^k} \eta\, d\mu_\varepsilon = \int_{\mathbf{R}^k} \eta\, d\mu$, then

$$\lim_{\varepsilon\to 0} \int_{\mathbf{R}^k} \zeta\, d\mu_\varepsilon = \int_{\mathbf{R}^k} \zeta\, d\mu \quad \forall\, \zeta \in C^0(\mathbf{R}^k) :\ |\zeta| \le \eta;$$

(c) if $\{\eta\mu_\varepsilon\}_{\varepsilon>0}$ is bounded, then

$$\lim_{\varepsilon\to 0} \int_{\mathbf{R}^k} \zeta\, d\mu_\varepsilon = \int_{\mathbf{R}^k} \zeta\, d\mu \quad \forall\, \zeta \in C^0(\mathbf{R}^k) :\ \lim_{|x|\to\infty} \frac{|\zeta|}{\eta} = 0.$$

Proofs of (a) and (b) can be found for instance in [2], Theorems 7.8.2 and 7.7.7, while (c) is a straightforward application of Monotone convergence Theorem.

4. Proof of Proposition 1

Let $\chi \in BV(H;\{0,1\})$; then $\chi^\Gamma \in L^1(\partial H, dx)$, and there exists a $|\nabla\chi|$-measurable function $\nu = (\nu_x, \nu_y) : H \to S^1$ such that

$$\int_H \chi \mathrm{div}\varphi = -\int_{\partial H} \varphi_2 \chi^\Gamma - \int_H \varphi\cdot\nu\, d|\nabla\chi| \quad \forall\, \varphi \in C^1_b(\overline{H};\mathbf{R}^2) \tag{4.1}$$

(here and after, we refer to [7], [9] for basic properties of BV functions). Choosing $\varphi = (0,1)$, it follows immediately that

$$E^0_\varepsilon(\chi) := \tfrac{1}{\varepsilon^2}\left(\int_H d|\nabla\chi| - \int_{\partial H}\chi^\Gamma\right) \geq 0. \tag{4.2}$$

The following lemma provides two integral estimates for h, which will be crucial in the sequel.

Lemma 1. *Let $h \in L^1(\mathbf{R};[0,\infty))$ and $\chi \in BV(H;\{0,1\})$ be such that*

$$\int \zeta(x)h(x)\,dx = \tfrac{1}{\varepsilon}\int_H \zeta(x)\chi(x,y)\,dx\,dy \quad \forall\,\zeta \in C^0_c(\mathbf{R}); \tag{4.3}$$

then for any $0<\delta<1$ and any $\xi \in C^1_c(\mathbf{R})$

$$\int \xi' h \;\leq\; \left[2E^0_\varepsilon(\chi)\left(\tfrac{1}{1-\delta}\int \xi^2 \;+\tfrac{\varepsilon^2}{\delta}\|\xi\|^2_\infty E^0_\varepsilon(\chi)\right)\right]^{1/2} \tag{4.4}$$

$$\int \frac{h}{h+\delta} \;\leq\; \tfrac{1}{1-\delta}\int_{\partial H}\chi^\Gamma + \tfrac{\varepsilon^2}{\delta}E^0_\varepsilon(\chi). \tag{4.5}$$

Proof. It follows from (4.1) that

$$\int_{\partial H}\xi\chi^\Gamma = -\int_H \xi\nu_y\,d|\nabla\chi| \leq \int_H \xi|\nu_y|\,d|\nabla\chi| \leq \int_H \xi\,d|\nabla\chi| \tag{4.6}$$

for any non negative $\xi \in C^0_b(\mathbf{R})$. Consider the marginal λ of $|\nabla\chi|$ on $\mathbf{R}_x$; it follows from (4.6) that $\chi^\Gamma \ll \lambda$, which by the Radon-Nykodym Theorem implies the existence of a λ- and $\mathcal{L}^1$-measurable function $\theta:\mathbf{R}\to\mathbf{R}$ such that

$$\int_{\partial H}\xi\chi^\Gamma = \int_H \xi\theta\,d|\nabla\chi| \quad \forall\,\xi \in C^0_b(\mathbf{R});$$

(4.6) entails $0\leq\theta\leq 1$, and by density we obtain

$$\int_G \chi^\Gamma = \int_{G\times(0,\infty)} \theta\,d|\nabla\chi| \tag{4.7}$$

for any λ- and $\mathcal{L}^1$-measurable set $G \subseteq \mathbf{R}$. Let now $\xi \in C_c^1(\mathbf{R})$; it holds

$$\begin{aligned} \varepsilon \int \xi' h &\overset{(4.3)}{=} \int_H \xi' \chi \overset{(4.1)}{=} -\int_H \xi \nu_x \, d|\nabla\chi| \\ &\leq \left(\int_H \xi^2 \, d|\nabla\chi| \right)^{1/2} \left(\int_H \nu_x^2 \, d|\nabla\chi| \right)^{1/2} =: (I_1)^{1/2} (I_2)^{1/2}. \end{aligned} \tag{4.8}$$

Since $\nu_x^2 = 1 - \nu_y^2 \leq 2(1 - |\nu_y|)$ $|\nabla\chi|$-a.e., we obtain

$$\begin{aligned} I_2 &= \int_H \nu_x^2 \, d|\nabla\chi| \leq 2 \left(\int_H d|\nabla\chi| - \int_H |\nu_y| \, d|\nabla\chi| \right) \\ &\overset{(4.6)}{\leq} 2 \left(\int_H d|\nabla\chi| - \int_{\partial H} \chi^\Gamma \right) = 2\varepsilon^2 E_\varepsilon^0(\chi). \end{aligned}$$

In order to estimate I_1, let $\delta \in (0,1)$ and consider the λ- and $\mathcal{L}^1$-measurable sets

$$\mathcal{U}_\delta := \{\theta > 1 - \delta\}, \quad \mathcal{A}_\delta := \mathbf{R} \setminus \mathcal{U}_\delta.$$

We claim that

$$\int_{\mathcal{A}_\delta \times (0,\infty)} d|\nabla\chi| \leq \tfrac{\varepsilon^2}{\delta} E_\varepsilon^0(\chi). \tag{4.9}$$

Indeed, it follows from (4.7) and the definition of $\mathcal{A}_\delta$ that

$$\tfrac{1}{1-\delta} \int_{\mathcal{A}_\delta} \chi^\Gamma = \tfrac{1}{1-\delta} \int_{\mathcal{A}_\delta \times (0,\infty)} \theta \, d|\nabla\chi| \leq \int_{\mathcal{A}_\delta \times (0,\infty)} d|\nabla\chi|,$$

so that

$$\begin{aligned} \delta \int_{\mathcal{A}_\delta \times (0,\infty)} d|\nabla\chi| &\leq \int_{\mathcal{A}_\delta \times (0,\infty)} d|\nabla\chi| - \int_{\mathcal{A}_\delta} \chi^\Gamma \leq \\ &\overset{(4.7)}{\leq} \left(\int_{\mathcal{A}_\delta \times (0,\infty)} d|\nabla\chi| - \int_{\mathcal{A}_\delta} \chi^\Gamma \right) + \left(\int_{\mathcal{U}_\delta \times (0,\infty)} d|\nabla\chi| - \int_{\mathcal{U}_\delta} \chi^\Gamma \right) \\ &= \int_H d|\nabla\chi| - \int_{\partial H} \chi^\Gamma = \varepsilon^2 E_\varepsilon^0(\chi), \end{aligned}$$

which proves (4.9). We write

$$I_1 = \int_H \xi^2 \, d|\nabla\chi| = \int_{\mathcal{U}_\delta \times (0,\infty)} \xi^2 \, d|\nabla\chi| + \int_{\mathcal{A}_\delta \times (0,\infty)} \xi^2 \, d|\nabla\chi| =: I_1' + I_1'';$$

in view of (4.9) we have

$$I_1'' \leq \|\xi\|_\infty^2 \tfrac{\varepsilon^2}{\delta} E_\varepsilon^0(\chi),$$

and using the definitions of $\mathcal{U}_\delta$ and θ we obtain

$$\begin{aligned} I_1' &\le \tfrac{1}{1-\delta}\int_{\mathcal{U}_\delta\times(0,\infty)} \xi^2\theta\, d|\nabla\chi| \le \tfrac{1}{1-\delta}\int_H \xi^2\theta\, d|\nabla\chi| \\ &= \tfrac{1}{1-\delta}\int_{\partial H} \xi^2\chi^\Gamma \le \tfrac{1}{1-\delta}\int \xi^2, \end{aligned}$$

which completes the proof of the first inequality.

To prove the second inequality we write

$$\int \frac{h}{h+\delta} = \int_H \frac{\chi}{\varepsilon(h+\delta)} = \int_H \frac{\chi^2}{\varepsilon(h+\delta)} = \int_H \chi\partial_y\xi,$$

where ξ is defined by

$$\xi(x,y) := \frac{1}{\varepsilon(h(x)+\delta)}\int_0^y \chi(x,\eta)\, d\eta.$$

Note that $0\le\xi\le 1$, $0\le\partial_y\xi\le 1/(\varepsilon\delta)$ and $\xi=0$ on ∂H. Therefore, using a density argument and integration by parts we obtain

$$\int \frac{h}{h+\delta} \le \int_H |\xi| d|\nabla\chi| \le \int_H d|\nabla\chi|. \tag{4.10}$$

The assertion now follows from (4.9) and the following inequality:

$$\begin{aligned} \int_{\mathcal{U}_\delta\times(0,\infty)} d|\nabla\chi| &\le \tfrac{1}{1-\delta}\int_{\mathcal{U}_\delta\times(0,\infty)} \theta\, d|\nabla\chi| \\ &\le \tfrac{1}{1-\delta}\int_H \theta\, d|\nabla\chi| \overset{(4.7)}{=} \tfrac{1}{1-\delta}\int_{\partial H}\chi^\Gamma. \end{aligned}$$

This completes the proof of the Lemma. □

The remaining part of the section concerns the proof of Proposition 1, which is performed separately for each item.

Proposition 1. (1) *Let* $\chi_\varepsilon \overset{w*}{\rightharpoonup} h$*; then* $E(h)\le \liminf_{\varepsilon\to 0} E_\varepsilon(\chi_\varepsilon)$.

Proof. We assume without loss of generality that $\liminf_{\varepsilon\to0} E_\varepsilon(\chi_\varepsilon)<\infty$, and consider a minimizing subsequence (still indexed by ε), so that

$$E_\varepsilon^0(\chi_\varepsilon) + \tfrac12\int_{\partial H}\chi_\varepsilon^\Gamma = E_\varepsilon(\chi_\varepsilon)\le C<\infty. \tag{4.11}$$

Define

$$h_\varepsilon(x) := \tfrac{1}{\varepsilon}\int_0^\infty \chi_\varepsilon(x,y)\, dy \in L^1(\mathbf{R};[0,\infty));$$

it follows from Lemma 1 that

$$\int \xi' h_\varepsilon \;\le\; \left[2E^0_\varepsilon(\chi_\varepsilon)\left(\tfrac{1}{1-\delta}\int \xi^2 \;+\tfrac{\varepsilon^2}{\delta}\|\xi\|^2_\infty E^0_\varepsilon(\chi_\varepsilon)\right)\right]^{1/2} \tag{4.12}$$

$$\int \frac{h_\varepsilon}{h_\varepsilon+\delta} \;\le\; \tfrac{1}{1-\delta}\int_{\partial H}\chi^\Gamma_\varepsilon + \tfrac{\varepsilon^2}{\delta}E^0_\varepsilon(\chi_\varepsilon) \tag{4.13}$$

for any $\xi \in C^1_c(\mathbf{R})$. In particular, (4.12) and (4.11) imply that $\{h_\varepsilon\}$ is uniformly bounded in $BV_{\text{loc}}(\mathbf{R})$, and therefore

$$h_\varepsilon \longrightarrow h \quad \text{in } L^1_{\text{loc}}(\mathbf{R}) \text{ and a.e.} \quad \text{as } \varepsilon \searrow 0 \tag{4.14}$$

(the limit h is identified in view of $\chi_\varepsilon \overset{w*}{\rightharpoonup} h$). Fatou's Lemma in turn implies that

$$\int \frac{h}{h+\delta} \le \liminf_{\varepsilon\to 0}\int \frac{h_\varepsilon}{h_\varepsilon+\delta}. \tag{4.15}$$

Using (4.14) and (4.15) we pass to the limit as $\varepsilon \searrow 0$ in (4.12) and (4.13), obtaining

$$\int \xi' h \;\le\; \left(\tfrac{2}{1-\delta}\left(\int \xi^2\right)\liminf_{\varepsilon\to 0} E^0_\varepsilon(\chi_\varepsilon)\right)^{1/2},$$

$$\int \frac{h}{h+\delta} \;\le\; \tfrac{1}{1-\delta}\liminf_{\varepsilon\to 0}\int_{\partial H}\chi^\Gamma_\varepsilon.$$

In order to pass to the limit as $\delta \searrow 0$, we observe that

$$|\{h>0\}| \le \liminf_{\delta\to 0}\int \frac{h}{h+\delta}.$$

Indeed, for any $\sigma > 0$ it holds $|\{h>\sigma\}| < \infty$ and $\frac{h}{h+\delta} \to 1$ in $\{h > \sigma\}$ as $\delta \to 0$; Fatou's Lemma then yields

$$|\{h>\sigma\}| = \int_{\{h>\sigma\}} \lim_{\delta\to 0}\frac{h}{h+\delta} \le \liminf_{\delta\to 0}\int \frac{h}{h+\delta},$$

and the arbitrariness of σ gives the inequality. Hence

$$\int \xi' h \;\le\; \left(2\left(\int \xi^2\right)\liminf_{\varepsilon\to 0} E^0_\varepsilon(\chi_\varepsilon)\right)^{1/2}, \tag{4.16}$$

$$\tfrac{1}{2}|\{h>0\}| \;\le\; \liminf_{\varepsilon\to 0}\tfrac{1}{2}\int_{\partial H}\chi^\Gamma_\varepsilon. \tag{4.17}$$

It follows from (4.16) that h' is a signed Radon measure, and that

$$\sup\left\{\int \xi h' \;\middle|\; \xi \in L^2(\mathbf{R}),\ \|\xi\|_2 \le 1\right\} \le \left(2\liminf_{\varepsilon\to 0} E^0_\varepsilon(\chi_\varepsilon)\right)^{1/2} < \infty;$$

hence $h' \in L^2(\mathbf{R})$, and

$$\tfrac{1}{2}\int |h'|^2 \le \liminf_{\varepsilon\to 0} E^0_\varepsilon(\chi_\varepsilon);$$

summing this inequality with (4.17) completes the proof of Proposition 1 (1). $\square$

Proposition 1. (3) *Let $h \in K$ such that $E(h) < \infty$, and let*

$$\chi_\varepsilon(x,y) = \begin{cases} 1 & 0 < y < \varepsilon h(x) \\ 0 & \text{else;} \end{cases}$$

then $E(h) = \lim_{\varepsilon \to 0} E_\varepsilon(\chi_\varepsilon)$.

Proof. Note that $h' \in L^2(\mathbf{R})$ in view of $E(h) < \infty$. Hence, for any $\varphi = (\varphi_1, \varphi_2) \in C_c^1(H; \mathbf{R}^2)$ such that $|\varphi| \leq 1$, it holds

$$\begin{aligned} \int_H \chi_\varepsilon \mathrm{div}\varphi &= \int \left(\int_0^{\varepsilon h(x)} \partial_x \varphi_1(x,y)\, dy + \int_0^{\varepsilon h(x)} \partial_y \varphi_2(x,y)\, dy \right) dx \\ &= \int \left[\partial_x \left(\int_0^{\varepsilon h(x)} \varphi_1(x,y)\, dy \right) - \varepsilon h'(x) \varphi_1(x, \varepsilon h(x)) + \varphi_2(x, \varepsilon h(x)) \right] dx \\ &= \int \left[\varphi_2(x, \varepsilon h(x)) - \varepsilon h'(x) \varphi_1(x, \varepsilon h(x)) \right] dx \\ &= \int_{\{h>0\}} \left[\varphi_2(x, \varepsilon h(x)) - \varepsilon h'(x) \varphi_1(x, \varepsilon h(x)) \right] dx \\ &\leq \int_{\{h>0\}} |\varphi(x, \varepsilon h(x))| \sqrt{(\varepsilon h')^2 + 1} \leq \int_{\{h>0\}} \sqrt{(\varepsilon h')^2 + 1}, \end{aligned}$$

where we have used Cauchy-Schwarz inequality in $\mathbf{R}^2$. The arbitrariness of φ implies that

$$\int_H d|\nabla \chi_\varepsilon| \leq \int_{\{h>0\}} \sqrt{(\varepsilon h')^2 + 1};$$

then

$$\begin{aligned} E_\varepsilon(\chi_\varepsilon) &= \tfrac{1}{\varepsilon^2} \left(\int_H d|\nabla \chi_\varepsilon| - \int_{\partial H} \chi_\varepsilon^\Gamma \right) + \tfrac{1}{2} \int_{\partial H} \chi_\varepsilon^\Gamma \\ &\leq \tfrac{1}{\varepsilon^2} \int_{\{h>0\}} \left(\sqrt{(\varepsilon h')^2 + 1} - 1 \right) + \tfrac{1}{2} |\{h > 0\}|. \end{aligned}$$

Using the inequality $\sqrt{w+1} - 1 \leq w/2$, we conclude that

$$E_\varepsilon(\chi_\varepsilon) \leq \tfrac{1}{2} \int |h'|^2 + \tfrac{1}{2} |\{h > 0\}| = E(h).$$

Since $\chi_\varepsilon \overset{w*}{\rightharpoonup} h$, in view of Proposition 1 (1) the proof is complete. $\square$

Proposition 1. (2) *If $\chi_\varepsilon \overset{w*}{\rightharpoonup} h$ and $\chi_\varepsilon^{(0)} \overset{w*}{\rightharpoonup} h^{(0)}$, then*

$$d(h^{(0)}, h)^2 \leq \liminf_{\varepsilon \to 0} \tfrac{1}{\varepsilon} d(\chi_\varepsilon^{(0)}, \chi_\varepsilon)^2.$$

Proof. We assume without loss of generality that $\liminf_{\varepsilon\to 0}\frac{1}{\varepsilon}d(\chi_\varepsilon^{(0)},\chi_\varepsilon)^2 < \infty$, and consider a minimizing sequence (still indexed by ε). Let $\overline{\mu}_\varepsilon$ be optimal transference plans for $\chi_\varepsilon^{(0)}$, χ_ε, and consider the marginal μ_ε of $\overline{\mu}_\varepsilon/\varepsilon$ on $\mathbf{R}^2_{x_0,x}$, that is the unique non negative Borel measure μ_ε on $\mathbf{R}^2$ such that

$$\frac{1}{\varepsilon}\int_{\mathbf{R}^4}\zeta(x_0,x)\,d\overline{\mu}_\varepsilon(x_0,y_0,x,y) = \int_{\mathbf{R}^2}\zeta(x_0,x)\,d\mu_\varepsilon(x_0,x) \quad \forall\,\zeta\in C_c^0(\mathbf{R}^2).$$

Since $\mu_\varepsilon(\mathbf{R}^2)=1$, in view of Lemma 3.1($a$) there exist a non negative finite Borel measure μ on $\mathbf{R}^2$ and a subsequence (still indexed by ε) such that $\mu_\varepsilon \overset{w*}{\rightharpoonup} \mu$. We prove that μ is an admissible transference plan for $h^{(0)}$, h. Let $\zeta\in C_c^0(\mathbf{R})$. Since

$$\int_{\mathbf{R}^2}|x_0-x|^2\,d\mu_\varepsilon = \frac{1}{\varepsilon}\int_{\mathbf{R}^4}|x_0-x|^2\,d\overline{\mu}_\varepsilon \le \frac{1}{\varepsilon}d(\chi_\varepsilon^{(0)},\chi_\varepsilon)^2 \le C, \tag{4.18}$$

it follows that $\{(1+|x_0-x|^2)\mu_\varepsilon\}_{\varepsilon>0}$ is bounded. Therefore, by Lemma 3.1(c)

$$\begin{aligned}\int_{\mathbf{R}^2}\zeta(x_0)\,d\mu &= \lim_{\varepsilon\to 0}\int_{\mathbf{R}^2}\zeta(x_0)\,d\mu_\varepsilon = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\mathbf{R}^4}\zeta(x_0)\,d\overline{\mu}_\varepsilon\\ &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_H \zeta(x_0)\chi_\varepsilon^{(0)}(x_0,y_0)\,dy_0\,dx_0,\end{aligned}$$

and since $\chi_\varepsilon^{(0)} \overset{w*}{\rightharpoonup} h^{(0)}$ we obtain

$$\int_{\mathbf{R}^2}\zeta(x_0)d\mu = \int \zeta(x_0)h^{(0)}(x_0)\,dx_0 \quad \forall\,\zeta\in C_c^0(\mathbf{R}).$$

Hence $h^{(0)}$ is the first marginal of μ, $\mu(\mathbf{R}^2)=1$ and the same argument (just interchange x_0 and x) then yields $\mu\in P(h^{(0)},h)$. Therefore

$$d(h^{(0)},h)^2 \le \int_{\mathbf{R}^2}|x_0-x|^2\,d\mu \le \liminf_{\varepsilon\to 0}\int_{\mathbf{R}^2}|x_0-x|^2\,d\mu_\varepsilon \overset{(4.18)}{\le} \liminf_{\varepsilon\to 0}\frac{1}{\varepsilon}d(\chi_\varepsilon^{(0)},\chi_\varepsilon)^2$$

which proves the result. □

In proving Proposition 1 (4) we shall need the following Lemma.

Lemma 4.1. *Let* $\chi_\varepsilon\in K_\varepsilon$, $h\in K$ *such that* $\chi_\varepsilon \overset{w}{\rightharpoonup} h$*; then*

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_H \zeta(x,y)\chi_\varepsilon(x,y)\,dy\,dx = \int \zeta(x,0)h(x)\,dx \quad \forall\,\zeta\in C_2^0(\mathbf{R}^2).$$

Proof. Let λ be the Borel measure on $\mathbf{R}^2$ defined by

$$\int_{\mathbf{R}^2}\zeta(x,y)\,d\lambda = \int \zeta(x,0)h(x)\,dx \quad \forall\zeta\in C_c^0(\mathbf{R}^2).$$

By assumption, and since $s \in K$,

$$\lim_{\varepsilon\to 0} \tfrac{1}{\varepsilon}\int_H (1+x^2+y^2)\chi_\varepsilon = \int (1+x^2)h = \int (1+x^2+y^2)\, d\lambda.$$

Hence, in view of Lemma 3.1(b) it suffices to show that $\chi_\varepsilon \overset{w*}{\rightharpoonup} \lambda$. Let $\zeta \in C_c^0(\mathbf{R}^2)$, and let $\zeta_n \in C_c^1(\mathbf{R}^2)$ such that $\zeta_n \to \zeta$ in $C^0(\mathbf{R}^2)$. We write

$$\tfrac{1}{\varepsilon}\int_H \zeta\chi_\varepsilon = \tfrac{1}{\varepsilon}\int_H (\zeta-\zeta_n)\chi_\varepsilon + \tfrac{1}{\varepsilon}\int_H [\zeta_n(x,y)-\zeta_n(x,0)]\chi_\varepsilon + \tfrac{1}{\varepsilon}\int_H \zeta_n(x,0)\chi_\varepsilon.$$

It holds

$$\tfrac{1}{\varepsilon}\int_H |\zeta-\zeta_n|\chi_\varepsilon \le \|\zeta-\zeta_n\|_\infty$$

and for any $\delta > 0$, using Young's inequality,

$$\tfrac{1}{\varepsilon}\int_H |\zeta_n(x,y)-\zeta_n(x,0)|\chi_\varepsilon \le \tfrac{\delta}{2}\|\partial_y\zeta_n\|_\infty^2 + \tfrac{1}{2\delta\varepsilon}\int_H y^2\chi_\varepsilon.$$

We pass to the limit as $\varepsilon \to 0$: recalling that $\chi_\varepsilon \overset{w}{\rightharpoonup} h$, and taking into account the arbitrariness of δ, we obtain

$$\limsup_{\varepsilon\to 0}\left|\tfrac{1}{\varepsilon}\int_H \zeta\chi_\varepsilon - \int \zeta_n(x,0)h\right| \le \|\zeta-\zeta_n\|_\infty,$$

and $\chi_\varepsilon \overset{w*}{\rightharpoonup} \lambda$ follows passing to the limit as $n\to\infty$. □

Proposition 1. (4) *Let $h \in K$ such that $\int h^3 < \infty$, and let*

$$\chi_\varepsilon(x,y) = \begin{cases} 1 & 0 < y < \varepsilon h(x) \\ 0 & \text{else;} \end{cases}$$

if $\chi_\varepsilon^{(0)} \overset{w}{\rightharpoonup} h^{(0)}$, then

$$d(h^{(0)},h)^2 = \lim_{\varepsilon\to 0}\tfrac{1}{\varepsilon} d(\chi_\varepsilon^{(0)},\chi_\varepsilon)^2.$$

Proof. Let $\lambda_\varepsilon^{(0)}$, $\lambda^{(0)}$ be the Borel measures on $\mathbf{R}^2$ defined by

$$\int_{\mathbf{R}^2} \zeta(x_0,y_0)\, d\lambda_\varepsilon^{(0)}(x_0,y_0) = \tfrac{1}{\varepsilon}\int_H \zeta(x_0,y_0)\chi_\varepsilon^{(0)}(x_0,y_0)\, dy_0\, dx_0,$$

$$\int_{\mathbf{R}^2} \zeta(x_0,y_0)\, d\lambda^{(0)}(x_0,y_0) = \int \zeta(x_0,0)h^{(0)}(x_0)\, dx_0.$$

The Borel measures λ_ε, λ are defined analogously through χ_ε and h. Using triangle inequality (**W2**, section 3), we write

$$\begin{aligned} \tfrac{1}{2\tau\varepsilon} d(\chi_\varepsilon^{(0)},\chi_\varepsilon)^2 &= d(\lambda_\varepsilon^{(0)},\lambda_\varepsilon)^2 \\ &\le \left[d(\lambda_\varepsilon^{(0)},\lambda^{(0)}) + d(\lambda^{(0)},\lambda) + d(\lambda,\lambda_\varepsilon)\right]^2. \end{aligned} \tag{4.19}$$

It follows from Lemma 4.1 that

$$\lim_{\varepsilon\to 0}\int_{\mathbf{R}^2}\zeta(x_0,y_0)\,d\lambda_\varepsilon^{(0)}=\int_{\mathbf{R}^2}\zeta(x_0,y_0)\,d\lambda^{(0)}\quad \forall\,\zeta\in C_2^0(\mathbf{R}^2),$$

which in view of **W2** implies

$$\lim_{\varepsilon\to 0} d(\lambda_\varepsilon^{(0)},\lambda^{(0)})=0. \tag{4.20}$$

From the definition of χ_ε it follows that

$$\begin{aligned}\tfrac{1}{\varepsilon}\int_H\zeta(x)\chi_\varepsilon &= \int\zeta(x)h(x)\,dx\quad \forall\varepsilon>0,\\ \tfrac{1}{\varepsilon}\int_H y^2\chi_\varepsilon &= \tfrac{\varepsilon^2}{3}\int h^3\longrightarrow 0\quad\text{as }\ \varepsilon\to 0;\end{aligned}$$

hence $\chi_\varepsilon \overset{w}{\rightharpoonup} h$, and again by Lemma 4.1 and **W2**

$$\lim_{\varepsilon\to 0} d(\lambda,\lambda_\varepsilon)=0. \tag{4.21}$$

Using (4.20) and (4.21) in (4.19) we obtain

$$\limsup_{\varepsilon\to 0}\tfrac{1}{2\tau\varepsilon}d(\chi_\varepsilon^{(0)},\chi_\varepsilon)^2\le d(\lambda^{(0)},\lambda)^2,$$

and in view of Proposition 1 (2) it remains to show that $d(\lambda^{(0)},\lambda)\le d(h^{(0)},h)$. Let μ be an optimal transference plan for $h^{(0)}$, h; we lift it to an admissible transference plan $\overline{\mu}$ for $\lambda^{(0)}$, λ defining

$$\int_{\mathbf{R}^4}\zeta(x_0,y_0,x,y)\,d\overline{\mu}=\int_{\mathbf{R}^2}\zeta(x_0,0,x,0)\,d\mu\quad\forall\zeta\in C_c^0(\mathbf{R}^4).$$

Indeed, we have

$$\begin{aligned}\int_{\mathbf{R}^4}\zeta(x_0,y_0)\,d\overline{\mu}&=\int_{\mathbf{R}^2}\zeta(x_0,0)\,d\mu=\int\zeta(x_0,0)h^{(0)}(x_0)\,dx_0=\int_{\mathbf{R}^2}\zeta(x_0,y_0)\,d\lambda^{(0)},\\ \int_{\mathbf{R}^4}\zeta(x,y)\,d\overline{\mu}&=\int_{\mathbf{R}^2}\zeta(x,0)\,d\mu=\int\zeta(x,0)h(x)\,dx=\int_{\mathbf{R}^2}\zeta(x,y)\,d\lambda,\end{aligned}$$

whence $\overline{\mu}\in P(\lambda^{(0)},\lambda)$, and therefore

$$d(\lambda^{(0)},\lambda)^2\le\int_{\mathbf{R}^4}|x_0-x|^2\,d\overline{\mu}=\int_{\mathbf{R}^2}|x_0-x|^2\,d\mu=d(h^{(0)},h)^2$$

which completes the proof of Proposition 1 (4). $\square$

5. Proof of the Main Result

It is enough to prove the assertion for $k=1$; indeed, once we have shown that $\chi_\varepsilon^{(1)} \overset{w}{\rightharpoonup} h^{(1)}$, the argument can then be iterated by selecting a subsequence at each step k. Since $\chi_\varepsilon^{(1)}$ are minimizers, it follows immediately from Proposition 1 (3) and (4) that

$$\tfrac{1}{2\tau\varepsilon} d(\chi_\varepsilon^{(0)}, \chi_\varepsilon^{(1)}) + E_\varepsilon(\chi_\varepsilon^{(1)}) \le C_1 \tag{5.1}$$

for a suitable positive constant C_1 independent of ε. Let

$$h_\varepsilon^{(1)}(x) := \tfrac{1}{\varepsilon} \int_0^\infty \chi_\varepsilon^{(1)}(x,y)\, dy \ \in L^1(\mathbf{R}), \qquad \int h_\varepsilon^{(1)} = 1;$$

arguing as in the proof of Proposition 1 (1), it follows from (5.1) and Lemma 1 that

$$h_\varepsilon^{(1)} \longrightarrow h^{(1)} \quad \text{in } \ L^1_{\mathrm{loc}}(\mathbf{R}) \tag{5.2}$$

for a subsequence (still indexed by ε).

Step 1. We are going to show that

$$\chi_\varepsilon^{(1)} \overset{w*}{\rightharpoonup} h^{(1)}. \tag{5.3}$$

By the definition of $h_\varepsilon^{(1)}$ and (5.2) we have

$$\tfrac{1}{\varepsilon} \int_H \zeta(x) \chi_\varepsilon^{(1)}(x,y)\, dy\, dx = \int \zeta(x) h_\varepsilon^{(1)}(x)\, dx \longrightarrow \int \zeta(x) h^{(1)}(x)\, dx \quad \forall\, \zeta \in C_c^0(\mathbf{R}),$$

so that it remains to show $h^{(1)} \in K$. Let $\overline{\mu}_\varepsilon$ be an optimal transference for $\chi_\varepsilon^{(0)}$, $\chi_\varepsilon^{(1)}$; it holds

$$\begin{aligned} \tfrac{1}{\varepsilon}\int_H (x^2+y^2)\chi_\varepsilon^{(1)}(x,y)\, dy\, dx &= \tfrac{1}{\varepsilon}\int_{\mathbf{R}^4} (x^2+y^2)\, d\overline{\mu}_\varepsilon \\ &\le \tfrac{1}{\varepsilon}\int_{\mathbf{R}^4} 2[(x_0-x)^2+(y_0-y)^2]\, d\overline{\mu}_\varepsilon + \tfrac{1}{\varepsilon}\int_{\mathbf{R}^4} 2(x_0^2+y_0^2)\, d\overline{\mu}_\varepsilon \\ &= \tfrac{2}{\varepsilon} d(\chi_\varepsilon^{(0)}, \chi_\varepsilon^{(1)})^2 + \tfrac{2}{\varepsilon}\int_H (x_0^2+y_0^2)\chi_\varepsilon^{(0)}(x_0,y_0)\, dy_0\, dx_0, \end{aligned}$$

and since $\chi_\varepsilon^{(0)} \overset{w}{\rightharpoonup} h^{(0)}$, using (5.1) we obtain

$$\limsup_{\varepsilon\to 0} \tfrac{1}{\varepsilon}\int_H (x^2+y^2)\chi_\varepsilon^{(1)}(x,y)\, dy\, dx \le 4\tau C_1 + 2\int x_0^2 h^{(0)}(x_0)\, dx_0 = C_2 < \infty.$$

In particular

$$\limsup_{\varepsilon\to 0} \int x^2 h_\varepsilon^{(1)}(x)\, dx = \limsup_{\varepsilon\to 0} \tfrac{1}{\varepsilon}\int_H x^2 \chi_\varepsilon^{(1)}(x,y)\, dy\, dx \le C_2,$$

which using also Lemma 3.1(c) implies that

$$\begin{aligned}\int h^{(1)}(x)\,dx &= \lim_{\varepsilon\to 0}\int h_\varepsilon^{(1)}(x)\,dx = 1,\\ \int x^2 h^{(1)}(x)\,dx &\le \liminf_{\varepsilon\to 0}\int x^2 h_\varepsilon^{(1)}(x)\,dx \le C_2.\end{aligned}$$

Thus $h^{(1)}\in K$ and (5.3) holds.

Step 2. Next we show that

$$\begin{aligned}&h^{(1)} \text{ minimizes}\\ &\qquad \tfrac{1}{2\tau}d(h^{(0)},h)^2+E(h)\\ &\text{among all } h:\mathbf{R}\to[0,\infty) \text{ such that } \textstyle\int h=1.\end{aligned}\tag{5.4}$$

Let $h:\mathbf{R}\to[0,\infty)$ such that $\int h = 1$. Since by (5.3) $\chi_\varepsilon^{(1)}\overset{w*}{\rightharpoonup} h^{(1)}$, Proposition 1 (1) and (2) give

$$\tfrac{1}{2\tau}d(h^{(0)},h^{(1)})^2+E(h^{(1)})\le \liminf_{\varepsilon\to 0}\left[\tfrac{1}{2\tau\varepsilon}d(\chi_\varepsilon^{(0)},\chi_\varepsilon^{(1)})^2+E_\varepsilon(\chi_\varepsilon^{(1)})\right]\overset{(5.1)}{\le} C_1,\tag{5.5}$$

so that we can assume without loss of generality

$$\tfrac{1}{2\tau}d(h^{(0)},h)<\infty,\quad E(h)<\infty.\tag{5.6}$$

The first bound in (5.6) implies $h\in K$; the second yields, via Gagliardo-Nirenberg inequality,

$$\int h^3\le\left(\int|h'|^2\right)^{\frac{2}{3}}\left(\int h\right)^{\frac{5}{3}}<\infty.$$

Proposition 1 (3) and (4) can then be applied, obtaining

$$\tfrac{1}{2\tau}d(h^{(0)},h)^2+E(h)=\lim_{\varepsilon\to 0}\left[\tfrac{1}{2\tau\varepsilon}d(\chi_\varepsilon^{(0)},\overline{\chi}_\varepsilon)^2+E_\varepsilon(\overline{\chi}_\varepsilon)\right]\tag{5.7}$$

where $\overline{\chi}_\varepsilon(x,y):=\chi_{(0,\varepsilon h(x))}(y)$. Recalling the minimizing property of $\chi_\varepsilon^{(1)}$ and (5.5), we conclude that

$$\begin{aligned}\tfrac{1}{2\tau}d(h^{(0)},h)^2+E(h) &\ge \liminf_{\varepsilon\to 0}\left[\tfrac{1}{2\tau\varepsilon}d(\chi_\varepsilon^{(0)},\chi_\varepsilon^{(1)})^2+E_\varepsilon(\chi_\varepsilon^{(1)})\right]\\ &\ge \tfrac{1}{2\tau}d(h^{(0)},h^{(1)})+E(h^{(1)})\end{aligned}$$

which proves (5.4).

Step 3. We show that both contributions to the minimal energy converge; that is

$$E(h^{(1)}) = \lim_{\varepsilon\to 0} E_\varepsilon(\chi_\varepsilon^{(1)}), \tag{5.8}$$

$$\tfrac{1}{2\tau} d(h^{(0)}, h^{(1)})^2 = \lim_{\varepsilon\to 0} \tfrac{1}{2\tau\varepsilon} d(\chi_\varepsilon^{(0)}, \chi_\varepsilon^{(1)})^2. \tag{5.9}$$

Applying the same argument of Step 2 with h replaced by $h^{(1)}$ we obtain

$$\tfrac{1}{2\tau} d(h^{(0)}, h^{(1)})^2 + E(h^{(1)}) = \lim_{\varepsilon\to 0} \left[\tfrac{1}{2\tau\varepsilon} d(\chi_\varepsilon^{(0)}, \chi_\varepsilon^{(1)})^2 + E_\varepsilon(\chi_\varepsilon^{(1)})\right].$$

By Proposition 1 (1) and (2)

$$\begin{aligned}\limsup_{\varepsilon\to 0} \tfrac{1}{2\tau\varepsilon} d(\chi_\varepsilon^{(0)}, \chi_\varepsilon^{(1)})^2 &\le \lim_{\varepsilon\to 0} \left[\tfrac{1}{2\tau\varepsilon} d(\chi_\varepsilon^{(0)}, \chi_\varepsilon^{(1)})^2 + E_\varepsilon(\chi_\varepsilon^{(1)})\right] - \liminf_{\varepsilon\to 0} E_\varepsilon(\chi_\varepsilon^{(1)}) \\ &\le \tfrac{1}{2\tau} d(h^{(0)}, h^{(1)})^2 \le \liminf_{\varepsilon\to 0} \tfrac{1}{2\tau\varepsilon} d(\chi_\varepsilon^{(0)}, \chi_\varepsilon^{(1)})^2,\end{aligned}$$

which proves (5.9); by the same argument one proves (5.8).

Step 4. To complete the proof of the Theorem it remains to show that $\chi_\varepsilon^{(1)} \overset{w}{\rightharpoonup} h^{(1)}$ as $\varepsilon \to 0$; i.e.

$$\lim_{\varepsilon\to 0} \tfrac{1}{\varepsilon} \int_H y^2 \chi_\varepsilon^{(1)}(x,y)\, dy\, dx \;=\; 0, \tag{5.10}$$

$$\lim_{\varepsilon\to 0} \tfrac{1}{\varepsilon} \int_H \zeta(x) \chi_\varepsilon^{(1)}(x,y)\, dy\, dx = \int \zeta(x) h^{(1)}(x)\, dx \quad \forall\, \zeta \in C_2^0(\mathbf{R}). \tag{5.11}$$

It follows from (5.9) that

$$\lim_{\varepsilon\to 0} \tfrac{1}{\varepsilon} \int_{\mathbf{R}^4} (|x_0 - x|^2 + |y_0 - y|^2)\, d\overline{\mu}_\varepsilon = d(h^{(0)}, h^{(1)})^2. \tag{5.12}$$

Let μ_ε denote the marginal of $\overline{\mu}_\varepsilon/\varepsilon$ on $\mathbf{R}^2_{x_0,x}$; arguing exactly as in the proof of Proposition 1 (2), it follows that there exists $\mu \in P(h^{(0)}, h^{(1)})$ and a subsequence such that $\mu_\varepsilon \overset{w*}{\rightharpoonup} \mu$, and

$$\begin{aligned}\int_{\mathbf{R}^2} |x_0 - x|^2\, d\mu &\le \liminf_{\varepsilon\to 0} \tfrac{1}{\varepsilon} \int_{\mathbf{R}^4} |x_0 - x|^2\, d\overline{\mu}_\varepsilon \\ &\le \lim_{\varepsilon\to 0} \tfrac{1}{\varepsilon} \int_{\mathbf{R}^4} (|x_0 - x|^2 + |y_0 - y|^2)\, d\overline{\mu}_\varepsilon \overset{(5.12)}{=} d(h^{(0)}, h^{(1)})^2.\end{aligned}$$

Hence μ is an optimal transference plan for $h^{(0)}$, $h^{(1)}$, and

$$\lim_{\varepsilon\to 0} \tfrac{1}{\varepsilon} \int_{\mathbf{R}^4} |x_0 - x|^2\, d\overline{\mu}_\varepsilon \;=\; \int_{\mathbf{R}^2} |x_0 - x|^2\, d\mu \tag{5.13}$$

$$\lim_{\varepsilon\to 0} \tfrac{1}{\varepsilon} \int_{\mathbf{R}^4} |y_0 - y|^2\, d\overline{\mu}_\varepsilon \;=\; 0. \tag{5.14}$$

The limit (5.10) follows immediately from (5.14) and the assumption $\chi_\varepsilon^{(0)} \overset{w}{\rightharpoonup} h^{(0)}$:

$$\begin{aligned}\tfrac{1}{\varepsilon}\int_H y^2\chi_\varepsilon^{(1)}(x,y)\,dy\,dx &= \tfrac{1}{\varepsilon}\int_{\mathbf{R}^4} y^2\,d\overline{\mu}_\varepsilon \le \tfrac{2}{\varepsilon}\int_{\mathbf{R}^4}|y_0-y|^2\,d\overline{\mu}_\varepsilon + \tfrac{2}{\varepsilon}\int_{\mathbf{R}^4} y_0^2\,d\overline{\mu}_\varepsilon \\ &= o_\varepsilon(1) + \tfrac{2}{\varepsilon}\int_H y_0^2\chi_\varepsilon^{(0)}(x_0,y_0)\,dy_0\,dx_0 = o_\varepsilon(1).\end{aligned}$$

To prove (5.11), note that

$$\int_{\mathbf{R}^2}(1+x_0^2)\,d\mu_\varepsilon = \tfrac{1}{\varepsilon}\int_{\mathbf{R}^4}(1+x_0^2)\,d\overline{\mu}_\varepsilon = \tfrac{1}{\varepsilon}\int_H(1+x_0^2)\chi_\varepsilon^{(0)}(x_0,y_0)\,dy_0\,dx_0.$$

Passing to the limit as $\varepsilon\to 0$, $\chi_\varepsilon^{(0)} \overset{w}{\rightharpoonup} h^{(0)}$ and $\mu \in P(h^{(0)},h^{(1)})$ give

$$\lim_{\varepsilon\to 0}\int_{\mathbf{R}^2}(1+x_0^2)\,d\mu_\varepsilon = \int(1+x_0^2)h^{(0)}(x_0)\,dx_0 = \int_{\mathbf{R}^2}(1+x_0^2)\,d\mu.$$

In addition

$$\lim_{\varepsilon\to 0}\int_{\mathbf{R}^2}|x_0-x|^2\,d\mu_\varepsilon \overset{(5.13)}{=} \int_{\mathbf{R}^2}|x_0-x|^2\,d\mu.$$

Therefore, applying Lemma 3.1(b) with $\eta = (1+x_0^2+|x_0-x|^2)$, we obtain for $\zeta\in C_2^0(\mathbf{R})$

$$\lim_{\varepsilon\to 0}\int_{\mathbf{R}^2}\zeta(x)\,d\mu_\varepsilon = \int_{\mathbf{R}^2}\zeta(x)\,d\mu = \int\zeta(x)h^{(1)}(x)\,dx$$

which gives (5.11). We have thus proved (5.10) and (5.11) for a subsequence; since the argument can be performed for any subsequence, Step 4 holds and the proof is complete.

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