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by

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Abstract. We study the Dirichlet problem for the parabolic equation

$$u_t = \Delta u^m, \ m > 0$$

in a bounded, non-cylindrical and non-smooth domain $\Omega \subset I\!\!R^{N+1}, N \geq 2$. Existence and boundary regularity results are established. We introduce a notion of parabolic modulus of left-lower (or left-upper) semicontinuity at the points of the lateral boundary manifold and show that the upper (or lower) Hölder condition on it plays a crucial role for the boundary continuity of the constructed solution. The Hölder exponent $\frac{1}{2}$ is critical as in the classical theory of the one-dimensional heat equation $u_t = u_{xx}$.

Key Words: Dirichlet problem; non-smooth domains; nonlinear diffusion; degenerate and singular parabolic equations; boundary regularity.

1 Introduction

Consider the equation

$$u_t = \Delta u^m, \tag{1.1}$$

where $u = u(x,t), x = (x_1, \ldots, x_N) \in I\!\!R^N, N \geq 2, t \in I\!\!R_+, \Delta = \sum_{i=1}^{N} \partial^2/\partial x_i^2, m > 0$. We study the Dirichlet problem (DP) for equation (1.1) in a bounded domain $\Omega \subset I\!\!R^{N+1}$. It can be stated as follows: given any continuous function on the parabolic boundary $\mathcal{P}\Omega$ of Ω, to find a continuous extension of this function to the closure of Ω which satisfies (1.1) in $\overline{\Omega}\backslash\mathcal{P}\Omega$.

The classical DP for the heat equation ($m = 1$ in (1.1)) is included in our problem. Another direction, this work fits in with, is the modern theory of nonlinear degenerate and singular parabolic equations. If $m > 1$, the equation (1.1) is a well-known porous medium equation, describing the flow of a compressible Newtonian fluid through a porous medium [20], while the singular case ($0 < m < 1$) arises (for example) in plasma physics [8]. A particular motivation for this work arises from the problem about the evolution of interfaces in problems for porous medium equation. Special interest concerns the cases when support of the initial data contains a corner or cusp singularity at some points. What about the movement of these kinds of singularities along the interface? To solve this problem, it is important, at the first stage, to develop general theory of boundary-value problems in non-cylindrical domains with boundary surfaces which has the same kind of behaviour as the interface. In many cases this may be non-smooth and characteristic (see e.g. [1-3]). Another motivation arises from the problem about the formation of singularities in Stefan problem [23].

We make now precise the meaning of solution to DP. Let Ω be bounded open subset of $I\!\!R^{N+1}, N \geq 2$. Let the boundary $\partial\Omega$ of Ω consists of the closure of a domain $B\Omega$ lying on $t = 0$, a domain $D\Omega$ lying on $t = T \in (0, \infty)$ and a (not necessarily connected) manifold $S\Omega$ lying in the strip $0 < t \leq T$. Denote

$$\Omega(\tau) = \{(x,t) \in \Omega : t = \tau\}$$

and assume that $\Omega(t) \neq \emptyset$ for $t \in (0,T)$. The set $\mathcal{P}\Omega = \overline{B\Omega}\bigcup S\Omega$ is called a parabolic boundary of Ω. Furthermore, the class of domains with described structure will be denoted by $\mathcal{D}_{0,T}$.

Let $\Omega \in \mathcal{D}_{0,T}$ is given and ψ is an arbitrary continous nonnegative function defined on $\mathcal{P}\Omega$. DP consists in finding a solution to equation (1.1) in $\Omega \bigcup D\Omega$ satisfying initial-boundary condition

$$u = \psi \quad \text{on } \mathcal{P}\Omega \tag{1.2}$$

Obviously, in general the equation (1.1) degenerates at points (x,t), where $u = 0$ and we cannot expect the considered problem to have classical solution. If $m \neq 1$, we shall follow the following notion of weak solution:

Definition 1.1 We shall say that the function $u(x,t)$ is a solution of DP (1.1), (1.2), if

(a) u is nonnegative and continuous in $\overline{\Omega}$, satisfying (1.2).

(b) for any t_0, t_1 such that $0 < t_0 < t_1 \leq T$ and for any domain $\Omega_1 \in \mathcal{D}_{t_0,t_1}$ such that $\overline{\Omega}_1 \subseteq \Omega \bigcup D\Omega$ and $\partial B\Omega_1, \partial D\Omega_1, S\Omega_1$ being sufficiently smooth manifolds, the following integral identity holds

$$\int\limits_{D\Omega_1} uf\,dx = \int\limits_{B\Omega_1} uf\,dx + \int\limits_{\Omega_1}(uf_t + u^m\Delta f)dxdt - \int\limits_{S\Omega_1} u^m\frac{\partial f}{\partial \nu}dxdt, \tag{1.3}$$

where $f \in C_{x,t}^{2,1}(\overline{\Omega}_1)$ is an arbitrary function that equals to zero on $S\Omega_1$ and ν is the outward-directed normal vector to $\Omega_1(t)$ at $(x,t) \in S\Omega_1$. If $m = 1$, however, the solution is understood in the classical sence.

After Wiener published his famous work [24], where he accomplished the long line investigations on the DP for Laplace equation in general domains, the DP for the heat equation was continously under the interest of many mathematicians in this century. The final result, namely the necessary and sufficient condition which is a quasigeometric characterization for a boundary point of an arbitrary bounded open subset of $I\!R^{N+1}$ to be regular for the heat equation has been established in [11], necessity being established earlier in [16]. Another necessary and sufficient condition had earlier been given in [17].

However, it should be mentioned that Wiener's criterion does not explicitly resolve the natural analytic question, which we impose in this paper for more general nonlinear equation (1.1). Namely, what about the relation between the solvability of the DP or regulartiy of the

boundary points and local modulus of continuity of the boundary manifolds. The importance of this question arises in view of applications which we mentioned earlier. Almost complete answer to this question was given by Petrowsky [21] in the case of one-dimensional linear heat equation $u_t = u_{xx}$. Results concerning one-dimensional reaction-diffusion equation $u_t = a(u^m)_{xx} + bu^\beta, a > 0, m > 0, b \in I\!R, \beta > 0$ were presented in recent papers by the author [4, 5]. Primarily applying the results of [4], a full description of the evolution of interfaces and of the local solution near the interface for all relevant values of parameters is presented in another recent paper [3]. However, the one-dimensional case is a very special case and the results which we present here are new even in the case of the classical heat equation.

The DP for the porous medium equation in cylindrical domain with smooth boundary was investigated in [7,13]. At the moment there is a complete well established theory of the boundary value problems in cylindrical domains for general second order nonlinear degenerate parabolic equations (which includes as a particular case (1.1) and (1.4) below) due to [6,7,9,10,13,22,25 etc.] (see the review article [14]).

The approach used in this paper may be well expressed by the citation from the classical work [24] on the DP for Laplace equation. As it was pointed out by Lebesgue and independently by Wiener "the Dirichlet problem divides itself into two parts, the first of which is the determination of a harmonic function corresponding to certain boundary conditions, while the second is the investigation of the behaviour of this function in the neighbourhood of the boundary". By using an approximation of both Ω and ψ, we also construct a limit solution as a limit of a sequence of classical solutions in regular domains. We then prove a boundary regularity by using barriers and a limiting process.

The main result of this paper on the existence and boundary continuity of the solution to DP is formulated in Theorem 2.1 (see also Corollary 2.1) of the Section 2. The important new element is that we introduce in this paper a notion of parabolic modulus of left-lower (or left-upper) semicontinuity of the lateral boundary manifold at the given point (Definition 2.1, Section 2). Our main assumption (Assumption $\mathcal{A}$ and (2.1), Section 2) consists in upper (or lower) Hölder condition on the parabolic modulus of left-lower (or left-upper) semincontinuity

at each point of the lateral boundary manifold. Moreover, as in the classical theory of one-dimensional heat equation, the critical Hölder exponent is equal to $\frac{1}{2}$. This assumption relates to the parabolic nature of the equation (1.1) and does not depend on m. At this point, it should be mentioned that equation (1.1) has no essential importance for our results, rather it is a suitable model example for three different class of parabolic equations, namely singular ($0 < m < 1$), degenerate ($m > 1$) and uniform ($m = 1$) parabolic equations. For example, by using our techniques the same results may be proved for the following reaction-diffusion-convection equation

$$u_t = a\Delta u^m + b \bigtriangledown \cdot u^\gamma + cu^\beta, \tag{1.4}$$

where $a, m, \gamma, \beta > 0, b, c \in I\!\!R$ (see Remark 3.1, Section 3). We believe that the same result is true for more general second order parabolic equations. However, in this paper we restrict ourselves to equation (1.1), in order to make the presentation of our barrier method for proving the boundary regularity less technical. It should also be mentioned that since our main result on the boundary regularity of a weak solution to equation (1.1) is of the local nature, similar result is true for an arbitrary bounded domain $\Omega \subset I\!\!R^N$.

It should also be mentioned that in this paper we restrict ourselves only with the existence and boundary regularity problems. We address issues regarding uniqueness of the constructed solution and related comparison theorems in a subsequent paper. The organisation of the paper is as follows: In Section 2 we outline the main result. In Section 3 we prove the main result (Theorem 2.1) from Section 2.

2   Statement of the Main Result

We shall use the usual notation:

$z = (x, t) = (x_1, \ldots x_N, t) \in I\!\!R^{N+1}, N \geq 2, x = (x_1, \overline{x}) = (x_1, x_2, \ldots, x_N) \in I\!\!R^N, \overline{x} = (x_2, \ldots, x_N) \in I\!\!R^{N-1}, |x|^2 = \sum_{i=1}^{N} |x_i|^2, |\overline{x}|^2 = \sum_{i=2}^{N} |x_i|^2$. For a point $z = (x, t) \in I\!\!R^{N+1}$ we denote by $Q(z; \delta)$ an open ball in $I\!\!R^{N+1}$ of radius $\delta > 0$ and with center in z.

Let $\Omega \in \mathcal{D}_{0,T}$ be a given domain. Assume that for arbitrary point $z_0 = (x^0, t_0) \in S\Omega$ (or $z_0 = (x^0, 0) \in \overline{S\Omega}$) there exists $\delta > 0$ and a conitnuous function ϕ such that, after a suitable

rotation of x-axes, we have

$$\overline{S\Omega} \bigcap Q(z_0, \delta) = \{z \in Q(z_0, \delta) : x_1 = \phi(\overline{x}, t)\}.$$

Suppose also that

$$\text{sign } (x_1 - \phi(\overline{x}, t)) = \text{ const for } z \in Q(z_0, \delta) \bigcap \Omega$$

Furthermore, we denote this constant by $d(z_0)$. Obviously, by introducing a new variable $x_1' = -x_1$, if necessary, we could have supposed that $d(z_0) = 1$. However, we describe the conditions for both cases $d(z_0) = \pm 1$ seperately, in order to distinguish these boundary points, which are similar to the left and right boundary points in the one-dimensional case.

Let $z_0 = (x^0, t_0) \in S\Omega$ be a given boundary point. For an arbitrary sufficiently small $\delta > 0$, consider a parabolic domain

$$P(\delta) = \{(\overline{x}, t) : |\overline{x} - \overline{x}^0| < \varepsilon_0(\delta + t - t_0)^{\frac{1}{2}}, t_0 - \delta < t < t_0\},$$

where $\varepsilon_0 > 0$ is an arbitrary fixed number.

Definition 2.1 Let

$$\omega^-(\delta) = max(\phi(\overline{x}^0, t_0) - \phi(\overline{x}, t) : (\overline{x}, t) \in \overline{P(\delta)}).$$

$$\omega^+(\delta) = min(\phi(\overline{x}^0, t_0) - \phi(\overline{x}, t) : (\overline{x}, t) \in \overline{P(\delta)}).$$

The function $\omega^-(\delta)$ (respectively $\omega^+(\delta)$) is called the parabolic modulus of left-lower (respectively left-upper) semicontinuity of the function ϕ at the point $(\overline{x}^0, t_0)$.

For suffuciently small $\delta > 0$ these functions are well-defined and converge to zero as $\delta \downarrow 0$. Our main assumption on the behaviour of the function ϕ near z_0 is as follows:

Assumption $\mathcal{A}$. There exists a function $F(\delta)$ which is defined for all positive sufficiently small δ; F is positive with $F(\delta) \downarrow 0$ as $\delta \downarrow 0$ and if $d(z_0) = 1$ (respectively $d(z_0) = -1$) then

$$\omega^-(\delta) \leq \delta^{\frac{1}{2}} F(\delta) \qquad (2.1)$$

(respectively $\omega^+(\delta) \geq -\delta^{\frac{1}{2}} F(\delta)$)

We prove in the next section that assumption $\mathcal{A}$ is sufficient for the regularity of the boundary point z_0. Namely, the constructed limit solution takes the boundary value $\psi(z_0)$ at the point $z = z_0$ continuously in $\overline{\Omega}$. It is well-known that in the case of the classical heat equation ($m = 1$ in (1.1)) boundary point $z_0 = (x^0, 0) \in \overline{S\Omega}$ is always regular (see e.g. [18, p. 172]). Hence, in this case the assumption $\mathcal{A}$ imposed on every boundary point $z_0 \in S\Omega$ is sufficient for solvability of the DP (see Corollary 2.1 below). It may easily be proved that the solution in this particular case is a unique classical solution.

However, in general to provide the regularity of the boundary point $z_0 = (x^0, 0) \in \overline{S\Omega}$ we need another assumption. Denote $x_1 = \overline{\phi}(\overline{x}) \equiv \phi(\overline{x}, 0)$.

Definition 2.2 Let

$$\omega_0^-(\delta) = max(\overline{\phi}(\overline{x}^0) - \overline{\phi}(\overline{x}) : |\overline{x} - \overline{x}^0| \leq \delta)$$

$$\omega_0^+(\delta) = min(\overline{\phi}(\overline{x}^0) - \overline{\phi}(\overline{x}) : |\overline{x} - \overline{x}^0| \leq \delta)$$

The function $\omega_0^-(\delta)$ (respectively $\omega_0^+(\delta)$) is called the modulus of lower (respectively upper) semicontinuity of the function $x_1 = \overline{\phi}(\overline{x})$ at the point $\overline{x} = \overline{x}^0$.

Assumption $\mathcal{B}$. There exists a function $F_1(\delta)$ which is defined for all positive sufficiently small δ; F_1 is positive with $F_1(\delta) \downarrow 0$ as $\delta \downarrow 0$ and if $d(z_0) = 1$ (respectively $d(z_0) = -1$) then

$$\omega_0^-(\delta) \leq \delta F_1(\delta) \tag{2.2}$$

(respectively $\omega_0^+(\delta) \geq -\delta F_1(\delta)$)

It may easily be verified that if we redefine $\overline{\phi}$ as $x_1 = \overline{\phi}(\overline{x}) \equiv \phi(\overline{x}, t_0)$ then assumption $\mathcal{B}$ is a consequence of the assumption $\mathcal{A}$ at the boundary point $z_0 = (x^0, t_0) \in S\Omega$. However, assumption $\mathcal{B}$ has a sense for the boundary points $z_0 = (x^0, 0) \in \overline{S\Omega}$ on the bottom of the lateral boundary manifold.

We prove in the next section that assumption $\mathcal{B}$ is sufficient for the regularity of the boundary point $z_0 = (x^0, 0) \in \overline{S\Omega}$. Namely, the constructed limit solution takes the boundary value $\psi(z_0)$ at the point $z = z_0$ continuously in $\overline{\Omega}$.

Thus our main theorem reads:

Theorem 2.1 DP (1.1), (1.2) is solvable in a domain Ω which satisfies the assumption $\mathcal{A}$ at every point $z_0 \in S\Omega$ and assumption $\mathcal{B}$ at every point $z_0 = (x^0, 0) \in \overline{S\Omega}$.

Corollary 2.1 There exists a unique classical solution to DP (1.1), (1.2) with $m = 1$, in a domain Ω which satisfies the assumption $\mathcal{A}$ at every point $z_0 \in S\Omega$.

It should be noted that our main result about the boundary regularity is of local nature and, consequently, an existence of different function $F(\delta)$ (or $F_1(\delta)$) for each boundary point in respective assumption $\mathcal{A}$ (or $\mathcal{B}$) is allowed. It may be easily observed that assumptions $\mathcal{A}$ and $\mathcal{B}$ coincide in the case of cylindrical domain Ω.

3 Proof of the Main Result

Step 1. Construction of the limit solution.

Consider a sequence of domains $\Omega_n \in \mathcal{D}_{0,T}, n = 1, 2, \ldots$ with $S\Omega_n, \partial B\Omega_n$ and $\partial D\Omega_n$ being sufficiently smooth manifolds. Assume that $\{S\Omega_n\}, \{\partial B\Omega_n\}$ and $\{\partial D\Omega_n\}$ approximate $S\Omega, \partial B\Omega$ and $\partial D\Omega$ respectively. Assume also that for arbitrary compact subset $\Omega^{(0)}$ of $\Omega \bigcup D\Omega$, there exists a number n_0 which depends on the distance between $\Omega^{(0)}$ and $\mathcal{P}\Omega$, such that $\Omega^{(0)} \subseteq \Omega_n \bigcup D\Omega_n$ for $n \geq n_0$. Let Ψ be a nonnegative and continuous function in $I\!\!R^{N+1}$ which coincides with ψ on $\mathcal{P}\Omega$. This continuation is always possible. Next we take $\psi_n = \Psi + n^{-1}, n = 1, 2, \ldots$ and consider a Dirichlet Problem (1.1), (1.2), in Ω_n, with ψ replaced by ψ_n. This is a nondegenerate parabolic problem and classical theory ([12,15,19]) imply the existence of a unique $C_{2+\alpha}$ solution. From maximum principle it follows that

$$n^{-1} \leq u_n(x, t) \leq M \text{ in } \overline{\Omega}_n, n = 1, 2, \ldots \tag{3.1}$$

where M is an upper bound for Ψ and $\psi_n, n = 1, 2, \ldots$ in some compact which contains $\overline{\Omega}$ and $\overline{\Omega}_n, n = 1, 2, \ldots$. Next we take a sequence of compact subsets $\Omega^{(k)}$ of $\Omega \bigcup D\Omega$ such that

$$\Omega = \bigcup_{k=1}^{\infty} \Omega^{(k)}, \Omega^{(k)} \subseteq \Omega^{(k+1)}, k = 1, 2, \ldots \tag{3.2}$$

By our construction, for each fixed k, there exists a number n_k such that $\Omega^{(k)} \subseteq \Omega_n \bigcup D\Omega_n$ for $n \geq n_k$. It is a well-known result of the modern theory of degenerate parabolic equations

(which includes (1.1) as a model example) that the sequence of uniformly bounded solutions $u_n, n \geq n_k$ to equation (1) is uniformly equicontinuous in a fixed compact $\Omega^{(k)}$ (see e.g. [9, Theorem 1], [10, Theorem 1 & Proposition 1]). From (3.2), by diagonalization argument and the Arzela-Ascoli theorem, we may find a subsequence n' and a limit function $\tilde{u} \in C(\Omega \bigcup D\Omega)$ such that $u_{n'} \to \tilde{u}$ as $n' \to +\infty$, pointwise in $\Omega \bigcup D\Omega$ and the convergence is uniform on compact subsets of $\Omega \bigcup D\Omega$. Now consider a function $u(x, t)$ such that $u(x, t) = \tilde{u}(x, t)$ for $(x, t) \in \Omega \bigcup D\Omega, u(x, t) = \psi$ for $(x, t) \in \mathcal{P}\Omega$. Obviously the function u satisfies the integral identity (1.3). It is also continuous in $B\Omega$, since above mentioned result on the equicontinuity of the sequence u_n is true up to some neighbourhood of every point $z \in B\Omega$ [10, Theorem 6.1]. Hence, the constructed function u is a solution of the Dirichlet Problem (1.1), (1.2), if it is continuous in $\mathcal{P}\Omega \backslash B\Omega$.

Step 2. Boundary regularity. Let $z_0 = (x^0, t_0) \in S\Omega$. We shall prove that z_0 is regular, namely that

$$\lim u(z) = \psi(z_0) \text{ as } z \to z_0, z \in \Omega \bigcup D\Omega \tag{3.3}$$

Without loss of generality assume that $d(z_0) = 1$. First, assume that $t_0 = T$. If $0 < \psi(z_0) < M$, we shall prove that for arbitrary sufficiently small $\varepsilon > 0$ the following two inequalities are valid

$$\liminf u(z) \geq \psi(z_0) - \varepsilon \text{ as } z \to z_0, z \in \Omega \bigcup D\Omega \tag{3.4}$$

$$\limsup u(z) \leq \psi(z_0) + \varepsilon \text{ as } z \to z_0, z \in \Omega \bigcup D\Omega \tag{3.5}$$

Since $\varepsilon > 0$ is arbitrary, from (3.4) and (3.5), (3.3) follows. If $\psi(z_0) = 0$ (or respectively $\psi(z_0) = M$), however, then it is sufficient to prove (3.5) (respectively (3.4)), since (3.4) (respectively (3.5)) follows directly from the fact that $0 \leq u \leq M$ in $\overline{\Omega}$. Let $\psi(z_0) > 0$. Take an arbitrary $\varepsilon \in (0, \psi(z_0))$ and prove (3.4). For arbitrary $\mu > 0$ consider a parabolic domain $P(\mu^{-2})$. We assume that $S\Omega$ in some neighbourhood of z_0 is represented by the function $x_1 = \phi(\overline{x}, t), (\overline{x}, t) \in P(\mu_0^{-2})$ with some $\mu_0 > 0$, where ϕ satisfies assumption $\mathcal{A}$ from Section 2. We also assume that $S\Omega_n$ in some neighbourhood of its point $z_n = (x_1^{(n)}, \overline{x}^0, T)$ is represented by the function $x_1 = \phi_n(\overline{x}, t), (\overline{x}, t) \in P(\mu_0^{-2})$, where $\{\phi_n\}$ is a sequence of sufficiently smooth

functions and $\phi_n \to \phi$ as $n \to +\infty$, uniformly in $P(\mu_0^{-2})$. We can also asssume that ϕ_n satisfies assumption $\mathcal{A}$ from Section 2 uniformly with respect to n. Namely, the parabolic modulus of left-lower semicointinuity of the function ϕ_n at the point $(\overline{x}^0, T)$ satisfies (2.1) uniformly with respect to n.

For arbitrary $\mu > 0$, consider a function

$$w_n(x, t) = f(\xi) \equiv M_1(\xi/h(\mu))^\alpha,$$

where

$$\xi = h(\mu) + \phi_n(\overline{x}^0, T) - x_1 - \mu[T - t + \varepsilon_0^{-2}|\overline{x} - \overline{x}^0|^2],$$

$$M_1 = \psi(z_0) - \varepsilon, \ h(\mu) = M_3\mu^{-1}F(\mu^{-2}),$$

$$M_3 = [(M_2/M_1)^{\frac{1}{\alpha}} - 1]^{-1}, M_2 = \psi(z_0) - \varepsilon/2,$$

and α is an arbitrary number such that $\alpha > m^{-1}$. If $m > 1$, then we assume also that $\alpha \leq (m - 1)^{-1}$. Then we set

$$V_n = \{(x, t) : \phi_n(\overline{x}, t) < x_1 < \phi_{1n}(\overline{x}, t), (\overline{x}, t) \in P(\mu^{-2})\},$$

$$\phi_{1n}(\overline{x}, t) = \phi_n(\overline{x}, t) + (1 + M_3)\mu^{-1}F(\mu^{-2}) - \mu[T - t + \varepsilon_0^{-2}|\overline{x} - \overline{x}^0|^2],$$

In the next lemma we clear the structure of V_n.

Lemma 3.1 If $\mu > 0$ is chosen such that $F(\mu^{-2}) \leq (1 + M_3)^{-1}$ then the parabolic boundary of V_n consists of two boundary surfaces $x_1 = \phi_n(\overline{x}, t)$ and $x_1 = \phi_{1n}(\overline{x}, t)$ (see Figure 1).

Proof. We have

$$\phi_{1n}(\overline{x}, t) - \phi_n(\overline{x}, t) = \mu[\delta_* + t - T - \varepsilon_0^{-2}|\overline{x} - \overline{x}^0|^2], \ \delta_* = (1 + M_3)\mu^{-2}F(\mu^{-2})$$

and $\delta_* \in (0, \mu^{-2}]$ if μ is chosen as in Lemma 3.1. Then it easily follows that $V_n = V_n^*$, where

$$V_n^* = \{(x, t) : \phi_n(\overline{x}, t) < x_1 < \phi_{1n}(\overline{x}, t), (\overline{x}, t) \in P(\delta_*)\}.$$

Obviously, the assertion of lemma is true for V_*. Lemma is proved.

In Figure 1 the domain V_n is described in the particular case when $\phi_n(\overline{x}, t) \equiv 0$, $N = 2$, $x_2^0 = 0$.
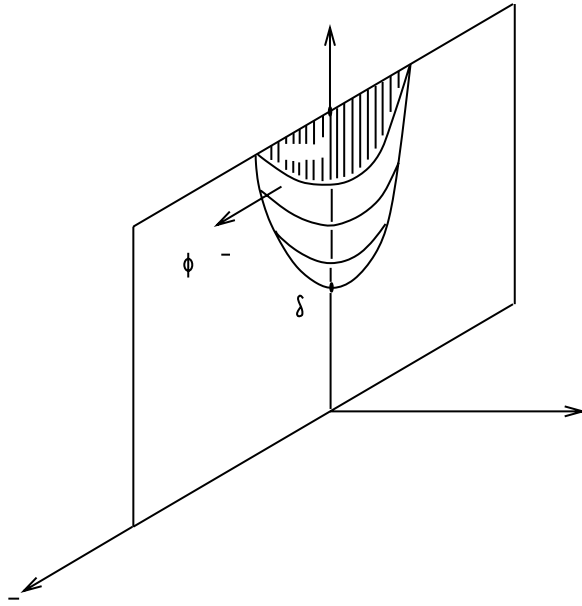
Figure 1. The domain V_n in a particular case when $\phi_n = 0, N = 2, x_2^0 = 0$.

In general, the structure of the domain V_n coincides with that given in Figure 1 if we change the variable x_1 with the new one $x_1' = x_1 - \phi_n(\overline{x}, t)$. More precisely, V_n is a domain in $I\!\!R^{N+1}$, lying in the strip $T - \delta_* < t < T$, its boundary consists of the single point lying on $\{t = T - \delta_*\}$, a domain DV_n lying on $\{t = T\}$ and a connected manifold SV_n lying in the strip $\{T - \delta_* < t \le T\}$. Boundary manifold SV_n consists of two boundary surfaces $x_1 = \phi_n(\overline{x}, t)$ and $x_1 = \phi_{1n}(\overline{x}, t)$.

Our purpose is to estimate u_n in V_n via the barrier function

$$\tilde{w}_n = \ max \ (w_n; (2n)^{-1}).$$

Obviously,

$$\tilde{w}_n = (2n)^{-1} \text{ for } x_1 \ge \theta_n(\overline{x}, t); \tilde{w}_n = w_n \text{ for } x_1 < \theta_n(\overline{x}, t),$$

where

$$\theta_n(\overline{x}, t) = (1 - (2M_1 n)^{-\frac{1}{\alpha}})h(\mu) + \phi_n(\overline{x}^0, T) - \mu[T - t + \varepsilon_0^{-2}|\overline{x} - \overline{x}^0|^2].$$

In the next lemma we estimate u_n via the barrier function $\tilde{w}_n$ on the parabolic boundary of V_n. For that the special structure of V_n plays an important role. Namely, our barrier function

takes the value $(2n)^{-1}$, which is less than a minimal value of u_n, on the part of the parabolic boundary of V_n which lies in Ω_n. Hence it is enough to compare u_n and $\tilde{w}_n$ on the part of the lateral boundary of Ω_n, which may easily be done in view of boundary condition for u_n.

Lemma 3.2 If $\mu > 0$ is chosen large enough, then

$$u_n > \tilde{w}_n \text{ on } \overline{SV_n}. \tag{3.6}$$

Proof. From (2.1) it follows that for $\mu > 0$ being large enough

$$\theta_n(\overline{x}, t) - \phi_{1n}(\overline{x}, t) < 0 \text{ for } (\overline{x}, t) \in \overline{P(\mu^{-2})},$$

and hence

$$\tilde{w}_n = (2n)^{-1} \text{ for } x_1 = \phi_{1n}(\overline{x}, t), (\overline{x}, t) \in \overline{P(\mu^{-2})}. \tag{3.7}$$

Without loss of generality, assume that $n > M_1^{-1}$. From (2.1) it also follows that

$$w_n = f(h(\mu) + \phi_n(\overline{x}^0, T) - \phi_n(\overline{x}, t) - \mu[T - t + \varepsilon_0^{-2}|\overline{x} - \overline{x}^0|^2]) \leq$$

$$f((M_3^{-1} + 1)h(\mu)) = M_2 \text{ for } x_1 = \phi_n(\overline{x}, t), (\overline{x}, t) \in \overline{P(\mu^{-2})},$$

and hence

$$\tilde{w}_n \leq M_2 \text{ for } x_1 = \phi_n(\overline{x}, t), (\overline{x}, t) \in \overline{P(\mu^{-2})} \tag{3.8}$$

We can also easily estimate u_n on SV_n. To estimate $u_n|_{x_1 = \phi_n(\overline{x}, t)}$, first we choose $n_1 = n_1(\varepsilon)$ so large that for $n \geq n_1$

$$\Psi|_{x_1 = \phi_n(\overline{x}, t)} > \Psi|_{x_1 = \phi(\overline{x}, t)} - \frac{\varepsilon}{8} \text{ for } (\overline{x}, t) \in \overline{P(\mu_0^{-2})}.$$

This is possible in view of uniform convergence of $\{\phi_n\}$ to ϕ in $\overline{P(\mu_0^{-2})}$. Then we choose $\mu > 0$ large enough in order that

$$\Psi|_{x_1 = \phi(\overline{x}, t)} > \psi(z_0) - \frac{\varepsilon}{8} \text{ for } (\overline{x}, t) \in \overline{P(\mu^{-2})}.$$

If μ and n are chosen like this, then we have

$$u_n|_{x_1 = \phi_n(\overline{x}, t)} > \psi(z_0) - \frac{\varepsilon}{4} \text{ for } (\overline{x}, t) \in \overline{P(\mu^{-2})}. \tag{3.9}$$

Thus, from (3.1) and (3.7) - (3.9), (3.6) follows. Lemma is proved.

Lemma 3.3 If $\mu > 0$ is chosen large enough, then at the points of V_n with $x_1 < \theta_n(\overline{x}, t)$, we have

$$Lw_n \equiv w_{n_t} - \Delta w_n^m < 0 \tag{3.10}$$

Proof At the points of V_n with $x_1 < \theta_n(\overline{x}, t)$, we have

$$Lw_n = \mu h^{-1}(\mu)\alpha M_1^{\frac{1}{\alpha}} f^{\frac{\alpha-1}{\alpha}} - h^{-2}(\mu)\alpha m(\alpha m - 1)M_1^{\frac{2}{\alpha}} f^{\frac{\alpha m-2}{\alpha}} \times$$

$$(1 + 4\mu^2\varepsilon_0^{-4}|\overline{x} - \overline{x}^0|^2) + 2\mu h^{-1}(\mu)\alpha m\varepsilon_0^{-2}(N-1)M_1^{\frac{1}{\alpha}} f^{\frac{\alpha m-1}{\alpha}}. \tag{3.11}$$

If $m > 1$ then from (3.11) and (3.8) it follows that

$$Lw_n \le h^{-2}(\mu)\alpha M_1^{\frac{1}{\alpha}} f^{\frac{\alpha-1}{\alpha}} S, \tag{3.12}$$

$$S = M_3 F(\mu^{-2}) - m(\alpha m - 1)M_1^{\frac{1}{\alpha}} f^{\frac{\alpha(m-1)-1}{\alpha}} + 2M_3 m\varepsilon_0^{-2}(N-1)F(\mu^{-2})\times$$

$$f^{m-1} \le M_3 F(\mu^{-2}) - m(\alpha m - 1)M_1^{\frac{1}{\alpha}} M_2^{m-1-\frac{1}{\alpha}} + 2M_3 m\varepsilon_0^{-2}(N-1)M_2^{m-1}F(\mu^{-2}).$$

Hence, if μ is chosen large enough, from (3.12), (3.10) follows. If $0 < m \le 1$, then from (3.11) and (3.8) we derive that

$$Lw_n \le h^{-2}(\mu)\alpha M_1^{\frac{1}{\alpha}} f^{\frac{\alpha m-1}{\alpha}} S, \tag{3.13}$$

$$S = M_3 F(\mu^{-2})f^{1-m} - m(\alpha m - 1)M_1^{\frac{1}{\alpha}} f^{-\frac{1}{\alpha}} + 2M_3 m\varepsilon_0^{-2}(N-1)\times$$

$$F(\mu^{-2}) \le M_3 F(\mu^{-2})M_2^{1-m} - m(\alpha m - 1)M_1^{\frac{1}{\alpha}} M_2^{-\frac{1}{\alpha}} + 2M_3 m\varepsilon_0^{-2}(N-1)F(\mu^{-2}).$$

If μ is chosen large enough, from (3.13), (3.10) again follows. Lemma is proved.

Thus $\tilde{w}_n$ is the maximum of two smooth subsolutions of the equation (1.1) in V_n. By the standard maximum principle, from Lemma 3.1, (3.6) and (3.10) we easily derive that

$$u_n \ge \tilde{w}_n \text{ in } \overline{V}_n$$

In the limit as $n' \to +\infty$, we have

$$u \ge \tilde{w} \text{ in } \overline{V}, \tag{3.14}$$

where

$$\tilde{w} = max(w; 0), \text{ in } \overline{V}$$

$$w(x,t) = f(h(\mu) + \phi(\overline{x}^0, T) - x_1 - \mu[T - t + \varepsilon_0^{-2}|\overline{x} - \overline{x}^0|^2]),$$

$$V = \{(x,t) : \phi(\overline{x}, t) < x_1 < \phi_1(\overline{x}, t), |\overline{x} - \overline{x}^0| < \varepsilon_0[\delta_* + t - T]^{\frac{1}{2}}, T - \delta_* < t < T\},$$

$$\phi_1(\overline{x}, t) = \phi(\overline{x}, t) + (1 + M_3)\mu^{-1}F(\mu^{-2}) - \mu[T - t + \varepsilon_0^{-2}|\overline{x} - \overline{x}^0|^2].$$

Obviously, we have

$$\lim_{z \to z_0, z \in \overline{V}} \tilde{w} = \lim_{z \to z_0, z \in \overline{\Omega}} \tilde{w} = \psi(z_0) - \varepsilon$$

Hence, from (3.14), (3.4) follows.

Assume now that $0 \leq \psi(z_0) < M$ and prove (3.5) for an arbitrary $\varepsilon > 0$ such that $\psi(z_0) + \varepsilon < M$. For arbitrary $\mu > 0$ consider a function

$$w_n(x,t) = f_1(\xi) \equiv [M^{\frac{1}{\alpha}} + \xi h^{-1}(\mu)(M_4^{\frac{1}{\alpha}} - M^{\frac{1}{\alpha}})]^\alpha,$$

where ξ is defined as before and

$$h(\mu) = M_6\mu^{-1}F(\mu^{-2}), M_4 = \psi(z_0) + \varepsilon,$$

$$M_5 = \psi(z_0) + \varepsilon/2, M_6 = (M^{\frac{1}{\alpha}} - M_4^{\frac{1}{\alpha}})(M_4^{\frac{1}{\alpha}} - M_5^{\frac{1}{\alpha}})^{-1},$$

and α is an arbitrary number such that $0 < \alpha < min\ (1; m^{-1})$. Similarly, consider the domains V_n (with M_3 replaced by M_6 in the expression of $\phi_{1n}(\overline{x}, t)$ and δ_*) and V_n^* (see Lemma 3.1). We then construct an upper barrier function as follows:

$$\tilde{w}_n = \ min\ (w_n; M).$$

Obviously,

$$\tilde{w}_n = M \text{ for } x_1 \geq \theta_n(\overline{x}, t); \tilde{w}_n = w_n \text{ for } x_1 < \theta_n(\overline{x}, t).$$

where

$$\theta_n(\overline{x}, t) = h(\mu) + \phi_n(\overline{x}^0, T) - \mu[T - t + \varepsilon_0^{-2}|\overline{x} - \overline{x}^0|^2].$$

Next, we prove an analog of the Lemma 3.2.

Lemma 3.4 If $\mu > 0$ is chosen large enough, then

$$u_n \leq \tilde{w}_n \text{ on } \overline{SV}_n. \tag{3.15}$$

Proof. From (2.1) it follows that for $\mu > 0$ being large enough

$$\theta_n(\overline{x}, t) - \phi_{1n}(\overline{x}, t) \leq 0 \text{ for } (\overline{x}, t) \in \overline{P(\mu^{-2})},$$

and hence

$$\tilde{w}_n = M \text{ for } x_1 = \phi_{1n}(\overline{x}, t), (\overline{x}, t) \in \overline{P(\mu^{-2})}. \tag{3.16}$$

From (2.1) it also follows that

$$w_n = f_1(h(\mu) + \phi_n(\overline{x}^0, T) - \phi_n(\overline{x}, t) - \mu[T - t + \varepsilon_0^{-2}|\overline{x} - \overline{x}^0|^2])$$

$$\geq f_1((M_6^{-1} + 1)h(\mu)) = M_5 \text{ for } x_1 = \phi_n(\overline{x}, t), (\overline{x}, t) \in \overline{P(\mu^{-2})},$$

and hence

$$\tilde{w}_n \geq M_5 \text{ for } x_1 = \phi_n(\overline{x}, t)), (\overline{x}, t) \in \overline{P(\mu^{-2})}. \tag{3.17}$$

Similarly, as in (3.9), we can establish that if $\mu > 0$ is large enough and $n \geq n_1(\varepsilon)$ then

$$u_n|_{x_1 = \phi_n(\overline{x}, t)} < \psi(z_0) + \frac{\varepsilon}{4} \text{ for } (\overline{x}, t) \in \overline{P(\mu^{-2})}. \tag{3.18}$$

Thus, from (3.1) and (3.16) - (3.18), (3.15) follows. Lemma is proved.

The next lemma is analog of the Lemma 3.3.

Lemma 3.5 If $\mu > 0$ is chosen large enough, then at the points of V_n with $x_1 < \theta_n(\overline{x}, t)$, we have

$$Lw_n > 0. \tag{3.19}$$

Proof. By using (3.17), at the points of V_n with $x_1 < \theta_n(\overline{x}, t)$, we have

$$Lw_n = -\mu h^{-1}(\mu)\alpha(M^{\frac{1}{\alpha}} - M_4^{\frac{1}{\alpha}})f_1^{\frac{\alpha-1}{\alpha}} + m\alpha(1 - \alpha m)h^{-2}(\mu) \times$$

$$(M^{\frac{1}{\alpha}} - M_4^{\frac{1}{\alpha}})^2 f_1^{\frac{\alpha m-2}{\alpha}}(1 + 4\mu^2\varepsilon_0^{-4}|\overline{x} - \overline{x}^0|^2) - 2\mu h^{-1}(\mu)\alpha m\varepsilon_0^{-2} \times$$

$$(N - 1)(M^{\frac{1}{\alpha}} - M_4^{\frac{1}{\alpha}})f_1^{\frac{\alpha m-1}{\alpha}} \geq h^{-2}(\mu)\alpha(M^{\frac{1}{\alpha}} - M_4^{\frac{1}{\alpha}})S, \tag{3.20}$$

$$S = -M_6 M_5^{\frac{\alpha-1}{\alpha}} F(\mu^{-2}) + m(1-\alpha m)(M^{\frac{1}{\alpha}} - M_4^{\frac{1}{\alpha}})M^{\frac{\alpha m-2}{\alpha}} - 2M_6 m \varepsilon_0^{-2}(N-1)M_5^{\frac{\alpha m-1}{\alpha}} F(\mu^{-2}).$$

Hence, if μ is chosen large enough, from (3.20), (3.19) follows. Lemma is proved.

Thus $\tilde{w}_n$ is the minimum of two smooth supersolutions of the equation (1) in V_n. By the standard maximum principle, from Lemma 3.1, (3.15) and (3.19) we easily derive that

$$u_n \le \tilde{w}_n \text{ in } \overline{V}_n.$$

In the limit as $n' \to \infty$, we have

$$u \le \tilde{w} \text{ in } \overline{V}, \tag{3.21}$$

where

$$\tilde{w} = min(w; M) \text{ in } \overline{V}$$

$$w(x,t) = f_1(h(\mu) + \phi(\overline{x}^0, T) - x_1 - \mu[T - t + \varepsilon_0^{-2}|\overline{x} - \overline{x}^0|^2])$$

and the domain V being defined as in (3.14). Obviously, we have

$$\lim_{z \to z_0, z \in \overline{V}} \tilde{w} = \lim_{z \to z_0, z \in \overline{\Omega}} \tilde{w} = \psi(z_0) + \varepsilon$$

Hence, from (3.21), (3.5) follows. Thus we have proved (3.3) for $z_0 = (x^0, T) \in S\Omega$ when $d(z_0) = 1$. The proof is similar when $d(z_0) = -1$.

Suppose now that $z_0 = (x^0, t_0) \in S\Omega$ with $t_0 < T$. Clearly, the same proof given in the case $t_0 = T$, implies the regularity of z_0 regarding subdomain $\Omega_- = \Omega \bigcap \{t < t_0\}$. Namely, (3.3) is valid for $z \in \overline{\Omega}_-$. Hence, it is enough to prove (3.3) for $z \in \Omega_+, \Omega_+ = \Omega \bigcap \{t > t_0\}$. The proof of this latter, however, is equivalent to the proof of regularity of the point $z_0 = (x^0, 0) \in \overline{S\Omega}$ under the assumption $\mathcal{B}$. That easily follows from the fact that assumption $\mathcal{B}$ (with redefined $\overline{\phi}(\overline{x}) \equiv \phi(\overline{x}, t_0)$) is a consequence of the assumption $\mathcal{A}$. Thus, to complete the proof, it remains just to prove (3.3) for $z_0 = (x^0, 0) \in \overline{S\Omega}$.

The proof is similar to that given above. Without loss of generality assume again that $d(z_0) = 1$. Let $\psi(z_0) > 0$. Take an arbitrary $\varepsilon \in (0, \psi(z_0))$ and prove (3.4). For arbitrary $\mu > 0$ consider a cylinder

$$R(\mu, \delta) = \{(\overline{x}, t) : |\overline{x} - \overline{x}^0| < \mu^{-1}, 0 < t < \delta\}.$$

We assume that $\overline{S\Omega}$ in some neighbourhood of z_0 is represented by the continuous function $x_1 = \phi(\overline{x}, t), (\overline{x}, t) \in \overline{R(\mu_0, \delta_0)}$ with some $\mu_0 > 0, \delta_0 > 0$, where $\overline{\phi}(\overline{x})$ satisfies assumption $\mathcal{B}$ (see (2.2)) from Section 2. We also assume that $\overline{S\Omega_n}$ in some neighbourhood of its point $A_n = (x_1^{(n)}, \overline{x}^0, 0)$ is represented by the function $x_1 = \phi_n(\overline{x}, t), (\overline{x}, t) \in \overline{R(\mu_0, \delta_0)}$, where $\{\phi_n\}$ is a sequence of sufficiently smooth functions and $\phi_n \to \phi$ as $n \to \infty$, uniformly in $\overline{R(\mu_0, \delta_0)}$. We can also assume that ϕ_n satisfies (2.2) uniformly with respect to n.

For arbitrary $\mu > \mu_0$ consider a function

$$w_n(x, t) = f(\xi) \equiv M_1(\xi/h(\mu))^\alpha,$$

where

$$\xi = h(\mu) + \phi_n(\overline{x}^0, 0) - x_1 + \mu[t - |\overline{x} - \overline{x}^0|^2],$$

$$M_1 = \psi(z_0) - \varepsilon, h(\mu) = M_3 \mu^{-1} F_1(\mu^{-1}),$$

$$M_2 = \psi(z_0) - \varepsilon/2, M_3 = 4[(M_2/M_1)^{\frac{1}{\alpha}} - 1]^{-1}$$

and α is an arbitrary number such that $\alpha > m^{-1}$. If $m > 1$, then we assume also that $\alpha \le (m-1)^{-1}$. Then we set

$$V_n = \{(x, t) : \phi_n(\overline{x}, t) < x_1 < \phi_{1n}(\overline{x}, t), (\overline{x}, t) \in R(\mu, \delta)\}$$

$$\phi_{1n}(\overline{x}, t) = \phi_n(\overline{x}^0, 0) + (1 - (2M_1 n)^{-\frac{1}{\alpha}})h(\mu) + \mu[t - |\overline{x} - \overline{x}^0|^2],$$

where $\delta = \delta(\mu) \in (0, \delta'], \delta' = min(\delta_1, \delta_2), \delta_1 = 2\mu^{-2} F_1(\mu^{-1})$ and $\delta_2 = \delta_2(\mu) \in (0, \delta_0]$ is chosen such that

$$\phi_n(\overline{x}, 0) - \phi_n(\overline{x}, t) \le \mu^{-1} F_1(\mu^{-1}) \text{ for } (\overline{x}, t) \in \overline{R(\mu_0, \delta_2)} \tag{3.22}$$

and for $n \ge n_1(\mu)$. The existence of δ_2 and n_1 follow from the following proposition.

Proposition 3.1 For arbitrary $\mu > \mu_0$ there exists $\delta_2 = \delta_2(\mu) \in (0, \delta_0]$ and $n_1 = n_1(\mu)$ such that (3.22) is valid for $n \ge n_1$.

Proof. Since $\{\phi_n\}$ converges to ϕ uniformly in $R(\mu_0, \delta_0)$, for arbitrary $\mu > \mu_0$ there exists a number $n_1 = n_1(\mu)$ such that for $n \geq n_1$, we have

$$\phi_n(\overline{x}, 0) - \phi_n(\overline{x}, t) \leq \phi(\overline{x}, 0) - \phi(\overline{x}, t) + \frac{1}{2}\mu^{-1}F_1(\mu^{-1}) \text{ in } \overline{R(\mu_0, \delta_o)} \tag{3.23}$$

Since ϕ is uniformly continuous in $\overline{R(\mu_0, \delta_0)}$, there also exists a number $\delta_2 = \delta_2(\mu) \in (0, \delta_0]$ such that

$$\phi(\overline{x}, 0) - \phi(\overline{x}, t) \leq \frac{1}{2}\mu^{-1}F_1(\mu^{-1}) \text{ in } \overline{R(\mu_0, \delta_2)} \tag{3.24}$$

From (3.23) and (3.24), (3.22) follows. Proposition is proved.

Furthermore we shall always suppose that $n \geq \ max \ (n_1; M_1^{-1})$. If $\mu > \mu_0$ is chosen large enough, from (2.2) and (3.22) it follows that

$$\phi_{1n}(\overline{x}, t) - \phi_n(\overline{x}, t) < \phi_n(\overline{x}^0, 0) - \phi_n(\overline{x}, 0) + \phi_n(\overline{x}, 0) - \phi_n(\overline{x}, t) + h(\mu) + \mu\delta_1 - \mu^{-1}$$

$$\leq \mu^{-1}[(M_3 + 4)F_1(\mu^{-1}) - 1] < 0 \text{ for } |\overline{x} - \overline{x}^0| = \mu^{-1}, 0 \leq t \leq \delta'.$$

Thus, the parabolic boundary of V_n consists of two boundary surfaces $x_1 = \phi_n(\overline{x}, t)$, $x_1 = \phi_{1n}(\overline{x}, t)$, and of the closure of a domain

$$V_n^0 = \{(x, 0) : \phi_n(\overline{x}, 0) < x_1 < \phi_{1n}(\overline{x}, 0), |\overline{x} - \overline{x}^0| < \mu^{-1}\}.$$

In the next lemma, which is analog of the Lemma 3.2, we estimate u_n via the barrier function w_n on the parabolic boundary $\mathcal{P}V_n$ of V_n.

Lemma 3.6 If $\mu > 0$ is chosen large enough, then

$$u_n > w_n \text{ on } \overline{\mathcal{P}V}_n. \tag{3.25}$$

Proof. We have

$$w_n = (2n)^{-1} \text{ for } x_1 = \phi_{1n}(\overline{x}, t). \tag{3.26}$$

From (2.2) and (3.22) it also follows that if μ is chosen large enough, then

$$w_n = f(h(\mu) + \phi_n(\overline{x}^0, 0) - \phi_n(\overline{x}, t) + \mu t - \mu|\overline{x} - \overline{x}^0|^2) \leq f(h(\mu) + \phi_n(\overline{x}^0, 0) - \phi_n(\overline{x}, 0) + \phi_n(\overline{x}, 0) -$$

$$\phi_n(\overline{x}, t) + \mu\delta_1) \leq f((4M_3^{-1} + 1)h(\mu)) = M_2 \text{ for } x_1 = \phi_n(\overline{x}, t), (\overline{x}, t) \in \overline{R(\mu, \delta)}. \tag{3.27}$$

From (3.27) it also follows that

$$\omega_n = f(h(\mu) + \phi_n(\overline{x}^0, 0) - x_1 - \mu|\overline{x} - \overline{x}^0|^2) \leq f(h(\mu) + \phi_n(\overline{x}^0, 0) - \phi_n(\overline{x}, 0)) \leq M_2 \text{ in} \overline{V}_n^0 \quad (3.28)$$

We can also easily estimate u_n on $\mathcal{P}V_n$. To estimate $u_n|_{x_1 = \phi_n(\overline{x}, t)}$, first we choose $n_2 = n_2(\varepsilon)$ so large that for $n \geq n_2$

$$\Psi|_{x_1 = \phi_n(\overline{x}, t)} > \Psi|_{x_1 = \phi(\overline{x}, t)} - \frac{\varepsilon}{8} \text{ for } (\overline{x}, t) \in \overline{R(\mu_0, \delta_0)}.$$

This is possible in view of uniform convergence of $\{\phi_n\}$ to ϕ in $\overline{R(\mu_0, \delta_0)}$. Then we choose $\mu > 0$ large enough and $\delta = \delta(\mu) > 0$ small enough in order that

$$\Psi|_{x_1 = \phi(\overline{x}, t)} > \psi(z_0) - \frac{\varepsilon}{8} \text{ for } (\overline{x}, t) \in \overline{R(\mu, \delta)}$$

and hence,

$$u_n|_{x_1 = \phi_n(\overline{x}, t)} > \psi(z_0) - \frac{\varepsilon}{4} \text{ for } (\overline{x}, t) \in \overline{R(\mu, \delta)} \quad (3.29)$$

Similarly, we can establish that if $\mu > 0$ is chosen large enough, there exists a number $n_2(\varepsilon)$ such that for $n \geq n_2$ we have

$$u_n > \psi(z_0) - \frac{\varepsilon}{4} \text{ in } \overline{V}_n^0 \quad (3.30)$$

Thus, from (3.1) and (3.26) - (3.30), (3.25) follows. The lemma is proved.

The next step consists in proving that for $\mu > 0$ being large enough

$$Lw_n < 0 \text{ in } V_n.$$

The proof coincides with that given above in Lemma 3.3. As before, by the standard maximum principle we then easily derive that

$$u_n \geq w_n \text{ in } \overline{V}_n.$$

In the limit as $n^{'} \to \infty$, we have

$$u \geq w \text{ in } \overline{V}, \quad (3.31)$$

where

$$w(x, t) = f(h(\mu) + \phi(\overline{x}^0, 0) - x_1 + \mu t - \mu|\overline{x} - \overline{x}^0|^2),$$

$$V = \{(x,t) : \phi(\overline{x}, t) < x_1 < \phi_1(\overline{x}, t), (\overline{x}, t) \in R(\mu, \delta)\},$$

$$\phi_1(\overline{x}, t) = \phi(\overline{x}^0, 0) + h(\mu) + \mu[t - |\overline{x} - \overline{x}^0|^2].$$

Obviously, we have

$$\lim_{z \to z_0, \, z \in \overline{V}} w = \lim_{z \to z_0, \, z \in \overline{\Omega}} w = \psi(z_0) - \varepsilon$$

Hence, from (3.31), (3.4) follows. To complete the proof it remains to prove (3.5) when $0 \leq \psi(z_0) < M$. To do that, we consider a barrier function

$$w_n(x, t) = f_1(\xi) \equiv [M^{\frac{1}{\alpha}} + \xi h^{-1}(\mu)(M_4^{\frac{1}{\alpha}} - M^{\frac{1}{\alpha}})]^\alpha,$$

where ξ is defined as before and

$$h(\mu) = M_6 \mu^{-1} F_1(\mu^{-1}), M_4 = \psi(z_0) + \varepsilon,$$

$$M_5 = \psi(z_0) + \varepsilon/2, M_6 = 4(M^{\frac{1}{\alpha}} - M_4^{\frac{1}{\alpha}})(M_4^{\frac{1}{\alpha}} - M_5^{\frac{1}{\alpha}})^{-1}$$

and α is an arbitrary number such that $0 < \alpha < min(1; m^{-1})$. The rest of the proof of (3.5) is similar to the given proof of (3.4) and to that given above in the case when $t_0 = T$, therefore we omit it. Thus we have completed the proof of the boundary continuity of the constructed limit solution. The Theorem 2.1 is proved. Corollary 2.1 is immediate (see Section 2).

Remark 3.1 The proof of the Theorem 2.1 in the case of the more general equation (1.4) almost completely coincides with that given above for the equation (1.1). The only difference consists in choosing an exponent α in respective barrier functions w_n. In this more general case it depends on the parameters m, β and γ. It should be also mentioned that if $c > 0$ and $\beta > 1$ in (1.4), we have to prevent blow up, say, imposing a restriction on the length of the time intervall: $T \in (0, T^*), T^* = M^{1-\beta}/c(\beta - 1), M = \sup \psi > 0$.

Remark 3.2 One may show by standard methods that the weak solution to DP is a classical solution in a neighbourhood of any interior point $z \in \Omega$, where $u(z) > 0$.

Acknowledgements

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