

Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

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systems

by

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Preprint no.: 77

2000



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Abstract The present paper contains an interpretation and generalization of Novikov's theory for Morse type inequalities for closed 1-forms in terms of concepts from Conley's theory for dynamical systems. We introduce the concept of a flow carrying a cocycle α , (generalized) α -flow for short, where α is a cocycle in bounded Alexander-Spanier cohomology theory. Gradient-like flows can then be characterized as flows carrying a trivial cocycle. We also define α -Morse-Smale flows that allow the existence of "cycles" in contrast to the usual Morse-Smale flows. α -flows without fixed points carry not only a cocycle, but a cohomology class, in the sense of [8], and we shall deduce a vanishing theorem for generalized Novikov numbers in that situation. By passing to a suitable cover of the underlying compact polyhedron adapted to the cocycle α , we construct a so-called π -Morse decomposition for an α -flow. On this basis, we can use the Conley index to derive generalized Novikov-Morse inequalities, extending those of M. Farber [13]. In particular, these inequalities include both the classical Morse type inequalities (corresponding to the case when α is a coboundary) as well as the Novikov type inequalities (when α is a nontrivial cocycle).

1 Introduction

Let X be an m -dimensional compact manifold and let f be a Morse function on X . Define $S(f)$ to be the union of the sets $S_i(f)$ which contain the

critical points of f with index i . Define

$$\begin{aligned} c_i &:= \#S_i(f) \\ b_i &:= \dim H_i(X, \mathbb{R}) \end{aligned}$$

Then the classical Morse inequalities are

$$\begin{aligned} c_i &\geq b_i \\ \sum_{j=0}^i (-1)^{i-j} c_j &\geq \sum_{j=0}^i (-1)^{i-j} b_j \end{aligned} \tag{1.1}$$

for $i = 0, 1, \dots, m$. The Morse inequalities provide an important connection between the analytic information and the topological information of X (see [17]). The method to obtain this result is to consider the change of the homotopy type of the level set

$$X^a = \{x \in X \mid f(x) \leq a\}$$

when a crosses the critical value. If c is a critical value, then there is the homotopy decomposition

$$X^{a+\varepsilon} \simeq X^{a-\varepsilon} \cup_{p_j \in f^{-1}(c) \cap S(f)} e(p_j)$$

where $e(p_j)$ is a j -cell corresponding to the critical point p_j of index j . Hence, X has a CW-decomposition

$$X = \cup_{p_j \in S(f)} e(p_j)$$

However, this decomposition only provides information about the dimension of the basis. It is not clear how each cell is attached. The successive work by R. Thom, S. Smale and J. W. Milnor in the 50's and 60's lead to the construction of the Morse complex which includes also the boundary operators.

Let us describe the Morse complex simply. Consider the negative gradient flow v of f :

$$\dot{v}(t) = -\nabla f(v(t)).$$

Then for each critical point $p \in S(f)$, there are the stable and unstable manifolds

$$\begin{aligned} W^u(p) &= \{x \in X \mid x \cdot t \rightarrow p, \text{ as } t \rightarrow -\infty\} \\ W^s(p) &= \{x \in X \mid x \cdot t \rightarrow p, \text{ as } t \rightarrow \infty\} \end{aligned}$$

R. Thom gave a disjoint decomposition of X by unstable manifolds

$$X = \cup_{p \in S(f)} W^u(p)$$

In [26], S. Smale obtained an important result for the gradient flow on a compact manifold. The result is that generically $W^u(q)$ intersects transversally with $W^s(p)$ for each point pair $(p, q) \in S(f) \times S(f)$. Therefore we can choose a generic Riemannian metric such that for each critical point pair $(q, p) \in S_i \times S_{i-1}$, $W^u(q) \cap$

$W^s(p)$ is 1-dimensional and is transversal to any level set $f^{-1}(a)$ between them. This condition for the flow is called the Morse-Smale condition.

Since $f^{-1}(a)$ is compact, $W^u(q) \cap W^s(p) \cap f^{-1}(a)$ is a finite set. Assign arbitrary orientations to the unstable manifolds of all the critical points. Now we can assign a number $\varepsilon(\gamma)$ to each trajectory $\gamma \in W^u(q) \cap W^s(p)$ which takes its value in $\{\pm 1\}$. Let $x \in \gamma \cap f^{-1}(a)$. Fix the direction $\dot{\gamma}(t)$, then $W_x^u(q)$, $W_x^s(p)$ and $T_x X$ will give the induced orientation to $T_x(f^{-1}(a) \cap W^u(q))$, $T_x(f^{-1}(a) \cap W^s(p))$ and $T_x(f^{-1}(a))$. If the orientation of $(T_x(f^{-1}(a) \cap W^u(q)), T_x(f^{-1}(a) \cap W^s(p)))$ is the same as the one of $T_x(f^{-1}(a))$, then we let $\varepsilon(\gamma) = 1$, otherwise, let $\varepsilon(\gamma) = -1$.

Define the incidence coefficient $n(q, p)$ to be

$$n(q, p) = \sum_{\gamma \in W^u(q) \cap W^s(p)} \varepsilon(\gamma).$$

Now we can define the boundary operator $\partial_i : S_i(f) \rightarrow S_{i-1}(f)$ as follows. For each critical point $q \in S_i(f)$,

$$\partial_i(q) = \sum_{p \in S_{i-1}(f)} n(q, p) \cdot p$$

It can be proved that $\partial_{i-1} \partial_i = 0$. Therefore $(S_i(f), \partial_i)$ becomes a complex. This is the Morse complex. The important thing is that the homology groups of the Morse complex are isomorphic to the standard homology groups of the underlying compact manifold. The reader can consult [24] for more information about the Morse homology theory.

In fact, the Morse complex was first used in [30] to prove the Morse inequalities (1.1). The Witten deformation technique used there provides an analytic way to prove the Morse type inequalities (see [4], [3], etc.), even the Novikov inequalities introduced below (see [6]).

Let us also remark that Morse theory has been generalized to the infinite dimensional case. In that situation, one needs to impose a compactness condition on the functional, the so-called Palais-Smale condition. See e. g. the books by K. C. Chang [7] and M. Struwe [29].

In the 80's, S. P. Novikov considered in [18] and [19] Morse type inequalities which are used to count the zeros of a closed Morse 1-form ω on X . By the Poincaré lemma, a closed 1-form is locally exact. For instance, we can assume that $\omega = df_p$ near a zero point p of ω . So closed Morse 1-form means that each zero point p of ω is a non-degenerate critical point of f_p and has a Morse index from f_p . These inequalities are similar to (1.1) formally. The difference is that the ordinary Betti number b_i at the right hand side of (1.1) is replaced by the Novikov number $b_i([\omega])$ which is determined by X and the cohomology class $[\omega]$. These Morse type inequalities involving Novikov numbers are called Novikov inequalities.

Novikov inequalities are obtained by constructing the Novikov complex (C_*, ∂)

with respect to the closed Morse 1-form ω . We can give a simple description of the Novikov complex for rank $\omega = 1$. In this case, ω can be lifted to the covering space \bar{X} with the deck transformation group being the infinite cyclic group $\mathbb{Z} = \langle t \rangle$ such that the pull-back form by the projection map is an exact form $d\bar{f}$ where \bar{f} is a Morse function on \bar{X} . Introduce the Novikov ring $\mathbb{Z}((t)) := \mathbb{Z}[[t]][t^{-1}]$. Now the module C_i is defined as the free $\mathbb{Z}((t))$ -module generated by $S_i(\omega)$, i.e., the set of all the zeros of ω with index i . As for the Morse complex, we can consider the flow generated by the dual vector field of ω satisfying the Morse-Smale condition and choose an orientation for each unstable manifold for each zero in $S(\omega)$. Consequently, we are able to define the boundary operator of the Novikov complex. Let $(q, p) \in S_i(\omega) \times S_{i-1}(\omega)$ and choose any lifting points \bar{q} and \bar{p} . Let $T_k(\bar{q}, \bar{p})$ be the finite set of all the trajectories joining \bar{q} and $\bar{p} \cdot t^k$. Define

$$n_k(q, p) = \sum_{\gamma \in T_k(\bar{q}, \bar{p})} \varepsilon(\gamma)$$

$$n(q, p) = \sum_k n_k(q, p) \cdot t^k$$

Note that $n(q, p) \in \mathbb{Z}((t))$. $n(q, p)$ is called the Novikov incidence coefficient.

Finally we can define the boundary operator $\partial_i : S_i(\omega) \rightarrow S_{i-1}(\omega)$ to be

$$\partial_i(q) = \sum_{p \in S_{i-1}(\omega)} n(q, p)p$$

There are some discussions about the properties of the Novikov incidence coefficients, for instance, the rationality and the growth properties of $n_k(q, p)$ (see [21], [22] and the references there).

The above construction of the Novikov complex stems from the analysis of the gradient flow generated by \bar{f} in the noncompact manifold \bar{X} . However, we can define the Novikov complex by triangulating the underlying manifold X and then studying the action of the boundary operators in the \mathbb{Z} -CW complex in the covering space \bar{X} . In more detail, we can let $C_*(X)$ be the simplicial complex of X . After lifting this triangulation to the covering space, we get $C_*(\bar{X})$ which can be viewed as a free \mathbb{Z} -module chain complex. Tensor $C_*\bar{X}$ with the Novikov ring $\mathbb{Z}((t))$, to get a complex $\mathbb{Z}((t)) \otimes_{\mathbb{Z}[t, t^{-1}]} C_*(\bar{X})$, because the boundary operator in $C_*(\bar{X})$ can be viewed as a matrix operator with each entry being a Laurent polynomial. This is also called the Novikov complex. It was proved in [23] that there is a closed Morse 1-form ω such that the second kind Novikov complex can be realized by the first Novikov complex given by ω . The construction of the Novikov complex with respect to a Morse closed 1-form implies the Novikov inequalities.

Since the Novikov ring is a principle ideal domain (see [16]), hence the homology module of the Novikov complex can be decomposed into two parts, the

free module part and the torsion part. The Novikov numbers are referred to be the ranks of the homology modules, and the number of the minimal generators of the torsion part.

The Novikov numbers can also be obtained from another chain complex with coefficient ring different from the Novikov ring. One way is to consider the ring of rational functions: $\mathcal{R} := (1 + t\mathbb{Z}[t])^{-1}\mathbb{Z}[t, t^{-1}]$. In [11], M. Farber has proved that the homology of $\mathcal{R} \otimes_{\mathbb{Z}[t, t^{-1}]} C_*(\bar{X})$ is isomorphic to $H_*((C_*, \partial))$. We will use this kind of definition of the Novikov numbers in this paper.

Recently, M. Farber ([14]) has used a noncommutative localization method to construct a universal complex, which can induce many kinds of Novikov complexes.

The Morse complex provides a way to construct the underlying manifold by attaching handles, and we know not only how many and which kinds of handles should be attached but also how to attach handles. Actually, we can get more refined Morse inequalities than (1.1) that contain the information from the torsion part of the homology group which is determined by the boundary operators. The construction of the Morse complex is based on the analysis of the gradient flow of the Morse function. From the point of view of dynamical systems, the information contained in the right hand side of (1.1) is nothing but the local information (Morse index) of the fixed points of the gradient flow of the Morse function f . Fixed points are only special invariant sets in a dynamical system. If we are given an arbitrary flow in a compact manifold and its invariant sets, can we get Morse-type inequalities that contain the local information of the invariant sets at the right hand side and the topological information of the underlying manifold at the left hand side? Of course, since a general flow is very complicated, it is very difficult to do so. However, C. C. Conley successfully generalized the concepts of nondegenerate critical points and the corresponding Morse index to the isolated invariant sets and the Conley index (see [9]). The isolated invariant sets contain many interesting invariant sets, e.g., the Bott type nondegenerate critical manifolds of the gradient flow of a smooth functions (see [4][5]), and the hyperbolic periodic orbits, or more general the hyperbolic sets. The Conley index characterizes the “local” topological information of the isolated invariant sets. C. C. Conley and E. Zehnder also obtained a generalization of (1.1) (see [10]). They consider the flow restricted to an isolated invariant set S and assume that S has a Morse decomposition $\{M_n, \dots, M_1\}$. It can be proved that each Morse set M_i is isolated and hence has a Conley index. Now the Morse type inequalities provide a connection between the Conley index of M_i and the Conley index of S .

However, in order to prove the Morse type inequalities, the flow must have a “global” gradient-like structure with respect to all the invariant sets in X . “Global” here means that if we collapse all the invariant sets into points, then the quotient flow is gradient-like. Only with a “global” gradient-like flow, we can get the Morse decomposition.

Now we return to the Novikov inequalities. We can construct a closed Morse

1-form such that its flow has no attractor and no repeller (see [19]). Therefore the flow generated by this 1-form is not a gradient-like flow in X though its lifting flow on \bar{X} is. Therefore, it is not possible to give a Morse decomposition for the whole manifold X .

After an introduction to the Morse theory for a gradient-like flow and the Novikov theory for a Morse closed 1-form, we reach the goal in the present paper. In this paper, we will introduce the concept, “flow carrying a cocycle α (α -flow and generalized α -flow)”, where α is a 1-dimensional cocycle in the bounded Alexander-Spanier cohomology which will be defined in section 3.1. When α is a coboundary, then this flow is a gradient-like flow. Conversely, if a flow is a gradient-like flow, then this flow carries a coboundary δg where g is a Lyapunov function. Moreover, in theorems 3.4.2 and 3.4.3, we give a sufficient and necessary condition for a flow to be a non gradient-flow if α is a nontrivial cocycle. The notion of α -flow contains many interesting flows. The gradient flow of a Morse function f is a df -flow. The flow generated by a closed 1-form ω is an ω -flow and the perturbation flows are also ω -flows. If an α -flow has no fixed point, then this flow is a flow “carrying a cohomology class” which is introduced in [8]. A generalized α -flow has a “global” α -flow structure with respect to the π -stable nonwandering sets. The π -stable nonwandering sets are those sets that when lifted to the covering space related to α , are also bounded sets. An example of a generalized α -flow is the α -Morse-Smale flow. In the definition of the Morse-Smale flow, one requires that such a flow has no “cycles” connecting the different hyperbolic fixed points or hyperbolic periodic orbits. The α -Morse-Smale flow is a Morse-Smale flow when lifted to the covering space with respect to the cocycle α . However, in the underlying manifold, it is possible that there exist “cycles” connecting different nonwandering sets or there exist $(\mathcal{U}(s), T)$ -chain cycles for arbitrary small $\varepsilon > 0$ connecting different nonwandering sets. The existence of such “cycles” or such chain cycles makes the flow a nongradient-like flow by theorem 3.4.2.

A flow carrying a cocycle has a π -Morse decomposition (Theorem 3.5.2). Hence there are the Novikov-Morse type inequalities, Theorem 4.5.1, if $\mathcal{I}\mathcal{N}_\pi = \mathcal{N}_\pi$. This theorem uses the Conley index to get the “local” topological information of the isolated invariant sets.

The inequalities (4.5.1) are a generalization of the Novikov inequalities for a closed Morse 1-form. Starting from these inequalities, we can recover many Novikov type inequalities found before. For example, we can consider the ω -flow generated by a smooth 1-form ω having Bott type nondegenerate zero sets, then (4.5.1) induces (4.6.2), which was essentially given in [13]. If we consider an α -Morse Smale flow, then we have Theorem 4.6.6 which provides new Novikov inequalities. Also, the existence of a flow carrying a cohomology class α implies the vanishing of the Novikov numbers $b_i([\omega])$. In addition, if we consider the trivial cocycle α , then Theorem 4.5.1 provides the Morse type inequalities, because all the representations involving α are trivial.

Since the isolated invariant sets are more complicated than a hyperbolic fixed point, hence it is not clear what the connecting orbits in the π -Morse decomposition are. The appendix gives another proof of the classical Novikov inequalities starting from the equality (4.5.1). These Novikov inequalities also contain the information about the torsion parts of the homology groups of the Novikov complex.

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Acknowledgement

References

Appendix

2 Basic results about flows

2.1 Basic concepts and structures of flows

Let X be a topological space. The continuous function from $X \times \mathbb{R} \rightarrow X$ given by $(x, t) \rightarrow x \cdot t$ is called a flow on X if for all $x \in X$, and $s, t \in \mathbb{R}$ it satisfies (1) $x \cdot 0 = x$ and (2) $(x \cdot s) \cdot t = x \cdot (s + t)$. For $Y \subset X$ and $\mathbb{R}' \subset \mathbb{R}$, let $Y \cdot \mathbb{R}'$ denote the set of points $x \cdot t$ such that $y \in Y$ and $t \in \mathbb{R}'$.

A subset $I \subset X$ is called invariant if $I \cdot \mathbb{R} = I$. If $N \subset X$ is a subset, we denote by $I(N)$ the invariant set in N : $I(N) = \{x \in N | x \cdot \mathbb{R} \subset N\}$. Clearly, $I(N)$ is invariant and it is closed if N is closed.

For $Y \subset X$, the ω -limit set $\omega(Y)$ of Y is defined to be the maximal invariant set in the closure of $Y \cdot [0, \infty]$. If $\overline{Y \cdot [0, \infty]}$ denotes the closure in X , then $\omega(Y) = I(\overline{Y \cdot [0, \infty]})$. Similarly, the ω^* -limit set $\omega^*(Y)$ of Y is defined to be the

maximal invariant set in the closure of $Y \cdot [0, -\infty]$.

The ω -limit set $\omega(Y)$ has some properties:

A. $\omega(Y)$ is a closed set.

B. $\omega(Y) = \cap \{\overline{Y \cdot [t, \infty)} \mid t \geq 0\}$.

C. Let S be a closed compact Hausdorff invariant set in X . If $Y \subset S$, then $\omega(Y)$ is a nonempty compact invariant set in S and is connected if Y is.

$\omega^*(Y)$ has analogous properties, since $\omega^*(Y)$ is the ω -limit set of the inverse flow. The above conclusions can be easily obtained from the definition of the ω -limit set.

Let S be an invariant set in X . The nonwandering set $N_\omega(S)$ consists of the points x in S with the property that if U is a neighborhood of x in S , then $x \in \omega(U)$. The nonwandering set is invariant and closed in S . Since the definition of $N_\omega(S)$ depends on the invariant set S , it is not generally true that $N_\omega(S) = N_\omega(N_\omega(S))$. For $x \in S, \omega(x) \subset N_\omega(S)$.

Attractor-repeller pairs

Let S be a compact Hausdorff invariant set in X . A subset $A \subset S$ is called an attractor (relative to S) if there is a neighborhood U of A in S such that $\omega(U) = A$. Similarly a set which is the ω^* -limit set of one of its neighborhoods is called a repeller. Let A be an attractor in S , and let $A^* := \{x \in S \mid \omega(x) \notin A\}$. Then A^* is a repeller in S which is called the complementary repeller in S . (A, A^*) is called an attractor-repeller pair in S . $C(A, A^*) = S \setminus A \cup A^*$ is called the set of connecting orbits to the pair, since the point $x \in C(A, A^*)$ if and only if $\omega(x) \in A$ and $\omega^*(x) \in A^*$.

There are two criteria to judge if a set is an attractor.

(1) Suppose $U \subset S$ and, for some $t > 0$, $(\overline{U}) \cdot t \subset \text{int}U$. Then $\omega(U)$ is an attractor contained in the interior of U .

(2) Let U be a compact set in S . If all its boundary points leave S in backward time, then the maximal invariant set in U is an attractor.

If (A, A^*) is an attractor-repeller pair in S and S' is an invariant subset in S , then $(A \cap S', A^* \cap S')$ is an attractor-repeller pair in S' .

Chain recurrent set

Let \mathcal{U} be a cover of S and let $x, x' \in S$. A (\mathcal{U}, t) -chain from x to x' is a sequence $\{x = x_1, \dots, x_{n+1} = x' \mid t_1, \dots, t_n\}$ such that $t_i \geq t$ and each pair $(x_i \cdot t_i, x_{i+1})$ ($i = 1, \dots, n$) belongs to the same open set in \mathcal{U} .

Let $Q(S) \subset S \times S$ be the set of pairs (x, x') where for any covering \mathcal{U} and any $t > 0$ there is a (\mathcal{U}, t) -chain from x to x' .

For $Y \subset S$, define

$$\begin{aligned}\Omega(Y) &:= \{x' \mid \text{there is } x \in Y \text{ s.t. } (x, x') \in Q(S)\} \\ \Omega^*(Y) &:= \{x' \mid \text{there is } x \in Y \text{ s.t. } (x', x) \in Q(S)\}\end{aligned}$$

$Q(S)$ is a closed, transitive relation on S , and if $(x, x') \in Q(S)$, then $(x \cdot I, x' \cdot I) \in Q(S)$, where I is any interval in the real line. For example, we prove the

last conclusion.

Proof Let \mathcal{U} be any open covering of S and take any positive t_0, t and t' . Suppose that $(x, x') \in Q(S)$, we want to prove that $(x \cdot t, x' \cdot t') \in Q(S)$.

Let $\bar{t} = \max\{t_0 + |t|, t_0 + |t'|\}$, and $\{x = x_1, \dots, x_{n+1} = x' | t_1, \dots, t_n\}$ be a (\mathcal{U}, \bar{t}) -chain from x to x' , then $\{x \cdot t = x_1, \dots, x_n, x_{n+1} = x' \cdot t' | t_1 - t, \dots, t_{n-1}, t_n + t'\}$ is a (\mathcal{U}, t_0) -chain from $x \cdot t$ to $x' \cdot t'$. \blacksquare

$\Omega(Y)$ and $\Omega^*(Y)$ have the following two properties:

(1) If Y is closed, $\Omega(Y)$ and $\Omega^*(Y)$ are closed invariant sets containing $\omega(Y)$ and $\omega^*(Y)$, resp.

(2) If Y is closed, $\Omega(Y)$ is the intersection of the attractors in S which contain $\omega(Y)$; similarly, $\Omega^*(Y)$ is the intersection of the repellers containing $\omega^*(Y)$.

The chain recurrent set $R(S)$ is defined to be the set of x such that $(x, x) \in Q(S)$, or equivalently $x \in \Omega(x)$. The relation of chain recurrent sets with the attractor-repeller pairs and the nonwandering sets are as follows:

- (1) $R(S) = \bigcap \{A \cap A^* | A \text{ is an attractor in } S\}$
- (2) $R(S) \supset N_\omega(S)$

Definition A invariant set S is called chain recurrent if $R(S) = S$. A flow in X is called a chain recurrent flow if its chain recurrent set is the whole space.

Morse decomposition

Let S be a compact Hausdorff invariant set in X . The intersection of an attractor and a repeller in S is called a Morse set (rel S).

Let $\emptyset = A_{n+1} \subset A_n \subset \dots \subset A_0 = S$ be a decreasing sequence of attractors in S and let $M_j = A_j \cap A_{j+1}^* (j = 1, 2, \dots, n)$. Then the ordered set $D = \{M_n, M_{n-1}, \dots, M_1\}$ is called a Morse decomposition of S . Clearly, the Morse sets in D are disjoint and for any $x \in S$, either x is in a Morse set, or there is i, j satisfying $i < j$ such that $\omega(x) \subset M_j$ and $\omega^*(x) \subset M_i$. This means that each point in a connection orbit of S goes from a Morse set with low index to a Morse set with high index.

The Morse decomposition is not unique. For example, let $1 \leq i < j \leq n$, consider the decreasing sequence of attractors: $\emptyset = A_{n+1} \subset A_n \subset \dots \subset A_0 = S$, then $D_{ji} = \{M_0, \dots, M_j, M_{ji}, M_{i-1}, \dots, M_1\}$ is also a decomposition of S . Here $M_{ji} = A_i \cap A_j^*$.

Conversely, let $\{M_n, \dots, M_1\}$ be any collection of disjoint invariant sets in S . If for any $x \in S$, there is $i < j$ such that $\omega^*(x) \subset M_i$ and $\omega(x) \subset M_j$, then it can be proved that $D = \{M_n, \dots, M_1\}$ is a Morse decomposition.

Let $M(D) = M_n \cup \dots \cup M_1$. Then for any decomposition, we have $M(D) = \bigcap_{i=1}^n \{A_i \cup A_i^*\}$. Hence $M(D) \supset R(S)$.

Suppose that the nonwandering set $N_\omega(S)$ has finitely many components, then these can be ordered and become a Morse decomposition with $M(D) = N_\omega(S)$.

But generally $M(D) \supset R(S) \supset N_\omega(S)$, hence in this case $M(D) = R(S) = N_\omega(S)$.

Gradient-like flows and chain recurrent flows

A flow is called gradient-like, if there is a continuous real valued function which is strictly decreasing on non constant solutions. Such a function is called a Lyapunov function. The concept of a gradient-like flow is a generalization of one of a gradient flow, i.e., a flow of the form:

$$\dot{v} = \text{grad}f(v)$$

where f is a C^1 -function defined on X . Clearly, $-f$ decreases along the nonconstant solutions as t increases.

A gradient-like flow need not be a gradient flow.

A flow is called a strongly gradient-like flow if its chain recurrent set is totally disconnected. A gradient-like flow with finitely many fixed points is a strongly gradient-like flow.

We have the basic structure theorem of flows (see [9]).

Theorem 2.1 *Every flow on a compact space is uniquely represented as the extension of a chain recurrent flow by a strongly gradient-like flow.*

This theorem means that if we collapse every connected component of the chain recurrent set to one point, then we get a quotient flow on the collapsed space which is a strongly gradient-like flow. The flow when restricted to every component of the chain recurrent set becomes a chain recurrent subflow.

2.2 Local structure of gradient-like flows

This section will provide a detailed description of the local structure near a fixed point of a gradient-like flow. The conclusion obtained here will be applied in sections 3.3 and 3.4.

Let v be a gradient-like flow on a compact metric space X having n fixed points p_i , $i = 1, \dots, n$. Then there exists an associated Lyapunov function $g(x)$ on X such that for a small $r > 0$, the sets $g(B_r(p_i))$, $i = 1, \dots, n$ do not intersect mutually.

Let p be a fixed point of the flow v , and let $B_{\frac{r}{2}}(p)$ be a closed ball centered at p with radius $\frac{r}{2}$. For $0 < s < \frac{r}{2}$, we define two sets on the sphere $\partial B_r(p)$,

$$\begin{aligned} B_{r,s}^+(p) &= \{x \in \partial B_r(p) \mid [x, x \cdot t] \cap \partial B_s(p) \neq \emptyset \text{ for } t > 0\} \\ B_{r,s}^-(p) &= \{x \in \partial B_r(p) \mid [x, x \cdot t] \cap \partial B_s(p) \neq \emptyset \text{ for } t < 0\} \end{aligned}$$

For any point $x \in B_{r,s}^+(p)$, let t_x denote the arrival time of the trajectory $[x, x \cdot t]$ ($t > 0$) to the sphere $\partial B_{\frac{r}{2}}(p)$. We have the following lemma.

Lemma 2.2.1 (1) $B_{r,s}^\pm(p)$ are closed sets on $\partial B_r(p)$.
(2) t_x is a lower semicontinuous function on $B_{r,s}^+(p)$.

Proof To prove (1), we need only to prove that the set $\partial B_r(p) - B_{r,s}^+(p)$ is open in $\partial B_r(p)$. Let $x \in \partial B_r(p) - B_{r,s}^+(p)$, then $x \cdot t(t > 0)$ does not intersect with $\partial B_s(p)$ and will finally drop into another fixed point q . In terms of the choice of $g(x)$, we can choose $a < b$ satisfying

$$B_r(p) \subset g^{-1}([b, +\infty)); B_r(q) \subset g^{-1}((-\infty, a]) \quad (2.2.1)$$

Since $x \cdot t(t > 0)$ drops into q , there exists a $T_x > 0$ such that $x \cdot T_x$ is in the interior of $B_r(q)$. Therefore there is a closed ball $D_\delta(x)$ such that for any $y \in D_\delta(x) \cap \partial B_r(p)$, $y \cdot T_x$ is in the interior of $B_r(q)$. We can choose δ small enough such that the set $D_\delta(x) \cdot [0, T_x] \cap \partial B_s(p) = \emptyset$. Since $y \cdot T_x \in g^{-1}((-\infty, a])$, $y \cdot [T_x, +\infty) \in g^{-1}((-\infty, a])$. Therefore $D_\delta(x) \cdot [0, +\infty) \cap \partial B_s(p) = \emptyset$ in view of (2.2.1). This shows that $\partial B_r(p) - B_{r,s}^+(p)$ is open in $\partial B_r(p)$.

To prove (2), let $x \in B_{r,s}^+(p)$ and let $t_x < +\infty$ be the arrival time. We need only to prove that for any small $\varepsilon > 0$, there exists a neighborhood $D_\delta(x)$ of x such that for any $y \in D_\delta(x) \cap B_{r,s}^+(p)$,

$$t_y \geq t_x - \varepsilon \quad (2.2.2)$$

Since t_x is the arrival time of the trajectory $x \cdot t(t > 0)$ to $\partial B_{\frac{r}{2}}(p)$, $x \cdot [0, t_x - \varepsilon]$ has a positive distance from $\partial B_{\frac{r}{2}}(p)$. By the continuity of the flow v , there is a neighborhood D_x such that $\bar{D}_x \cdot [0, t_x - \varepsilon]$ has a positive distance from $\partial B_{\frac{r}{2}}(p)$ as well. Hence for $y \in D_x \cap B_{r,s}^+(p)$, (2.2.2) is true. \blacksquare

Since the Lyapunov function $g(x)$ is strictly decreasing along any nonconstant trajectory, there exists a $\delta_x > 0$ such that $g(x) - g(x \cdot t_x) = 2\delta_x$. By lemma 2.2.1 (2) and the property of $g(x)$, the function $g(y) - g(y \cdot t_y)$ is lower semicontinuous with respect to y on $B_{r,s}^+(p)$. Hence for each $x \in B_{r,s}^+(p)$, there exists a neighborhood U_x in $B_{r,s}^+(p)$ such that $\forall y \in U_x$, $g(y) - g(y \cdot t_y) > \delta_x$. This implies that $\delta_{r,s}^+(p) := \min_{x \in B_{r,s}^+(p)} \delta_x$ is positive and $\forall x \in B_{r,s}^+(p)$, $g(x) - g(x \cdot t_x) > \delta_{r,s}^+(p)$. In the same way, we can obtain a $\delta_{r,s}^-(p) > 0$ such that $\forall x \in B_{r,s}^-(p)$, there is $g(x \cdot (-t_x)) - g(x) > \delta_{r,s}^-(p)$.

Let $0 < s_1 < s_2 < \frac{r}{2}$, then we have the following conclusions:

- (1) $B_{r,s_1}^+(p) \subset B_{r,s_2}^+(p); B_{r,s_1}^-(p) \subset B_{r,s_2}^-(p)$
- (2) $\delta_{r,s_2}^\pm(p) \leq \delta_{r,s_1}^\pm(p)$
- (3) There are $s_0 > 0$, such that for any $0 < s < s_0$, there is $B_{r,s}^+(p) \cap B_{r,s}^-(p) = \emptyset$

The first two conclusions are obvious. For (3), we can choose $s_0 > 0$ satisfying $\text{osc}_{x \in B_{s_0}(p)} g(x) < \min \delta_{r,s_0}^\pm(p)$. If there is a $0 < s < s_0$ such that

$B_{r,s}^+(p) \cap B_{r,s}^-(p) \neq \emptyset$, then this means that there is a trajectory in $\bar{B}_r(p)$ which has non empty intersection with $\partial B_r(p)$ and $\partial B_s(p)$. But this is absurd, because $g(x)$ is strictly decreasing along any trajectory. After traveling from $\partial B_s(p)$ to $\partial B_r(p)$, then back to $\partial B_s(p)$, the value of $g(x)$ will decrease at least $\delta_{r,s}^+(p) + \delta_{r,s}^-(p)$ which contradicts the fact that $\text{osc}_{x \in B_s(p)} g(x) \leq \text{osc}_{x \in B_{s_0}(p)} g(x) < \min \delta_{r,s}^\pm(p)$. Hence for $0 < s < s_0$, (3) holds.

We introduce some notations that we will use later in this paper.

Let v be a flow on a compact metric space (X, d) , whose nonwandering set has finitely many components. Denote them by $A = \{A_n, \dots, A_1\}$. Define $B_r(A_i) (i = 1, \dots, n)$ to be the open neighborhood of A_i having distance less than r to A_i . Now consider the flow v restricted to the punctured space $X \setminus \cup_{i=1}^n B_r(A_i)$. Then all the trajectories can be classified into three types:

- (1) trajectories with domain $[a, b]$, $-\infty < a < b < \infty$;
- (2) trajectories with domain $[a, +\infty)$ or $(-\infty, b]$, $-\infty < a, b < \infty$;
- (3) $(-\infty, +\infty)$.

We denote the sets of three type trajectories by $\Gamma_1^A(r), \Gamma_2^A(r)$ and $\Gamma_3^A(r)$ respectively. If A_i is a point p_i in X for $i = 1, \dots, n$, then we denote $\Gamma_i^A(r)$ simply by $\Gamma_i(r)$.

Since the flow v can be viewed as the action of the Lie group \mathbb{R} on the compact metric space X , we assume in this paper that this action is Lipschitz with a finite Lipschitz constant. Therefore we can talk about the tangent vector of a trajectory of v and integrate the tangent vector to get the length of a segment of a trajectory.

3 Flows carrying a cocycle α

3.1 Bounded Alexander-Spanier cohomology

In his paper [8], Churchill introduced a bounded Alexander-Spanier cohomology theory. If we consider the category of compact pairs, this bounded Alexander-Spanier cohomology theory is identical with the usual Alexander-Spanier theory. One can see the proof of equivalence in [8]. In this section we will formulate such a cohomology theory.

Let \mathcal{R} be an ordered ring and X be any set. A function $\alpha : X \rightarrow \mathcal{R}$ is called bounded if there is an element $M \in \mathcal{R}^+$, called a bound for α , such that $|\alpha(x)| \leq M$ for any $x \in X$. Let $X^{q+1} = X \times \dots \times X$ ($q+1$ times, $q \geq 0$), and $\bar{C}^q(X, \mathcal{R})$ be the module of all bounded function from X^{q+1} to \mathcal{R} with addition and scalar multiplication defined pointwise. A coboundary homomorphism $\delta : \bar{C}^q(X, \mathcal{R}) \rightarrow \bar{C}^{q+1}(X, \mathcal{R})$ is defined as

$$\delta\alpha(x_0, \dots, x_{q+1}) = \sum_{i=0}^{q+1} (-1)^i \alpha(x_0, \dots, \hat{x}_i, \dots, x_{q+1})$$

where $(x_0, \dots, x_{q+1}) \in X^{q+2}$ and $(x_0, \dots, \hat{x}_i, \dots, x_{q+1})$ denotes the $(q+1)$ -tuple $(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_{q+1})$. It is easy to prove that $\delta\delta = 0$ and so $(\bar{C}^*(X, \mathcal{R}), \delta)$ is a complex.

An element $\alpha \in \bar{C}^q(X, \mathcal{R})$ is said to be locally zero, if α as a function from X^{q+1} to \mathcal{R} vanishes in a neighborhood of the diagonal of X^{q+1} . All the locally zero elements of $\bar{C}^q(X, \mathcal{R})$ constitute a submodule of $\bar{C}^q(X, \mathcal{R})$, denoted by $C_0^q(X, \mathcal{R})$. $(C_0^*(X, \mathcal{R}), \delta)$ is a subcomplex, since if α is locally zero in X^{q+1} , then $\delta\alpha$ is locally zero in X^{q+2} . Define $C^*(X, \mathcal{R})$ to be the quotient cochain complex of $(\bar{C}^*(X, \mathcal{R}), \delta)$ by $(C_0^*(X, \mathcal{R}), \delta)$. Now the bounded Alexander-Spanier cohomology group $\mathcal{H}^*(X, \mathcal{R})$ is the cohomology group of the cochain complex $(C^*(X, \mathcal{R}), \delta)$.

For any topological space Y and any continuous map $f : X \rightarrow Y$, there is the induced homomorphism $f^\# : \bar{C}^*(Y, \mathcal{R}) \rightarrow \bar{C}^*(X, \mathcal{R})$ defined by

$$f^\#\alpha(x_0, \dots, x_q) = \alpha(f(x_0), \dots, f(x_q)),$$

where $\alpha \in \bar{C}^q(Y, \mathcal{R})$ and $(x_0, \dots, x_q) \in Y^{q+1}$. It is easily seen if $\alpha \in C_0^*(Y, \mathcal{R})$, then $f^\#\alpha \in C_0^*(X, \mathcal{R})$. Therefore $f^\#$ induces a homomorphism from $C^*(Y, \mathcal{R})$ to $C^*(X, \mathcal{R})$, and we still denote it by $f^\#$.

Let i be the inclusion map from a subspace A to X , then we have the cochain map $i^\# : C^*(X, \mathcal{R}) \rightarrow C^*(A, \mathcal{R})$. Let $C^*(A, X, \mathcal{R}) := \ker i^\#$. Now the bounded Alexander-Spanier cohomology module $\mathcal{H}^*(A, X, \mathcal{R})$ is defined to be the cohomology module of the cochain complex $C^*(A, X, \mathcal{R})$.

Let (B, Y) be a topological pair and let $f : (A, X) \rightarrow (B, Y)$ be continuous, then the cochain map $f^\# : C^*(B, Y) \rightarrow C^*(A, X)$ induces the homomorphism $f^* : \mathcal{H}^*(B, Y, \mathcal{R}) \rightarrow \mathcal{H}^*(A, X, \mathcal{R})$.

As mentioned in the beginning of this section, the cohomology theory constructed above is identical to the usual Alexander-Spanier cohomology theory when restricted to the category of compact pairs. Therefore it satisfies the Eilenberg-Steenrod axioms and the continuity which is a special property of Alexander-Spanier cohomology theory.

Let $[\alpha] \in \mathcal{H}^1(X, \mathcal{R})$, then a cocycle $\alpha \in [\alpha]$ is a bounded function from X^2 to \mathcal{R} with $\delta\alpha$ being locally zero. If $[\alpha] = 0$, then the representation α is called coboundary and there is a bounded function $\beta : X \rightarrow \mathcal{R}$ such that $\delta\beta - \alpha$ is locally zero. If this is the case, we write $\delta\beta \simeq \alpha$.

Let $[\alpha] \in \mathcal{H}^1(X, \mathcal{R})$ and A be a path connected component of X . Suppose that $[\alpha]|_A$ is zero and there are two bounded function β and β' on A such that $\delta\beta \simeq \delta\beta' \simeq \alpha$, then $\beta - \beta'$ is constant on A . Namely, because $\delta(\beta - \beta')$ is locally zero on A , for any point $x \in A$ there is a small closed neighborhood such that $\beta - \beta'$ is constant on it. Since A is path connected, any two points on A can be joined by a chain of such closed neighborhoods. Hence $\beta - \beta'$ is a constant on A . Define $I_\alpha(x, y) := \beta(y) - \beta(x)$, where $x, y \in A$. It is obvious that $I_\alpha(x, y)$ depends only on $[\alpha]$.

Now let X be a topological space. Let $C(X)$ be the group of all the continuous

maps from X to S^1 . ($C(X)$ can also be seen as the set of continuous functions of absolute value 1 defined on X . then we can talk about the angle of $f(x)$). Let $R(X)$ be the subgroup of $C(X)$ consisting of all the continuous maps having the form $\exp 2\pi iH(x)$ with $H(x)$ being a real-valued continuous function on X .

Proposition 3.1.1[8][25] *If X is a compact polyhedron, then the map $J : C(X)/R(X) \rightarrow \mathcal{H}^1(X, \mathbb{R})$ is a group embedding, and the range of J spans $\mathcal{H}^1(X, \mathbb{R})$.*

Let γ be a curve in X and let $\Delta_\gamma \arg f$ be the change of angular variable of f along γ . It is well-known that for any closed curve γ in X , $\frac{1}{2\pi} \Delta_\gamma \arg f$ is an integer depending only on the homology class of γ and the coset of f in the group $C(X)/R(X)$. Hence $C(X)/R(X)$ is in fact the integral 1-dimensional cohomology group.

Take a group of closed curves $\{l_1, \dots, l_m\}$ ($m = \dim X$) such that the homology classes $\{[l_1], \dots, [l_m]\}$ constitute a basis of $\mathcal{H}_1(X, \mathbb{R})$. With respect to this basis, we have the dual basis $\{[f_1], \dots, [f_m]\}$. These circle-valued maps f_1, \dots, f_k are called rank 1 integral cocycles.

3.2 Integration of cocycles along chains

Let (X, d) be a compact metric space with metric d . For any point $p \in X$, we can choose a sequence of open balls, $\bar{B}_1 \supset B_1 \supset \bar{B}_2 \supset \dots$ such that $\bigcap_{i=1}^\infty B_i = \{p\}$. Since the bounded Alexander-Spanier cohomology has continuity, it follows that $\lim_{i \rightarrow \infty} H^1(\bar{B}_i) = H^1(\{p\}) = 0$. Therefore for a fixed cocycle α , there is a small open ball B_p such that $\alpha|_{\bar{B}_p \times \bar{B}_p}$ is a coboundary. Then there is a bounded function β_p defined on \bar{B}_p satisfying $\delta\beta_p \simeq \alpha|_{\bar{B}_p \times \bar{B}_p}$. Since X is compact, for any arbitrary small $r > 0$, there are finitely many open balls $\{B_j\}_{j=1}^l$ with radius r covering X . Furthermore, on every closed ball \bar{B}_j , there is β_j such that $\alpha|_{\bar{B}_j \times \bar{B}_j} \simeq \delta\beta_j$ and $|\beta_j|$ has an upper bound M_j on \bar{B}_j . We call this covering an r -covering and denote it by $\mathcal{U}(r)$. Define $M_\alpha(r) = \max_j M_j$.

Take a curve $\gamma : [a, b] \rightarrow X$, and choose a partition $a = t_0 < t_1 < \dots < t_k = b$ such that for $0 \leq j \leq k-1$, $\gamma([t_j, t_{j+1}]) \subset B_{i_j}$ for some $1 \leq i_j \leq l$. Now we can define the integration of α along γ to be

$$\int_\gamma \alpha = \sum_{j=0}^{k-1} \beta_{i_j}(\gamma(t_{j+1})) - \beta_{i_j}(\gamma(t_j))$$

Notice when $\gamma([a, b]) \subset B_{i_0}$, then $\int_\gamma \alpha = I_\alpha(\gamma(a), \gamma(b))$.

Proposition 3.2.1[8] *The integral $\int_\gamma \alpha$ is independent of the partition of γ , the covering $\{B_i\}_{i=1}^l$ and the functions $\{\beta_i\}$ selected on $\{\bar{B}_i\}$. It depends only on the cocycle α and the relative homotopy class of γ .*

Let v be a flow on X and choose an r -covering \mathcal{U} for small r . Let $\gamma = \{x_0 = x, x_1, \dots, x_k = y | t_0, \dots, t_{k-1}\}$ be a (\mathcal{U}, T) -chain from x to y . We define the integral of α along the chain γ as

$$\int_{\gamma} \alpha = \sum_{j=0}^{k-1} \int_{[x_j, x_j \cdot t_j]} \alpha + \sum_{j=0}^{k-1} \beta_{i_j}(x_{j+1}) - \beta_{i_j}(x_j \cdot t_j),$$

where $(x_j \cdot t_j, x_{j+1})$ lies in B_{i_j} for some $B_{i_j} \in \mathcal{U}$.

If we connect point pairs $(x_j \cdot t_j, x_{j+1})$ ($j = 1, \dots, k-1$) by any curve lying in B_{i_j} , we get an induced curve $\tilde{\gamma}$ from x to y . It is obvious that

$$\int_{\gamma} \alpha = \int_{\tilde{\gamma}} \alpha$$

It is easily seen that the integration $\int_{\gamma} \alpha$ has the following properties:

- (1) If $\lambda \in \mathcal{R}$, then $\int_{\gamma} \lambda \alpha = \lambda \int_{\gamma} \alpha$.
- (2) If γ is a trivial curve, i.e., represented by one point, then $\int_{\gamma} \alpha = 0$.
- (3) Let γ_1 and γ_2 be two curves for which the end point of γ_1 is the starting point of γ_2 . Let $\gamma_1 * \gamma_2$ denote the curve γ_1 followed by γ_2 , then

$$\int_{\gamma_1 * \gamma_2} \alpha = \int_{\gamma_1} \alpha + \int_{\gamma_2} \alpha$$

Proposition 3.2.2 *Let α be a cocycle on the compact metric space (X, d) . Let γ be a curve on M with length $l(\gamma) \leq L$, then there is a constant C_L depending only on L and $\{M_j\}$ such that*

$$\left| \int_{\gamma} \alpha \right| \leq C_L$$

Proof With respect to α there is an r -covering $\mathcal{U} = \{B_j\}$ such that $\alpha|_{\bar{B}_j \times \bar{B}_j} \simeq \delta \beta_j$ and $|\beta_j|$ has upper bound M_j on \bar{B}_j . Since γ is a curve with length L , we can use at most $\lceil \frac{L}{2r} \rceil + 1$ r -balls to cover r . Therefore

$$\left| \int_{\gamma} \alpha \right| \leq 2 \left(\lceil \frac{L}{2r} \rceil + 1 \right) \max_{B_j \in \mathcal{U}} M_j$$

■

Corollary 3.2.3 *Let v be a flow on a compact metric space (X, d) and let*

α be a cocycle on it. If $\gamma(T)$ is a trajectory with time interval $T \leq T_0$, then there is a constant C_{T_0} depending on T_0 and $\{M_j\}$ such that

$$\left| \int_{\gamma(T)} \alpha \right| \leq C_{T_0}$$

Proof For any $\gamma(T)$ with time interval $T \leq T_0$, we have

$$l(\gamma(t)) = \int_{\gamma(T)} |\dot{\gamma}(t)| dt \leq T_0 \max |\dot{\gamma}(t)|,$$

hence by applying proposition 3.2.2, we get the conclusion. \blacksquare

3.3 Integration of cocycles in a gradient-like flow

In this section we assume that v is a gradient-like flow with finitely many fixed points $\{p_j; j = 1, 2, \dots, n\}$. Let α be a cocycle on X . Then there is an associated r -covering $\mathcal{U}(r)$ such that when restricted to the closure of each open ball in $\mathcal{U}(r)$, α is a coboundary. Let r be small enough such that the Lyapunov function $g(x)$ corresponding to the flow v has mutually non-intersecting images $g(B_r(p_j))$ for $j = 1, 2, \dots, n$.

Define $\delta_0(r) = \min_{1 \leq j \leq n} \delta_{r, \frac{r}{4}}^\pm(p_j)$. Since $g(x)$ is uniformly continuous, there exists $s_0 \leq \frac{r}{4}$ satisfying $\text{osc}_{B_{s_0}} g(x) < \frac{1}{3} \delta_0(r)$. By our choice and in view of the analysis of the local structure of gradient-like flows (see section 2.2), it is easy to see that the following inequality holds:

$$3 \max_x \text{osc}_{y \in B_{s_0}(x)} g(y) < \delta_0(r) \leq \delta_{r, \frac{r}{4}}^\pm(p_j) \leq \delta_{r, s_0}^\pm(p_j) \quad (3.3.1)$$

for $j = 1, 2, \dots, n$.

Proposition 3.3.1 *Let v be a gradient-like flow with finitely many fixed points. α is a cocycle with an associated r -covering. Let $\mathcal{U}(s)$ be an s -covering with $s < s_0 < \frac{r}{4}$ for some s_0 satisfying (3.3.1). Then there exist $M > 0$ and $T(s_0) > 0$ depending on r but not on s such that for any $(\mathcal{U}(s), T(s_0))$ -chain $\tilde{\gamma}$, we have*

$$\left| \int_{\tilde{\gamma}} \alpha \right| \leq M \quad (3.3.2)$$

Proof Firstly we will prove that given any $s \leq r$, for any trajectory $\gamma \in \Gamma_1(s)$ with domain $[a, b]$, there exists a constant $T(s)$ satisfying $b - a \leq T(s)$.

Let $x \in \partial B_s(p_j)$, then the trajectory $x \cdot t$ or $x \cdot (-t)$ for $t \geq 0$ will flow into some different fixed point of v . (There does not exist any trajectory joining one point to itself.) Without loss of generality, we assume that $(x \cdot t)(t \geq 0)$ flows into the

point p_k for $k \neq j$. (It is possible that the trajectory may pass through some ball $B_s(p_l)$, but this does not influence the result below.) Hence there exists a $T_x > 0$ such that $x \cdot T_x \cap B_{\frac{s}{2}}(p_k) \neq \emptyset$. By the continuity of the flow, there exists a small closed ball $D_{s_x}(x)$ which satisfies that for any point $y \in D_{s_x}(x) \cap \partial B_s(p_j)$, the trajectory $[y, y \cdot T_x] \cap \partial B_{\frac{3s}{4}}(p_k) \neq \emptyset$. This shows that for any trajectories starting from $D_{s_x}(x) \cap \partial B_s(p_j)$ their time intervals are not greater than T_x , and covering the sphere $\partial B_s(p_j)$ by finitely such small closed balls, then it is easy to see that any trajectory in $\Gamma_1(s)$ starting from $\partial B_s(p_j)$ has time interval not greater than a constant $T_j(s)$. Let $T(s) = \max_{1 \leq j \leq n} T_j(s)$, then all the trajectories in $\Gamma_1(s)$ have time interval not greater than $T(s)$.

Let $\gamma = \{x_0, y_0, x_1, y_1, \dots, x_{k-1}, y_{k-1}, x_k | t_0, \tau_0, \dots, t_{k-1}, \tau_{k-1}\}$ be a chain. It is called an (r, s_0) -chain if it satisfies the following conditions:

- (1) $x_j \in \partial B_r(p_{l_j})$ for $j = 0, 1, \dots, k-1$ and $x_k \in B_r(p_{l_k})$;
- (2) $y_j \cdot \tau_j \in \partial B_r(p_{l_{j+1}})$ for $j = 0, 1, \dots, k-1$;
- (3) the point pair $(x_j \cdot t_j, y_j) \in B_s(m_j)$, for $j = 0, 1, \dots, k-1$, where those $B_s(m_j)$'s are elements in the s -covering $\mathcal{U}(s)$ of X .
- (4) $B_s(m_j), j = 0, 1, \dots, k-1$ and $B_s(p_{l_j}), j = 0, 1, \dots, k$, are $2k-1$ open balls and any two of them do not intersect.
- (5) $t_j, \tau_j \leq T(s_0)$, where $T(s_0)$ is an upper bound for the time interval of all the trajectories in $\Gamma_1(s_0)$.

Now compute the integration of α along the (r, s_0) -chain γ .

$$\begin{aligned}
\left| \int_{\gamma} \alpha \right| &\leq \sum_{j=0}^{k-1} \left| \int_{[x_j, x_j \cdot t_j]} \alpha \right| + \sum_{j=0}^{k-1} \left| \int_{[y_j, y_j \cdot \tau_j]} \alpha \right| \\
&\quad + \sum_{j=0}^{k-1} \left| \int_{[x_j \cdot t_j, y_j]} \alpha \right| + \sum_{j=0}^{k-1} \text{osc}_{x \in B_r(p_{l_{j+1}})} \beta_{p_{l_{j+1}}} \\
&= I + II + III + IV
\end{aligned} \tag{3.3.3}$$

Since $0 < t_j, \tau_j \leq T(s_0)$, applying corollary 3.2.3, we obtain

$$I + II \leq 2kC_1$$

where C_1 is a constant depending only on $\max |\dot{v}(t)|, \alpha$ and r .

Connect the point pair $(x_j \cdot t_j, y_j)$ with a line segment, then its length is at most $2s$. Using proposition 3.2.2, we have

$$III \leq kC_2$$

Here C_2 depends only on s_0, α and r .

For IV ,

$$IV \leq 2kM_{\alpha}(r)$$

Combining the above estimates, we have

$$\left| \int_{\gamma} \alpha \right| \leq kC \quad (3.3.4)$$

where C depends on r, s_0, α and $\max |\dot{v}(t)|$.

Now the proof of the proposition is changed to the problem to reduce each $(\mathcal{U}(s), T(s_0))$ -chain to an (r, s_0) -chain, while keeping the integration of α invariant.

Let $\tilde{\gamma} = \{x_0, x_1, \dots, x_k | t_0, \dots, t_{k-1}\}$ be a $(\mathcal{U}(s), T(s_0))$ -chain. Consider the following cases.

(a) If the chain $\tilde{\gamma}$ is contained in a ball $\bar{B}_r(p_{i_0})$ for $0 \leq i_0 \leq n$, then

$$\begin{aligned} \left| \int_{\tilde{\gamma}} \alpha \right| &= |\beta_{p_{i_0}}(x_k) - \beta_{p_{i_0}}(x_0)| \\ &\leq \text{osc}_{x \in B_r(p_{i_0})} \beta_{p_{i_0}}(x) \leq 2M_\alpha(r) \end{aligned}$$

(b) There is $1 \leq k_0 \leq k$ such that the subchain $\tilde{\gamma}_{0k_0} = \{x_0, x_1, \dots, x_{k_0} | t_0, \dots, t_{k_0-1}\}$ is contained in $B_r(p_{i_0})$ but the point $x_{k_0} \cdot t_{k_0}$ is not in $\cup_{j=0}^n \bar{B}_r(p_j)$. Hence there is a first intersection point \bar{x}_0 of $[x_{k_0}, x_{k_0} \cdot t_{k_0}]$ and $\partial B_r(p_{i_0})$. Let \bar{t}_0 satisfy $\bar{x}_0 \cdot \bar{t}_0 = x_{k_0} \cdot t_{k_0}$ and let $\bar{y}_0 = x_{k_0+1}$. Denote the ball containing the point pair $(\bar{x}_0 \cdot \bar{t}_0, \bar{y}_0)$ by $B_s(m_0)$. Since $t_{k_0} \geq T(s_0)$ and $\tilde{\gamma}_{0k_0} \subset B_r(p_{i_0})$, this implies that $x_{k_0} \in B_{s_0}(p_{i_0})$, hence the trajectory $[\bar{x}_0, \bar{x}_0 \cdot \bar{t}_0]$ does not intersect with $B_{s_0}(p_{i_0})$. Otherwise, $[x_{k_0}, x_{k_0} \cdot t_{k_0}]$ starts from $B_{s_0}(p_{i_0})$, intersects $\partial B_r(p_{i_0})$, then goes back to $B_{s_0}(p_{i_0})$. This conclusion contradicts (3.3.1) when we check the change of the Liapounov function $g(x)$ along $[x_{k_0}, x_{k_0} \cdot t_{k_0}]$. Therefore $[\bar{x}_0, \bar{x}_0 \cdot \bar{t}_0]$ is part of a trajectory in $\Gamma_1(s_0)$, and we have $\bar{t}_0 \leq T(s_0)$. Since $\bar{x}_0 \cdot \bar{t}_0 = x_{k_0} \cdot t_{k_0}$ is not in $\bar{B}_r(p_{i_0})$, by the choice of s , we know that $\bar{y}_0 := x_{k_0+1} \notin B_{s_0}(p_{i_0})$. Hence $\bar{y}_0 \cdot t_{k_0+1}$ meets firstly $B_{s_0}(p_{l_1})$ for some fixed point p_{l_1} . Let $\bar{y}_0 \cdot \bar{\tau}_0$ be the last intersection point of $[\bar{y}_0, \bar{y}_0 \cdot t_{k_0+1}]$ with $\partial B_r(p_{i_1})$. In the same way, we can prove that $\bar{\tau}_0 \leq T(s_0)$. Joining x_0 and \bar{x}_0 by a curve $l_{x_0 \bar{x}_0}$, then the subchain $\tilde{\gamma}_{0k_0}$ is reduced to a curve $l_{x_0 \bar{x}_0}$ combined with a chain $\{\bar{x}_0, \bar{y}_0 | \bar{t}_0, \bar{\tau}_0\}$ and it is clear that the reduction keeps the integration of α invariant.

(c) There is a $2 \leq k_0 \leq k$ such that the subchain $\tilde{\gamma}_{0, k_0-1} = \{x_0, x_1, \dots, x_{k_0-1} | t_0, t_1, \dots, t_{k_0-2}\}$ and the trajectory $[x_{k_0-1}, x_{k_0-1} \cdot t_{k_0-1}]$ is contained in $B_r(p_{i_0})$ but $x_{k_0} \notin B_r(p_{i_0})$. Then we take a point $\bar{x}_0 \in \partial B_r(p_{i_0}) \cap B_s(m_0)$, where $B_s(m_0)$ is the ball containing $(x_{k_0-1} \cdot t_{k_0-1}, x_{k_0})$. Since $t_{k_0} \geq T(s_0)$, the trajectory $[x_{k_0}, x_{k_0} \cdot t_{k_0}]$ will firstly meet $B_{s_0}(p_{i_1})$. We denote x_{k_0} by \bar{y}_0 and represent the last intersection point of $[x_{k_0}, x_{k_0} \cdot t_{k_0}]$ with $\partial B_r(p_{i_1})$ by $\bar{y}_0 \cdot \bar{\tau}_0$ for some $\bar{\tau}_0 > 0$. Due to the same argument as in (b), we know that $\bar{\tau}_0 \leq T(s_0)$. So $\tilde{\gamma}_{0k_0}$ is reduced to the combination of a curve $l_{x_0 \bar{x}_0}$ and a chain $\{\bar{x}_0, \bar{y}_0 | \bar{t}_0, \bar{\tau}_0\}$. It is clear that such inductions also keep the integration of α invariant. Note that x_{k_0-1} must be in $B_{s_0}(p_{i_0})$.

(d) There is a subchain $\tilde{\gamma}_{0k_0} = \{x_0, x_1, \dots, x_{k_0} | t_0, \dots, t_{k_0-1}\}$ contained in $B_r(p_{i_0})$ and the trajectory $[x_{k_0}, x_{k_0} \cdot t_{k_0}]$ goes through $B_r(p_{i_1})$. Let $\bar{x}_0 = \bar{y}_0$ be

the first intersection point of $[x_{k_0}, x_{k_0} \cdot t_{k_0}]$ with $\partial B_r(p_{i_1})$. Also we can prove that $\bar{\tau}_0 \leq T(s_0)$ as in (b) and (c). Then the chain $\tilde{\gamma}_{0k_0}$ followed by $[x_{k_0}, \bar{y}_0 \cdot \bar{\tau}_0]$ is reduced to the combination of $l_{x_0 \bar{x}_0}$ with the chain $\{\bar{x}_0, \bar{y}_0 | 0, \bar{\tau}_0\}$, while the integration of α is invariant under change.

(e) If $x_0 \notin \bigcap_{j=1}^n B_r(p_j)$, then $x_0 \cdot t_0$ will firstly meet some $B_s(p_{i_0})$. Let $x'_0 = x_0 \cdot t'_0$ be the last intersection point of $[x_0, x_0 \cdot t_0]$ with $\partial B_r(p_{i_0})$. In the same way, we can prove $t'_0 \leq T(s_0)$.

Now using the above steps (b)-(e) continuously, any $(\mathcal{U}(s), T(s_0))$ -chain $\tilde{\gamma} = \{x_0, x_1, \dots, x_k | t_0, \dots, t_{k-1}\}$ can be reduced to the combination of a curve $l_{x_0 \bar{x}_0}$ (or $[x_0, x_0 \cdot t'_0]$ in case (e)) with an (r, s_0) -chain $\bar{\gamma} = \{\bar{x}_0, \bar{y}_0, \dots, \bar{x}_l | \bar{t}_0, \bar{\tau}_0, \dots, \bar{\tau}_{l-1}\}$ if we can prove that (4) in the definition of a (r, s_0) -chain holds.

If (4) is not true, then there exists an (r, s_0) -cycle $\tilde{\gamma}_c = \{\bar{x}_{i_0}, \bar{y}_{i_0}, \dots, \bar{y}_{i_l}, \bar{x}_{i_0} | \bar{t}_{i_0}, \bar{\tau}_{i_0}, \dots, \bar{\tau}_{i_l}\}$ where the point pair $(\bar{y}_{i_l} \cdot \bar{\tau}_{i_l}, \bar{x}_{i_0}) \in \partial B_r(p_{i_0})$. Now extend the trajectory $\bar{y}_{i_j} \cdot t (t \geq 0)$ for $j = 0, \dots, l$. Since $\bar{\tau}_c$ is obtained by reducing the $(\mathcal{U}(s), T(s_0))$ -chain, hence $\bar{y}_{i_j} \cdot t$ will go through $B_{s_0}(p_{i_{j+1}})$. We let $\bar{y}_{i_j} \cdot \tau_{i_j} (\tau_{i_j} > \bar{\tau}_{i_j})$ be a point in $B_{s_0}(p_{i_{j+1}})$. Take the inverse process corresponding to (2)-(4), then there is a trajectory starting from some point $x_{i_{j-1}} \in B_{s_0}(p_{i_{j-1}})$ to some point $x_{i_{j-1}} \cdot t_{i_{j-1}}$ in the s -ball containing $\bar{x}_{i_{j-1}} \cdot t_{i_{j-1}}$ and \bar{y}_{i_j} . Therefore from the cycle $\tilde{\gamma}_c$ we get a $(\mathcal{U}(s), T(s_0))$ -chain $\tilde{\gamma}_c = \{x_{i_0}, \bar{y}_{i_0}, \dots, \bar{y}_{i_l}, x_{i_0} | t_{i_0}, \tau_{i_0}, \dots, t_{i_l}, \tau_{i_l}\}$.

Consider the change of the function $g(x)$ along $\tilde{\gamma}_c$. On one hand, since the two ends of $\tilde{\gamma}_c$ are x_{i_0} and $\bar{y}_{i_l} \cdot \tau_{i_l}$, by (3.3.1)

$$\left| \int_{\tilde{\gamma}_c} \delta g \right| \leq \text{osc}_{x \in B_{s_0}(p_{i_0})} g(x) < \frac{1}{3} \delta_0(r)$$

On the other hand, each $\bar{y}_{i_j} \cdot t (0 < t \leq \tau_{i_j}) (j = 0, \dots, l)$ goes across $B_r(p_{i_j}) \setminus B_{\frac{r}{4}}(p_{i_j})$, we have

$$\int_{\tilde{\gamma}_c} \delta g \geq (l+1) \delta_{r, \frac{r}{4}}^+ - 2(l+1) \max_x \text{osc}_{y \in B_s(x)} g(y) > \delta_0(r) - \frac{2}{3} \delta_0(r) = \frac{1}{3} \delta_0(r)$$

This is absurd. The contradiction shows that the chain $\tilde{\gamma}$ we get from a $(\mathcal{U}(s), T(s_0))$ -chain $\tilde{\gamma}$ is indeed an (r, s_0) -chain. Let $\bar{\gamma} = \{\bar{x}_0, \bar{y}_0, \dots, \bar{x}_l | \bar{t}_0, \bar{\tau}_0, \dots, \bar{\tau}_{l-1}\}$. Since v is gradient-like, the index l is less than n . Therefore applying the estimate (3.3.4) and the fact that the reduction from $\tilde{\gamma}$ to $\bar{\gamma}$ keeps the integration of α , we have

$$\left| \int_{\tilde{\gamma}} \alpha \right| = \left| \int_{\bar{\gamma}} \alpha \right| \leq nC.$$

where C depends on r, s_0, α and $\max |\dot{v}(t)|$. Proposition 3.3.1 now is proved. \blacksquare

3.4 Flows carrying a cocycle α

Consider a fixed cocycle α on X . With respect to α there is an r -covering for

small r such that the restriction of α to the closure of every r -ball is a coboundary. Let v be a flow on X having n fixed points $\{p_i\}_{i=1}^n$. Consider the flow v restricted to the punctured space $X \setminus \bigcup_{i=1}^n B_r(p_i)$. Recall that in the end of section 2.2, we have defined three sets $\Gamma_1(r), \Gamma_2(r)$ and $\Gamma_3(r)$. Now we have the following definitions.

Definition 3.4.1 *The flow v defined on the compact metric space (X, d) is said to be an α -flow, if there exist a cocycle α , a small $r > 0$ and a $T_0 > 0$ such that for some $\rho > 0$ and $0 \leq \lambda < 1$, the following conditions are satisfied:*

- (1) v is gradient-like in $B_r(p_i)$
- (2) $\max_{(x,y) \in \bar{B}_r(p_i) \times \bar{B}_r(p_i)} |I_\alpha(x, y)| \leq \lambda\rho$, for $1 \leq i \leq n$.
- (3) for any trajectory $\gamma \in \Gamma_1(r)$,

$$\int_\gamma \alpha \geq \rho$$

(4) if $\gamma(T_0)$ denotes any sub-trajectory of $\gamma \in \Gamma_2(r) \cup \Gamma_3(r)$ with time interval T_0 , then

$$\int_{\gamma(T_0)} \alpha \geq \rho$$

Similarly, if we replace the fixed points by connected nonwandering sets of v and replace the set $\Gamma_i(r)$ by $\Gamma_i^A(r)$ which is also defined at the end of section 2.2, then we can get the definition of generalized α -flows with respect to the nonwandering set A .

Definition 3.4.2 *The flow v defined on the compact metric space (X, d) is said to be a generalized α -flow with respect to the nonwandering set $A = \{A_n, \dots, A_1\}$, if there exist a cocycle α , a small $r > 0$ and a $T_0 > 0$ such that for some $\rho > 0$ and $0 \leq \lambda < 1$, the following conditions are satisfied:*

(1) There is a Lyapunov function $g_i(x)$ defined in $B_r(A_i)$ for $i = 1, \dots, n$, such that $g_i(x)$ is constant on A_i .

- (2) $\max_{(x,y) \in \bar{B}_r(A_i) \times \bar{B}_r(A_i)} |I_\alpha(x, y)| \leq \lambda\rho$, for $1 \leq i \leq n$.
- (3) for any trajectory $\gamma \in \Gamma_1^A(r)$,

$$\int_\gamma \alpha \geq \rho$$

(4) if $\gamma(T_0)$ is any sub-trajectory of $\gamma \in \Gamma_2^A(r) \cup \Gamma_3^A(r)$ having time interval T_0 , then

$$\int_{\gamma(T_0)} \alpha \geq \rho$$

Definition 3.4.3 *A flow v is called carrying a cocycle α if v is an α -flow or a*

generalized α -flow.

We will give examples and some propositions to show that the (generalized) α -flows include many interesting and important flows. Let us give some examples first.

Example 3.4.1 Let v be the gradient flow generated by a Morse function f on a smooth manifold X . We say that the flow v is a df -flow. In fact, in this case every function β_j defined on the closed ball B_j in the r -covering can be taken as the restriction of f to $B_r(p)$. Now let $S(f)$ be the critical point set and p_i denote a critical point. Let $\rho = \frac{1}{2} \min |f(p_i) - f(p_j)|$, where the minimum is taken over all pairs $(p_i, p_j) \in S_i(f) \times S_j(f)$, $i \neq j$, and choose r small enough such that $\max_{p_i \in S(f)} \text{osc}_{B_r(p_i)} |f| \leq \frac{\rho}{2}$, then the flow v satisfies conditions (1) and (2). To prove (3), let $\gamma \in \Gamma_1(r)$ be a segment of a trajectory joining $B_r(p_i)$ and $B_r(p_j)$. Since v is a gradient-flow, the p_i and p_j must be two critical points with different index i and j . Then

$$\begin{aligned} \int_{\gamma} df &= f(\gamma(b)) - f(\gamma(a)) \\ &= f(p_i) - f(p_j) + f(\gamma(b)) - f(p_i) + f(p_j) - f(\gamma(a)) \\ &\geq f(p_i) - f(p_j) - 2 \max_{p_i \in S(f)} \text{osc}_{B_r(p_i)} |f| \\ &\geq 2\rho - \rho = \rho, \end{aligned}$$

hence (3) is proved. Since v is a gradient flow, $\Gamma_2(r) \cup \Gamma_3(r)$ is empty. Hence the condition (4) is trivial.

Example 3.4.2 Let v be the flow generated by the vector field V dual to the closed Morse 1-form ω . Take an r -covering of X such that ω is exact when restricted to the closed ball $\bar{B}_r(p)$, for instance, let $\omega = df_p|_{\bar{B}_r(p)}$. By the preceding discussion, we know that $I_{\omega}(x, y)$ and $\int_{\gamma} \omega$ are independent of the choice of the r -covering and the related functions $\{f_p\}$. Therefore

$$\begin{aligned} \max_{(x,y) \in \bar{B}_r(p_i) \times \bar{B}_r(p_i)} |I_{\omega}(x, y)| &= |\text{osc}_{x \in \bar{B}_r(p_i)} f_{p_i}(x)| \\ &\leq 2r \max_{x \in \bar{B}_r(p_i)} |df_{p_i}| = 2r \max_{x \in \bar{B}_r(p_i)} |\omega(x)| \end{aligned} \quad (3.4.1)$$

On the other hand, there exists a $r_0 > 0$ such that a good coordinate system can be chosen in $B_{r_0}(p_i)$ for any $i = 1, \dots, n$, to make the dual vector field have the form $(-x_1, \dots, -x_s, x_{s+1}, \dots, x_m)$. Furthermore, r_0 can be chosen small enough such that

$$r_0 < \min_{i \neq j} d(B_{r_0}(p_i), B_{r_0}(p_j)),$$

where $p_i \in S_i(\omega)$.

Let $r < \frac{r_0}{2}$, then by analyzing the explicit expression of the trajectory in $B_{r_0}(p_i), i = 1, \dots, n$, it is easy to see that for any $\gamma \in \Gamma_1(r)$,

$$\int_a^b |\dot{\gamma}(t)| dt \geq \frac{r_0}{2}.$$

Using the above inequality, for any $\gamma \in \Gamma_1(r)$, we have

$$\begin{aligned} \int_{\gamma} \omega &= \sum_j f_{x_{i_j}}(\gamma(t_{j+1})) - f_{x_{i_j}}(\gamma(t_j)) \\ &= \sum_j \int_{t_j}^{t_{j+1}} |df_{x_{i_j}}|^2 = \int_a^b |\omega(\gamma(t))|^2 dt \\ &\geq \min_{x \notin \cup_i B_r(p_i)} |\omega(x)| \cdot \int_a^b |\dot{\gamma}(t)| dt \\ &\geq \min_{x \notin \cup_i B_r(p_i)} |\omega(x)| \cdot C_{r_0} \end{aligned} \tag{3.4.2}$$

where $C_{r_0} = \frac{r_0}{2}$

Therefore if we can choose r, ρ satisfying

$$4r \max_{x \in \cup_i \bar{B}_r(p_i)} |\omega(x)| \leq \rho \leq C_{r_0} \min_{x \notin \cup_i B_r(p_i)} |\omega(x)| \tag{3.4.3}$$

then conditions (2) and (3) are satisfied ((1) is obvious). Let $|\cdot|_0$ be the Euclidean metric, then there exist constants C_0 and C_1 such that for any vector field V near any point $x \in X$,

$$C_0 |V|_0 \leq |V| \leq C_1 |V|_0$$

Then

$$\min_{x \notin \cup_i B_r(p_i)} |\omega(x)| \geq C_0 r$$

and

$$\max_{x \in \cup_i \bar{B}_r(p_i)} |\omega(x)| \leq C_1 r$$

Hence (3.4.3) holds if we can choose ρ satisfying

$$4C_1 r^2 \leq \rho \leq C_{r_0} C_0 r \tag{3.4.4}$$

This can be done by choosing r small enough. Now we can choose r and ρ such that the conditions (2) and (3) are true.

For the condition (4), we can choose T_0 large enough such that for fixed r and ρ ,

$$\rho \leq \min_{x \notin \cup_i B_r(p_i)} |\omega(x)|^2 T_0 \leq \int_{\gamma(T_0)} |\omega(\gamma(t))|^2 dt = \int_{\gamma(T_0)} \omega$$

The above argument shows that the flow v satisfies our definition, i.e., v is a ω -flow, which fits with the fact that v is generated by the vectorfield dual to ω .

Remark Example 3.4.1 can also be treated as in the example 3.4.2, but in this case every trajectory is in the set $\Gamma_1(r)$. In the case of Example 3.4.2, it is possible for some closed 1-forms that the flows they generate have trajectories in $\Gamma_2(r) \cup \Gamma_3(r)$.

Example 3.4.3 We let ω be a closed 1-form as in the example 3.4.2. Now we consider a family of flows depending on the parameter $\varepsilon, 0 \leq \varepsilon < 1$:

$$\dot{v}_\varepsilon(t) = V(v(t)) + \varepsilon U(v(t)),$$

where $V(x)$ is the dual vector field of ω and $U(x)$ is a vector field vanishing on a small neighborhood of each zero point of ω and satisfying the norm inequality:

$$|U(x)| \leq |V(x)| = |\omega(x)|.$$

We claim that the flow v_ε is also an ω -flow. This can be seen as follows. Firstly (3.4.1) holds. Secondly we study the integration of ω along $\gamma_\varepsilon \in \Gamma_1^\varepsilon(r)$, we have

$$\begin{aligned} \int_{\gamma_\varepsilon} \omega &= \sum_j f_{x_{i_j}}(\gamma_\varepsilon(t_{j+1})) - f_{x_{i_j}}(\gamma_\varepsilon(t_j)) \\ &= \sum_j \int_{t_j}^{t_{j+1}} df_{x_{i_j}}(\dot{\gamma}_\varepsilon(t_j)) \\ &= \sum_j \int_{t_j}^{t_{j+1}} \langle \omega(\gamma_\varepsilon(t)), V(\gamma_\varepsilon(t)) + \varepsilon U(\gamma_\varepsilon(t)) \rangle dt \\ &\geq (1 - \varepsilon) \int_a^b |\omega(\gamma_\varepsilon(t))|^2 dt \end{aligned}$$

Since $U(x)$ vanishes in a small neighborhood of each zero point, the argument behind (3.4.2) also holds in this case. Hence (3.4.3) should be modified as

$$4r \max_{x \in \cup_i \bar{B}_r(p_i)} |\omega(x)| \leq \rho \leq (1 - \varepsilon) C_{r_0} \min_{x \notin \cup_i B_r(p_i)} |\omega(x)|$$

Thus we can choose r, ρ small enough such that the conditions (2) and (3) in the definition are true.

For (4), we should choose T_0 large enough such that

$$\rho \leq (1 - \varepsilon) \min_{x \notin \cup_i B_r(p_i)} |\omega(x)|^2 T_0 \leq \int_{\gamma_\varepsilon(T_0)} \omega$$

Therefore, if we choose r, ρ, T suitably, the conditions (1)-(4) are also true for the flow v_ε .

Example 3.4.4 This example will provide a method for changing a gradient flow of a Morse function to an α -flow with α being a nontrivial cocycle.

It is well-known that in any closed smooth manifold M_0 there exists a Smale function with exactly one local maximum point and one local minimum point. Hence the gradient flow of such a Smale function has only one minimal attractor and one minimal repeller. Now take small m -disc D_0^m and D_1^m whose center points are the minimum point and the maximum point respectively. Now cutting the two discs and connecting the two holes by a “bridge” $S^{m-1} \times D^1$, we get a new manifold M_1 . Take the flow on $S^{m-1} \times D^1$ to be the one generated by the vector field $(0, \pm \frac{d}{dt})$ and let the flow go from the part of the original minimum point to the part of the original maximum point. After connecting the vector fields smoothly in the connecting collar, we get an α -flow on M_1 (This fact can be shown in the same way as in Example 3.4.9.). It is obvious that this flow on M_1 has no attractor or repeller and the set $\Gamma_2(r) \cup \Gamma_3(r) \neq \emptyset$.

If a gradient flow of a Morse function has more than two minimum or maximum points, we can proceed as above and connect some minima and maxima pairwise to construct the α -flow with nontrivial cocycle α and having attractors and repellers.

Example 3.4.5 (Flows carrying a cohomology class) We consider an extreme case in the definition of an α -flow v , i.e., that v has no fixed point in X . In this case, the set $\Gamma_1(r) \cup \Gamma_2(r) = \emptyset$ and the only condition that makes v an α -flow is that there exist constants $\rho > 0$ and $T_0 > 0$ such that for any trajectory $\gamma(T_0)$ with time interval T_0 , we have

$$\int_{\gamma(T_0)} \alpha \geq \rho$$

Now the α -flow v becomes a so called “flow carrying a cohomology class” as introduced by R.C.Churchill in his paper [8]. The reason that the flow is called “carrying a cohomology class” is that the above condition is independent of the choice of the representative in the cohomology class $[\alpha]$. In fact, if $\alpha_1 \in [\alpha]$ is another cocycle, then there exists a coboundary $\beta \in \mathcal{H}^1(X, \mathbb{R})$, such that

$$\alpha_1 - \alpha \simeq \beta$$

and so

$$\begin{aligned} \int_{\gamma(kT_0)} \alpha_1 &= \int_{\gamma(kT_0)} \alpha + \beta(e(\gamma(kT_0))) - \beta(s(\gamma(kT_0))) \\ &\geq k\rho - 2M_\beta \end{aligned}$$

where $s(\gamma)$ and $e(\gamma)$ are the start point and the end point of the trajectory γ ,

and M_β is the bound of β . Hence if we choose $k > \lceil \frac{2M_\beta + \rho}{\rho} \rceil + 1$, then we have

$$\int_{\gamma(kT_0)} \alpha_1 \geq \rho.$$

The existence of a flow carrying a cohomology class in a manifold will induce the vanishing theorem for the Novikov numbers. This result will be given in section 4.5.

The flow in the following example is a simple example of such an α -flow which also carries a cohomology class. It was first explained in this way in [8].

Example 3.4.6 Let S^1 be the unit circle. We will define a nontrivial cocycle α on S^1 as follows. Define two open sets in S^1 :

$$U_1 = \left\{ e^{i\theta} \mid -\frac{5\pi}{8} < \theta < \frac{5\pi}{8} \right\}$$

$$U_2 = \left\{ e^{i\theta} \mid \frac{3\pi}{8} < \theta < \frac{13\pi}{8} \right\}$$

Then $U_1 \cup U_2$ is an open covering of S^1 . Define two bounded functions in U_1 and U_2 . Let $\beta_2 = 0$ in U_2 and let

$$\beta_1(e^{i\theta}) = \begin{cases} 1 & \theta \in [0, \frac{5\pi}{8}] \\ 0 & \theta \in [-\frac{5\pi}{8}, 0) \end{cases}$$

Now we can define a nontrivial cocycle $\alpha : S^1 \times S^1 \rightarrow \mathbb{R}$,

$$\alpha(e^{i\theta_1}, e^{i\theta_2}) = \begin{cases} \beta_i(e^{i\theta_2}) - \beta_i(e^{i\theta_1}) & \text{if } |\theta_1 - \theta_2| < \frac{\pi}{4}, i = 1, 2 \\ 0 & \text{if } |\theta_1 - \theta_2| \geq \frac{\pi}{4} \end{cases}$$

Let $\pi_i : T^m = S^1 \times \cdots \times S^1 \rightarrow S^1$ be the projection map which projects from the m -dimensional torus to the i -th factor space. Define $\alpha_i(x, y) : T^m \times T^m \rightarrow \mathbb{R}$ to be the pull-back map $(\pi_i \times \pi_i)^* \alpha$. It is easy to see that $[\alpha_i] (i = 1, \dots, m)$ is a linearly independent basis of the bounded Alexander-Spanier cohomology group $\mathcal{H}^1(X, \mathbb{R})$

Let $\omega = (\omega_1, \dots, \omega_m)$ be a constant irrational vector field in \mathbb{R}^m . Here “irrational” means that $\omega_i (i = 1, \dots, m)$ is an irrational number. Then this vector field can be projected to the torus T^m such that ω is a vector field on T^m . Consider the irrational flow generated by ω ,

$$\dot{v} = \omega(v)$$

Taking any point x on T^m and integrating α_i along $[x, x \cdot t]$, we can get

$$\int_{[x, x \cdot t]} \alpha_i = \left[\frac{\omega_i t}{2\pi} \right].$$

This shows that the irrational flow is an $\alpha_i(-\alpha_i)$ -flow for $i = 1, \dots, m$, if $\omega_i > 0(\omega_i < 0)$. In particular, it is a flow carrying cohomology class $[\alpha_i](-[\alpha_i])$ if $\omega_i > 0(\omega_i < 0)$.

Example 3.4.7 In Example 3.4.1 and Example 3.4.2, we have considered the flow generated by a dual vector field of a Morse function or a closed Morse 1-form. In those cases, the fixed points of the flows are non-degenerate fixed points. In some cases, those flows generated by a function or a closed 1-form may have Bott type nondegenerate critical manifolds. For instance, let Z be a non-degenerate critical manifold of a function f . This means the following condition holds:

- (1) $\text{grad}f(x) = 0, \forall x \in Z$.
- (2) At any point $x \in Z$, there is a decomposition $T_x(M) = T_x^\perp(Z) \otimes T_x(Z)$ and $d^2f(x)$ as bilinear form is nondegenerate along the vertical tangent subspace $T_x^\perp(Z)$.

Since locally a closed 1-form ω is the differential of a smooth function, hence the Bott type nondegenerate zero set of the closed 1-form ω can be defined as above.

Therefore using the same argument as in Example 3.4.2, it can be proved that such a flow generated by ω is a generalized ω -flow with respect to all the Bott type nondegenerate zero sets.

Example 3.4.8 In the paper [27], S.Smale introduced the Morse-Smale flow. We give the definition below.

A flow v on a manifold M is called a Morse-Smale flow if it satisfies:

- (1) The chain recurrent set of v consists of a finite number of hyperbolic closed orbits and hyperbolic fixed points. (The reader can see the meaning of “hyperbolic” in section 4.3)
- (2) The unstable manifold of any closed orbit or fixed point has transversal intersection with the stable manifold of any closed orbit or fixed point.

It was proved that such flows have “global” gradient-like structures and have a Morse decomposition which induces the Morse inequalities.

However, in some cases, although the nonwandering set of the flow contains only the hyperbolic periodic orbits and the hyperbolic fixed points, the flow is not a Morse-Smale flow because of the existence of “cycles” which consist of some orbits “connecting” different invariant sets and form a closed curve. We can give a definition of such flows when restricted to the category of flows carrying a cocycle.

α -Morse-Smale flow Let v be a generalized α -flow with respect to the nonwandering set $A = \{A_n, \dots, A_1\}$. If A contains only the hyperbolic orbits or hyperbolic fixed points, then v is called an α -Morse-Smale flow.

The following example provides a concrete α -Morse-Smale flow.

Example 3.4.9 Let W be a compact oriented connected manifold with bound-

ary $\partial W = \partial_+ W \cup \partial_- W$. W is called a flow manifold if it satisfies the following two conditions:

- (1) $\partial_+ W \cap \partial_- W = \emptyset$;
- (2) The Euler characteristic number $\chi(W) = \chi(\partial_- W)$.

In his paper [1] and [2], D.Asimov has proved the facts that if W is a flow manifold such that $\dim(W) \neq 3$ and W is not a Möbius band, then W has a round handle decomposition and a manifold with a round handle decomposition has a non singular Morse-Smale vector field, i.e., the flow generated by this vector field has only hyperbolic periodic orbits and at the boundary $\partial_- W(\partial_+ W)$, the vector field points outward (inward). Actually, he obtained the following result (see [15]):

Let $f : W \rightarrow [a, b]$ be a Morse function with two critical points p and q of index k and $k + 1$ respectively, such that the unstable manifold of q does not intersect the stable manifold p , then there is a non-singular Morse-Smale vector field X on W satisfying,

- (1) $X = -\nabla f$ on a neighborhood of ∂W , and
- (2) the flow of X has exactly one closed orbit and this orbit has index k and is untwisted.

Now we can use this fact to construct many α -Morse-Smale flows. A simple example is that we can reverse the flow and then double the manifold to get a closed manifold $2W$ and a flow on it such that the nonwandering set of the flow are the untwisted hyperbolic periodic orbits with index k and index $n - 1 - k$. This is actually an α -flow. The cocycle α can be defined as follows. Take a metric d on $2W$, and define two open sets on $2W$ to be

$$\begin{aligned} U_1 &= \{x \in 2W \mid d(x, \partial_- W) \leq 2\varepsilon\} \\ U_2 &= \{x \in 2W \mid d(x, \partial_- W) \geq \varepsilon\}. \end{aligned}$$

Then $U_1 \cup U_2$ is a covering of $2W$. Denote the two half parts of $2W$ by W_1 and W_2 . Define $\beta_2(x) = 0$ on U_2 and define $\beta_1(x)$ in U_1 to be

$$\beta_1(x) = \begin{cases} 0 & \text{if } d(x, W_1) = 0 \\ 1 & \text{if } d(x, W_1) > 0 \end{cases}$$

Now the cocycle $\alpha : 2W \times 2W \rightarrow \mathbb{R}$ is defined as

$$\alpha(x, y) = \begin{cases} \beta_i(y) - \beta_i(x) & \text{if } d(x, y) < \varepsilon, \text{ for } i = 1, 2 \\ 0 & \text{if } d(x, y) \geq \varepsilon \end{cases}$$

In the same way we can piece together many flow manifolds with boundary to get α -Morse-Smale flows that possesses hyperbolic periodic orbits and hyperbolic fixed points with different indexes.

Theorem 3.4.1 *Let v be an α -flow on the compact metric space (X, d) . If*

α is a trivial cocycle, then v is a gradient-like flow. Conversely, if v is a gradient-like flow, then v is a δg -flow with g being the associated Lyapunov function of the flow v .

Proof Firstly we will show that if v is an α -flow with α being a trivial cocycle, then $\Gamma_2(r) \cup \Gamma_3(r)$ is empty. In fact, if we assume that there exists a trajectory $\gamma \in \Gamma_2(r) \cup \Gamma_3(r)$, then for large T the integral $\int_{\gamma(T)} \alpha$ can be larger than any pre-given number. Now using a shortest curve K joining the two ends of $\gamma(T)$, then $\int_{\gamma(T)*K} \alpha = 0$, since α is a trivial cocycle. This gives $\int_{\gamma(T)} \alpha = |\int_K \alpha|$, however, this is not possible, since $|\int_K \alpha|$ is bounded for any finitely long curve K .

Secondly, we will show that there is no trajectory in $\Gamma_1(r)$ forming a "cycle". If this case happened, then there would exist a $(\mathcal{U}(r), T)$ -cycle $\tilde{\gamma} = \{p'_{i_0}, \dots, p'_{i_k} = p'_{i_0} | t_0, \dots, t_{k-1}\}$ where the point pair $(p'_{i_j}, p'_{i_{j+1}}) \subset B_r(p_{i_j})$ for $j = 0, \dots, k-1$. Since α is a trivial cocycle, we get $\int_{\tilde{\gamma}} \alpha = 0$. On the other hand, we have

$$\begin{aligned} \int_{\tilde{\gamma}} \alpha &\geq \sum_{j=0}^{k-1} \int_{[p'_{i_j}, p'_{i_{j+1}} | t_j]} \alpha - \sum_{j=0}^{k-1} \text{osc}_{B_r(p_{i_j})} \beta_{i_j} \\ &\geq k\rho - \lambda k\rho > 0 \end{aligned}$$

This is a contradiction. Therefore this result with the first one implies that for any $x \in X$, its limit sets $\omega(x)$ and $\omega^*(x)$ are in $\cup_{i=1}^n B_r(p_i)$, furthermore in $\{p_i, i = 1, \dots, n\}$ in view of (1) of the definition 3.4.1. Hence v is a gradient-like flow.

For the second conclusion, since each gradient-like flow v has an associated Lyapunov function g , if we define $\alpha : X^2 \rightarrow \mathbb{R}$ to be

$$\alpha(x, y) = g(y) - g(x), \quad \forall (x, y) \in X^2$$

then it is easy to see that v is a δg -flow. ■

Theorem 3.4.2 *Let v be an α -flow on the compact metric space (X, d) . Let $\{s_i\} \rightarrow 0$ as $i \rightarrow \infty$. If α is a nontrivial cocycle and for any $M > 0$ and $T > 0$ there is a $(\mathcal{U}(s_i), T)$ -chain $\tilde{\gamma}_i$ for each s_i such that*

$$\left| \int_{\tilde{\gamma}_i} \alpha \right| \geq M \tag{3.4.5}$$

then v is not a gradient-like flow. Furthermore, if it is known that for any $M > 0$ there is a trajectory $\tilde{\gamma}$ satisfying (3.4.5) or there is an oriented cycle $\tilde{\gamma}_0$ consisting of some orbits joining fixed points where the direction is determined by the forward direction of the flow v such that

$$\left| \int_{\tilde{\gamma}} \alpha \right| \geq 1 \tag{3.4.6}$$

then v is not a gradient-like flow.

Proof The first conclusion is a direct corollary of proposition 3.3.1. and the third one is obvious. We only consider the second case. From proposition 3.2.3, if the time intervals of the trajectories have an upper bound, then for any cocycles on X the absolute value of the integration, $|\int_{\tilde{\gamma}} \alpha|$ has a uniform bound. Therefore the trajectory $\tilde{\gamma}$ with respect to the arbitrary large M has arbitrary large time interval and it is the chain needed for the hypothesis in the first conclusion. ■

Theorem 3.4.3 *Let v be a α -flow with a non-trivial cocycle α on a compact metric space (X, d) . If v is not a gradient-like flow, then for any small sequence $\{s_i\}$ which tends to zero as $i \rightarrow \infty$, and for any $M > 0, T > 0$ there is a $(\mathcal{U}(s_i), T)$ -chain $\tilde{\gamma}_i$ for each s_i such that*

$$\left| \int_{\tilde{\gamma}} \alpha \right| \geq M \quad (3.4.7)$$

Proof If v is not a gradient-like flow, then there is a non fixed point x_0 in the chain recurrent set of v . Therefore for any s and $T > 0$ there is a $(\mathcal{U}(s), T)$ -chain $\tilde{\gamma} = \{x_0, \dots, x_k = x_0 | t_0, \dots, t_{k-1}\}$.

Now we consider two possibilities:

- (1) If $x_0 \cdot t \notin \cup_{i=1}^n B_r(p_i)$

In this case, we have the integration:

$$\int_{[x, x \cdot t]} \alpha \geq \left[\frac{t}{T_0} \right] \rho$$

Since x_0 is a point in the chain recurrent set of v , there is a $(\mathcal{U}(s), T)$ -chain $\tilde{\gamma} = \{x_0, \dots, x_k = x_0 | t_0, \dots, t_{k-1}\}$, and we can modify the chain $\tilde{\gamma}$ to get a $(\mathcal{U}(s), kT)$ -chain $\tilde{\gamma}_1 = \{x_0, x_0 | kT\}$. We can let $\tilde{\gamma}$ to be a $(\mathcal{U}(s_0), T)$ -chain for small $s_0 < s$ such that there exists a x'_0 in the ball $B_s(m_{i_1})$ containing the point pair $(x_0 \cdot T, x_1)$ satisfying $x'_0 \cdot (-T) \in B_s(p_{i_0})$ and $x'_0 \cdot T \in B_s(m_{i_2})$. The point is that a $(\mathcal{U}(s_0), T)$ -chain for small s_0 can provide a chain $(\mathcal{U}(s), T)$ for large s , while decreasing the number of trajectories it contains. Hence we can obtain a $(\mathcal{U}(s), T)$ -chain $\tilde{\gamma}_1 = \{x_0, x_0 | T\}$ for any s and T . The integration of α along $n\tilde{\gamma}_1$ has the following estimates:

$$\int_{n\tilde{\gamma}_1} \alpha \geq n \left(\left[\frac{T}{T_0} \right] \rho - \text{osc}_{B_s} \beta_{x_0} \right) \geq n \left(\left[\frac{T}{T_0} \right] \rho - 2M_\alpha \right)$$

Therefore if we choose $T \geq \frac{2M_\alpha \cdot T_0}{\rho} + 1$ and n large enough, then (3.4.7) is true.

- (2) If $x_0 \cdot t_0 \in \cup_{i=1}^n B_r(p_i)$ for some t_0

In this case we can replace x_0 by $x_0 \cdot t_0$, since $x_0 \cdot t_0$ is also a point in the chain

recurrent set of the flow v . So we assume that $x_0 \in B_r(p_{i_0})$. Using the same argument as in (1), there exists an $s_0 > 0$ depending on x_0 such that for any $s < s_0$, there is a $(\mathcal{U}(s), T)$ -chain $\tilde{\gamma} = \{x_0, \dots, x_k = x_0 | t_0, \dots, t_{k-1}\}$ satisfying $(x_j \cdot t_j, x_{j+1}) \in B_s(m_{j+1}) \subset B_r(p_{i_{j+1}})$ for $j = 0, 1, \dots, k-1$. Thus we have

$$\int_{n\tilde{\gamma}} \alpha \geq n(k\rho - \lambda k\rho)$$

If n is large enough, the integration will be larger than any given number. So (3.4.7) holds. \blacksquare

In view of Theorem 3.4.1-3.4.3, the following corollary is obvious.

Corollary 3.4.4 *Let v be an α -flow on the compact metric space (X, d) . If for any chain $\tilde{\gamma}$,*

$$\left| \int_{\tilde{\gamma}} \alpha \right| \leq M$$

for some $M > 0$, then v is a gradient-like flow.

Example 3.4.10 Theorem 3.4.2 and 3.4.3 give a sufficient and necessary condition (3.4.5) for an α -flow to be a non gradient-like flow on X . The following simple example shows that an α -flow with α being a nontrivial cocycle can be a gradient-like flow if it does not satisfy the condition (3.4.5).

Let S^1 be the unit circle with the standard metric. Choose $r > 0$ small enough and consider the following four open sets

$$\begin{aligned} U_1 &= \left\{ \theta \mid \left| \theta - \frac{\pi}{2} \right| < 2r \right\} \\ U_2 &= \left\{ \theta \mid \left| \theta - \frac{3\pi}{2} \right| < 2r \right\} \\ U_3 &= \left\{ \theta \mid \frac{\pi}{2} + r < \theta < \frac{3\pi}{2} - r \right\} \\ U_4 &= \left\{ \theta \mid -\frac{\pi}{2} + r < \theta < \frac{\pi}{2} - r \right\} \end{aligned}$$

Then $\cup_{i=1}^4 U_i$ is a covering of S^1 . Define bounded functions $\beta_i (i = 1, 2, 3, 4)$ on U_i as follows

$$\beta_i(x) = 0, \text{ if } i = 1, 2$$

and

$$\beta_3(e^{i\theta}) = \begin{cases} 0 & \text{if } \frac{\pi}{2} + r < \theta < \pi \\ 1 & \text{if } \pi \leq \theta < \frac{3\pi}{2} - r \end{cases}$$

$$\beta_4(e^{i\theta}) = \begin{cases} 2 & \text{if } -\frac{\pi}{2} + r < \theta \leq 0 \\ 0 & \text{if } 0 < \theta < \frac{\pi}{2} - r \end{cases}$$

Consequently, we can define a cocycle $\alpha(x, y) : S^1 \times S^1 \rightarrow \mathbb{R}$ to be

$$\alpha(e^{i\theta_1}, e^{i\theta_2}) = \begin{cases} \beta_i(e^{i\theta_2}) - \beta_i(e^{i\theta_1}) & \text{if } |\theta_1 - \theta_2| < r, \text{ for } i = 1, 2, 3, 4 \\ 0 & \text{if } |\theta_1 - \theta_2| \geq r \end{cases}$$

This cocycle is a nontrivial cocycle. Since if we let γ be the oriented curve starting from the point $\theta = \frac{\pi}{2}$ and then going around the circle in the clockwise direction, then for any integer l , we have

$$\int_{l\gamma} \alpha = l.$$

Now we consider the gradient-like flow v on S^1 , that has two fixed points at $\theta = \pm\frac{\pi}{2}$, and flows from the point $\theta = \frac{\pi}{2}$ to the point $\theta = -\frac{\pi}{2}$. Then it is easy to check that v carries the cocycle α with the parameter $\lambda = 0, \rho = 1$ and $\Gamma_2(r) \cup \Gamma_3(r) = \emptyset$ in the definition 3.4.1.

Example 3.4.11 Let v be a flow on S^1 which has three fixed points at $\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$, and the forward direction of the flow is the anticlockwise direction. It is easy to see that v carries a nontrivial cocycle and is a non gradient-like flow.

Example 3.4.12 Let v be the flow in Example 3.4.2, i.e., generated by a closed Morse 1-form ω . If $[\omega]$ is a nontrivial cohomology class, then v is a non-gradient-like flow.

This result can be seen as follows. We can lift ω to a covering space \bar{X} such that the pull-back form by the projection map is an exact form, say, $d\bar{f}$. Then the lifting flow \bar{v} is the gradient flow of \bar{f} on any cobordism $\bar{f}^{-1}[a, b]$ for $-\infty < a < b < \infty$. Since \bar{f} is strictly decreasing along the nonconstant trajectory of $-\bar{v}$, there does not exist a global attractor in the interior of $\bar{f}^{-1}[a, b]$. Therefore, there is always a trajectory or a chain consisting of orbits joining fixed points such that the start point is at the level set $\bar{f}^{-1}(b)$ but the end point is at $\bar{f}^{-1}(a)$. Since $|a|, |b|$ can be arbitrarily large, hence by Theorem 3.4.2, v on the underlying manifold X is not a gradient-like flow.

Remark However, for general α -flows, things would be complicated, since in the definition of the α -flow, we use a “global” integration. Hence this allows some “fluctuation” to occur in the flow

3.5 π -Morse decomposition of α -flows

From proposition 3.1.1, we know that there is a group embedding $J : C(X)/R(X) \rightarrow \mathcal{H}^1(X; \mathbb{R})$ whose image spans $\mathcal{H}^1(X; \mathbb{R})$. Let $\{f_i\}_{i=1}^m$ be functions in $C(X)$ such that the image $\alpha_i = J(f_i)$ ($i = 1, 2, \dots, m$) is the cocycle representing the rank 1, indivisible integral cohomology class $[\alpha_i]$, while making

$\{[\alpha_i]\}_{i=1}^m$ linearly independent. Let $[l_i]$ be the integral homology class dual to $[\alpha_i]$ for $i = 1, \dots, m$, i.e.,

$$\int_{[l_j]} [\alpha_i] = \int_{l_j} \alpha_i = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Let π be a subgroup of the fundamental group of X spanned by $\{l_1, \dots, l_s\}$. Since X is a compact polyhedron, there is a covering space \bar{X} with deck transformation group π . Let $P_\pi : \bar{X} \rightarrow X$ be the projection map. Take a function $f_i : X \rightarrow S^1$ for $1 \leq i \leq s$. There is a function $\hat{f}_i := f_i \circ P_\pi : \bar{X} \rightarrow S^1$. Since the pair

$$\begin{aligned} \langle (f_i)_* \cdot (P_\pi)_* \cdot \pi_1(\bar{X}, \bar{x}_0), d\theta \rangle &= \langle (P_\pi)_* \pi_1(\bar{X}, \bar{x}_0), (f_i)^*(d\theta) \rangle \\ &= 2\pi \langle (P_\pi)_* \pi_1(\bar{X}, \bar{x}_0), \alpha_i \rangle = 0, \end{aligned}$$

for $1 \leq i \leq s$, each \hat{f}_i has a lifting map $\bar{f}_i : \bar{X} \rightarrow \mathbb{R}$ such that $\hat{f}_i(\bar{x}) = \exp 2\pi i \bar{f}_i(\bar{x})$.

Assume that 1 is a regular value of f_i for $1 \leq i \leq s$. Define $\bar{N}_i(k) = \bar{f}_i^{-1}(k)$, $\bar{W}_i(k) = \bar{f}_i^{-1}([k, k+1])$ and $N_i = f_i^{-1}(1)$, then $P_\pi^{-1}(N_i) = \cup_{k=-\infty}^{\infty} \bar{N}_i(k)$ and $l_i \cdot \bar{N}_i(0) = \bar{N}_i(1)$. Define $\bar{X}_0 = \{\bar{x} \in \bar{X}; 0 \leq \bar{f}_i(\bar{x}) \leq 1, 1 \leq i \leq s\}$, then \bar{X}_0 is a fundamental domain in \bar{X} . Let $\bar{N}_{i0}(1) := \bar{X}_0 \cap \bar{N}_i(1)$ and let $\bar{N}_{i0}(0) := \bar{X}_0 \cap \bar{N}_i(0)$. Then the boundary of \bar{X}_0 is $\cup_{i=1}^s (\bar{N}_{i0}(0) \cup \bar{N}_{i0}(1))$. The functions $\{f_i; 1 \leq i \leq s\}$ divide the boundary of \bar{X}_0 into two parts:

$$\bar{N}_0^+ = \cup_{i=1}^s \bar{N}_{i0}(1); \bar{N}_0^- = \cup_{i=1}^s \bar{N}_{i0}(0)$$

Definition 3.5.1 (π -Morse decomposition) *Let v be a flow in a compact polyhedron with metric d . Let \bar{v} be the lifting flow of v in the covering space \bar{X} which has a deck transformation group π as above. If v satisfies the following conditions:*

(1) *The nonwandering set \bar{A} of \bar{v} has finitely many connected components $\{\bar{A}_n, \dots, \bar{A}_1\}$ in the fundamental domain \bar{X}_0 .*

(2) *$\{\bar{N}_0^+, \bar{A}_n, \dots, \bar{A}_1\}$ is a relative Morse decomposition of \bar{v} in \bar{X}_0 , where “relative” means that $\forall \bar{x} \in \bar{X}_0$, $\omega(\bar{x}) \cup \omega^*(\bar{x}) \not\subset \cup_{i=1}^n \bar{A}_i$, the trajectory $\bar{x} \cdot t$ goes through $\bar{N}_0^+(\bar{N}_0^-)$ at forward (backward) time and $\omega(\bar{x}) \cup \omega^*(\bar{x}) \not\subset \bar{X}_0$.*

Then we say that v has a π -Morse decomposition.

Lemma 3.5.1 *Let v be a flow in a compact space X and let $P_\pi : \bar{X} \rightarrow X$ be the covering projection mentioned above. If \bar{A} is a nonwandering set in \bar{X} , then $P_\pi(\bar{A})$ is also a nonwandering set in X .*

Proof Let \bar{A} be a nonwandering set in \bar{X} and let $\bar{x} \in \bar{A}$. Take an open neighborhood \bar{U} of \bar{x} such that $P_\pi : \bar{U} \rightarrow P_\pi(\bar{U})$ is a local homeomorphism. Since $\bar{x} \in \omega(\bar{U})$, there is a sequence $\bar{x}_i \in \bar{U}$ and $t_i \in \mathbb{R}, t_i \rightarrow \infty$, such that for

any smaller neighborhood \bar{V} of \bar{x} , when i becomes large enough, then $\bar{x}_i \cdot t_i \in V$. Fix \bar{x}_i and project the integral curve $\bar{x}_i \cdot t$ to X , then we get an integral curve $P_\pi(\bar{x}_i \cdot t) = P_\pi(\bar{x}_i) \cdot t$ starting at $P_\pi(\bar{x}_i) \in P_\pi(\bar{U})$ ending at $P_\pi(\bar{x}_i \cdot t) \in P_\pi(V)$. Therefore $P(\bar{x})$ is a nonwandering point in X , which shows that $P_\pi(\bar{A})$ is a nonwandering set in X . \blacksquare

By lemma 3.5.1, the set $P_\pi(\bar{A}_i)(i = 1, \dots, n)$ is a nonwandering set of v in X , and is called a π -**stable nonwandering set** of v . Denote the set $\{P_\pi(\bar{A}_n), \dots, P_\pi(\bar{A}_1)\}$ by $\mathcal{N}_\pi(v)$.

Now we let v be an α -flow on X and the cocycle α is nontrivial with rank s . Without loss of generality we can assume that the cohomology class $[\alpha]$ has the following representation:

$$[\alpha] = \sum_{i=1}^s \lambda_i [\alpha_i], \quad \lambda_i > 0 \quad (3.5.1)$$

for $i = 1, \dots, s$. (3.5.1) implies that there is a bounded function $\beta : X \rightarrow \mathbb{R}$ such that

$$\alpha - \sum_{i=1}^s \lambda_i \alpha_i \simeq \delta \beta \quad (3.5.2)$$

We still use π to represent the subgroup of the fundamental group of X spanned by $\{l_1, \dots, l_s\}$.

Theorem 3.5.2 *Let v be an α -flow with α being a nontrivial cocycle having rank s on a compact polyhedron X with metric d . Then v has a π -Morse decomposition with rank $\pi = s$ and the π -stable nonwandering set is $\mathcal{N}_\pi(v) = \{p_i\}_{i=1}^n$.*

Proof As before, we denote the lifting flow of v in \bar{X} by \bar{v} and denote the lifting trajectory of any γ by $\bar{\gamma}$. To show that v has a π -Morse decomposition, we should prove the following facts:

(1) There is no oriented cycle in \bar{X}_0 .

(2) For any $\bar{x} \in \bar{X}_0$, either $\bar{x} \cdot t$ goes through $\bar{N}_0^+(\bar{N}_0^-)$ and then out of \bar{X}_0 forever at forward(backward) time, or $\omega(x) \cup \omega^*(x)$ is contained in the union of the fixed points in \bar{X}_0 .

Since an oriented cycle in \bar{X}_0 when projected down to X will be an oriented cycle in X , therefore for the proof of (1), we will show that any oriented cycle in X can be “untied” when lifting to the covering space \bar{X} , i.e., the lifting trajectories go from the infinitely far place to the infinitely far place.

Let γ_c be an oriented cycle consisting of fixed points $p_{i_0}, \dots, p_{i_k} = p_{i_0}$ and trajectories $[p_{i_0}, p_{i_1}], \dots, [p_{i_{k-1}}, p_{i_k}]$. Consider the integration of α along γ_c . Ap-

plying the definition that v is an α -flow, we have

$$\int_{\gamma_c} \alpha \geq \sum_{j=0}^{k-1} \int_{[p_{i_j}, p_{i_{j+1}}]_r} \alpha - \sum_{j=0}^{k-1} \text{osc}_{x \in B_r(p_{i_j})} \beta_{p_{i_j}}(x) \geq k\rho - \lambda k\rho > 0$$

where $[p_{i_j}, p_{i_{j+1}}]_r \in \Gamma_1(r)$ is a sub-trajectory of $[p_{i_j}, p_{i_{j+1}}]$.

Notice that

$$\sum_{i=1}^s \lambda_i \int_{\gamma_c} \alpha_i = \int_{\gamma_c} \alpha > 0$$

Hence there is an index $1 \leq i \leq s$ such that

$$\int_{\gamma_c} \alpha_i \geq 1$$

Here we use the fact that α_i is an integral cocycle. Let $\bar{\gamma}_c$ be the lifting curve of γ_c on \bar{X} . It is easy to see that the following holds:

$$\int_{\bar{\gamma}_c \cap \bar{W}_i(0)} \delta \bar{f}_i = \frac{1}{2\pi} \Delta_{\gamma_c} f_i = \int_{\gamma_c} \alpha_i \geq 1$$

and furthermore, for any integer $m > 0$,

$$\int_{\bar{\gamma}_c \cap (\cup_{k=0}^{m-1} \bar{W}_i(k))} \delta \bar{f}_i \geq m$$

This shows $\bar{\gamma}_c$ goes from the infinitely far place to infinitely far place. Therefore (1) is proved.

To prove (2), we only need to prove for any $x \in X$ with its limit set $\omega(x)$ ($\omega^*(x)$) not in the union of all the fixed points of v , that the lifting flow $\bar{x} \cdot t$ will pass through \bar{N}_0^+ (\bar{N}_0^-) then go to infinity at forward (backward) time.

Assume that $\omega(x) \notin \cup_{i=1}^n \{p_i\}$. There are two cases which may occur for the trajectory $x \cdot t$ ($t > 0$).

Case 1. $x \cdot t$ meets infinitely many balls $\{B_r(p_{i_0}), B_r(p_{i_1}), \dots, B_r(p_{i_l}), \dots\}$. Then the integration of α is

$$\int_{[x, x \cdot t]} \alpha \geq \sum_{k=0}^{l(t)-1} \int_{[x, x \cdot t]_k} \alpha - \sum_{k=0}^{l(t)-1} \text{osc}_{x \in B_r(p_{i_k})} \beta_{p_{i_k}}(x)$$

where $l(t)$ is an integer representing the number of balls that $[x, x \cdot t]$ has met and $[x, x \cdot t]_k \in \Gamma_1(r)$ is a trajectory of $[x, x \cdot t]$ between $B_r(p_{i_k})$ and $B_r(p_{i_{k+1}})$.

We have the estimate

$$\int_{[x, x \cdot t]} \alpha \geq l(t)(1 - \lambda)\rho \tag{3.5.3}$$

Since $\alpha \simeq \sum_{i=1}^s \lambda_i \alpha_i + \delta \beta$ and $\beta : X \rightarrow \mathbb{R}$ is a bounded function (let its bound be M_β), then

$$\begin{aligned} \sum_{i=1}^s \lambda_i \int_{[x, x \cdot t]} \alpha_i &\geq l(t)(1 - \lambda)\rho - |\beta(x \cdot t) - \beta(x)| \\ &\geq l(t)(1 - \lambda)\rho - 2M_\beta \end{aligned} \quad (3.5.4)$$

If t is large enough such that $l(t) > \frac{2M_\beta + C}{(1 - \lambda)\rho}$, then we have

$$\sum_{i=1}^s \lambda_i \int_{[x, x \cdot t]} \alpha_i > C$$

for arbitrary large C . Now we can use the same argument as in (1) to show that the trajectory $[x, x \cdot t]$ must pass through \bar{N}_0^+ and then travel to the infinitely far place of \bar{X} .

Case 2. After meeting finitely many balls, $x \cdot t$ travels in the region $X \setminus \cup_{i=1}^n B_r(p_i)$.

In this case, we have (assume that $[x, x \cdot t]$ meets no balls).

$$\int_{[x, x \cdot t]} \alpha \geq \left[\frac{t}{T_0} \right] \rho$$

In the same way as in case 1, we can prove that $[x, x \cdot t]$ will pass through \bar{N}_0^+ and then go to infinity.

We can also discuss the case that $\omega^*(x) \notin \cup_{j=1}^n \{p_j\}$. In that case the trajectory $x \cdot t$ will pass through \bar{N}_0^- and then go to infinity as $t \rightarrow -\infty$.

Now we have proved (2), and the theorem is proved. \blacksquare

Using the same argument as in the proof of Theorem 3.5.2, we get

Theorem 3.5.3 *Let v be a generalized α -flow with respect to the nonwandering set $A = \{A_n, \dots, A_1\}$ on a compact polyhedron X with metric d . If α is a nontrivial cocycle having rank s , then v has a π -Morse decomposition with respect to A with rank $\pi = s$ and the π -stable nonwandering set is $\mathcal{N}_\pi(v) = A$.*

4 Novikov-Morse Theory

4.1 Deformation of complexes relative to α -flows

In this section, we let X be an m -dimensional compact polyhedron. Let v be an α -flow on X , where α is a nontrivial cocycle.

By Theorem 3.5.2, the flow v has a π -Morse decomposition, where π is a subgroup of the fundamental group of X determined by the cocycle α . Assume that

π is spanned by $G_+ := \{l_1, \dots, l_s\}$, where s is the rank of α . Let \bar{X} be the covering space of X with deck transformation group π . As in section 3.5, the boundary of the fundamental domain \bar{X}_0 , $\partial\bar{X}_0$ consists of two parts, $\bar{N}_0^+ = \cup_{i=1}^s \bar{N}_{i0}(1)$, and $\bar{N}_0^- = \cup_{i=1}^s \bar{N}_{i0}(0)$ and satisfying for $i = 1, 2, \dots, s$,

$$l_i \cdot \bar{N}_{i0}(0) = \bar{N}_{i0}(1)$$

Define π_+ to be the monoid constructed by (G_+, e) with the group action from π .

Now the triangulation of X provides a triangulation of \bar{X}_0 such that $\partial\bar{X}_0$ is an $m - 1$ -dimensional subcomplex. Define R to be the subcomplex formed by those $m - 1$ -cells in \bar{N}_0^- . For any p -dimensional cell e in R , we can define a map κ from R to G_+ such that $\kappa(e)$ is a replacement map sending e to $\kappa(e) \cdot e \in \bar{N}_0^+$.

Now we fix the triangulation of $\partial\bar{X}_0$ and modify the inner simplices in \bar{X}_0 . In detail, we will replace those cells supported on the boundary by cylindrical cells.

Note that $\partial\bar{X}_0$ is a simplicial complex, we can retract $\partial\bar{X}_0$ a small distance ε into the interior (if we choose a metric). ε is so small such that each cell in $\partial\bar{X}_0$ does not degenerate during the movement. In this way, we actually get a CW-decomposition of \bar{X}_0 with the cells around the boundary being cylindrical cells. Denote the $m - 1$ -dimensional subcomplex obtained by retracting \bar{N}_0^+ by distance ε by R_I .

Lifting \bar{X}_0 to the universal covering \tilde{X} , we can get a fundamental domain \tilde{X}_0 in \tilde{X} . At this point, we forget all the information about the flow.

Now the lifted part of the boundary $\partial\bar{X}_0$ forms some part of the boundary $\partial\tilde{X}_0$. Hence the elements of G_+ are still replacement maps. We can view $\kappa(e)$ for $e \in R$ as a replacement map acting on \tilde{R} , the lifting of R .

Define a set \tilde{N}_0 in \tilde{X}_0 to be the space getting rid of all the cylindrical cells supported on R_I having the form: $e \times \varepsilon I, I = (0, 1)$. Lifting \bar{N}_0 to \tilde{X}_0 , we get \tilde{N}_0 . Let $i_- : R \rightarrow \tilde{R} \cap \partial\tilde{X}_0$ and $i_+ : R \rightarrow \tilde{R}_I \cap \partial\tilde{N}_0$ be two injections. Then i_{\pm} are injections from R to \tilde{N}_0 . Define f_q to be a map from $C_q(R)$ to $C_q(\tilde{N}_0)$:

$$f_q = \kappa \cdot (i_-)_*(\cdot) - (i_+)_*(\cdot)$$

where $\kappa \cdot (i_-)_*(\cdot)$ is a \mathbb{Z} -linear extension of the map defined on the basis as follows: $\forall e \in C_q(R)$,

$$\kappa \cdot (i_-)_*(e) = \kappa(e) \cdot (i_-)_*(e)$$

Here $\kappa(e)$ is the element in G_+ .

$\kappa \cdot (i_-)_*(\cdot)$ also commutes with the differential d . Take a simplex $e \in C_q(R)$

and let $d_R e = \sum_{e'} e'$, then we have

$$\begin{aligned} \kappa \cdot (i_-)_*(d_R e) &= \sum_{e'} \kappa \cdot (i_-)_*(e') = \sum_{e'} \kappa(e') \cdot (i_-)_*(e') \\ &= \kappa(e) \cdot \left(\sum_{e'} (i_-)_*(e') \right) = \kappa(e) \cdot d_{\tilde{N}_0} (i_-)_*(e) \\ &= d_{\tilde{N}_0} (\kappa \cdot (i_-)_*(e)) \end{aligned}$$

Therefore $f = \{f_q = \kappa \cdot (i_-)_* - i_+\}$ is a chain map from $C_*(R)$ to $C_*(\tilde{N}_0)$.

Consequently, we can construct the algebraic mapping cone $(\text{con}(f)_*, d_c)$, whose q -th term and differential are

$$\text{con}(f)_q = C_{q-1}(R) \oplus C_q(\tilde{N}_0)$$

$$d_c = \begin{pmatrix} -d_R & 0 \\ f & d_{\tilde{N}_0} \end{pmatrix}$$

Now our aim is to construct the chain equivalence between $(\text{con}(f)_*, d_c)$ and $(C_*(\tilde{X}_0), d_{\tilde{X}_0})$.

Since the basis of the \mathbb{Z} -module chain complex $C_*(\tilde{X}_0)$ contains two kinds of cells, the cylindrical cells supported on $i_+(R)$ and the other cells in \tilde{N}_0 . Each element $\beta \in C_q(\tilde{X}_0)$ can be represented linearly and uniquely in the following form:

$$\beta = (-1)^{q-1} (i_+)_*(e) \times \varepsilon I + s$$

where $e \in C_{q-1}(R)$, $I = (0, 1)$ and $s \in C_q(\tilde{N}_0)$.

Define $\theta_q : \text{con}(f)_q \rightarrow C_q(\tilde{X}_0)$ as

$$\theta_q(e \oplus s) = (-1)^{q-1} (i_+)_*(e) \times \varepsilon I + s = \beta$$

and define $\eta_q : C_q(\tilde{X}_0) \rightarrow \text{con}(f)_q$ as

$$\eta_q(\beta) = e \oplus s$$

Clearly, $\theta = \{\theta_q\}$ and $\eta = \{\eta_q\}$ are chain maps satisfying $\theta \cdot \eta = \text{Id}_{\tilde{X}_0}$ and $\eta \cdot \theta = \text{Id}_c$

Proposition 4.1.1 *Let $(\text{con}(f)_*, d_c)$, θ and η be defined as above, then the following diagram commutes and θ defines a chain equivalence between $(\text{con}(f)_*, d_c)$ and $(C_*(\tilde{X}_0), d_{\tilde{X}_0})$.*

$$\begin{array}{ccc} \text{con}(f)_q & \xrightarrow{d_c} & \text{con}(f)_{q-1} \\ \downarrow \theta_q & & \theta_{q-1} \downarrow \\ C_q(\tilde{X}_0) & \xrightarrow{d_{\tilde{X}_0}} & C_{q-1}(\tilde{X}_0) \end{array}$$

Proof Let $e \oplus s \in C_{q-1}(R) \oplus C_q(\tilde{N}_0)$, then

$$\begin{aligned} & \theta_{q-1} \cdot d_c(e \oplus s) \\ &= \theta_{q-1}((-d_R e) \oplus (\kappa(e)(i_-)_*(e) - (i_+)_*(e) + d_{\tilde{N}_0} s)) \\ &= -(-1)^{q-2}(i_+)_*(d_R e) \times \varepsilon I + \kappa(e)(i_-)_*(e) - (i_+)_*(e) + d_{\tilde{N}_0} s \end{aligned}$$

On the other hand,

$$\begin{aligned} & d_{\tilde{X}_0} \cdot \theta_q(e \oplus s) \\ &= d_{\tilde{X}_0}((-1)^{q-1}(i_+)_*(e) \times \varepsilon I + s) \\ &= (-1)^{q-1}(i_+)_*(d_R e) \times \varepsilon I + (i_+)_*(e) \times \varepsilon - (i_+)_*(e) \times 0 + d_{\tilde{N}_0} s \\ &= (-1)^{q-1}(i_+)_*(d_R e) \times \varepsilon I + \kappa(e) \cdot (i_-)_*(e) - (i_+)_*(e) + d_{\tilde{N}_0} s \end{aligned}$$

Hence the diagram is commutative, and the claim is true. \blacksquare

Let $\rho_e : \pi_1(X) \rightarrow \pi$ be the extension of π by the normal group $\pi_1(\tilde{X})$. Since π_+ is a monoid in π , the set $\rho_e^{-1}(\pi_+)$ is also a monoid of $\pi_1(X)$. We denote $\rho_e^{-1}(\pi_+)$ by $\pi_1(X)_+$. $\mathbb{Z}\pi_1(X)_+$ is a subring of $\mathbb{Z}\pi_1(X)$.

Now we can tensor $C_*(R), C_*(\tilde{X}_0)$ and $\text{con}(f)_*$ with the ring $\mathbb{Z}\pi_1(X)_+$, then we have $\mathbb{Z}\pi_1(X)_+$ -module chain complexes $(\mathbb{Z}\pi_1(X)_+ \otimes C_*(R), I \otimes d_R), (\mathbb{Z}\pi_1(X)_+ \otimes C_*(\tilde{X}_0), I \otimes d_{\tilde{X}_0})$, and $\mathbb{Z}\pi_1(X)_+ \otimes \text{con}(f)_*, I \otimes d_c$, where

$$I \otimes d_c = \begin{pmatrix} -I \otimes d_R & 0 \\ I \otimes f & I \otimes d_{\tilde{N}_0} \end{pmatrix}$$

and $I \otimes f = I \otimes (\kappa \cdot (i_-)_* - (i_+)_*)$ is a $\mathbb{Z}\pi_1(X)_+$ -chain map from $\mathbb{Z}\pi_1(X)_+ \otimes C_*(R)$ to $\mathbb{Z}\pi_1(X)_+ \otimes C_*(\tilde{N}_0)$.

Let $\tilde{R}_+ = \cup_{g \in \pi_1(X)_+} g \cdot i_-(R)$, then $C_*(\tilde{R}_+)$ becomes a $\mathbb{Z}\pi_1(X)_+$ -module chain complex. Let $\tilde{X}_+ = \cup_{g \in \pi_1(X)_+} g \cdot \tilde{X}_0$, then $C_*(\tilde{X}_+)$ is also a $\mathbb{Z}\pi_1(X)_+$ -module chain complex. Define a chain equivalence $s_* : \mathbb{Z}\pi_1(X)_+ \otimes C_*(\tilde{X}_0) \rightarrow C_*(\tilde{X}_+)$ as follows,

for $g \otimes e \in \mathbb{Z}\pi_1(X)_+ \otimes C_*(\tilde{X}_0)$,

$$s_*(g \otimes e) = g \cdot e$$

Extending the maps $(i_{\pm})_*, f$ equivariantly to $(\tilde{i}_{\pm})_*, \tilde{f}$, acting on $C_*(\tilde{R}_+)$, e.g., let $g \in \pi_1(X)_+, e \in C_*(R)$, then $g \cdot e \in C_*(\tilde{R}_+)$ and $\tilde{f}(g \cdot e) = g \cdot \tilde{f}(e) = g \cdot f(e) = g \cdot [\kappa \cdot (i_-)_*(e) - (i_+)_*(e)]$.

The following diagram is commutative

$$\begin{array}{ccc} \mathbb{Z}\pi_1(X)_+ \otimes C_*(R) & \xrightarrow{I \otimes f} & \mathbb{Z}\pi_1(X)_+ \otimes C_*(\tilde{N}_0) \\ \downarrow s_* & & \downarrow s_* \\ C_*(\tilde{R}) & \xrightarrow{\tilde{f}} & C_*(\tilde{N}_+) \end{array} \quad (4.1.1)$$

where $\tilde{N}_+ = \cup_{g \in \pi_1(X)_+} g \cdot \tilde{N}_0$.

Consequently, we have the following commutative diagram with exact rows and with the vertical chain maps s_* being $\mathbb{Z}\pi_1(X)_+$ -isomorphic.

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathbb{Z}\pi_1(X)_+ \otimes C_{*-1}(R) & \longrightarrow & \mathbb{Z}\pi_1(X)_+ \otimes \text{con}(f)_* & \longrightarrow & \mathbb{Z}\pi_1(X)_+ \otimes C_*(\tilde{N}_0) \longrightarrow 0 \\
& & \downarrow s_* & & \downarrow ss_* & & \downarrow s_* \\
0 & \longrightarrow & C_{*-1}(\tilde{R}_+) & \longrightarrow & \text{con}(\tilde{f})_* & \longrightarrow & C_*(\tilde{N}_+) \longrightarrow 0
\end{array}$$

where $\text{con}(\tilde{f})_* = C_{*-1}(\tilde{R}_+) \oplus C_*(\tilde{N}_+)$ is a $\mathbb{Z}\pi_1(X)_+$ -module chain complex with differential,

$$\tilde{d}_c = \begin{pmatrix} -d_{\tilde{R}_+} & 0 \\ \tilde{f} & d_{\tilde{N}_+} \end{pmatrix}$$

Here the fact that $ss_* := s_* \oplus s_*$ is a chain map is justified by the commutative diagram (4.1.1).

From the Five-Lemma for complexes, ss_* is a chain equivalence. Thus we get the following corollary from proposition 4.1.1.

Corollary 4.1.2 *The two $\mathbb{Z}\pi_1(X)_+$ -module chain complexes $(C_*(\tilde{X}_+), d_{\tilde{X}_+})$ and $(\text{con}(\tilde{f})_*, \tilde{d}_c)$ are chain equivalent.*

Similarly, if we extend the maps $\kappa \cdot (i_-)_*$, $(i_+)_*$ and f to $\mathbb{Z}\pi_1(X)$ -equivariant maps from $C_*(\tilde{R})$ to $C_*(\tilde{N})$, we have

Corollary 4.1.3 *The two $\mathbb{Z}\pi_1(X)$ -module chain complexes $(C_*(\tilde{X}), \tilde{d}_{\tilde{X}})$ and $(\text{con}(\tilde{f})_*, \tilde{d}_c)$ are chain equivalent. Here the q -th term and differential of the mapping cone $(\text{con}(\tilde{f})_*, \tilde{d}_c)$ is*

$$\text{con}(\tilde{f})_q = C_{q-1}(\tilde{R}) \oplus C_q(\tilde{N})$$

$$\tilde{d}_c = \begin{pmatrix} -d_{\tilde{R}} & 0 \\ \tilde{f} & d_{\tilde{N}} \end{pmatrix}$$

where $\tilde{f} = \kappa \cdot \widetilde{(i_-)_*} - \widetilde{(i_+)_*}$.

4.2 Monodromy representations and Novikov numbers

In this section, we will construct the Novikov complex associated with a cycle α and establish some properties of this Novikov complex.

Abelization of fundamental groups

Recall the group π is relative to the α -flow. Take the generator set G_+ of π , $G_+ = \{l_1, \dots, l_s\}$. There is a natural homomorphism

$$\rho_A : \pi \longrightarrow \mathbb{Z}^s, \quad s = \text{rank} \pi$$

which sends l_i to $(0, \dots, 0, e_i, 0, \dots, 0)$, where e_i is the unit 1.

Combining ρ_A with the extension homomorphism, we get a group homomorphism:

$$\rho_{\pi_1} = \rho_A \cdot \rho_e : \pi_1(X) \longrightarrow \mathbb{Z}^s$$

This group homomorphism can be extended to a ring homomorphism:

$$\rho_{\pi_1} : \mathbb{Z}[\pi(X)] \longrightarrow \mathbb{Z}[\mathbb{Z}^s]$$

However, there is a natural ring homomorphism ρ_q from $\mathbb{Z}[\mathbb{Z}^s]$ to the Laurent polynomial ring $Q_s = \mathbb{Z}[t_i, t_i^{-1}; i = 1, 2, \dots, s]$. ρ_q is defined as follows. Let $g = \sum z_j E_j$, where $E_j = (a_1, \dots, a_s)$ is an s -dimensional vector with integral entries, then

$$\rho_q(g) = \sum z_j t^{E_j} = \sum z_j t_1^{a_1} \cdots t_s^{a_s}$$

Hence combined with ρ_{π_1} , we get the representation

$$\rho_{\pi_1 q} = \rho_q \cdot \rho_{\pi_1} : \mathbb{Z}[\pi_1(X)] \longrightarrow Q_s$$

In fact, $\rho_{\pi_1 q}$ is fully determined by the group π and its representation. Restricting $\rho_{\pi_1 q}$ to the subring $\mathbb{Z}[\pi_1(X)_+]$, we can get a ring homomorphism

$$\rho_P = \rho_{\pi_1 q}|_{\mathbb{Z}[\pi_1(X)_+]} : \mathbb{Z}[\pi_1(X)_+] \longrightarrow P_s = \mathbb{Z}[t_1, \dots, t_s]$$

Monodromy representations

Let \tilde{E} be a local system of free abelian groups on the compact polyhedron X , then \tilde{E} is determined by its monodromy representation $\rho_{\tilde{E}}$:

$$\rho_{\tilde{E}} : \pi_1(X, x_0) \longrightarrow \text{Aut}(\tilde{E}_0) = GL(k, \mathbb{Z})$$

where \tilde{E}_0 is the fibre of the free abelian group at x_0 and $k = \text{rank}(\tilde{E}_0)$. Let $E = \tilde{E} \otimes \mathbb{C}$, then E is a complex flat vector bundle with the holonomy ρ_E

$$\rho_E : \pi_1(X, x_0) \longrightarrow GL(k; \mathbb{Z}) \otimes \mathbb{C}$$

Now the tensor product of the representations $\rho_q \otimes \rho_{\tilde{E}}$ gives a representation of a $\mathbb{Z}[\pi_1(X)]$ -ring to the linear space $(Q_s)^k$, where Q_s is the polynomial space with s variables over \mathbb{Z} .

Since $\rho_{\tilde{E}}$ is an anti-homomorphism, i.e., $\forall g, g' \in \mathbb{Z}[\pi_1(X)], \rho_E(g \cdot g') = \rho_E(g') \cdot \rho_E(g)$, hence $\rho_P \otimes \rho_{\tilde{E}}$ gives a right $\mathbb{Z}[\pi_1(X)]_+$ -module structure on P_s^k . With the

P_s -module structure of itself, P_s^k becomes a $(P_s, \mathbb{Z}[\pi_1(X)_+])$ -bimodule.

Define $D_* = P_s^k \otimes_{\mathbb{Z}[\pi_1(X)_+]} C_*(\tilde{X}_+)$, then D_* is a P_s -module chain complex.

Evaluation representations

Take any complex s -vector $a = (a_1, \dots, a_s) \in \mathbb{C}^s$. The complex number field \mathbb{C} can be given a P_s -module structure, whose module structure is provided by the action: for a polynomial $P(t_1, \dots, t_s)$, $P(t_1, \dots, t_s) \cdot x = P(a_1, \dots, a_s) \cdot x = P(a) \cdot x$ for $x \in \mathbb{C}$. We denote the P_s -module of \mathbb{C} evaluated at $t = a$ by \mathbb{C}_a . Similarly for any $a \in (\mathbb{C}^*)^s$, \mathbb{C} can be viewed as a Q_s -module. If p is a prime number, then the field \mathbb{Z}_p also has a P_s -module structure which is given by the evaluation at $t = 0$. We consider the complexes $\mathbb{C}_a \otimes_{P_s} D_*$ and $\mathbb{Z}_p \otimes_{P_s} D_*$. The following theorem for $s = 1$ is given in the paper [13]. Here we consider the case $s \geq 1$ and use a different argument, which is based on section 4.1.

Theorem 4.2.1 *Let $D_* = P_s^k \otimes_{\mathbb{Z}[\pi_1(X)_+]} C_*(\tilde{X}_+)$. We have*

(1) *For any nonzero complex vector $a \in (\mathbb{C}^*)^s$, the homology $H_*(\mathbb{C}_a \otimes_{P_s} D_*)$ is isomorphic to $H_*(X, a^\alpha \otimes E)$, which is viewed as the homology of the presheaf $a^\alpha \otimes E$ on X .*

(2) *Let p be a prime number and let \mathbb{Z}_p have the P_s -module structure which is provided by the evaluation at $t = (t_1, \dots, t_s) = 0$. Then the homology $H_*(\mathbb{Z}_p \otimes_{P_s} D_*)$ is isomorphic to $H_*(\bar{X}_0, |R_I| \times \varepsilon \bar{I}; \mathbb{Z}_p \otimes P_\pi^* \tilde{E})$, where \tilde{E} is a local system on X and $P_\pi^* E$ is the pull-back local system on \bar{X}_0 by the projection $P_\pi : \bar{X} \rightarrow X$.*

(3) *$H_*(\mathbb{C}_0 \otimes_{P_s} D_*)$ is isomorphic to $H_*(\bar{X}_0, |R_I| \times \varepsilon \bar{I}; P_\pi^* E)$.*

Proof Since the proof of (3) is the same as that of (2), we only give the proofs of (1) and (2).

Proof of (1)

Since the $\mathbb{Z}[\pi_1(X)_+]$ -basis of $C_*(\tilde{X}_+)$ is finite, all the complexes related to $C_*(\tilde{X}_+)$ are finitely generated, and hence all the homology groups are finitely generated.

For $a \in (\mathbb{C}^*)^s$, we have the isomorphism

$$\begin{aligned} \mathbb{C}_a \otimes_{P_s} D_* &\cong \mathbb{C}_a \otimes_{P_s} ((P_s)^k \otimes_{\mathbb{Z}[\pi_1(X)_+]} C_*(\tilde{X}_+)) \\ &\cong (\mathbb{C}_a \otimes_{Q_s} (Q_s)^k) \otimes_{\mathbb{Z}[\pi_1(X)]} (\mathbb{Z}[\pi_1(X)] \otimes_{\mathbb{Z}[\pi_1(X)_+]} C_*(\tilde{X}_+)) \\ &\cong (\mathbb{C}_a \otimes_{\mathbb{Z}} \mathbb{Z}^k) \otimes_{\mathbb{Z}[\pi_1(X)]} C_*(\tilde{X}) \\ &\cong \mathbb{C}^k \otimes_{\mathbb{Z}[\pi_1(X)]} C_*(\tilde{X}) \end{aligned}$$

Here the representation of $\mathbb{Z}[\pi_1(X)]$ is given by

$$g \longrightarrow \rho_{P(a)}(g) \otimes \rho_E(g)$$

Hence the homology of $\mathbb{C}^k \otimes_{\mathbb{Z}[\pi_1(X)]} C_*(\tilde{X})$ is the same as the homology of the presheaf $a^\alpha \otimes E$ on M that corresponds to the flat vector bundle produced by the above holonomy representation. (1) is proved.

Proof of (2):

$$\mathbb{Z}_p \otimes_{P_s} D^* \cong \mathbb{Z}_p \otimes_{P_s} (P_s^k \otimes_{\mathbb{Z}[\pi_1(X)_+]} C_*(\tilde{X}_+)) \cong \mathbb{Z}_p^k \otimes_{\mathbb{Z}[\pi_1(X)_+]} C_*(\tilde{X}_+)$$

Here the representation of $\mathbb{Z}[\pi_1(X)_+]$ on \mathbb{Z}_p^k is

$$g \longrightarrow \rho_{P(0)}(g) \otimes \rho_E(g) \quad (4.2.1)$$

Since $\rho_{P(0)}(g) = \rho_{P(0)} \cdot \rho_A \rho_e(g)$, except in the case that $g \in \mathbb{Z}[\pi_1(X)_+]$ satisfies $\rho_e(g) = 0 \in \mathbb{Z}^k$, the evaluation representation will make the final representation vanish. Hence (4.2.1) becomes

$$\begin{aligned} g &\longrightarrow \rho_E(g), \text{ if } \rho_e(g) = 0 \\ g &\longrightarrow 0, \text{ if } \rho_e(g) \neq 0 \end{aligned}$$

By corollary 4.1.2, in order to prove (2), we need to prove that $\mathbb{Z}_p^k \otimes_{\mathbb{Z}[\pi_1(X)_+]} \text{con}(\tilde{f})_*$ is equivalent to $(\mathbb{Z}_p \otimes P_\pi^* \tilde{E}) \otimes C_*(\tilde{X}_0, |R_I| \times \varepsilon \bar{I})$. Now we have

$$\mathbb{Z}_p^k \otimes_{\mathbb{Z}[\pi_1(X)_+]} \text{con}(\tilde{f})_q = (\mathbb{Z}_p^k \otimes_{\mathbb{Z}[\pi_1(X)_+]} C_{q-1}(\tilde{R}_+)) \oplus (\mathbb{Z}_p^k \otimes_{\mathbb{Z}[\pi_1(X)_+]} C_q(\tilde{N}_+))$$

$$I \otimes_{\mathbb{Z}[\pi_1(X)_+]} d_c = \begin{pmatrix} -I \otimes_{\mathbb{Z}[\pi_1(X)_+]} \otimes \tilde{d}_{\tilde{R}_+} & 0 \\ I \otimes_{\mathbb{Z}[\pi_1(X)_+]} \tilde{f} & I \otimes_{\mathbb{Z}[\pi_1(X)_+]} \tilde{d}_{\tilde{N}_+} \end{pmatrix}$$

If we take $\tilde{E} \otimes C_*(R)$ as the complexes of R with twisted coefficients \tilde{E} , then $\mathbb{Z}_p^k \otimes_{\mathbb{Z}[\pi_1(X)_+]} C_*(\tilde{R}_+) \cong \mathbb{Z}_p \otimes (\tilde{E} \otimes C_*(R))$. Similarly $\mathbb{Z}_p^k \otimes_{\mathbb{Z}[\pi_1(X)_+]} C_*(\tilde{N}_+) \cong \mathbb{Z}_p \otimes (P_\pi^* \tilde{E} \otimes C_*(\tilde{X}_0, |R_I| \times \varepsilon \bar{I}))$. Therefore

$$\mathbb{Z}_p^k \otimes_{\mathbb{Z}[\pi_1(X)_+]} \text{con}(\tilde{f})_* \cong (\mathbb{Z}_p \otimes (\tilde{E} \otimes C_{*-1}(R))) \oplus (\mathbb{Z}_p \otimes (P_\pi^* \tilde{E} \otimes C_*(\tilde{X}_0, |R_I| \times \varepsilon \bar{I})))$$

and its differential is the same as

$$d_c = \begin{pmatrix} -d_R & 0 \\ -(i_+)_* & d_{\tilde{X}_0} \end{pmatrix}$$

The conclusion of (2) now comes from the following lemma.

Lemma 4.2.2 *Let N be a compact manifold with a piece of boundary denoted by N_+ . Let R be a topological space having a continuous map $i_+ : R \longrightarrow N_+$ which is a homotopy equivalence, then the homology of the mapping cone $\mathcal{C} = \text{con}((i_+)_* : C_*(R) \rightarrow C_*(N))$ is isomorphic with that of $C_*(N, N_+)$.*

Proof We have the short exact sequence of complexes

$$0 \longrightarrow C_*(N_+) \xrightarrow{j} C_*(N) \longrightarrow C_*(N, N_+) \longrightarrow 0$$

Then we have the long exact sequence of homology groups:

$$\cdots \xrightarrow{j_*} H_q(N) \longrightarrow H_q(N, N_+) \longrightarrow H_{q-1}(N_+) \xrightarrow{j_*} H_{q-1}(N) \longrightarrow \cdots$$

Similarly, from the short exact sequence of the mapping cone

$$0 \longrightarrow C_*(N) \longrightarrow \text{con}(i_+)_* \longrightarrow C_{*-1}(R) \longrightarrow 0$$

we get the long exact sequence

$$\begin{aligned} \cdots \longrightarrow H_q(R) \xrightarrow{(i_+)_*} H_q(N) \longrightarrow H_q(\text{con}(i_+)) \longrightarrow H_{q-1}(R) \longrightarrow \\ \longrightarrow H_{q-1}(N) \longrightarrow \cdots \end{aligned}$$

The two long exact sequences yield the following two short exact sequences

$$\begin{aligned} 0 \longrightarrow \text{Coker}(j_* : H_q(N_+) \rightarrow H_q(N)) \longrightarrow H_q(N, N_+) \longrightarrow \\ \longrightarrow \text{Ker}(j_* : H_{q-1}(N_+) \rightarrow H_{q-1}(N)) \longrightarrow 0 \end{aligned}$$

and

$$\begin{aligned} 0 \longrightarrow \text{Coker}((i_+)_* : H_q(R) \rightarrow H_q(N)) \longrightarrow H_q(\text{con}(i_+)_*) \longrightarrow \\ \longrightarrow \text{Ker}((i_+)_* : H_{q-1}(R) \rightarrow H_{q-1}(N)) \longrightarrow 0 \end{aligned}$$

Note that $(i_+)_* : H_q(R) \rightarrow H_q(N)$ is the combination of the isomorphisms $H_q(R) \rightarrow H_q(N_+)$ and $j_* : H_q(N_+) \rightarrow H_q(N)$, and therefore using the Five-Lemma, we get for any $q \geq 0$

$$H_q(N, N_+) \cong H_q(\text{con}(i_+)_*)$$

■

Novikov numbers

In Theorem 4.2.1, we have considered the complex $D_* = P_s^k \otimes_{\mathbb{Z}[\pi_1(X)_+]} C_*(\tilde{X}_+)$.

In this part, we always let the vector bundle E that appears in the previous section be a trivial line bundle. Therefore the complex D_* we consider here has the form $D_* = P_s \otimes_{\mathbb{Z}[\pi_1(X)_+]} C_*(\tilde{X}_+)$. Since the representation of $\mathbb{Z}[\pi_1(X)_+]$ in P_s is completely determined by the cohomology class $[\alpha]$, we denote the homology group $H_*(\mathbb{C}_a \otimes_{P_s} D_*)$ as $H_*(X, a^\alpha)$, or in other words, view the homology group $H_*(\mathbb{C}_a \otimes_{P_s} D_*)$ as the homology group of the presheaf a^α on X which is given by the monodromy representation $\rho_P : \mathbb{Z}[\pi_1(X)_+] \rightarrow P_s = \mathbb{Z}[t_1, \dots, t_s]$.

Definition $b_i([\alpha]) := \text{rank}(H_i(D_*))$ for $i = 0, 1, \dots, m$ are called the Novikov numbers.

The following is essentially given in [20] and the proof here is essentially the same as in the paper [12] except that we consider here $s \geq 1$ and X a compact polyhedron. For the sake of completeness, we give the proof here.

Theorem 4.2.3 *Let X be a compact polyhedron. Define a function for fixed i to be*

$$a \in (\mathbb{C}^*)^s \longrightarrow \dim_{\mathbb{C}} H_i(X, a^\alpha),$$

then it has the following properties:

(1) *It is generically constant, more precisely, except on a proper algebraic subvariety \mathcal{L} in $(\mathbb{C}^*)^s$, the dimension $\dim_{\mathbb{C}} H_i(X, a^\alpha)$ is constant and this constant is just the Novikov number $b_i([\alpha])$ we defined above.*

(2) *For any point $\hat{a} \in \mathcal{L}$,*

$$\dim_{\mathbb{C}} H_i(X, \hat{a}^\alpha) > b_i([\alpha])$$

Proof we denote by $B_i(a)$ the rank of the following linear map

$$I \otimes d : \mathbb{C}_a \otimes_{P_s} D_i \longrightarrow \mathbb{C}_a \otimes_{P_s} D_{i-1}$$

It is easy to see that this map $I \otimes d$ can be represented by a matrix with entries being polynomials with s variables on $(\mathbb{C}^*)^s$. Therefore, we know that except on a proper algebraic variety in $(\mathbb{C}^*)^s$, $B_i(a)$ is constant and on that algebraic variety $B_i(a)$ is smaller than the generic constant.

Consider the following truncated complex,

$$\dots \longrightarrow \mathbb{C}_a \otimes D_i \longrightarrow \mathbb{C}_a \otimes D_{i-1} \longrightarrow 0,$$

and using the Euler-Poincaré formula, we obtain

$$\sum_{j=0}^{\infty} (-1)^j \text{rank}(\mathbb{C}_a \otimes D_{i-1+j}) = \sum_{j=0}^{\infty} (-1)^j \dim_{\mathbb{C}}(H_j(\mathbb{C}_a \otimes D_{i-1+j}))$$

Therefore we obtain

$$\begin{aligned} \dim_{\mathbb{C}} H_i(X; a^\alpha) &= \sum_{j=0}^{\infty} (-1)^j \text{rank}(\mathbb{C}_a \otimes D_{i+j}) \\ &\quad - \sum_{j=1}^{\infty} (-1)^j \text{rank}(H_{i+j}(\mathbb{C}_a \otimes D_*)) - B_i(a) \end{aligned}$$

Now it is easy to see that (1) and (2) are true.

Remark If $s = 1$ and X is a closed manifold, then it was proved in the paper [11] that the Novikov number we defined here is the same as the Novikov number which is defined as the dimension of the homology group of the Novikov complex and which was originally introduced by S.S.Novikov in his papers [18],[19] and [20] (see the introduction).

4.3 Morse type inequalities for a filtration

In this part, we always assume \mathcal{R} to be a commutative Noetherian ring, and C be a f.g. finite \mathcal{R} -module chain complex. Suppose that C has a filtration of finite length: $\{0\} \subset C_{n+1} \subset \cdots \subset C_1 \subset C_0 = C$. We will give a Morse type inequality for C under the above filtration.

First, we consider a triple (τ', τ'', τ) of complexes, satisfying the inclusion relations $\tau' \subset \tau'' \subset \tau$. Then we have a short exact sequence of complexes

$$0 \longrightarrow \tau''/\tau' \longrightarrow \tau/\tau' \longrightarrow \tau/\tau'' \longrightarrow 0 \quad (4.3.1)$$

Define $C(\mu, \tau) = \tau/\mu$ to be the quotient complex of complexes τ and μ . Let $H_*(\mu, \tau) = H_*(C(\mu, \tau))$ be the homology groups of the quotient complex $C(\mu, \tau)$. We get from (4.3.1) the long exact sequence of the homology groups of the triple (τ', τ'', τ) ,

$$\cdots \longrightarrow H_1(\tau'', \tau) \xrightarrow{\delta_1} H_0(\tau', \tau'') \longrightarrow H_0(\tau', \tau) \longrightarrow H_0(\tau'', \tau) \xrightarrow{\delta_0} 0 \quad (4.3.2)$$

where $\delta_j : H_j(\tau'', \tau) \longrightarrow H_{j-1}(\tau', \tau'')$, $j = 0, 1, \cdots$ is the connecting homomorphism.

Introduce some notations as follows

$$\begin{aligned} b_j(\mu, \tau) &= \text{rank}(H_j(\mu, \tau)) \\ d_j(\tau', \tau'', \tau) &= \text{rank}(\text{im} \delta_j) \\ p(\mu, \tau, ; t) &= \sum_{j \geq 0} b_j(\mu, \tau) t^j \\ q(\tau', \tau'', \tau; t) &= \sum_{j \geq 0} d_j(\tau', \tau'', \tau) t^j \end{aligned}$$

Then we have

Lemma 4.3.1 *If (τ', τ'', τ) is the triple of f.g. \mathcal{R} -module chain complexes in (4.3.1), then*

$$p(\tau'', \tau; t) - p(\tau', \tau; t) + p(\tau', \tau''; t) = (1+t)q(\tau', \tau'', \tau; t) \quad (4.3.3)$$

Proof Truncating the long exact sequence of (4.3.2), we obtain the exact sequence for $j = 0, 1, \dots$ (we define $\delta_{-1} = 0$),

$$0 \rightarrow \text{im}(\delta_j) \rightarrow H_j(\tau', \tau'') \rightarrow H_j(\tau', \tau) \rightarrow H_j(\tau'', \tau) \rightarrow \text{im}\delta_{j-1} \rightarrow 0$$

Now it is easy to get the result by using the standard method. ■

For the given filtration of C , we let $(\tau', \tau'', \tau) = (C_{n+1}, C_j, C_{j-1})$, $j = 1, \dots, n$. By lemma 4.3.1, we have for $j = 1, \dots, n$

$$\begin{aligned} p(C_j, C_{j-1}; t) - p(C_{n+1}, C_{j-1}; t) + p(C_{n+1}, C_j; t) \\ = (1+t)q(C_{n+1}, C_j, C_{j-1}; t) \end{aligned}$$

Hence we get

Proposition 4.3.2 *If C is a f.g. finite \mathcal{R} -module chain complex with filtration $\{0\} \subset C_{n+1} \subset C_n \subset \dots \subset C_0 = C$, then we have*

$$\sum_{j=1}^{n+1} p(C_j, C_{j-1}; t) = p(C_{n+1}, C; t) + (1+t) \sum_{j=1}^n q(C_{n+1}, C_j, C_{j-1}; t)$$

Morse type inequalities for a filtration of a G -CW complex

Let G be a topological group acting on a CW-complex X . Suppose that X is a finite G -CW complex (i.e., X/G is a finite CW-complex) with a G -invariant filtration

$$\emptyset \subset X_{n+1} \subset X_n \subset \dots \subset X_0 = X \quad (4.3.4)$$

Remark For a G -CW complex X , there is a natural filtration filtered by its own skeleton, e.g.,

$$\emptyset = X^{-1} \subset X_0 \subset \dots \subset X^p \subset \dots \cup_{p \geq 0} X^p = X$$

The p -th chain group is $C_p(X) = H_p(X^p, X^{p-1})$, and its basis consists of the p -dimensional equivariant cells. The differential $d_p : C_p(X) \rightarrow C_{p-1}(X)$ is the connecting homomorphism of the triple (X^p, X^{p-1}, X^{p-2}) .

Let \mathcal{R} be a commutative Noetherian ring and $\rho : \mathbb{Z}G \rightarrow \text{End}(\mathcal{R})$ be a representation of the group G in \mathcal{R} . Then (4.3.4) induces a filtration of the \mathcal{R} -module chain complex $\mathcal{R} \otimes_{\mathbb{Z}G} C_*(X)$,

$$0 \subset \mathcal{R} \otimes_{\mathbb{Z}G} C_*(X_{n+1}) \subset \cdots \subset \mathcal{R} \otimes_{\mathbb{Z}G} C_*(X_0) = \mathcal{R} \otimes_{\mathbb{Z}G} C_*(X) \quad (4.3.5)$$

Denote $\mathcal{R} \otimes_{\mathbb{Z}G} C_*(X_j)$ by D_j for $j = 1, \dots, n+1$, then (4.3.5) becomes

$$0 \subset D_{n+1} \subset \cdots \subset D_0 = D \quad (4.3.6)$$

Since X is a finite G -CW complex, the \mathcal{R} -module chain complex D is finitely generated. Hence we have

Proposition 4.3.3 *Let \mathcal{R} be a commutative Noetherian ring with a G -representation. Let X be a finite G -CW complex with a G -invariant filtration (4.3.4), then*

$$\sum_{j=1}^{n+1} p(D_j, D_{j-1}; t) = p(D_{n+1}, D; t) + (1+t) \sum_{j=1}^n q(D_{n+1}, D_j, D_{j-1}; t) \quad (4.3.7)$$

where $D_j = \mathcal{R} \otimes_{\mathbb{Z}G} C_*(X_j)$, and $p(\cdot, \cdot; t)$, $q(\cdot, \cdot, \cdot; t)$ are defined as above, e.g.,

$$p(D_j, D_{j-1}; t) = \sum_{l \geq 0} b_l(D_j, D_{j-1}) t^l$$

$$b_l(D_j, D_{j-1}) = \text{rank}(H_l(\mathcal{R} \otimes_{\mathbb{Z}G} C_*(X_j, X_{j-1})))$$

Comparison of Poincaré polynomials for prime ideals

Let \mathcal{R} be the ring mentioned above and C a chain complex over the ring \mathcal{R} . Take a prime ideal \mathcal{P} in \mathcal{R} , and we can define a chain complex $C(\mathcal{P})$ over the quotient field $Q(\mathcal{R}/\mathcal{P})$ of the entire ring \mathcal{R}/\mathcal{P} ,

$$C(\mathcal{P}) := Q(\mathcal{R}/\mathcal{P}) \otimes_{\mathcal{R}} C$$

Define the j -th Betti number of $C(\mathcal{P})$:

$$b_j(C; \mathcal{P}) = b_j(\emptyset, C; \mathcal{P}) = \dim_{Q(\mathcal{R}/\mathcal{P})}(H_j(Q(\mathcal{R}/\mathcal{P}) \otimes_{\mathcal{R}} C))$$

Let \mathcal{Q} be another prime ideal such that $\mathcal{P} \subset \mathcal{Q}$. We can define the torsion number $T_j(\mathcal{P}, \mathcal{Q})$ of $C(\mathcal{P})$ with respect to \mathcal{Q} as:

$$T_j(\mathcal{P}, \mathcal{Q}) = \text{rank}_{\mathcal{P}}(\text{imd}_{j+1}(\mathcal{P})) - \text{rank}_{\mathcal{Q}}(\text{imd}_{j+1}(\mathcal{Q}))$$

where $d_j(\mathcal{P})$ represents the boundary operator in $C(\mathcal{P})$.

In the same way, we define the Poincaré polynomial of $C(\mathcal{P})$ as:

$$P(C; t, \mathcal{P}) := \sum_{j \geq 0} b_j(C, \mathcal{P}) t^j$$

The following result is proved in [13]. However, since we need more information than provided in [13], we will give a different proof below.

Proposition 4.3.4 *Let \mathcal{R} be a commutative Noetherian ring and C a chain complex on \mathcal{R} . If \mathcal{P}, \mathcal{Q} are two prime ideals in \mathcal{R} such that $\mathcal{P} \subset \mathcal{Q}$, then there is a polynomial $Q(\mathcal{P}, \mathcal{Q}; t)$ with nonnegative integer coefficients such that*

$$P(C; t, \mathcal{Q}) = P(C; t, \mathcal{P}) + (1 + t)Q(\mathcal{P}, \mathcal{Q}; t) \quad (4.3.8)$$

where $(1 + t)Q(\mathcal{P}, \mathcal{Q}; t) = \sum_{j \geq 0} (T_j(\mathcal{P}, \mathcal{Q}) + T_{j-1}(\mathcal{P}, \mathcal{Q})) t^j$.

Proof For any $j = 0, 1, \dots$, we have two short exact sequences.

$$\begin{aligned} 0 &\longrightarrow \ker d_j(\mathcal{P}) \longrightarrow C_j(\mathcal{P}) \longrightarrow \operatorname{im} d_j(\mathcal{P}) \longrightarrow 0 \\ 0 &\longrightarrow \operatorname{im} d_{j+1}(\mathcal{P}) \longrightarrow \ker d_j(\mathcal{P}) \longrightarrow H_j(C(\mathcal{P})) \longrightarrow 0 \end{aligned}$$

Therefore

$$\begin{aligned} \operatorname{rank}_{\mathcal{P}} C_j(\mathcal{P}) &= \operatorname{rank}_{\mathcal{P}}(\ker d_j(\mathcal{P})) + \operatorname{rank}_{\mathcal{P}}(\operatorname{im} d_j(\mathcal{P})) \\ &= \operatorname{rank}_{\mathcal{P}}(H_j(C(\mathcal{P}))) + \operatorname{rank}_{\mathcal{P}}(\operatorname{im} d_j(\mathcal{P})) + \operatorname{rank}_{\mathcal{P}}(\operatorname{im} d_{j+1}(\mathcal{P})) \end{aligned}$$

In the same way, for the prime ideal \mathcal{Q} , we have

$$\operatorname{rank}_{\mathcal{Q}} C_j(\mathcal{Q}) = \operatorname{rank}_{\mathcal{Q}}(H_j(C(\mathcal{Q}))) + \operatorname{rank}_{\mathcal{Q}}(\operatorname{im} d_j(\mathcal{Q})) + \operatorname{rank}_{\mathcal{Q}}(\operatorname{im} d_{j+1}(\mathcal{Q})).$$

Since C_j is free module on \mathcal{R} , we have

$$\operatorname{rank}_{\mathcal{P}} C_j(\mathcal{P}) = \operatorname{rank}_{\mathcal{Q}} C_j(\mathcal{Q})$$

Thus, by taking the difference we get

$$b_j(C; \mathcal{Q}) - b_j(C; \mathcal{P}) = T_j(\mathcal{P}, \mathcal{Q}) + T_{j-1}(\mathcal{P}, \mathcal{Q})$$

and then

$$P(C; t, \mathcal{Q}) - P(C; t, \mathcal{P}) = \sum_{j \geq 0} (T_j(\mathcal{P}, \mathcal{Q}) + T_{j-1}(\mathcal{P}, \mathcal{Q})) t^j$$

Note that the Euler-Poincaré equality holds:

$$P(C; -1, \mathcal{P}) = P(C; -1, \mathcal{Q})$$

which implies that the right hand side can be written as $(1+t)Q(\mathcal{P}, \mathcal{Q}; t)$ with $Q(\mathcal{P}, \mathcal{Q}; t)$ a polynomial with nonnegative integer coefficients.

4.4 Isolated invariant sets and Conley index

In this section, we will explain the concepts of isolated invariant sets and Conley index which were introduced by C.Conley (see [9],[10]) as a generalization of critical points and Morse index of a Morse function. Using the Conley index pair, we can generalize the Novikov inequalities for counting the zeros of a closed Morse 1-form to the Novikov-Morse type inequalities for the isolated α -flows which will be defined in section 4.5.

Isolated invariant sets

Let a flow v be defined on the space X and let $N \subset X$ be a compact set. N is called an isolated invariant set of the flow if the invariant set $I(N) \subset \text{int}N$.

It is easy to prove that if $D = \{M_n, \dots, M_1\}$ is a Morse decomposition of an isolated invariant set S , then each Morse set $M_i (i = 1, \dots, n)$ is an isolated set.

Index pair

Let S be an isolated invariant set. A compact pair (N_0, N_1) in X is called an index pair for S , if

- (1) $N_1 \setminus N_0$ is an isolated neighborhood for S .
- (2) N_0 is positively invariant relative to N_1 .
- (3) if $v \in N_1$ and $v \cdot \mathbb{R}^+ \not\subset N_1$, then there is a $t \geq 0$ such that $v \cdot [0, t] \subset N_1$ and $v \cdot t \in N_0$.

Here condition (2) means that if $v \in N_0$ and $v \cdot [0, t] \subset N_1$, then $v \cdot [0, t] \subset N_0$. (3) implies that for any point $v \in N_1$, either v flows into the invariant set or flows out of N_1 at finite time through the "exit set" N_0 .

The index pair has the following two important properties:

Homotopy invariance

Let (N_0, N_1) and (\bar{N}_0, \bar{N}_1) be two index pairs of an isolated invariant set S . Then the two index pairs as space pairs are homotopically equivalent.

Continuation principle

Since we do not use this property in this paper, we refer the interested reader to the fourth chapter of [9].

Because of the homotopy invariance, the index pair defined by C.Conley is called the Conley index, and we define $h(S)$ to be the homotopy type of the index pair $(*, N_1/N_0)$ of the isolated invariant set S .

Computation of the Conley index

Sum formula

If S_1 and S_2 are isolated invariant sets with $S_1 \cap S_2 = \emptyset$, then $S_1 \cup S_2$ is an isolated invariant set and $h(S_1 \cup S_2) = h(S_1) \vee h(S_2)$, where \vee denotes the join of pointed topological spaces.

This formula can be achieved as follows. Let (N_0, N_1) and (\bar{N}_0, \bar{N}_1) be the index pairs of S_1 and S_2 respectively, then $(N_0 \cup \bar{N}_0, N_1 \cup \bar{N}_1)$ is the index pair for the isolated invariant set $S_1 \cup S_2$. Hence $h(S_1 \cup S_2) = (*, N_1 \cup \bar{N}_1 \setminus N_0 \cup \bar{N}_0) = h(S_1) \vee h(S_2)$

Product formula

Let S_1 and S_2 be isolated invariant sets with index pairs (N_0, N_1) and (\bar{N}_0, \bar{N}_1) respectively, then $S_1 \times S_2$ is an isolated invariant set with index pair $(N_0, N_1) \times (\bar{N}_0, \bar{N}_1) = (\partial N_0 \times \bar{N}_1 \cup \partial \bar{N}_0 \times N_1, N_1 \times \bar{N}_1)$. Hence $h(S_1 \times S_2) = (*, N_1 \times \bar{N}_1 \setminus \partial N_0 \times \bar{N}_1 \cup \partial \bar{N}_0 \times N_1) = (*, N_1 \setminus N_0) \wedge (*, \bar{N}_1 \setminus \bar{N}_0) = h(S_1) \wedge h(S_2)$. Here “ \wedge ” is the smash product between two pointed spaces.

Index of hyperbolic fixed points of the flow

Let x_0 be a fixed point of the flow $\dot{v} = V(v)$. Consider its linearized equation at x_0 ,

$$\dot{v} = DV(x_0) \cdot v \tag{4.4.1}$$

x_0 is called a hyperbolic fixed point if the real part of each eigenvalue of the coefficient matrix $DV(x_0)$ is not zero. Hence the tangent space $T_{x_0}M = E^+ \oplus E^-$, where $E^+(E^-)$ represents the eigenspace of the eigenvalues having positive(negative) real part. Let $k = \text{rank}(E^+)$ and D^k be the k -dimensional unit disc in E^+ . Since the local unstable manifold and the local stable manifold are transversal at the hyperbolic fixed point x_0 , the Conley index $h(v_0) = (*, D^k / \partial D^k) = (*, S^k)$.

Index of an orientable hyperbolic periodic orbit

Let x_0 be any point in the hyperbolic periodic orbit v_0 and let $T_{x_0}^\perp M$ be the tangent subspace vertical to v_0 . Then the derivative of the Poincaré map defines the stable eigenspace E^- and the unstable eigenspace E^+ . Let $k = \dim E^+$, then this means that the eigenspace of the unstable manifold is orientable, the Conley index $h(v_0) = (*, S^1 \amalg \{*\}) \wedge (\partial D^k, D^k)$, where $(*, S^1 \amalg \{*\})$ is the Conley index of the flow $\dot{\theta} = 1$ in S^1 . Therefore,

$$\begin{aligned} h(v_0) &\cong (*, S^1 \amalg \{*\}) \wedge (*, S^k) \\ &\cong (*, S^1 \amalg \{*\}) \wedge (*, S^1) \wedge (*, S^{k-1}) \\ &\cong (*, S^2 \vee S^1) \wedge (*, S^{k-1}) \\ &\cong (*, S^k) \vee (*, S^{k+1}) \end{aligned}$$

i.e., the index of v_0 is the sum of a pointed k -sphere and a pointed $k + 1$ -sphere.

Index of an unorientable hyperbolic periodic orbit

Let P_{x_0} be the Poincaré map at a point of the hyperbolic orbit. Without loss of generality, we can assume that $dP_{x_0} \in O(m-1)$. Now the unstable manifold is the integral submanifold of E^+ . If the unstable manifold is unorientable, $E^+ \times_P S^1$ is a twisted vector bundle on S^1 twisted by dP_{x_0} . We can connect dP_{x_0} with $-I \in O(m-1)$ by a path, so the homotopy type of $E^+ \times_{-I} S^1$ is the same as

$E^+ \times_P S^1$. Hence we can assume that dP_{x_0} reverses only 1 eigensubspace and the Conley index of v_0 is the product of a pointed $k - 1$ -sphere with a Möbius band collapsing its boundary, hence

$$h(v_0) = (*, \mathbb{R}P^2 \amalg \{*\}) \wedge (*, S^{k-1})$$

Index of a critical manifold

Let Z be a non-degenerate critical manifold of a smooth function f . This means that the following two conditions:

(1) $\text{grad}f(x) = 0, \forall x \in Z$.

(2) at any point $x \in Z$, there is a decomposition $T_x M = T_x^\perp(Z) \oplus T_x(Z)$ and $d^2f(x)$ as bilinear form is nondegenerate along the vertical tangent subspace $T_x^\perp(Z)$

The subspace $T_x^\perp(Z)$ can be decomposed again into the direct sum of E_x^+ and E_x^- which correspond to the eigenspaces of positive eigenvalues and negative eigenvalues of $d^2f|_{T_x^\perp(Z)}$.

If Z is connected, $\text{rank}E_x^-$ does not depend on the point x . Hence $\text{ind}(Z) = \dim(\text{rank}E_x^-)$ is an invariant of Z and is called an index of Z with respect to the Morse function f .

Let $k = m - \dim Z - \text{ind}Z$ and let D_x^k, S_x^{k-1} be the unit disc and unit sphere in E_x^+ . We let $D^k(Z), S^{k-1}(Z)$ represent the fibre bundles on Z with fibres D^k and S^{k-1} respectively.

Now giving a metric to the manifold M , there is an ε -small closed tubular neighborhood $U(Z)$ such that $U(Z)$ is homeomorphic to the normal bundle $T^\perp(Z)$. Let $\partial U^+(Z)$ be the intersection of the boundary $\partial U(Z)$ with the unstable manifold $U^+(Z)$ of the gradient flow of f . Then $(*, U(Z) \setminus \partial U^+(Z))$ is the index pair of Z which is homotopic to $(*, U^+(Z) \setminus \partial U^+(Z))$. The latter pointed space is homeomorphic to $(*, D^k(Z) \setminus S^{k-1}(Z))$. Let $\widetilde{o}(Z)$ be the local system on Z such that $\mathbb{C} \otimes \widetilde{o}(Z)$ is the orientation bundle $o(Z)$ on Z . Let \widetilde{E} be a local system of abelian groups on M and $E = \widetilde{E} \otimes \mathbb{C}$. Applying the Leray-Hirsch theorem (see[28]) to the fibre-bundle pair $(S^{k-1}(Z), D^k(Z))$, we have

$$\begin{aligned} H_*(*, D^k(Z) \setminus S^{k-1}(Z); \widetilde{o}(Z) \otimes E|_Z) &\cong H_*(S^{k-1}(Z), D^k(Z); \widetilde{o}(Z) \otimes E|_Z) \\ &\cong H_*(Z, \widetilde{o}(Z) \otimes E|_Z) \otimes H_*(*, S^k; \mathbb{C}) \cong H_{*-k}(Z, \widetilde{o}(Z) \otimes E|_Z) \end{aligned}$$

Hence

$$H_*(h(Z), \widetilde{o}(Z) \otimes E|_Z) \cong H_{*-k}(Z, \widetilde{o}(Z) \otimes E|_Z)$$

4.5 Novikov-Morse type inequalities for flows carrying a cocycle

Let v be a flow on a compact polyhedron X having a π -Morse decomposition. Assume that the π -stable nonwandering set of v , $\mathcal{N}_\pi(v) = \{A_n, \dots, A_1\}$.

We have the following definitions.

Definition 4.5.1 (Isolated π -stable nonwandering sets) *Let the flow v be as above. A π -stable nonwandering set A_i in A is called an isolated π -stable nonwandering set if A_i is also an isolated invariant set of v .*

We denote the set containing all the isolated π -stable nonwandering sets by $\mathcal{IN}_\pi(v)$.

Definition 4.5.2 *Let v be a generalized α -flow with respect to $A = \{A_n, \dots, A_1\}$. Then by Theorem 3.5.3, v has an associated π -Morse decomposition and $\mathcal{N}(v) = A$. If each A_i is an isolated π -stable nonwandering set, then v is called an isolated generalized α -flow or an isolated flow that carries a cocycle α .*

It is easy to see that all the previous examples are isolated (generalized) α -flows.

Before formulating our main result in this section, we introduce some algebraic notations.

Ideals

Let $P_s = \mathbb{Z}[t_1, \dots, t_s]$ be the polynomial ring with s variables over the integers \mathbb{Z} . Define a prime ideal I in P_s to be $I = 1 + \langle t_1, \dots, t_s \rangle$, where $\langle t_1, \dots, t_s \rangle$ is the ideal generated by t_1, \dots, t_s . Then the zero set of I is a codimension 1 arithmetic variety in \mathbb{C}^s . Let $a \in (\mathbb{C}^*)^s$ be not in the zero set of I . Define I_a to be the prime ideal in the polynomial ring P_s consisting of the polynomials vanishing at a . By the choice of a , the free terms of all the polynomials $f(t) \in I_a$ are divisible by some prime number p . Therefore we obtain

$$I_a \subset I_p = \langle p \rangle + \langle t_1, \dots, t_s \rangle$$

Theorem 4.5.1 *Let X be a compact polyhedron with a metric d . Let v be a flow on X having a π -Morse decomposition with rank $\pi = s$ and satisfying $\mathcal{IN}_\pi(v) = \mathcal{N}_\pi(v)$. Let \tilde{E} be a local system of free abelian groups and let $E = \mathbb{C} \otimes \tilde{E}$. If $a \in (\mathbb{C}^*)^s$ is not in the arithmetic variety associated to the prime ideal $I = 1 + \langle t_1, \dots, t_s \rangle \subset P_s$, then there is a prime number p relative to a such that*

$$\begin{aligned} \sum_{A \in \mathcal{IN}_\pi(v)} p(h(A); t, \mathbb{Z}_p \otimes J_A^* E) &= p(X; t, a^p \otimes E) \\ &+ (1+t)Q_1(I_a, I_p; t) + (1+t)Q_2(I_p; t) \end{aligned} \quad (4.5.1)$$

where J_A is the inclusion map from the isolated neighborhood of an isolated invariant set to X , $Q_1(I_a, I_p; t)$ and $Q_2(I_p; t)$ are all polynomials with nonnegative

integer coefficients, and a^φ is a complex line bundle determined by the representation of the group φ .

Proof Since v has a π -Morse decomposition, the lifting flow \bar{v} of v has a relative Morse decomposition

$$D = \{\bar{N}_0^+, \bar{A}_n, \dots, \bar{A}_1\}$$

and it is easy to see that $\bar{A}_i (i = 1, \dots, n)$ are isolated invariant sets in \bar{X}_0 because of that $A_i (i = 1, \dots, n)$ are isolated invariant sets. Now we use the following lemma about the existence of a filtration of a Morse decomposition which was given in [10], with a slight improvement here.

Lemma 4.5.2 *Let S be an isolated invariant set in a compact space X and let $\{M_n, \dots, M_1\}$ be a Morse decomposition of S . Then there exists an increasing sequence of compact sets $N_n \subset N_{n-1}, \dots, \subset N_0$ such that, for any $j \geq i$, the pair (N_{j-1}, N_i) is an index pair for M_{j_i} . In particular, (N_n, N_0) is an index pair for S , and (N_j, N_{j-1}) is an index pair for M_j . Those compact sets N_j can be chosen to be CW-complexes if the space X is a CW-complex.*

Applying this lemma, we can get a filtration $\bar{R}_I \times \varepsilon \bar{I} \subset \bar{N}_n \subset \bar{N}_{n-1} \subset \dots \subset \bar{N}_1 \subset \bar{X}_0$. Let \bar{U} be a set in \bar{X}_0 . Let the lifting of \bar{U} in the fundamental domain of the universal covering be \tilde{U}_0 . We can define a semi-equivariant set in the universal covering \tilde{X} to be $\tilde{U}_+ := \cup_{g \in \pi_1(X)_+} g \cdot \tilde{U}_0$. With that notations we get a $\pi_1(X)_+$ -equivariant filtration in \tilde{X} :

$$\widetilde{\bar{R}_I \times \varepsilon \bar{I}} \subset (\tilde{N}_n)_+ \subset (\tilde{N}_{n-1})_+ \subset \dots \subset \tilde{X}_+ \quad (4.5.2)$$

This filtration of spaces induces the following filtration of complexes:

$$0 \subset P_s^k \otimes_{\mathbb{Z}[\pi_1(X)_+]} C_*(\widetilde{\bar{R}_I \times \varepsilon \bar{I}}) \subset P_s^k \otimes_{\mathbb{Z}[\pi_1(X)_+]} C_*((\tilde{N}_n)_+) \dots \subset P_s^k \otimes_{\mathbb{Z}[\pi_1(X)_+]} C_*(\tilde{X}_+)$$

Here the representation of the subring $\mathbb{Z}[\pi_1(X)_+]$ in the space P_s^k is described in section 4.2.

Continue to tensor the above filtration by $\mathbb{Z}_p \otimes_{P_s}$ and use proposition 4.3.3, we get the Poincaré polynomial

$$\begin{aligned} & \sum_{j=1}^{n+1} p(\mathbb{Z}_p \otimes J^*E \otimes C_*(\bar{N}_j), \mathbb{Z}_p \otimes J^*E \otimes C_*(\bar{N}_{j-1}); t) \\ &= p(\mathbb{Z}_p \otimes J^*E \otimes C_*(\bar{R}_I \times \varepsilon \bar{I}), \mathbb{Z}_p \otimes J^*E \otimes C_*(\bar{X}_0); t) \\ &+ (1+t) \sum_{j=1}^n q(\mathbb{Z}_p \otimes J^*E \otimes C_*(\bar{R}_I \times \varepsilon \bar{I}), \\ & \mathbb{Z}_p \otimes J^*E \otimes C_*(\bar{N}_j), \mathbb{Z}_p \otimes J^*E \otimes C_*(\bar{N}_{j-1}); t) \end{aligned} \quad (4.5.3)$$

where we define $\bar{N}_0 = \mathbb{Z}_p \otimes J^*E \otimes C_*(\bar{X}_0)$ and $\bar{N}_{n+1} = \mathbb{Z}_p \otimes J^*E \otimes C_*(\bar{R}_I \times \varepsilon\bar{I})$. Denote the sum term on the right hand side of (4.5.3) by $Q_2(\mathbb{Z}_p; t)$, then (4.5.3) can be written as

$$\begin{aligned} & \sum_{j=1}^{n+1} p(\mathbb{Z}_p \otimes J^*E \otimes C_*(\bar{N}_j, \bar{N}_{j-1}); t) \\ &= p(\mathbb{Z}_p \otimes J^*E \otimes C_*(\bar{R}_I \times \varepsilon\bar{I}), \mathbb{Z}_p \otimes J^*E \otimes C_*(\bar{X}_0); t) \\ &+ (1+t)Q_2(\mathbb{Z}_p; t) \end{aligned} \quad (4.5.4)$$

Since a is not in the arithmetic variety associated to the prime ideal I , we have $I_a \subset I_p$, where I_p is defined before this theorem. By Theorem 4.2.1, (2), we have

$$\begin{aligned} & p(\mathbb{Z}_p \otimes J^*E \otimes C_*(\bar{R}_I \times \varepsilon\bar{I}), \mathbb{Z}_p \otimes J^*E \otimes C_*(\bar{X}_0); t) \\ &= p(\mathbb{Z}_p \otimes_{P_s} (P_s^k \otimes_{\mathbb{Z}[\pi_1(X)_+]} C_*(\tilde{X}_+)); t) \\ &= p(\mathbb{Z}_p \otimes_{P_s} D_*; t) \\ &= \sum_{j \geq 0} b_j(\mathbb{Z}_p \otimes_{P_s} D_*) t^j \\ &= \sum_{j \geq 0} \dim_{Q(P_s/I_p)} H_j(Q(P_s/I_p) \otimes_{P_s} D_*) t^j \\ &= p(D_*; t, I_p) \end{aligned}$$

Now by proposition 4.3.4,

$$p(D_*; t, I_p) = p(D_*; t, I_a) + (1+t)Q_1(I_a, I_p; t) \quad (4.5.5)$$

where

$$(1+t)Q_1(I_a, I_p; t) = \sum_{j \geq 0} (T_j(I_a, I_p) + T_{j-1}(I_a, I_p)) t^j. \quad (4.5.6)$$

But by Theorem 4.2.1, we have

$$\begin{aligned} p(D_*; t, I_a) &= \sum_{j \geq 0} \dim_{Q(P_s/I_a)} H_j(Q(P_s/I_a) \otimes_{P_s} D_*) t^j \\ &= \sum_{j \geq 0} \dim_{\mathbb{C}} (H_j(\mathbb{C}_a \otimes_{P_s} D_*)) t^j \\ &= \sum_{j \geq 0} \dim_{\mathbb{C}} (H_j(X; a^\varphi \otimes E)) t^j \end{aligned} \quad (4.5.7)$$

Combining (4.5.4), (4.5.5) and (4.5.7) and using the notation $p(X; t, a^\alpha \otimes E)$ for $p(D_*; t, I_a)$, we get

$$\begin{aligned} & \sum_{j=1}^{n+1} p(\mathbb{Z}_p \otimes J^*E \otimes C_*(\bar{N}_j, \bar{N}_{j-1}); t) \\ &= p(X; t, a^\alpha \otimes E) + (1+t)Q_1(I_a, I_p; t) + (1+t)Q_2(I_p; t) \end{aligned} \quad (4.5.8)$$

Note that $(\bar{N}_j, \bar{N}_{j-1})$ is the index pair for the isolated invariant set \bar{A}_j and hence is the index pair for the isolated invariant set $A_j \in \mathcal{IN}_\pi(v)$. (4.5.8) induces the equality (4.5.1) ■

Corollary 4.5.3 (Euler-Poincaré formula) *Under the hypothesis of Theorem 4.5.1,*

$$\sum_{A \in \mathcal{IN}_\pi(v)} p(h(A); -1, \mathbb{Z}_p \otimes J_A^* E) = p(X, -1, a^\varphi \otimes E) \quad (4.5.9)$$

In particular, if E is a trivial line bundle, then

$$\sum_{A \in \mathcal{IN}_\pi(v)} p(h(A); -1, \mathbb{Z}_p) = \chi(X) \quad (4.5.10)$$

for any prime number p . Here $\chi(X)$ is the Euler characteristic number of the compact polyhedron.

Proof We only need to prove the second conclusion. If E is a trivial line bundle, then

$$\begin{aligned} p(X; -1, a^\alpha) &= \sum_{j \geq 0} \dim_{\mathbb{C}}(H_j(X; a^\alpha))(-1)^j \\ &= \sum_{j \geq 0} \dim_{\mathbb{C}}(H_j(\mathbb{C}_a \otimes_{P_s} D_*))(-1)^j \\ &= \sum_{j \geq 0} \text{rank}_{P_s}(P_s \otimes_{\mathbb{Z}[\pi_1(X)_+]} C_j(\tilde{X}_+))(-1)^j \\ &= \sum_{j \geq 0} \text{rank}_{\mathbb{Z}} C_j(X)(-1)^j = \chi(X) \end{aligned}$$

Here we use the fact D_j is a free module over the ring P_s . ■

If v is a generalized α -flow with respect to the nonwandering set $A = \{A_n, \dots, A_1\}$, then by Theorem 3.5.3 v has a π -Morse decomposition with rank $\pi = \text{rank } \alpha$, and $A = \mathcal{N}_\pi(v)$. Therefore the following theorem is the direct consequence of Theorem 4.5.1.

Theorem 4.5.4 *Let X be a compact polyhedron with a metric d . Let v be a generalized α -flow with respect to the nonwandering set $A = \{A_n, \dots, A_1\}$ and assume that each A_i is an isolated invariant set in X . Let \tilde{E} be a local system of free abelian groups and let $E = \mathbb{C} \otimes \tilde{E}$. If $a \in (\mathbb{C}^*)^s$ is not in the arithmetic variety associated to the prime ideal $I = 1 + \langle t_1, \dots, t_s \rangle \subset P_s$, then there is a prime number p relative to a such that*

$$\begin{aligned} \sum_{A_i \in A} p(h(A_i); t, \mathbb{Z}_p \otimes J_{A_i}^* E) &= p(X; t, a^\alpha \otimes E) \\ &+ (1+t)Q_1(I_a, I_p; t) + (1+t)Q_2(I_p; t) \end{aligned} \quad (4.5.11)$$

where J_{A_i} is the inclusion map from the isolated neighborhood of an isolated invariant set to X , and $Q_1(I_a, I_p; t)$ and $Q_2(I_p; t)$ are both polynomials with non-negative integer coefficients.

4.6 Applications to special flows carrying a cocycle

In this section, we will start from the general formula (4.5.1) and use the Conley index introduced in section 4.3 to get some Novikov-Morse type inequalities for some special important flows.

If v is a generalized α -flow with respect to the isolated nonwandering set $A = \{A_n, \dots, A_1\}$ with α being a trivial cocycle, then v has actually a “global” gradient-like structure in view of Theorem 3.4.1. Therefore the formula (4.5.1) is a Morse type inequality for an isolated invariant set which has the Morse decomposition $A = \{A_n, \dots, A_1\}$. This result was given in the paper [10].

Now we consider the case that α is a non-trivial cocycle.

The important example is the ω -flow generated by a closed Morse 1-form ω (see Example 3.4.2). One can also consider the Bott type Novikov inequalities. The following two theorems were given in [13] for $s = 1$.

Theorem 4.6.1 *Let X be an oriented closed smooth manifold and ω a rank s closed Morse 1-form. Let \tilde{E} be a local system of free abelian group and $E = \mathbb{C} \otimes \tilde{E}$. Assume that $a \in (\mathbb{C}^*)^s$ is not in the arithmetic variety associated to the prime ideal $I = 1 + \langle T_1, \dots, T_s \rangle \subset P_s$. Then the number $c_j(\omega)$ of zeros of ω having index j satisfies*

$$\begin{aligned} c_j(\omega) &\geq \frac{\dim_{\mathbb{C}} H_j(M, a^\omega \otimes E)}{\dim E} \\ \sum_{i=0}^j (-1)^i c_{j-i}(\omega) &\geq \sum_{i=0}^j (-1)^i \frac{\dim_{\mathbb{C}} H_{j-i}(M, a^\omega \otimes E)}{\dim E} \end{aligned} \quad (4.6.1)$$

for $j = 0, 1, \dots, m$.

Theorem 4.6.2 *Let ω be a closed 1-form with Bott type nondegenerate zero sets with rank s . Let the quantities \tilde{E}, E and a be the same as in Theorem 4.6.1. Then there is a prime p such that*

$$\begin{aligned} &\sum_Z \sum_{j \geq 0} \dim_{\mathbb{Z}_p} H_j(Z, \mathbb{Z}_p \otimes \tilde{E}|_Z \otimes o(Z)) t^{j + \text{ind}(Z)} \\ &= p(X, t; a^\omega \otimes E) + (1 + t)Q(E, t) \end{aligned} \quad (4.6.2)$$

where $o(Z)$ is the orientation bundle on Z and the sum in the left hand side is taken for any zero manifold of ω and $Q(E, t)$ is a polynomial with nonnegative coefficients.

Proof of Theorem 4.6.1 and 4.6.2 From Example 3.4.2 and 3.4.7, we know the flow generated by the dual vector field of ω is a flow carrying the cocycle ω . Since the fixed points of these flow are just the (Bott type) nondegenerate zero points(sets), by section 4.3, we have the following result.

(1) For a zero point x_0 ,

$$\begin{aligned} H_l(h(x_0), \mathbb{Z}_p \otimes J^*E) &\cong H_l(*, S^j; \mathbb{Z}_p \otimes J^*E) \\ &\cong \begin{cases} \otimes_1^{\dim E} \mathbb{C} \otimes \mathbb{Z}_p & l = j \\ 0 & l \neq j \end{cases} \end{aligned}$$

(2) For a Bott type zero set Z , it has

$$H_*(h(Z), \widetilde{o(Z)} \otimes E|_Z) \cong H_{*-k}(Z, \widetilde{o(Z)} \otimes E|_Z)$$

Now applying formula (4.5.1), we get the conclusions. ■

In view of Example 3.4.3, the formulas (4.6.1) and (4.6.2) not only hold for the closed 1-forms, they also hold for a family of flows which is a small perturbation of the flow generated by ω because the Conley indices of fixed point sets are invariant under the perturbation. Hence we have

Corollary 4.6.3 *Let v_ε be a family of flows with parameter ε in Example 3.4.3. Under the hypothesis of Theorem 4.6.1 and 4.6.2, the formula (4.6.1) holds if ω is a closed Morse 1-form and the formula (4.6.2) holds if ω is a closed 1-form having Bott type zero sets.*

If we let E be the trivial line bundle and let each entry a_i of $a = (a_1, \dots, a_s) \in (\mathbb{C}^*)^s$ be a transcendental number, then from Theorem 4.6.1 we have the following corollary

Corollary 4.6.4 (Classical Novikov inequality) *Let X be an oriented closed smooth manifold and ω a rank 1 Morse closed 1-form. Then the numbers $c_j(\omega)$ of zeros of ω having index j satisfy*

$$\begin{aligned} c_j(\omega) &\geq b_j([\omega]) \\ \sum_{i=0}^j (-1)^i c_{j-i}(\omega) &\geq \sum_{i=0}^j (-1)^i b_{j-i}([\omega]) \end{aligned} \tag{4.6.3}$$

for $j = 0, 1, \dots, m$.

In the appendix, we will give a proof of the refined Novikov inequalities that include the information of the torsion part.

In Example 3.4.5, we have considered a flow carrying a cohomology class. The existence of such flow gives a vanishing theorem for the Novikov numbers.

Theorem 4.6.5 (Vanishing theorem) *Let X be a compact polyhedron with a metric d . If there exists a flow carrying a cohomology class $[\alpha]$ on X , then*

$$b_i([\alpha]) = 0, \quad \forall i = 0, 1, \dots, m$$

Theorem 4.6.6 *Let v be a α -Morse-Smale flow on a closed oriented manifold X and let its nonwandering set be $A = \{A_n, \dots, A_1\}$ consisting of the hyperbolic fixed points and hyperbolic periodic orbits. Let c_j be the number of hyperbolic fixed points with index j , a_j be the number of the hyperbolic periodic orbits with index j , and $\mu_j = c_j + a_j + a_{j+1}$. Then*

$$\begin{aligned} \mu_j &\geq b_j([\alpha]) \\ \sum_{i=0}^j (-1)^{j-i} \mu_i &\geq \sum_{i=0}^j (-1)^{j-i} b_i([\alpha]) \end{aligned} \quad (4.6.4)$$

for $j = 0, 1, \dots, m$.

Proof The conclusion is obtained by applying Theorem 4.5.4 with E being the orientation bundle of X and the calculation of the homology of the Conley indices of the hyperbolic fixed points and the orientable or unorientable hyperbolic periodic orbits.

For the calculation of the Conley indices, we can use the results in section 4.3. ■

Acknowledgement

We thank Michael Farber for explaining his work on the Novikov inequalities for us and for stimulating discussions. The first author also wants to thank Guofang Wang for his discussions with him.

References

- [1] D. Asimov, Round handles and non-singular Morse-Smale flows, *Ann. Math.*, 102(1975) 41-54
- [2] _____, Homotopy of non-singular Morse-Smale flows, *Ann. Math.*, 102(1975) 55-65

- [3] J.-M. Bismut, The Witten complex and the degenerate Morse inequalities, *J. Diff. Geom.* 23(1986) 207-240
- [4] R. Bott, Non degenerate critical manifolds, *Ann. Math.* 60(1954) 248-261
- [5] _____, Morse theory indomitable, *Publ. Math. IHES.* 68(1988), 99-114
- [6] M. Braverman, M. Farber, Novikov type inequalities for differential forms with non-isolated zeros, *Math. Proc. Cambridge Philos. Soc.* 122 (1997), no. 2, 357–375.
- [7] K.C.Chang, Infinite dimensional Morse theory and multiple solution problems. Birkhäuser 1993
- [8] R. Churchill, Invariant sets which carry cohomology. *J. Diff. Equ.* 13(1973), 523-550
- [9] C. C. Conley, Isolated invariant sets and the Morse index, *CBMS Reg. Conf. Series in Math.* 38, 1978, A. M. S., Providence, R. I.
- [10] C. C. Conley, E. Zehnder, Morse-type index theory for flows and periodic solutions for Hamiltonian equations, *Comm. Pure and Appl. Math.*, Vol 37(1984), 207-253
- [11] M. Farber, Exactness of the Novikov inequalities, *Func. Anal.Appl.* 19(1985), 40-48
- [12] _____, Lusternik-Schnirelman theory for closed 1-forms, *Math. DG/9811113*, 18Nov1998
- [13] _____, Counting zeros of closed 1-forms, *Math. DG/9903133* 23Mar1999
- [14] _____, Morse-Novikov critical point theory, Cohn localization and Dirichlet units, *Math. DG/9911157* 20Nov1999
- [15] J. Franks, The periodic structure of non-singular Morse-Smale flows, *Comm. Math. Helv.* 53(1978) 279-294
- [16] H. Hofer and D. Salamon, Floer homology and Novikov rings, Floer memorial volume, Birkhäuser (1995) 483-524
- [17] J. W. Milnor, Morse theory, *Ann. Math.*, Studies 51, Princeton Univ. Press, 1963
- [18] S. P. Novikov, Multivalued functions and functionals. An analog of Morse theory, *Dokl. Akad. Nauk SSSR*, 260, No.1(1981) 31-35

- [19] _____, The Hamiltonian formalism and a multivalued analog of Morse theory, Usp. Mat. Nauk, 37, No.5 (1982) 3-49
- [20] _____, Bloch homology, critical points of functions and closed 1-forms, Soviet. Math. Dokl. 33(1986) 551-555
- [21] A. Pajitnov, Morse theory of closed 1-forms, LNM. 1474, Springer, Berlin, 1991
- [22] _____, C^0 -generic properties of boundary operators in Novikov complex, arXiv.Math.DG/9812157 29Dec1998
- [23] _____, Surgery on the Novikov complex, K-Theory 10(1996) 323-412
- [24] M. Schwarz, Morse homology, Prog. in Math., Birkhäuser, 1993
- [25] S. Schwartzman, Asymptotic cycles, Ann. Math., Vol 66, No.2, (1957) 270-284
- [26] S. Smale, On gradient dynamical systems, Ann. Math. 74(1961) 199-206
- [27] S. Smale, Morse inequalities for a dynamical system, Bull. A.M.S.66, (1960) 43-49
- [28] E. Spanier, Algebraic Topology, McGraw-Hill, New York, 1966
- [29] M. Struwe, Variational methods : applications to nonlinear partial differential equations and Hamiltonian systems, Springer, 1996
- [30] E. Witten, Supersymmetry and Morse theory, J. Diff. Geom. 17(1982) 661-692

Appendix

In this section we will provide a proof of the classical refined Novikov inequalities which includes also the information from the torsion parts of the homology groups. The starting point is the equality (4.5.1). If the α -flow we consider is generated by the dual vector field of the closed Morse 1-form, then the connecting homomorphism is easily determined.

Let X be a CW-complex with subcomplex Y and let X^k be the k -skeleton of X , and define

$$X_Y^k = X^k \cup Y.$$

Then we have a filtration relative to Y ,

$$Y = X_Y^{-1} \subset X_Y^0 \subset \cdots \subset X_Y^m = X$$

This filtration induces the filtration of the complex $C_*(X)$:

$$C_*(X_Y^{-1}) \subset C_*(X_Y^0) \subset \cdots \subset C_*(X_Y^m) = C_*(X)$$

Let G be a field and set the subcomplex $C_j = G \otimes C_*(X_Y^{m-j})$, $j = 0, 1, \dots, m, m+1$. Then we obtain the filtration

$$0 \subset C_{m+1} \subset C_m \subset \cdots \subset C_0.$$

For this filtration of complexes, we get from proposition 4.3.3 the following formula:

$$\sum_{j=1}^{m+1} p(C_j, C_{j-1}; t) = p(C_{m+1}, C; t) + (1+t) \sum_{j=1}^m q(C_{m+1}, C_j, C_{j-1}; t), \quad (1)$$

where

$$\begin{aligned} p(C_j, C_{j-1}; t) &= \sum_{l \geq 0} b_l(C_j, C_{j-1}) t^l \\ q(C_{m+1}, C_j, C_{j-1}; t) &= \sum_{l \geq 0} d_l(C_{m+1}, C_j, C_{j-1}) t^l \\ d_l(C_{m+1}, C_j, C_{j-1}) &= \text{rank}(\text{im} \delta_l) \end{aligned}$$

and $\delta_l : H_l(C_j, C_{j-1}) \rightarrow H_{l-1}(C_{m+1}, C_j)$ is the connecting homomorphism of the triple of complexes (C_{m+1}, C_j, C_{j-1}) .

Because of the choice of the filtration, we have

$$\begin{aligned} H_l(C_j, C_{j-1}) &\cong H_l(G \otimes C_*(X_Y^{m-j}, X_Y^{m-j+1})) \\ &\cong \begin{cases} \otimes_1^{k_{m-j+1}} G & l = m - j + 1 \\ 0 & l \neq m - j + 1 \end{cases} \end{aligned} \quad (2)$$

where k_{m-j+1} is the number of $m - j + 1$ -cells in X but not in Y .

We also have

$$H_{l-1}(C_{m+1}, C_j) \cong H_{l-1}(G \otimes G_*(Y, X^{m-j})).$$

The connecting homomorphism δ_l can be identified with the homomorphism going from $H_l(G \otimes C_*(X_Y^{l-1}, X_Y^l))$ to $H_{l-1}(G \otimes C_*(Y, X^{l-1}))$. Consider the commutative diagram

$$\begin{array}{ccc} H_l(G \otimes C_*(X_Y^{l-1}, X_Y^l)) & & \\ \downarrow \delta_l & \searrow d_l & \\ H_{l-1}(G \otimes C_*(Y, X_Y^{l-1})) & \xrightarrow{i_*} & H_{l-1}(G \otimes C_*(X_Y^{l-2}, X_Y^{l-1})) \end{array}$$

Here $i : (Y, X_Y^{l-1}) \longrightarrow (X_Y^{l-2}, X_Y^{l-1})$ is the inclusion map of the space pair. Therefore $d_l = i_* \cdot \delta_l$ and d_l is just the boundary operator in the cellular complex $G \otimes W_*(X, Y)$ with coefficients in G . Thus

$$\text{rank}_G(\text{imd}_l) = \text{rank}_G(\text{im}(i_* \cdot \delta_l)) \leq \text{rank}_G(\text{im}\delta_l) \quad (3)$$

and the inequality holds if and only if i_* is an injection.

Therefore using (1)-(3), we have

$$\sum_{j=0}^m k_j t^j \geq p(G \otimes W_*(Y, X); t) + (1+t) \sum_{j=1}^m \text{rank}_G(\text{imd}_j) t^j \quad (4)$$

where $d_j : G \otimes W_j(Y, X) \longrightarrow G \otimes W_{j-1}(Y, X)$ is the boundary operator.

Let $X = \bar{X}_0$, $Y = \bar{R}_I \times \varepsilon \bar{I}$ and $G = \mathbb{Z}_p$, (4) becomes

$$\sum_{j=0}^m k_j t^j \geq p(\mathbb{Z}_p \otimes W_*(Y, X); t) + (1+t) \sum_{j=1}^m \text{rank}_{\mathbb{Z}_p}(\text{imd}_j) t^j \quad (5)$$

Now we continue the progress from (4.5.4), to get

$$\begin{aligned} \sum_{j=0}^m k_j t^j &\geq p(X; t, a^\alpha) + (1+t) \sum_{j=1}^m \text{rank}_{\mathbb{Z}_p}(\text{imd}_j) t^j \\ &\quad + \sum_{j \geq 0} (T_j(I_a, I_p) + T_{j-1}(I_a, I_p)) t^j \end{aligned} \quad (6)$$

Note that

$$T_j(I_a, I_p) = \text{rank}_{\mathbb{C}}(\text{imd}_{j+1}(I_a)) - \text{rank}_{\mathbb{Z}_p}(\text{imd}_{j+1}(I_p)),$$

where $d_j(I_a) : Q(P_s/I_a) \otimes_{P_s} D_j \rightarrow Q(P_s/I_a) \otimes_{P_s} D_{j-1}$ and $d_j(I_p) : Q(P_s/I_p) \otimes_{P_s} D_j \rightarrow Q(P_s/I_p) \otimes_{P_s} D_{j-1}$ are the boundary operators.

Therefore, from (6), we have

$$\sum_{j=0}^m k_j t^j \geq p(X; t, a^\alpha) + \sum_{j \geq 0} (\text{rank}_{\mathbb{C}}(\text{imd}_j(I_a)) + \text{rank}_{\mathbb{C}}(\text{imd}_{j-1}(I_a))) t^j \quad (7)$$

Define the minimal number of the generators of the torsion part of $H_j(X, a^\omega)$ by $q_j([\omega])$.

Now we can prove the following classical Novikov inequality

Theorem (Novikov inequality) *Let ω be a closed Morse 1-form on a closed oriented manifold X . Let $c_j(\omega)$ be the number of zero points of ω with index j , then*

$$c_j(\omega) \geq b_j([\omega]) + q_j([\omega]) + q_{j-1}([\omega]),$$

for $j = 0, 1, \dots, m$.

Proof Lift the 1-form ω to the covering space \bar{X} such that ω becomes a Morse function \bar{f} . By considering the handle decomposition of the fundamental domain \bar{X}_0 with respect to \bar{f} , it is easy to see that $k_j = c_j$ in (7). Since $\text{rank}(\text{im}d_j(I_a)) \geq q_j([\omega])$, (7) then induces the result. ■