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Time-space discretization of the nonlinear hyperbolic system
$$u_{tt} = div(\sigma(Du) + Du_t)$$

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TIME-SPACE DISCRETIZATION OF THE NONLINEAR HYPERBOLIC SYSTEM $\mathbf{u_{tt} = div(\sigma(Du) + Du_t)}$

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Abstract. The numerical treatment of the hyperbolic system of nonlinear wave equations with linear viscosity, $u_{tt} = \operatorname{div}(\sigma(Du) + Du_t)$, is studied for a large class of globally Lipschitz continuous functions σ, including non-monotone stress-strain relations. The analyzed method combines an implicit Euler scheme in time with Courant (continuous and piecewise affine) finite elements in space for general time steps with varying meshes. Explicit a priori error bounds in $L^\infty(L^2)$, $L^2(W^{1,2})$, and $W^{1,2}(L^2)$ are established for the solutions of the fully discrete scheme.

1. Introduction

In this paper we study the numerical treatment of the nonlinear hyperbolic system

$$u_{tt} = \operatorname{div}(\sigma(Du) + Du_t) \text{ in } \Omega \times (0,T) \tag{1.1}$$

subject to the boundary and initial conditions

$$\begin{aligned} u &= 0 \quad \text{on } \partial\Omega\times(0,T), \\ u &= u_0 \quad \text{in} \quad \Omega\times\{0\}, \\ u_t &= v_0 \quad \text{in} \quad \Omega\times\{0\}. \end{aligned}$$

Here, u is a vector-valued mapping from $\Omega \subset \mathbb{R}^n$ into $\mathbb{R}^m$ and the initial data satisfy $u_0 \in W_0^{1,2}(\Omega;\mathbb{R}^m)$ and $v_0 \in L^2(\Omega;\mathbb{R}^m)$.

The physical interest in this equation lies in the fact that it describes for $m = n$ the evolution of a viscoelastic body with reference configuration Ω. Non-monotone stress-strain relations, modeled by $\sigma = D\Phi$ for non-convex energy density functions Φ, are of main interest in simulations of solid-solid phase transitions, see, e.g., [1, 9, 10, 4, 5]. This equation arises also in the two-dimensional scalar case ($n = 2$ and $m = 1$) for the out-of-plane displacement field of an anti-plane shear deformation [11]. Numerical experiments have been reported in [7].

Inspired by the uniqueness proof for Lipschitz continuous stresses σ in [10, 6], we present in this paper the a priori error analysis for an approximating scheme for the system (1.1) that combines continuous finite elements in spaces with a discontinuous Galerkin approximation in time. We obtain estimates for the approximation error of

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the deformation u, the deformation gradient Du and the velocity field $v = u_t$ under very general assumptions. In particular, the time step size k_j has only to satisfy the condition $k_j \le Qk_{j+1}$ for a global constant $Q > 0$ and the spatial triangulations are only assumed to be quasiuniform at each time step with a typical diameter h_j of the elements. Our analysis reveals an interesting coupling of k_j and h_j through the quotient h_j^2/k_j which suggests that the time steps should not be too small compared to the square of the spatial discretization parameter h_j. Our general result, Theorem 4.1, implies immediately the following convergence estimates (see Section 2 for the precise definitions).

Theorem 1.1. *Let Ω be a bounded polygonal domain in $\mathbb{R}^n$. Fix $T > 0$ and define discrete times $0 = t_0 < t_1 < \cdots < t_N = T$. Suppose that $\mathcal{T}_j$ is a regular triangulation of Ω for $j = 0, \ldots, N$ such that $\mathcal{T}_j$ is a refinement of $\mathcal{T}_{j-1}$. Assume that $\mathcal{S}_{0,j}(\mathcal{T}_j)$ is the space of all continuous functions that vanish on $\partial\Omega$ and are affine on the elements in $\mathcal{T}_j$. Let $U_j \in \mathcal{S}_{0,j}$ be the solution of the implicit Euler scheme defined in Section 2.3 and let*

$$k_j = t_j - t_{j-1}, \quad k = \max_{j=1,\ldots,N} k_j, \quad Q = \max_{j=2,\ldots,N} \frac{k_j}{k_{j-1}}, \quad h = \max_{j=0,\ldots,N} h_j.$$

Assume, furthermore, that σ is globally Lipschitz continuous and that the solution u of the system (1.1) belongs to $W^{1,\infty}(W^{2,2}) \cap W^{2,2}(L^2)$. Finally, define the discretization errors e_j and δ_j at the time step t_j by

$$e_j = u(t_j) - U_j, \quad \delta_j = v(t_j) - \frac{1}{k_j}\big(U_j - U_{j-1}\big).$$

Then there exist constants c_1 and c_2 such that the following holds. If $c_1 k < 1$, then

$$\max_{\nu=1,\ldots,N} \|e_\nu\|^2 + \sum_{\nu=1}^{N} k_\nu \big(\|\delta_\nu\|^2 + \|De_\nu\|^2\big) \le c_2(T + T^2 + h^4 + h^8)\exp\big(c_1 T\big)\|u\|_{h,k}^2,$$

where

$$\|u\|_{h,k}^2 = k^2 \| \, |u_{tt}| + |Du_{tt}| + |Du_t| \, \|_{L^2(L^2)}^2 + h^2(1 + \max_{j=1,\ldots,N} \frac{h_j^2}{k_j})\|u\|_{W^{1,\infty}(W^{2,2})}^2.$$

The constants c_1 and c_2 depend only on the Lipschitz constant of σ, the shape of the triangles in the triangulations $\mathcal{T}_j$, and on k_j via Q, but neither on h_j nor on u.

Remarks. 1) The statement of the theorem assumes tacitly that the initial data can be approximated sufficiently well, see estimate (2.7) below.

2) The assumption $c_1 k \le 1$ implies that $2k_j \mathrm{Lip}(\sigma) < 1$ for $j = 1, \ldots, N$. This condition ensures that the discrete scheme has a unique solution, see Theorem 2.2.

The paper is organized as follows. We define the discrete scheme in Section 2 and prove existence and uniqueness of the discrete solution. Section 3 contains a series of estimates for the solutions of the approximating scheme which are used in Section 4 to prove the general convergence result in Theorem 4.1 which contains Theorem 1.1 as a special case.

2. The discrete scheme

In this section, we introduce the relevant notation and define the discrete scheme. Then we prove existence and uniqueness of the discrete solutions and derive an identity in the spirit of Galerkin orthogonality. This relation replaces in our convergence analysis the identity

$$(u-\overline{u})_{tt} = \operatorname{div}\big(\sigma(Du)-\sigma(D\overline{u})+D(u-\overline{u})_t\big)$$

for the difference of two solutions u and $\overline{u}$ of the system (1.1) and from which one easily deduces uniqueness of solutions for Lipschitz continuous σ, see Section 2.2.

2.1. **Notation.** We assume that $\Omega \subset \mathbb{R}^n$ is a polygonal domain with boundary $\Gamma = \partial\Omega$ and exterior normal ν to Γ. We use the standard notation for the Lebesgue spaces $L^p(\Omega;\mathbb{R}^m)$ with norm $\|\cdot\|_p$, and we write $(\cdot,\cdot)$ for the inner product in L^2. The Sobolev spaces $W^{k,p}(\Omega;\mathbb{R}^m)$ are equipped with the standard norm $\|\cdot\|_{k,p}$ and the seminorm $|\cdot|_{k,p}$, respectively. We frequently abbreviate $X(0,T;Y(\Omega))$ by $X(Y)$ if the corresponding domain Ω, time interval $(0,T)$ and the range of the functions is clear from the context. Thus $L^2(L^2)$ denotes, for example, both $L^2(0,T;L^2(\Omega))$ and $L^2(0,T;L^2(\Omega;\mathbb{R}^m))$. The space of all real $m\times n$ matrices $\mathbb{M}^{m\times n}$ is equipped with the Frobenius norm, $|A|^2 = \operatorname{tr}(A^TA)$, where A^T denotes the transpose of the matrix A, and with the inner product $F:G$ that is induced by the scalar product in $\mathbb{R}^{mn}$.

Let $0=t_0<t_1<\cdots<t_N=T$ be a partition of the time interval $[0,T]$ into N subintervals $I_j=(t_{j-1},t_j)$ of length $k_j=t_j-t_{j-1}$, $j=1,\dots,N$. Suppose that $(\mathcal{T}_j)_{j=0,\dots,N}$ is a family of regular triangulations in the sense of [2] with maximal mesh-size h_j and that the union of all elements in $\mathcal{T}_j$ is equal to $\overline{\Omega}$. We denote by $\mathcal{S}_{0,j}=\mathcal{S}_{0,j}(\mathcal{T}_j)$ the finite element space of continuous functions $u_h:\Omega\to\mathbb{R}^m$ that have zero boundary values on Γ and are affine on the elements in $\mathcal{T}_j$. We use the interpolation operator Π_j onto $\mathcal{S}_{0,j}$ due to Scott and Zhang [13] which satisfies the projection property

$$\Pi_j v = v \quad \text{for all } v\in\mathcal{S}_{0,j}, \tag{2.1}$$

the stability estimate

$$\|D\Pi_j u\| \le c_S\|Du\| \quad \text{for all } u\in W_0^{1,2}(\Omega), \tag{2.2}$$

and the approximation estimate

$$\|u-\Pi_j u\| + h_j\|Du-D\Pi_j u\| \le c_A h_j^2\|D^2u\| \quad \text{for all } u\in W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega). \tag{2.3}$$

Throughout the paper, u denotes the unique solution of the system (1.1) guaranteed by Theorem 2.1 below. We define the discrete solution $U_j\in\mathcal{S}_{0,j}$ at the time step t_j in Section 2.3 below and we use $V_j=(U_j-U_{j-1})/k_j$ for $j=1,\dots,N$ as an approximation for the discrete velocities.

The goal of our analysis is to estimate the errors in u and in u_t. To simplify the notation, we set $v=u_t$, $u_j=u(t_j)$, $v_j=v(t_j)$, and

$$e_j = u_j-U_j = u(t_j)-U_j, \quad \delta_j = v_j-V_j = v(t_j)-V_j \quad \text{for } j=1,\dots,N.$$

2.2. **Existence and uniqueness for Lipschitz continuous stress functions.** Our analysis relies on the existence result [6] for the system (1.1) which requires that $\sigma(F) = \partial\Phi(F)/\partial F$ where the stored energy function Φ has the following three properties (H1), (H2), and (H3).

(H1) $\Phi \in C^2(\mathbb{M}^{m\times n})$.

(H2) There exist constants $\overline{c}$, $\overline{C} > 0$ and $p \geq 2$ such that

$$\overline{c}|F|^p - \overline{C} \leq \Phi(F) \leq \overline{C}\big(|F|^p + 1\big), \quad |\sigma(F)| \leq \overline{C}(|F|^{p-1} + 1)$$

for all $F \in \mathbb{M}^{m\times n}$.

(H3) There exists a constant $K > 0$ such that

$$-K|F - G|^2 \leq \big(\sigma(F) - \sigma(G)\big) : (F - G) \quad \text{for all } F, G \in \mathbb{M}^{m\times n}.$$

Hypothesis (H3) follows, for example, from monotonicity or global Lipschitz continuity of σ. In the latter case one can choose $K = \mathrm{Lip}(\sigma)$.

In this situation, the following existence result holds (see [4, 5] for related results).

Theorem 2.1 ([6, Theorem 4.1])**.** *Under the foregoing assumptions, the system (1.1) has a weak solution*

$$u \in L^\infty(0,\infty; W_0^{1,2}(\Omega;\mathbb{R}^m)) \cap W^{1,\infty}(0,\infty; L^2(\Omega;\mathbb{R}^m))$$
$$\cap\, W^{1,2}_{loc}([0,\infty); W^{1,2}(\Omega;\mathbb{R}^m)) \cap W^{2,2}_{loc}([0,\infty); W^{-1,2}(\Omega;\mathbb{R}^m)),$$

i.e., $u(\cdot,0) = u_0$, $u_t(\cdot,0) = v_0$, *and, for all* $\zeta \in C_0^\infty(\Omega\times(0,\infty);\mathbb{R}^m)$,

$$\int_0^\infty \int_\Omega \big((\sigma(Du) + Du_t) : D\zeta - u_t\cdot\zeta_t\big)\mathrm{d}x\,\mathrm{d}t = 0. \tag{2.4}$$

Moreover, u satisfies the dissipation inequality

$$E[u(t), u_t(t)] - E[u_0, v_0] \leq -\int_0^t \int_\Omega |Du_t|^2 \mathrm{d}x\,\mathrm{d}s$$

for almost every $t > 0$ where the total energy is given by

$$E[u,v] = \int_\Omega \big(\Phi(Du) + \frac{1}{2}|v|^2\big)\mathrm{d}x \qquad \square.$$

If σ is globally Lipschitz continuous, then the following inequalities imply uniqueness of the weak solution u (see [6, 10]). Suppose that u and $\overline{u}$ are solutions with the same initial and boundary conditions. If we test the difference of the two equations

$$u_{tt} = \mathrm{div}\,\big(\sigma(Du) + Du_t\big), \quad \overline{u}_{tt} = \mathrm{div}\,\big(\sigma(D\overline{u}) + D\overline{u}_t\big)$$

by $u - \overline{u}$ and integrate in space and time, then we obtain

$$\partial_T \frac{1}{2}\Big(\int_0^T \int_\Omega |Du - D\overline{u}|^2 \mathrm{d}x\,\mathrm{d}t + \int_\Omega |u(t) - \overline{u}(t)|^2 \mathrm{d}x\Big) \tag{2.5}$$
$$\leq \mathrm{Lip}(\sigma) \int_0^T \int_\Omega \big(|Du - D\overline{u}|^2 + |u_t - \overline{u}_t|^2\big)\mathrm{d}x\,\mathrm{d}t.$$

Similarly, if we use $u_t - \overline{u}_t$ as a test function, we get

$$\partial_T \frac{1}{2}\int_0^T \int_\Omega |u_t - \overline{u}_t|^2 \mathrm{d}x\,\mathrm{d}t \leq \frac{1}{4}\mathrm{Lip}^2(\sigma) \int_0^T \int_\Omega |Du - D\overline{u}|^2 \mathrm{d}x\,\mathrm{d}t. \tag{2.6}$$

The asserted uniqueness follows by applying Gronwall's inequality (see, e.g., [12]) to the sum of the two inequalities. A discrete version of this Gronwall argument is used in Section 3 as the key ingredient in Theorem 4.1.

2.3. **Definition of the implicit scheme.** In order to define the discrete scheme, let U_0, $V_0 \in \mathcal{S}_0$ denote given approximations to u_0 and v_0. We assume that

$$\|e_0\| = \|u_0 - U_0\| \le c_A h_0 \|Du_0\|,$$

and additionally for $u_0 \in W^{2,2}(\Omega;\mathbb{R}^m)$ and $v_0 \in W^{1,2}(\Omega;\mathbb{R}^m)$ that

$$\|e_0\| + h_0\|De_0\| \le c_A h_0^2 \|D^2 u_0\|, \quad \|\delta_0\| = \|v_0 - V_0\| \le c_A h_0 \|Dv_0\|. \tag{2.7}$$

We then define successively the discrete solution U_j at time t_j by minimizing the variational integral (2.8) below. Since we allow a variable step-size in the time discretization, we cannot discretize the second derivatives with a second difference quotient, and we use a backwards difference quotient for the discrete velocities instead.

Theorem 2.2. *Suppose that $\sigma = D\Phi$ is Lipschitz continuous with Lipschitz constant $\mathrm{Lip}(\sigma)$ and that k is small enough such that $2k\mathrm{Lip}(\sigma) \le 1$. Then there exists for $j = 1, \dots, N$ a unique solution U_j of the variational problem: Minimize*

$$\int_\Omega \big(\Phi(DU) + \frac{1}{2k_j}|DU - DU_{j-1}|^2 + \frac{1}{2}\big|\frac{1}{k_j}(U - U_{j-1}) - V_{j-1}\big|^2\big)\mathrm{d}x \tag{2.8}$$

among all functions $U \in \mathcal{S}_{0,j}$. The minimizer U_j is a solution of the corresponding Euler-Lagrange system in weak form, i.e., a solution of

$$\int_\Omega \big(\big(k_j\sigma(DU_j) + D(U_j - U_{j-1})\big) : DW_j + (V_j - V_{j-1}) \cdot W_j\big)\mathrm{d}x = 0 \tag{2.9}$$

for all $W_j \in \mathcal{S}_{0,j}$.

Proof. We only need to show that the variational integral has a convex integrand. Existence and uniqueness of solutions follows then from the direct method in the calculus of variations (see, e.g., [3]). By assumption,

$$0 \le \frac{1}{k_j}|A - B|^2 - \mathrm{Lip}(\sigma)|A - B|^2 \le \big(\sigma(A) + \frac{1}{k_j}A - \big(\sigma(B) + \frac{1}{k_j}B\big)\big) : (A - B),$$

that is, $\sigma(F) + k_j^{-1}F$ is monotone and hence $\Phi(F) + |F|^2/(2k_j)$ convex. □

Remark. The structural assumption $\sigma = D\Phi$ guarantees the existence of the solution u and U_j of the continuous and the discretized system. The error analysis below is entirely based on the Galerkin orthogonality (2.8) and does not rely on this assumption.

2.4. **Discrete orthogonality.** The following version of the Galerkin orthogonality is an important ingredient in the proof of Theorem 4.1.

Proposition 2.1. *Suppose that u is the unique solution of the system (1.1) and that $\{U_j\}$ is the unique approximation constructed in (2.8). Then*

$$(\delta_j - \delta_{j-1}, W_j) + \int_{I_j} \big(\sigma(Du) - \sigma(DU_j), DW_j\big)\mathrm{d}t + (De_j - De_{j-1}, DW_j) = 0 \tag{2.10}$$

for $j = 1, \dots, N$ and for all $W_j \in \mathcal{S}_{0,j}$.

Proof. The idea is to test the weak formulation (2.4) by $\chi_j W_j$ in order to get an analogue of (2.9); χ_j denotes the characteristic function of the time interval I_j. Let $W_j^\ell \in C_0^\infty(\Omega;\mathbb{R}^m)$ be a sequence of smooth functions with $\|W_j - W_j^\ell\|_{W^{1,2}(\Omega)} \to 0$ as $\ell \to \infty$ and choose $\psi_\mu \in C_0^\infty(I_j)$ with $\psi_\mu \equiv 1$ on $(t_{j-1}+\mu, t_j-\mu)$. Then

$$\int_0^\infty \int_\Omega \big((\sigma(Du) + Du_t) : DW_j^\ell(x)\psi_\mu(t) - u_t \cdot W_j^\ell(x)\frac{\partial}{\partial t}\psi_\mu(t)\big)\,\mathrm{d}x\,\mathrm{d}t = 0$$

for $\mu \in (0, k_j/4)$ and $\ell \in \mathbb{N}$. This expression converges for $\ell \to \infty$ and $\mu \to 0$ to

$$\int_{I_j} \big(\sigma(Du(t,x)) + Du_t(t,x), DW_j\big)\,\mathrm{d}t + (v_j - v_{j-1}, W_j) = 0.$$

The assertion of the proposition follows by subtracting this equation from the Euler-Lagrange equation (2.9). □

2.5. **A discrete Gronwall inequality.** Our convergence result is based on the following discrete Gronwall inequality.

Lemma 2.1 ([8, Lemma 1.4.2])**.** *Suppose that $a > 0$ and that $\{b_\nu\}$, $\{\tau_\nu\}$ are sequences of non-negative real numbers. Assume that the sequence $\{\varphi_\nu\}$ satisfies*

$$\varphi_0 \le a \quad \textit{and} \quad \varphi_\nu \le a + \sum_{j=1}^{\nu} b_j + \sum_{j=0}^{\nu-1} \tau_j \varphi_j$$

for $n \ge 1$. Then

$$\varphi_\nu \le \big(a + \sum_{j=1}^{\nu} b_j\big)\,\exp\big(\sum_{k=0}^{\nu-1}\tau_k\big). \qquad \square$$

3. Estimates for the solution of the discrete system

In our estimates, we will frequently express a difference $(e_j - e_{j-1})/k_j$ of spatial errors as an error in velocities, δ_j. The resulting correction term K_j is characterized in the following lemma.

Lemma 3.1. *Let K_j, $j = 1, \dots, N$, be given by*

$$K_j = \frac{1}{k_j}\int_{t_{j-1}}^{t_j} (s - t_{j-1}) u_{tt}(s)\mathrm{d}s\,.$$

Then, for $j = 1, \dots, N$,

$$\frac{1}{k_j}(e_j - e_{j-1}) = \delta_j + K_j \tag{3.1}$$

and

$$\|K_j\|^2 \le \frac{k_j}{3}\int_{t_{j-1}}^{t_j}\int_\Omega u_{tt}^2 \mathrm{d}x\,\mathrm{d}t\,, \quad \|DK_j\|^2 \le \frac{k_j}{3}\int_{t_{j-1}}^{t_j}\int_\Omega |Du_{tt}|^2 \mathrm{d}x\,\mathrm{d}t\,. \tag{3.2}$$

Proof. It follows from Taylor's formula that

$$u(t_j, x) = u(t_{j-1}, x) + k_j u_t(t_j, x) - \int_{t_{j-1}}^{t_j} (s - t_{j-1}) u_{tt}(s,x)\mathrm{d}s\,.$$

This allows us to estimate

$$\begin{aligned}\frac{1}{k_j}(e_j - e_{j-1}) &= \frac{U_j - U_{j-1}}{k_j} - \frac{u(t_j) - u(t_{j-1})}{k_j} \\ &= V_j - v(t_j) + K_j = \delta_j + K_j.\end{aligned}$$

Finally, by Hölder's inequality,

$$\begin{aligned}\|K_j\|^2 &= \int_\Omega \Big(\frac{1}{k_j}\int_{t_{j-1}}^{t_j}(s-t_j)u_{tt}(s)\mathrm{d}s\Big)^2\mathrm{d}x \\ &\le \frac{1}{k_j^2}\Big(\int_{t_{j-1}}^{t_j}(s-t_{j-1})^2\mathrm{d}s\Big)\Big(\int_{t_{j-1}}^{t_j}\int_\Omega u_{tt}^2\mathrm{d}x\,\mathrm{d}t\Big) \\ &= \frac{k_j}{3}\int_{t_{j-1}}^{t_j}\int_\Omega u_{tt}^2\mathrm{d}x\,\mathrm{d}t\,.\end{aligned}$$

This concludes the proof of the lemma. □

The next proposition is a discrete analogue of (2.6). The idea is to use the orthogonality (2.10) with $\Pi_j\delta_j = \delta_j + (\Pi_j\delta_j - \delta_j)$ and to recover the structure of (2.6) plus approximation errors.

Proposition 3.1. *The following estimate holds for $j = 1, \dots, N$:*

$$\begin{aligned}&\frac{1}{2}\|\delta_j\|^2 - \frac{1}{2}\|\delta_{j-1}\|^2 + \frac{1}{4}\|\delta_j - \delta_{j-1}\|^2 \\ &\quad\le 2c_S^2\mathrm{Lip}^2(\sigma)k_j\|De_j\|^2 - \frac{k_j}{2}\|D\delta_j\|^2 + \|\delta_j - \Pi_j\delta_j\|^2 \\ &\qquad + \frac{5k_j}{2}\|D\delta_j - D\Pi_j\delta_j\|^2 + \frac{5k_j}{2}\|DK_j\|^2 + c_S^2\mathrm{Lip}^2(\sigma)\,k_j^2\|Du_t\|^2_{L^2(I_j;L^2)}.\end{aligned}$$

Proof. It follows from (2.10) with $W_j = \Pi_j\delta_j$ that

$$\begin{aligned}\frac{1}{2}\|\delta_j\|^2 - \frac{1}{2}\|\delta_{j-1}\|^2 + \frac{1}{2}\|\delta_j - \delta_{j-1}\|^2 &= (\delta_j, \delta_j - \delta_{j-1}) \\ &= (\delta_j - \delta_{j-1}, \delta_j - \Pi_j\delta_j) - (De_j - De_{j-1}, D\Pi_j\delta_j) \\ &\quad - \int_{I_j}\big(\sigma(Du) - \sigma(DU_j), D\Pi_j\delta_j\big)\mathrm{d}t\,.\end{aligned}$$

We denote the three terms on the right-hand side by T_1, T_2, and T_3. Then

$$T_1 \le \frac{1}{4}\|\delta_j - \delta_{j-1}\|^2 + \|\delta_j - \Pi_j\delta_j\|^2,$$

and by (3.1)

$$\begin{aligned}T_2 &= -k_j\big(D\delta_j + DK_j, D\delta_j - D\delta_j + D\Pi_j\delta_j\big) \\ &\le k_j\big(-\frac{3}{4}\|D\delta_j\|^2 + \frac{5}{2}\|DK_j\|^2 + \frac{5}{2}\|D\delta_j - D\Pi_j\delta_j\|^2\big).\end{aligned}$$

We have by Hölder's inequality, Young's inequality, and the stability estimate (2.2) that

$$\begin{aligned}
|T_3| &\le \int_{I_j} \|\sigma(Du)-\sigma(DU_j)\|\,\|D\Pi_j\delta_j\|\mathrm{d}t \\
&\le \sqrt{k_j}\|\sigma(Du)-\sigma(DU_j)\|_{L^2(I_j;L^2)}\,\|D\Pi_j\delta_j\| \\
&\le c_S^2\mathrm{Lip}^2(\sigma)\int_{I_j}\int_\Omega |Du-DU_j|^2\mathrm{d}x\,\mathrm{d}t + \frac{k_j}{4c_S^2}\|D\Pi_j\delta_j\|^2 \\
&\le 2c_S^2\mathrm{Lip}^2(\sigma)\int_{I_j}\int_\Omega \big(|Du-Du_j|^2+|Du_j-DU_j|^2\big)\mathrm{d}x\,\mathrm{d}t + \frac{k_j}{4}\|D\delta_j\|^2.
\end{aligned}$$

Since

$$|Du-Du_j|^2 = \Big|\int_t^{t_j} Du_t(s)\mathrm{d}s\Big|^2 \le (t_j-t)\int_t^{t_j}|Du_t(s)|^2\mathrm{d}s$$

and

$$\int_{t_{j-1}}^{t_j}(t_j-t)\int_t^{t_j}|Du_t|^2\mathrm{d}s\,\mathrm{d}t \le \int_{t_{j-1}}^{t_j}(t_j-t)\mathrm{d}t\int_{t_{j-1}}^{t_j}|Du_t|^2\mathrm{d}s = \frac{k_j^2}{2}\int_{t_{j-1}}^{t_j}|Du_t|^2\mathrm{d}t\,,$$

we obtain

$$\int_{I_j}\int_\Omega |Du-Du_j|^2\mathrm{d}x\,\mathrm{d}t \le \frac{k_j^2}{2}\int_{I_j}\int_\Omega |Du_t|^2\mathrm{d}x\,\mathrm{d}t\,.$$

The assertion of the proposition follows easily from the foregoing estimates. □

In the next proposition we derive the analogue of the estimate (2.5) for the discrete scheme.

Proposition 3.2. *The following estimate holds for $j=1,\dots,N$:*

$$\begin{aligned}
&\frac12\|De_j\|^2-\frac12\|De_{j-1}\|^2+\frac14\|De_j-De_{j-1}\|^2 \\
&\quad\le (c_S^2\mathrm{Lip}^2(\sigma)+\frac12)k_j\|De_j\|^2-(\delta_j-\delta_{j-1},e_j)+\frac{k_j}{2}\|D\delta_j\|^2 \\
&\qquad+\frac14\|\delta_j-\delta_{j-1}\|^2+\|e_j-\Pi_je_j\|^2+k_j\|De_j-D\Pi_je_j\|^2 \\
&\qquad+\frac{k_j}{2}\|DK_j\|^2+\frac12\,c_S^2\mathrm{Lip}^2(\sigma)k_j^2\,\|Du_t\|^2_{L^2(I_j;L^2)}.
\end{aligned}$$

Proof. We obtain from (2.10) with $W_j=\Pi_je_j$ that

$$\begin{aligned}
\frac12\|De_j\|^2-\frac12\|De_{j-1}\|^2+\frac12\|De_j-De_{j-1}\|^2 &= \big(De_j-De_{j-1},De_j-D\Pi_je_j\big) \\
&\quad-(\delta_j-\delta_{j-1},\Pi_je_j)-\int_{I_j}\big(\sigma(Du)-\sigma(DU_j),D\Pi_je_j\big)\mathrm{d}t\,.
\end{aligned}$$

We denote the three terms on the right-hand side by T_1, T_2, and T_3. By Lemma 3.1,

$$\begin{aligned}
T_1 &= k_j\big(D\delta_j+DK_j,De_j-D\Pi_je_j\big) \\
&\le k_j\big(\frac12\|D\delta_j\|^2+\frac12\|DK_j\|^2+\|De_j-D\Pi_je_j\|^2\big),
\end{aligned}$$

and by Young's inequality

$$\begin{aligned} T_2 &= (\delta_j - \delta_{j-1}, e_j - \Pi_j e_j) - (\delta_j - \delta_{j-1}, e_j) \\ &\le -(\delta_j - \delta_{j-1}, e_j) + \frac{1}{4}\|\delta_j - \delta_{j-1}\|^2 + \|e_j - \Pi_j e_j\|^2. \end{aligned}$$

Finally, T_3 can be estimates as in the proof of Proposition 3.1 by

$$\begin{aligned} T_3 &\le \sqrt{k_j}\mathrm{Lip}(\sigma)\|Du - DU_j\|_{L^2(I_j;L^2)}\|D\Pi_j e_j\| \\ &\le \frac{1}{2}\, c_S^2 \mathrm{Lip}^2(\sigma)\, \|Du - DU_j\|^2_{L^2(I_j;L^2)} + \frac{k_j}{2c_S^2}\|D\Pi_j e_j\|^2 \\ &\le \frac{1}{2}\, c_S^2 \mathrm{Lip}^2(\sigma) k_j^2\, \|Du_t\|^2_{L^2(I_j;L^2)} + (c_S^2\mathrm{Lip}^2(\sigma) + \frac{1}{2})k_j\|De_j\|^2. \end{aligned}$$

The assertion of the proposition follows from the foregoing inequalities. □

We now combine the estimates in Propositions 3.1 and 3.2 to obtain an estimate on the time interval I_j.

Corollary 3.1. *Let $\mathcal{A}_j$ denote the approximation errors,*

$$\mathcal{A}_j = \|\delta_j - \Pi_j\delta_j\|^2 + \frac{5}{2}\, k_j\|D\delta_j - D\Pi_j\delta_j\|^2 + \|e_j - \Pi_j e_j\|^2 + k_j\|De_j - D\Pi_j e_j\|^2,$$

and $\mathcal{R}_j$ the terms depending on the regularity of u,

$$\mathcal{R}_j = \frac{k_j}{2}\, \|K_j\|^2 + 3k_j\|DK_j\|^2 + \frac{3}{2}\, c_S^2\mathrm{Lip}^2(\sigma)k_j^2\|Du_t\|^2_{L^2(I_j;L^2)}.$$

Then the following estimate holds for $j = 1, \dots, N$:

$$\begin{aligned} &\frac{1}{2}\|\delta_j\|^2 - \frac{1}{2}\|\delta_{j-1}\|^2 + \frac{1}{2}\|De_j\|^2 - \frac{1}{2}\|De_{j-1}\|^2 + \frac{1}{4}\|De_j - De_{j-1}\|^2 \\ &\le \big(3c_S^2\mathrm{Lip}^2(\sigma) + \frac{1}{2}\big)k_j\|De_j\|^2 - (\delta_j - \delta_{j-1}, e_j) + \mathcal{A}_j + \mathcal{R}_j - \frac{k_j}{2}\, \|K_j\|^2. \qquad \square \end{aligned} \tag{3.3}$$

Remark. The discrete Gronwall inequality will be applied to the sum of (3.3) in j. The sum of the terms $-(\delta_j - \delta_{j-1}, e_j)$ on the right-hand side is estimated by a discrete summation by parts which leads to the (discrete) time integral of $\|\delta_j\|^2$.

Proposition 3.3. *Let*

$$q_j = \frac{k_j}{k_{j-1}}, \quad Q_\nu = \max_{j=2,\dots,\nu} q_j, \quad c_3 = \max\big\{3c_S^2\mathrm{Lip}^2(\sigma) + \frac{1}{2}, Q_N + \frac{1}{2}\big\}. \tag{3.4}$$

Then the following estimate holds for $\nu = 1, \dots, N$:

$$\begin{aligned} \frac{1}{2}\|\delta_\nu\|^2 + \frac{1}{2}\|De_\nu\|^2 \le c_3 \sum_{j=1}^{\nu} k_j\big(\|\delta_j\|^2 + \|De_j\|^2\big) - (\delta_\nu, e_\nu) + (\delta_0, e_1) \\ + \sum_{j=1}^{\nu}(\mathcal{A}_j + \mathcal{R}_j) + \frac{1}{2}\|\delta_0\|^2 + \frac{1}{2}\|De_0\|^2. \end{aligned}$$

Proof. We take the sum of the inequality (3.3) for $j = 1, \dots, \nu$ and obtain

$$\frac{1}{2}\|\delta_\nu\|^2 + \frac{1}{2}\|De_\nu\|^2 \le \big(3c_S^2\mathrm{Lip}^2(\sigma) + \frac{1}{2}\big)\sum_{j=1}^{\nu} k_j\|De_j\|^2 - \sum_{j=1}^{\nu}\big(\delta_j - \delta_{j-1}, e_j\big)$$
$$+ \sum_{j=1}^{\nu}(\mathcal{A}_j + \mathcal{R}_j - \frac{k_j}{2}\|K_j\|^2) + \frac{1}{2}\|\delta_0\|^2 + \frac{1}{2}\|De_0\|^2.$$

With a discrete summation by parts in the second term on the right-hand side, we deduce that

$$-\sum_{j=1}^{\nu}\big(\delta_j - \delta_{j-1}, e_j\big) = -(\delta_\nu, e_\nu) + (\delta_0, e_1) + \sum_{j=1}^{\nu-1}\big(\delta_j, e_{j+1} - e_j\big)$$
$$= -(\delta_\nu, e_\nu) + (\delta_0, e_1) + \sum_{j=1}^{\nu-1} k_{j+1}\big(\delta_j, \delta_{j+1} + K_{j+1}\big)$$
$$\le -(\delta_\nu, e_\nu) + (\delta_0, e_1) + \sum_{j=1}^{\nu-1} k_{j+1}\big(\|\delta_j\|^2 + \frac{1}{2}\|\delta_{j+1}\|^2 + \frac{1}{2}\|K_{j+1}\|^2\big).$$

The assertion follows from this inequality since $k_{j+1}\|\delta_j\|^2 = q_{j+1}k_j\|\delta_j\|^2$. □

In order to apply Gronwall's inequality, we need a further summation of the inequality in Proposition 3.3. The term $-(\delta_\nu, e_\nu)$ corresponds to the spatial integral of $u_t u = \partial_t|u|^2/2$ which fits naturally in the formulation of Gronwall's inequality in Section 2.2. This is not the case in the implicit time discretization used here.

Proposition 3.4. *Let*

$$\varphi_\nu = \|e_\nu\|^2 + \sum_{j=1}^{\nu} k_j\big(\|\delta_j\|^2 + \|De_j\|^2\big). \tag{3.5}$$

Then

$$\frac{1}{2}\varphi_N \le \big(c_3 + \frac{1}{2}\big)\sum_{\nu=1}^{N} k_\nu\varphi_\nu + T(\delta_0, e_1) + \frac{1}{2}\|e_0\|^2$$
$$+ \frac{T}{2}\big(\|\delta_0\|^2 + \|De_0\|^2\big) + 2\sum_{\nu=1}^{N} k_\nu \sum_{j=1}^{\nu}\big(\mathcal{A}_j + \mathcal{R}_j\big).$$

Proof. We multiply the inequality in the assertion of Proposition 3.3 by k_ν and take the sum of the resulting inequalities from $\nu = 1$ to N. This leads to

$$\frac{1}{2}\sum_{\nu=1}^{N} k_\nu\big(\|\delta_\nu\|^2 + \|De_\nu\|^2\big) \le c_3\sum_{\nu=1}^{N} k_\nu\sum_{j=1}^{\nu} k_j\big(\|\delta_j\|^2 + \|De_j\|^2\big)$$
$$-\sum_{\nu=1}^{N} k_\nu(\delta_\nu, e_\nu) + T(\delta_0, e_1) + \sum_{\nu=1}^{N} k_\nu\sum_{j=1}^{\nu}\big(\mathcal{A}_j + \mathcal{R}_j\big) + \frac{T}{2}\big(\|\delta_0\|^2 + \|De_0\|^2\big).$$

In view of Lemma 3.1,

$$\begin{aligned}
-\sum_{\nu=1}^{N} k_\nu\,(\delta_\nu, e_\nu) &= -\sum_{\nu=1}^{N}(e_\nu - e_{\nu-1} - k_\nu K_\nu, e_\nu)\\
&= -\frac{1}{2}\sum_{\nu=1}^{N}\big(\|e_\nu\|^2 - \|e_{\nu-1}\|^2 + \|e_\nu - e_{\nu-1}\|^2\big) + \sum_{\nu=1}^{N} k_\nu (K_\nu, e_\nu)\\
&\le -\frac{1}{2}\,\|e_N\|^2 + \frac{1}{2}\,\|e_0\|^2 + \frac{1}{2}\sum_{\nu=1}^{N} k_\nu\big(\|e_\nu\|^2 + \|K_\nu\|^2\big).
\end{aligned}$$

This implies

$$\begin{aligned}
&\frac{1}{2}\Big\{\|e_N\|^2 + \sum_{\nu=1}^{N} k_\nu\big(\|\delta_\nu\|^2 + \|De_\nu\|^2\big)\Big\}\\
&\le \big(c_3 + \frac{1}{2}\big)\sum_{\nu=1}^{N} k_\nu\Big\{\|e_\nu\|^2 + \sum_{j=1}^{\nu} k_j\big(\|\delta_j\|^2 + \|De_j\|^2\big)\Big\}\\
&\quad + T(\delta_0, e_1) + \frac{1}{2}\|e_0\|^2 + \frac{T}{2}\big(\|\delta_0\|^2 + \|De_0\|^2\big) + 2\sum_{\nu=1}^{N} k_\nu \sum_{j=1}^{\nu}\big(\mathcal{A}_j + \mathcal{R}_j\big).
\end{aligned}$$

This implies the assertion of the proposition. □

It only remains to estimate the term $T(\delta_0, e_1)$ on the right-hand side. This is done in the next lemma.

Lemma 3.2. *Let φ_ν be the quantity defined in (3.5). Then*

$$\big|T(\delta_0, e_1)\big| \le k_1 \sum_{\nu=1}^{N} k_\nu \varphi_\nu + 2k_1 T\mathcal{R}_1 + \frac{3T}{4}\,\|\delta_0\|^2 + T\|e_0\|^2.$$

Proof. It follows from (3.1) that

$$\begin{aligned}
\big|(\delta_0, e_1)\big| &\le \big|(\delta_0, e_1 - e_0)\big| + \big|(\delta_0, e_0)\big| \le \big|k_1(\delta_0, \delta_1 + K_1)\big| + \big|(\delta_0, e_0)\big|\\
&\le \frac{1}{4}\|\delta_0\|^2 + k_1^2\|\delta_1\|^2 + \frac{1}{4}\|\delta_0\|^2 + k_1^2\|K_1\|^2 + \frac{1}{4}\|\delta_0\|^2 + \|e_0\|^2.
\end{aligned}$$

By definition, $k_1\|K_1\|^2 \le 2\mathcal{R}_1$ and $k_1\|\delta_1\|^2 \le \varphi_\nu$ for $\nu = 1, \dots, N$. Thus

$$T\big|(\delta_0, e_1)\big| \le \frac{3T}{4}\,\|\delta_0\|^2 + T\|e_0\|^2 + k_1\sum_{\nu=1}^{N} k_\nu\varphi_\nu + 2k_1\sum_{\nu=1}^{N} k_\nu\mathcal{R}_1.$$

This implies the assertion of the lemma. □

4. General convergence result

We are now in a position to state and prove the convergence result for the approximation of solutions of the system (1.1) by the implicit Euler scheme defined in Section 2.3. Recall that $\mathcal{A}_j$, $\mathcal{R}_j$ and φ_ν have been defined in Proposition 3.1 and Corollary 3.4, respectively. Moreover, we set

$$\mathcal{A}(T) = 8\big(1 + k_1\big)\sum_{\nu=1}^{N} k_\nu \sum_{j=1}^{\nu}\mathcal{A}_j, \quad \mathcal{R}(T) = 8\big(1 + k_1\big)\sum_{\nu=1}^{N} k_\nu \sum_{j=1}^{\nu}\mathcal{R}_j,$$

and

$$a = 2(2T+1)\|e_0\|^2 + 2T\|De_0\|^2 + 5T\|\delta_0\|^2. \tag{4.1}$$

Finally, we define $c_4 = c_4(\mathrm{Lip}(\sigma), Q_N)$ and $c_5 = c_5(\mathrm{Lip}(\sigma), Q_N)$ by

$$c_4 = 4\big(c_3 + \frac{1}{2} + k_1\big), \quad c_5 = \max\big\{\frac{3}{2}c_S^2\mathrm{Lip}^2(\sigma), 1\big\}, \tag{4.2}$$

respectively.

Theorem 4.1. *Suppose that Ω, $\mathcal{T}_j$ and $\mathcal{S}_{0,j}$ satisfy the assumptions in Section 2.1, and that there exists a $Q_N > 0$ such that $k_{j-1} \le Q_N k_j$ for $j = 2, \dots, N$. Assume, furthermore, that σ is globally Lipschitz continuous, and that the unique solution of the system (1.1) belongs to $W^{1,2}(W^{1,2})$. Suppose that $k < 1/c_4$. Then*

$$\max_{\nu=1,\dots,N} \|e_\nu\|^2 + \sum_{\nu=1}^N k_\nu\big(\|\delta_\nu\|^2 + \|De_\nu\|^2\big) \le \big(a + \mathcal{A}(T) + \mathcal{R}(T)\big)\exp\big(c_4 T\big).$$

Moreover, if $u \in W^{1,\infty}(W^{2,2}) \cap W^{2,2}(L^2)$, then

$$\mathcal{R}(T) \le 12c_5 T\, k^2\Big(\|u_{tt}\|^2_{L^2(I_j;L^2)} + \|Du_{tt}\|^2_{L^2(I_j;L^2)} + \|Du_t\|^2_{L^2(I_j;L^2)}\Big),$$

and

$$\mathcal{A}(T) \le 60c_A^2T^2h^2 \max_{j=1,\dots,N}\big(\frac{h_j^2}{k_j} + 1\big)\|u\|^2_{W^{1,\infty}(W^{2,2})} + \mathcal{C}(T),$$

where $\mathcal{C}(T)$ is the coarsening error,

$$\mathcal{C}(T) = 12T\sum_{j=1}^N \Big\{\frac{2}{k_j^2}\big\|U_{j-1} - \Pi_j U_{j-1}\big\| + \frac{5}{k_j}\big|U_{j-1} - \Pi_j U_{j-1}\big|^2_{1,2}\Big\}$$

which vanishes if $\mathcal{T}_j$ is a refinement of $\mathcal{T}_{j-1}$. Finally, if the approximation estimate (2.7) holds, then

$$a \le c_A^2 h_0^2\big((4T+2)h_0^2 + 4T\big)\big(\|D^2u_0\|^2 + \|Dv_0\|^2\big).$$

Proof. Based on the estimates in Section 3, we first show that φ_ν satisfies the assumptions in the discrete Gronwall inequality of Lemma 2.1. It follows from Proposition 3.4 and Lemma 3.2 that

$$\begin{aligned}\frac{1}{2}\varphi_N \le \big(c_3 + \frac{1}{2} + k_1\big)\sum_{\nu=1}^N k_\nu\varphi_\nu + 2\big(1+k_1\big)\sum_{\nu=1}^N k_\nu\sum_{j=1}^\nu\big(\mathcal{A}_j + \mathcal{R}_j\big)\\ + \big(T + \frac{1}{2}\big)\|e_0\|^2 + \frac{T}{2}\|De_0\|^2 + \frac{5T}{4}\|\delta_0\|^2.\end{aligned}$$

By assumption, $k_N \le k$ and thus $c_4k_N = 4\big(c_3 + 1/2 + k_1\big)k_N \le 1$. This allows us to absorb the term $\big(c_3 + \frac{1}{2} + k_1\big)k_N\varphi_N \le \varphi_N/4$ on the left-hand side. We obtain

$$\begin{aligned}\varphi_N \le c_4\sum_{\nu=1}^{N-1} k_\nu\varphi_\nu + 8(1+k_1)\sum_{\nu=1}^N k_\nu\sum_{j=1}^\nu\big(\mathcal{A}_j + \mathcal{R}_j\big)\\ + 2(2T+1)\|e_0\|^2 + 2T\|De_0\|^2 + 5T\|\delta_0\|^2.\end{aligned}$$

It follows that the assumptions in the discrete Gronwall inequality in Lemma 2.1 are satisfied with a as defined in (4.1) and

$$b_\nu = 8(1+k_1)k_\nu \sum_{j=1}^{\nu} \big(\mathcal{A}_j + \mathcal{R}_j\big), \quad \tau_\nu = c_4 k_\nu, \quad \nu = 1,\dots,N.$$

Thus

$$\varphi_N \le \big(a + \sum_{\nu=1}^{N} b_\nu\big)\, \exp(c_4 T) = \big(a + \mathcal{A}(T) + \mathcal{R}(T)\big)\, \exp(c_4 T).$$

We now estimate $\mathcal{A}(T)$ and $\mathcal{R}(T)$. By definition of $\mathcal{R}_j$ in Corollary 3.1 and by (3.2) we infer

$$\mathcal{R}_j \le c_5 k_j^2 \Big(\|u_{tt}\|^2_{L^2(I_j;L^2)} + \|Du_{tt}\|^2_{L^2(I_j;L^2)} + \|Du_t\|^2_{L^2(I_j;L^2)}\Big).$$

This implies

$$\sum_{\nu=1}^{N} k_\nu \sum_{j=1}^{\nu} \mathcal{R}_j \le c_5 T\, k^2 \big(\|u_{tt}\|^2_{L^2(L^2)} + \|Du_{tt}\|^2_{L^2(L^2)} + \|Du_t\|^2_{L^2(L^2)}\big).$$

Since $kc_4 \le 1$ implies $k_1 \le 1/2$ we obtain

$$\mathcal{R}(T) \le 12 c_5 T\, k^2 \Big(\|u_{tt}\|^2_{L^2(I_j;L^2)} + \|Du_t\|^2_{L^2(I_j;L^2)} + \|Du_t\|^2_{L^2(L^2)}\Big).$$

It remains to estimate the approximation error. By definition of e_j and (2.1),

$$e_j - \Pi_j e_j = u(t_j) - U_j - \Pi_j\big(u(t_j) - U_j\big) = u(t_j) - \Pi_j u(t_j).$$

The approximation estimate (2.3) implies

$$\|e_j - \Pi_j e_j\|^2 + k_j \|De_j - D\Pi_j e_j\|^2 \le c_A^2 h_j^2 \big(h_j^2 + k_j\big)\, \|D^2 u(t_j)\|^2.$$

Similarly,

$$\delta_j - \Pi_j \delta_j = v(t_j) - \Pi_j v(t_j) + \frac{1}{k_j}\big(U_{j-1} - \Pi_j U_{j-1}\big).$$

This estimate and (2.2)-(2.3) yield

$$\begin{aligned}\|\delta_j - \Pi_j\delta_j\|^2 + \frac{5}{2}k_j\|D\delta_j - D\Pi_j\delta_j\|^2 &\le 2c_A^2 h_j^2\big(h_j^2 + \frac{5}{2}k_j\big)\,\|D^2 u_t(t_j)\|^2 \\ &+ \frac{2}{k_j^2}\|U_{j-1} - \Pi_j U_{j-1}\|^2 + \frac{5}{k_j}|U_{j-1} - \Pi_j U_{j-1}|^2_{1,2}.\end{aligned}$$

We conclude from the foregoing estimates that

$$\begin{aligned}\sum_{j=1}^{N} \mathcal{A}_j &\le 5\sum_{j=1}^{N} h_j^2 c_A^2 \big(h_j^2 + k_j\big)\,\big(\|D^2 u(t_j)\|^2 + \|D^2 u_t(t_j)\|^2\big) + \frac{1}{12T}\,\mathcal{C}(T) \\ &\le 5c_A^2 T h^2 \max_{j=1,\dots,N}\big(\frac{h_j^2}{k_j} + 1\big)\,\|u\|^2_{W^{1,\infty}(W^{2,2})} + \frac{1}{12T}\,\mathcal{C}(T).\end{aligned}$$

Hence

$$\mathcal{A}(T) \le 12\, T \sum_{j=1}^{N} \mathcal{A}_j \le 60 c_A^2 T^2 h^2 \max_{j=1,\dots,N}\big(1 + \frac{h_j^2}{k_j}\big)\,\|u\|^2_{W^{1,\infty}(W^{2,2})} + \mathcal{C}(T).$$

Clearly, if $\mathcal{T}_j$ is a refinement of $\mathcal{T}_{j-1}$, then, by (2.1), $\mathcal{C}(T) = 0$. Finally, if (2.7) holds, then

$$a \le c_A^2\big(2(2T+1)h_0^4 + 2Th_0^2\big)\|D^2u_0\|^2 + 4Th_0^2\|Dv_0\|^2.$$

This proves the assertion of the theorem. □

Proof of Theorem 1.1. This is a special case of Theorem 4.1. □

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