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by

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LOCAL ESTIMATES FOR A CLASS OF FULLY NONLINEAR EQUATIONS ARISING FROM CONFORMAL GEOMETRY

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1. Introduction

Conformal deformations play an important role in the global geometry. In general, such deformations are guided by certain partial differential equations. Yamabe problem is one of the examples. In this paper, we are interested in a class of fully nonlinear differential equations related to the deformation of conformal metrics.

Let (M, g_0) be a compact connected smooth Riemannian manifold of dimension $n \geq 3$, and let $[g_0]$ denote the conformal class of g_0. The Schouten tensor of the metric g is defined as

$$S_g = \frac{1}{n-2}\left(Ric_g - \frac{R_g}{2(n-1)} \cdot g\right),$$

where Ric_g and R_g are the Ricci tensor and scalar curvature of g respectively. This tensor is connected to the study of conformal invariants, in particular conformally invariant tensors and differential operators (e.g., see [6] and references therein). In [16], The following σ_k*-scalar curvatures* of g were considered by Viaclovsky in [16]:

$$\sigma_k(g) := \sigma_k(g^{-1} \cdot S_g),$$

where σ_k is the kth elementary symmetric function, $g^{-1}\cdot S_g$ is locally defined by $(g^{-1}\cdot S_g)^i_j = g^{ik}(S_g)_{kj}$. When $k = 1$, σ_1-scalar curvature is just the scalar curvature R (upto a constant multiple). σ_k can also be viewed as a function of the eigenvalues of symmetric matrices, that is a function in $\mathbb{R}^n$. According to Gårding [7],

$$\Gamma_k^+ = \{\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{R}^n \mid \sigma_j(\Lambda) > 0, \forall j \leq k\},$$

is a natural class for σ_k. A metric g is said to be in Γ_k^+ if $\sigma_j(g)(x) > 0$ for $j \leq k$ and $x \in M$.

The case of $k = 1$, deforming scalar curvature R to a constant in its conformal class is known as the Yamabe problem, the final solution was obtained by Schoen in [12] (see also [1] and [15]). We refer [10] for the literature on Yamabe problem. There is a recent interest in deforming σ_k-scalar curvature in its conformal class. This type of problem was

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considered by Viaclovsky [16] and [18]. If $g = e^{-2u} g_0$, the problem is equivalent to solve the following fully nonlinear equation introduced in [16]:

$$\sigma_k^{1/k}\left(\nabla^2 u + du\otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0}\right) = e^{-2u}. \tag{1}$$

Here, and in the rest of the paper, we will always work with the background metric g_0. More generally, one would like to consider equation of the form

$$\sigma_k^{1/k}\left(\nabla^2 u + du\otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0}\right) = fe^{-2u}, \tag{2}$$

for a nonnegative function f.

The equation (1) is a type of fully nonlinear equation when $k \geq 2$. To solve the problem, one needs to establish a priori estimates for the solutions of these equations. One may immediately find out that such a priori estimates can not exist in general. On the standard sphere there is a non-compact family of solutions to equation (1). In solving the Yamabe problem, the blow-up (or rescaling) technique plays a very important role to rule out the exceptional case of standard sphere. This kind of technique can be applied since there exist *local estimates* in the Yamabe problem, which corresponds to a semilinear elliptic equation. The main objective of this paper is to establish the similar *local estimates* for the fully nonlinear equation (1). These are the local derivative estimates upto second order for the solutions, the crucial step is the local C^1 estimates. These *local estimates* bear some direct consequences to uniqueness and existence of equation (1) by following similar steps as in Schoen's work ([13, 14]) in the Yamabe problem. We will persue these elsewhere.

We note that *local estimates* in general do not hold for fully nonlinear equations. Pogorelov [11] constructed an example for Monge-Ampère equation which there is no interior estimates when the dimension $n \geq 3$.

There have been some recent developments related to the equation (2). Viaclovsky investigated variational and uniqueness properties of the equation in [16] and [17]. In [18], he obtained global C^2 estimates for equation (2) depending on global C^0 bounds. When $k = n$, he proved global C^0 bounds and the existence, under some geometric conditions. In an important case $n = 4$ and $k = 2$, Chang, Gursky and Yang obtained a global a priori estimate in [4] by geometric arguments for the equation (2) when the manifold is not conformally equivalent to the standard 4-sphere, which in turn gives the existence of the solutions for equation (1) in the special case $n = 4$ and $k = 2$.

Now, we state our main results.

Theorem 1. *Suppose f is a positive function on M. Let $u \in C^4$ be an admissible solution (See Definition 1) of (2) in B_r, the geodesic ball of radius r in a Riemannian manifold (M, g_0). Then, there exists a constant $c > 0$ depending only on r, $\|g_0\|_{C^4(B_r)}$ and $\|f\|_{C^2(B_r)}$ (independent of $\inf f$), such that*

$$\|u\|_{C^2(B_{r/2})} \leq c(1 + e^{-2\inf_{B_r} u}).$$

As a consequence, we have the following ϵ-convergence.

Corollary 1. *There exists a constant $\varepsilon_0 > 0$ such that for any sequence of solutions u_i of (2) in B_1 with*

$$\int_{B_1} e^{-nu} dvol(g_0) \leq \varepsilon_0,$$

either

(1) *There is a subsequence u_{i_l} uniformly converges to $+\infty$ in any compact subset in B_1, or*
(2) *There is a subsequence u_{i_l} converges strongly in $C^{1,\alpha}_{loc}(B_1)$, $\forall 0 < \alpha < 1$. If f is smooth and strictly positive in B_1, then u_{i_l} converges strongly in $C^m_{loc}(B_1)$, $\forall m$.*

If u is an entire solution of

$$\sigma_k^{1/k}\left(\nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g_E\right) = e^{-2u}, \qquad in\ \mathbb{R}^n$$

with finite volume $\int_{\mathbb{R}^n} e^{-nu} dvol(g_E) < \infty$, then $\inf_{\mathbb{R}^n} u > -\infty$. Here g_E is the standard metric of $\mathbb{R}^n$.

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2. A Harnack inequality

We begin this section by recalling some basic properties of elementary symmetric functions. Let $\Lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbf{R}^n$. The k-th elementary symmetric functions is defined as

$$\sigma_k(\Lambda) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}.$$

Set $\sigma_0 = 1$ and $\sigma_q = 0$ for $q > n$. One can use another equivalent definition of Γ_k^+:

$$\Gamma_k^+ = \{\text{component of } \{\sigma_k > 0\} \text{ containing the positive cone}\}.$$

A real symmetric $n \times n$ matrix A is said to lie in Γ_k^+ if its eigenvalues lie in Γ_k^+.

Let $\Lambda_i = (\lambda_1, \cdots, \check{\lambda_i}, \cdots, \lambda_n) = (\lambda_1, \lambda_2, \cdots, \lambda_{i-1}, \lambda_{i+1}, \cdots, \lambda_n)$ and $\Lambda_{ij} = (\lambda_1, \cdots, \check{\lambda_i}, \cdots, \check{\lambda_j}, \cdots, \lambda_n)$ for $i \neq j$. Therefore, $\sigma_q(\Lambda_i)$ ($\sigma_q(\Lambda_{ij})$ resp.) means the sum of the terms of $\sigma_q(\Lambda)$ not containing the factor λ_i (λ_i and λ_j resp.). We list the following well known properties of σ_k and Γ_k^+ (e.g., see [8], [7] and [2])

Proposition 1. *Newton-MacLaurin inequality*

$$(n - q + 1)(q + 1)\sigma_{q-1}(\Lambda)\sigma_{q+1}(\Lambda) \leq q(n - q)\sigma_q^2(\Lambda). \tag{3}$$

If $\Lambda \in \Gamma_k^+$,

$$\frac{n!k}{(k-1)!(n-k+1)!(n-k+1)}\sigma_k^{k-1}(\Lambda) \leq \sigma_{k-1}^k(\Lambda). \tag{4}$$

Γ_k^+ is an open convex cone. Let $F=\sigma_k^{1/k}$, then the matrix $\frac{\partial F}{\partial A_{ij}}$ is positive definite for $A\in\Gamma_k^+$ and by (4),

$$\sum_j F^{jj}\geq 1, \tag{5}$$

where A_{ij} are the entries of A. The function F is concave in Γ_k^+. If $A=(A_{ij})$ is diagonal with $A=\Lambda$. Then, $\forall l$ fixed,

$$F^{ll}=\frac{1}{n-k+1}\sum_{i\geq 1}F^{ii}-\frac{1}{k}F^{1-k}\lambda_l\sigma_{k-2}(\Lambda_l). \tag{6}$$

Furthermore, if $\Lambda\in\Gamma_q^+$, then $\Lambda_i\in\Gamma_{q-1}^+, \forall q=0,1,\cdots,n,\ i=1,2,\cdots,n.$

Definition 1. *Let $W=(\nabla^2u+du\otimes du-\frac{|\nabla u|^2}{2}g_0+\sigma_{g_0})$, for $u\in C^2$, we say u is an admissible solution of equation (2) if W is in Γ_k^+.*

The following is the local C^1 estimates.

Proposition 2. *Let $u\in C^3$ be an admissible solution of (2) in B_r for some $r>0$. There exists a constant $c>0$ depending only on k, n, r, $\|g_0\|_{C^3(B_r)}$ and $\|f\|_{C^1(B_r)}$ such that*

$$|\nabla u|^2(x)<c(1+e^{-2\inf_{B_r}u}), \qquad \textit{for } x\in B_{r/2}. \tag{7}$$

Proof: We may assume $r=1$. Let ρ be a test function $\rho\in C_0^\infty(B_1)$ such that

$$\begin{array}{rcll} \rho & \geq & 0, & \text{in } B_1,\\ \rho & = & 1, & \text{in } B_{1/2},\\ |\nabla\rho(x)| & \leq & 2b_0\rho^{1/2}(x), & \text{in } B_1,\\ |\nabla^2\rho| & \leq & b_0, & \text{in } B_1. \end{array} \tag{8}$$

Here $b_0>1$ is a constant .

Set $H=\rho|\nabla u|^2$ and assume that H achieves its maximum at x_0. After appropriate choice of the normal coordinates at x_0, we may assume that W is diagonal at the point. Let w_{ij} be the entries of W, we have at x_0,

$$\begin{aligned} w_{ii}&=u_{ii}+u_i^2-\frac{1}{2}|\nabla u|^2+S_{ii},\\ u_{ij}&=-u_iu_j-S_{ij}, \qquad \forall i\neq j, \end{aligned} \tag{9}$$

where S_{ij} are entries of S_{g_0} and $u_i=\nabla_iu=\frac{\partial u}{\partial x_i}$. Since x_0 is the maximum point of H, we have $H_i(x_0)=0$, i.e.,

$$\sum_{l=1}^n u_{il}u_l=-\frac{\rho_i}{2\rho}|\nabla u|^2, \tag{10}$$

where $u_{il} = \nabla_l \nabla_i u$, i.e., the covariant derivative of $\nabla_i u$ in the direction of $\frac{\partial}{\partial x_l}$ with respect to the metric g_0. Similarly, we will use the notations u_{ijl} and u_{ijlm} to denote higher order covariant derivatives. By the choice of the test function ρ, we have at x_0

$$|\sum_{l=1}^{n} u_{il}u_l| \leq b_0 \rho^{-1/2} |\nabla u|^2. \tag{11}$$

We may assume that

$$H(x_0) \geq A_0^2 b_0^2,$$

i.e., $\rho^{-1/2} \leq \frac{1}{A_0 b_0}|\nabla u|$, and

$$|S_{g_0}| \leq A_0^{-1} |\nabla u|^2,$$

where A_0 is a large fixed number to be chosen later, otherwise we are done. Thus, from (11) we have

$$|\sum_{l=1}^{n} u_{il}u_l| \leq \frac{|\nabla u|^3}{A_0}(x_0). \tag{12}$$

Since x_0 is the maximum point of H, the matrix

$$(H_{ij}) = \left((-2\frac{\rho_i \rho_j}{\rho} + \rho_{ij})|\nabla u|^2 + 2\rho u_{lij} u_l + 2\rho u_{il} u_{jl} \right)$$

is nonpositive definite. Set

$$F^{ij} = \frac{\partial {\sigma_k}^{1/k}}{\partial w_{ij}}.$$

(F^{ij}) is a diagonal matrix at x_0 as W is diagonal.

We denote $\lambda_i = w_{ii}$ and $\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$. In what follows, we denote C (which may vary from line to line) as a constant depending only on $\|f\|_{C^1(B_1)}$, k, n, and $\|g_0\|_{C^3(B_1)}$ ($\|f\|_{C^2(B_1)}$ and $\|g_0\|_{C^4(B_1)}$ in the next section). By Proposition 1 and (12),

$$0 \geq F^{ij} H_{ij} = F^{ij} \left\{ \left(-2\frac{\rho_i \rho_j}{\rho} + \rho_{ij} \right) |\nabla u|^2 + 2\rho u_{lij} u_l + 2\rho u_{il} u_{jl} \right\}. \tag{13}$$

The first term in (13) is bounded from below by $10 b_0^2 \sum_{i \geq 1} F^{ii} |\nabla u|^2$. Let us denote R_{ijlm} the curvature tensor of g_0. Since

$$u_{lij} = u_{ijl} + \sum_m R_{lijm} u_m,$$

the second term in (13) can be estimated as follows,

$$
\begin{aligned}
\sum_{i,j,l} F^{ij} u_{ijl} u_l &\geq \sum_{i,j,l} F^{ij} u_{ijl} u_l - C|\nabla u|^2 \sum_i F^{ii} \\
&= \sum_{i,j,l} \{F^{ij}(w_{ij})_l u_l - F^{ij}(u_i u_j - \frac{|\nabla u|^2}{2}\delta_{ij})_l u_l\} - C|\nabla u|^2 \sum_i F^{ii} \\
&= \sum_l F_l u_l - 2\sum_{i,j,l} F^{ij} u_{il} u_j u_l + \sum_{i,k,l} F^{ii} u_{kl} u_k u_l - C|\nabla u|^2 \sum_i F^{ii} \\
&= \sum_l e^{-2u}(f_l u_l - 2f|\nabla u|^2) - 2\sum_{i,l} F^{ii} u_{il} u_l u_i + \sum_{i,l} F^{ii} u_{il} u_l u_i \\
&\quad -C|\nabla u|^2 \sum_i F^{ii} \\
&\geq -C(1+e^{-2u})|\nabla u|^2 - \sum_i F^{ii} \frac{|\nabla u|^4}{A_0}.
\end{aligned} \tag{14}
$$

Here, we have used $\sum_i F^{ii} w_{ii} = F$.

We need the following crucial Lemma.

Lemma 1. *We may pick constant A_0 sufficient large (depending only on k, n, and $\|g_0\|_{C^3(B_1)}$), such that,*

$$
\sum_{i,j,l} F^{ij} u_{il} u_{jl} \geq A_0^{-\frac{3}{4}} |\nabla u|^4 \sum_{i\geq 1} F^{ii}. \tag{15}
$$

Assuming the lemma, the Proposition can be proved as follows.

Inequalities (13), (14), (5) and (15) yield

$$
\begin{aligned}
0 &\geq -10b_0^2|\nabla u|^2 \sum_j F^{jj} - Ce^{-2u}\rho|\nabla u|^2 + \left(-\frac{(n+2)^2}{A_0} + A^{-\frac{3}{4}}\right)\rho|\nabla u|^4 \sum_j F^{jj} \\
&\geq \sum_j F^{jj} \left\{-10nb_0^2|\nabla u|^2 - Ce^{-2\inf u}|\nabla u|^2 + \left(-\frac{(n+2)^2}{A_0} + A_0^{-\frac{3}{4}}\right)\rho|\nabla u|^4\right\}.
\end{aligned} \tag{16}
$$

Choosing A_0 large enough so that $A_0 > 2((n+2)^2)^4$ and multipling (16) by ρ, we get

$$
H^2 \leq C(1+e^{-2\inf u})H,
$$

thus

$$
|\nabla u(x)|^2 \leq C(1+e^{-2\inf_{x\in B_1} u}) \quad \text{for } x \in B_{1/2}. \tag{17}
$$

Now we verify the Lemma.

Proof of Lemma 1. Set $\tilde{u}_{ij} = u_{ij} + S_{ij}$, we estimate that,

$$\sum_{i,j,l} F^{ij} u_{il} u_{jl} \geq \frac{1}{2} \sum_{i,l} F^{ii} \tilde{u}_{il}^2 - C \frac{1}{A_0^2} |\nabla u|^4 \sum_i F^{ii}.$$

Hence, to prove the Lemma we only need to check the

claim: *We may pick constant A_0 sufficient large (depending only on k, n, and $\|g_0\|_{C^3(B_1)}$), such that,*

$$\sum_{i,l} F^{ii} \tilde{u}_{il}^2 \geq A_0^{-\frac{5}{8}} \sum_i F^{ii} |\nabla u|^4. \tag{18}$$

By (9), we have

$$\begin{aligned} \sum_{i,l} F^{ii} \tilde{u}_{il}^2 &= \sum_i F^{ii} \tilde{u}_{ii}^2 + \sum_{i \neq l} F^{ii} u_i^2 u_l^2 \\ &= \sum_i F^{ii} \left\{ \tilde{u}_{ii}^2 + u_i^2 (|\nabla u|^2 - u_i^2) \right\} \\ &= \sum_i F^{ii} (w_{ii}^2 - 2u_i^2 w_{ii} + w_{ii} |\nabla u|^2 + \frac{|\nabla u|^4}{4}). \end{aligned} \tag{19}$$

Set $I = \{1, 2, \cdots, n\}$. Recall that at x_0, by (12), we have for any $i \in I$,

$$|u_i \left(u_{ii} - (|\nabla u|^2 - u_i^2) \right) - \sum_l S_{il} u_l| = |\sum_l u_{il} u_l| \leq \frac{1}{A_0} |\nabla u|^3.$$

This implies that

$$|u_i \left(u_{ii} - (|\nabla u|^2 - u_i^2) \right)| \leq \frac{2}{A_0} |\nabla u|^3. \tag{20}$$

Set $\delta_0 = A_0^{-1/4}$. We divide I into two subsets I_1 and I_2, where

$$I_1 = \{ i \in I | u_i^2 \geq \delta_0 |\nabla u|^2 \} \quad \text{and} \quad I_2 = \{ i \in I | u_i^2 < \delta_0 |\nabla u|^2 \}.$$

For any $i \in I_1$, by (20) we can deduce that

$$\left| w_{ii} - \frac{|\nabla u|^2}{2} \right| < 2\delta_0^3 |\nabla u|^2 < 2\delta_0^2 |\nabla u|^2. \tag{21}$$

We divide the proof of the main claim (18) into four cases.

Case 1. $k = n$.

In this case, by (19), we have

$$\begin{aligned} \sum_{i,l} F^{ii} \tilde{u}_{il}^2 &= \sum_i F^{ii} \left(w_{ii}^2 + \frac{|\nabla u|^4}{4} \right) + \frac{n-2}{n} F |\nabla u|^2 \\ &\geq \sum_i F^{ii} \frac{|\nabla u|^4}{4}. \end{aligned}$$

Here we have used $F^{ii}w_{ii} = \frac{1}{n}F$ for each i. Note that this is true only for this case.

Case 2. $\tilde{u}_{ii}^2 + u_i^2(|\nabla u|^2 - u_i^2) \geq \delta_0^2|\nabla u|^4$.

By (19), we have

$$\sum_{i,l} F^{ii}\tilde{u}_{il}^2 \geq \frac{1}{2}A_0^{-\frac{1}{2}}\sum_i F^{ii}|\nabla u|^4.$$

Case 3. There is j_0 satisfying

$$\tilde{u}_{jj}^2 \leq \delta_0^2|\nabla u|^4 \quad \text{and} \quad u_j^2 < \delta_0|\nabla u|^2. \tag{22}$$

We may assume that $j_0 = n$. We consider the subcases $k \leq n-2$ and $k = n-1$ separately.

Subcase 3.1. $k \leq n-2$.

Since $w_{nn} = \tilde{u}_{nn} + u_n^2 - |\nabla u|^2/2$, (22) implies that

$$\left| w_{nn} + \frac{|\nabla u|^2}{2} \right| < 2\delta_0|\nabla u|^2 = 2A_0^{-\frac{1}{4}}|\nabla u|^2. \tag{23}$$

For $j \in I_2$, it is clear that

$$w_{jj}^2 - 2u_j^2 w_{jj} = (w_{jj} - u_j^2)^2 - u_j^4 \geq -\delta_0^2|\nabla u|^4.$$

Hence, we have

$$\sum_{j\in I_2} F^{jj}(w_{jj}^2 - 2u_j^2 w_{jj}) \geq -A_0^{-\frac{1}{2}}|\nabla u|^4 \sum_i F^{ii}. \tag{24}$$

Using (20)–(24), we have

(25)

$$\begin{aligned}
\sum_{i,l} F^{ii}\tilde{u}_{il}^2 &= \sum_i F^{ii}(w_{ii}^2 - 2u_i^2 w_{ii} + w_{ii}|\nabla u|^2 + \frac{|\nabla u|^4}{4}) \\
&= \sum_i F^{ii}(w_{ii}^2 - 2u_i^2 w_{ii}) + \frac{|\nabla u|^4}{4}\sum_i F^{ii} + F|\nabla u|^2 \\
&\geq \sum_{i\in I_1} F^{ii}(w_{ii}^2 - 2u_i^2 w_{ii}) + \sum_{j\in I_2, j\neq n} F^{ii}(w_{jj}^2 - 2u_i^2 w_{jj}) \\
&\quad + F^{nn}(w_{nn}^2 - 2u_n^2 w_{nn}) + \frac{|\nabla u|^4}{4}\sum_i F^{ii} \\
&\geq \sum_{i\in I_1} F^{ii}\left(\frac{|\nabla u|^4}{4} - 2u_i^2\frac{|\nabla u|^2}{2}\right) + F^{nn}\frac{|\nabla u|^4}{4} + (1 - 32\delta_0^2)\frac{|\nabla u|^4}{4}\sum_i F^{ii} \\
&\geq \tilde{F}^1\frac{|\nabla u|^4}{4} - \tilde{F}^1|\nabla u|^4 + F^{nn}\frac{|\nabla u|^4}{4} + (1 - 32\delta_0^2)\frac{|\nabla u|^4}{4}\sum_i F^{ii},
\end{aligned}$$

where $\tilde{F}^1 = \max_{i\in I_1} F^{ii}$.

Now from (21) and (23), $w_{ll} > 0 > w_{nn}$ for any $l \in I_1$, when A_0 is large. By Proposition 1,

$$F^{nn} \leq F^{ll}, \forall l \in I_1. \tag{26}$$

Since $w_{jj} \geq 0, \forall j \in I_1$, by (6), $\tilde{F}^1 \leq \frac{1}{n-k+1} \sum_i F^{ii}$. Back to (25), we get

$$\begin{aligned}
\sum_{i,l} F^{ii}\tilde{u}_{il}^2 &= -\tilde{F}^1 \frac{|\nabla u|^4}{2} + (1 - 32\delta_0^2)\frac{|\nabla u|^4}{4}\sum_i F^{ii} \\
&\geq \left(1 - \frac{2}{n-k+1} - 32A_0^{-\frac{1}{2}}\right)\frac{|\nabla u|^4}{4}\sum_i F^{ii} \\
&\geq \left(\frac{n-k-1}{n-k+1} - 32A_0^{-\frac{1}{2}}\right)\frac{|\nabla u|^4}{4}\sum_i F^{ii}.
\end{aligned}$$

The claim (18) is valid for this subcase if we just simply pick

$$A_0 \geq (64(n-k+1))^2.$$

Subcase 3.2. $k = n-1$.

If we pick A_0 large enough, from (23) we have $w_{nn} < 0$. Since $(w_{ij}) \in \Gamma_{n-1}^+$, there is at most one negative eigenvalue. Thus, $w_{ii} \geq 0$ for any $i < n$.

¿From Proposition 1, for any $i \neq n$, $(w_{ii}, w_{nn}) \in \Gamma_1^+$ and $(w_{ii}, w_{jj}, w_{nn}) \in \Gamma_2^+$ for $i \neq j$ and $i, j < n$. This implies that

$$w_{ii} + w_{nn} > \frac{-w_{ii}w_{nn}}{w_{jj}}, \quad \text{for } i \neq j \text{ and } i, j < n. \tag{27}$$

We first show that the order of I_1 is 1. Assume by contradiction that there are at least two distinct $i, j \in I_1$. By (27) and (21), we have

$$w_{ii} + w_{nn} > \frac{-w_{ii}w_{nn}}{w_{jj}} \geq (1 - 4\delta_0)|\nabla u|^2.$$

On the other hand,

$$w_{ii} + w_{nn} < 2(A_0^{-\frac{3}{4}} + A_0^{-\frac{1}{4}})|\nabla u|^2.$$

This is a contradiction.

We may now assume $I_1 = \{1\}$. Let $I_2' = I_2 \backslash \{n\}$. By (27), we have

$$\begin{aligned}
\sum_{j\in I_2'}\sum_l F^{jj}\tilde u_{jl}^2 &= F^{2-n}\sum_{j\in I_2'}\sum_l (w_{11}+w_{nn})\sigma_{n-3}(\Lambda_{1jn})\tilde u_{jl}^2\\
&= F^{2-n}\sum_{j\in I_2'}(w_{11}+w_{nn})\sigma_{n-3}(\Lambda_{1jn})\left(w_{jj}^2-2u_j^2w_{jj}+w_{jj}|\nabla u|^2+\frac{|\nabla u|^4}{4}\right)\\
&\ge F^{2-n}\sum_{j\in I_2'}(w_{11}+w_{nn})\sigma_{n-3}(\Lambda_{1jn})\left(w_{jj}+(1-2\delta_0)\frac{|\nabla u|^2}{2}\right)^2\\
&\ge F^{2-n}\sum_{j\in I_2'}\sigma_{n-3}(\Lambda_{1jn})\left(w_{jj}+(1-2\delta_0)\frac{|\nabla u|^2}{2}\right)|w_{11}w_{nn}|.\\
&\ge F^{2-n}\sum_{j\in I_2'}\sigma_{n-3}(\Lambda_{1jn})\left(w_{jj}+(1-2\delta_0)\frac{|\nabla u|^2}{2}\right)(1-2\delta_0)\frac{|\nabla u|^4}{4}.
\end{aligned}$$

We claim that

$$F^{2-n}\sum_{j\in I_2'}\sigma_{n-3}(\Lambda_{1jn})(w_{jj}+(1-2\delta_0)\frac{|\nabla u|^2}{2})\ge c_1\sum_i F^{ii},$$

for some constant $c_1>0$. As in (26), we have

$$F^{ii}\le F^{nn}\qquad \text{for } i<n.$$

Also, for $1<j<n$, $w_{jj}+w_{nn}>0$ and $|w_{nn}+|\nabla u|^2/2|\le 2\delta_0|\nabla u|^2$, it follows $(w_{jj}+w_{11})<\frac{2}{1-2\delta_0}w_{jj}$. Therefore,

$$F^{n-2}F^{nn}=(w_{jj}+w_{11})\sigma_{n-3}(\Lambda_{1jn})<\frac{2}{1-2\delta_0}w_{jj}\sigma_{n-3}(\Lambda_{1jn}).$$

Together with $F^{jj}\le F^{nn}$, we have

$$w_{jj}\sigma_{n-3}(\Lambda_{1jn})\ge cF^{n-2}\sum_i F^{ii},\qquad \text{for } j\in I_2'.$$

The claim is verified, so is the lemma for case 3.

Case 4. $k\le n-1$ and there is no $j\in I$ satisfying (22).

We may assume that there is i_0 such that $\tilde u_{i_0i_0}^2\le\delta_0^2|\nabla u|^4$, otherwise we are in the Case 2. Recall that $\tilde u_{ii}=u_{ii}+S_{ii}$. Since there is no $i\in I$ satisfying (22), we have

$$\tilde u_{i_0i_0}^2\le\delta_0^2|\nabla u|^4\quad\text{and}\quad u_{i_0}^2\ge\delta_0|\nabla u|^2. \tag{28}$$

Assume that $i_0=1$. By (20) we have

$$u_1^2\ge(1-2\delta_0)|\nabla u|^2$$

and $w_{11}>0$. Now it is clear that $(|\nabla u|^2-u_j^2)\ge(1-2\delta_0)|\nabla u|^2$ for all $j>1$, and there is no other $j\in I, j\neq 1$ satisfying (28) if A_0 is large enough. Also, by the assumption in this case, no $j\in I$ satisfying (22). Thus, if for some $j>1$, $\tilde u_{jj}^2\le\delta_0^2|\nabla u|^4$, we must have

$u_j^2(|\nabla u|^2 - u_j^2) \geq \delta_0^2 |\nabla u|^4$. That is, $\tilde{u}_{jj}^2 + u_j^2(|\nabla u|^2 - u_j^2) \geq \delta_0^2 |\nabla u|^4$ for $j > 1$. By (19), it implies that

$$\sum_{i,l} F^{ii}\tilde{u}_{il}^2 \geq \sum_{i\geq 2} F^{ii}(\tilde{u}_{ii}^2 + u_i^2(|\nabla u|^2 - u_i^2)) \geq \delta_0^2 |\nabla u|^4 \sum_{i\geq 2} F^{ii}.$$

Since $\sigma_{k-2}(\Lambda_1) > 0$ and $w_{11} > 0$, it follows from (6) that,

$$\sum_{j\geq 2} F^{ii} \geq \frac{n-k}{n-k+1}\sum_{i\geq 1} F^{ii}. \tag{29}$$

The claim (18) is valid in this case. The proof of the lemma is complete. ■

Corollary 2. *Let u be a solution of (2) in B_r, then*

$$\max_{x\in B_{r/2}} u \leq c(1 + e^{-2\inf_{x\in B_r} u}),$$

for some constant $c > 0$ depending only on r, $\|f\|_{C^1(B_r)}$ and $\|g_0\|_{C^3(B_r)}$.

Remark 1. *Let u satisfy (1) and $v = e^u$. The function v satisfies*

$$\sigma_k^{1/k}\left(v\cdot\nabla^2 v - \frac{1}{2}|\nabla v|^2 g_0 + v^2 S_{g_0}\right) = f.$$

(7) is equivalent to

$$\left|\frac{\nabla v}{v}\right|(x) \leq c + c(\inf_{x\in B_r} v)^{-2} \qquad \textit{for } x \in B_{r/2}.$$

3. Local C^2 estimates

Theorem 1 follows from Proposition 2 and the next proposition.

Proposition 3. *Let $k \geq 2$, suppose $u \in C^4$ be an admissible solution of (2) in B_r. Then, there exists a constant $c > 0$ depending only on r, $\|g_0\|_{C^4(B_r)}$ and $\|f\|_{C^2(B_r)}$ such that*

$$|\nabla^2 u|(x) < c(1 + e^{-2\inf_{B_r} u}), \qquad \textit{for } x \in B_{r/2}. \tag{30}$$

Proof: Again, we assume $r = 1$. Since u is admissible, and $W \in \Gamma_k^+$ for $k \geq 2$, there is a constant c depending only on k and n, $|w_{ij}| \leq c\sum_i w_{ii}$. In turn,

$$|\nabla^2 u|(x) \leq (1+c)(\Delta u + \sum_i |S_{ii}| + |\nabla u|^2)(x). \tag{31}$$

By Proposition 1, we only need to get an upper bound for Δu. Let ρ be chosen as before, set

$$G = \rho(\Delta u + |\nabla u|^2).$$

We estimate the maximum of G. Let $y_0 \in M$ be the maximum point of G. Without loss of generality, we assume $G(y_0) > 1$. Moreover, by Proposition 2 we may assume that $|\nabla u|^2 \le \Delta u$, i.e.,

$$0 < \rho \Delta u(y_0) \le G(y_0) \le 2\rho \Delta u(y_0).$$

By choosing proper coordinates, we may assume that (u_{ij}) is diagonal at y_0, namely $u_{ij} = u_{ii}\delta_{ij}$. Now at y_0, we have

$$0 = G_j(y_0) = \frac{\rho_j}{\rho} G + \rho \sum_{l \ge 1} (u_{llj} + 2u_l u_{lj}), \qquad \text{for any } j, \tag{32}$$

and,

$$G_{ij} \quad = \quad \frac{\rho\rho_{ij} - 2\rho_i\rho_j}{\rho^2} G + \rho \sum_{l \ge 1} (u_{llij} + 2u_{li}u_{lj} + 2u_l u_{lij}).$$

Since y_0 is a maximum point of G,

$$\begin{aligned} 0 &\ge \sum_{i,j \ge 1} F^{ij} G_{ij} \\ &\ge \sum_{i,j \ge 1} F^{ij} \frac{\rho\rho_{ij} - 2\rho_i\rho_j}{\rho^2} G + \rho \sum_{i,j,l \ge 1} F^{ij} (u_{ijll} + 2u_{li}u_{lj} + 2u_l u_{lij}) \\ &\quad - C\rho \sum_i |u_{ii}| \sum_i F^{ii}, \end{aligned}$$

where the last term comes from the commutators related the curvature tensor of g_0 and its derivatives. By the concavity of $\sigma_k^{\frac{1}{k}}$ and (31),

$$\begin{aligned} 0 &\ge \sum_{i,j \ge 1} F^{ij} \frac{\rho\rho_{ij} - 2\rho_i\rho_j}{\rho^2} G + \rho \sum_{i,j,l \ge 1} F^{ij} (u_{jill} + 2u_{li}u_{lj} + 2u_l u_{lij}) \\ &\quad - C\rho \sum_i |u_{ii}| \sum_i F^{ii} \\ &= \sum_{i,j \ge 1} \left\{ F^{ij} \frac{\rho\rho_{ij} - 2\rho_i\rho_j}{\rho^2} G + \rho F^{ij} (w_{ij})_{ll} - \rho F^{ij} (u_i u_j - \frac{1}{2} u_k^2 \delta_{ij} + S_{ij})_{ll} \right\} \\ &\quad + \rho \sum_{i,j,l \ge 1} F^{ij} (2u_{li}u_{lj} + 2u_l u_{lij}) - C\rho \sum_i |u_{ii}| \sum_i F^{ii} \\ &\ge \sum_{i,j \ge 1} F^{ij} \frac{\rho\rho_{ij} - 2\rho_i\rho_j}{\rho^2} G + \rho \sum_l F_{ll} - 2\rho \sum_{i,j,l} F^{ij} u_i u_{jll} \\ &\quad + \rho \sum_{i,j,l} F^{ii} (u_{kl}^2 + u_k u_{kll}) + 2\rho \sum_{i,j,l \ge 1} F^{ij} u_l u_{lij} \\ &\quad - C(1+G) \sum_i F^{ii}. \end{aligned} \tag{33}$$

Now we estimate the terms on the right hand side. ¿From our construction of ρ, we have

$$\sum_{i,j\geq 1} F^{ij}\frac{\rho\rho_{ij}-2\rho_i\rho_j}{\rho^2}G \leq 10b_0\sum_{i,j\geq 1}F^{ij}\frac{1}{\rho}G.$$

By equation (2) and Proposition 2, we have

$$\begin{aligned}\sum_{l\geq 1}\rho F_{ll} &= \sum_{l\geq 1}\rho(f_{ll}-2u_{ll}-2f_l u_l+4u_l^2)e^{-2u}\\ &\geq -(C+G)e^{-2u}\geq -(C+G)e^{-2\inf_{B_1}u}\\ &\geq -CGe^{-2\inf_{B_1}u},\end{aligned}$$

for some constant $C>1$ depending only on $\|f\|_{C^2(B_1)}$. By (32), we have for any $l\geq 1$,

$$\sum_l \rho u_{jll} = -\frac{\rho_j}{\rho}G-2\rho\sum_l u_l u_{jl}.$$

Together with (33), we obtain,

$$\begin{aligned}-\sum_{i,j,l,\geq 1}\rho F^{ij}u_i u_{jll} &\geq -C\frac{1}{\rho}(\sum_i F^{ii})(|H|^{1/2}G+2\rho H\sum_l |u_{ll}|)\\ &\geq -C\frac{1}{\rho}(\sum_i F^{ii})(HG+G+2n(n-2)^2HG)\\ &\geq -C\frac{1}{\rho}(\sum_i F^{ii})(H+1)G.\end{aligned}$$

Similarly, we get

$$\rho\sum_{i,k,l\geq 1}F^{ii}u_k u_{kll}\geq -C\frac{1}{\rho}(\sum_i F^{ii})(H+1)G.$$

Now by (2) we compute

$$\begin{aligned}\rho\sum_{i,j,l\geq 1}F^{ij}u_l u_{lij} &\geq \rho\sum_{i,j,l\geq 1}F^{ij}u_l u_{ijl}-CH\sum_i F^{ii}\\ &= \rho F_l u_l-\rho\sum_{i,j,l\geq 1}F^{ij}u_l(u_iu_j-\frac{1}{2}|\nabla u|^2\delta_{ij}+S_{ij})_l-CH\sum_i F^{ii}\\ &\geq -C(1+e^{-2u})H\sum_i F^{ii}-CH\frac{1}{\rho}G\sum_i F^{ii}.\end{aligned}$$

As $\sum F^{ii} \geq 1$, and $G(y_0) \geq 1$, the above and Proposition 2 yield

$$
\begin{aligned}
0 \;\geq\; & -C\sum F^{ii}\frac{1}{\rho}G - CGe^{-2u} - 2CHe^{-2u} - \frac{C}{\rho}\sum F^{ii}(H+1)G \\
& +\rho\sum_{i\geq 1} F^{ii}u_{kl}^2 \\
\geq\; & -\frac{C}{\rho}\sum F^{ii}(G+Ge^{-2u}+HG) + \rho\sum_{i,j,l\geq 1} F^{ii}u_{jl}^2 \\
\geq\; & -\frac{C}{\rho}\sum F^{ii}(G+Ge^{-2u}+HG) + \frac{1}{\rho}\frac{1}{n}\sum_i F^{ii}\rho^2(\Delta u)^2 \\
\geq\; & -\frac{C}{\rho}\sum F^{ii}\left\{(G+Ge^{-2u}+HG) - \frac{1}{2n}G^2\right\}.
\end{aligned}
\tag{34}
$$

It follows from (34) that at y_0, $G \leq C(1+e^{-2\inf_{B_1} u})$. ∎

Remark 2. *Since the estimates in Theorem 1 are independent of the lower bound of f, $C^{1,1}$ regularity estimates can be deduced for the solutions of degenerate equation (2) with nonnegative function f. If f is positive and $\inf u$ is bounded from below, we will have higher regularity estimates for the solution u by Evans-Krylov theorem (*[5] *and* [9]*).*

Remark 3. *The arguments in the proofs of Propositions 2 and 3 can be generalized to deal with the equation of the form*

$$
\sigma_k\left(\nabla^2 u + du\otimes du - \frac{|\nabla u|^2}{2}g_0 + S_{g_0}\right) = f^{k-1}(e^{-u}, x), \tag{35}
$$

with the function $f(s,x) > 0$ satisfying the following structure conditions: there is a constant C and a function $h(s,x)$ with $h_s(s,x) \geq 0$, such that $\forall (s,x) \in R\times M$,

$$
|\nabla_{s,x} f(s,x)| \leq C f^{\frac{1}{2}}(s,x), \quad |sf_{s,x}(s,x)| + +|s^2 f_{ss}(s,x)| + |f_{xx}(s,x)| \leq h(s,x). \tag{36}
$$

Namely,

Theorem 2. *Suppose f satisfies the structural conditions (36). Let $u \in C^4$ be an admissible solution of (35) in B_r, the geodesic ball of radius r in a Riemannian manifold (M, g_0). Then, there exists a constant $c > 0$ depending only on r and $\|g_0\|_{C^4(B_r)}$, such that*

$$
\|u\|_{C^2(B_{r/2})} \leq c(1 + \sup_{x\in M} h(e^{-\inf_{B_r} u}, x) + e^{-\inf_{B_r} u}).
$$

Proof: The basic observation is that the proof of Proposition 2 can carry through without major changes, in the proof of Proposition 3 we may use the following fact to overcome the term $F_{ll} = (f^{\frac{k-1}{k}})_{ll}$ in (33).

Fact: $\sum_i F^{ii} \geq \frac{1}{k}\sigma_k^{-\frac{1}{k(k-1)}}(\Lambda)\sigma_1^{\frac{1}{k-1}}(\Lambda)$.

By Proposition 1 and the definition $F=\sigma_k^{1/k}$, we only need to prove

$$\sigma_{k-1}^{k-1}\geq\sigma_1\sigma_k^{k-2}.$$

This is a consequence of the Newton-MacLaurin inequality (3) as follows,

$$\begin{aligned}\sigma_1\sigma_2^2\sigma_3^4\cdots\sigma_{k-2}^{2(k-3)}\sigma_{k-1}^{k-3}\sigma_k^{k-2} &= (\sigma_1\sigma_3)(\sigma_2\sigma_4)^2\cdots(\sigma_{k-2}\sigma_k)^{k-2}\\ &< \sigma_2^2\sigma_3^4\cdots\sigma_{k-2}^{2(k-3)}\sigma_{k-1}^{2(k-2)}.\end{aligned}$$

■

Finally, Corollary 1 follows from Theorem 1 and the next Proposition.

Proposition 4. *There exists a constant $\varepsilon_0>0$ such that any solution u of (2) in B_1 with*

$$\int_{B_1}e^{-nu}dvol(g_0)\leq\varepsilon_0$$

satisfies

$$\inf_{B_{1/2}}u\geq -c_{\varepsilon_0},$$

for some constant $c_{\varepsilon_0}>0$ depending only on ε_0.

Proof: We make use of a rescaling argument as in [13], together with Theorem 1, to prove this Proposition.

Assume by contradiction that there is a sequence of solutions u_i of (2) in B_1 such that

$$\int_{B_1}e^{-nu_i}dvol(g_0)\to 0,\qquad \text{as } i\to\infty$$

and

$$\inf_{B_{1/2}}u_i\to-\infty, \tag{37}$$

as $i\to\infty$.

Consider the function $(3/4-r)^2\sup_{B_r}e^{-nu_i}:(0,3/4)\to[0,\infty)$. As the function is continuous, there is $r_0^i\in(0,3/4)$ such that

$$\left(\frac{3}{4}-r_0^i\right)^2\sup_{B_{r_0^i}}e^{-nu_i}=\sup_{0<r<3/4}\left(\frac{3}{4}-r\right)^2\sup_{B_r}e^{-nu_i}.$$

Moreover, there exists $z_0^i\in\overline{B}_{r_0}$ such that $e^{-nu_i(z_0^i)}=\sup_{B_{r_0^i}}e^{-nu_i(z)}$. Let $s_0^i=(3/4-r_0^i)/2$. From the definition,

$$\sup_{B_{s_0}(z_0^i)}e^{-nu_i}\leq\sup_{B_{s_0+r_0}(z_0^i)}e^{-nu_i}\leq 4e^{-nm_i}, \tag{38}$$

where $m_i=u_i(z_0^i)$. Consider the rescaled function $v^i(y)=u_i(\exp_{z_0^i}e^{m_i}y)-m_i$ in $B_{e^{-m_i}s_0^i}$. v^i satisfies equation of type (2).

By (38), we have,

$$\int_{B_{e^{-m_i}s_0}} e^{-nv_i} = \int_{B_{s_0}(z_0^i)} e^{-nu_i} \to 0, \quad \text{as } i \to \infty$$

and

$$v_i(0) = 0 \quad \text{and} \quad v_i(x) \geq -\frac{1}{n}\log 4.$$

From (37), one may check that $e^{-m_i}s_0^i \geq a_0 > 0$ for any i. Now by Proposition 2, or Corollary 2, $\sup v^i$ is uniformally bounded in $B_{e^{-m_i}s_0^i/2}$. This is a contradiction. ■

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