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HEAT TRACE ASYMPTOTICS DEFINED BY TRANSFER BOUNDARY CONDITIONS

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Abstract. We compute the first 5 terms in the short-time heat trace asymptotics expansion for an operator of Laplace type with transfer boundary conditions using the functorial properties of these invariants.

1. Introduction

Let $M := (M^+, M^-)$ be a pair of compact smooth manifolds of dimension m which have a common smooth boundary $\Sigma := \partial M^+ = \partial M^-$. A structure Ξ over M will be a pair of corresponding structures $\Xi := (\Xi^+, \Xi^-)$ over the manifolds $M^\pm$. Let g be a Riemannian metric on M; we assume henceforth that $g^+|_\Sigma = g^-|_\Sigma$, but do not assume any matching condition on the normal derivatives. Let V be a smooth vector bundle over M; we do not assume any relationship between $V^+|_\Sigma$ and $V^-|_\Sigma$; in particular, we can consider the situation when we have $\dim V^+ \neq \dim V^-$. Let D be an operator of Laplace type on $C^\infty(V)$. The operator D determines a natural connection ∇ and a natural 0^{th} order operator E so that [5]:

$$D = -\left(g^{ij}\nabla_i\nabla_j + E\right).$$

Let the inward unit normals $\nu^\pm$ of $\Sigma \subset M^\pm$ determine ν; note that $\nu^+ = -\nu^-$. Assume given auxiliary impedance matching terms $\mathcal{S} = \{S^{++}, S^{+-}, S^{-+}, S^{--}\}$ where $S^{\varepsilon\varrho} : V^\varrho|_\Sigma \to V^\varepsilon|_\Sigma$. The transfer boundary operator $\mathcal{B}_T(\mathcal{S})$ is defined by:

$$\mathcal{B}_T(\mathcal{S})\phi := \left\{ \begin{pmatrix} \nabla^+_{\nu^+} + S^{++} & S^{+-} \\ S^{-+} & \nabla^-_{\nu^-} + S^{--} \end{pmatrix} \begin{pmatrix} \phi^+ \\ \phi^- \end{pmatrix} \right\}\Bigg|_\Sigma . \tag{1.1}$$

The terms S^{+-} and S^{-+} connect the structures on M^+ and M^- and are crucial to our investigation. These boundary conditions arise physically in heat transfer problems (see to Carslaw and Jaeger [7]), some problems of quantum mechanics [1], and in conformal field theory [2]. More on various spectral problems appearing in the string theory context can be found in [12].

Let $D_{\mathcal{B}_T(\mathcal{S})}$ be the associated realization of D with the boundary condition $\mathcal{B}_T(\mathcal{S})\phi = 0$. Let Q be a smooth endomorphism of V which we use to localize the heat trace. As $t \downarrow 0$, there is a complete asymptotic expansion with locally computable coefficients:

$$\mathrm{Tr}_{L^2}\left(Qe^{-tD_{\mathcal{B}_T(\mathcal{S})}}\right) \sim \sum_{n\geq 0} a_n(Q, D, \mathcal{B}_T(\mathcal{S}))t^{(n-m)/2}. \tag{1.2}$$

In a formal limiting case $S^{++} - S^{-+} = S^{--} - S^{+-} \to \infty$ while $v = 2(S^{++} + S^{+-})$ is kept finite one arrives at transmittal boundary conditions: $\phi^+ = \phi^-$, $\nabla_{\nu^+}\phi^+ + \nabla_{\nu^-}\phi^- = v\phi^+$. The heat trace asymptotics for these boundary conditions have been studied in [3, 9, 11]. Some other particular cases of the boundary operator (1.1) have been considered in [4, 10].

Let R_{ijkl} be the components of the Riemann curvature tensor, let Ω be the curvature of ∇, and let the second fundamental forms $L^\pm$ of $\Sigma \subset M^\pm$ determine L.

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We let Roman indices i, j, k, and l range from 1 to m and index a local orthonormal frame for the tangent bundle of M and let Roman indices a, b, c range from 1 to $m-1$ and index a local orthonormal frame for the tangent bundle of Σ. We adopt the Einstein convention and sum over repeated indices. Let $\operatorname{Tr}^\pm$ be the fiber trace in $V^\pm$, let ';' denote multiple covariant differentiation with respect to the Levi-Civita connection on M and ∇, and let ':' denote multiple covariant differentiation with respect to the Levi-Civita connection of Σ and ∇. Let $S=(S^{++},S^{--})$.

Local formulae which decouple can be written in the following format:

Definition 1.1. *Let $\mathcal{E}(\nabla^*R,\nabla^*E,\nabla^*\Omega)$ and $\mathcal{F}(\nabla^*R,\nabla^*E,\nabla^*\Omega,\nabla^*L,\nabla^*S)$ be local invariants on M and ∂M, respectively. Set:*

$$\begin{aligned}&\textstyle\int_M \operatorname{Tr}(\mathcal{E}) := \int_{M^+}\operatorname{Tr}^+(\mathcal{E}^+)+\int_{M^-}\operatorname{Tr}^-(\mathcal{E}^-),\\ &\textstyle\int_{\partial M}\operatorname{Tr}(\mathcal{F}) := \int_{\partial M^+}\operatorname{Tr}^+(\mathcal{F}^+)+\int_{\partial M^-}\operatorname{Tr}^-(\mathcal{F}^-)=\int_\Sigma\{\operatorname{Tr}^+(\mathcal{F}^+)+\operatorname{Tr}^-(\mathcal{F}^-)\}.\end{aligned}$$

What is crucial is that the invariants $\mathcal{E}^\pm$ and $\mathcal{F}^\pm$ involve only structures on $M^\pm$. We illustrate these two types in the following examples:

$$\begin{aligned}&\textstyle\int_M \operatorname{Tr}(QR_{ijji}E)=\int_{M^+}\operatorname{Tr}^+(Q^+R^+_{ijji}E^+)+\int_{M^-}\operatorname{Tr}^-(Q^-R^-_{ijji}E^-),\\ &\textstyle\int_{\partial M}\operatorname{Tr}(QSL_{aa})=\int_{\partial M^+}\operatorname{Tr}^+(Q^+S^{++}L^+_{aa})+\int_{\partial M^-}\operatorname{Tr}^-(Q^-S^{--}L^-_{aa}).\end{aligned}$$

There are, however, invariants which intertwine the two structures and which do not decouple; for example, the following invariant is a 'mixed' invariant which measures the interactions of these two structures:

$$\textstyle\int_\Sigma\{\operatorname{Tr}^+(Q^+S^{+-}S^{-+})+\operatorname{Tr}^-(Q^-S^{-+}S^{+-})\}.$$

The main result of this letter is the following:

Theorem 1.2. *With transfer boundary conditions, we have that:*

(1) $a_0(Q,D,\mathcal{B}_T(\mathcal{S}))=(4\pi)^{-m/2}\int_M\operatorname{Tr}(Q)$.

(2) $a_1(Q,D,\mathcal{B}_T(\mathcal{S}))=(4\pi)^{(1-m)/2}\frac14\int_{\partial M}\operatorname{Tr}(Q)$.

(3) $a_2(Q,D,\mathcal{B}_T(\mathcal{S}))=(4\pi)^{-m/2}\frac16\int_M\operatorname{Tr}\{Q(R_{ijji}+6E)\}$
$+(4\pi)^{-m/2}\frac16\int_{\partial M}\operatorname{Tr}\{Q(2L_{aa}+12S)+3Q_{;\nu}\}$.

(4) $a_3(Q,D,\mathcal{B}_T(\mathcal{S}))$
$$\begin{aligned}&=(4\pi)^{(1-m)/2}\tfrac{1}{384}\textstyle\int_{\partial M}\operatorname{Tr}\{Q(96E+16R_{ijji}-8R_{a\nu\nu a}+13L_{aa}L_{bb}\\ &\qquad+2L_{ab}L_{ab}+96SL_{aa}+192S^2+Q_{;\nu}(6L_{aa}+96S)+24Q_{;\nu\nu})\}\\ &\quad+(4\pi)^{(1-m)/2}\tfrac{1}{384}\textstyle\int_\Sigma\{\operatorname{Tr}^+(192Q^+S^{+-}S^{-+})+\operatorname{Tr}^-(192Q^-S^{-+}S^{+-})\}.\end{aligned}$$

(5) $a_4(Q,D,\mathcal{B}_T(\mathcal{S}))$
$$\begin{aligned}&=(4\pi)^{-m/2}\tfrac{1}{360}\textstyle\int_M\operatorname{Tr}\{Q(60E_{;kk}+60R_{ijji}E+180E^2+30\Omega^2\\ &\qquad+12R_{ijji;kk}+5R_{ijji}R_{kllk}-2R_{ikjk}R_{iljl}+2R_{ijkl}R_{ijkl})\}\\ &\quad+(4\pi)^{-m/2}\tfrac{1}{360}\textstyle\int_{\partial M}\operatorname{Tr}\{Q(240E_{;\nu}+42R_{ijji;\nu}+24L_{aa:bb}+120EL_{aa}\\ &\qquad+20R_{ijji}L_{aa}+4R_{a\nu a\nu}L_{bb}-12R_{a\nu b\nu}L_{ab}+4R_{abcb}L_{ac}\\ &\qquad+\tfrac{40}{3}L_{aa}L_{bb}L_{cc}+8L_{ab}L_{ab}L_{cc}+\tfrac{32}{3}L_{ab}L_{bc}L_{ac}+360(SE+ES)\\ &\qquad+120SR_{ijji}+144SL_{aa}L_{bb}+48SL_{ab}L_{ab}+480S^2L_{aa}+480S^3\\ &\qquad+120S_{:aa})+Q_{;\nu}(180E+30R_{ijji}+12L_{aa}L_{bb}+12L_{ab}L_{ab}\\ &\qquad+72SL_{aa}+240S^2)+Q_{;\nu\nu}(24L_{aa}+120S)+30Q_{;ii\nu}\}\end{aligned}$$

$$
\begin{aligned}
&+(4\pi)^{-m/2}\tfrac{1}{360}\textstyle\int_\Sigma \operatorname{Tr}^+\{480(Q^+S^{++}+S^{++}Q^+)S^{+-}S^{-+}\\
&\qquad +480Q^+S^{+-}S^{--}S^{-+}\\
&\qquad +(288Q^+L_{aa}^+ +192Q^+L_{aa}^- +240Q^+_{;\nu^+})S^{+-}S^{-+}\}\\
&+(4\pi)^{-m/2}\tfrac{1}{360}\textstyle\int_\Sigma \operatorname{Tr}^-\{480(Q^-S^{--}+S^{--}Q^-)S^{-+}S^{+-}\\
&\qquad +480Q^-S^{-+}S^{++}S^{+-}\\
&\qquad +(288Q^-L_{aa}^- +192Q^-L_{aa}^+ +240Q^-_{;\nu^-})S^{-+}S^{+-}\}.
\end{aligned}
$$

We may decompose the heat trace invariants in the form:

$$a_n(Q,D,\mathcal{B}_T(\mathcal{S}))=a_n^M(Q,D)+a_n^{\partial M}(Q,D,S)+a_n^\Sigma(Q,D,\mathcal{S}). \tag{1.3}$$

The invariants a_n^M and $a_n^{\partial M}$ decouple and can be expressed as local integrals of the form given in Definition 1.1; the invariant a_n^Σ involves integrals of mixed structures. Theorem 1.2 reflects this decomposition. We shall prove Theorem 1.2 by analyzing the 3 terms appearing in Equation (1.3) separately. Here is a brief guide to the remainder of this letter. In Section 2, we apply results of Branson and Gilkey [5] concerning the heat trace asymptotics with Robin boundary conditions to determine a_n^M and $a_n^{\partial M}$. In Section 3, we express a_n^Σ in terms of certain invariants with universal undetermined coefficients (see Lemma 3.1); these new terms which measure the interaction between the structures on $M^\pm$ are the heart of the matter. The proof of Theorem 1.2 is then completed in Sections 4 and 5 by determining the universal coefficients of Lemma 3.1. In Section 4, we derive a new functorial property by doubling the manifold; in Section 5, we use conformal variations. We refer to [8] for an analogous computation of the heat content asymptotics with transfer boundary conditions.

2. Robin boundary conditions

Let D be an operator of Laplace type on a compact Riemannian manifold N with smooth boundary ∂N and let S be an auxiliary endomorphism defined on the boundary. Robin boundary conditions are defined by the operator:

$$\mathcal{B}_R(S)\phi:=(\nabla_\nu\phi+S\phi)|_{\partial N}.$$

If we take $S^{+-}=0$ and $S^{-+}=0$, then the boundary conditions decouple so

$$
\begin{aligned}
a_n(Q,D,\mathcal{B}_T(\mathcal{S})) &= a_n(Q^+,D^+,\mathcal{B}_R(S^{++}))+a_n(Q^-,D^-,\mathcal{B}_R(S^{--}))\\
&= a_n(Q,D,\mathcal{B}_R(S)).
\end{aligned}
$$

Thus we may use Branson-Gilkey-Vassilevich [6] (Theorem 4.1) to determine the invariants $a_n^M(Q,D)$ and $a_n^{\partial M}(Q,D,S)$. Furthermore, we see that all the terms in the mixed integrals defining $a_n^\Sigma(Q,D,\mathcal{B}_T(\mathcal{S}))$ must contain either S^{+-} or S^{-+} and hence, since we are taking traces and have not identified V^+ with V^-, both of these terms must appear in every mixed monomial as these are the only structures relating M^+ to M^-.

As the boundary integrands describing a_n^Σ are homogeneous of weight $n-1$ and as the variables S^{**} have weight 1, monomials which contain both S^{+-} and S^{-+} have weight at least 2 and thus do not appear in the expansion of a_n for $n\le 2$. This completes the proof of Theorem 1.2 (1)-(3).

3. The mixed invariants

We can identify the general form of the invariants a_n^Σ for $n\le 4$ as follows:

Lemma 3.1. *There exist universal constants so that:*

(1) $a_3^\Sigma(Q,D,\mathcal{B}_T(\mathcal{S}))$
$$=(4\pi)^{-m/2}\tfrac{1}{384}\textstyle\int_\Sigma \alpha_0\{\operatorname{Tr}^+(Q^+S^{+-}S^{-+})+\operatorname{Tr}^-(Q^-S^{-+}S^{+-})\}.$$

(2) $a_4^\Sigma(Q,D,\mathcal{B}_T(\mathcal{S}))=(4\pi)^{-m/2}\frac{1}{360}\int_\Sigma\{$
$\frac12 c_1\mathrm{Tr}^+(Q^+S^{++}S^{+-}S^{-+})+\frac12 c_1\mathrm{Tr}^-(Q^-S^{--}S^{-+}S^{+-})$
$+\frac12 c_2\mathrm{Tr}^+(S^{++}Q^+S^{+-}S^{-+})+\frac12 c_2\mathrm{Tr}^-(S^{--}Q^-S^{-+}S^{+-})$
$+\alpha_2\mathrm{Tr}^+(S^{++}S^{+-}Q^-S^{-+})+\alpha_2\mathrm{Tr}^-(S^{--}S^{-+}Q^+S^{+-})$
$+\alpha_3L^+_{aa}\mathrm{Tr}^+(Q^+S^{+-}S^{-+})+\alpha_3L^-_{aa}\mathrm{Tr}^-(Q^-S^{-+}S^{+-})$
$+\alpha_4L^-_{aa}\mathrm{Tr}^+(Q^+S^{+-}S^{-+})+\alpha_4L^+_{aa}\mathrm{Tr}^-(Q^-S^{-+}S^{+-})$
$+\alpha_5\mathrm{Tr}^+(Q^+_{;\nu^+}S^{+-}S^{-+})+\alpha_5\mathrm{Tr}^-(Q^-_{;\nu^-}S^{-+}S^{+-})\}$.

(3) $c_1=c_2$.

Proof. We observe first that the heat trace coefficient must be symmetric with respect to interchanging the labels "+" and "−". Since we have written down a complete basis of invariants of weight 2 and 3 which contain both S^{-+} and S^{+-}, assertions (1) and (2) now follow.

We generalize an argument from [5] to prove assertion (3). If D, Q and S^{**} are real, then $\mathrm{Tr}\left(Qe^{-tD}\right)$ is real. This shows that all universal constants given above are real. Suppose now that the bundles $V^\pm$ are equipped with Hermitian inner products and that the operators $D^\pm$ are formally self-adjoint. This means that the associated connections $\nabla^\pm$ are unitary and the endomorphisms $E^\pm$ are symmetric. Suppose that S^{++} and S^{--} are self-adjoint, and that S^{+-} is the adjoint of S^{-+}. It then follows that D is self-adjoint. Therefore, $\mathrm{Tr}\left(Qe^{-tD}\right)$ is real; this implies necessarily that $c_1=c_2$. □

We remark in passing that it is exactly this argument which shows that the term $\int_M\mathrm{Tr}(720QSE)$ appearing in [5] for scalar Q must be replaced by the term $\int_M\mathrm{Tr}(360Q(SE+ES))$ for endomorphism valued Q [6].

Since $c_1=c_2$, the lack of commutativity involved in dealing with endomorphisms plays no role; thus it suffices to consider the scalar case where everything is commutative. We assume therefore for the remainder of this letter that the bundles $V^\pm=M^\pm\times\mathbb{C}$ are trivial line bundles and that the operators $D^\pm$ are scalar. Thus we may drop 'Tr' from the notation. We set $\alpha_1:=c_1=c_2$ - the symmetrization term then becomes

$$(4\pi)^{-m/2}\tfrac{1}{360}\int_\Sigma\alpha_1(Q^+S^{++}S^{+-}S^{-+}+Q^-S^{--}S^{-+}S^{+-}).$$

4. Doubling the manifold

In Section 2, we related the heat trace asymptotics for transfer and Robin boundary conditions by taking $S^{+-}=S^{-+}=0$. We now give a different relationship between transfer and Robin boundary conditions related to doubling the manifold.

Lemma 4.1. *Let $M^\pm:=M^0$ be a m-dimensional Riemannian manifold with boundary $\partial M^0=\Sigma$ and let $D^\pm=D^0$ be a scalar operator of Laplace type. Fix an angle $0<\theta<\frac{\pi}{2}$. Let S^{++} and S^{+-} be arbitrary. Set:*

$$\begin{aligned}
S^{-+}&:=S^{+-},\\
S^{--}&:=S^{++}+(\tan\theta-\cot\theta)\,S^{+-},\\
S_\phi&:=S^{++}+\tan\theta\,S^{+-}=S^{--}+\cot\,\theta\,S^{-+},\\
S_\psi&:=S^{++}-\cot\,\theta S^{+-}=S^{--}-\tan\,\theta\,S^{-+}.
\end{aligned}$$

Then:

$$\begin{aligned}
a_n(Q,D,\mathcal{B}_T(\mathcal{S}))&=a_n(\cos^2\theta\,Q^++\sin^2\theta\,Q^-,D^0,\mathcal{B}_{R(S_\phi)})\\
&+\ a_n(\sin^2\theta\,Q^++\cos^2\theta\,Q^-,D^0,\mathcal{B}_{R(S_\psi)}).
\end{aligned}$$

Proof. If $u, v \in C^\infty(V^0)$, define $u^\phi, v^\psi \in C^\infty(M)$ by setting

$$\begin{aligned} u^\phi(x^+) &= \quad \cos\theta\, u(x), \quad u^\phi(x^-) = \sin\theta\, u(x) \\ v^\psi(x^+) &= -\sin\theta\, v(x), \quad v^\phi(x^-) = \cos\theta\, v(x). \end{aligned}$$

The conditions $\mathcal{B}_T(\mathcal{S})u^\phi = 0$ and $\mathcal{B}_T(\mathcal{S})v^\psi = 0$ are equivalent to the conditions:

$$\begin{aligned} &(\nabla_{\nu^0} + S^{++} + \tan\theta\, S^{+-})u|_{\partial M^0} = 0, \quad (\nabla_{\nu^0} + S^{--} + \cot\theta\, S^{-+})u|_{\partial M^0} = 0, \\ &(\nabla_{\nu^0} + S^{++} - \cot\theta\, S^{+-})v|_{\partial M^0} = 0, \quad (\nabla_{\nu^0} + S^{--} - \tan\theta\, S^{-+})v|_{\partial M^0} = 0, \end{aligned}$$

or equivalently to the conditions $(\nabla_{\nu^0} + S_\phi)u|_{\partial M^0} = 0$ and $(\nabla_{\nu^0} + S_\psi)v|_{\partial M^0} = 0$.

Let $\{\lambda_i, u_i\}$ and $\{\mu_j, v_j\}$ be discrete spectral resolutions for D^0 for Robin boundary conditions $\mathcal{B}_R(S_\phi)$ and $\mathcal{B}_R(S_\phi)$. Since

$$Du_i^\phi = \lambda_i u_i^\phi, \quad Dv_j^\psi = \mu_j v_j^\psi, \quad \mathcal{B}_T(\mathcal{S})u_i^\phi = 0, \quad \text{and } \mathcal{B}_T(\mathcal{S})v_j^\psi = 0,$$

and since $\{u_i^\phi, v_j^\psi\}$ is a complete orthonormal basis for $L^2(M)$, $\{\lambda_i, u_i^\phi\} \cup \{\mu_j, v_j^\psi\}$ is a discrete spectral resolution of D with transfer boundary conditions $\mathcal{B}_T(\mathcal{S})$. Thus we may compute:

$$\begin{aligned} \mathrm{Tr}_{L^2}(Qe^{-tD_{\mathcal{B}_T(\mathcal{S})}}) &= \textstyle\int_M \sum_i Qe^{-t\lambda_i}|u_i^\phi|^2 + \int_M \sum_j Qe^{-t\mu_j}|v_j^\psi|^2\} \\ &= \textstyle\int_{M^0} \sum_i (\cos^2\theta\, Q^+ + \sin^2\theta\, Q^-)|u_i|^2 e^{-t\lambda_i} \\ &+ \textstyle\int_{M^0} \sum_j (\sin^2\theta\, Q^+ + \cos^2\theta\, Q^-)|v_j|^2 e^{-t\mu_j} \\ &= \mathrm{Tr}_{L^2}(\cos^2\theta\, Q^+ + \sin^2\theta\, Q^-)e^{-tD^0_{\mathcal{B}_R(\phi)}} \\ &+ \mathrm{Tr}_{L^2}(\sin^2\theta\, Q^+ + \cos^2\theta\, Q^-)e^{-tD^0_{\mathcal{B}_R(\psi)}}. \qquad \square \end{aligned}$$

We use Lemma 4.1 as follows. We set $Q^- = 0$. (The case $Q^- \neq 0$ may be used as a check, but no additional information is obtained.) We use [5] (Theorem 1.2), Lemma 3.1, and Lemma 4.1 to derive the following relations,

$$\begin{aligned} &192Q^+(\cos^2\theta S_\phi^2 + \sin^2\theta S_\psi^2) = 192Q^+(S^{++}S^{++} + S^{+-}S^{+-}) \\ &\quad = 192Q^+S^{++}S^{++} + \alpha_0 Q^+S^{+-}S^{+-}, \\ &480Q^+(\cos^2\theta S_\phi^3 + \sin^2\theta S_\psi^3) \\ &\quad = 480Q^+(S^{++}S^{++}S^{++} + 3S^{++}S^{+-}S^{+-} + S^{+-}S^{+-}S^{+-}[\tan\theta - \cot\theta]) \\ &\quad = 480Q^+S^{++}S^{++}S^{++} + \alpha_1 Q^+S^{++}S^{+-}S^{+-} \\ &\qquad + \alpha_2 Q^+[S^{++} + S^{+-}(\tan\theta - \cot\theta)]S^{+-}S^{+-}, \\ &480Q^+L_{aa}(\cos^2\theta S_\phi^2 + \sin^2\theta S_\psi^2) = 480Q^+L_{aa}(S^{++}S^{++} + S^{+-}S^{+-}) \\ &\quad = 480Q^+L_{aa}S^{++}S^{++} + (\alpha_3 + \alpha_4)Q^+L_{aa}S^{+-}S^{+-}, \\ &240Q^+_{;\nu^+}(\cos^2\theta S_\phi^2 + \sin^2\theta S_\psi^2) = 240Q^+_{;\nu^+}(S^{++}S^{++} + S^{+-}S^{+-}) \\ &\quad = 240Q^+_{;\nu^+}S^{++}S^{++} + \alpha_5 Q^+_{;\nu_+}S^{+-}S^{+-}. \end{aligned}$$

This implies that:

$$192 = \alpha_0, \quad 960 = \alpha_1, \quad 480 = \alpha_2, \quad 480 = \alpha_3 + \alpha_4, \quad \alpha_5 = 240. \tag{4.1}$$

5. Conformal variations

The missing information about $\{\alpha_3, \alpha_4\}$ is obtained via conformal transformations. As before, we deal only with the scalar situation. Given (M, D) and $\psi^+ \in C^\infty(M^+)$, we vary the structures on M^+ to define the one-parameter family of operators

$$D(\varepsilon) := (e^{2\varepsilon\psi^+} D^+, D^-)$$

with associated structures $g^+(\varepsilon) := e^{2\varepsilon\psi^+} g^+$, $\nabla^+(\varepsilon)$, and $E^+(\varepsilon)$. To ensure that $g^+(\varepsilon)|_\Sigma = g^-|_\Sigma$, we assume ψ^+ vanishes on Σ. Let $Q = (Q^+, 0)$, $\psi := (\psi^+, 0)$, and

$$\mathcal{S}(\varepsilon) := \mathcal{B}_T(\mathcal{S}(0)) - \nabla(\varepsilon)\mathrm{Id}.$$

The following Lemma is a purely formal computation; see [5] for details.

Lemma 5.1. *Adopt the notational conventions established above. Then*

(1) $\partial_\varepsilon|_{\varepsilon=0} a_n(1, D(\varepsilon), \mathcal{B}_T(\mathcal{S}(\varepsilon))) = (m-n) a_n(\psi, D, \mathcal{B}_T(\mathcal{S}))$.

(2) $\partial_\varepsilon|_{\varepsilon=0} a_n(e^{-2\varepsilon\psi} Q, D(\varepsilon), \mathcal{B}_T(\mathcal{S}(\varepsilon))) = 0$ *for* $m = n + 2$.

We use the following relations to apply Lemma 5.1:

$$\begin{array}{ll} \partial|_{\varepsilon=0} S^{++}(\varepsilon) = \frac{m-2}{2}\psi^+_{;\nu^+}, & S^{+-}(\varepsilon) = S^{+-}(0), \\ S^{-+}(\varepsilon) = S^{-+}(0), & S^{--}(\varepsilon) = S^{--}(0), \\ \partial_\varepsilon|_{\varepsilon=0} L^+_{aa}(\varepsilon) = -(m-1)\psi^+_{;\nu^+}, & \partial_\varepsilon|_{\varepsilon=0}\{\nabla^+_{\nu^+}(\varepsilon)(e^{-2\varepsilon\psi}Q)\} = -2Q\psi^+_{;\nu^+}. \end{array}$$

Clearly Lemma 5.1 (1) yields no new information as the localizing function is continuous on Σ and thus cannot separate the contributions from α_3 and α_4. In fact, comparing the coefficient of the invariant $\psi^+_{;\nu^+} S^{+-} S^{-+}$, one obtains

$$\tfrac{m-2}{2}(\alpha_1 + \alpha_2) - (m-1)(\alpha_3 + \alpha_4) = (m-4)\alpha_5$$

which is consistent with Equation (4.1). However, Lemma 5.1 (2) with $m = 6$ yields the additional relation:

$$2\alpha_1 - 5\alpha_3 - 2\alpha_5 = 0.$$

We use Equation (4.1) to complete the proof of Theorem 1.2 by computing:

$$\alpha_3 = 288, \quad \alpha_4 = 192. \tag{5.1}$$

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