Local stress regularity in scalar non-convex variational problems

by

Carsten Carstensen and Stefan Müller

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LOCAL STRESS REGULARITY IN SCALAR NON-CONVEX VARIATIONAL PROBLEMS

CARSTEN CARSTENSEN AND STEFAN MÜLLER

ABSTRACT. Motivated by relaxation in the calculus of variations, this paper addresses convex but not necessarily strictly convex minimization problems. A class of energy functionals is described for which any stress field $\sigma$ in $L^q(\Omega)$ with $\text{div}\sigma$ in $W^{1,p'}(\Omega)$ (from Euler Lagrange equations and smooth lower order terms) belongs to $W^{1,q}_{\text{loc}}(\Omega)$. Applications include the scalar double-well potential, an optimal design problem, a vectorial double-well problem in a compatible case, and Hencky elastoplasticity with hardening. If the energy density depends only on the modulus of the gradient we also show regularity up to the boundary.

1. Introduction

Given a volume term $f \in L^q_{\text{loc}}(\Omega)$ and Dirichlet data $u_0 \in W^{1,p}(\Omega)$ let the admissible displacements $\mathcal{A}$ be a nonvoid closed convex subset of $W^{1,p}(\Omega)$ with $u_0 + W^{1,p}_0(\Omega) \subseteq \mathcal{A} \subseteq W^{1,p}(\Omega)$. The task to

\[
\text{minimize } E(u) := \int_{\Omega} W(Du) \, dx - \int_{\Omega} f u \, dx \quad \text{amongst } u \in \mathcal{A}
\]

may fail to have a solution in $\mathcal{A}$. Typically, infimizing sequences exist and are bounded in the seminorm of $W^{1,p}(\Omega)$ and weakly convergent towards some $u$ in $\mathcal{A}$; but, $u$ may fail to minimize the energy $E$ as the functional $E : \mathcal{A} \to \mathbb{R}$ is not (sequentially) weakly lower semicontinuous owing to its non-convexity.

Nevertheless, $u$ describes the macroscopic, space-averaged state and so is of interest. Relaxation results in the calculus of variations show that $u$ can be computed as a solution of the relaxed problem,

\[
\text{minimize } RE(u) := \int_{\Omega} \varphi(Du) \, dx - \int_{\Omega} f u \, dx \quad \text{amongst } u \in \mathcal{A}.
\]

In the general case, $\varphi$ is the quasiconvexification of $W$ [Dac89, Rou97]; the arguments of this paper are essentially restricted to the situation

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where $\varphi$ is the convex envelope of $W$ and so is easier to compute or to approximate.

It was observed in [Fri94, CP97a] for scalar problems and recently in [BKK00] in the general case, that the stress fields $\sigma_j := DW(Du_j)$ of an infimizing sequence $u_j$ converge in a weak sense. The limit $\sigma$ is given as the stress of a relaxed functional $\varphi$ of $W$, i.e., $\sigma = D\varphi(Du)$. Hence, the stress field associated with (1.1) can be computed from (1.2); for the regularity of $\sigma$, it suffices to study (1.2).

This paper establishes local regularity of the stress variable $\sigma$ under minimal conditions on $u$. We consider a class of convex (but not necessarily strictly convex) $C^1$ functions $\varphi$ with

$$ |D\varphi(A) - D\varphi(B)|^2 \leq c(1 + |A|^s + |B|^s) \times (D\varphi(A) - D\varphi(B)) : (A - B) $$

(1.3)

for all $A, B \in M^{m \times n}$ ($M^{m \times n}$ denotes the real $m \times n$ matrices) and a multiplicative constant $c$. Our interpretation of (1.3) is as follows: As a function of the two variables $(A, B)$, the left-hand side has zeros where the right-hand side has, but, off the diagonal in $M^{m \times n} \times M^{m \times n}$, they are of higher order. This local bound plus sufficient growth conditions for a proper choice of $s \geq 0$ yield (1.3). Note carefully that (1.3) implies convexity of $\varphi$ but not strict convexity.

Theorem 2.1 of Section 2 asserts that the monotonicity condition (1.3) and $\text{div} D\varphi(Du) = f$ in $W^{1,q}(\Omega)$ for some solution $u$ of (1.2) yield $\sigma = D\varphi(Du)$ in $W^{1,q}_{\text{loc}}(\Omega)$. Examples follow for the scalar two-well potential in Section 3 and for a relaxed energy density of an optimal design problem in Section 4.

A symmetric variant of (1.1)-(1.2) where $Du$ (for $n = m$) is replaced by the symmetric part $\varepsilon(u) := \text{sym} Du$,

$$ \minimize RE(u) := \int_\Omega \varphi(\varepsilon(u)) \, dx - \int_\Omega f \, u \, dx \text{ amongst } u \in A, $$

(1.4)

is discussed in Section 5. Emphasis is put on robustness of the stress in the Lamé constant $\lambda \to \infty$ involved in the elastic contribution of the models. Applications to Hencky elastoplasticity and a vector two-well example in Section 6 and 7 conclude this paper.

Throughout this paper, $M^{m \times n}$ denotes the real $m \times n$ matrices endowed with the Euclidean scalar product $; : = \sum_{j=1}^m \sum_{k=1}^n A_{jk} B_{jk}$ and induced (Frobenius matrix) norm $| \cdot |, |A| := (A : A)^{1/2}$. We use standard notation for Sobolev and Lebesgue spaces and norms resp. seminorms.
2. Abstract Stress Regularity Result

Let $\Omega$ be an open set in $\mathbb{R}^n$ and let $\varphi : \mathbb{M}^{m \times n} \to \mathbb{R}$ be $C^1$ and let $D \varphi$ be its derivative. Suppose that there exist constants $1 < p < \infty$, $1 < r < \infty$, $0 \leq s < \infty$ and $0 < c_1$, such that, for all $A, B \in \mathbb{M}^{m \times n}$,

\begin{equation}
|D \varphi(A) - D \varphi(B)|^r \leq c_1 (1 + |A|^s + |B|^s) \times (D \varphi(A) - D \varphi(B)) : (A - B). 
\end{equation}

**Theorem 2.1.** Assume furthermore that $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ and $\sigma := D \varphi(Du)$ satisfy, for $p' := p/(p - 1)$ and $q := r/(1 + s/p)$,

$\sigma \in L^{q}_{\text{loc}}(\Omega; \mathbb{M}^{m \times n})$ and $\text{div} \sigma \in W^{1,p'}_{\text{loc}}(\Omega; \mathbb{R}^m)$.

Suppose $p' \leq q$ and $r \leq 2$. Then

$\sigma \in W^{1,q}_{\text{loc}}(\Omega; \mathbb{M}^{m \times n})$.

**Remarks 2.1.** (a) The point is that (2.1) implies that $\varphi$ is convex; but $\varphi$ need not to be strictly convex since the lower bound is in terms of stress differences but not in terms of $|A - B|$.

(b) The assumptions on $u$ can be localized to $u \in W^{1,p'}_{\text{loc}}(\Omega; \mathbb{R}^n)$ by choosing another $\Omega$.

**Proof.** Given any direction $M \in \mathbb{M}^{m \times n}$, $|M| = 1$, and some $\eta \in \mathcal{D}(\Omega)$ with supp $\eta \subset \omega \subset \overline{\omega} \subset \Omega$ for some bounded open set $\omega$ which lies compactly in $\Omega$. Set $\alpha := 1/(r - 1)$, and $\beta := r/(r - 1)$. Let $0 < h < h_0 := \text{dist}(\text{supp} \eta; \partial \omega)$ and define, for almost all $x \in \omega$,

$\tau(x) := (\sigma(x + hM) - \sigma(x))/h$,

$e(x) := (u(x + hM) - u(x))/h$,

$\delta(x) := De(x)$.

A standard argument in the approximation of weak derivatives by difference quotients shows

\begin{equation}
\|e\|_p := \|e\|_{L^p(\omega)} \leq c_2 \|u\|_{W^{1,p}(\Omega)}
\end{equation}

with an $h$-independent constant $c_2$ (that depends on $\omega$ and $\Omega$). Here and throughout the proof, $\| \cdot \|_t := \| \cdot \|_{L^t(\omega)}$ denotes the $L^t(\omega)$-norm with respect to the subdomain $\omega$ of $\Omega$.

Owing to $u \in W^{1,p}(\Omega)$, $\|e\|_{L^p(\omega)}$ is bounded $h$-independently. A careful estimation using Hölder’s inequality and

$\text{div} \sigma \in W^{1,p'}(\omega; \mathbb{M}^{m \times n})$ resp. $\sigma \in L^{p'}(\omega; \mathbb{M}^{m \times n})$
as well as $q' \leq p$ yields, in analogy with (2.2), the $h$-independent bound

$$\|e^{q/r} \|_{1+p/s}^{r/q} + \|e\|_p + \|e\|_{q'} + \|\eta^\beta \text{ div } \tau\|_{p'} + \|\eta\|_{W^{1,\infty}(\Omega)} \leq c_3,$$

where $q(x) := 1 + |Du(x)|^r + |Du(x + hM)|^r$ and $\eta$ is fixed.

To verify the assertion, we have to bound $|\tau|_{L^q(K)}$ by $h$-independent quantities for each compact $K \subset \Omega$ (below $K$ is a compact subset of the interior of $\text{supp} \, \eta$).

Owing to the estimate (2.1) (with $A := Du(x + hM)$ and $B := Du(x)$) we have,

$$|\tau|^r \leq c_1 h^{2-r} q \tau : \delta \quad \text{a.e. in } \omega. \tag{2.4}$$

Raising (2.4) to the power $q/r$, then multiplying with $\eta^{\alpha q}$, and finally integrating the result over $\Omega$, we infer

$$\|\eta^\alpha \tau\|_q^q \leq c_1 h^{2-r} \|e^{q/r}\|_{1+p/s}^{r/q} \int_\Omega \eta^{\alpha q} q \tau : \delta \, dx. \tag{2.5}$$

Applying Hölder’s inequality (for $r/q$ resp. $(r/q)' = 1 + p/s$) and raising the result to the power $r/q$ we obtain (since $0 \leq \tau : \delta$ and $\alpha r = \beta$)

$$\|\eta^\alpha \tau\|_q^r \leq c_1 h^{2-r} \|e^{q/r}\|_{1+p/s}^{r/q} \int_\Omega \eta^{\beta} \tau : \delta \, dx. \tag{2.6}$$

Since $\delta = D\epsilon$ on $\omega$, an integration by parts proves (recall $h \leq h_0$)

$$\int_\Omega \eta^{\beta} \tau : \delta \, dx = - \int_\Omega \epsilon \cdot \text{div}(\eta^{\beta} \tau) \, dx \leq \beta \|\eta\|_{1,\infty} \|\eta^{\beta-1} \tau \epsilon\|_1 + \|\epsilon\|_p \|\eta^{\beta} \text{ div } \tau\|_{p'}. \tag{2.7}$$

Since $\beta - 1 = \alpha$, Hölder’s inequality leads to

$$\|\eta^{\beta-1} \tau \epsilon\|_1 \leq \|\epsilon\|_q \|\eta^\alpha \tau\|_q. \tag{2.8}$$

The combination of (2.6)-(2.8) with (2.3) and $r \leq 2$ proves

$$\|\eta^\alpha \tau\|_q^r \leq c_1 c_2 h^{2-r} (1 + \|\eta^\alpha \tau\|_q). \tag{2.9}$$

With Young’s inequality $(ab \leq (ac)^r/r + (b/c)^{r'}/r'$ for positive $a, b, c$) we observe from (2.9), $r \leq 2$, and $q > 1$ that $\|\eta^\alpha \tau\|_q$ is bounded $h$-independently; hence,

$$\limsup_{h \to 0} \|\eta^\alpha \tau\|_{L^q(\Omega)} < \infty \quad \text{for all } \eta \in \mathcal{D}(\Omega).$$

The proof is finished. \qed

This section is concluded with the simple example of a $p$-Laplace equation to illustrate the growth condition in (2.1).
Example 2.1. Given $2 \leq p < \infty$, let $\varphi(F) := |F|^p/p$ for $F \in \mathbb{R}^n$, $m = 1$. Then, $D\varphi(F) = |F|^{p-2}F$ and so, for fixed $B \in \mathbb{R}^n$ and $A \in \mathbb{R}^n$ with $|A| \to \infty$,

$$\frac{|D\varphi(A) - D\varphi(B)|^2}{(D\varphi(A) - D\varphi(B) \cdot (A - B))} \approx |D\varphi(A)|/|A| = |A|^{p-2}.$$ 

Indeed, it is known (e.g., by a combination of Lemmas 2.1 to 2.3 in [CK01]) that, for any $A, B \in \mathbb{R}^n$,

$$\frac{|D\varphi(A) - D\varphi(B)|^2}{(D\varphi(A) - D\varphi(B) \cdot (A - B))} \leq (1 + \max\{1, p - 2\}^2)(|A|^{p-2} + |B|^{p-2}).$$

As a corollary of Theorem 2.1, we therefore obtain local regularity of the stress field, i.e., $\sigma := D\varphi(Du) \in W^{1,p'}_{loc}(\Omega; \mathbb{R}^n)$, for a minimizer $u \in W^{1,p}(\Omega)$ of (1.2) with $f \in W^{1,p'}_{loc}(\Omega)$.

3. An application to the scalar 2-well problem

This section concerns the scalar double-well problem where

$$(3.1) \quad W : \mathbb{R}^n \to \mathbb{R}, \quad F \mapsto |F - F_1|^2 |F - F_2|^2$$

for the given two distinct wells $F_1, F_2 \in \mathbb{R}^n$, $F_1 \neq F_2$. The scalar problem (1.1) with (3.1) (for $m = 1$) can be deduced from the Ericksen-James energy density in an anti-plane shear model; the version for $n = 1$, due to O. Bolza [Bol06] (cf. also [You69]), is the model example in non-convex minimization.

Proposition 3.1 ([CP97a]). Let $a := (F_2 - F_1)/2$ and $b := (F_1 + F_2)/2$. The convex envelope $\varphi$ of (3.1) is

$$\varphi(F) := \max\{|F - b|^2 - |a|^2, 0\}^2 + 4(|a|^2 |F - b|^2 - |a \cdot (F - b)|^2)$$

and satisfies (2.1) with $r = 2$, $s = 2$, and $c_1 = 4 \max\{2, |F_1 - F_2|^2\}$.

Corollary 3.2. Adopt notation of Proposition 3.1 and let $u$ be a minimizer of (1.2). Then $\sigma := D\varphi(Du)$ belongs to $W^{1,4/3}_{loc}(\Omega; \mathbb{R}^n)$.

Proof. The assertion follows from Theorem 2.1 and Proposition 3.1 since the Euler Lagrange equations of the minimization problem (1.2) provide $-\text{div} \, \sigma = f \in W^{1,4/3}(\Omega)$.

Remark 3.1. Further estimates in [CP97a] allow one to control other quantities. In particular,

$$\max\{0, |B - Du|^2 - |A|^2\} \in H^{1}_{loc}(\Omega) \text{ and } M \cdot Du \in H^{1}_{loc}(\Omega)$$

for all directions $M$ perpendicular to $A$. 
4. An Application to an Optimal Design Problem

The relaxed model for an optimal design problem derived in [GKR86] has the form (1.2) where \( \varphi(F) = \psi(|F|) \). Given positive parameters \( 0 < t_1 < t_2 \) and \( 0 < \mu_2 < \mu_1 \) with \( t_1 \mu_1 = t_2 \mu_2 \), the \( C^1 \) function \( \psi : [0, \infty) \rightarrow [0, \infty) \) is defined by \( \psi(0) = 0 \) and

\[
\psi'(t) := \begin{cases} 
\mu_1 t & \text{if } 0 \leq t \leq t_1, \\
t_1 \mu_1 = t_2 \mu_2 & \text{if } t_1 \leq t \leq t_2, \\
\mu_2 t & \text{if } t_2 \leq t.
\end{cases}
\]

**Proposition 4.1 (\cite{CP97a}).** The function \( \varphi(F) = \psi(|F|) \) satisfies (2.1) with \( r = 2 \), \( s = 0 \), and \( c_1 = 1/\mu_1 \).

Therefore, Theorem 2.1 yields local stress regularity for minimizers of (1.2) when \( f \in H^1_{loc}(\Omega) \).

**Corollary 4.2.** Adopt notation of Proposition 4.1 and let \( u \) be a minimizer of (1.2) in \( \mathcal{A} := H^1_0(\Omega) \). Then \( \sigma := D\varphi(Du) \) belongs to \( W^{1,2}_{loc}(\Omega; \mathbb{R}^n) \).

The rest of this section is devoted to establish regularity up to the boundary.

**Theorem 4.3.** Suppose that \( f \in W^{1,2}_0(\Omega) \) and that \( \Omega \) is a \( C^{2,1} \) domain. If \( u \) is a minimizer of (1.2), then \( \sigma := D\varphi(Du) \) belongs to \( W^{1,2}(\Omega; \mathbb{R}^n) \).

The remaining part of this section is devoted to a proof of Theorem 4.3 via a local reflection argument. Owing to the local regularity of Corollary 4.2, it remains to prove \( \sigma \in W^{1,2}(\Omega \cap B(x_0, \delta); \mathbb{R}^n) \) for each point \( x_0 \) on the boundary \( \partial\Omega \) and some small \( \delta > 0 \). Without loss of generality, we suppose \( x_0 = 0 \) and that the Cartesian coordinate system at hand directly allows a \( C^{2,1} \) parameterization.

**Definition 4.1.** Let \( \chi : B'_0 \rightarrow \mathbb{R} \) be a (scalar) \( C^{2+\alpha} \) function where \( B_0 := B(0, \delta_0) \subset \mathbb{R}^n \) and \( B'_0 := \{ x' \in \mathbb{R}^{n-1} : |x'| < \delta_0 \} \subset \mathbb{R}^{n-1} \) denotes the \( \delta_0 \)-ball around \( x_0 = 0 \) in \( n \) and \( (n - 1) \) dimensions, respectively. Suppose that \( \chi \) parameterizes the boundary \( \Gamma := \partial\Omega \) near \( x_0 = 0 \), i.e.,

\[
\Gamma \cap B_0 = \{(x', \chi(x')) \in B_0 : x' \in B'_0 \},
\]

\[
\Omega \cap B_0 = \{(x', x_n) \in B_0 : x' \in B', x_n > \chi(x') \},
\]

\[
B_0 \setminus \overline{\Omega} = \{(x', x_n) \in B_0 : x' \in B', x_n < \chi(x') \}.
\]

Let \( \nu \) be the unit normal vector on \( \Gamma \) and set

\[
\Psi(x) := (x', \chi(x')) - x_n \nu(x', \chi(x'))
\]

for all \( x =: (x', x_n) \in B_0 \subset B'_0 \times \mathbb{R} \).
Lemma 4.4. The pull-back metric \( g := D\Psi^T D\Psi : B_0 \to M_{\text{sym}}^{n \times n} \) is \( C^{1+\alpha} \) and, with \( E := I_{n-1} + e_n \otimes D\chi(c') - x_n D_x \nu(x', \chi(x')) \in M_{\text{sym}}^{n \times (n-1)} \) for the \( n \times (n-1) \) unit matrix \( I_{n-1} \), given by

\[
g(x) = \begin{pmatrix} E^T E & 0 \\ 0 & 1 \end{pmatrix} \in M_{\text{sym}}^{n \times n} \text{ for } (x', x_n) \in B.
\]

Proof. The \( C^1 \) property follows from direct calculations; the derivative \( D_x \nu(x', \chi(x')) \) replaces the \( n \times (n-1) \) matrix of \( \nu_j(x', \chi(x')) \) differentiated by \( x_1, \ldots, x_{n-1} \). Since \( \partial \chi / \partial x_k \) is tangential on \( \Gamma \) for \( k = 1, 2, \ldots, n-1 \) there holds \( \nu \cdot \partial \chi / \partial x_k = 0 \); whence \( \nu^T D\chi = 0 \). From \( |\nu|^2 = 1 \), we deduce \( \nu^T D\nu = 0 \). Those orthogonalities yield the block structure asserted for \( g(x) \). \( \Box \)

Definition 4.2. Suppose that \( \delta \) is small enough, \( 0 < \delta < \delta_0 \), such that \( \Psi(B_+) =: \omega \subset \Omega \), \( B_+ := \{(x', x_n) \in B : \pm x_n > 0 \} \) where \( B := B(0, \delta) \) and \( B' := \{x' \in \mathbb{R}^{n-1} : |x'| < \delta \} \) denote the \( \delta \)-ball around \( x_0 = 0 \) in \( n \) and \( (n-1) \) dimensions, respectively. For any \( x = (x', x_n) \in B_+ \) set \( Sx := (x', -x_n) \in B_- \) and

\[
\begin{align*}
\tilde{u}(x) &= -\tilde{u}(Sx) := u(\Psi(x)), \\
\tilde{\sigma}(x) &= -\tilde{\sigma}(Sx) S := \sigma(\Psi(x)) \operatorname{cof} D\Psi, \\
\tilde{f}(x) &= -\tilde{f}(Sx) := (\det g(x))^{1/2} f(\Psi(x)), \\
\tilde{g}(x) &= \tilde{g}(Sx) := g(x).
\end{align*}
\]

Lemma 4.5. There holds \( \tilde{u} \in W^{1,2}(B), \tilde{f} \in W^{1,2}(B), \tilde{\sigma} \in H(\text{div}, B), \)

\[
\tilde{\sigma} = D\varphi(\nabla \tilde{u} \tilde{g}^{-1/2}) \operatorname{cof} \tilde{g}^{1/2} \text{ in } B,
\]

and

\[
\text{div } \tilde{\sigma} = \tilde{f} \text{ in } \mathcal{D}'(B).
\]

Proof. A polar decomposition \( QU = D\Psi(x) \) shows \( g(x) = U^2, g(x)^{1/2} = U \). Since \( Q = D\Psi(x)g(x)^{-1/2} \) is orthonormal,

\[
|\nabla u(\Psi(x))| = |\nabla u(\Psi(x)) D\Psi(x) g(x)^{-1/2}| = |\nabla \tilde{u}(x) g(x)^{-1/2}|.
\]

Since \( \varphi(\cdot) = \psi(|\cdot|) \) solely depends on the modulus, this shows, at \( \xi := \Psi(x), x \in B_+ \),

\[
\sigma(\xi) = D\varphi(\nabla u(\xi)) = \psi'(|\nabla u(\xi)|) \text{sign} \nabla u(\xi) = \psi'(|\nabla \tilde{u}(x) g(x)^{-1/2}|) \text{sign}(\nabla \tilde{u}(x) D\Psi^{-1}(x)) = D\varphi(\nabla \tilde{u}(x) g^{-1/2}(x)) Q^T.
\]

Since \( \operatorname{adj} D\tilde{\psi}(x) = \operatorname{cof} g^{1/2}(x) Q^T \), this proves the asserted identity for \( \tilde{\sigma} \) in \( B_+ \).
By assumption, \( \text{div } \sigma = f \) in \( \mathcal{D}'(\Omega) \), and elementary transformations show, for all \( \eta \in \mathcal{D}(B_\pm) \) with the test function \( \eta \circ \Psi^{-1} \),

\[
\int_{B_+} \tilde{f}(x) \eta(x) \, dx = \int_{\omega} f(\xi) \eta(\Psi^{-1}(\xi)) \, d\xi
\]

\[
= -\int_{\omega} \nabla \eta(\Psi^{-1}(\xi)) \cdot D\Psi^{-1}(\xi) \sigma(\xi) \, d\xi.
\]

The substitution of \( \tilde{\sigma} \) and a re-transformation give

\[
\int_{B_+} \tilde{f}(x) \eta(x) \, dx = \int_{B_+} \tilde{\sigma}(x) \cdot \nabla \eta(x) \, dx.
\]

This proves \( \text{div } \tilde{\sigma} = \tilde{f} \) in \( \mathcal{D}'(B_+) \).

The block structure of \( g \) shows that \( g^\alpha \) commutes with \( S = \text{diag}(1, \ldots, 1, -1) \), i.e., \( S g^\alpha = g^\alpha S \) for \( \alpha \in \mathbb{R} \). Since \( \varphi(\cdot) \) depends solely on the modulus, \( D\varphi \) commutes with \( S \) as well, i.e., \( D\varphi(-S \cdot) = -SD\varphi(\cdot) \).

Then, for \( x \in B_-, \xi \in B_+, x = S\xi, \)

\[
\text{cof } \tilde{g}^{1/2}(x) \, D\varphi(\tilde{g}^{-1/2}(x) \nabla \tilde{u}(x)) = \text{cof } \tilde{g}^{1/2}(\xi) \, D\varphi(-\tilde{g}^{-1/2}(\xi) \nabla \tilde{u}(\xi))
\]

\[
= -S \text{cof } \tilde{g}^{1/2}(\xi) \, D\varphi(\tilde{g}^{-1/2}(\xi) \nabla \tilde{u}(\xi)) = -\tilde{\sigma}(\xi) S = \tilde{\sigma}(x).
\]

Thus, \( \tilde{\sigma} = \text{cof}(\tilde{g}^{1/2}) \, D\varphi(\tilde{g}^{-1/2} \nabla \tilde{u}) \) holds almost everywhere in \( B \).

Owing to \( \tilde{f} = 0 = \tilde{u} \) on \( \overline{B_+} \cap \overline{B_-} = B \cap (B' \times \{0\}) \), \( \tilde{u} \) and \( \tilde{f} \) belong to \( W^{1,2}(B) \). Notice that \( g \in C(B) \). Clearly, \( \tilde{\sigma} \in L^2(B) \) and \( \tilde{\sigma}|_{B_\pm} \in H(\text{div}, B_\pm) \). Hence it remains to prove \( \text{div } \tilde{\sigma} = \tilde{f} \) in \( \mathcal{D}'(B) \).

Given \( \eta \in \mathcal{D}(B) \) set \( \alpha := (\eta + \eta \circ S)/2 \) and \( \beta := (\eta - \eta \circ S)/2 \). Since \( \nabla \alpha(x) = (\nabla \eta(x) + \nabla \eta(Sx)S)/2 = \nabla \alpha(Sx)S \),

\[
\int_{B} \tilde{\sigma} \cdot \nabla \alpha \, dx = \int_{B_+} (\tilde{\sigma}(x) + \tilde{\sigma}(Sx)) \cdot \nabla \alpha(x) \, dx = 0.
\]

Since \( \beta = 0 \) on \( B' \times \{0\} \) and \( \nabla \beta(Sx) = -\nabla \beta(x)S \), a transformation to \( B_+ \) and an integration by parts in \( B_+ \) lead to

\[
\int_{B} \tilde{\sigma} \cdot \nabla \eta \, dx = \int_{B_+} \tilde{\sigma}(x) \cdot \nabla \beta(x) \, dx + \int_{B_+} \tilde{\sigma}(Sx) \cdot \nabla \beta(Sx) \, dx
\]

\[
= 2 \int_{B_+} \tilde{\sigma}(x) \cdot \nabla \beta(x) \, dx = 2 \int_{B_+} \tilde{f}(x) \beta(x) \, dx = \int_{B} \tilde{f}(x) \beta(x) \, dx.
\]

Hence \( -\text{div } \tilde{\sigma} = \tilde{f} \) in \( \mathcal{D}'(B) \) and the proof of Lemma 4.5 is finished. \( \square \)

The proceeding results on the reflection near \( x_0 \) at the boundary provides \( \tilde{\sigma} \) perturbed by the metric \( \tilde{g}^{-1/2} \). Hence, Theorem 2.1 does not directly lead to \( \tilde{\sigma} \in W^{1,2}_{\text{loc}}(B; \mathbb{M}^{n \times n}) \) which then shows \( \sigma \in W^{1,2}(\Omega \cap \mathbb{R}^n) \).
\(B(x_0, \delta/2; \mathbb{R}^n)\) and so concludes the proof of Theorem 4.3. Instead, we need to follow the proof of Theorem 2.1 for the perturbed situation.

**Proof of Theorem 4.3.** Given \(x \in B, h > 0, M \in \mathbb{R}^n, |M| = 1\) set for brevity, \(x_2 := x + hM, x_0 := x - hM,\) and \(x_1 := x\) and, for \(j = 1, 2,\)

\[
F_j := \nabla \tilde{u}(x_j), \quad \sigma_j := \tilde{\sigma}(x_j), \quad U_j := \tilde{g}^{-1/2}(x_j),
\]

\[
V_j := \cof U_j^{-1}, \quad \Sigma_j := \sigma_j \det U_j, \quad T_j := \Sigma_j U_j^{-1}.
\]

Moreover, let \(a \leq Cb\) be abbreviated as \(a \lesssim b\) if \(C\) is a generic constant that is independent of (sufficient small) \(\delta > 0, h > 0.\) The constant \(C > 0,\) however, may depend on \(g, U_j, V_j,\) e.g., through \(\|\tilde{g}\|_{W^{1,\infty}(B)}, \|\cof \tilde{g}\|_{W^{1,\infty}(B)}, \|\tilde{g}^{-1}\|_{W^{1,\infty}(B)}, \|\eta\|_{W^{1,\infty}(B)}\). Then,

\[
|\sigma_2 - \sigma_1|^2 \leq |D\varphi(F_2 U_2 V_2 - D\varphi(F_1 U_1 V_1))|^2 \lesssim |V_2 - V_1|^2 |D\varphi(F_1 U_1)|^2 + |D\varphi(F_2 U_2) - D\varphi(F_1 U_1)|^2.
\]

With Proposition 4.1, the above notation, and the identity \(T_j U_j = \Sigma_j\) we infer

\[
|D\varphi(F_2 U_2) - D\varphi(F_1 U_1)|^2 \lesssim (D\varphi(F_2 U_2) - D\varphi(F_1 U_1)) \cdot (F_2 U_2 - F_1 U_1)
\]

\[
= (T_2 - T_1) \cdot (F_2 U_2 - F_1 U_1)
\]

\[
= (\Sigma_2 - \Sigma_1) \cdot (F_2 - F_1) + (T_2 - T_1) \cdot (F_1 + F_2)(U_2 - U_1)
\]

\[
+ T_1 \cdot F_1(U_2 - U_1) - T_2 \cdot F_2(U_2 - U_1).
\]

Given \(\eta \in \mathcal{D}(B), 0 \leq \eta \leq 1,\) which equals one in a neighborhood of \(x_0 = 0\) and provided \(|h|\) sufficiently small, the combination of the last two estimates is multiplied by \(\eta^2/h^2\) and integrated over \(\text{supp} \eta.\) With the notation \(\tilde{\tau}(x) := (\tilde{\sigma}(x_2) - \tilde{\sigma}(x_1))/h\) and \(\tilde{c}(x) := (\tilde{u}(x_2) - \tilde{u}(x_1))/h,\) we deduce,

\[
\int \eta^2(x) |\tilde{\tau}(x)|^2 dx \lesssim \int \eta^2 |\tilde{\sigma}(x)|^2 dx
\]

\[
+ 1/h^2 \int \eta^2 (\Sigma_2 - \Sigma_1) \cdot (F_2 - F_1) dx
\]

\[
+ 1/h \int \eta^2 |T_2 - T_1| \left(\|\nabla \tilde{u}(x)\| + \|\nabla \tilde{u}(x + hM)\|\right) dx
\]

\[
+ 1/h^2 \int \eta^2 T_1(U_2 - U_1) \cdot F_1 dx
\]

\[
- 1/h^2 \int \eta^2 T_2(U_2 - U_1) \cdot F_2 dx
\]

\[
=: I + II + III + IV - V.
\]
Term I is bounded since \( \tilde{\sigma} \in L^2(B) \). Term II is recast into
\[
II = \frac{1}{h^2} \int \eta^2(x) \cdot (F_2 - F_1) \, dx
\]
\[
= \int \eta^2(x) \det \tilde{g}^{-1/2}(x) \tilde{\tau}(x) \cdot \nabla \tilde{e}(x) \, dx
\]
\[
+ \int \eta^2(x) \left( \det \tilde{g}^{-1/2}(x_2) - \det \tilde{g}^{-1/2}(x_1) \right) \, dx
\]
\[
\times \tilde{\sigma}(x_2) \cdot \nabla \tilde{e}(x) \, dx.
\]
Since \( \tilde{\eta}^2 \det \tilde{g}^{-1/2} \in H^1(B) \) is a feasible test function we have
\[
\int \eta^2 \det \tilde{g}^{-1/2} \tilde{\tau} \cdot \nabla \tilde{e} \, dx = - \int \tilde{\eta} \cdot \nabla (\eta^2 \det \tilde{g}^{-1/2}) \, dx
\]
\[
+ \int \tilde{\eta}(x) \eta^2(x) \det \tilde{g}^{-1/2}(x) \left( f(x_2) - f(x_1) \right) \, dx
\]
\[
\lesssim \| \tilde{\eta} \|_{L^2(B)} \left( \| f \|_{L^2(B)} + \| \tilde{\eta} \|_{L^2(B)} \right)
\]
with the abbreviations \( \| \cdot \|_p := \| \cdot \|_{L^p(B)} \) and \( \| \cdot \|_{1,p} := \| \cdot \|_{W^{1,p}(B)} \). A shift in the variable \( x_2 \) in the term \( \nabla \tilde{e}(x) \) yields
\[
\int \eta^2(x) \left( \det \tilde{g}^{-1/2}(x_2) - \det \tilde{g}^{-1/2}(x_1) \right) \, dx
\]
\[
= - \int \left( \eta^2(x_1) - \eta^2(x_0) \right) \, dx
\]
\[
\times \left( \det \tilde{g}^{-1/2}(x_1) - \det \tilde{g}^{-1/2}(x_0) \right) \tilde{\sigma}(x) \cdot \nabla \tilde{u}(x) \, dx
\]
\[
- \int \eta^2(x) \left( \det \tilde{g}^{-1/2}(x_2) - \det \tilde{g}^{-1/2}(x_1) \right) \tilde{\tau}(x) \cdot \nabla \tilde{u}(x) \, dx
\]
\[
- \int \eta^2(x) \left( \det \tilde{g}^{-1/2}(x_2) - \det \tilde{g}^{-1/2}(x_1) \right) \, dx
\]
\[
\times \left( \det \tilde{g}^{-1/2}(x_2) - 2 \det \tilde{g}^{-1/2}(x_1) \right)
\]
\[
\lesssim \| \tilde{\sigma} \|_{L^2(B)} \cdot \| \tilde{\eta} \|_{L^2(B)}.
\]
In the last step we employed \( g \in C^{1,1} \) and so required that \( \partial \Omega \) is \( C^{2,1} \). Altogether,
\[
II \lesssim \| \tilde{\eta} \|_{L^2(B)} \left( \| f \|_{L^2(B)} + \| \tilde{\sigma} \|_{L^2(B)} + \| \tilde{\eta} \|_{L^2(B)} \right).
\]
Since \( T_j = D\varphi(\nabla \tilde{u}(x_j)) = \sigma_j \text{cof } U_j \) similar arguments lead to
\[
III = \frac{1}{h} \int \eta^2 |T_2 - T_1| \left( |\nabla \tilde{u}(x_1)| + |\nabla \tilde{u}(x_2)| \right) \, dx
\]
\[
\lesssim \left( \| \tilde{\eta} \|_{L^2(B)} + \| \tilde{\sigma} \|_{L^2(B)} \right) \| \tilde{u} \|_{L^2(B)}.
\]
A shift in the variable \( x_2 \) in term \( V \) and similar arguments result in

\[
IV - V = \int \eta^2 T_1 (\tilde{g}^{-1/2}(x_2) - 2\tilde{g}^{-1/2}(x_1) + \tilde{g}^{-1/2}(x_0))/h^2 \cdot \nabla \tilde{u}(x) \, dx
+ \int (\eta^2(x_1) - \eta^2(x_2))/h \cdot T_1 (\tilde{g}^{-1/2}(x) - \tilde{g}^{-1/2}(x_0))/h \cdot \nabla \tilde{u}(x) \, dx
\lesssim \| \tilde{\sigma} \|_2 \| \tilde{u} \|_{1,2}.
\]

Absorbing \( \| \eta \tau \|_2 \) in II and III one concludes the proof.

5. A Symmetric Variant for Geometrically Linear Models

This section concerns Theorem 2.1 for symmetrized gradients. Some (geometrically) linear models in elasticity involve the symmetric Green strain

\[
\varepsilon(u) := \text{sym } Du := ((\partial u_j/\partial x_k + \partial u_k/\partial x_j) : j, k = 1, \ldots, n)
\]

for \( m = n \). For the ease of this presentation we focus on \( p = r = q = 2 \) and \( s = 0 \) as in linear elasticity but emphasize robustness with respect to the incompressible limit \( \lambda \to \infty \) (see below).

Let \( M_{sym}^{n \times n} \) denote the symmetric real \( n \times n \) matrices. The fourth-order elasticity tensor \( C : M_{sym}^{n \times n} \to M_{sym}^{n \times n} \) is defined by

\[
CE := \lambda \text{tr}(E) I + 2\mu E \quad \text{for } E \in M_{sym}^{n \times n}
\]

for the positive Lamè constants \( \lambda, \mu \), the trace \( \text{tr}(E) := \sum_{j=1}^n E_{jj} \), and the \( n \times n \) unit matrix \( I \). Since \( C \) is positive definite, there exist an inverse \( C^{-1} \) and their square roots \( C^{1/2} \) and \( C^{-1/2} \). The norm

\[
| E |_C := (E : CE)^{1/2} = | C^{1/2} E | \quad \text{for } E \in M_{sym}^{n \times n}
\]

is induced by the energy scalar product with respect to \( C \) in \( M_{sym}^{n \times n} \).

Suppose that \( \varphi : M_{sym}^{n \times n} \to \mathbb{R} \) is \( C^1 \) and that, for its derivative \( D\varphi \), there exists a constant \( c_4 \) such that, for all \( A, B \in M_{sym}^{n \times n} \),

\[
| D\varphi(A) - D\varphi(B) |_{C^{-1}} \leq c_4 (D\varphi(A) - D\varphi(B)) : (A - B).
\]

**Theorem 5.1.** Assume furthermore that

\[
u \in H^1(\Omega; \mathbb{R}^n) \quad \text{and} \quad \sigma := D\varphi(\varepsilon(u))
\]

satisfy

\[
\sigma \in L^2_{loc}(\Omega; M_{sym}^{n \times n}) \quad \text{and} \quad \text{div} \sigma \in H^1_{loc}(\Omega; \mathbb{R}^n).
\]

Then

\[
\sigma \in H^1_{loc}(\Omega; M_{sym}^{n \times n}).
\]
Moreover, if \( \omega_0 \subset \subset \omega_1 \subset \subset \Omega \) for nonvoid open sets \( \omega_0 \) and \( \omega_1 \), there exists a \( \lambda \)-independent constant \( c_5 > 0 \) such that

\[
(5.2) \quad \| \sigma \|_{H^1(\omega_0)} \leq c_5 (\| u \|_{H^1(\omega_1)} + \| \sigma \|_{L^2(\omega_1)} + \| \text{div} \sigma \|_{H^1(\omega_1)}).
\]

Remarks 5.1. (a) Korn’s inequality does not play an explicit role in the proof. It is used, however, in applications to guarantee \( u \in H^1(\Omega) \) (and so the boundedness of \( e \) in \( L^2_{\text{loc}}(\Omega) \) in the proof).

(b) The fourth order elasticity tensor could be more general; for the assertion \( \sigma \in H^1_{\text{loc}}(\Omega, \mathbb{M}^{n \times n}_{\text{sym}}) \) it is sufficient that \( \mathbb{C} \) is a linear, continuous, and positive definite operator.

(c) The constant \( c_5 \) in (5.2) depends on \( c_4, \mu, \omega_0, \) and \( \omega_1 \); but neither on \( \sigma \) nor on \( u \).

(d) The functional \( \varphi : \mathbb{M}^{n \times n}_{\text{sym}} \to \mathbb{R} \) may depend on \( \mathbb{C} \) and \( \lambda \); the constant \( c_5 \) depends on \( \varphi \) only through \( c_4 \) and stays \( \lambda \)-independent as long as \( c_4 \) does.

Proof of Theorem 5.1. The proof follows the arguments of the proof of Theorem 2.1 where the differential operator \( D \) is replaced by the symmetric variant \( \varepsilon \), e.g., \( \delta := \varepsilon(e) \). This results in

\[
\| \eta \| \mathbb{C}^{-1/2} \|_2 \leq c_0 \int_{\omega} \eta^2 \varepsilon(e) : \tau \, dx = -c_0 \int_{\omega} \eta \, \text{div}(\tau \eta^2) \, dx
\]

\[
(5.3) \quad \leq c_6 \| \eta \|_2 \| \eta \|_{W^{1,\infty}(\Omega)} \| 2 \| \eta \tau \|_2 + \| \eta \, \text{div} \tau \|_2 \)
\]

\[
\leq c_7 (\| u \|_{H^1(\omega)} + \| \text{div} \sigma \|_{H^1(\omega)} + \| \eta \|_2)^2
\]

for some \( (h, \lambda, \mu) \)-independent constant \( c_7 > 0 \). The first assertion follows (with a \( \lambda \)-depending constant) from this and

\[
\| \eta \tau \|_2 \leq (2\mu + \lambda)^{1/2} \| \mathbb{C}^{-1/2} \tau \|_2.
\]

In order to prove (5.2) we are given \( \omega_0 \subset \subset \omega_1 \) and suppose that \( \omega \) is a bounded Lipschitz domain between \( \omega_0 \) and \( \omega_1 \), \( \omega_0 \subset \subset \omega \subset \subset \omega_1 \). Assume that \( \eta \in D(\omega) \) satisfies \( 0 \leq \eta \leq 1 \) and equals \( \eta = 1 \) on \( \omega_0 \). Then, we introduce the deviator \( \text{dev}(\tau) := \tau - \text{tr}(\tau)/n \mathbb{I} \) and rewrite (5.3) as

\[
\| \eta \| \mathbb{C}^{-1/2} \|_2 = \frac{\| \text{dev}(\eta \tau) \|_2^2}{2\mu} + \| \text{tr}(\eta \tau) \|_2^2 \]

\[
(5.4) \quad \leq c_7 \left( \| u \|_{H^1(\omega)} + \| \text{div} \sigma \|_{H^1(\omega)} + \| \text{dev}(\eta \tau) \|_2^2 + \| \text{tr}(\eta \tau) \|_2^2 \right)
\]

The \( \lambda \)-independent bound requires an extra argument using the Stokes problem [BF91, GR86] where integral means must be factored out. With the center \( \xi \) of mass of \( \omega \) and \( e_1 = (1, 0, \ldots, 0) \) we define the
constant
\[ \tau_0 := \int_{\omega} \text{tr}(\eta \tau) dx / |\omega| \in \mathbb{R} \]

(|\omega| \text{ is the measure of } \omega) \text{ and the function } v_1 \in H^1(\omega; \mathbb{R}^n) \text{ by}
\[ v_1(x) := \tau_0((x - \xi) \cdot e_1) e_1 \text{ for } x \in \omega. \]

Then, \( \text{div } v_1 = \tau_0 \) and \( \int_{\omega}(\tau_0 - \text{tr}(\eta \tau)) dx = 0. \) The solvability of the Stokes equations guarantees the existence of \( v_2 \in H^1_0(\omega; \mathbb{R}^n) \) with \( \text{div } v_2 = \tau_0 - \text{tr}(\eta \tau) \) and the bound
\[ \| v_2 \|_{H^1(\omega)} \leq c_8 \| \tau_0 - \text{tr}(\eta \tau) \|_2 \leq c_8 \| \text{tr}(\eta \tau) \|_2. \]

Then, with \( c_9 > 0, \)
\[ v := v_1 - v_2 \in H^1(\omega; \mathbb{R}^n)) \text{ satisfies} \]
\[ (5.5) \quad \text{div } v = \text{tr}(\eta \tau) \quad \text{and} \quad \| v \|_{H^1(\omega)} \leq c_9 \| \text{tr}(\eta \tau) \|_2. \]

Recall \( \text{tr}(\eta \tau) / n I := \eta \tau - \text{dev}(\eta \tau) \) and deduce
\[ \| \text{tr}(\eta \tau) \|_2^2 = \int_{\omega} \text{tr}(\eta \tau) \text{div } v dx = \int_{\omega} \text{tr}(\eta \tau) I : Dv dx \]
\[ = n \int_{\omega} (\eta \tau - \text{dev}(\eta \tau)) : Dv dx. \]

Cauchy inequalities and integration by parts result in
\[ \frac{1}{n} \| \text{tr}(\eta \tau) \|_2^2 \leq \| Dv \|_2 \| \text{dev}(\eta \tau) \|_2 - \int_{\Omega} v \cdot (\tau \nabla \eta + \eta \text{div } \tau) dx \]
\[ \leq \| v \|_{H^1(\omega)} \left( \| \text{dev}(\eta \tau) \|_2 + \| \eta \text{div } \tau \|_2 \right) \]
\[ - \int_{\Omega} v \cdot \tau \nabla \eta dx. \]

To recast the last term with a summation by parts, let \( \otimes \) denote the dyadic product and set
\[ V_h(x) := \frac{1}{h} \left( (v \otimes \nabla \eta)(x) - (v \otimes \nabla \eta)(x - hM) \right) \in \mathbb{M}^{n \times n} \quad \text{for a.e } x \in \omega. \]

Since \( (v \otimes \nabla \eta)_{jk} = v_j \partial \eta / \partial x_k \) belongs to \( H^1(\omega) \) we have
\[ (5.7) \quad \lim_{h \to 0} \| V_h \|_2 \leq \| v \otimes \nabla \eta \|_{H^1(\omega)} \leq \| v \|_{H^1(\omega)} \| \eta \|_{W^{2,\infty}(\omega)}. \]

Since \( \eta \in D(\omega) \) is fixed, we infer (for sufficiently small \( h \)) with (5.7)
\[ - \int_{\Omega} v \cdot \tau \nabla \eta dx = \int_{\Omega} V_h : \sigma dx \]
\[ \leq \| \sigma \|_{L^2(\omega)} \| V_h \|_{L^2(\omega)} \leq c_{10} \| \sigma \|_{L^2(\omega)} \| v \|_{H^1(\omega)}. \]
Proof. Set with $\xi \in E$ and abbreviate $\alpha$ a continuous and monotonously decreasing function $\xi$. Another application of (5.8) yields finally

\[ \| \operatorname{dev}(\eta \tau) \|_2 \leq \| \operatorname{dev}(\eta \tau) \|_2 + \| \tau \|_2 + \| \sigma \|_{L^2(\omega_1)}. \]

We return to (5.4) and substitute $\| \operatorname{tr}(\eta \tau) \|_2$ with the bound (5.8) on the right-hand side of (5.4). The resulting estimate reads

\[ \frac{\| \operatorname{dev}(\eta \tau) \|_2^2}{2\mu} + \frac{\| \operatorname{tr}(\eta \tau) \|_2^2}{n^2(2\mu/n + \lambda)} \leq c_{12} \left( \| u \|_{H^1(\omega)}^2 + \| \operatorname{div}(\sigma) \|_{H^1(\omega)}^2 + \| \sigma \|_{L^2(\omega_1)}^2 + \| \operatorname{dev}(\eta \tau) \|_2^2 \right) \]

and allows us to absorb $\| \operatorname{dev}(\eta \tau) \|_2$ with Young’s inequality. Hence

\[ c_{13} \| \eta \|_{C^{-1/2}} \| \tau \|_2 \leq \| u \|_{H^1(\omega)} + \| \operatorname{div}(\sigma) \|_{H^1(\omega)} + \| \sigma \|_{L^2(\omega_1)}. \]

Another application of (5.8) yields finally

\[ c_{14} \| \eta \|_\sigma \tau \|_2 \leq \| u \|_{H^1(\omega)} + \| \operatorname{div}(\sigma) \|_{H^1(\omega)} + \| \sigma \|_{L^2(\omega_1)}. \]

The proof is then concluded as in Theorem 2.1. \qed

6. An Application to Hencky Elastoplasticity with Hardening

One time step within an elastoplastic evolution problem leads to Hencky’s model. For various hardening laws and von-Mises yield condition, the minimization problem takes the form (1.4). After an elimination of internal variables [ACZ99] the problem reads, in the notation of the previous section,

\[ (6.1) \quad \varphi(E) := \frac{1}{2} E : \mathbb{C} E - \frac{1}{4\mu} \max\{0, |\operatorname{dev} \mathbb{C} E| - \sigma_y\}^2/(1 + \eta) \]

for $E \in \mathbb{M}^{n\times n}_{\text{sym}}$, $\mathbb{C}$ is the fourth-order elasticity tensor, $\sigma_y > 0$ is the yield stress and $\eta > 0$ is the modulus of hardening. The model of perfect plasticity corresponds to $\eta = 0$ [Tem83].

**Proposition 6.1.** We have, for all $A, B \in \mathbb{M}^{n\times n}_{\text{sym}}$,

\[ (6.2) \quad |D\varphi(A) - D\varphi(B)|_{C^{-1}} \leq (D\varphi(A) - D\varphi(B)) : (A - B). \]

**Proof.** Set $\xi(x) := 1 - \max\{0, 1 - \sigma_y/(2\mu x)\}/(1 + \eta)$ to define the continuous and monotonously decreasing function $\xi : [0, \infty) \rightarrow (0, 1]$ with $\xi(0) = 1 \geq \xi(x) > \eta/(1 + \eta) > 0$ for $0 < x < \infty$. Then,

\[ D\varphi(E) = (\lambda + 2\mu/n) \operatorname{tr}(E) I + 2\mu \xi(|\operatorname{dev} E|) \operatorname{dev} E \]

for all $E \in \mathbb{M}^{n\times n}_{\text{sym}}$. Without loss of generality, we suppose $a := |\operatorname{dev} A| \leq b := |\operatorname{dev} B|$ and abbreviate $\alpha := \xi(a)$ and $\beta := \xi(b)$. First we calculate

\[ 2\mu \delta := |D\varphi(A) - D\varphi(B)|_{C^{-1}}^2 - (D\varphi(A) - D\varphi(B)) : (A - B) \]

Using this in (5.6) and the estimate (5.5) to bound $\| v \|_{H^1(\omega)}$, we deduce

\[ (5.8) \quad c_{11} \| \operatorname{tr}(\eta \tau) \|_2 \leq \| \operatorname{dev}(\eta \tau) \|_2 + \| \tau \|_2 + \| \sigma \|_{L^2(\omega_1)}. \]
and then have to show that
\[ \delta = \left| \text{dev}(\xi(a)A - \xi(b)B) \right|^2 - \text{dev}(\xi(a)A - \xi(b)B) : \text{dev}(A - B) \]
is non-positive. To see \( \delta \leq 0 \), observe that \( 0 \leq (1-\alpha)\beta + \alpha(1-\beta) \).

Expanding the squares and collecting terms we infer in combination with Cauchy’s inequality
\[ \delta = (\xi(a)a - \xi(b)b)^2 - (\xi(a)\alpha - \xi(b)\beta)(a - b) \]
\[ + (\text{dev}(A) : \text{dev}(B) - ab)((1-\alpha)\beta + \alpha(1-\beta)) \]
\[ \leq (\xi(a)a - \xi(b)b)^2 - (\xi(a)\alpha - \xi(b)\beta)(a - b) \]
\[ = (\xi(a)a - \xi(b)b)((\alpha - 1)a - (\beta - 1)b) \].

An elementary analysis shows that \( x\xi(x) \geq 0 \) is monotonously increasing in \( 0 \leq x < \infty \) while \( x(\xi(x) - 1) \leq 0 \) is monotonously decreasing. As a consequence, \( a \leq b \) implies \( \xi(a)a \leq \xi(b)b \) and \( (\xi(a) - 1)a \geq (\xi(b) - 1)b \).

Taking this in the last estimate of \( \delta \) into account we conclude \( \delta \leq 0 \).

We therefore have the following consequence of Theorem 5.1.

**Corollary 6.2.** If \( u \) is a minimizer of (1.2) in \( A \subseteq H^1(\Omega) \) and \( f \in H^1_{\text{loc}}(\Omega) \) then \( \sigma := D\varphi(\varepsilon(u)) \) belongs to \( W^{1,2}_{\text{loc}}(\Omega; \mathbb{R}^n) \).

**Remarks 6.1.**
(a) The corollary is essentially due to Seregin [Ser93].
(b) The case \( \eta = 0 \) corresponds to perfect plasticity [Tem83] and is excluded from our analysis. Then, \( u \) only belongs to \( BD(\Omega) \), the space of bounded deformations.

7. **An application to a vector 2-well problem**

Given two distinct wells \( E_1 \) and \( E_2 \) in \( M^{n \times n}_{\text{sym}} \) with minimal energies \( W_1^0 \) and \( W_2^0 \) in \( \mathbb{R} \), we have a quadratic elastic energy
\[ (7.1) \quad W_j(E) := \frac{1}{2}(E - E_j) : C(E - E_j) + W_j^0 \quad \text{for all } E \in M^{n \times n}_{\text{sym}}. \]

Energy minimization balances the configuration of the two phases and so the strain energy density \( W \) is modeled by the minimum
\[ (7.2) \quad W(E) = \min\{W_1(E), W_2(E)\} \quad \text{for all } E \in M^{n \times n}_{\text{sym}}. \]

The two wells (transformation strains) are said to be compatible if the following condition holds
\[ (7.3) \quad E_1 = E_2 + \frac{1}{2}(a \otimes b + b \otimes a) \quad \text{for some } a, b \in \mathbb{R}^n. \]
The constant $\gamma$ is given by a certain projection onto the space of symmetric matrices and satisfies $0 < \gamma \leq \frac{1}{2} |E_2 - E_1|^2$ and, in the compatible case (7.3), takes its upper bound $\gamma = \frac{1}{2} |E_2 - E_1|^2$.

The quasiconvexification $\varphi$ of $W$ reads, owing to [Koh91],

$$\varphi(E) = \begin{cases} 
W_2(E) & \text{if } W_2(E) + \gamma \leq W_1(E), \\
\frac{1}{2} (W_2(E) + W_1(E)) - \frac{1}{4\gamma} (W_2(E) - W_1(E))^2 - \frac{1}{4} & \text{if } |W_2(E) - W_1(E)| \leq \gamma, \\
W_1(E) & \text{if } W_1(E) + \gamma \leq W_2(E).
\end{cases}$$

(7.4)

**Lemma 7.1** ([CP97b]). In the compatible case (7.3), we have, for all $A, B \in M_{\text{sym}}^{n \times n}$,

$$|D\varphi(A) - D\varphi(B)|_{C^{-1}}^2 \leq \left( D\varphi(A) - D\varphi(B) \right) : (A - B).$$

(7.5)

We therefore have the following consequence of Theorem 5.1.

**Corollary 7.2** ([Ser93]). If $u$ is a minimizer of (1.4) in $A \subseteq H^1(\Omega)$ and $f \in H^1_{\text{loc}}(\Omega)$ then the stress $\sigma := D\varphi(\varepsilon(u))$ belongs to $W^{1,2}_{\text{loc}}(\Omega; M_{\text{sym}}^{n \times n})$.

**Remarks 7.1.** (a) The corollary is due to Seregin [Ser93, Theorem 2.2]; besides the local stress regularity, he shows that the strain tensor locally has bounded mean oscillation and investigates the pure phase area.

(b) In case of incompatible wells (i.e., if (7.3) fails), Lemma 7.1 fails (as it guarantees convexity of $\varphi$). Due to Seregin [Ser96], $\varphi(\text{sym } F)$ can be rewritten as the sum of a convex function (which then satisfies an estimate of the form (7.5)) and a linear combination of second order minors of $F$. Then, up to cofactor matrices of the gradient $F$ (stress free if pure Dirichlet boundary conditions are imposed), the stress belongs to $W^{1,2}_{\text{loc}}(\Omega; M_{\text{sym}}^{n \times n})$. The interpretation of $\det Du$ as a constant pressure may be formally correct (as the model is in material coordinates) but is doubtful from the physical point of view: A linearisation is behind (7.1) and so material and spatial coordinates coincide and incompressibility reads $\text{div } u = 0$ and not $\det Du = 1$.

(c) A time-discretized model for hysteresis of [MTL] leads to a similar variational problem. From a stress estimate in [CP00], we obtain an analogue of Lemma 7.1 and can conclude $\sigma \in W^{1,2}_{\text{loc}}(\Omega; M_{\text{sym}}^{n \times n})$ as well.

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References


[Bol06] Oskar Bolza. A fifth necessary condition for a strong extremum of the integral \( \int_{x_0}^{x_1} f(x, y, y') \, dx \). *Trans. Amer. Math. Soc.*, 7(2):314–324, 1906.


Institute for Applied Mathematics and Numerical Analysis, Vienna University of Technology, Wiedner Hauptstrasse 8-10, A-1040 Vienna, Austria

E-mail address: Carsten.Carstensen@tuwien.ac.at

Max Planck Institute for Mathematics in the Sciences, Inselstr. 22-26, D-04103 Leipzig, Germany.

E-mail address: Stefan.Mueller@mis.mpg.de