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**Drift-diffusion limits of kinetic models
for chemotaxis: a generalization**

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DRIFT-DIFFUSION LIMITS OF KINETIC MODELS FOR CHEMOTAXIS: A GENERALIZATION

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ABSTRACT. We study a kinetic model for chemotaxis introduced by Othmer, Dunbar, and Alt [22], which was motivated by earlier results of Alt, presented in [1], [2]. In two papers by Chalub, Markowich, Perthame and Schmeiser, it was rigorously shown that, in three dimensions, this kinetic model leads to the classical Keller-Segel model as its drift-diffusion limit when the equation of the chemo-attractant is of elliptic type [4], [5]. As an extension of these works we prove that such kinetic models have a macroscopic diffusion limit in both two and three dimensions also when the equation of the chemo-attractant is of parabolic type, which is the original version of the chemotaxis model.

INTRODUCTION

In [16] and [17] Keller and Segel introduced and studied a model for aggregation of the cellular slime mold *Dictyostelium discoideum* due to cyclic AMP which is an attractive chemical signal for the amoebae. The model is of advection-diffusion type and consists of two coupled parabolic equations

$$(1) \quad \frac{\partial \rho}{\partial t} = \nabla \cdot (D(\rho, S)\nabla \rho - \chi(\rho, S)\rho \nabla S),$$

$$(2) \quad \frac{\partial S}{\partial t} = D_0 \Delta S + \phi(\rho, S).$$

Here $\rho = \rho(x, t)$ denotes the cell density and $S = S(x, t)$ is the density of the chemo-attractant. The cells are attracted by the chemical and χ denotes their chemotactic sensitivity. The substance S diffuses and is also produced by the amoebae. Typically $\phi(\rho, S)$ is given by

$$(3) \quad \phi(\rho, S) = \alpha \rho - \beta S, \quad \alpha, \beta \geq 0.$$

where $-\beta S$ is the loss term due to decay or external chemical reactions. The first rigorous derivation of the macroscopic chemotaxis equations from microscopic models, namely interacting stochastic many particle systems, was given in [26].

In [4] a kinetic model of the equation (1) was discussed with a reduced version of the equation (2) which is the Poisson equation without decay term

$$(4) \quad -\Delta S = \alpha \rho.$$

The following kinetic equation for the oriented cell density $f = f(x, v, t) \geq 0$ is considered in [4, page 3]

$$(5) \quad \frac{\partial f}{\partial t} + v \cdot \nabla_x f = \int_V (T[S]f' - T^*[S]f)dv',$$

where x, v , and t indicate position, velocity, and time, respectively. Here the abbreviations $f' = f(x, v', t)$, $T[S] = T[S](x, v, v', t)$ and $T^*[S] = T[S](x, v', v, t)$ are used. The cell density ρ fulfills

$$(6) \quad \rho(x, t) = \int_V f(x, v, t)dv,$$

where V is the set of admissible velocities which is assumed to be compactly supported (e.g. spherically symmetric balls, spheres, or spherical shells).

Using stochastic models for the motion of bacteria and leukocytes Alt formally derived (1) from a transport equation which is similar to (5), [1, section 8], [2, section 3]. Later a general formulation of this velocity-jump process was presented and studied in [22, section 3]. In [23] and [24] Othmer and Hillen studied the formal diffusion limit of a transport equation of (5) by moment expansions, which is the generalization of earlier Alt's works [1], [2], and showed its limit becomes chemotaxis equations (1), (2) under specific assumptions on turning kernel (see e.g. [24, see page 1237-1240]). Based on their results [24] a rigorous proof of their limit was given in [4]. After using diffusive scaling of time and space, the non-dimensional form of (5) leads to [4, page 4]

$$(7) \quad \epsilon^2 \frac{\partial f_\epsilon}{\partial t} + \epsilon v \cdot \nabla_x f_\epsilon = -\mathcal{T}_\epsilon[S_\epsilon](f_\epsilon), \quad x \in \mathbb{R}^n, v \in V, t > 0$$

where

$$\mathcal{T}_\epsilon[Z](g) = \int_V (T_\epsilon^*[Z]g - T_\epsilon[Z]g') dv'.$$

The diffusion limit $\epsilon \rightarrow 0$ was studied with respect to initial conditions

$$(8) \quad f_\epsilon(x, v, 0) = f_0(x, v), \quad x \in \mathbb{R}^n, v \in V,$$

and coupled to the equation (2) for the chemo-attractant. The authors proved in [4] that the coupled nonlinear system (7), (8), and (2) results in Keller-Segel type equations for chemotaxis as its macroscopic drift-diffusion limit under suitable conditions turning kernel in three dimension (compare e.g. [4, Theorem 5] and [5, Theorem 2]). In [4] and [5], the authors also proved that for suitable turning kernels, blow up can be prevented on the kinetic level for fixed $\epsilon > 0$. However, there seem to be some technical difficulties to prove the limit in two dimension, although a similar result is expected to hold as in three dimensions. The method of proofs in [4] is mainly based on the potential estimate for S in (2) where S has the following Newtonian potential representation in \mathbb{R}^3 , i.e.

$$S(x, t) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\rho(y, t)}{|x - y|} dy.$$

In this article, we consider the transport equation (7) with initial condition (8) coupled to

$$(9) \quad \frac{\partial S_\epsilon}{\partial t} - \Delta S_\epsilon = \rho_\epsilon = \int_V f_\epsilon dv$$

instead of (4).

Our main result is the existence of a macroscopic diffusion limit of the kinetic model in both two and three dimensions. More precisely, under the same assumptions on the turning kernel $K[S]$ as given in [4], we prove that the coupled nonlinear system (7), (8), and (9) converges to Keller-Segel type equations for $\epsilon \rightarrow 0$ (compare Theorem 3.4). We can also show that certain kernels excludes blow up of the solutions in finite time on the kinetic level (compare Theorem 2.5). Our main tool is the potential estimate for the heat operator for S . More precisely, we use the following representation formula

$$(10) \quad S_\epsilon(x, t) = \Gamma * \rho_\epsilon(x, t) = \int_0^t \int_{\mathbb{R}^n} \frac{1}{(4\pi(t-s))^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4(t-s)}} \rho_\epsilon(y, s) dy ds.$$

For simplicity, throughout this article, the decay term is assumed to be zero, i.e. $\beta = 0$ in (3). Our result is true however also for non-zero decay term, see also the discussion in Remark 3.6.

The plan of this paper is as follows: In section 2, we introduce notations used in this article and briefly review derivations of the macroscopic equation presented in [4]. In section 3, we prove that the kinetic model has a global solution for (7)-(10) under the same assumptions on the turning kernel as in [4]. In section 4, we present the proof of existence of the diffusion limit for a short time interval.

1. PRELIMINARIES

We first introduce notations which will be used throughout this article and also recall some observations presented in [4].

- $z_0 = (x_0, t_0)$ denotes an arbitrary point in \mathbb{R}^{n+1} , where $x_0 \in \mathbb{R}^n$ and $t_0 \in [0, \infty)$.
- By Γ we denote the fundamental solution of the heat equation in $\mathbb{R}^n \times \mathbb{R}$

$$\Gamma(x, t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}.$$

- For $\Omega \subset \mathbb{R}^n$ and $1 \leq q \leq \infty$, $L^q(\Omega)$ denotes the Banach space of measurable functions with

$$\|u\|_{L^q(\Omega)} = \left(\int_{\Omega} |u(x)|^q dx \right)^{1/q}, \quad q < \infty \quad \text{and} \quad \|u\|_{L^\infty(\Omega)} = \text{ess sup}_{\Omega} |u|.$$

Let $L_+^q(\Omega) = \{f \in L^q(\Omega) : f \geq 0\}$.

- Let $Q = \Omega \times (a, b)$. For $1 \leq q \leq \infty$, $L^q(Q)$ denotes the Banach space of all measurable functions with the finite norm

$$\|u\|_{L^q(Q)} = \left(\int_a^b \int_{\Omega} |u(x, t)|^q dx dt \right)^{1/q}.$$

- For $1 \leq q \leq \infty$, $W^{k,q}(\Omega)$ denotes the usual Sobolev space; i.e., $W^{k,q}(\Omega) = \{u : D^\alpha u \in L^q(\Omega), 0 \leq |\alpha| \leq k\}$.
- $C^\alpha(\Omega)$ denotes the Banach space of functions that are Hölder continuous with the exponent $\alpha \in (0, 1)$, and $C^{k+\alpha}(\Omega)$ consists of all functions whose all derivatives up to k -th order are Hölder continuous with the exponent $\alpha \in (0, 1)$.
- $u \in L_{\text{loc}}^p(Q)$ means $u \in L^p(Q')$ for all $Q' \Subset Q$.
- By $C = C(\alpha, \beta, \dots)$ we denote a constant depending on the prescribed quantities α, β, \dots . The domain Ω considered in this article is \mathbb{R}^2 or \mathbb{R}^3 .

To make this note self-contained, we review the formal derivation of the macroscopic equation from the kinetic model presented in [4] (compare the details in [4, page 5-7]). Since the integral of $\mathcal{T}_\epsilon[S](f)$ with respect to the velocity vanishes, the macroscopic conservation equation is obtained

$$(11) \quad \frac{\partial \rho_\epsilon}{\partial t} + \nabla \cdot J_\epsilon = 0,$$

where $J_\epsilon(x, t) = \epsilon^{-1} \int_V v f_\epsilon(x, v, t) dv$ is the flux density. The turning kernel is assumed to have the following asymptotic expansion $T_\epsilon[S] = T_0[S] + \epsilon T_1[S] + O(\epsilon^2)$. Then the turning operator can be expanded in a similar way and

$$\mathcal{T}_k[S](f) = \int_V (T_k^*[S]f - T_k[S]f') dv'.$$

By asymptotic expansion of $f_\epsilon = f_0 + \epsilon f_1 + O(\epsilon^2)$ and $S_\epsilon = S_0 + \epsilon S_1 + O(\epsilon^2)$, the equation for the leading order terms can be obtained from (7):

$$(12) \quad \mathcal{T}_0[S_0](f_0) = 0, \quad S_0 = \rho_0 * \Gamma, \quad \rho_0 = \int_V f_0 dv.$$

Comparing coefficients in (7) results in

$$(13) \quad v \cdot \nabla_x f_0 = -\mathcal{T}_0[S_0](f_1) - \mathcal{T}_1[S_0](f_0) - \mathcal{T}_{0S}[S_0, S_1](f_0)$$

where $\mathcal{T}_{0S}[S_0, S_1]$ is a turning operator and its kernel is the Frechet derivative of T_0 with respect to S , evaluated at S_0 in the direction S_1 . Here, for clarity, we recall the assumptions on the leading order turning operator presented in [4, (A0) page 6].

Assumption 1.1. *There exists a bounded velocity distribution $F(v) > 0$, such that $T_0^*[S]F = T_0[S]F'$ and*

$$\int_V vF(v)dv = 0, \quad \int_V F(v)dv = 1.$$

The turning rate $T_0[S]$ is bounded, and there exists a constant $\gamma = \gamma[S] > 0$ such that $T_0[S]/F \geq \gamma$ for all $(v, v') \in V \times V, x \in \mathbb{R}^n$, and $t > 0$.

Let us recall two useful lemmas proven in [4].

Lemma 1.2. *Let $\chi : \mathbb{R} \rightarrow \mathbb{R}, g : V \rightarrow \mathbb{R}$, and let*

$$\phi_\epsilon^S[S] = \frac{T_\epsilon[S]F' + T_\epsilon^*[S]F}{2}, \quad \phi_\epsilon^A[S] = \frac{T_\epsilon[S]F' - T_\epsilon^*[S]F}{2},$$

denote the symmetric and, respectively, antisymmetric part of $T_\epsilon[S]F'$. Then

$$\begin{aligned} \int_V \int_V T_\epsilon(Fg)\chi(g)dv &= \frac{1}{2} \int_V \int_V \phi_\epsilon^S[S](g - g')(\chi(g) - \chi(g'))dv' dv \\ &\quad + \frac{1}{2} \int_V \int_V \phi_\epsilon^A[S](g + g')(\chi(g) - \chi(g'))dv' dv. \end{aligned}$$

The same holds for $T_k[S]$ with analogous definitions of $\phi_k^S[S]$ and $\phi_k^A[S]$.

Proof. See Lemma 1 in [4]. □

With $g = f/F$ and $\chi = \text{id}$ one obtains

Lemma 1.3. *Let the assumption 1.1 hold. Then, the entropy equality*

$$\int_V T_0[S](f) \frac{f}{F} dv = \frac{1}{2} \int_V \int_V \phi_0^S[S] \left(\frac{f}{F} - \frac{f'}{F'} \right)^2 dv' dv \geq 0$$

holds. For $g \in L^2(V; dv/F)$, the equation $T_0[S](f) = g$ has a unique solution $f \in L^2(V; dv/F)$ satisfying $\int_V f dv = 0$ if and only if $\int_V g dv = 0$.

Proof. See Lemma 2 in [4]. □

The kernel of $T_0[S]$ is spanned by F , thus from (12) and by using the entropy equality one obtains $f_0(x, v, t) = \rho_0(x, t)F(v)$ where ρ_0 has to be determined. Since the equilibrium distribution is independent of S , (13) leads to

$$(14) \quad T_0[S_0](f_1) = -vF \cdot \nabla \rho_0 - \rho_0 T_1[S_0](F).$$

Therefore, f_1 can be written as follows

$$f_1(x, v, t) = -\kappa(x, v, t) \cdot \nabla \rho_0(x, t) - \Theta(x, v, t) \rho_0(x, t) + \rho_1(x, t)F(v),$$

where $\kappa = \kappa[S_0]$, $\Theta = \Theta[S_0]$ are solutions of $T_0[S_0](\kappa) = vF$, $T_0[S_0](\Theta) = T_1[S_0](F)$, respectively, and ρ_1 is a new unknown. For the flux density, we have the asymptotic expansion $J_\epsilon = \int_V v f_1 dv + O(\epsilon)$. Therefore, passing to the limit $\epsilon \rightarrow 0$, the conservation equation (11) becomes the following convection-diffusion equation

$$(15) \quad \partial_t \rho_0 - \nabla \cdot (D[S_0] \nabla \rho_0 - \Gamma[S_0] \rho_0) = 0,$$

where the diffusive tensor and the convection field are given by

$$D[S_0](x, t) = \int_V v \otimes \kappa[S_0](x, v, t) dv, \quad \Gamma[S_0](x, t) = - \int_V v \Theta[S_0](x, v, t) dv.$$

Here tensor notation is used, i.e. $u \otimes v = (u_i v_j)_{i,j=1,\dots,n}$. Thus, the formal limit of (7) and (10) is the equation (15) coupled to

$$(16) \quad S_0 = \rho_0 * \Gamma = \int_0^t \int_{\mathbb{R}^n} \frac{1}{(4\pi(t-s))^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4(t-s)}} \rho_0(y, s) dy ds.$$

2. GLOBAL SOLUTIONS OF KINETIC MODEL

In this section we show that solutions of the coupled system (7)-(10) do not blow up for fixed $\epsilon > 0$ if the turning kernel satisfies a certain structure condition. Without loss of generality, let us set $\epsilon = 1$ in (7). We first recall some well-known facts needed for our purpose.

Theorem 2.1. (Young's inequality) *Suppose $1 \leq p, q, r \leq \infty$. If $f \in L^p(\mathbb{R}^n)$ and $g \in L^r(\mathbb{R}^n)$, then $f * g \in L^q(\mathbb{R}^n)$ and*

$$\|f * g\|_{L^q(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^r(\mathbb{R}^n)}, \quad \frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1.$$

Proof. See e.g. [7, page 232-233]. □

Lemma 2.2. (Gronwall's inequality) *Let g and h be positive functions. Suppose that f is continuous and satisfies*

$$f(t) \leq g(t) + h(t) \int_0^t f(s) ds.$$

Then

$$(17) \quad f(t) \leq g(t) + h(t) \int_0^t g(s) e^{\int_s^t h(\tau) d\tau} ds.$$

Proof. Computations are straightforward, and thus the details are omitted (see e.g. [6]). □

The next lemma shows L^p and L^∞ -estimate of S in terms of f .

Lemma 2.3. *Suppose S to be a solution of (9) in \mathbb{R}^n with $n = 2, 3$.*

(1) *In the case $n = 2$, S satisfies the following estimates*

$$(18) \quad \|S(\cdot, t)\|_{L^p(\mathbb{R}^2)} \leq C p t^{\frac{1}{2}} \|f_0(\cdot, \cdot)\|_{L^1(\mathbb{R}^2 \times V)} \quad 1 \leq p < \infty,$$

$$(19) \quad \|S(\cdot, t)\|_{L^\infty(\mathbb{R}^2)} \leq C t^{\frac{1}{2}} \sup_{0 \leq s \leq t} \|f(\cdot, \cdot, s)\|_{L^2(\mathbb{R}^2 \times V)}.$$

(2) *In the case $n = 3$, S satisfies the following estimates*

$$(20) \quad \|S(\cdot, t)\|_{L^p(\mathbb{R}^3)} \leq C \frac{2p}{3-p} t^{-\frac{1}{2} + \frac{3}{2p}} \|f_0(\cdot, \cdot)\|_{L^1(\mathbb{R}^3 \times V)} \quad 1 \leq p < 3,$$

$$(21) \quad \|S(\cdot, t)\|_{L^\infty(\mathbb{R}^3)} \leq C t^{\frac{1}{4}} \sup_{0 \leq s \leq t} \|f(\cdot, \cdot, s)\|_{L^2(\mathbb{R}^3 \times V)}.$$

Proof. We first consider the case $n = 2$. For given p with $1 \leq p < \infty$, using Young's inequality and then change variables $y = \frac{\tilde{y}}{\sqrt{t-s}}$, we have

$$\begin{aligned} \|S(\cdot, t)\|_{L^p(\mathbb{R}^2)} &\leq \int_0^t \|\Gamma(\cdot, t-s)\|_{L^p(\mathbb{R}^2)} \|\rho(y, s)\|_{L^1(\mathbb{R}^2)} \\ &\leq C \int_0^t \frac{1}{(t-s)^{1-\frac{1}{p}}} ds \|f_0\|_{L^1(\mathbb{R}^2 \times V)} = C p t^{\frac{1}{2}} \|f_0\|_{L^1(\mathbb{R}^2 \times V)} \end{aligned}$$

where $C = C(p)$. In the last equality we used $1 - 1/p < 1$ for all $1 \leq p < \infty$. We also used that $\|f(\cdot, \cdot, t)\|_{L^1(\mathbb{R}^2 \times V)} = \|f_0(\cdot, \cdot)\|_{L^1(\mathbb{R}^2 \times V)}$ due to the macroscopic conservation equation (11). Similarly, for $n = 3$ we have

$$\begin{aligned} \|S(\cdot, t)\|_{L^p(\mathbb{R}^3)} &\leq \int_0^t \|\Gamma(\cdot, t-s)\|_{L^p(\mathbb{R}^3)} \|\rho(y, s)\|_{L^1(\mathbb{R}^3)} \\ &\leq C \int_0^t \frac{1}{(t-s)^{\frac{3}{2}-\frac{3}{2p}}} ds \|f_0\|_{L^1(\mathbb{R}^3 \times V)} = C \frac{2p}{3-p} t^{-\frac{1}{2} + \frac{3}{2p}} \|f_0\|_{L^1(\mathbb{R}^3 \times V)}, \end{aligned}$$

where $1 \leq p < 3$ was used in the last calculation. Since estimates (19) and (21) can easily be seen by similar computations, we omit the details. This completes the proof. \square

The structure condition for $T[S]$ assumed in [4, (A1) page 12] is

Assumption 2.4. *There exists $C > 0$ such that for all $x \in \mathbb{R}^n$, $v, v' \in V, t \in \mathbb{R}^+$ and $S \in W^{1,\infty}(\mathbb{R}^n)$, nonnegative turning kernel T satisfies*

$$(22) \quad T[S](x, v, v', t) \leq C(1 + S(x + v, t) + S(x - v', t)).$$

Under the Assumption 2.4, the next theorem shows the global existence of solutions for system (7)-(10).

Theorem 2.5. *Let the Assumption 2.4 hold. Assume that $f_0 \in L^1_+ \cap L^\infty(\mathbb{R}^n \times V)$ where $n = 2$ or 3 . Then there exist global solutions $f(\cdot, \cdot, t) \in L^1_+ \cap L^\infty(\mathbb{R}^n \times V)$ and $S(\cdot, t) \in L^p(\mathbb{R}^n)$ for all $2 \leq p \leq +\infty$ for all $t \in [0, \infty)$ of the nonlinear system (7)-(10) for any fixed $\epsilon > 0$.*

Proof. Here, without loss of generality, we assume $\epsilon = 1$. Mass is conserved for ρ , thus $\|\rho(\cdot, t)\|_{L^1(\mathbb{R}^n)} = \|f_0\|_{L^1(\mathbb{R}^n \times V)}$. Since the turning kernel is nonnegative, we have

$$\partial_t f(x, v, t) + v \cdot \nabla_x f(x, v, t) \leq \int_V T[S](x, v, v', t) f(x, v', t) dv'.$$

Using the Assumption 2.4, we get

$$\begin{aligned} f(x, v, t) &\leq f_0(x - vt, v) + C \int_0^t \rho(x - vs, t - s) ds \\ &\quad + C f_1(x, v, t) + C f_2(x, v, t), \end{aligned}$$

where

$$\begin{aligned} \partial_t f_1(x, v, t) + v \cdot \nabla_x f_1(x, v, t) &= \int_V S(x + v, t) f(x, v', t) dv', \\ \partial_t f_2(x, v, t) + v \cdot \nabla_x f_2(x, v, t) &= \int_V S(x - v', t) f(x, v', t) dv'. \end{aligned}$$

with initial conditions $f_1(x, v, 0) = 0, f_2(x, v, 0) = 0$. For the first term f_1 , one can easily see that

$$f_1(x, v, t) = \int_0^t S(x - vs + v, t - s) \rho(x - vs, t - s) ds.$$

After simple calculations, we obtain the following estimate

$$(23) \quad \|f_1(\cdot, \cdot, t)\|_{L^p(\mathbb{R}^n \times V)} \leq C \sup_{0 \leq s \leq t} \|S(\cdot, s)\|_{L^p(\mathbb{R}^n)} \int_0^t \|\rho(\cdot, t - s)\|_{L^p(\mathbb{R}^n)} ds.$$

In a similar way, f_2 can be written as follows.

$$\begin{aligned} f_2(x, v, t) &= \int_0^t \int_V S(x - vs - v', t - s) f(x - vs, v', t - s) dv' ds \\ &= \int_0^t S(\cdot, t - s) * f(x - vs, \cdot, t - s)(x - vs) ds. \end{aligned}$$

Using Young's inequality, we obtain the following pointwise estimate for $S * f$

$$(24) \quad \begin{aligned} &|S(\cdot, t - s) * f(x - vs, \cdot, t - s)(x - vs)| \\ &\leq \|S(\cdot, t - s) * f(x - vs, \cdot, t - s)\|_{L^\infty(\mathbb{R}^n)} \\ &\leq \|S(\cdot, t - s)\|_{L^p(\mathbb{R}^n)} \|f(x - vs, \cdot, t - s)\|_{L^q(V)}, \end{aligned}$$

where q is the Hölder conjugate of p , i.e. $q = p/(p - 1)$. Here we note that $q \leq p$ if $p \geq 2$. Since V is compact, we have

$$\|f(x - vs, v', t - s)\|_{L^q(V)} \leq C \|f(x - vs, v', t - s)\|_{L^p(V)},$$

where $C = C(V)$. Therefore, f_2 satisfies

$$\|f_2(\cdot, \cdot, t)\|_{L^p(\mathbb{R}^n \times V)} \leq C \sup_{0 < s < t} \|S(\cdot, s)\|_{L^p(\mathbb{R}^n)} \int_0^t \|f(\cdot, \cdot, t-s)\|_{L^p(\mathbb{R}^n \times V)} ds,$$

where $p \geq 2$. Using $\|\rho(\cdot, t)\|_{L^p(\mathbb{R}^n \times V)} \leq C(V)\|f(\cdot, \cdot, t)\|_{L^p(\mathbb{R}^n \times V)}$ and summing up the above estimates, we obtain for $p \geq 2$

$$(25) \quad \begin{aligned} & \|f(\cdot, \cdot, t)\|_{L^p(\mathbb{R}^n \times V)} \leq \|f_0(\cdot, \cdot)\|_{L^p(\mathbb{R}^n \times V)} \\ & + C(1 + \sup_{0 \leq s \leq t} \|S(\cdot, s)\|_{L^p(\mathbb{R}^n)}) \int_0^t \|f(\cdot, \cdot, s)\|_{L^p(\mathbb{R}^n \times V)}, \end{aligned}$$

where $C = C(V)$. Up to this point, all calculations are independent of dimensions. To estimate, however, the L^p -norm of S , we need to consider the different cases separately, depending on the dimension. We start with two dimensional case.

• **The two dimensional case: \mathbb{R}^2**

Using the estimate (18), for all $2 \leq p < \infty$ we obtain

$$\begin{aligned} & \|f(\cdot, \cdot, t)\|_{L^p(\mathbb{R}^2 \times V)} \leq C\|f_0(\cdot, \cdot)\|_{L^p(\mathbb{R}^2 \times V)} \\ & + C(1 + t^{\frac{1}{p}})\|f_0\|_{L^1(\mathbb{R}^2 \times V)} \int_0^t \|f(\cdot, \cdot, s)\|_{L^p(\mathbb{R}^2 \times V)}. \end{aligned}$$

Therefore, applying Gronwall's inequality, we have for $2 \leq p < \infty$.

$$(26) \quad \|f(\cdot, \cdot, t)\|_{L^p(\mathbb{R}^2 \times V)} \leq C\|f_0(\cdot, \cdot)\|_{L^p(\mathbb{R}^2 \times V)}(1 + t^{\frac{1}{p}} \exp(Ct^{\frac{1+p}{p}})),$$

where $C = C(f_0, V, p)$. Next we will show the L^∞ -estimate of f . Note first that due to estimate (19), we get

$$(27) \quad \|S(\cdot, t)\|_{L^\infty(\mathbb{R}^2)} \leq Ct^{\frac{1}{2}} \sup_{0 \leq s \leq t} \|f(\cdot, \cdot, s)\|_{L^2(\mathbb{R}^2 \times V)} \leq C(t^{\frac{1}{2}} + t \exp(Ct^{\frac{3}{2}})).$$

The last inequality in (27) is due to the estimate (26) when $p = 2$. Letting $p = \infty$ in (23), we have

$$\|f_1(\cdot, \cdot, t)\|_{L^\infty(\mathbb{R}^2 \times V)} \leq \sup_{0 \leq s \leq t} \|S(\cdot, s)\|_{L^\infty(\mathbb{R}^2)} \int_0^t \|\rho(\cdot, t-s)\|_{L^\infty(\mathbb{R}^2)} ds.$$

On the other hand, taking $p = \infty$ and $q = 1$ in (24), we have

$$\begin{aligned} & \|f_2(\cdot, \cdot, t)\|_{L^\infty(\mathbb{R}^2 \times V)} \leq C \sup_{0 \leq s \leq t} \|S(\cdot, s)\|_{L^\infty(\mathbb{R}^2)} \int_0^t \|f(\cdot, \cdot, t-s)\|_{L^1(\mathbb{R}^2 \times V)} ds \\ & = Ct \sup_{0 \leq s \leq t} \|S(\cdot, s)\|_{L^\infty(\mathbb{R}^2)} \|f_0(\cdot, \cdot)\|_{L^1(\mathbb{R}^2 \times V)} ds. \end{aligned}$$

Therefore, combining the above estimates and using (27), we have

$$\begin{aligned} & \|f(\cdot, \cdot, t)\|_{L^\infty(\mathbb{R}^2 \times V)} \leq C(1 + t \sup_{0 \leq s \leq t} \|S(\cdot, s)\|_{L^\infty(\mathbb{R}^2)}) \|f_0(\cdot, \cdot)\|_{L^1(\mathbb{R}^2 \times V)} \\ & + C(1 + \sup_{0 \leq s \leq t} \|S(\cdot, s)\|_{L^\infty(\mathbb{R}^2)}) \int_0^t \|f(\cdot, \cdot, s)\|_{L^\infty(\mathbb{R}^2 \times V)} \\ & \leq C[1 + t^{\frac{3}{2}} + t^2 \exp(Ct^{\frac{3}{2}})] + C[1 + t^{\frac{1}{2}} + t \exp(Ct^{\frac{3}{2}})] \int_0^t \|f(\cdot, \cdot, s)\|_{L^\infty(\mathbb{R}^2 \times V)}. \end{aligned}$$

Gronwall's inequality implies

$$\begin{aligned} & \|f(\cdot, \cdot, t)\|_{L^\infty(\mathbb{R}^2 \times V)} \leq C[1 + t^{\frac{3}{2}} + t^2 \exp(Ct^{\frac{3}{2}})] + C[1 + t^{\frac{1}{2}} + t \exp(Ct^{\frac{3}{2}})] \\ & \left\{ \int_0^t (1 + s^{\frac{3}{2}} + s^2 \exp(Cs^{\frac{3}{2}})) \exp\left(\int_s^t 1 + \tau^{\frac{1}{2}} + \tau \exp(C\tau^{\frac{3}{2}}) d\tau\right) ds \right\}. \end{aligned}$$

After simple calculations and simplifications, we obtain

$$(28) \quad \|f(\cdot, \cdot, t)\|_{L^\infty(\mathbb{R}^2 \times V)} \leq C(1 + \exp(Ct \exp(Ct^{\frac{3}{2}}))),$$

where $C = C(f_0, V)$.

• **The three dimensional case: \mathbb{R}^3**

Note first that, due to (20) and (25), for any p with $2 \leq p < 3$ we obtain

$$(29) \quad \|f(\cdot, \cdot, t)\|_{L^p(\mathbb{R}^3)} \leq C(1 + t^{-\frac{1}{2} + \frac{3}{2p}} \exp(Ct^{\frac{1}{2} + \frac{3}{2p}})),$$

where $C = C(f_0, V, p)$ and we used Gronwall's inequality (17). In particular, when $p = 2$, we have

$$(30) \quad \|f(\cdot, \cdot, t)\|_{L^2(\mathbb{R}^3)} \leq C(1 + t^{\frac{1}{4}} \exp(Ct^{\frac{5}{4}}))$$

Using (21) and (30), we have

$$(31) \quad \|S(\cdot, t)\|_{L^\infty(\mathbb{R}^3)} \leq C(t^{\frac{1}{4}} + t^{\frac{1}{2}} \exp(Ct^{\frac{5}{4}})).$$

Following a similar procedure as in the two dimensional case, we obtain the following L^∞ -estimate of f .

$$\begin{aligned} \|f(\cdot, \cdot, t)\|_{L^\infty(\mathbb{R}^3 \times V)} &\leq C(1 + t \sup_{0 \leq s \leq t} \|S(\cdot, s)\|_{L^\infty(\mathbb{R}^3)}) \|f_0(\cdot, \cdot)\|_{L^1(\mathbb{R}^3 \times V)} \\ &\quad + C(1 + \sup_{0 \leq s \leq t} \|S(\cdot, s)\|_{L^\infty(\mathbb{R}^3)}) \int_0^t \|f(\cdot, \cdot, s)\|_{L^\infty(\mathbb{R}^3 \times V)} ds. \end{aligned}$$

Therefore, with the aid of (31), one can have

$$\begin{aligned} \|f(\cdot, \cdot, t)\|_{L^\infty(\mathbb{R}^3 \times V)} &\leq C(1 + t^{\frac{3}{4}} + t^{\frac{3}{2}} \exp(Ct^{\frac{5}{4}})) \\ &\quad + C(1 + t^{\frac{1}{4}} + t^{\frac{1}{2}} \exp(Ct^{\frac{5}{4}})) \int_0^t \|f(\cdot, \cdot, s)\|_{L^\infty(\mathbb{R}^3 \times V)}. \end{aligned}$$

Applying Gronwall's inequality and simplifying it, we obtain

$$(32) \quad \|f(\cdot, \cdot, t)\|_{L^\infty(\mathbb{R}^3 \times V)} \leq C[1 + \exp(Ct^{\frac{1}{2}} \exp(Ct^{\frac{5}{4}}))],$$

where $C = C(f_0, V)$. This completes the proof. \square

In the proof of Theorem 2.5, L^p and L^∞ estimates for f and S are obtained. Since such estimates are of independent interest, we restate them in the next corollary.

Corollary 2.6. *Let f and S be solutions of the nonlinear system (7)-(10) for fixed $\epsilon > 0$ (here $\epsilon = 1$). Then, under the same assumption as in Theorem 2.5, f and S satisfy the following estimates for all $t \in [0, \infty)$.*

• *Two dimensional case:*

$$\begin{aligned} \|f(\cdot, \cdot, t)\|_{L^p(\mathbb{R}^2 \times V)} &\leq C(1 + t^{\frac{1}{p}} \exp(Ct^{\frac{1+p}{p}})), \quad 2 \leq p < \infty, \\ \|f(\cdot, \cdot, t)\|_{L^\infty(\mathbb{R}^2 \times V)} &\leq C(1 + \exp(Ct \exp(Ct^{\frac{3}{2}}))), \\ \|S(\cdot, t)\|_{L^\infty(\mathbb{R}^2)} &\leq C(t^{\frac{1}{2}} + t \exp(Ct^{\frac{3}{2}})). \end{aligned}$$

• *Three dimensional case:*

$$\begin{aligned} \|f(\cdot, \cdot, t)\|_{L^p(\mathbb{R}^3 \times V)} &\leq C(1 + t^{\frac{1}{2} + \frac{3}{2p}} \exp(Ct^{\frac{1}{2} + \frac{3}{2p}})) \quad 2 \leq p < 3, \\ \|f(\cdot, \cdot, t)\|_{L^\infty(\mathbb{R}^3 \times V)} &\leq C[1 + \exp(Ct^{\frac{1}{2}} \exp(Ct^{\frac{5}{4}}))], \\ \|S(\cdot, t)\|_{L^\infty(\mathbb{R}^3)} &\leq C(t^{\frac{1}{4}} + t^{\frac{1}{2}} \exp(Ct^{\frac{5}{4}})). \end{aligned}$$

Proof. This is a restatement of (26), (27), (28), (29), (31), and (32). \square

Remark 2.7. Before we obtained L^p -estimates of f for certain ranges of p which depend on the dimension. By using interpolation arguments, we can also have similar estimates for other exponents p in both cases. Under the same assumption as in Theorem 2.5, when $n = 2$, f satisfies

$$\|f(\cdot, \cdot, t)\|_{L^p(\mathbb{R}^2 \times V)} \leq C(1 + t^{\frac{p-1}{p}} \exp(Ct^{\frac{3(p-1)}{p}})), \quad 1 \leq p < 2.$$

Indeed, since $1 \leq p < 2$, we can interpolate the L^p -norm of f in terms of the L^1 and L^2 -norm of f . By similar arguments, in case $n = 3$, one can have

$$\|f(\cdot, \cdot, t)\|_{L^p(\mathbb{R}^3 \times V)} \leq C \exp(1 + t^{\frac{5(p-1)}{2p}} \exp(Ct^{\frac{5(p-1)}{2p}})), \quad 1 \leq p < 2,$$

$$\|f(\cdot, \cdot, t)\|_{L^p(\mathbb{R}^3 \times V)} \leq C[1 + \exp(Ct^{\frac{p-2}{2p}} \exp(t^{\frac{5(p-2)}{4p}}))] \quad 3 \leq p < \infty.$$

The last inequality is obtained by interpolating the left-hand side by the L^2 - and L^∞ -norm of f . Since computations are straightforward, and thus the details are omitted. \square

3. LOCAL EXISTENCE OF DIFFUSION LIMITS

In this section, the diffusion limit for kinetic models of the form (7)-(10) is presented. First, in next lemma, we review estimates for S , which are known from potential theory. Since the proofs are straightforward, the details are omitted (see e.g. [18, Chap. 4] and [19, Chap. 4, 6]).

Lemma 3.1. Let $I = [0, T) \subset \mathbb{R}$ and $0 < T < \infty$. Suppose $\rho \in L^\infty(I; L^1(\mathbb{R}^n)) \cap L^\infty(I; L^q(\mathbb{R}^n))$ where $q > n$.

(1) In the case $n = 2$

$$S \in L^\infty(I; W^{1,p}(\mathbb{R}^2)) \cap L^\infty(I; C^{1+\alpha}(\mathbb{R}^2)), \quad 1 \leq p < \infty, \quad 0 < \alpha \leq \frac{q-2}{q},$$

and S satisfies the following estimate

$$\|S\|_{L^\infty(I; W^{1,p}(\mathbb{R}^2))} + \|S\|_{L^\infty(I; C^{1+\alpha}(\mathbb{R}^2))} \leq C(\|\rho\|_{L^\infty(I; L^1(\mathbb{R}^2))} + \|\rho\|_{L^\infty(I; L^q(\mathbb{R}^2))}).$$

(2) In the case $n = 3$

$$S \in L^\infty(I; W^{1,p}(\mathbb{R}^3)) \cap L^\infty(I; C^{1+\alpha}(\mathbb{R}^3)), \quad 1 \leq p < \infty, \quad 0 < \alpha \leq \frac{q-3}{q},$$

and S satisfies the following estimates

$$\|S\|_{L^\infty(I; W^{1,p}(\mathbb{R}^3))} + \|S\|_{L^\infty(I; C^{1+\alpha}(\mathbb{R}^3))} \leq C(\|\rho\|_{L^\infty(I; L^1(\mathbb{R}^3))} + \|\rho\|_{L^\infty(I; L^q(\mathbb{R}^3))}).$$

We need similar assumptions on $\phi_\epsilon^S[S]$ and $\phi_\epsilon^A[S]$ as in [4].

Assumption 3.2. There exist $\gamma > 0$ and a non-decreasing function $\Lambda \in L^\infty_{\text{loc}}$, such that

$$\phi_\epsilon^S[S] \geq \gamma(1 - \epsilon\Lambda(\|S\|_{W^{1,\infty}(\mathbb{R}^n))})FF'$$

$$\int_V \frac{\phi_\epsilon^A[S]^2}{F\phi_\epsilon^S[S]} dv' \leq \epsilon^2\Lambda(\|S\|_{W^{1,\infty}(\mathbb{R}^n)}),$$

where $F \in L^\infty(V)$ is a positive velocity distribution satisfying Assumption 1.1, and ϕ_ϵ^S and ϕ_ϵ^A are defined in Lemma 1.2.

Theorem 3.3. Let the Assumption 1.1 and Assumption 3.2 hold and let $q > n$ with $n = 2, 3$. Assume further that

$$f_0 \in \chi_q = L^1(\mathbb{R}^n \times V) \cap L^q(\mathbb{R}^n \times V; \frac{dx dv}{F^{q-1}}).$$

Then there exists a $t^* > 0$, independent of ϵ , such that the solution f_ϵ, S_ϵ satisfies

$$f_\epsilon \in L^\infty((0, t^*); \chi_q),$$

$$S_\epsilon \in L^\infty((0, t^*); W^{1,p}(\mathbb{R}^n) \cap C^{1+\alpha}(\mathbb{R}^n)), \quad 1 \leq p < \infty, \quad \alpha = \frac{q-n}{q},$$

$$(33) \quad r_\epsilon = \frac{f_\epsilon - \rho_\epsilon F}{\epsilon} \in L^2((0, t^*); \mathbb{R}^n \times V : \frac{dx dv dt}{F}).$$

Proof. This can be shown by following the same procedure given in the proof of Theorem 4 in [4], and therefore, the details are omitted. \square

Now we are ready to prove the existence of the diffusion limit in a short time interval.

Theorem 3.4. *Let the assumption of Theorem 3.3 hold. Assume further that for families (S_ϵ) , which are uniformly bounded in $L^\infty_{\text{loc}}([0, \infty); C^{1+\alpha}(\mathbb{R}^n))$ for some α as $\epsilon \rightarrow 0$ with $0 < \alpha \leq 1$, such that S_ϵ and ∇S_ϵ converges to S_0 and ∇S_0 , respectively, in $L^p_{\text{loc}}([0, \infty); \mathbb{R}^n)$ for some $p > n/(n-1)$ with $n = 2, 3$, we have the convergence*

$$(34) \quad \begin{aligned} T_\epsilon[S_\epsilon] &\rightarrow T_0[S_0] && \text{in } L^p_{\text{loc}}([0, \infty); \mathbb{R}^n \times \bar{V} \times \bar{V}), \\ \frac{\mathcal{T}_\epsilon[S_\epsilon](F)}{\epsilon} &= \frac{2}{\epsilon} \int_V \phi_\epsilon^A[S_\epsilon] dv' \rightarrow \mathcal{T}_1[S_0](F) && \text{in } L^p_{\text{loc}}([0, \infty); \mathbb{R}^n \times \bar{V}). \end{aligned}$$

Then solution f_ϵ and S_ϵ of (7)-(10) satisfy

$$\begin{aligned} f_\epsilon &\rightarrow \rho_0 F && \text{in } L^\infty((0, t^*); \chi_q) \text{ weak } *, \\ S_\epsilon &\rightarrow S_0 && \text{in } L^q_{\text{loc}}((0, t^*); \mathbb{R}^n), \quad 1 \leq q < \infty, \\ \nabla S_\epsilon &\rightarrow \nabla S_0 && \text{in } L^q_{\text{loc}}((0, t^*); \mathbb{R}^n), \quad 1 \leq q < \infty. \end{aligned}$$

Proof. Since the proof is similar to that of Theorem 5 in [4], we present only a brief sketch of the procedure. First we note, due to (33), that

$$J_\epsilon = \frac{1}{\epsilon} \int_V v f_\epsilon dv = \int_V v r_\epsilon dv \in L^2((0, t^*); L^2(\mathbb{R}^n))$$

uniformly in ϵ . Recalling the cell conservation equation (11), one can easily see that

$$\partial_t(\nabla S_\epsilon) \in L^2((0, t^*); L^2_{\text{loc}}(\mathbb{R}^n))$$

by considering the gradient of the convolution of (10). The strong convergence follows combining the above estimate and the parabolic regularity for the convolutions defining S_ϵ and ∇S_ϵ from ρ_ϵ . Therefore, the kinetic equation (7) leads to

$$\epsilon \frac{\partial f_\epsilon}{\partial t} + v \cdot \nabla_x f_\epsilon = -\rho_\epsilon \frac{\mathcal{T}[S_\epsilon](F)}{\epsilon} - \mathcal{T}_\epsilon[S_\epsilon](r_\epsilon).$$

By assumption (34) and passing to the limit, we obtain

$$\mathcal{T}_0[S_0](r_0) = -vF \cdot \nabla \rho_0 - \rho_0 \mathcal{T}_1[S_0](F).$$

This equation can be solved as (14). The limit of the cell conservation equation is $\partial_t \rho_0 + \nabla \cdot J_0 = 0$ with $J_0 = \int_V v r_0 dv$. This completes the proof. \square

Remark 3.5. *In Theorem 3.4, the local existence of the diffusion limit was shown. However, we do not know whether the limit exists globally in time or blows up in finite time. There are fewer results about regularity questions for the full parabolic-parabolic system (1)-(3), in particular, in case the domain is \mathbb{R}^n when compared to the large amount of blow-up results that are known for the parabolic-elliptic system (see e.g. [3], [8], [9], [10], [11], [20] for unbounded domains, and for sake of space compare the survey paper [14] for the results on bounded domains). To the best of the authors' knowledge, blow up results in finite time for the full parabolic-parabolic system are only due to Herrero and Velázquez, [12] when the domain is a disk in two dimension (compare also the related results in [13], [15], and [25] for blow up which might happen in either finite or infinite time). It seems, however, to be an open problem whether or not solutions of the full parabolic-parabolic system blow up in finite time in the whole space $\mathbb{R}^n, n = 2, 3$. Recently, in [21], it was shown that self-similar solutions exist globally in time in two dimension for suitable initial conditions, i.e. $\int_{\mathbb{R}^2} u_0 < 8\pi$.*

Remark 3.6. *As mentioned earlier, in this article, we consider the chemo-attractant equation with no decay term. Our results, however, can easily be extended to the case with non-zero decay term. Indeed, S has similar estimates as in the case with zero decay term because the fundamental solution for this case is of the following form*

$$(4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{|x|^2}{4t} - \beta t\right).$$

Furthermore, we can also extend our arguments to more general types of production and degradation of the external stimulus. More precisely, one can consider $\phi(\rho, S)$ in (3) of the form

$$\phi(\rho, S) = \alpha(\rho, S)\rho - \beta(\rho, S)S.$$

If $\alpha(\cdot, \cdot)$ is positive, $\beta(\cdot, \cdot)$ is nonnegative and both functionals are bounded, then one obtains the same estimates for S as in Lemma 2.3. Since the rest of our arguments follows the same procedures, we omit the details. \square

Remark 3.7. *In [4, section 3] and [5, section 2] specific examples for the 3D turning kernel are presented. For instance, if turning kernel is $T_\epsilon = \varphi(S(x, t), S(x + \epsilon v + t))$ where φ is strictly positive and increasing with respect to second argument. This kernel satisfies the structure condition (22). So global existence of solutions for the kinetic model with an elliptic equation for S is known for fixed $\epsilon > 0$. The drift-diffusion limit was also proven locally in time. By using asymptotic expansion, we have*

$$T_\epsilon[S] = T_0[s] + T_1[S] + O(\epsilon^2), \quad T_0[S] = \varphi(S, S), \quad T_1[S] = \varphi_2(S, S)v \cdot \nabla S,$$

where φ_2 is the partial derivative of φ with respect to the second argument. In this case, the chemotactic sensitivity and diffusive coefficient can be computed explicitly (see [5, page 5])

$$\chi(S_0) = \frac{\varphi_2(S_0, S_0)d}{\varphi(S_0, S_0)|V|}, \quad D(S_0) = \frac{d}{\varphi(S_0, S_0)|V|^2}, \quad d = \frac{1}{3} \int_V |v|^2 dv.$$

In general, χ and D are variables of S_0 , but they may become constants with a particular choice of φ as indicated in [5, see Example 1]. More precisely, if φ is of the form

$$\varphi(S, \tilde{S}) = \psi(\tilde{S} - S), \quad \psi(x) = C_1 \frac{x}{\sqrt{1+x^2}} + C_2, \quad C_2 > C_1 > 0,$$

then one can easily see that φ is strictly positive and increasing, and $\varphi(S_0, S_0) = C_2$, $\varphi_2(S_0, S_0) = C_1$, which immediately implies that χ and D are constants. Therefore, in such case, the macroscopic equation is

$$\partial_t \rho_0 - \nabla \cdot (D \nabla \rho_0 - \chi \rho_0 \nabla S) = 0, \quad -\Delta S_0 = \rho_0, \text{ in } \mathbb{R}^3,$$

with constant coefficients D and χ . Blowup for the solutions of this system for $\chi = D$ was proven, compare the reference list in remark 3.5. In our case the macroscopic equation for (7) and (10) becomes

$$\partial_t \rho_0 - \nabla \cdot (D \nabla \rho_0 - \chi \rho_0 \nabla S) = 0, \quad \partial_t S_0 - \Delta S_0 = \rho_0, \quad \text{in } \mathbb{R}^n, n = 2, 3.$$

We also obtain global existence of solutions for the kinetic model and local existence of solutions for the drift-diffusion limit. However, we do not know global existence or blow up in finite time for the macroscopic equation as indicated in Remark 3.5. \square

Remark 3.8. *Results on the direct regularizations of blowup solutions of the parabolic Keller-Segel model are discussed in [27]. Velázquez considered the model*

$$(35) \quad \begin{aligned} u_t &= \Delta u - \nabla \cdot (G_\epsilon(u) \nabla v), & (x, t) &\in \mathbb{R}^2 \times \mathbb{R}^+ \\ -\Delta v &= u, & (x, t) &\in \mathbb{R}^2 \times \mathbb{R}^+ \end{aligned}$$

where $G_\epsilon(u) = \frac{1}{\epsilon} Q(\epsilon u)$ and $Q(\xi)$ is an increasing function which satisfies

$$\begin{aligned} Q(\xi) &= \xi + O(\xi^2) \text{ as } \xi \rightarrow 0, \\ Q(\xi) &\sim \text{const. as } \xi \rightarrow \infty, \end{aligned}$$

e.g. $Q(\xi) = \xi/(1 + \xi)$. For $\epsilon > 0$ the solutions of (35) are globally defined in time under general assumptions on the initial data. If $\epsilon = 0$ then $G_\epsilon(u) = G_0(u) = u$, so the system reads

$$u_t = \Delta u - \nabla \cdot (u \nabla v), \quad -\Delta v = u,$$

and solutions blow-up in finite time if their initial mass is large enough. It is then a rather natural question to try to understand the asymptotics of the solutions of (35) when ϵ approaches zero. This has been addressed in [27] by using formal asymptotic expansions. With this method it has been seen that there are solutions of (35) which have a finite amount of mass concentrated in a neighborhood of a set of points $x_i(t)$. In the limit $\epsilon \rightarrow 0$ the function u that characterizes the solution of (35) approaches to a continuous, bounded density plus a set of moving Dirac masses placed at the points $x_i(t)$ and having masses $M_i(t)$. If one denotes u as the smooth part of the solution in the limit $\epsilon \rightarrow 0$ it turns out that, according to the analysis done in [27], the joint dynamics of the singular parts of the solution plus the smooth part is given by

$$\begin{aligned} u_t &= \Delta u - \nabla \cdot (u \nabla v) + \frac{1}{2\pi} \sum_{j=1}^N M_j(t) \frac{(x - x_j(t))}{|x - x_j(t)|^2} \cdot \nabla u, \quad (x, t) \in \mathbb{R}^2 \times \mathbb{R}^+ \\ -\Delta v &= u, \quad (x, t) \in \mathbb{R}^2 \times \mathbb{R}^+ \\ \dot{x}_i(t) &= \Gamma(M_i(t)) A_i(t), \quad t > 0, \quad i = 1, 2, \dots, N \\ A_i(t) &= - \sum_{j=1, j \neq i}^N \frac{M_j(t)(x_i(t) - x_j(t))}{2\pi |x_i(t) - x_j(t)|^2} + \nabla v(x_i(t), t), \quad t > 0 \\ \frac{dM_i(t)}{dt} &= u(x_i(t), t) M_i(t), \quad t > 0, \quad i = 1, 2, \dots, N, \end{aligned} \tag{36}$$

where $\Gamma(M)$ can be computed in terms of $Q(\xi)$. Moreover, the question of describing how the solutions of (35) that are initially of order one can develop regions with high mass densities by means of the blow-up mechanism for (35) with $\epsilon = 0$ has also been considered in [27]. Therefore, system (36) can be considered as a way of continuing the solutions of (35) beyond the blow-up time for $\epsilon = 0$. The well posedness of this model is analyzed in [28].

An interesting question is if and how the "kinetic regularizations" compare to certain "regularizations of the parabolic model".

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