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FROM 1970 UNTIL PRESENT: THE KELLER-SEGEL MODEL IN CHEMOTAXIS AND ITS CONSEQUENCES

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Abstract. This article summarizes various aspects and results for some general formulations of the classical chemotaxis models also known as Keller-Segel models. It is intended as a survey of results for the most common formulation of this classical model for positive chemotactical movement and offers possible generalizations of these results to more universal models. Furthermore it collects open questions and outlines mathematical progress in the study of the Keller-Segel model since the first presentation of the equations in 1970.

Key words. Chemotaxis equations, steady state analysis, global existence, finite time blow-up, invariant sets, self-similar solutions, traveling waves

AMS subject classifications. 35B30, 35J20, 35J25, 35J65, 35K50, 35K57, 92C17

1. Introduction. Mathematical analysis of biological phenomena has become more and more important in understanding these complex processes. Thus, the number of mathematicians studying biological and medical phenomena and problems is continuously increasing in recent years. One such biological topic is the movement of population densities or the movement of single particles. Changes in the environment of mobile species can influence its movement. For example, humans sense their environment and given a particular situation or the state of the environment, they make their decisions as to where to move. For example we might be attracted by a tantalizing smell and move towards it, since we expect a delicious food, or we move away from a place if there is a repellent odor. Animals and humans also use this effects (for example) to attract mating partners with special colorful feathers or with enticing perfumes etc. In [85] one can find the silk moth *Bombyx mori* as an example of a species that uses a special odor to attract a mating partner. During mating season the female moth secretes a scent caused by a pheromone *bombykol* which attracts the male to move in direction of the increasing concentration of this scent. This helps the male moths to find the female. Before presenting another example where changes in the environment affect the movement of a mobile species let me cite the following anecdote from the German news magazine "DER SPIEGEL" 36/1998 [45] that is said to have happened in the late 1950s at the University of Princeton:

> "*The genius was stunned. In the late 1950s Albert Einstein watched disbelievingly a film of the young scientist John Tyler Bonner at Princeton University. The star of the movie was an unimpressive tiny creature: an amoeba called Dictyostelium.(...) As soon as "Dicty",(...), starts to become hungry it undergoes a miraculous metamorphosis.(...) The Dictys become one.(...) Einstein's question is still unsolved: Why does Dicty undergo a deadly intermezzo as a complex multicellular organism to live then alone and autistically?*" (Quoted from [45] translated by the author.)

The cellular slime mold Dictyostelium discoideum was discovered by K. B. Raper in 1935 and in the subsequent years aroused the interest of many scientists. Nowadays Dictyostelium discoideum is a model organism for biomedical research of the National Institutes of Health (NIH). One reason for the growing interest in this cellular slime mold was caused by the fact that "*development in Dictyostelium discoideum results only in two terminal cell types, but processes of morphogenesis and pattern formation occur as in many higher organisms*" (see [103, page 354]). This raised the hope of biologists that studying this cellular slime mold might aid in the understanding of the secret of cell differentiation. But what initiates the change from a single cell organism to a complex multicellular organism? And how does this process take place?

During its life cycle a myxamoebae population of the Dictyostelium grows by cell division as long as there is sufficient nourishment. When the food resources are exhausted the myxamoebae spread over the entire domain available to them. After a while one cell starts to exude cyclic Adenosine Monophosphate (cAMP) which attracts the other myxamoebae. The myxamoebae begin to move towards the so-called founder cell and are also stimulated to emit cAMP. The myxamoebae aggregate and start to differentiate. At the end of the aggregation the myxamoebae form a pseudoplasmoid, in which every myxamoebae maintains its individual integrity. This pseudoplasmoid moves towards light sources. After a time a fruiting body is formed and spores are spread. Thus the life cycle begins again. For more details on the life cycle of the Dictyostelium we refer to [15], for example.

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A reaction to an external stimulus is generally called taxis and is then specified by describing the reason for the reaction. Therefore, there are many different tactical responses such as chemotaxis, galvanotaxis and phototaxis. In this article I will focus on chemotactical movement of mobile species which can lead to various different pattern formations.

Chemotaxis is the influence of chemical substances in the environment on the movement of mobile species. This can lead to strictly oriented movement or to partially oriented and partially tumbling movement. The movement towards a higher concentration of the chemical substance is termed positive chemotaxis and the movement towards regions of lower chemical concentration is called negative chemotactical movement. Thus, the Bombyx mori and the Dictyostelium discoideum are two species that move in a chemotactically positive manner towards the higher concentration of the bombykol resp. the cAMP. The substances that lead to positive chemotaxis are chemoattractants and those leading to negative chemotaxis are so-called repellents.

Chemotaxis is an important means for cellular communication. Communication by chemical signals determines how cells arrange and organize themselves, like for instance in development or in living tissues. A large number of examples for mobile species behaving in a chemotactical manner are known. In addition to the above mentioned two examples, I would also like to draw attention to a third species, the bacterial strain *Rhizobia meliloti*. As described in [37], this bacterial strain responds chemotactically to root exudates isolated from the soil of leguminous plants. The bacterial strain in the surrounding soil of the plants are guided to nodules in the roots of nitrogen-fixing plants by a chemical gradient. Therefore, they play an important role in agricultural ecology.

One aspect during positive oriented chemotactical movement is the formation of cells (amoebae, bacteria, etc) amounts during the responds of the species population to the change of the chemical concentrations in the environment. Such aggregation patterns often require a certain threshold number of individuals. Therefore, depending on the case in question, that is the species being observed, such threshold phenomena should be reflected in the model. For example aggregation in Dictyostelium is only possible if the total number of myxamoebae in the population is larger than a threshold number of myxamoebae. In [26] the threshold value of $5 \cdot 10^4$ myxamoebae per cm^2 is given for Dictyostelium discoideum. This chemotactical effect has been observed in experiments to demonstrate chemotaxis of bacteria (see for example [116]). Positive and negative chemotaxis can be studied in petri dish cultures. If the bacteria are placed in the center of the dish of agar that contains an attractant, the bacteria will exhaust the local supply and then move outward following the attractant gradient they have created. This results in an expanding ring of bacteria. To demonstrate negative chemotaxis one can place a disk of repellent in a petri dish of semisolid agar and bacteria. The bacteria will then move away from the repellent. This movement away from the repellent will lead to the creation of a clear zone around the disk.

Alternatively, one can demonstrate chemotaxis by observing bacteria in the chemical gradient produced when a thin capillary tube is filled with an attractant and lowered into a bacterial suspension. While the attractant diffuses, the bacteria collect and move up the tube. The observed positive chemotactical effect in this experiment is the formation of bacteria (myxamoebae, cells, etc.) bands. Such and similar experiments have been carried out for example by Adler [1]. Adler's observations correspond to the formation of traveling waves and pulses that spread through the population density. Thus an interesting question is whether or not the mathematical models describing chemotactical movement have traveling wave solutions.

These phenomena have motivated a large number of scientists to study chemotaxis and to use the mathematical language to describe the observed phenomena. The intention of the present survey is to collect the results for a classical model describing chemotactical movement, to expose the lines of research.

The outline of the present article is as follows:

In the second section two different approaches for modeling chemotaxis will be inculded. This section will also introduce the "classical" chemotaxis model by Keller and Segel, the center of our considerations for the remainder of the paper. The third section is devoted to steady-state analysis for this classical model by Keller and Segel done so far. It will be shown that all the effects demonstrated in the analysis depend on the functional forms of the three main processes during chemotactical movement. They are:

a) The sensing of the chemoattactant which has an effect on the oriented movement of the species.
b) The production of the chemoattractant by a mobile species or an external source.
c) The degradation of the chemoattractant by a mobile species or an external effect.

Within the context of steady-state analysis the focus will be on a linear chemotactic sensitivity function but will also collect the results for different versions of the Keller-Segel model. When appropriate I will summarize results for other sensitivity functions in a table at the end a section. Section 4 will deal with the possibility of an explosion of the solution in finite time in the case of a linear sensitivity function. Here I point out the different lines of research in chronological order. This can be accomplished without without losing clarity in the results. Section 5 addresses questions asked in [67] on the possibility of explosion of the solution in finite time in the case of a linear sensitivity function. This section is then followed by generalizations of these results for other more general versions of the classical model in Section 6. In the seventh section of the present article I will present some comparison results for some general versions of the Keller-Segel model proved by Wolfgang Alt in his Habilitation [3]. This section, however, will be somewhat technical. I will then turn to self-similar solutions and to results on traveling wave solutions for Keller-Segel type systems in Section 8. At some places in the text questions will be formulated that arise from the results stated in the article. These questions are partially answered in subsequent sections, but some are still open problems and might be worth further study. Finally I will close this summary of results for the Keller-Segel model in Section 9 with some brief comments on other approaches and models for chemotaxis.

2. Different perspectives to model chemotactical movement and the formulation of the classical chemotaxis equations. Modeling chemotactical movement of mobile species can be done from two different perspectives: either from the microscopic or from the macroscopic perspective. Both approaches have been used over the years and the derivation of the macroscopic equations from the microscopic, or to be precise the validation of the passage to the limiting equations is still a topic that is studied by a large number of scientists and depending on the model is still an open problem.

2.1. The macroscopic perspective. The first approach that should be presented here is the macroscopic perspective where one considers the whole population respectively the density of the population at one place and one time directly. This approach leads to a continuous reaction-diffusion model where the diffusion of the population density is modeled with Fourier's and Fick's laws and in which the reactions are viewed as functions of the population density and possibly some external signal or control substance.

In the year 1970 Evelyn Fox Keller and Lee A. Segel used this perspective to present a system of four strongly coupled parabolic partial differential equations, which describes the aggregation of cellular slime molds like the Dictyostelium discoideum. Let $u(t,x)$ denote the myxamoebae density of the cellular slime molds and $v(t,x)$ denote a chemoattractant concentration at time t in point x. To model the aggregation of a cellular slime population they assume in [69] the following basic processes that take place during the aggregation phase:

(a) The chemoattractant is produced per amoeba at a rate $f(v)$.
(b) There exist an extracellular enzyme that degrades the chemoattractant. The concentration of the is enzyme at time t in point x is denoted by $p(t,x)$. This enzyme is produced by the myxamoebae at a rate $g(c,p)$ per amoeba.
(c) The chemoattractant and the enzyme react to form a complex $\mathcal{E}$ of concentration η which dissociates into a free enzyme plus the degraded product.

$$v + p \underset{r_{-1}}{\overset{r_1}{\rightleftarrows}} \mathcal{E} \overset{r_2}{\to} p + \text{ degraded product}$$

(d) The chemoattractant, the enzyme and the complex diffuse according to Fick's law.

The balance of the myxamoebae density $u(t,x)$ in any control volume D (which holds for example in the special case of Dictyostelium discoideum aggregation) implies the equation

$$\frac{d}{dt}\int_D u(t,x)\ dx = -\int_{\partial D} (J^{(u)}(t,x)\cdot n(x))dS. \tag{2.1}$$

Here $J^{(u)}(t,x)$ denotes the flow of the myxamoebae density. This flow contains according to Fick's law a part that is proportional to the density gradient and according to Fourier's law for the heat flow a part that is proportional to the chemoattractant gradient. Thus we see that: $J^{(u)}(t,x) = k_2\nabla v - k_1\nabla u$. As a chemical substance the chemoattractant diffuses and we get

$$\frac{d}{dt}\int_D v(t,x)\ dx = Q^{(v)}(t,D) - \int_{\partial D} (J^{(v)}(t,x)\cdot n(x))dS, \tag{2.2}$$

where $Q^{(v)}(t,D)$ denotes the produced chemoattractant $v(t,x)$ per domain and time volume. The flow $J^{(v)}(t,x)$ is given by: $J^{(v)}(t,x) = -k_c \nabla v$. Assuming the analogous equations for the enzyme and the complex, and taking the basic processes into account we derive at the following system:

$$\left.\begin{array}{rcll} u_t &=& \nabla(k_1(u,v)\nabla u - k_2(u,v)\nabla v), & x\in\Omega,\ t>0 \\ v_t &=& k_c\Delta v - r_1 vp + r_{-1}\eta + uf(v), & x\in\Omega,\ t>0 \\ p_t &=& k_p\Delta p - r_1 vp + (r_{-1}+r_2)\eta + ug(v,p), & x\in\Omega,\ t>0 \\ \eta_t &=& k_\eta\Delta\eta + r_1 vp - (r_{-1}+r_2)\eta, & x\in\Omega,\ t>0 \\ \partial u/\partial n &=& \partial v/\partial n \ = \ \partial p/\partial n \ = \ \partial\eta/\partial n \ = \ 0, & x\in\partial\Omega,\ t>0 \\ u(0,x) &=& u_0(x),\ v(0,x) \ = \ v_0(x), & x\in\Omega, \\ p(0,x) &=& p_0(x),\ \eta(0,x) \ = \ \eta_0(x), & x\in\Omega, \end{array}\right\} \tag{2.3}$$

where r_{-1}, r_1 and r_2 are constants representing the reaction rates mentioned in assumption (c). Here Ω denotes a bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega$.

Simplifying the chemical processes in the life cycle via assuming that the complex is in a steady state with regard to the chemical reaction and that the total concentration of the free and the bounded enzyme is a constant one gets a simplified formulation of this original Keller-Segel model that has already been proposed by E. F. Keller and L. A. Segel themselves to reduce their original system of four strongly coupled parabolic equations to a model that is as simple as possible. Thus their motto that "*it is useful for the sake of clarity to employ the simplest reasonable model*" (see [69, page 403]) leads them to the following system of "only two" strongly coupled nonlinear parabolic equations:

$$\left.\begin{array}{rcll} u_t &=& \nabla(k_1(u,v)\nabla u - k_2(u,v)\nabla v), & x\in\Omega,\ t>0 \\ v_t &=& k_c\Delta v - k_3(v)v + uf(v), & x\in\Omega,\ t>0 \\ \partial u/\partial n &=& \partial v/\partial n \ = \ 0, & x\in\partial\Omega,\ t>0 \\ u(0,x) &=& u_0(x),\ v(0,x) \ = \ v_0(x), & x\in\Omega. \end{array}\right\} \tag{2.4}$$

However, it might be necessary to remember the original system if one tries to describe certain pattern formations during the aggregation of some particular species. It is possible that the reduction to two equations that was done in [69] was too restrictive to cover all observable generated patterns during the aggregation of mobile species.

2.2. The microscopic approach. From the microscopic perspective one interprets the movement of species populations as a consequence of microscopic irregular movement of single members of the considered population that results in a macroscopic regular behaviour of the whole population. This then leads in a parabolic limit to reaction-diffusion processes, however, in this case the passage to the continuum limit of the microscopic problem and thus studying the resulting, continuous partial differential equations has to be valid and justified. Usually it is assumed that in a particles population each single particle moves around in a random walk. Leaving the justification of the limiting process open, this approach gives us at least a formal way to derive reaction-diffusion processes from the microscopic point of view.

For example in [112] H. G. Othmer and A. Stevens used the microscopic perspective and started with a continuous-time, discrete-space random walk for a *single particle* in one space dimension. Restricting themselves to one step jumps and assuming that the conditional probability $p_i(t)$ that a walker is at $i\in\mathbb{Z}$ at time t – conditioned on the fact that it begins at $i=0$ at $t=0$ – evolves according to the continuous time master equation

$$\frac{\partial p_i}{\partial t} = \mathcal{T}^+_{i-1}(W)\ p_{i-1} + \mathcal{T}^-_{i+1}(W)\ p_{i+1} - (\mathcal{T}^+_i(W) + \mathcal{T}^-_i(W))\ p_i. \tag{2.5}$$

Here $\mathcal{T}^\pm_i(\cdot)$ are the transition probabilities per unit time for a one-step jump to $i\pm1$, and $(\mathcal{T}^+_i(W)+\mathcal{T}^-_i(W))^{-1}$ is the mean waiting time at the i^{th} site. It is assumed that these are nonnegative and suitably smooth functions of their arguments. The vector W is given by $W = (\cdots, w_{-i-1/2}, w_{-i}, w_{-i+1/2}, \cdots, w_o, w_{1/2}, \cdots)$. For generality and in context with a self-attracting reinforced random walk analyzed by Davis [28] the density of the control species w is defined on the embedded lattice of half the step size. As (2.5) is written, the jump probabilities may depend on the entire state and on the entire distribution of the control species. Since there is no explicit dependence on the previous state the process may appear to be Markovian, but if the evolution of w_i depends on p_i then there is an implicit history dependence, and thus the jump process by itself is not Markovian. However, the composite process for the evolution (p,w) is a Markov process. There are three distinct types of models that are considered in [112], which differ in the dependence of the transition rates on w,: (i) strictly local models in which for example $\mathcal{T}^\pm_i$ are equal, (ii) barrier models in which for example $\mathcal{T}^\pm_i(W) = \mathcal{T}_i(w_{i\pm1/2})$, and (iii) gradient models for example with

$\mathcal{T}^+_{i-1}(W) = \alpha + \beta(\tau(w_i) - \tau(w_{i-1}))$ and $\mathcal{T}^-_{i+1}(W) = \alpha + \beta(\tau(w_i) - \tau(w_{i+1}))$ for $\alpha \geq 0$ and a function τ of the control substance.

Considering a grid of mesh size h and setting $x = ih$ the *formal* expansion of the righthand side of equation (2.5) as a function of x to second order in h leads

1. in case (i) to $\frac{\partial p}{\partial t} = h^2 \frac{\partial^2}{\partial x^2}(\mathcal{T}(w)p) + \mathcal{O}(h^4)$ and so with an assumed scaling $\lim_{h\to 0, \lambda \to 0} \lambda h^2 = D$, where λ has dimension t^{-1}, to the limiting problem $\frac{\partial p}{\partial t} = D \frac{\partial^2}{\partial x^2}(\mathcal{T}(w)p)$,
2. in case (ii) with the same scaling to $\frac{\partial p}{\partial t} = D \frac{\partial}{\partial x}\left(\mathcal{T}(w)\frac{\partial p}{\partial x}\right)$,
3. and in case (iii) once again with the same scaling as before to $\frac{\partial p}{\partial t} = D \frac{\partial}{\partial x}\left[p\left(\frac{\alpha}{p}\frac{\partial p}{\partial x} - 2\beta\tau'(w)\frac{\partial w}{\partial x}\right)\right]$.

Assuming various possible evolution equations of the control substance w this leads formally from a random walk of a single particle to a limiting diffusion equation *for the probability of one particle to be located at x in time t*. Of course one can also study these equations as an ad hoc approach for particle densities, but their derivations are then only formal approaches and by far not rigorous. The key problem to derive the limiting equations from the multi particle random walk is the interaction of the particles via the control species. A rigorous derivation of limiting equations in these cases is not done yet. Simulations of these are presented in [112] and [137].

The first rigorous derivation of chemotaxis equations from a microscopic model, namely an interacting stochastic many-particle system, has been done in [135] and [139]. In [139] Stevens proved that for large enough particle numbers the dynamics of the below given interacting particle systems are well described by the solution of chemotaxis systems which for this case describe population densities. Explicit error estimates are also given. For the derivation it was assumed that every particle interacts mainly with those of the other particles which are located in a certain neighbourhood of itself. The neighbourhood is macroscopically small and microscopically large. As a consequence the interaction range between the particles is shrinking as the number of particles goes to ∞, while the number of particles in the shrinking neighbourhoods is also growing to ∞.

So let the subscript u mark the terms related to bacteria and the subscript v mark the terms related to the chemical substance slime particles. Let $S(M,t) = S_u(M,t) + S_v(M,t)$ denote the set of all particles in a M-particle system. the particles are numbered consecutively by taking a new number for each new-born particle. $P^k_M(t) \in \mathbb{R}^d$, $k \in S(M,t)$ describes the position of the kth particle at time $t \geq 0$. Furthermore let δ_x denote the Dirac measure at $x \in \mathbb{R}^d$, $d \in \mathbb{N}$. Stevens considered the following measure valued empirical processes:

$$t \mapsto S_{M_u}(t) = \frac{1}{M}\sum_{k\in S_u(M,t)} \delta_{P^k_M(t)},\ t \mapsto S_{M_v}(t) = \frac{1}{M}\sum_{k\in S_v(M,t)} \delta_{P^k_M(t)} \text{ and } S_M(t) = S_{M_u}(t) + S_{M_v}(t).$$

The dynamics of the particles depend on the following smoothed versions of S_{M_u}, S_{M_v}:

$$\hat{s}_{M_u} = \left(S_{M_u} * W_M * \hat{W}_M\right)(x),\ \hat{s}_{M_v} = \left(S_{M_v} * W_M * \hat{W}_M\right)(x)$$

where W_M and $\hat{W}_M$ are moderately scaled functions of a fixed symmetric function W_1 (e.g. a Gaussian): $W_M = M^\alpha W_1(M^{\alpha/d}x)$ and $\hat{W}_M = M^{\hat{\alpha}} W_1(M^{\hat{\alpha}/d}x)$, where α and $\hat{\alpha}$ are constants that for technical reasons have to fulfill certain smallness conditions (see [139, page 4]). Setting up the corresponding Fokker-Planck equations for each particle and taking the particle interaction into account she ends up with the following equation

$$dP^k_M(t) = \chi_M\left(t, P^k_M(t)\right)\nabla\hat{s}_{M_v}\left(t, P^k_M(t)\right)dt + \sqrt{2\mu}\,dW^k(t), \tag{2.6}$$

where $W^k(\cdot)$ are independent $\mathbb{R}^d$-valued standard Brownian motions, $\mu > 0$ is a constant and $\chi_M(t,x)$ is given by the equation $\chi_M(t,x) = \chi(\hat{s}_{M_u}(t,x), \hat{s}_{M_v}(t,x))$ with a smooth function $\chi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$.

Under some technical assumptions she ends up with the weak formulation of the classical chemotaxis system

$$\left.\begin{array}{rcl} u_t &=& \nabla(\mu\nabla u - \chi(u,v)u\nabla v) \\ v_t &=& \eta\Delta v - \gamma(u,v)v + \beta(u,v)u \end{array}\right\}$$

where $S_{M_u} \to u$, $S_{M_v} \to v$ as $M \to \infty$ in probability, $\eta > 0$ is a constant and $\gamma(\cdot,\cdot)$ and $\beta(\cdot,\cdot)$ are smooth, positive functions on $\mathbb{R}^+ \times \mathbb{R}^+$. The derivation of the limit dynamics is done by extensions of techniques used by Oelschläger [108]. For further results on these aspects we refer the interested reader to [112, 138, 135, 136] and [139].

Another paper that should be mentioned in the context of the derivation of the Keller-Segel equations as a model for population densities from the kinetic equations is [4]. Denoting the density of individuals moving at (t, x) in direction θ and having started their run at time τ by a smooth function $\sigma(t, x, \theta, \tau)$ W. Alt started in [4] with the differential-integral system

$$\frac{\partial}{\partial t}\sigma(t,x,\theta,\tau) + \frac{\partial}{\partial \tau}\sigma(t,x,\theta,\tau) + \theta\nabla_x(c(t,x)\sigma(t,x,\theta,\tau)) = -\beta(t,x,\theta,\tau)\sigma(t,x,\theta,\tau) \tag{2.7}$$

for $\tau > 0$, $\theta \in S^{n-1}$(= the unit sphere in n-dimensional space), and speed $c(t, x)$ of an individuum from the beginning of the run that stops at time t and point x, with a given probability $\beta(t, x, \theta, \tau)$ per unit time. Here we have that

$$\sigma(t,x,\eta,0) = \int_0^\infty \int_{S^{N-1}} \beta(t,x,\theta,\tau)\sigma(t,x,\theta,\tau)k(t,x,\theta,\eta)\ d\theta\ d\tau$$

for each $\eta \in S^{N-1}$, where $k(t, x, \theta, \eta)$ denotes the given probability of a new chosen direction η after an individuum has stopped a run with direction θ at (t, x). W. Alt shows that under some additional boundedness assumptions and hypotheses relating the size of some appearing parameters the density

$$\overline{u}(t,x) := \int_{S^{N-1}} \overline{\sigma}(t,x,\theta)\ d\theta = \int_{S^{N-1}} \left(\int_0^\infty \sigma(t,x,\theta,\tau)\ d\tau \right)\ d\theta$$

satisfies the first equation of the Keller-Segel model.

Last but not least I should mention the results from C. S. Patlak [114] in this section. In his paper from 1953 C. S. Patlak derives the partial differential equation of the random walk problem with persistence of direction and external bias. Here persistence of direction or internal bias meants that the probability a particle travels in a given direction is not necessarily the same for all directions, but depends only on the particle's previous direction of motion. External bias means that the probability a particle travels in a given direction is dependent upon an external force on the particle. However, instead of speaking of the probability that a particle is at a point, Patlak speaks of a large number of particles moving around and therefore of the density of the particles about a point as a measure of the required probability. Thus in his picture of a random walk he speaks of a particle traveling in a straight line for a certain length of time τ with an average speed c before turning, where the turning means a change in direction of the particle's motion. To make the idea of a random walk completely explicit - as opposed to diffusion - Patlak assumes that the particles have negligible interactions with each other and thus collision between the particles can be ignored. So let me list up the assumptions that Patlak uses throughout his paper:

1. The particles have negligible interaction with each other.
2. Each time the particle turns it start off afresh, with no "memory" of its previous c and τ.
3. The amount of time spent in turning is negligible compared to the time the particle spends traveling between turns.
4. During a unit length of time the number of particles in each small reagion, as well as the distributions of c and τ, remain approximately the same.

For the net displacement of a particle Patlak assumes that the probability of travel in any direction after turning and the distance of travel in a given direction are not necessarily the same for all directions. Now using the assumptions above he derives a modified Fokker-Planck equation. From this the partial differential equation for the random walk with persistence and external bias is obtained, which is more or less the first equation of the Keller-Segel model. Even though these results are older than the paper by Keller and Segel system (2.4) is known as "*the classical chemotaxis model*" resp. as "*the Keller-Segel model in chemotaxis*".

Since it is not the goal of the present paper to go into the precise details of the derivations and approaches of [4, 112, 114, 138] and [135] we now leave this topic of the different possibilities to model chemotaxis and move to the main goal of the present paper, namely a review of the achieved results for system (2.4) which - as we have seen - can be derived from the macroscopic and microscopic perspective on chemotactical movement.

3. Linear stability analysis for the uniform distribution and nonconstant steady state solutions. Studying the steady state problem of the model (2.4) is already a challenging mathematical problem, showing a large variety of interesting aspects and uses a lot of astute mathematical techniques. Some tools used for the steady state analysis for the Keller-Segel model performed until now were techniques from the calculus of variations to show the existence of nonconstant stationary solutions and the existence of spike solutions. One tool used in this context is for example the mountain pass theorem by Ambrosetti and Rabinowitz [142, Theorem 6.1., page 109]. But let us proceed step by step to illustrate the way of progress on this topic.

In their paper from 1970 E. F. Keller and L. A. Segel studied in the case of two spatial dimensions the stability of a uniform state (u_0, v_0) for the species and the chemical attractant. Studying the effect of small (time dependent) perturbations of these uniform distributions they found by Taylor expansions in u and v of the right hand sides of the equations in (2.4) around the uniform state the following instability condition. The uniform distribution is unstable if

$$\frac{k_2(u_0,v_0)f(v_0)}{k_1(u_0,v_0)\left(k_3(v_0)+v_0k_3'(v_0)\right)}+\frac{u_0f'(v_0)}{k_3(v_0)+v_0k_3'(v_0)}>1, \text{ or equivalent if } \frac{k_2(u_0,v_0)v_0}{k_1(u_0,v_0)u_0}+\frac{u_0f'(v_0)}{k_3(v_0)+v_0k_3'(v_0)}>1, \quad (3.1)$$

since a uniform state (u_0, v_0) satisfies the equality $u_0 f(v_0) = v_0 k_3(v_0)$. Here Keller and Segel call the uniform solution stable if the time dependent perturbations of the uniform distribution decrease with time. On the other hand they call the uniform distribution unstable, if these perturbations lead to solutions of (2.4) that increase in time.

Even though Keller's and Segel's stability analysis of the uniform state in [69] and the presented instability criterion is valid for a very general formulation of the system, the next "landmark" in the studies of the Keller-Segel model was the paper by V. Nanjundiah [102]. In that paper Nanjundiah performs a non-linear stability analysis for some versions of the Keller-Segel model in space dimension $N = 2$. In a linear stability analysis he first re-derives the instability criterion of the uniform distribution of myxamoebae and cAMP. Then his non-linear stability analysis for the system given by the equations

$$\left.\begin{array}{rcll} u_t &=& \nabla(\nabla u - u\nabla\Phi(v)), & x\in\Omega,\ t>0 \\ v_t &=& k_c\Delta v - \gamma v + \tilde{\alpha}u, & x\in\Omega,\ t>0 \\ \partial u/\partial n &=& \partial v/\partial n \ = \ 0, & x\in\partial\Omega,\ t>0 \\ u(0,x) &=& u_0(x),\ v(0,x) \ = \ v_0(x), & x\in\Omega, \end{array}\right\}$$

where k_c, γ, $\tilde{\alpha}$ are positive constants strictly larger than zero and $\Phi(v)$ denoting a chemotactical sensitivity function, leads him to one key statement that is mentioned in [102, page 102] for the case of *a linear or a logarithmic chemotactical sensitivity function* $\Phi(v)$ is the following:

> "*The end-point (in time) of the aggregation is such that the cells are distributed in the form of δ-function concentrations.*"

We want to sketch Nanjundiah's arguments leading to the above expectation for the case of a linear sensitivity function. So V. Nanjundiah considered in this case the following steady state system:

$$\left.\begin{array}{rcll} 0 &=& \nabla(\nabla u - u\nabla v), & x\in\Omega, \\ 0 &=& \Delta v - v + u, & x\in\Omega, \\ 0 &=& \partial u/\partial n \ = \ \partial v/\partial n, & x\in\partial\Omega. \end{array}\right\} \quad (3.2)$$

For a general solution (u, v) of this system we see that the mean values over Ω of u and v are equal to the same constant. The first equation of (3.2) motivates us to define a new function $\psi(x)$ by $u(x) = \psi(x)e^{v(x)}$. In general ψ is strictly positive and only at those points equal to zero, where u is equal to zero. We now conclude from the first equation that $0 = \nabla\left(\nabla\psi e^v\right)$ and therefore ψ satisfies the equation $\Delta\psi + \nabla\psi\nabla v = 0$ in $\Omega \subset \mathbb{R}^2$ with Neumann boundary data at $\partial\Omega$. If we restrict ourselves to functions (u, v) that are both finite everywhere we see that $\psi = ue^{-v}$. However the equation for ψ implies that this function cannot attain a critical point in Ω, since at such a point the gradient vanishes and $\Delta\psi$ would be either strictly positive or strictly negative. According to the boundary conditions Hopf's maximum principle [31, Hopf's Lemma, page 330] implies that ψ is equal to a constant. Thus $u(x) = \text{const} \cdot e^{v(x)}$. This however implies that u and v attain their extrema at the same point in Ω, since u is a monotonic function of v and as a consequence from the first equation of (3.2) we see that $\nabla u - u\nabla v = 0$, i.e. the population current vanishes everywhere in Ω. A result, independent from the reaction terms of the second equation.

However this result contains the assumption that the functions u and v are finite everywhere in Ω and therefore ψ is finite in Ω. So if the time dependent equations describe aggregation, such an assumption then has to break down in the points where the aggregation takes place, i.e. in the aggregation centers. From the fact that for the time dependent problem the L^1-norm of the solutions is uniformly bounded by the L^1-norm of the initial data we see that the set of points where aggregation takes place has to be a set of measure zero.

In [102] V. Nanjundiah also elaborated the fact that the singularities can only be of δ-function type. Therefore he remarked that the trivial solution satisfies the equations (3.2) pointwise. The mass condition on the solution can only be satisfied if the solution has singularities. Since we have from the previous arguments that $u = Ke^v$ we get for v the problem $\Delta v + Ke^v - v = 0$ with homogeneous Neumann boundary data on $\partial\Omega$. From this equation one can derive all possible steady state solutions. Furthermore a uniform solution $v \equiv const =: L$ always exists, where L is defined by the mean value of v over Ω. Figure 3.1 shows the connection between K and L. We see that possible non-constant

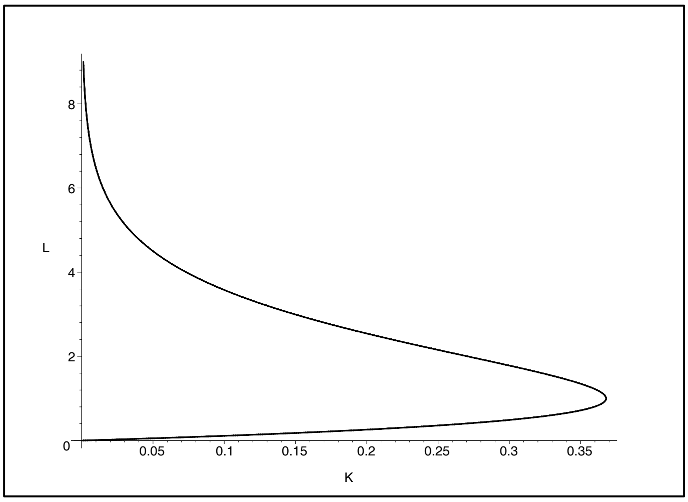

FIG. 3.1. *The connection between the values of the constants K and L.*

steady state solutions can have value of K in the range of the interval $[0, Le^{-L}]$. For $K = 0$ there are two possible values, namely $L = 0$ and $L \to \infty$.

Furthermore V. Nanjundiah showed that all solutions between the one with $K = 0$ and the uniform solution are unstable if the uniform one itself is unstable by imposing small time dependent fluctuations on an arbitrary solution at time $t = 0$. In view of (3.1) the uniform solution is unstable if $u = v = v_0$ with $v_0 > 1$ is true.

Nanjundiah's paper was followed by two papers which contain conjectures for the assymptotic behaviour of the solution of the Keller-Segel model for the space dimensions $N = 1$, $N = 2$ and $N \geq 3$. In [24] S. Childress and J. K. Percus pointed out that the arguments used by V. Nanjundiah are independent of the dimension of space in which aggregation occurs. However they showed that singular behaviour of the solution is in fact a phenomenon that is space dependent. In their paper they restricted themselves to the (as they called it) minimal system given by the equations

$$\left.\begin{array}{rcll} u_t & = & \nabla(\nabla u - \chi u \nabla v), & x \in \Omega,\ t > 0 \\ v_t & = & k_c \Delta v - \gamma v + \tilde{\alpha} u, & x \in \Omega,\ t > 0 \\ 0 & = & \partial u/\partial n \ = \ \partial v/\partial n, & x \in \partial\Omega,\ t > 0 \\ u(0,x) & = & u_0(x),\ v(0,x) \ = \ v_0(x), & x \in \Omega, \end{array}\right\} \tag{3.3}$$

which (as mentioned before) is due to some simplifying assumptions done by V. Nanjundiah in [102] and is nowadays the most common formulation of the chemotaxis equations. Their studies and their performed asymptotic expansion analysis (see [24, page 236-237]) lead to the following possible time asymptotic behaviour for the solution of system (3.3):

> "*In particular, for the special model we have investigated, collapse cannot occur in a one-dimensional space; may or may not in two dimensions, depending upon the cell population; and must, we surmise, in three or more dimensions under a pertubation of sufficiently high symmety.*"

Here Childress and Percus refer that aggregation proceeds to the formation of δ-functions in the cell density as *chemotactic collapse*. Their analysis and conjecture for $N = 2$ and Ω a disc was confirmed by the result in [25], where S. Childress gives an asymptotic expansion describing the imminent collapse of a radially symmetric aggregate of chemotactic cells. However the studies of the stationary problem continued independently from this conjecture and the report on the time independent problem should be closed first until the time asymptotic behaviour of the solution of (3.3) becomes the main subject of the present considerations in the upcoming sections.

The papers by Childress and Percus were followed by the studies of stationary solutions done by R. Schaaf. In [124] she analyzed solutions of the system

$$\left.\begin{array}{rcll} 0 & = & \nabla(k_1(u,v)\nabla u - k_2(u,v)\nabla v), & x\in\Omega,\ t>0 \\ 0 & = & k_c\Delta v + g(u,v), & x\in\Omega,\ t>0 \\ \partial u/\partial n & = & \partial v/\partial n\ =\ 0, & x\in\partial\Omega,\ t>0 \\ u(0,x) & = & u_0(x),\ v(0,x)\ =\ v_0(x), & x\in\Omega. \end{array}\right\} \tag{3.4}$$

with general nonlinearities satisfying the conditions

1. $\Omega\subset\mathbb{R}^N$ is a bounded open region with smooth boundary.
2. $k_1, k_2 : \mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ are twice continuously differentiable and the ODE

$$\frac{d}{ds}r(s) = k_2(r,s)/k_1(r,s)$$

has a unique solution $r:\mathbb{R}^+\to\mathbb{R}^+$ for any initial condition $r(s_0)=r_0,\ s_0,r_0\in\mathbb{R}^+$.

3. $g:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}$ is twice continuously differentiable and $g^{-1}(\{0\})\neq\emptyset$.

via bifurcation techniques. Furthermore a criterion for bifurcation of stable nonhomogeneous aggregation patterns is given. In [124] R. Schaaf focused on the properties of stationary solutions of the Keller-Segel model with homogeneous Neumann boundary data in a very general setting. She shows that the stationary problem of the Keller-Segel model in a more general setting than the cases studied by V. Nanjundiah can also be reduced to a parameter-dependent single scalar equation. More precisely she shows the following theorem:

Theorem 3.1 (Schaaf). $(u,v)\in\{w\in\mathcal{X}\mid w(\overline{\Omega})\subset\mathbb{R}^+\}\times\{w\in\mathcal{X}\mid w(\overline{\Omega})\subset\mathbb{R}^+\}$ *is a solution of (3.4) iff, for* $\lambda\in\mathbb{R}^+$,

$$u(x)=\varphi(v(x),\lambda)\text{ for all }x\in\overline{\Omega}\text{ and }k_c\Delta v+g(\varphi(v(x),\lambda),v)=0. \tag{3.5}$$

Here the space $\mathcal{X}$ is defined as $\{w\in\mathcal{Z}\mid\partial w/\partial n=0\}$ where $\mathcal{Z}$ is the space $C^{2,\beta}(\overline{\Omega},\mathbb{R})$ with $0<\beta<1$ for $n>1$ and $C^2(\overline{\Omega},\mathbb{R})$ for $N=1$. The function $\varphi(s,\lambda)$ is given by $r(s)$ with

$$\frac{d}{ds}r(s)=k_2(r,s)/k_1(r,s),\ r(1)=\lambda.$$

Then bifurcation methods are used in [124] to find natural bifurcation points. Furthermore R. Schaaf gives a stability analysis for the constant stationary solutions of the Keller-Segel model.

For $k_1(u,v)=1$, $k_2(u,v)=\chi u$ and $g(u,v)=-\gamma v+\alpha u$ the stationary solutions of the Keller-Segel model solve the equation

$$k_c\Delta v-\gamma v+\alpha\lambda\exp(\chi v)=0\text{ in }\Omega \tag{3.6}$$

with homogeneous Neumann boundary data. Of course the question of positive, nontrivial solutions for this equation arises. The existence of nontrivial radially symmetric solutions for this equation has been shown in [11, Proposition 1] under the assumption that $\gamma>0$, but Biler did not consider the nonsymmetric case (The Neumann boundary data implies that there are no solutions provided $\gamma=0$.).

Using variational techniques introduced by M. Struwe and G. Tarantello in [143] J. Wei and G. Wang [149], T. Senba and T. Suzuki [129, 130] and in [62] myself proved for $\Omega\subset\mathbb{R}^2$ independently the existence of nontrivial solutions of (3.6) without symmetry assumptions for $\gamma\geq1$ and $4\pi<\alpha\chi\lambda/k_c$. The existence of nontrivial solutions of (3.6) in the case that $\alpha\chi\lambda/k_c<4\pi$ follows from arguments that will be mentioned later in the present paper.

The idea of the existence proof for $\gamma\geq1$ and $4\pi<\alpha\chi\lambda/k_c$ is based on the studies of the functional

$$\mathcal{F}_{\alpha\chi\lambda/k_c}(v):=\frac{1}{2}\int_\Omega|\nabla v|^2+\gamma v^2dx-\frac{\alpha\chi\lambda}{k_c}\log\left(\frac{1}{|\Omega|}\int_\Omega e^v\,dx\right),$$

where $v \in \mathcal{D} := \{v \in H^1(\Omega) \; | v$ has mean value equal to zero over $\Omega\}$. One easily notices that $v \equiv 0$ is a strict local minimum for $\mathcal{F}_{\alpha\chi\lambda/k_c}$ in the case that $\gamma > \frac{\alpha\chi\lambda}{|\Omega| k_c} - \mu_1$, where μ_1 is the first eigenvalue of the Laplacian with homogeneous Neumann boundary data. Then one recognizes that for a smooth domain Ω and $\alpha\chi\lambda > 4k_c\pi$ there is a sequence $\{v_\varepsilon\}_{\varepsilon \geq 0} \subset \mathcal{D}$ with

$$v_\varepsilon(x) = \log\left(\frac{\varepsilon^2}{(\varepsilon^2 + \pi|x - x_0|^2)^2}\right) - \frac{1}{|\Omega|}\int_\Omega \log\left(\frac{\varepsilon^2}{(\varepsilon^2 + \pi|x - x_0|^2)^2}\right) dx,$$

where x_0 is an arbitrary point on $\partial\Omega$, such that $\mathcal{F}_{\alpha\chi\lambda/k_c}(v_\varepsilon) \to -\infty$ and $||\nabla v_\varepsilon||_{L^2(\Omega)} \to \infty$ as $\varepsilon \to 0$. As a consequence there exists a $v_0 \in \mathcal{D}$ Such that $\mathcal{F}_{\alpha\chi\lambda/k_c}(v_0) < 0$ and $||v_0||_{H^1(\Omega)} \geq 1$. One now defines

$$\mathcal{P} \equiv \{p : [0,1] \to \mathcal{D} \mid p \text{ is continuous and } p(0) = 0,\; p(1) = v_0\} \text{ and sets } k_{\alpha\chi\lambda/k_c} \equiv \inf_{p\in\mathcal{P}} \max_{t\in[0,1]} \mathcal{F}_{\alpha\chi\lambda/k_c}(p(t))$$

for all $\alpha\chi\lambda/k_c \geq 4\pi$. Using the fact that the mapping $\alpha\chi\lambda/k_c \mapsto \frac{k_c k_{\alpha\chi\lambda/k_c}}{\alpha\chi\lambda}$ is monotone decreasing for all $\alpha\chi\lambda/k_c \geq 4\pi$ we see that it is differentiable for almost every $\alpha\chi\lambda/k_c \geq 4\pi$. The rest of the proof then consists of the construction of a Palais-Smale sequence for $\mathcal{F}_{\alpha\chi\lambda/k_c}$ that contains a subsequence that converges strongly in $H^1(\Omega)$ to a critical point of $\mathcal{F}_{\alpha\chi\lambda/k_c}$. The construction of the Palais-Smale sequence can be done exactly as in the paper by M. Struwe and G. Tarantello [143]. The existence of the nontrivial critical point of the Functional $\mathcal{F}_{\alpha\chi\lambda/k_c}$ over the set $\mathcal{D}$ allows us to conclude the existence of a nontrivial solution of equation (3.6). This is easy to be seen. If one introduces the new function $w := \chi v - (\chi \int_\Omega v \; dx)/|\Omega|$ we get from (3.6) the Euler-Lagrange equation of the minimizing problem $\inf \mathcal{F}_{\alpha\chi\lambda/k_c}(v)$ over the set $\mathcal{D}$. Thus the existence of a nontrivial critical point of the functional gives us also the existence of a nontrivial steady state solution of the Keller-Segel model with a linear sensitivity function.

With different methods than those just mentioned Y. Kabeya and W.-M. Ni proved in [68] also the existence of positive nontrivial stationary solutions of (3.6). Furthermore they showed the following results:

Theorem 3.2 (Kabeya & Ni). *Let $\Omega \subset \mathbb{R}^2$. Suppose that $t = \lambda e^{\chi t}$ has two positive solutions. Then there exists a non constant solution v_ε of* (3.6) *provided $\varepsilon := \sqrt{k_c/\gamma}$ is sufficiently small. Moreover, there exist constants $C_1 > 0$, $C_2 > 0$, $\delta > 0$, $K > 0$ and $\varepsilon > 0$ such that:*

$$\sup_\Omega v_\varepsilon \;\leq\; C_1,\;\; \inf_\Omega v_\varepsilon \;\leq\; C_2 e^{-\frac{\delta}{\varepsilon}} \text{ and } \int_\Omega (\varepsilon^2|\nabla v_\varepsilon|^2 + \gamma v_\varepsilon^2)dx \;\geq\; K\varepsilon^2$$

for any $0 < \varepsilon < \varepsilon_0$. Furthermore for sufficiently small $\varepsilon > 0$ the solution v_ε has exactly one local maximum point in $\overline{\Omega}$, which must lie on the boundary $\partial\Omega$.

This theorem is similar to results that have been established for the stationary Keller-Segel model with a logarithmic chemotactical sensitivity. In this case the transformation introduced by R.Schaaf in [124] leads to the problem

$$d\Delta w - w + w^p = 0, \text{ in } \Omega \text{ with } \partial w/\partial n = 0, \text{ on } \partial\Omega. \tag{3.7}$$

In [79, 104] and [105] the authors prove the existence of stationary solutions of this equation for $\Omega \subset \mathbb{R}^N$ with $N \geq 2$ and $1 < p < (N+2)/(N-2)$ if $N \geq 3$ and $1 < p < \infty$ if $N = 2$. Their results for this problem can be summarized as follows:

Theorem 3.3 (Ni & Takagi). *Let w_d be a least energy solution of* (3.7), *i.e. a critical point of*

$$J_d(v) := \int_\Omega \frac{1}{2}(d|\nabla v|^2 + v^2) - \frac{1}{p+1} v_+^{p+1} dx$$

such that $J_d(w_d) = c_d$ where $c_d := \inf_{h\in\Gamma} \max_{0\leq t\leq 1} J_d(h(t))$, in which $\Gamma := \{h \in C([0,1]; W^{1,2}(\Omega)) \mid h(0) = 0, h(1) = e\}$

and $e \not\equiv 0$ is a nonnegative function in $W^{1,2}(\Omega)$ with $J_d(e) = 0$. The w_d has at most one local maximum in $\overline{\Omega}$ and this is attained in exactly one point which must lie on the boundary, provided that d is sufficiently small. If $P_d \in \partial\Omega$ is the unique point at which $\max w_d$ *is achieved. Then $\lim_{d\to 0} H(P_d) = \max_{P\in\partial\Omega} H(P)$ where $H(p)$ denotes the mean curvature of $\partial\Omega$ at P.*

Of course many generalizations of this result have been published in the recent years (see for example the papers by [46] and [119]), but it is not the goal of the present paper to mention all these results. Therefore I leave this to the interested reader and turn now to the time dependent problem. However it is recommended to recall the presented results when looking at the time asymptotic behaviour of the solution in the upcoming section. Recalling the results of the present section will help to understand the results for the time asymptotics of the solution and will help to understand which behaviour one might expect for the solution. Before we now definitively turn to the time dependent problem let us explain some terms used in the present section. We have seen that there are different effects that one can expect. In some cases we spoke of aggregation and in other cases of a special form of aggregation namely the formation of δ-singularities. This was sometimes called chemotactical collapse. Before we turn to the time dependent model we therefore now introduce three important effects in the mathematical language.

Definition 3.4. *Let $(u(t,x), v(t,x))$ be a solution of (2.4) for the corresponding initial data $(u_0(x), v_0(x))$. We say that the model describes* **aggregation**, *if* $\liminf_{t\to\infty} ||u(t,x)||_{L^\infty(\Omega)} > ||u_0(x)||_{L^\infty(\Omega)}$ *and* $||u(t,x)||_{L^\infty(\Omega)} < \text{konst}$ *for all* t. *The solution* **blows up** *resp. is a* **blow-up solution** *if* $||u(t,x)||_{L^\infty(\Omega)}$ *or* $||v(t,x)||_{L^\infty(\Omega)}$ *becomes unbounded in either finite or infinite time, i.e. there exists a time* T_{max} *with* $0 < T_{max} \le \infty$ *such that*

$$\limsup_{t\to T_{max}} ||u(t,x)||_{L^\infty(\Omega)} = \infty \text{ or } \limsup_{t\to T_{max}} ||v(t,x)||_{L^\infty(\Omega)} = \infty.$$

We will speak of **chemotactical collapse** *if* $\limsup_{t\to\infty} ||u(t,x)||_{L^\infty(\Omega)} < ||u_0(x)||_{L^\infty(\Omega)}$.

Remark that this definition is almost identical to that given in [112] beside the difference in the allowed blow-up time for a blow-up solution. Furthermore remark that the three cases do not exclude themselves, i.e. more than one case can happen for the same solution. At the end of this section let us summarize the mentioned results in the following table:

Table 3.1
Collection of results for the stationary Keller-Segel models.

Observation	References
For model (2.4) the uniform distribution (u_0, v_0) becomes unstable if $\frac{k_2(u_0,v_0)v_0)}{k_1(u_0,v_0)u_0} + \frac{u_0 f'(v_0)}{k_3(v_0)+v_0 k_3'(v_0)} > 1$.	[69]
All solutions between the one with $K = 0$ (as defined in this section) and the uniform solution are unstable if the uniform one itself is unstable.	[102]
The stationary problem of the Keller-Segel model can be reduced to the parameter-dependent single scalar equation (3.5).	[102, 124]
There exist non-constant stationary solutions of the Keller-Segel model; for example for a linear and for a logarithmic chemotactical sensitivity function.	[11, 62, 68, 79, 104, 105, 129] [130] and [149]

4. The time dependent problem: The case of a linear chemotactical sensitivity. The conjectures and observations by V. Nanjundiah, S. Childress and J. K. Percus have been the initiating motivations for a large number of researchers to study the time asymptotic behaviour of the solution of the system (3.3). There is still an avalanche of publications running and I am pretty sure that by the date of publication of the present paper the number will have already increased once again. Parallel to the results of this section various papers were published in which versions of the Keller-Segel model with a different chemotactic sensitivity function were studied. I will mention these results in an upcoming section. Thus we will only focus on results for (3.3) in this section resp. the upcoming subsections.

4.1. Early results on the time asymptotics and conjectures. The first step in the analysis of the conjectures of Childress and Percus was done by W. Jäger and S. Luckhaus in 1992. In [67] they introduced the transformation

$$U(t,x) := \frac{|\Omega| u(t,x)}{\int\limits_\Omega u(t,x)dx} \text{ and } V(t,x) := v(t,x) - \frac{1}{|\Omega|}\int\limits_\Omega v(t,x)dx$$

which leads to the system:

$$\left.\begin{array}{rcll} U_t &=& \nabla(\nabla U - \chi U \nabla V), & x \in \Omega,\ t > 0 \\ \frac{1}{k_c}(V_t + \gamma V) &=& \Delta V + \frac{\tilde{\alpha}}{k_c}(U-1), & x \in \Omega,\ t > 0 \\ \partial U/\partial n &=& \partial V/\partial n \ = \ 0, & x \in \partial\Omega,\ t > 0 \\ U(0,x) &=& U_0(x),\ V(0,x) \ = \ V_0(x). & x \in \Omega, \end{array}\right\} \tag{4.1}$$

Jäger and Luckhaus then assume that $\tilde{\alpha} = k_c \alpha$, the constants χ, k_c, $\tilde{\alpha}$ are of the order $\frac{1}{\varepsilon}$ with ε and γ and α are of order 1. Thus they get for small ε resp. for $\varepsilon \to 0$ the system

$$\left.\begin{array}{rcll} U_t &=& \nabla(\nabla U - \chi U \nabla V), & x \in \Omega,\ t > 0 \\ 0 &=& \Delta V + \frac{\tilde{\alpha}}{k_c}(U-1), & x \in \Omega,\ t > 0 \\ \partial U/\partial n &=& \partial V/\partial n \ = \ 0, & x \in \partial\Omega,\ t > 0 \\ U(0,x) &=& U_0(x) & x \in \Omega. \end{array}\right\} \tag{4.2}$$

Their result in space dimension $N = 2$ for system (4.2) is summarized as follows:

Theorem 4.1 (Jäger & Luckhaus). *Let Ω be a bounded open set in $\mathbb{R}^2$, $\partial\Omega$ is a C^1-boundary, $U_0(x)$ is C^1 and satisfies the boundary condition.*

1. *There exists a critical number $c(\Omega)$ such that $\alpha\chi\overline{U_0(x)} < c(\Omega)$ implies that there exists a unique, smooth positive solution to (4.2) for all time.*
2. *Let Ω be a disk. There exists a positive number c^* with the following property: If $\alpha\chi\overline{U_0(x)} > c^*$ then radially symmetric positive initial values can be constructed such that explosion of $U(t,x)$ happens in the center of the disc in finite time.*

Here the notation $\overline{U_0(x)}$ is used for the mean value of $U_0(x)$ over the domain Ω. More precisely Jäger and Luckhaus show the following Proposition which implies *1.* of the previous Theorem 4.1.

Proposition 4.2 (Jäger & Luckhaus). *Let Ω be a domain satisfying the smoothness assumptions from the Theorem above. Let $U(t,x)$ be a smooth positive solution to (4.2) and t^* the maximal time of existence, $0 < t^* \le \infty$. There exists a positive number $c_1(\Omega)$ such that $t^* < \infty$ implies*

$$\lim_{k\to\infty} \overline{\lim_{t\to t^*}} \chi\alpha\overline{U_0(x)} \int_\Omega (U(t,x)-k)_+ dx \ge c_1(\Omega).$$

Proposition 4.2 is shown by multiplying the first equation of (4.2) with $\varphi = (U-k)_+^{m-1}$ where $k \ge 0$ and $m > 1$. Then the second equation of (4.2) allows to estimate the term

$$-\int_\Omega U\nabla V \nabla (U-k)_+^{m-1} dx = -\int_\Omega \nabla V \nabla \left(\frac{m-1}{m}(U-k)_+^m + k(U-k)_+^{m-1}\right) dx \text{ from above by } c(k,m)\int_\Omega (U-k)_+^m dx.$$

If the statement of Proposition 4.2 did not hold this estimate would allow us to find the following inequality

$$\frac{d}{dt}\int_\Omega (U-k)_+^m dx \le c_2(\Omega,k) + c_3(k)\int_\Omega (U-k)_+^m dx \text{ for all } t \ge t_1$$

which would give us a bound for the L^m-norms of $U(t,x)$ and the global existence would follow by standard regularity arguments for solutions of elliptic and parabolic equations.

For the proof of the blow-up statement 2. of Theorem 4.1, W. Jäger and S. Luckhaus studied the function

$$M(t,\rho) := \int_0^{\sqrt{\rho}} (U(t,r)-1)r\ dr \text{ for } r = |x|,\ 0 \le \rho \le R.$$

Using the equations of (4.2) they found that $M(t,\rho)$ has to solve the following initial boundary value problem:

$$\frac{\partial}{\partial t}M = 4\rho\frac{\partial^2}{\partial\rho^2}M + \alpha\chi\overline{U_0(x)}\frac{\partial}{\partial\rho}M + \alpha\chi\overline{U_0(x)}M, \text{ with } M(0,\rho) \ = \int_0^{\sqrt{\rho}} (U_0(r)-1)r\ dr \text{ and } M(t,0) \ = \ M(t,R) \ = \ 0.$$

Constructing a subsolution $W(t,\rho)$ for this problem such that $W(t,\rho) \leq M(t,\rho)$ for all t, ρ and

$$\lim_{t \to T_{finite}} \sup_{\rho < \epsilon} W(t,\rho) \geq \omega > 0$$

for each $\epsilon > 0$ they proved that the solution has to blow up at time T_{finite} in the center of the disk. Furthermore Jäger and Luckhaus asked for more information about the blow-up behaviour of the solution of (4.2). In a remark [67, page 820] they formulated the following questions:

> "*It would be interesting to know more about the set of explosion points at t^*. The solution may globally exist as weak solutions. The development of singularities after a finite time t^* is another important topic to be studied.*"

Even though it was not the next paper in the chronology I now mention the results from M. A. Herrero and J. J. L. Velázquez [49] from 1996 and M. A. Herrero, E. Medina and J. J. L. Velázquez [52, 53] from 1997 and 1998 since they studied system (4.2) in those papers. In [49, 54] they focused on the possible formation of δ-function singularities in finite time in space dimension $N = 2$. Using asymptotic expansion methods in [49] their result was the following:

THEOREM 4.3 (Herrero & Velázquez). *Let $R > 0$, and let $\Omega_R = \{x \in \mathbb{R}^2 : |x| < R\}$. Then there exist radial solutions of* (4.2) *defined in an interval* $(0,T)$ *with* $T > 0$*, and such that:*

$$u(t,r) \to \frac{8\pi k_c}{\chi\alpha}\delta(0) + \psi(r) \text{ as } t \to T, \tag{4.3}$$

in the sense of measures, where $\delta(0)$ is the Dirac measure centered at $r = 0$, and:

$$\psi(r) = \frac{C}{r^2} e^{-2|\log(r)|^{1/2}} \left(2|\log(r)|\right)^{\frac{1}{2\sqrt{2}|\log(r)|^{1/2}} - \frac{1}{2}} (1 + o(1)) \tag{4.4}$$

as $r \to 0$, where C is a positive constant depending on χ. At $t = T$, the profile near $r = 0$ is given by:

$$u(t,r) = \frac{8\pi k_c}{\chi\alpha}\delta(0) + \psi(r); \;\; \psi(r) \text{ as in (4.4)}. \tag{4.5}$$

Moreover, if we set $S(t) = (T-t)(\sup_{\Omega} u(t,r)) \equiv (T-t)u(0,t)$, one has that $\lim_{t\to T} S(t) = \infty$. More precisely, there holds:

$$S(t) = C_1(T-t)^{-1}|\log(T-t)|^{1 - \frac{1}{\sqrt{|\log(T-t)|}}} \text{ as } t \to T, \text{ for some } C_1 > 0.$$

So they found solutions that form in finite time a δ-singularity in the center of the disk in $\mathbb{R}^2$. Furthermore they investigated a result for the whole space in the three dimensional case in [52, 53] and [54]. There they studied self-similar solutions and could formulate the following statements.

THEOREM 4.4 (Herrero, Medina & Velázquez).

1. *Consider* (4.2) *in space dimension $N = 3$ with $\Omega = \mathbb{R}^3$. Then, for any $T > 0$ and any constant $C > 0$, there exists a radial solution $(u(t,r), v(t,r))$ of (4.2) that is smooth for all times $0 < t < T$, blows up at $r = 0$ and $t = T$, and is such that:*

$$\int_{|x| \leq r} u(T,s)\, ds \to C.$$

2. *Consider* (4.2) *in space dimension $N = 3$ with $\Omega = \mathbb{R}^3$. For any $T > 0$ there exists a sequence $\{\delta_n\}_{n\in\mathbb{N}}$ with* $\lim_{n\to\infty} \delta_n = 0$*, and a sequence of radial solutions $(u_n(t,r), v_n(t,r))$, that blows up at $r = 0$ and $t = T$, and are such that $u_n(t,r)$ is self-similar, and*

$$u_n(t,r) \sim \left(\frac{8\pi k_c}{\chi\tilde{\alpha}} + \delta_n\right)(4\pi r^2)^{-1} \text{ as } r \to 0.$$

For this solution

$$N(t,r) := \int_{|x| \leq r} u(T,s)\, ds \to 0 \text{ as } r \to 0$$

3. *No radial, self-similar solution of (4.2) exists such that $N(t,r) < \infty$ as $r \to 0$ when $N = 2$, resp. when $\Omega = \mathbb{R}^2$.*

Thus one slowly got more and more insights for the Keller-Segel model but at this point several questions still remained open. To name a few beside the questions raised in [67] we list the following.

1. What happens if one drops the assumption of radial symmetry of the solution and how does in the case of blow-up the blow-up profile of the solutions look like?
2. Is it possible to prove blow up results also for the full system (3.3)?
3. Can one give the precise value for the threshold value which decides whether the solution might blow up or not?

As in the section before I summarize the results of this section in a table, too.

Table 4.1
Possible time asymptotical behaviour of the solutions of the simplified model 4.2.

Dimension	Observation	References
N=2	There exists a critical value $c(\Omega)$ such that a unique, smooth positive solution to (4.2) exists globally in time if $\alpha\chi\overline{U_0(x)} < c(\Omega)$.	[67]
	Let Ω be a disk. Then there exists a positive number c^* such that there exists radially symmetric positive initial data with the following property: If $\alpha\chi\overline{U_0(x)} > c^*$ then radially symmetric positive initial values can be constructed such that explosion of $U(t,x)$ happens in the center of the disc in finite time.	
	There exists radially initial data such that the solution of (4.2) forms in the center of a disk Ω a δ-function singularity described in Theorem 4.3 in finite time.	[49, 54]
	When $\Omega = \mathbb{R}^2$, then no radial, self-similar solutions of (4.2) exist such that $\int\limits_{\lvert x\rvert \le r} u(T,s)\; ds < \infty$ as $r \to 0$.	[52, 54]
N=3	Let $\Omega = \mathbb{R}^3$. Then there exists, for any $T > 0$ and any constant $C > 0$, a radial solution $(u(t,r), v(t,r))$ of (4.2) that is smooth for all times $0 < t < T$, blows up at $r = 0$ and $t = T$, and is such that: $\int\limits_{\lvert x\rvert \le r} u(T,s)\; ds \to C$.	[52, 53, 54]
	For any $T > 0$ there exists a sequence $\{\delta_n\}_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} \delta_n = 0$, and a sequence of radial solutions $(u_n(t,r), v_n(t,r))$, that blows up at $r = 0$ and $t = T$, and are such that $u_n(t,r)$ is self-similar, and $u_n(t,r) \sim \left(\frac{8\pi k_c}{\chi\tilde{\alpha}} + \delta_n\right)(4\pi r^2)^{-1}$ as $r \to 0$. For this solution $\int\limits_{\lvert x\rvert \le r} u(T,s)\; ds \to 0$ as $r \to 0$.	

4.2. Progress and further questions. After W. Jäger's and S. Luckhaus' paper in 1992 the next step was performed by T. Nagai in [87]. In his 1995 article "*Blow-up of radially symmetric solutions to a chemotaxis system*" [87] he proved the following result for the simplified system

$$\left.\begin{array}{rcll} u_t &=& \nabla(\nabla u - \chi u \nabla v), & x \in \Omega,\ t > 0 \\ 0 &=& k_c \Delta v - \gamma v + \tilde{\alpha} u, & x \in \Omega,\ t > 0 \\ \partial u/\partial n &=& \partial v/\partial n \ = \ 0, & x \in \partial\Omega,\ t > 0 \\ u(0,x) &=& u_0(x), & x \in \Omega. \end{array}\right\} \tag{4.6}$$

Theorem 4.5 (Nagai).
1. *Suppose that $N = 1$, or $N = 2$ and $\tilde{\alpha}\chi \int\limits_{B(0,R)} u_0(x)dx < 4\omega_N$ with radially symmetric $u_0(x)$. Then $T_{max} = \infty$ and $\sup\limits_{t\ge 0} \left\{ \lvert\lvert u(t,\cdot)\rvert\rvert_{L^\infty(B(0,R))} + \lvert\lvert v(t,\cdot)\rvert\rvert_{L^\infty(B(0,R))} \right\} < \infty$.*

2. Let $N \geq 2$ and u_0 be radially symmetric. If

$$0 > 2N(N-1)\left(\frac{1}{\omega_N}\int_\Omega u_0(x)dx\right)^{2/N}\left(\frac{1}{\omega_N}\int_{B(0,R)} u_0(x)|x|^N dx\right)^{(N-2)/N} - \frac{N}{2}\tilde{\alpha}\chi\left(\frac{1}{\omega_n}\int_\Omega u_0(x)dx\right)^2$$
$$+\tilde{\alpha}\chi N R^{-N}\left(\frac{1}{\omega_n}\int_\Omega u_0(x)dx\right)\left(\frac{1}{\omega_N}\int_{B(0,R)} u_0(x)|x|^N dx\right)$$
$$+\tilde{\alpha}\chi\gamma\begin{cases}\frac{1}{e}\left(\frac{1}{\omega_N}\int_\Omega u_0(x)dx\right)^{3/2}\left(\frac{1}{\omega_N}\int_{B(0,R)} u_0(x)|x|^N dx\right)^{1/2}, & \text{if } N=2\\ \frac{N}{2(N-2)}\left(\frac{1}{\omega_N}\int_\Omega u_0(x)dx\right)^{(2N-2)/N}\left(\frac{1}{\omega_N}\int_{B(0,R)} u_0(x)|x|^N dx\right)^{2/N}, & \text{if } N\geq 3\end{cases}$$

where ω_N denotes the area of the unit sphere S^{N-1} in R^N, then $T_{max} < \infty$ and

$$\limsup_{t\to T_{max}} ||u(t,\cdot)||_{L^\infty(B(0,R))} = \infty.$$

Furthermore the radially symmetric solution $(u(t,r), v(t,r))$ of (4.6) satisfies $u(t,r) + v(t,r) \leq K(n)$ for $\frac{1}{n} \leq r \leq R$ and $0 \leq t < T_{max}$ where $K(n)$ denotes a generic positive constant depending on $n \in \mathbb{N}$ such that $K(n) \to \infty$ as $n \to \infty$. Thus the blow-up can only occur at the point $r = 0$.

While the first statement is easy to check the second is based on some subtle estimates of the expression

$$M_N(t) := \frac{1}{\omega_N}\int_{B(0,R)} u(t,x)|x|^N dx.$$

The global existence proof of solutions of (4.6) in one space dimension performed in [87] illustrates in a nice way how one tries to show the existence of the solution global in time in higher space dimensions. Therefore we first demonstrate this proof here. If we integrate for $N = 1$ the second equation of (4.6) on $(-R, x)$ we get

$$v_x(t,x) = \gamma\int_{-R}^{x} v(t,y)dy - \alpha\int_{-R}^{x} u(t,y)dy.$$

Thus we see that

$$|v_x(t,x)| \leq \alpha\int_{-R}^{R} u_0(x)dx \text{ on } \Omega\times(0,T_{max}).$$

For $x \in \Omega = (-R, R)$ we now obtain

$$2Rv(t,x) = \int_{-R}^{R} v(t,y)dy + \int_{-R}^{R}\int_{y}^{x} v_x(t,z)dz\, dy \text{ and therefore } 0 \leq v(t,x) \leq \frac{\alpha}{2R}\left(\frac{1}{\gamma} + 4R^2\right)\int_\Omega u_0(x)dx.$$

Thus $||v(t,\cdot)||_{L^\infty(\Omega)} \leq const$ and $||v_x(t,\cdot)||_{L^\infty(\Omega)} \leq const$ for all $0 < t < T_{max}$. Multiplying now the first equation of (4.6) with u^p for $p \geq 1$ and integrating the equation over Ω yields

$$\frac{1}{p+1}\frac{d}{dt}\int_\Omega u^{p+1}dx = \frac{-2p}{p+1}\int_\Omega |\nabla u^{(p+1)/2}|^2 dx + \frac{p(p+1)}{2}\cdot const\cdot\int_\Omega u^{p+1}dx.$$

Now the bound of the L^∞-norm of the solution can be obtained by application of N. D. Alikakos' version of the Moser iteration introduced in [2]. One therefore sees that the question whether the solution exists globally in time or not depends crucially on a uniform bound for the L^∞-norm of the gradient of v. This simplified version of the Keller-Segel model has been extensively studied by Nagai and his coauthors. Once again we cannot follow the chronology since the different versions of the Keller-Segel model have been studied parallel. Thus I concentrate on the results on system (4.6) in this subsection and turn to the results for the full Keller-Segel model (3.3) resp. (4.1) later on. The simplified versions allow to decouple the system. Therefore techniques are available in these cases which are not at hand for the full parabolic version. For the simplified version (4.6) recent results from Nagai, Senba, Suzuki et al. give more information about the blow-up profile of the solution and the non symmetric blow up. However their proofs are very technical and desire fine estimates that are difficult to demonstrate in a simple way. Thus I restrict myself to present their results in Table 4.2 and 4.3.

Table 4.2
Possible time asymptotical behaviour of the solutions of the simplified model 4.6 with $\gamma > 0$.

Dimension	Observation	References
$N=1$	The solution of the Keller-Segel model exists globally in time and is uniformly bounded for all $t \geq 0$.	[87]
$N=2$	If $\tilde{\alpha}\chi \int_{\Omega} u_0(x)dx < 4\pi$ then the classical solution of the Keller-Segel model exists globally in time and is uniformly bounded for all $t \geq 0$. If Ω is a circle and u_0 is radially symmetric or satisfies $u(x) = u(-x)$ in Ω, then this statement holds if $\tilde{\alpha}\chi \int_{\Omega} u_0(x)dx < 8\pi$.	[87, 89, 90, 91] [94, 97] and [126]
	Let $x_0 \in \Omega$. If $\tilde{\alpha}\chi \int_{\Omega} u_0(x)dx > 8\pi$ and if $\int_{\Omega} u_0(x)\|x-x_0\|^2dx$ is sufficiently small, then the corresponding solution of (4.6) and (4.2) blows up in finite time.	[98]
	Assume that $\partial\Omega$ has a line segment $\mathcal{L}_0$, and that Ω lies on one side of a line $\mathcal{L}$ containing $\mathcal{L}_0$. If furthermore $\tilde{\alpha}\chi \int_{\Omega} u_0(x)dx > 4\pi$ and if $\int_{\Omega} u_0(x)\|x-x_0\|^2dx$ is sufficiently small for a point $x_0 \in \mathcal{L}_0$ that is not an end-point of $\mathcal{L}_0$, then the corresponding solution of (4.6) and (4.2) blows up in finite time.	[98]
	If Ω is a circle, u_0 is radially symmetric and if $\tilde{\alpha}\chi \int_{\Omega} u_0(x)dx > 4\omega_2$, then there exists a constant C_1 depending on $\frac{1}{\omega_2}\int_{\Omega} u_0(x)dx$ such that if $0 < \frac{1}{\omega_2}\int_{\Omega} u_0(x)\|x\|^2dx < C_1$ then u blows up in finite time.	[87]
	If $\tilde{\alpha}\chi \int_{\Omega} u_0(x)dx < 8\pi$ and $T_{max} < \infty$ then there exists a point in $x_0 \in \partial\Omega$ such that $\limsup\limits_{t\to T_{max}} \int_{\Omega\cap B(x_0,\epsilon)} u(t,x)dx \geq 2\pi/a_*\tilde{\alpha}$ for any $\epsilon > 0$, where a_* is a root of $a_* - \chi/2 - \|\|u_0\|\|_{L^1(\Omega)}\tilde{\alpha}a_*/16\pi = 0$ such that $a_* < \chi$.	[92, 93]
	Suppose $T_{max} < \infty$. Then there exist for any isolated blow-up point x_0 two positive constants δ, $m \geq m_*$ and a non-negative function $f \in L^1(\overline{B(x_0,\delta)\cap\Omega}) \cap C(\overline{B(x_0,\delta)\cap\Omega} \setminus \{x_0\})$ such that $u(t,\cdot)$ converges weakly in the Banach space of all Radon measures on $\overline{B(x_0,\delta)\cap\Omega}$ to $m\delta_{x_0} + f$ as $t \to T_{max}$, where m_* is equal to $4\pi/\tilde{\alpha}\chi$ if $x_0 \in \partial\Omega$ and equal to $8\pi/\tilde{\alpha}\chi$ if $x_0 \in \Omega$.	[92, 93, 127] and [132]
	Suppose $\Omega = \mathbb{R}^2$ and let $x_0 \in \mathbb{R}^2$. If $\tilde{\alpha}\chi \int_{\mathbb{R}^2} u_0(x)dx > 8\pi$ and if $\int_{\mathbb{R}^2} u_0(x)\|x-x_0\|^2dx$ is sufficiently small, then the corresponding solution of (4.6) blows up in finite time.	[95]

Table 4.3
Possible time asymptotical behaviour of the solutions of the simplified model 4.6 with $\gamma > 0$.

Dimension	Observation	References
$N \geq 3$	If Ω is a sphere and u_0 is radially symmetric, then there exists a constant C_1 depending on $\frac{1}{\omega_N} \int_\Omega u_0(x)dx$ such that if $0 < \frac{1}{\omega_N} \int_\Omega u_0(x)\|x\|^N dx < C_1$ then u blows up in finite time.	[87]
	Suppose $\Omega = \mathbb{R}^N$ and let $x_0 \in \mathbb{R}^N$. If $\int_{\mathbb{R}^N} u_0(x)\|x - x_0\|^N dx$ is sufficiently small, then the corresponding solution of (4.6) blows up in finite time.	[95]

Of course one can now draw conclusions on the possible number of blow-up points. However I will mention these conclusions a little bit later. Thus at this point let us turn again to a different line of research.

4.3. Analysis of the system (4.1). Similar to their result for the simplified system (3.3) M. A. Herrero and J. J. L. Velázquez achieved a very important contribution on the blow-up profile of the solution of the full parabolic systems (3.3) and the system (4.1) with $\gamma = 0$ in their papers [50] and [51]. Using once again asymptotic expansion theory they were able to describe the blow-up profile of the system (3.3) and proved therefore the possibility of a δ-function formation in finite time for radially symmetric solutions as it was conjectured by Nanjundiah [102] and Childress and Percus [24]. Their main result for system (3.3) is summarized as follows:

Theorem 4.6 (Herrero & Velázquez). *Let $R > 0$, and let $\Omega_R = \{x \in \mathbb{R}^2 : |x| < R\}$. Then there exist radial solutions of* (3.3) *defined in an interval* $(0, T)$ *with* $T > 0$*, and such that:*

$$u(t, r) \to \frac{8\pi k_c}{\tilde{\chi}\tilde{\alpha}} \delta(0) + \psi(r) \text{ as } t \to T, \tag{4.7}$$

in the sense of measures, where $\delta(0)$ is the Dirac measure centered at $r = 0$, and:

$$\psi(r) = \frac{C}{r^2} e^{-2|\log(r)|^{1/2}} (1 + o(1)) \tag{4.8}$$

as $r \to 0$, where C is a positive constant depending on χ. At $t = T$, the profile near $r = 0$ is given by:

$$u(t, r) = \frac{8\pi k_c}{\tilde{\chi}\tilde{\alpha}} \delta(0) + \psi(r); \;\; \psi(r) \text{ as in } (4.8). \tag{4.9}$$

Moreover, if we set $S(t) = (T - t)(\sup_\Omega u(t, r)) \equiv (T - t)u(0, t)$, one has that $\lim_{t \to T} S(t) = \infty$. More precisely, there holds:

$$S(t) = C_1 (T - t)^{-1} e^{\sqrt{2|\log(T-t)|}} \text{ as } t \to T, \text{ for some } C_1 > 0. \tag{4.10}$$

The studies of the asymptotic behaviour of the solution in the non-symmetric case began with the results of [11, 43, 90] and [153]. In [11, 43, 90] the authors introduce independently from each other a Lyapunov functional for the system (3.3) resp. (4.1) which became an important tool in the then following studies of the time asymptotic behaviour of the solution of the systems (3.3) resp.(4.1). This Lyapunov function is given by

$$F(u(t), v(t)) := \frac{1}{2\tilde{\alpha}\chi} \int_\Omega k_c |\nabla v(t)|^2 + \gamma v^2(t) + u(t) \log(u(t)) - u(t)v(t) dx. \tag{4.11}$$

Using a Moser-Trudinger type inequality originally formulated by Chang and Yang in [23] the analysis of this functional shows the following:

1. The functional $F(u, v)$ is bounded from below, if $\tilde{\alpha}\chi \int_\Omega u_0(x)dx \leq 4\pi$.
2. The functional $F(u, v)$ is no longer bounded from below, if $\tilde{\alpha}\chi \int_\Omega u_0(x)dx > 4\pi$.

3. For radially symmetric functions the functional $F(u,v)$ is bounded from below, if $\tilde{\alpha}\chi \int\limits_\Omega u_0(x)dx \leq 8\pi$. and is no longer bounded from below, if $\tilde{\alpha}\chi \int\limits_\Omega u_0(x)dx > 8\pi$.

Now two different lines of research became recognizable. One considered system (3.3) and used more PDE based methods to prove global existence and finite time blow-up results for this system and the other was concerned with system (4.1) and used methods more related to the calculus of variations. Once again one has to follow these two lines separately to get a clear picture of the achieved results. Let us first have a closer look at the results for (3.3)

4.3.1. Results for system (3.3). Since the question of the well-posedness of a negative cross-diffusion system is not trivial I first turn to the results on the local existence of a solution and possible regularity results. Here one should basically mention A. Yagi [153] and T. Nagai, T. Senba and K. Yoshida [90] whose results can be summarized as follows:

THEOREM 4.7. *Let Ω be a bounded, smooth domain in $\mathbb{R}^2$. Assume $u_0,\ v_0 \in H^{1+\epsilon_0}(\Omega)$ for some $0 < \epsilon_0 \leq 1$ and $u_0(x) \geq <,\ v_0(x) \geq 0$ on Ω. Let T_{max} be the maximal existence time of $(u(t), v(t))$.*

1. (Yagi) *System* (3.3) *has a non-negative solution (u, v) satisfying*

$$u, v \in C([0, T_{max}) : H^{1+\epsilon_1}(\Omega)) \cap C^1((0, T_{max}) : L^2(\Omega)) \cap C((0, T_{max}) : H^2(\Omega))$$

for any $0 < \epsilon_1 < \min\{\epsilon_0, 1/2\}$. Moreover (u, v) has further regularity properties:

$$u \in C^1((0, T_{max}) : H^1(\Omega)),\ \ v \in C^{\frac{1}{4}}((0, T_{max}) : H^3(\Omega)) \cap C^{\frac{5}{4}}((0, T_{max}) : H^1(\Omega)).$$

2. (Yagi) *If $T_{max} < \infty$, then*

$$\lim_{t \to T_{max}} (||u(t,\cdot)||_{H^{1+\epsilon_0}(\Omega)} + ||v(t,\cdot)||_{H^{1+\epsilon_0}(\Omega)}) = \infty,$$
$$\limsup_{t \to T_{max}} ||u(t,\cdot)||_{L^p(\Omega)} = \infty \ for\ any\ 1 < p \leq \infty,$$
$$\limsup_{t \to T_{max}} ||v(t,\cdot)||_{H^{1+\epsilon}(\Omega)} = \infty \ for\ any\ 0 < \epsilon \leq \epsilon_0.$$

3. (Nagai, Senba & Yoshida) *If*

$$\int\limits_\Omega u_0(x)dx < \frac{4\Theta k_c}{\tilde{\alpha}\tilde{\chi}},$$

where $\Theta = 8\pi$ for $\Omega = \{x \in \mathbb{R}^2 : |x|^2 < R\}$ and (u_0, v_0) is radial in x and $\Theta = 4\pi$ otherwise, then the solution (u, v) of (3.3) *exists globally in time and $\sup\limits_{t \geq 0}\{||u(t,\cdot)||_{L^\infty(\Omega)} + ||v(t,\cdot)||_{L^\infty(\Omega)}\} < \infty$.*

The local existence and regularity results summarized in Theorem 4.7 above have been achieved by using semi-group theory. A. Yagi also proved similar local existence results for more general forms of the system (2.4) in [153] and I will turn to these results later. The bound of the L^∞-norm of the solution can once more be achieved by application of N. D. Alikakos' version of the Moser iteration introduced in [2]. Once again the basic and most important step is to find a uniform L^∞-norm estimate of $\nabla v(t,x)$ for all $t \geq 0$. Nagai, Senba and Yoshida succeeded in finding such a bound in the case where the functional $F(u,v)$ is bounded from below. A. Yagi studied in [153] which norms of the solution have to blow-up if the solution exists only for a finite maximal time of existence T_{finite}. However beside the results of Herrero and Velázquez in [50, 51] there are no results, that show the existence of initial data such that the corresponding solution of (3.3) has to blow up in finite time. However there are results under the assumption that there is a solution which blows-up in finite time. Let us therefore now turn to those results, that studies the blow-up profile and behaviour of such a solution.

Under the **main assumption** that there is a solution of the Keller-Segel model that blows up in finite time T_{finite} such that

$$\inf_{0 \leq t < T_{finite}} F(u(t), v(t)) > 0 \text{ or} \tag{4.12}$$
$$\lim_{t \to T_{finite}} F(u(t), v(t)) = -\infty$$

Nagai, Senba and Suzuki proved in [96, 98] the following results.

THEOREM 4.8 (Nagai, Senba & Suzuki). *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial\Omega$. Furthermore let $\mathcal{B}$ denote the set of all those points x_0 in $\overline{\Omega}$ such that there is a sequence $\{x_k\}_{k\in\mathbb{N}} \subset \overline{\Omega}$ and a sequence $\{t_k\}_{k\in\mathbb{N}} \subset [0, T_{finite})$ with $u(t_k, x_k) \to \infty$, $t_k \to T_{finite}$ and $x_k \to x_0$ as $k \to \infty$. $\mathcal{B}_I \subset \mathcal{B}$ denotes the set of all isolated blow-up points, i.e $x_0 \in \mathcal{B}_I$, iff there exists a $R > 0$ such that*

$$\sup_{0\le t<T_{finite}} ||u(t,\cdot)||_{L^\infty((B(x_0,R)\setminus B(x_0,r))\cap\Omega)} < \infty$$

for any $r \in (0, R)$ with $B(x_0, R) := \{x \in \mathbb{R}^2 \mid |x_0 - x| < R\}$. Then the following statements hold:

1. *Given $x_0 \in \mathcal{B}_I$, we have $0 < R << 1$, $m \ge m^*$, and*

$$f \in L^1(B(x_0, P) \cap \Omega) \cap C(\overline{B(x_0, R) \cap \Omega} \setminus \{x_0\})$$

satisfying $f \ge 0$ and $u(t,\cdot)dx$ converges weakly to $m\delta_{x_0}(dx) + f\ dx$ as $t \to T_{finite}$ in the set of Radon measures on $\overline{B(x_0, R) \cap \Omega}$, where

$$m^* := \begin{cases} 8\pi, & x_0 \in \Omega \\ 4\pi, & x_0 \in \partial\Omega. \end{cases}$$

2. *If (4.12) occurs, then $\mathcal{B} = \mathcal{B}_I$.*
3. *If (3.3) is radially symmetric and $T_{max} < \infty$ then $\mathcal{B} = \{0\}$.*

These results imply that in the case of a finite time blow-up of the solution the set of isolated blow-up points has finite cardinality and that

$$1 < 2 \times \sharp(\mathcal{B}_I \cap \Omega) + \sharp(\mathcal{B}_I \cap \partial\Omega) \le \frac{||u_0||_{L^1(\Omega)}}{4\pi}.$$

However a better lower bound of the quantity is of interest in the non radially symmetric case with $||u_0||_{L^1(\Omega)} > 8\pi$. Where does the blow-up occur? Is there only one blow-up point in the interior of Ω or are there two blow-up points at the boundary $\partial\Omega$ in this case?

Beside the previous results Senba and Suzuki established in [131] the following results using rearrangement and symmetrization arguments. These results are similar to those achieved independently and by other methods in [62] and [63] for the system (4.1).

THEOREM 4.9 (Senba & Suzuki). *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial\Omega$.*

1. *If Ω is the unit disk, $\tilde{\alpha}\chi||u_0||_{L^1(\Omega)} < 8\pi k_c$ and $u_0(x) = u_0(-x)$, $v_0(x) = v_0(-x)$ hold, then the solution of (3.3) exists globally in time and satisfies the equations in the classical sense, i.e. the solution is sufficiently smooth.*
2. *If $T_{max} < \infty$ then*

$$\lim_{t\to T_{max}} ||u(t)\log u(t)||_{L^1(\Omega)} = \lim_{t\to T_{max}} ||u(t)v(t)||_{L^1(\Omega)} = \lim_{t\to T_{max}} ||\nabla v(t)||^2_{L^2(\Omega)} = \lim_{t\to T_{max}} \int_\Omega e^{av(t)}dx = \infty,$$

where $a > 1$.

3. *If Ω is simply connected , $\tilde{\alpha}\chi||u_0||_{L^1(\Omega)} < 8\pi k_c$, and $T_{max} < \infty$ then*

$$\lim_{t\to T_{max}} \int_{\partial\Omega} e^{v(t)/2}dx = \infty.$$

The last statement in particular implies together with the previous statements on the number of isolated blow-up points in Theorem 4.8 that if there is a solution that blows up in finite time for $4\pi k_c < \tilde{\alpha}\chi||u_0||_{L^1(\Omega)} < 8\pi k_c$ then the blow-up has to happen at the boundary of the domain. However, at this stage it has to be pointed out that these results do not give the existence of a solution that blows-up in either finite or infinite time. These results always use the existence of a solution that blows up in finite time as an assumption, but do not prove that those solutions in fact really exist.

Beside the analysis of the Keller-Segel system (3.3) on a bounded domain T. Nagai also studied the problem on the whole space $\mathbb{R}^2$. In this situation he could prove that for $\tilde{\alpha}\chi \int_{\mathbb{R}^2} u_0(x)dx < 4\pi k_c$ the solution exists globally in time, once again via analyzing the functional $F(u,v)$ for $\Omega = \mathbb{R}^2$ this time. Furthermore he found several decay properties of the solution but for those results we refer the reader to [95].

Throughout this subsection we focused the two dimensional case and left out the other space dimensions. So what is known for the cases $N = 1$ and $N \geq 3$? For the case $N = 1$ the paper by K. Osaki and A. Yagi [109] fills the gap of the missing global existence proof for (3.3). Furthermore they show there that in this case the solution converges to a stationary solution as $t \to \infty$. For the case of higher space dimensions $N \geq 3$ and a bounded domain $\Omega \subset \mathbb{R}^N$ I am aware of any result on the time asymptotic behaviour of the solution. The local existence of a solution can be established in such cases using the results of H. Amann [6, 7] for example. This has been mentioned for example in [61] and [115].

4.3.2. Results for system (4.1). Independent from the previous research line and parallel to those results there were the results for the system (4.1). Under different regularity assumptions on the domain and the solution than those assumed in [90] and [153] H. Gajewski and K. Zacharias proved in [43] the local existence of a weak solution of (4.1) where they defined a weak solution of (4.1) in the following way.

Definition 4.10 (Gajewski & Zacharias). *A pair of functions* $(U(t,x), V(t,x))$ *with*

$$\begin{array}{ll} U \in L^\infty(0,T; L^\infty_+(\Omega)) \cap L^2(0,T; H^1(\Omega)), & U_t \in L^2(0,T; (H^1(\Omega))^*), \\ V \in L^\infty(0,T; L^\infty(\Omega)) \cap C(0,T; H^1(\Omega)), & V_t \in L^2(0,T; L^2(\Omega)) \end{array}$$

is called a weak solution of (4.1) *if for all* $h \in L^2(0,T; H^1(\Omega))$ *the following identities hold:*

$$0 = \int_0^T \langle U_t, h \rangle \; dt + \int_0^T \int_\Omega (\nabla U - U \nabla V) \cdot \nabla h \; dx \; dt,$$

$$0 = \int_0^T \int_\Omega V_t h \; dx \; dt + \int_0^T \int_\Omega (k_c \nabla V \cdot \nabla h + (\gamma V - \alpha\chi(U-1)) \cdot h) \; dx \; dt.$$

Their existence result is:

Theorem 4.11 (Gajewski & Zacharias). *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded domain and its boundary is piecewise from the class* C^2*. For* $U_0 \in L^\infty_+(\Omega)$ *and* $V_0 \in W^{1,p}(\Omega)$*,* $p > 2$*, and appropriate* $T > 0$ *there is a unique weak solution of* (4.1) *with* $U(0) = U_0$*,* $V(0) = V_0$*. Moreover, for* $0 \leq t < T$ *it holds* $t \mapsto U(t) \in L^\infty_+$ *and the function* $t \mapsto ||\nabla V(t)||^2_{L^2(\Omega)}$ *is absolutely continuous on* $[0,T]$*.*

For (4.1) the Lyapunov function F takes the following form:

$$F(U(t),V(t)) := \frac{1}{2\alpha\chi} \int_\Omega k_c |\nabla V(t)|^2 + \gamma V^2(t) \; dx + \int_\Omega U(t)(\log(U(t)) - 1) - 1 - U(t)V(t) dx.$$

In fact Gajewski and Zacharias showed that one can bound F by a functional only depending on V, namely

$$F(U(t),V(t)) \geq \mathcal{F}(V(t)) = \frac{1}{2\alpha\chi} \int_\Omega k_c |\nabla V(t)|^2 + \gamma V(t)^2 \; dx - |\Omega| \log\left(\frac{1}{|\Omega|} \int_\Omega e^{V(t)} dx \right).$$

Using the Moser-Trudinger type inequality by Chang and Yang in [23] it is possible to show that $\mathcal{F}(V)$ is lower semicontinuous and coercive on the set $\mathcal{D} := \{V \in H^1(\Omega) \mid V \text{ has mean value zero over the domain } \Omega\}$ if $\alpha\chi|\Omega| < 4\Theta k_c$, where Θ denotes the smallest interior angle of the piecewise smooth domain Ω. Therefore the calculus of variations guarantees the existence of a minimizer of $\mathcal{F}$ over the set $\mathcal{D}$. As a conclusion we get the boundedness of the functional F from below in this case. The boundedness of the Lyapunov functional and the fact that for

$$U(x) = \frac{|\Omega| V(x)}{\left(\int_\Omega e^{V(t)} dx \right)}$$

the equality $F(U,V) = \mathcal{F}(V)$ holds lead Gajewski and Zacharias to:

THEOREM 4.12 (Gajewski & Zacharias). *Let $\alpha\chi|\Omega| < 4\Theta k_c$. Then there exist a sequence $t_k \to \infty$ and functions U^*, V^* such that $U(t_k) \to U^*$ in $L^2(\Omega)$, $V(t_k) \rightharpoonup V^*$ in $H^1(\Omega)$, $F(U(t_k),V(t_k)) \to F(U^*,V^*)$. Moreover the identity*

$$U^* = \frac{|\Omega| e^{V^*}}{\left(\int\limits_\Omega e^{V^*} dx\right)}$$

holds, and V^* is the solution of the boundary value problem

$$-k_c \Delta V^* + \gamma V^* = \alpha\chi(U^* - 1) \text{ in } \Omega, \; \frac{\partial V^*}{\partial n} = 0 \text{ on } \partial\Omega.$$

As one can see the previous result does not only hold for subsequences as it has been shown in [62, Theorem 3, page 408]. Furthermore the steady state might also be nontrivial in the case where $\alpha\chi|\Omega| < 4\Theta k_c$. Gajewski and Zacharias presented in [43, Proposition 5.3, page 109] an example in which $\Omega := \{(x,y) : 0 < x < a, \; 0 < y < b\}$ denotes a rectangle where

$$ab < \frac{2\pi k_c}{\alpha\chi} \text{ and } a^2 > \frac{\pi^2 k_c}{\alpha\chi(\log(4) - 1) - \gamma} > 0$$

and the initial data $(U_0(x), V_0(x))$ is given by $U_0(x) = 1 + \cos\left(\frac{\pi x}{a}\right)$ and $V_0(x) = \cos\left(\frac{\pi x}{a}\right)$. We then see that $F(U_0,V_0) < 0 = F(1,0)$ and thus there has to be a nontrivial stationary solution of system (4.1) also in this case and not only in the cases mentioned in Section 3.

The boundedness of the Lyapunov functional $F(U,V)$ by the functional $\mathcal{F}(V)$ has several consequences that are demonstrated in [62]. However using the same sequence as in the proof of the existence of a nontrivial steady state solution in section 3 one can show for $\alpha\chi|\Omega| > 4k_c\pi$ there is a sequence of functions $\{(U_\varepsilon, V_\varepsilon)\}_{\varepsilon \geq 0}$ such that $F(U_\varepsilon, V_\varepsilon) \to -\infty$ as $\varepsilon \to 0$. Furthermore, if $\Omega \subset \mathbb{R}^2$ is simply connected and if $\alpha\chi|\Omega| < 8\pi$, $p \in (1, 8\pi k_c/\alpha\chi|\Omega|)$ is arbitrary but fixed and $q > 1$ is such that $1 = 1/p + 1/q$ then one can bound the functional $\mathcal{F}(V)$ in the following way:

$$\mathcal{F}(V) \geq \int\limits_\Omega \left(\frac{k_c}{2\alpha\chi} - \frac{p|\Omega|}{16\pi}\right) |\nabla V|^2 + \frac{\gamma}{2\alpha\chi} V^2 \; dx - \frac{2|\Omega|}{q} \log\left(\int\limits_{\partial\Omega} e^{qV/2} \; dS\right) + K(p,q,\alpha,\chi,k_c,|\Omega|),$$

where $K(p,q,\alpha,\chi,k_c,|\Omega|)$ denotes a constant depending on p, q, α, χ, k_c and $|\Omega|$. However this has the consequence that we have the following blow-up result which summarizes the results from [60, 62, 63, 64].

THEOREM 4.13 (Horstmann & Wang). *Let $\Omega \subset \mathbb{R}^2$ be a smooth, simply connected domain and $\gamma > 0$. Furthermore assume that $4k_c\pi < \alpha\chi|\Omega|$ and that $\alpha\chi|\Omega|/k_c \neq 4\pi m$ for $m \in \mathbb{N}$, then there exist a constant $-\infty < \hat{K} \leq 0$ and initial data (U_0, V_0), such that $\hat{K} > F(U_0,V_0)$ and the corresponding solution of (4.1) blows up in finite or infinite time. For these blow-up solutions the following statements hold:*

1.

$$\lim_{t \to T_{max}} ||U(t,x)||_{L^2(\Omega)} = \lim_{t \to T_{max}} ||U(t,x)||_{L^\infty(\Omega)} = \infty$$

2.

$$\lim_{t \to T_{max}} ||U(t) \log U(t)||_{L^1(\Omega)} = \lim_{t \to T_{max}} \int\limits_\Omega U(t,x)V(t,x) \; dx = \infty$$

3.

$$\lim_{t \to T_{max}} ||\nabla V(t,x)||_{L^2(\Omega)} = \lim_{t \to T_{max}} \int\limits_\Omega e^{V(t,x)} \; dx = \lim_{t \to T_{max}} ||V(t,x)||_{L^\infty(\Omega)} = \infty$$

4. If $4\pi k_c < \alpha\chi|\Omega| < 8\pi k_c$ and Ω is a simply connected domain, then

$$\lim_{t \to T_{max}} \int_{\partial\Omega} e^{qV(t,x)/2} dS = \infty$$

for every $q \in (8\pi k_c/(8\pi k_c - \alpha\chi|\Omega|), \infty)$.

There are technical reasons why one has to exclude the multiples of 4π in the theorem. From the biological point of view this makes no sense and there is a hint in [68] that in the second theorem the statements are in fact true if one only assumes $4k_c\pi < \alpha\chi|\Omega|$. For $\gamma = 0$ there is a similar result for radially symmetric initial data in [64]. Since this proof is easy to illustrate I give this theorem and the sketch of the proof, too. So we can formulate our blow-up result.

THEOREM 4.14 (Horstmann). *Let $\Omega = B(0,R) \subset \mathbb{R}^2$. Further assume that $\gamma = 0$, $8k_c\pi < \alpha\chi|\Omega|$ and $\alpha\chi|\Omega|/k_c \neq 8\pi q$, $q \in \mathbb{N}$, then there exist radially symmetric initial data (U_0, V_0) and let a constant $\hat{K}$, such that $\hat{K} > F(U_0, V_0)$ and the corresponding solution of* (4.1) *blows up in finite or infinite time.*

The proof of Theorem 4.14 is easily demonstrated. First one shows the existence of the constant $\hat{K}$ via contradiction. Thus one assumes that there is no such constant. Therefore there exists for $\Omega = B(0,R) \subset \mathbb{R}^2$ a sequence $(v_m)_{m\in\mathbb{N}} \in \mathcal{D}$ of solutions of the equation

$$\left.\begin{array}{rcll} -k_c\Delta v_m &=& \alpha\chi\left(|\Omega|e_m^v \Big/ \int_\Omega e_m^v dx - 1\right), & \text{in } \Omega \\ \partial v_m/\partial n &=& 0, & \text{on } \partial\Omega, \end{array}\right\}$$

$$\text{with} \int_\Omega |\nabla v_m|^2 \, dx < \infty \ \forall m \in \mathbb{N}, \ \lim_{m\to\infty} \int_\Omega |\nabla v_m|^2 dx = \infty \text{ and } \lim_{m\to\infty} \log\left(\frac{1}{|\Omega|}\int_\Omega e^{v_m} dx\right) = \infty.$$

These sequences can therefore be identified as a sequence of stationary solutions of the system (4.1) in the radially symmetric setting and with $\gamma = 0$. Using the transformation

$$w_m = v_m - \log\left(\frac{1}{|\Omega|}\int_\Omega e^{v_m}\right) - \frac{\alpha\chi}{4k_c}|x|^2$$

the function w_m solves the problem

$$\left.\begin{array}{rcll} -\Delta w_m &=& \frac{\alpha\chi}{k_c} e^{w_m + (\alpha\chi/4k_c)|x|^2}, & \text{in } \Omega \\ \partial w_m/\partial n &=& -\frac{\alpha\chi}{2k_c}(x \cdot n(x)), & \text{on } \partial\Omega, \end{array}\right\}$$

$$\text{with} \int_\Omega |\nabla w_m|^2 \, dx < \infty \ \forall m \in \mathbb{N}, \ \lim_{m\to\infty} \int_\Omega |\nabla w_m|^2 dx = \infty \ \text{ and } \frac{1}{|\Omega|}\int_\Omega e^{w_m + \frac{\alpha\chi}{4k_c}|x|^2} dx = 1 \ \forall m \in \mathbb{N}.$$

Using the results from [75] and the Sobolev imbedding theorems we see that v_m is in fact $C^{2,\beta}(\overline{\Omega})$ provided $\partial\Omega$ is Lipschitz and thus w_m also belongs to $C^{2,\beta}(\overline{\Omega})$. According to a result from Brézis and Merle [20] there exists a subsequence $(w_{m_i})_{m_i\in\mathbb{N}}$ for these $(w_m)_{m\in\mathbb{N}}$ such that one of the following three alternatives holds:

1. The sequence $(w_{m_i})_{m_i\in\mathbb{N}}$ is uniformly bounded in $L^\infty_{loc}(\Omega)$.
2. For each compact subset $\mathcal{K} \subset \Omega$ we have:

$$\sup_{\mathcal{K}} w_{m_i} \to -\infty \text{ uniformly, as } m_i \to \infty.$$

3. There exists a blow-up set $\mathcal{BS} = \{p_1, ..., p_m\} \subset \Omega$ and sequences $(x^j_{m_i})_{j\in\{1,...,m\}} \subset \Omega$ such that for $m_i \to \infty$, $x^j_{m_i} \to p_j$, $w_{m_i}(x^j_{m_i}) \to \infty$ for $j = 1...m$. Furthermore, on each compact subset $\mathcal{K} \subset \Omega \setminus \mathcal{BS}$ we have

$$\sup_{\mathcal{K}} w_{m_i} \to -\infty, \text{ as } m_i \to \infty$$

and

$$\frac{\alpha\chi}{k_c}e^{w_{m_i}+(\alpha\chi/4k_c)|x|^2} \to \sum_{j=1}^{m} 8\pi q_j \delta_{x=p_j}$$

in the sense of measure, where $q_j \in \mathbb{N}$.
(See [78] for the statement about the q_j.)

However as it has been done in [64] one can show that none of these alternatives is possible for such a sequence of stationary solutions. Therefore such a sequence cannot exist and one can conclude that there exists a constant $\hat{K} \in \mathbb{R}$ ($\hat{K} \leq 0$), such that for all radially symmetric stationary solutions (U, V) of system (4.1) $F(U, V) \geq \hat{K} > -\infty$ holds. Now let us choose a ε_0 arbitrary but fixed, such that $\hat{K} > \mathcal{F}(V_{\varepsilon_0}(x))$ where

$$V_{\varepsilon_0}(x) = \log\left(\frac{\varepsilon_0^2}{(\varepsilon_0^2 + \pi|x|^2)^2}\right) - \frac{1}{|\Omega|}\int\limits_{\Omega} \log\left(\frac{\varepsilon_0^2}{(\varepsilon_0^2 + \pi|x|^2)^2}\right) dx.$$

We see that $V_{\varepsilon_0}(x) \in W^{1,\infty}(\Omega)$. Now let us set

$$U_{\varepsilon_0}(x) = \frac{|\Omega| e^{V_{\varepsilon_0}(x)}}{\int\limits_{\Omega} e^{V_{\varepsilon_0}}(x) dx}.$$

We see that $U_{\varepsilon_0} \in L^{\infty}_{+}(\Omega)$ and that $F(U_{\varepsilon_0}(x), V_{\varepsilon_0}(x)) = \mathcal{F}(V_{\varepsilon_0}(x)) < \hat{K}$. Choosing $U_0(x) = U_{\varepsilon_0}(x)$ and $V_0(x) = V_{\varepsilon_0}(x)$ the corresponding solution of the Keller-Segel model (4.1) has to blow up in finite or infinite time.

Of course there are questions directly connected with the above results. The two most important are:

1. Is the blow-up time for the blow-up solution from Theorem 4.13 and Theorem 4.14 finite?
2. Suppose $T_{max} < \infty$, does either $\inf\limits_{0 \leq t < T_{max}} F(U(t), V(t)) > -\infty$ or $\lim\limits_{t \to T_{max}} F(U(t), V(t)) = -\infty$ hold?

H. Gajewski and K. Zacharias gave in [43, p. 94 & 95] an example for initial data for system (4.1) such that the solution blows up in the corner of a rhombic domain. They considered the domain

$$\Omega = \left\{ (x, y) \mid \frac{|x|}{a} + \frac{|y|}{b} < 1,\ a = \sqrt{\frac{\tan(\Theta/2)}{2}},\ b = \frac{1}{\sqrt{2\tan(\Theta/2)}} \right\}$$

with an acute opening angle $\Theta \leq \pi/2$. For $\alpha = \chi = k_c = 1$ they used the initial data $u_0(x) = U_0(x)/\overline{U_0(x)}$, where

$$U_0(x) = \frac{8(1+\sigma)}{\sigma} \exp\left(-\frac{|x|}{\sigma}\right)$$

with $0 < \sigma < 1$ and as $V_0(x)$ the solution of the boundary value problem

$$\Delta V_0 + \overline{U_0}\,(u_0 - 1) = 0 \text{ in } \Omega,\ \frac{\partial V_0}{\partial n} = 0 \text{ on } \partial\Omega.$$

The corresponding solution $(U(t), V(t))$ of the equations (4.1) blows up in finite time in a corner of the domain. Furthermore their numerical calculations showed that for this solution the Lyapunov functional $F(U(t), V(t)) \to -\infty$ in finite time. There is also another numerical example given in [42] where the initial data is such that the function $u(x, 0)$ already has its maximum in the corner with the smallest interior angle of the rhombic domain. The solution then blows up in this corner in finite time.

In Table 4.4 some known results with their references are summarized once again.

Table 4.4
Possible time asymptotical behaviour of the solutions of (3.3) and of (4.1).

Dimension	Observation	References
$N=1$	The solution of the Keller-Segel model exists globally in time and converges to a stationary solution as $t \to \infty$.	[109]
$N=2$	If $\alpha\chi \int_\Omega u_0(x)dx < 4\pi k_c$, then the solution exists globally in time and its L^∞-norm is uniformly bounded for all times. Furthermore it converges to a stationary solution as $t \to \infty$.	[11, 43, 61] and [90]
	If $4\pi k_c < \alpha\chi \int_\Omega u_0(x)dx < 8\pi k_c$, then there exist initial data such that the corresponding solution of the Keller-Segel model blows up at the boundary of Ω either in finite or in infinite time.	[61, 65, 131]
	If $8\pi k_c < \alpha\chi \int_\Omega u_0(x)dx$, then there exist initial data such that the corresponding solution of the Keller-Segel model blows up either in finite or in infinite time.	[51, 60, 64] and [65]
	Furthermore there exist radially symmetric initial data such that $u(t,x)$ forms a δ-singularity in finite time in the center of a disc Ω.	[51]
	Given a blow-up solution and an isolated blow-up point x_0, we have $0 < R << 1$, $m \geq m^*$, and $f \in L^1(B(x_0,P)\cap\Omega)\cap C(\overline{B(x_0,R)\cap\Omega}\setminus\{x_0\})$ satisfying $f \geq 0$ and $u(t,\cdot)dx$ converges weakly to $m\delta_{x_0}(dx) + f\ dx$ as $t \to T_{finite}$ in the set of Radon measures on $\overline{B(x_0,R)\cap\Omega}$, where m^* is either 8π, for $x_0 \in \Omega$ or 4π for $x_0 \in \partial\Omega$.	[96]
	If the blow-up time is finite and (4.12) holds there exist only isolated blow-up points.	
	If (3.3) is radially symmetric and $T_{max} < \infty$ then the set of blow-up point consists only of the origin$\{0\}$.	
	If $\Omega = \mathbb{R}^2$ and $\tilde{\alpha}\chi \int_{\mathbb{R}^2} u_0(x)dx < 4\pi k_c$ then the solution of (3.3) exists globally in time.	[95]
N=3	Let Ω be a smoothly bounded domain in $\mathbb{R}^3$. For sufficiently smooth initial data, satisfying the boundary data there exists a unique solution of (3.3) locally in time. Furthermore for all $T > 0$ there exists a constant C_T, such that if the initial data satisfies $\|v_0\|_{H^2(\Omega)} < C_T$, $\|u_0\|_{L^\infty(\Omega)} < C_T$ and $\|\nabla u_0\|_{L^2(\Omega)} < C_T$, then the problem (3.3) has a unique solution on $[0,T]\times\Omega$.	[16]

4.4. Results for related systems. There are some related results that should be mentioned at this point of this overview. P. Biler studied in [11] system (3.3) with different boundary conditions. For system (3.3) his local existence result is pretty much the same as the local existence result of Gajewski and Zacharias [43]. However for the system

$$\left.\begin{array}{rcll} u_t &=& \nabla(\nabla u - \tilde{\chi} u \nabla v), & x \in \Omega,\ t > 0 \\ 0 &=& k_c \Delta v - \gamma v + \tilde{\alpha} u, & x \in \Omega,\ t > 0 \\ 0 &=& \partial u/\partial n - \tilde{\chi} u \partial v/\partial n,\ x \in \partial\Omega,\ t > 0, & v(t,x) \ = \ K_\gamma * (\tilde{\alpha} u(t,x)), \\ u(0,x) &=& u_0(x), & x \in \Omega, \end{array}\right\} \tag{4.13}$$

where K_γ denotes the Bessel potential, Biler proves the following finite time blow up result:

THEOREM 4.15 (Biler). *If $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded star-shaped domain (with respect to the origin), then for $U_0(x)$ with sufficiently large $||u_0(x)||_{L^1(\Omega)} = M$, there is no global in time solution of* (4.13).

Results similar to those results in [63, 64] have been proven by G. Wolansky in [151] for the system

$$\left.\begin{array}{rcll} 0 & = & \nabla(\nabla u - \tilde{\chi} u \nabla v), & x \in \Omega,\ t > 0 \\ v_t & = & k_c \Delta v + \tilde{\alpha} u, & x \in \Omega,\ t > 0 \\ 0 & = & \partial u/\partial n - \tilde{\chi} u \partial v/\partial n,\ v(t,x)\ =\ 0, & x \in \partial\Omega,\ t > 0 \\ u(0,x) & = & u_0(x),\ v(0,x)\ =\ v_0(x) & x \in \Omega. \end{array}\right\} \tag{4.14}$$

This system contains an elliptic equation for the myxamoebae density and a parabolic equation for the cAMP-concentration similar to the second equation in (3.3). Even though he also has a Lyapunov function the techniques he used to prove his blow up result can only be applied to a system with Dirichlet boundary conditions for the second equation. They fail in the case of Neumann boundary conditions as treated in [63, 64]. In [151] G. Wolansky is led to an equation for the stationary solutions of his model which is similar to problem (3.6). However in his case the equation is equipped with Dirichlet boundary data, which allows him to use different arguments (more precisely the moving plane method see [31, pp. 521-522]) to exclude the first alternative of the Brézis and Merle [20] result. In the case that is mentioned in the previous section and [63, 64] one has to use different techniques to get rid of the possible alternatives stated by Brézis and Merle in [20].

5. Comparison of the questions asked by Jäger and Luckhaus with the results so far. Let us now take some time and let us see which questions of those asked in [67] have been answered up to now and which remain open. W. Jäger and S. Luckhaus asked about more information on the set of blow-up points. We have seen in the previous sections that there exists the possibility of blow-up point in the interior and at the boundary of a domain $\Omega \subset \mathbb{R}^2$. Also the upper bound of the possible number of blow-up points is sharp and known, however a better lower bound is still needed. Also the location of the boundary blow-up points should be studied more carefully. For smooth domains the boundary blow-up point should be the point of the greatest curvature which would correspond with the numerical calculations of H. Gajewski and K. Zacharias in [43] for piecewise smooth domains and with the results and hints from the steady state analysis resp. the shape of the least energy solutions.

The question whether the solution exists globally in time as a weak solution can be negated. However it might be possible to study the problem for a different formulation of a solution like L^1-solutions. But as far as I know there has not been any attempt to do so up to now. For the third question we turn to an own subsection.

5.1. What happens after blow-up?. In connection with the question "*What happens after the blow up of the solution?* " that was already asked in [67], we have seen the solution does not exist globally in time as a weak H^1-solution, but is there a notation of a measure valued solution or L^1-solution for the Keller-Segel model? With such a notation, which would be natural since the solution belongs to $L^1(\Omega)$ for all times, it would make sense to study the possible movement of the aggregation centers in the considered domain.

Using a different ansatz than that just mentioned, J.J.L. Velázquez made the first step to give an answer to the question what will happen after blow-up in [147, 148] using a different approach than the idea of introducing a new notation of a solution.

In [146] J. J. L. Velazquez studied the question wether aggregation at the interior of the boundary of the domain $\Omega \subset \mathbb{R}^2$ takes place in a stable manner, or, if on the contrary, solutions exhibit a tendency to move towards the boundary. His result is that after small pertubations of the solution found in [51], the new solution will blow up in a manner entirely similar but in a slightly shifted point of Ω at a slightly different time. Thus his computations indicate that the possibility of aggregates with high density of u moving quickly towards the boundary does not exist.

He then studies in [147, 148] the system

$$\left.\begin{array}{rcll} u_t & = & \nabla(\nabla u - G_\epsilon(u)\nabla v), & x \in \mathbb{R}^2, \quad t > 0 \\ 0 & = & \Delta v + u, & x \in \mathbb{R}^2, \quad t > 0, \end{array}\right\} \tag{5.1}$$

where $G_\epsilon(u) = \frac{1}{\epsilon} Q(\epsilon u)$ for a small parameter $\epsilon > 0$ and an increasing function $Q(\xi)$ satisfying $Q(s) = s + O(s^2)$ as $s \to 0$ and $Q(s) \sim L$ as $s \to \infty$, where $L > 0$ is a given number. For $Q(s) = \frac{s}{1+s}$ and $\epsilon = 0$ the system becomes formally the system (4.2). Using this system Velázquez studies the motion of regions with high densities of u in [147] and derives a system of equation that describes the dynamics of these regions of high densities of u. The well-posedness of the derived system for the dynamics of the high density regions is established in [148]. According to the formal limit as $\epsilon \to 0$ his results can therefore somehow be interpreted as a formal explanation of the behaviour of the solution of (4.2) after blow up.

6. More general formulations of the chemotaxis equations. The original formulation of the Keller-Segel model allowed more general functional forms than we assumed in the last section. Even though the question whether the given functional form represents the situation in Dictyostelium aggregation in an appropriate way should be discussed, the system is adequate to describe chemotactical movement of mobile species. A number of possible plausible functional forms has been proposed by E.F. Keller in [74]. There she proposed several functional forms that will also be discussed in the upcoming subsections. Furthermore she discussed the possible existence of traveling wave solutions, a topic which will be in the center of our observations later during this paper. Since there is a large number of different examples for species that move positive chemotactically and also a large variety of different models for the chemotactical sensing of the particular species (see for example [80] for a model of the cAMP production and sensing mechanism in Dictyostelium discoideum) it is useful to try to find a more general theory that contains a larger class of possible models. Let us see what results are available in this cases. So let us now turn to more general formulations of the system without having a particular example in mind. So we focus on the following system of two nonlinear parabolic partial differential equations, which is given by

$$\left.\begin{array}{rcll} u_t &=& \nabla(k(u,v)\nabla u - h(u,v)\nabla v), & x\in\Omega,\quad t>0 \\ v_t &=& k_c\Delta v - f(v)v + g(u,v), & x\in\Omega,\quad t>0 \end{array}\right\} \tag{6.1}$$

for $\Omega \subset \mathbb{R}^N$ completed with either

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega\times\{t>0\}, \tag{6.2}$$

$$\text{or } u = 0,\ v = 0 \text{ on } \partial\Omega\times\{t>0\}, \tag{6.3}$$

$$\text{or } k(u,v)\frac{\partial u}{\partial n} - h(u,v)\frac{\partial v}{\partial n} = 0,\ v = 0 \text{ on } \partial\Omega\times\{t>0\} \tag{6.4}$$

as boundary conditions and initial data $u(0,x) = u_0(x)$ and $v(0,x) = v_0(x)$ $x \in \Omega$. Here k_c once again is a positive constant. For the functions appearing in the model the following conditions have been considered to be reasonable: $k(r,s) > 0$ for all $(r,s) \in \mathbb{R}\times\mathbb{R}$, the function f satisfies $f(s) \geq$ const for all $s \in \mathbb{R}$ and $\frac{\partial}{\partial r}g(r,s) \neq 0$ holds for all $(r,s) \in \mathbb{R}\times\mathbb{R}$.

The question whether a solution to such problems exist locally in time has been studied in [61] using results by H. Amann [6, 7] and in [153] using other techniques. As it was mentioned in the previous section a Lyapunov function is a helpful tool for analyzing the time asymptotic behaviour of the solution. So one wonders under which conditions the present system has a Lyapunov function. Therefore we turn to this question next.

6.1. Lyapunov functions. For the rest of the present paper we will use the following notations:

$$F(v) := \int_0^v f(s)s\ ds \text{ and } G(u,v) := -\int_0^v g(u,s)\ ds.$$

At some places of the present paper we will assume that

$$\int_\Omega F(v)\ dx \geq k_1 \int_\Omega v^2\ dx \tag{6.5}$$

is true, where k_1 is a nonnegative constant (If we have homogeneous Neumann boundary data we assume $k_1 > 0$!).

Theorem 6.1 (Horstmann). *If there exists a function $R(u)$ such that*

$$\frac{h(u,v)}{k(u,v)}\left[\frac{\partial^2}{\partial u^2}G(u,v) + \frac{d^2}{du^2}R(u)\right] + \frac{\partial^2}{\partial u\partial v}G(u,v) = 0,$$

then there exists a Lyapunov function for system (6.1)*, provided $\frac{\partial^2}{\partial u^2}G(u,v) + \frac{d^2}{du^2}R(u) \geq 0$ holds true for the solution of* (6.1)*. In the case of boundary condition* (6.3) *we have to assume additionally that $\frac{\partial}{\partial u}G(0,0) = 0 = \frac{dR}{du}(0)$. The Lyapunov function for system* (6.1) *is then given by*

$$\mathcal{H}(u(t),v(t)) := \int_\Omega \frac{k_c}{2}|\nabla v(t)|^2 + F(v(t)) + R(u(t)) + G(u(t),v(t))dx.$$

A large number of examples is given in [61]. Let us here only give two examples for which a Lyapunov function $\mathcal{H}(u,v)$ exists.

1. Let consider (6.1) with $h(u,v) = u$, $g(u,v) = \frac{1}{2}u^2 e^{-v}$, $k(u,v) = 1$, $f(v)$ arbitrary. Then we have the Lyapunov function

$$\mathcal{H}(u(t),v(t)) := \int_\Omega \frac{k_c}{2}|\nabla v(t)|^2 + F(v(t)) + \frac{1}{2}u^2(t)e^{-v(t)}dx, \text{ i.e. } R(u) = u^2/2.$$

2. One can also find a whole class of other examples where a Lyapunov function exists. Let us suppose that we study system (6.1) together with (6.2). Let $h(u,v) = h_2(u)\phi(v)$ and $g(u,v) = \phi(v)\int_0^u h_1(s)\, ds$,

$$k(u,v) = \tilde{k}(u) + \frac{h_2(u)}{h_1(u)}\frac{d}{du}h_1(u)\int_0^v \phi(s)ds$$

and $f(v)$ be arbitrary. We see that there is a function $R(u)$ such that

$$\frac{d^2}{du^2}R(u) = \frac{k(u,v)h_1(u)}{h_2(u)} + \frac{d}{du}h_1(u)\int_0^v \phi(s)\, ds.$$

Of course we see in this example that the right hand side has to be independent of v. If this is the case, then there exists a Lyapunov function $\mathcal{H}(u,v)$ of the type given above, which is possibly unbounded from below. This example includes the systems studied in [43] and [115]. In [115] we have $h(u,v) = u\phi(v)$ (with $\phi(v) > 0$), $g(u,v) = u\phi(v)$, $k(u,v) = 1$ and $f(v) = \text{const} > 0$. Finally we get in this case $R(u) = u\log(u)$. For further results concerning some special cases of this type of systems see [115].

In fact this result allows to make statements for a larger class of nonlinearities in $g(u,v)$ than those studied before for system (6.1) (as far as the author knows). Under certain additional assumptions one can now formulate results for the time asymptotic behaviour of the solution. Therefore we now make the following main assumption for the rest of this section.

$$\textbf{Main assumption:} \int_\Omega G(u,v) + R(u) \geq k_2 \int_\Omega |\nabla v|^2\, dx + \text{const with } \frac{k_c}{2} + k_2 > 0. \tag{6.6}$$

In some special cases of (6.1) one can show that the solution of (6.1) converges to a possibly nontrivial steady state as $t \to \infty$ (see [43, 61] and [115]). The results of W. Alt [3], R. Schaaf [124] and K. Post [115] concerning the Keller-Segel model in chemotaxis seem to indicate that such behaviour can also be expected in a more general setting. The following theorem summarizes our results on this aspect.

Theorem 6.2 (Horstmann). *Suppose that $(u(t),v(t))$ is a weak solution of* (6.1) *and that* (6.5) *as well as our main assumption* (6.6) *is satisfied. Furthermore let either*

1. $\left[\frac{\partial^2}{\partial u^2}G(u,v) + \frac{d^2}{du^2}R(u)\right]/k(u,v) \leq k_3$ *and* $\frac{\partial}{\partial u}G(u,v) + \frac{d}{du}R(u) > k_4$ *or*
2. $0 < \left[\frac{\partial^2}{\partial u^2}G(u,v) + \frac{d^2}{du^2}R(u)\right]\exp\left(\frac{\partial}{\partial u}G(u,v) + \frac{d}{du}R(u)\right) \leq k_5 k(u,v)$ *and*

$$\sqrt{\frac{k(u(t),v(t))}{\frac{\partial^2}{\partial u^2}G(u,v) + \frac{d^2}{du^2}R(u(t))}} \in L^2(\Omega)$$

for all $t \geq 0$

be true for the solution $(u(t),v(t))$ of (6.1)*. Let additionally f be Hölder continuous with Hölder exponent $\beta \leq 1$ such that $0 < \beta \leq 1$ if $N \leq 3$ or $\beta < 2/N$ if $N > 3$. Finally assume that $|f(v)| \leq K_f$ for all $v \in \mathbb{R}$. Then there exist a sequence $(t_k)_{k\in\mathbb{N}}$ and two functions v^* and g^* such that $v(t_k) \rightharpoonup v^*$ in $H^1(\Omega)$ resp. in $H^1_0(\Omega)$), $f(v(t_k))v(t_k) \to f(v^*)v^*$ in $L^2(\Omega)$ and $g(u(t_k),v(t_k)) \rightharpoonup g^*$ in $L^2(\Omega)$. Furthermore*

$$\int_\Omega k_c \nabla v^* \nabla\varphi + f(v^*)v^*\varphi\, dx = \int_\Omega g^*\varphi\, dx$$

for all $\varphi \in H^1(\Omega)$ *(resp.* $\varphi \in H_0^1(\Omega)$*). Finally we see that*

$$\exp\left(\frac{-\left[\frac{\partial}{\partial u}G(u(t_k),v(t_k))+\frac{d}{du}R(u(t_k))\right]}{2}\right)\to \text{const}$$

in $L^2(\Omega)$ *if 1. holds and, respectively,*

$$\exp\left(\frac{\left[\frac{\partial}{\partial u}G(u(t_k),v(t_k))+\frac{d}{du}R(u(t_k))\right]}{2}\right)\to \text{const}$$

in $L^2(\Omega)$ *if 2. holds.*

The previous given first example satisfies assumption 1. of Theorem 6.2 while the systems studied in [43] and [115] satisfy assumption 2. of Theorem 6.2. The proof of this theorem goes along the line of the proof of Theorem 5.2 in [43, page 107] and can be found in detail in [61].

Furthermore one can also formulate certain conditions under which some L^p-estimates for the solution are possible. This has also been done in [61].

More general forms of the Keller-Segel model (2.4) have also been studied by A. Yagi in [153] for the case of two space dimensions and in the case of one spatial dimension by K. Osaki and A. Yagi in [109]. In [109] the authors study the Keller-Segel model (2.4) with $k_1(u,v) = const$, $g(u,v) = \alpha u - \gamma v$ and $k_2(u,v) = u\chi(v)$ where $\chi(s)$ is a smooth function of $s \in (0,\infty)$ satisfying

$$|\chi^{(i)}(s) < | \leq const \cdot \left(s + \frac{1}{s}\right)^r,$$

for $0 < s < \infty$, $i = 0,1,2$ with some positive constant and exponent r. In [109] they show that there exists a compact set of finite fractal dimension which attracts the solutions exponentially.

A. Yagi studied in [153] the two dimensional case of the Keller-Segel model under the assumptions that $k_1(u,v) = c_0 + c_1u + c_2v$ with a positive constant $c_0 > 0$ and non-negative constants c_1, c_2, and assuming that $k_2(u,v) = u\chi(v)$ with $0 \leq \chi(s) \leq b_0\left(1+\frac{1}{s}\right)$, and that $\chi(s)$, $k_3(s)$, $f(s)$ are smooth functions of $s \in \mathbb{R}^+$ satisfying $0 \leq f(s) \leq b_1$, $b_2 \leq k_3(s) \leq b_3\left(s^{p_0}+1\right)$, where b_0, b_1, b_2, b_3 are positive constants strictly larger than zero and the exponent $p_0 > 0$. For the initial data he assumed that $u_0(x) > 0$ on $\overline{\Omega}$, $v_0(x) \geq \mu_0 > 0$ on $\overline{\Omega}$ belong to $H^{1+\epsilon_0}(\Omega)$ with some exponent $0 < \epsilon_0 \leq 1$ and a positive constant μ_0. Using semigroup theory Yagi established the local in time existence of an unique, positive, classical solution in the same space as it has already been mentioned in the case of a linear sensitivity function in a previous section. Furthermore he determines blow-up norms of the maximal solution.

6.1.1. Results on finite time blow-up. As it has been already mentioned in the section on the steady state solutions of the Keller-Segel model Nanjundiah's conjecture also contained a statement on the time asymptotical behaviour of the solution of the Keller-Segel model with a logarithmic chemotactical sensitivity function. Therefore we will first look at the results for this conjecture and related results. In this subsection we consider (2.4) with $k_1(u,v) = 1$, $g(u,v) = \frac{1}{\varepsilon}(u-v)$, $k_c = \frac{1}{\varepsilon}$ and $k_2(u,v)$ either $\chi\frac{u}{v}$ or $\chi p u v^{p-1}$ for $p > 0$. For $\varepsilon = 0$ it is easy to show in the same way as it has been done in the case of a linear chemotactical sensitivity function that the solution exists globally in time and that the L^∞-norm of the solution is uniformly bounded for all times. Thus the interesting cases are once more the higher dimensional ones. So let us summarize these cases:

1. Let $k_2(u,v) = \chi p u v^{p-1}$ for $p > 0$.
   (a) Let $N = 2$ and $\varepsilon = 0$. If $0 < p < 1$, then the solution of the Keller-Segel model exists globally in time and is uniformly bounded. If Ω is a disk, u_0 is radially symmetric, $\int_\Omega u_0(x)|x|^2dx$ is sufficiently small and $p > 1$, then the corresponding solution of (2.4) blows up in finite time.(See [88, 89, 126].)
   (b) Let $N \geq 3$ and $\varepsilon = 0$. If Ω is a disk, u_0 is radially symmetric, $\int_\Omega u_0(x)|x|^{(N-2)p+2}dx$ is sufficiently small and $p > 0$, then $T_{max} < \infty$ and the corresponding solution of (2.4) blows up in finite time.(See [88, 89, 126].)
2. Let $k_2(u,v) = \chi\frac{u}{v}$.

(a) Let $N = 2$ and $\varepsilon = 0$. If Ω is a disk, u_0 is radially symmetric, then the solution is globally bounded in time. (See [88, 89, 126].)
(b) Let $N \geq 3$ and $\varepsilon = 0$. If Ω is a disk, u_0 is radially symmetric and $\chi < 2/(N-2)$, then the solution is globally bounded in time. (See [88, 89, 126].)
(c) Let $N = 2$ and $\varepsilon = 1$. If $\chi < 1$, then the solution of (2.4) exists globally in time and for $T > 0$ there exists a constant $C_T < \infty$ such that

$$\sup_{0 \leq t \leq T} \left(||u(t,\cdot)||_{L^\infty(\Omega)} + ||v(t,\cdot)||_{L^\infty(\Omega)} \right) < C_T.$$

(See [91].)
(d) Let $N = 2$ and $\varepsilon = 1$. If Ω is a disk, the initial data $(u_0(x), v_0(x))$ is radially symmetric and $\chi < 5/2$, then the solution exists globally in time. (See [91].)
(e) Let $N \geq 3$ and $\varepsilon = 0$. If Ω is a disk, u_0 is radially symmetric, $\int_\Omega u_0(x)|x|^2 dx$ is sufficiently small and $\chi > 2N/(N-2)$, then the solution of (2.4) blows up in finite time. (See [88, 89, 126].)

Similar results have been obtained by K. Post studying (2.4) with $k_1(u,v) = 1$, $k_2(u,v) = u\phi'(v)$ $g(u,v) = u\phi'(v) - v$ and $k_c = 1$. For the precise details I refer the reader to [115].

Rascle and Ziti analyzed in [118] the system

$$\left.\begin{array}{rcll} u_t & = & \nabla(\mu \nabla u - \chi u v^{-\beta} \nabla v), & x \in \Omega,\ t > 0 \\ v_t & = & -kuv^m, & x \in \Omega, \end{array}\right\}$$

where the constants χ,$k > 0$. They constructed self-similar solutions for this system assuming that $m < \beta = 1$. For $\mu = 0$ and one space dimension they observed that the bacterial density concentrated in finite time at the origin. For two space dimensions and initial data for the bacterial density which is zero at the origin they derived chemotactic rings concentrated around the origin after finite time. In higher space dimensions they achieved blow-up of the solution by an initial singularity of the chemoattractant in the origin.

For $\mu > 0$ Rascle and Ziti observed in one space dimension that there are smooth initial data leading to finite time blow-up of the solution, while they were unable to construct self-similar solutions in space dimension larger or equal to two for reasonable initial conditions.

6.1.2. Prevention of overcrowding. There are different points of view whether blow-up in chemotaxis is relevant or not. In fact for the chemotaxis system introduced and derived in the Davis' case by Othmer and Stevens for a single particle in [112] blow-up in finite time seems to correspond with the fact that the particle is trapped respectively localizes in finite time at one particular place (see also [135] for more comments on that aspect). Thus blow-up really makes sense for their model. Furthermore, blowing up of the solution only describes the concentration of the particle populations in some aggregation centers. Since the Keller-Segel model only wants to describe the aggregation phase of chemotactical movement and not the formation of a fruiting body the blow-up question is surely worth studying. As J. J. L. Velázquez wrote in [147, pp. 1-2]:

> "*Blow-up usually takes place in physical or biological models if they are approximations of more realistic models, usually containing small parameter (say $\epsilon > 0$), that cannot exhibit singular behaviours unless this parameter is set to zero. Suppose that for $\epsilon = 0$ the limit problem can develop singularities in finite time. The behaviour of the complete model for $\epsilon > 0$ usually is similar to that of the limit model away from the singularities. However, the features of the problem with $\epsilon > 0$ but small are usually very different from that of the limit problem near the singularities. The presence of blow-up just indicates that the approximations that lead to the simpler model where blow-up takes place are not valid anymore near the singularity and that the whole dynamics of the complete model needs to be taken into account here.* "

However there are also other models of Keller-Segel type which exclude the possibility of blow-up solutions directly by introducing some mechanisms, that provide to strong aggregations or where the chemical production and decay directly is such that blow-up is impossible. For example in the case of the linear chemotactical sensitivity function $g(u(t,x), v(t,x)) \in L^4(\Omega)$ for all $t \geq 0$ with an uniform bound for all $t \geq 0$ guarantees the global existence of the solution of system (6.1) (see for example [61]). One model containing such prevention of an overcrowding of the chemotactical species has been proposed by T. Hillen and K. J. Painter in [56]. They considered the system (6.1) on a C^3-differentiable, compact Riemannian manifold $\mathcal{M}$ under the assumptions that $k(u,v) = 1$, $h(u,v) = u\beta(u)\chi(v)$ where β, χ are three times continuous differentiable functions satisfying $\chi > 0$, $\beta(0) > 0$ and there exists a $\overline{u} > 0$ such that $\beta(\overline{u}) = 0$ and $\beta(u) > 0$ for $0 < u < \overline{u}$. They assumed that the function $f(v) \equiv 0$ and that $g(u,v) = g_1(u,v)u - g_2(u,v)v$ is twice continuously differentiable with a bounded death rate $g_2 \geq \delta > 0$ and a birthrate $g_1 \geq 0$. In their paper they prove the global existence of the solution in this case and present numerical simulations for the time evolution of the system in one and two space dimensions. They also show the potential pattern variety of the final steady state patterns for their version of model (6.1).

6.1.3. Chemotaxis equations with population growth. An extremely large number of models describing chemotactical movement for species in a reproduction stage can be found in the literature. In general they are based on some version of the Keller-Segel equations with an additional growth term in the first equation. For example A. Bonami, D. Hilhorst, E. Logak and M. Mimura consider in [13, 14] the following versions of the classical model:

$$\left.\begin{array}{rcll} u_t & = & \nabla(k_1(u)\nabla u - u\nabla\chi(v)) + f(u), & x \in \Omega,\ t > 0 \\ \varepsilon v_t & = & k_c\Delta v - \gamma v + \tilde{\alpha}u, & x \in \Omega,\ t > 0 \end{array}\right\} \tag{6.7}$$

where $k_1(u) = 1$ and $f(u) = u(1-u)(u-a)$ with a constant $0 < a < 1$ (see also [48] for results related to this system). However also different functional forms for $f(u)$ are thinkable. For example in [17] one finds the proposed functional form $f(u) = au$ for a positive constant $a > 0$ and in [29] system 6.7 is studied with $k_1(u) = u^m$ and $f(u) = u(1-u^p)$ with $m > 1$ and $p \geq 1$. Some effects of such growth terms on the various possible patterns that one can observe during the evolution of the solution will be mentioned in Section 8.5 of the present paper.

For some results on a more general chemotaxis growth model we refer the interested reader to [150]. There X. Wang studied in one space dimension the steady state solutions of the system

$$\left.\begin{array}{rcll} u_t & = & \nabla(\lambda\nabla u - \chi u\nabla\phi(v)) + (kf(u) - \theta - \beta v)v, & x \in [0,1],\ t > 0 \\ v_t & = & \Delta v - f(u)v, & x \in [0,1],\ t > 0 \\ u_x & = & \chi u(\phi(v))_x \text{ at } x = 0, 1,\ v_x(0) \ = \ 0,\ v_x(1) = h(1-u(1)), & \end{array}\right\} \tag{6.8}$$

where λ, k, θ, h and β are positive constants and $\chi \geq 0$ for different possible growth terms $f(u) \in C^3([0,\infty))$ and chemotactic sensitivity functions $\phi(v) \in C^5([0,\infty))$ satisfying $f(0) = 0$, $f'(s) > 0$ and $\phi'(s) > 0$ for $s \in [0,\infty)$. He also proves the global existence and boundedness of the solution for those different growth factors for the population density.

7. The comparison principles by W. Alt for chemotaxis equations. In his (unfortunately almost unknown) Habilitation [3] from 1980 Wolfgang Alt studies quasilinear parabolic and elliptic systems including the chemotaxis equations by Keller and Segel with and without growth terms and for single and many species populations. I restrict myself to mention only some results from the very nice and interesting work from 1980 although more general results might hold and are shown in [3]. However I present W. Alt's results in an own separated section, since it is a little bit difficult to get this reference.

The time-dependent Keller-Segel system is included in the class of quasilinear parabolic cross-diffusion systems and the steady state problem belongs to the class of quasilinear elliptic systems. Important tools in the studies of elliptic and parabolic equations of second order are comparison and maximum principles to prove qualitative properties of the solution like boundedness or blow-up phenomena of the solution by constructing suitable super- and subsolutions for the considered problems. Also for existence results for elliptic and parabolic problems comparison principles have been used to apply Perron's method. Wolfgang Alt presents such comparison principles in [3] which also hold for coupled systems of the following general form:

$$B_0(\mathbf{u}(y))D_0\mathbf{u}(y) = \sum_{i,j=1}^{n}\frac{\partial}{\partial y_i}\left(a_{i,j}(y)A(\mathbf{u}(y))\frac{\partial}{\partial y_j}\mathbf{u}(y)\right) - \sum_{k=1}^{l}B_k(\mathbf{u}(y))\sum_{\mu=1}^{m}b_{k,\mu}(y)D_\mu\mathbf{u}(y) + C(y,\mathbf{u}(y),D\mathbf{u}(y)) + F(y,\mathbf{u}(y)) \tag{7.1}$$

on a domain Λ in $\mathbb{R}^d$ and for a function $\mathbf{u} \in C^2(\Lambda, \mathbb{R}^M)$ resp. a distribution $\mathbf{u} \in \mathcal{D}'(\Lambda, \mathbb{R}^M)$, where

1. The notations

$$D_\mu\mathbf{u}(y) := X_\mu(y)\cdot\nabla\mathbf{u}(y) \text{ and } Du(y) := (D_1\mathbf{u}(y), ..., D_m\mathbf{u}(y)) \in \mathbb{R}^{m,M}$$

are used for vectorfields $X_\mu = (\beta_1^\mu, ..., \beta_n^\mu) \in C^{1,1}_{loc}(\overline{\Lambda}, \mathbb{R}^M)$ $(\mu = 0, 1, ..., m)$ and $y \in \overline{\Lambda}$.

2. The $a_{i,j}$ are defined as $a_{i,j}(y) = \sum_{\nu,\mu=1}^{m} a_{\nu,\mu}(y)\beta_i^\mu(y)\beta_j^\mu(y)$ with continuous functions $a_{\mu,\nu} = a_{nu,\mu}$ and $(a_{\mu,\nu}) \geq 0$ on $\overline{\Lambda}$, and satisfy for all $y \in \overline{\Lambda}$ and all $\xi \in \mathbb{R}^m$ the inequality

$$\sum_{\nu,\mu=1}^{m} \xi_\nu \xi_\mu a_{\nu,\mu}(y) \geq \zeta(y)|\xi|^2$$

with a on $\overline{\Lambda}$ lower semicontinuous positive function ζ.
3. The functions $b_{i,\mu}$ are continuous on $\overline{\Lambda}$.
4. The matrix-functions $A, B_i : \mathbb{R}^M \to \mathbb{R}^{M\times M}$ are continuous and $\det(A) \geq 0$ and $B_0 \geq 0$ on $\mathbb{R}^M$.
5. $F : \overline{\Lambda} \times \mathbb{R}^M \to \mathbb{R}^M$ and $C : \overline{\Lambda} \times \mathbb{R}^M \times \mathbb{R}^{m,M} \to \mathbb{R}^M$ are measurable in $y \in \Lambda$, continuous in $z \in \mathbb{R}^M$ and $w \in \mathbb{R}^{m,M}$, and uniformly bounded on $\overline{\Lambda} \times \mathcal{K}$, with a compact set $\mathcal{K} \subset \mathbb{R}^M$. For example the function C can be given by $\mathbb{R}^M$-valued bilinear forms $C_{\mu,\nu}$ like:

$$C(y, z, w) = \sum_{\mu,\nu=1}^{m} C_{\mu,\nu}[w^\mu, w^\nu]$$

with $w = (w^1, ..., w^m) \in \mathbb{R}^{M,m}$.

Although W. Alt's results hold for systems in this general setting I restrict myself to systems appropriate to model chemotaxis and present his results if possible in versions for those problems. From the applicational point of view one would like to know whether the considered model remains bounded for all times or not. Thus the existence of invariant sets for the system is an interesting topic worth studying. Alt presents such results in his Habilitation. Therefore the first result presented here is the following invariance theorem for parabolic systems (see [3, Satz 1.25, page 31 & 32]):

Theorem 7.1 (Alt). *Let $p \geq 0$ be in $C^{1,1}_{loc}(\mathbb{R}^M)$ and let $\mathcal{M}$ be defined as the set $\mathcal{M} := \{z \in \mathbb{R}^M \mid p(z) = 0\} \neq \emptyset$. Furthermore let us assume that $\mathbf{u}$ is a weak solution of the parabolic problem*

$$\begin{array}{rcll} \mathbf{u}_t &=& \sum_{j=1}^{N} \frac{\partial}{\partial x_j}\left(A(\mathbf{u})\frac{\partial}{\partial x_i}\mathbf{u}\right) + F(\mathbf{u}), & in\ \Omega \times (0,\tau)\ (\Omega \subset \mathbb{R}^N \\ A(\mathbf{u})\frac{\partial \mathbf{u}}{\partial n} &=& \psi(\mathbf{u}), & on\ \partial\Omega_N \times [0,\tau] \\ \mathbf{u}(x,0) &=& \mathbf{u}_0(x), & on\ \partial\Omega_D \times [0,\tau], \end{array}$$

where $\partial\Omega_N$ and $\partial\Omega_D$ denote disjunct subsets of $\partial\Omega$. For the boundary conditions we assume that there exists a continuous family of symmetric $M \times M$-matrices $A^*(z) \geq 0$, $z \in \mathbb{R}^M$ and that there exist vector functions $\theta_j : \partial\Omega_N \times \mathbb{R}^M \to \overline{\mathbb{R}}$ such that

$$\psi_j(x,z) = A^*(z)\theta_j(x,z) \ and\ \theta_j^T(x,z)\cdot A^*(z)\cdot\theta_j(x,z) \leq const_{\mathcal{K}}$$

for all z in a compact subset $\mathcal{K}$ of $\mathbb{R}^M$ and $x \in \partial\Omega_N$, where $\psi_0 = \psi$ and $\psi_j = \frac{\partial}{\partial z_j}\psi$, for $j = 1, ..., M$. Furthermore we suppose that there exists a neighbourhood $\mathcal{U}$ of $\mathcal{M}$ such that

$$\nabla p \cdot \psi \leq 0,\ \nabla p \cdot F \leq 0\ as\ well\ as\ \nabla^2 p \cdot A \geq 0\ holds\ on\ \mathcal{U} \setminus \mathcal{M},\ if\ \nabla^2 p\ exists.$$

Then $\mathbf{u}(\cdot, 0) \subset \mathcal{M}$ implies $\mathbf{u}(\cdot, t) \subset \mathcal{M}$ for all $t \in [0,\tau]$.

But not only the question whether the solution remains in a certain set for all times if the initial data belongs to this set is an interesting question. One would like to know more about the behaviour of the solution as $t \to \infty$. As demonstrated in the previous section the existence of Lyapunov functionals is a helpful tool in the studies of the time asymptotic as $t \to \infty$. In [3] W. Alt presents results for the existence of Lyapunov functionals which is different from the results of Theorem 6.1 of this paper. To be precise W. Alt proved the following Corollary:

Corollary 7.2 (Alt). *Assume that the assumptions of Theorem 7.1 are fulfilled. The Dini-derivative of the functional*

$$E(t) := \int_\Omega p(\mathbf{u}(t,x))dx,\ \ t \in [0,\tau]$$

satisfies for all weak solution of problem (7.1) *which have values in* $\mathcal{U}$ *the inequality*

$$\frac{d}{dt}E(t) \le -\int_\Omega \sum_{j=1}^N \left(\frac{\partial}{\partial x_j}\mathbf{u}\right) \nabla^2 p(\mathbf{u}) A(\mathbf{u}) \left(\frac{\partial}{\partial x_j}\mathbf{u}\right) dx + \int_\Omega \nabla p(\mathbf{u}) \cdot F(\mathbf{u}) dx + \int_{\partial\Omega} \nabla p(\mathbf{u}) \cdot \psi(\mathbf{u}) dS$$

almost everywhere in $[0,\tau]$. *If for all* $z \in \mathcal{U} \setminus \mathcal{M}$ *either* $\nabla p \cdot \psi < 0$ *or* $\nabla p \cdot F < 0$ *holds, then* E *is a Lyapunov functional for* $\mathcal{M}$, *i.e.* $\frac{d}{dt}E(t) < 0$ *as long as* $E(t) > 0$. *If for all* $z \in \mathcal{U} \setminus \mathcal{M}$ *either* $\nabla p \cdot \psi < 0$ *or* $\nabla p \cdot F < 0$ *or* $\nabla^2 p \cdot A < 0$ *holds, then* E *is a Lyapunov functional for* $\mathcal{M} \cup \mathcal{N}_0$, *where* $\mathcal{N}_0$ *contains all constants* $z_0 \in \mathbb{R}^M$, *that are zero of* $\nabla p \cdot \psi$ *and* $\nabla p \cdot F$.

Now, let me demonstrate this results by applying Theorem 7.1 to an example (see [3, Beispiel 1.41, page 38]):

Consider the weak solution of the taxis-system

$$\begin{array}{rcll} u_t &=& \nabla(k_1(u)(\nabla u - k_2(u,v)\nabla v)) + f(u,v), & \text{in } \Omega \times \{\text{t} \ge 0\} \\ v_t &=& k_c \Delta v + g(u,v), & \text{in } \Omega \times \{\text{t} \ge 0\} \\ 0 &=& k_1(u)\frac{\partial u}{\partial n} - h(u,v) & \text{on } \partial\Omega \times \{\text{t} \ge 0\} \\ 0 &=& \frac{\partial v}{\partial n}, & \text{on } \partial\Omega \times \{\text{t} \ge 0\}, \end{array} \tag{7.2}$$

where all coefficient functions are continuous and $k_1(u), k_c \ge 0$, $k_1(u) = 0$ for $u \ge \overline{u}$. Furthermore let $f(u,v), h(u,v) \le 0$ for $u \ge \overline{u}$ and all $v \in \mathbb{R}$ and on every compact subset of $\mathbb{R}^2$ the inequalities

$$|h(u,\cdot)|^2, |\frac{\partial}{\partial u}h(u,\cdot)|^2, |\frac{\partial}{\partial v}h(u,\cdot)|^2 \le k_1(u)$$

hold and assume that the initial data (u_0, v_0) satisfy $u_0 \le \overline{u}$.

We see that this system satisfies the conditions of Theorem 7.1 with

$$\mathcal{M} = \{(u,v) \in \mathbb{R}^2 \mid u \le \overline{u}\},\ p(u,v) = \frac{1}{2}\left(\max\{0, u - \overline{u}\}\right)^2 \text{ and } A(u,v) = \begin{pmatrix} k_1(u) & -k_1(u)k_2(u,v) \\ 0 & k_c \end{pmatrix}.$$

Now we have to check the conditions for the boundary data $\psi = (h,0)^T$. We set

$$A^* = \begin{pmatrix} k_1 & 0 \\ 0 & \frac{1}{2}(k_c + k_1 k_2^2) \end{pmatrix}.$$

Thus we have $\psi = A^*\theta$ with $\theta = (h/k_1, 0)^T$. We see that $\theta^T A^* \theta$ can be bounded uniformly and that the analoguous staements also hold for the partial derivatives $\frac{\partial}{\partial u}\psi$ and $\frac{\partial}{\partial v}\psi$. Thus we conclude with Theorem 7.1 the following:

If the initial data (u_0, v_0) satisfy $u_0 \le \overline{u}$, then the solution of system (7.2) satisfies $u(\cdot,t) \le \overline{u}$ for all $t \ge 0$.

In contrast to this result for weak solutions the next Theorem [3, Satz 2.40] is a strong (local) comparison principle for classical solutions of system (7.1). Before citing this result we have to introduce "*sets of comparison*" and "*comparison surfaces*".

Definition 7.3. *For* $i = 1,..,I$ *and a domain* $\mathcal{V} \subset \mathbb{R}^M$ *let the functions* ϱ_i *belong to the class* $C^{1,1}_{loc}(\mathcal{V}, \mathbb{R}^M)$, *where* ϱ_i *is piecewise of class* $C^{2,1}_{loc}(\mathcal{V}, \mathbb{R}^M)$ *and the functions* ς_i *belong to the class* $C^{1,1}_{loc}(\overline{\mathcal{V}}, \mathbb{R})$. *Then we define for each* $y \in \overline{\mathcal{V}}$ *the sets of comparison* $\mathcal{M}_y := \{z \in \mathcal{V} \mid \varrho_i(z) \le \varsigma_i(z),\ i = 1, ..., I\}$ *and for* $i = 1, ..., I$ *the* ϱ_i*-boundary* $\partial_i \mathcal{M}_y := \{z \in \partial\mathcal{M}_y \subset \mathcal{V} \mid \varrho_i(z) = \varsigma_i(z)\}$.

Definition 7.4. *Let* $(\mathcal{V}_y)_{y \in \Lambda}$ *denote a continuous family of sets* $\mathcal{V}_y \subset \mathcal{V} \subset \mathbb{R}^M$, *with* $\mathcal{V}$ *as in the previous definition. Furthermore let there be given functions* $\varrho \in C^{1,1}_{loc}(\overline{\mathcal{V}})$ *and* $\varsigma \in C^{1,1}_{loc}(\Lambda)$ *such that for each compact set*

$K \subset \mathbb{R}^M$ and for each relatively compact set $\Lambda^* \subset \Lambda$ there exist positive constants c and δ and a continuous family of projections

$$\pi_y : \mathcal{M}_y^\delta := \mathcal{V}_y \cap K \cap \{\varsigma(y) - \delta \leq \varrho \leq \varsigma(y)\} \to \mathcal{V}_y \cap \{\varrho = \varsigma(y)\}$$

with

$$|\pi_y(z) - z| \leq c|\varrho(z) - \varsigma(y)| \text{ and } \nabla\varrho(z) \neq 0$$

for all $y \in \Lambda^*$ and $z \in \mathcal{M}_y^\delta$. Furthermore let there be a finite number of sets $\mathcal{V}^\kappa$ with $\mathcal{V} = \bigcup_{\kappa=1}^{k_0} \mathcal{V}^\kappa$ such that $\varrho|_{\overline{\mathcal{V}^\kappa}} \in C^{2,1}_{loc}(\overline{\mathcal{V}^\kappa})$ and

$$\pi_y(\overline{\mathcal{V}^\kappa} \cap \mathcal{V}_y) \subset \overline{\mathcal{V}^\kappa}$$

for all $\kappa = 1, .., k_0$ and $y \in \Lambda^*$ is satisfied. Then we will call the surfaces $\mathcal{V}_y \cap \{\varrho = \varsigma(y)\}$ surfaces of comparison.

Now let us turn to the strong comparison result for the equation

$$\begin{aligned} B_0(\mathbf{u}(y))D_0\mathbf{u}(y) = A(\mathbf{u}(y)) \sum_{\nu,\mu=1}^{m} a_{\nu,\mu}(y) D_\nu D_\mu \mathbf{u}(y) - \sum_{i=1}^{l} B_i(\mathbf{u}(y)) \sum_{i,\mu=1}^{m} b_{i,\mu}(y) D_\mu \mathbf{u}(y) \\ + \sum_{\nu,\mu=1}^{m} C_{\nu,\mu}(y, \mathbf{u}(y))(D_\mu \mathbf{u}(y) \cdot D_\nu \mathbf{u}(y)) + F(y, \mathbf{u}(y)) \end{aligned} \tag{7.3}$$

where we additionally assume that A, B_i, F and $C_{\nu,\mu}$ are locally Lipschitz continuous in $\mathbf{u}$ and uniformly continuous in $y \in \overline{\Lambda}$, and that the functions $b_{i,\mu}$ are bounded on $\overline{\Lambda}$:

Theorem 7.5 (Alt). *Let us assume that $(\mathcal{M}_y)_{y \in \overline{\Lambda}}$ is a family of sets of comparison. Additionally we assume that for each $z \in \mathcal{V}$, for which $y \in \overline{\Lambda}$ and $i \in \{1, ..., I\}$ exist such that $z \in \partial_i \mathcal{M}_y$, the following properties are satisfied:*

$$\nabla\varrho_i(z) \neq 0, \ \nabla\varrho_i(z) \notin \sum_{s \neq i, z \in \partial_s \mathcal{M}_y} \mathbb{R} \cdot \nabla\varphi_s(z)$$

and the points of discontinuity of $\nabla^2 \varrho_i$ lie on a finite system of smooth surfaces in $\mathcal{V}$, which intersect the surface $\{z \in \mathcal{V} \mid \varrho_i(z) = \varsigma_i(z)\}$ in each such $z \in \partial_i \mathcal{M}_y$ transversally. Let us suppose that for each pair (ϱ_i, ς_i) and the families of sets $(\mathcal{V}_y^i)_{y \in \Lambda}$ with

$$\mathcal{V}_y^i := \{z \in \mathcal{V} \mid \varrho_l(z) \leq \varsigma_l(z), \ l \in \{1, ..., I\} \setminus \{i\}\}$$

there exist continuous functions ζ_i and ι_i on $\Lambda \times \mathcal{V}$ such that the following three properties are satisfied on $\partial_i \mathcal{M}_y$, $y \in \Lambda$:

1. *There are continuous functions λ^i, λ^i_0, ..., λ^i_s on $\mathcal{V}$ such that for all $z \in \mathcal{V}_y$ with $\varrho_i(z) = \varsigma_i(y)$ and $y \in \Lambda$ such that*

$$\nabla\varrho_i(z) \cdot A(z) = \lambda^i(z)\nabla\varrho_i(z), \ \lambda^i(z) > 0 \tag{7.4}$$
$$\nabla\varrho_i(z) \cdot B_0(z) = \lambda^i_0(z)\nabla\varrho_i(z), \ \lambda^i_0(z) > 0 \tag{7.5}$$
$$\nabla\varrho_i(z) \cdot B_l(z) = \lambda^i_l(z)\nabla\varrho_i(z) + p_l(z), \ \lambda^i_l(z) \in \mathbb{R} \tag{7.6}$$
$$\tag{7.7}$$

   and $p_l(z) \in [\nabla\varrho_i(z)]^\perp \subset \mathbb{R}^M$, $l = 1, ..., s$.
2. *Let $\mathcal{C}(y, z) \subset \mathbb{R}^{m,M}$, $(z \in \mathcal{V})$ denote for every $y \in \Lambda$ a (uniform in $y \in \Lambda^*$) locally Lipschitz continuous, given family of sets. The matrix $W(y) \in \mathbb{R}^{m,m}$ denotes the positive definite root of the symmetric matrix $(a_{\nu,\mu}(y))$ and $\wp_i$ denotes for each $i \in \{1, .., I\}$ a continuous vector field on $\mathcal{V}$ such that $\wp_i(z) \cdot \nabla\varrho_i(z) = 1$ for all $z \in \mathcal{V}$. Furthermore let us assume that for all $y \in \Lambda$, $z \in \mathcal{V}_y$ with $\varrho_i(z) = \varsigma_i(y)$ and*

$$w \in P_z(W(y) \cdot \mathcal{C}(x, z)) := ((W(y) \cdot \mathcal{C}(x, z)) - ((W(y) \cdot \mathcal{C}(x, z)) \cdot \nabla\varrho_i(z))\wp_i(z))$$

$$\lambda^i(z)\sum_{k=1}^{m}\min_{\kappa=1,\dots,k_0\ ;\ z\in\overline{\mathcal{V}^\kappa}}\left\{w_k\cdot\nabla^2\varrho_i(z)\cdot w_k\right\}$$
$$+\sum_{j=1}^{l}\sum_{k=1}^{m}\left(\sum_{\mu=1}^{m}b_{j,\mu}(y)W^{-1}_{\mu,k}(y)\right)w_j(z)\cdot w_k$$
$$-\sum_{\nu,\mu,k,l=1}^{m}W^{-1}_{\mu,k}(y)W^{-1}_{\nu,l}(y)\nabla\varrho_i(z)\cdot C_{\mu,\nu}(y,z)|w_k,w_l]\geq\zeta_i(x,z)|w|^2-\iota_i(y,z) \tag{7.8}$$

3. *Let $\mathcal{S}(y_0,\Lambda)$ denote the set of all points $y\in\Lambda$ such that there exists a continuous, piecewise $C^{1,1}$ curve γ with $\gamma(0)=y_0$ and $\gamma(t)\in\Lambda$ for all $t>0$, which is the integral-curve to one of the vector fields $-X_0,\pm X_1,\dots,\pm X_m$. The generalized Hessian matrix $H^+_{\mu,\nu}\varsigma_i(x)$ of the second derivatives $(D_\mu D_\nu\varsigma_i)$ of the function ς_i is then defined as follows:*

   Denote for all $\xi\in\mathbb{R}^m$ by γ_ξ the solution curve of the vector field $\sum_{\mu=1}^{m}\xi_\mu X_\mu$ with $\gamma_\xi(0)=y$. Then we set

$$\sum_{\mu,\nu=1}^{m}\xi_\mu\xi_\nu H^+_{\mu,\nu}\varsigma_i(y):=d_t^+\frac{d}{dt}(\varsigma_i\circ\gamma_\xi)\ |_{t=0}$$

   with

$$d_t^+g(0):=\limsup_{t\to0}\frac{g(t)-g(0)}{t}$$

   for a Lipschitz continuous function g on $\mathbb{R}$.
   Then, for all $y\in\Lambda$ and $z\in\mathcal{V}^i_y$ with $\varrho_i(z)=\varsigma_i(y)$ the following comparison condition should hold:

$$\nabla\varrho_i(z)\cdot F(y,z)+\iota_i(y,z)\ \leq\ \lambda_0^i(z)D_0\varsigma_i(y)-\lambda^i(y)\sum_{\mu,\nu=1}^{m}a_{\mu,\nu}(y)H^+_{\mu,\nu}\varsigma_i(y)+\sum_{j=1}^{l}\sum_{k=1}^{m}\lambda_k^i(z)b_{k,\mu}(y)D_\mu\varsigma_i(y)$$
$$+\sum_{\mu,\nu=1}^{m}\left(a_{\mu,\nu}\lambda^i(y)\min_{\kappa=1,\dots,k_o\ ;\ z\in\overline{\mathcal{V}^\kappa}}(\wp_i\cdot\nabla^2\varrho_i\cdot\wp_i)(z)+\nabla\varrho(z)\cdot C_{\mu,\nu}(y,z)[\wp_i(z),\wp_i(z)]\right)D_\mu\varsigma_i(y)D_\nu\varsigma_i(y)$$
$$-\sum_{\mu,\nu=1}^{m}\left(\frac{c_i^2(y,z)}{\zeta_i(y,z)}\delta_{\mu,\nu}\right)D_\mu\varsigma_i(y)D_\nu\varsigma_i(y)$$

   Here $c_i(y,z)$ defines the function

$$c_i(y,z)\ =\ \sup_{w\in P_z(W(y)\cdot\mathcal{C}(x,z))}\left\{\frac{1}{|w|}\sum_{k,\nu=1}^{m}\max_{k\ ;\ z\in\overline{\mathcal{V}^k}}\left(W_{k,\mu}(y)\lambda^i(y)\wp_i(y)\nabla^2\varrho_i(z)w_k\right)\right.$$
$$\left.-\frac{1}{2}\sum_{\mu=1}^{m}W^{-1}_{\mu,k}(y)\nabla\varrho(z)\cdot(C_{\mu,\nu}(y,z)[w_k,\wp_i(y)]+C_{\mu,\nu}(y,z)[\wp_i(z),w_k])\right\}^{1/2} \tag{7.9}$$

   The case $\zeta_i=0$ and $c>0$ is allowed, if $D\varsigma_i$ can be chosen identically equal to zero.

Then for every solution $\mathbf{u}\in C^2(\Lambda,\mathbb{R}^M)$ of (7.3) with $D\mathbf{u}(y)\in\mathcal{C}(y,\mathbf{u}(y))$ and $\mathbf{u}(y)\in\mathcal{M}_y$ for all $y\in\Lambda$ we have the following statements:

1. *If there is a $i\in\{1,\dots,I\}$ and a $y_i\in\Lambda$ with $\mathbf{u}(y_i)\in\partial_i\mathcal{M}_{y_i}$ then $\mathbf{u}(t)\in\partial_i\mathcal{M}_t$ for all $t\in\overline{\mathcal{S}(y_i,\Lambda)}$.*
2. *Let $J\subset\{1,\dots,I\}$, such that for every $i\in J$ there exists a $y_i\in\Lambda$ with $\mathbf{u}(y_i)\in\partial_i\mathcal{M}_{y_i}$, then $\mathbf{u}(t)\in\bigcap_{i\in J}\partial_i\mathcal{M}_t$ for all $t\in\bigcap_{i\in J}\overline{\mathcal{S}(y_i,\Lambda)}$.*

Theorem 7.5 is very general and technical. However it allows together with a re-formulation of Hopf's maximum principle for system (see [3, Lemma 3.2, page 63]) to find additional hypotheses under which the solution of certain boundary value problems remain in the interior of the sets of comparison. Furthermore it is possible to prove a general invariance theorem for parabolic systems. Here we only mention an application of these theorems to autonomous parabolic Neumann boundary value problems. Therefore let us once again consider the following problem:

$$\begin{array}{rcll} \mathbf{u}_t & = & \sum_{j=1}^{N} \frac{\partial}{\partial x_j}\left(A(\mathbf{u})\frac{\partial}{\partial x_i}\mathbf{u}\right) + F(\mathbf{u}), & \text{in } \Omega\times(0,\tau)\ (\Omega\subset\mathbb{R}^{\mathrm{N}} \\ A(\mathbf{u})\frac{\partial \mathbf{u}}{\partial n} & = & \psi(\mathbf{u}), & \text{on } \partial\Omega_{\mathrm{N}}\times[0,\tau] \end{array}$$

where we assume that $F,\ \psi\in C^{0,1}_{loc}(\mathcal{V},\mathbb{R}^M)$ and $A\in C^{1,1}_{loc}(\mathcal{V},\mathbb{R}^{M,M})$ with $det(A)\geq 0$.

Theorem 7.6 (Alt). *Let $\varrho_i\in C^{1,1}_{loc}(\mathcal{V},\mathbb{R})$, $\mathcal{V}\subset\mathbb{R}^M$ $(i=1,...,I)$ denote given functions and let $Q\subset\mathbb{R}^I$ be also given. For $q\in Q$ the set $\mathcal{M}_q$ is defined as*

$$\mathcal{M}_q := \{z\in\mathcal{V}\mid \varrho_i(z)\leq q_i,\ i=1,...,I\}.$$

Let $q^0\in Q$ be given, such that the initial condition $u(0,x)\in\mathcal{M}_{q^0}$ is satisfied for all $x\in\overline{\Omega}\times\{t=0\}$. Furthermore we assume that there exists a solution $\omega\in C^1([0,\tau_0],\mathbb{R}^I)$ of the system of differential inequalities

$$\frac{d}{dt}\omega_i(t)\geq \sup_{z\in\partial_i\mathcal{M}_q}\{\nabla\varrho_i(z)\cdot F(z)\},\ i=1,..,I,\ 0\leq t\leq\tau_0$$

such that $\omega_i(0)\geq q_i^0$, $i=1,...,I$ and a positive, continuous function ζ_0 on $[0,\tau_0]$, such that the following conditions are satisfied for

$$Q := \{q\in\mathbb{R}^I\mid \exists t\in[0,\tau_0]\ with\ \omega_i(t)\leq q_i\leq\omega_i(t)+\zeta_0(t)\ for\ all\ i=1,...,I\}:$$

$\nabla\varrho_i(z)\neq 0$ holds for all $z\in\mathbb{R}^M$ with a $q\in Q$ and $z\in\partial_i\mathcal{M}_q$. Furthermore $\nabla\varrho_i(z)\notin\sum_{s\neq i,z\in\partial_s\mathcal{M}_q}\mathbb{R}\cdot\nabla\varphi_s(z)$ holds on the "edges" of $\partial\mathcal{M}_q$ and ϱ is piecewise $C^{2,1}$. For all $z\in\partial_i\mathcal{M}_q$ with $q\in Q$ and $i\in\{1,..,I\}$ we have:

1. *$\nabla\varrho_i(z)A(z)=\lambda_i(z)\nabla\varrho_i(z),\ \lambda_i(z)>0$*
2. *$\nabla^2\varrho_i(z)A(z)\geq 0$ on $[\nabla\varrho(z)]^\perp$, for all $\kappa=1,...,k_0$ with $z\in\overline{\mathcal{V}^\kappa}$.*
3. *$\nabla\varrho_i(z)\cdot\psi(z)\leq 0$.*

Then $u(t,x)\in\mathcal{M}_{\omega(t)}$ for all $(t,x)\in\Omega\times[0,\tau_0]$.

This Theorem allows us to make statements on the time asymptotic behaviour of some special cases of the solution of Keller-Segel type models. For example (see also [3, Beispiel 5.18, pages 109 - 111]) let us consider the system

$$\left.\begin{array}{rcll} u_t & = & \nabla\left(\frac{k_1}{1+u}(\nabla u-\chi u)\nabla v)\right)+(\beta-\alpha u), & in\ \Omega\times\{t\geq 0\} \\ v_t & = & \Delta v-\delta v+\gamma u, & in\ \Omega\times\{t\geq 0\} \\ \frac{\partial u}{\partial n} & = & 0\ =\ \frac{\partial v}{\partial n}, & on\ \partial\Omega\times\{t\geq 0\}, \end{array}\right\} \tag{7.10}$$

where $k_1>1$ and χ, β, α, δ, γ are positive constants. We define $m_0=k_1-1>\frac{\beta}{\alpha}$ and

$$\varrho(u,v)=v-\frac{m_0}{k_1\chi}\log\left(\frac{u}{m_0}\right)+\frac{1}{k_1\chi}(u-m_0).$$

Then

$$\nabla\varrho(u,v)=\begin{pmatrix}\frac{1}{k_1\chi}\left(1-\frac{m_0}{u}\right)\\ 1\end{pmatrix}\ \text{and}\ \nabla\varrho(u,v)\cdot F(u,v)=\frac{1}{k_1\chi}(\beta-\alpha u)(1-\frac{m_0}{u})+\gamma u-\delta v.$$

We now set

$$\varrho_1(u,v)=-v,\ \varrho_2(u,v)=\varrho(u,v),\ \varrho_3(u,v)=v+\frac{1}{k_1\chi}\left(u-m_0-m_0\log\left(\frac{u}{m_0}\right)\right)$$

and

$$q\in\mathbb{R}^3\ \text{with}\ q_1=-a^-,\ q_2=a^+-\frac{1}{k_1\chi}\left(m_0\log\left(\frac{\beta}{\alpha m_0}\right)-\frac{\beta}{\alpha}+m_0\right)\ \text{and}\ q_3=a^+$$

for some $0 \le a^- < a^+$. The set of comparison $\mathcal{M}_q$ is then convex. Now let us additionally assume that

$$0 < \gamma < \frac{\alpha - \delta}{k_1 \chi}\left(1 - \frac{m_0}{\beta}\alpha\right)$$

holds. Let (u, v) denote the solution of the system (7.10) and let $a_0^+ \ge \gamma m_0/\delta$ be minimal and $a_0^- > 0$ be maximal such that with q^0 defined analogously as above $u(0, x) \in \mathcal{M}_{q^0}$ is satisfied for all $x \in \Omega$. If we set

$$a^+(t) = a_0^+ e^{-\delta t} + \frac{\gamma m_0}{\delta}\left(1 - e^{-\delta t}\right),\ t \ge 0$$

and choose a^- as the solution of the ODE

$$\frac{d}{dt}a^-(t) = \gamma \underline{u}(t) - \delta a^-(t),\ a^-(0) = a_0^-$$

where $\underline{u}(t) < \beta/\alpha$ is the uniquely defined by the equation

$$m_0 \log\left(\frac{\alpha \underline{u}(t)}{\beta}\right) + \frac{\beta}{\alpha} - \underline{u}(t) = -k_1 \chi a^+(t)$$

then we see for the analogously defined $q(t)$ that $u(t, x) \in \mathcal{M}_{q(t)}$ for all $(t, x) \in \Omega \times [0, \infty)$. The function $a^+(t)$ converges as $t \to \infty$ monotone decreasing to $a_\infty^+ = \gamma m_0/\delta$, and the function $a^-(t)$ converges as $t \to \infty$ monotone increasing to a $a_\infty^- \le \frac{\gamma\beta}{\delta\alpha} < a_\infty^+$ which is the unique solution of the equation

$$m_0 \log\left(\frac{\delta\alpha a^-(t)}{\gamma\beta}\right) - \left(\frac{\delta}{\alpha} - k_1\chi\right) a^-(t) + \frac{\beta}{\alpha} + \frac{k_1 \chi m_0 \gamma}{\delta} = 0.$$

Thus we see that the set $\mathcal{M}_{q_\infty}$ with the corresponding q_∞ is a global attracting set for all positive solutions of (7.10).

Wolfgang Alt's Habilitation from 1980 contains many results more. However a complete presentation of his nice results would expand the present work too much. Thus this short presentation of some of his results should be enough. I refer the interested reader to [3] for more results an more applications and examples of the invariance principles derived by W. Alt.

8. Self-similar solutions and traveling waves. Not only the question whether the solution of (3.3) blows up in finite time or exists globally in time, or the shape of the blow-up profile are mathematically interesting questions. The mathematical fascination of the Keller-Segel model, may also be caused by the fact that one can apply several mathematical tools resp. theories to show different aspects of the qualitative behaviour of the solution or to prove the existence of different types of solutions of the various possible forms of the model.

8.1. Self-similar solutions. Beside the results on self-similar solutions for the Keller-Segel model by Herrero, Medina and Velázquez [52, 53] that have been already mentioned in a previous section and the results on self-similar blow-up in [18, 19] in which the case of spatial dimension $N > 2$ is studied, there is another paper on that particular topic that should be summarized at this point. In 1999 Y. Mizutani, N. Muramoto and K. Yoshida studied in [82] the question whether positive self-similar radial solutions

$$u(t, x) = \frac{1}{t}\varphi\left(\frac{|x|}{\sqrt{t}}\right),\ v(t, x) = \psi\left(\frac{|x|}{\sqrt{t}}\right)$$

of the problem

$$\left.\begin{array}{rcll} u_t &=& \nabla(\nabla u - \chi u \nabla v), & x \in \mathbb{R}^2,\ t > 0 \\ \varepsilon v_t &=& \Delta v + \tilde{\alpha} u, & x \in \mathbb{R}^2,\ t > 0 \end{array}\right\}$$

exist or not. Substitution of the previous expressions gives the system

$$\left.\begin{array}{rcl} 0 &=& (\varphi' - \chi\varphi\psi')' + \frac{1}{r}(\varphi' - \psi\varphi\psi') + \frac{r}{2}\varphi' + \varphi \\ 0 &=& \psi'' + \frac{1}{r}\psi' + \frac{\varepsilon r}{2}\psi' + \tilde{\alpha}\varphi \\ 0 &=& \varphi'(0) \ = \ \psi'(0) \end{array}\right\} \tag{8.1}$$

for the functions (φ, ψ). For $r > 0$ the first equation thus results in $\left[2r\left(\varphi' - \chi\varphi\psi'\right) + r^2\varphi\right]' = 0$ and therefore in $\varphi = \lambda e^{-r^2/4} e^{\chi\psi}$, where $\lambda = \varphi(0)e^{-\chi\psi(0)} > 0$. Now substituting this into the second equation we see that ψ solves

$$0 = \psi'' + \left(\frac{1}{r} + \frac{\varepsilon r}{2}\right)\psi' + \tilde{\alpha}\lambda e^{-r^2/4} e^{\chi\psi} \text{ under the constraint } \int_0^\infty r\psi(r)\, dr < \infty.$$

Mizutani's, Muramoto's and Yoshida's and T. Nagai's [81] results are the following:

Theorem 8.1. *Let* $0 < \tilde{\alpha}\chi\lambda \log\left(\frac{\varepsilon}{\varepsilon-1}\right) < 1/e$.

1. (Mizutani & Nagai) *Then there exists an* $0 < a_* < 1$ *such that* (8.1) *with* $\psi(0) = a_*$ *admits a positive solution satisfying the mass constraint. Furthermore there exists a* μ^* *such that if* $\tilde{\alpha}\chi\lambda > \mu^*$, *there are no positive solutions of* (8.1).
2. (Mizutani, Muramoto & Yoshida) *Then there exists an* $1 < a^*$ *such that* (8.1) *with* $\psi(0) = a^*$ *admits a positive solution satisfying the mass constraint. Furthermore* $\psi(0)$ *tends to infinity as* $\tilde{\alpha}\chi\lambda \log\left(\frac{\varepsilon}{\varepsilon-1}\right) \to 0$.

Using variational techniques like the famous mountain pass theorem (see for instance [142]) Mizutani, Muramoto und Yoshida showed both the statements in [82].

Without directly assuming that the self-similar solutions are radially symmetric Naito et al. showed in [99] that the ansatz $u(x,t) = \frac{1}{t}\varphi\left(\frac{x}{\sqrt{t}}\right)$ and $v(x,t) = \psi\left(\frac{x}{\sqrt{t}}\right)$ leads to the system:

$$\left.\begin{array}{rcll} 0 &=& \nabla(\nabla\varphi - \varphi\nabla\psi) + \frac{x}{2}\nabla\varphi + \varphi, & x \in \mathbb{R}^2 \\ 0 &=& \Delta\psi + \frac{\varepsilon x}{2}\nabla\psi + \varphi, & x \in \mathbb{R}^2 \\ 0 &\leq& \varphi, \psi \text{ in } \mathbb{R}^2 \text{ and } \varphi(x), \psi(x) \to 0 \text{ as } |x| \to \infty. & \end{array}\right\} \quad (8.2)$$

The existence of a solution of this system has been proven in [83, Theorem 1.1 and Theorem 1.2, page 429]. In [99] they showed that any classical solution of (8.2) is radially about the origin and satisfies $\varphi, \psi \in L^1(\mathbb{R}^2)$. Furthermore they showed that the solution set of (8.2) can be expressed as a one-parameter family

$$\mathcal{S} = \{(\varphi(s), \psi(s)) : s \in \mathbb{R}\}.$$

If $\lambda(s) := ||\varphi(s)||_{L^1(\mathbb{R}^2)}$, then the solution $(\varphi(s), \psi(s))$ and $\lambda(s)$ satisfy the following properties:

1. $s \mapsto (\varphi(s), \psi(s)) \in C^2(\mathbb{R}^2) \times C^2(\mathbb{R}^2)$ and $s \mapsto \lambda(s) \in \mathbb{R}$ are continuous.
2. $(\varphi(s), \psi(s)) \to (0,0)$ in $C^2(\mathbb{R}^2) \times C^2(\mathbb{R}^2)$ and $\lambda(s) \to 0$ as $s \to -\infty$.
3. $||\psi(s)||_{L^\infty(\mathbb{R}^2)} \to \infty$, $\lambda(s) \to 8\pi$, and $\varphi(s)dx \rightharpoonup 8\pi\delta_0(dx)$ in the sense of measure as $s \to \infty$, where $\delta_0(dx)$ denotes Dirac's $\delta-$function with support in the origin.
4. $0 < \lambda(s) < 8\pi$ for $s \in \mathbb{R}$, if $0 < \varepsilon \leq 1/2$, and $0 < \lambda(s) \leq \max\{4\pi^3/3, 4\pi^3\varepsilon^2/3\}$ for $s \in \mathbb{R}$, if $\varepsilon > 1/2$.

These properties result in the consequence that there exists a critical value $8\pi \leq \lambda^* \leq \max\{4\pi^3/3, 4\pi^3\varepsilon^2/3\}$ such that for $\lambda \in (0, \lambda^*)$ there exists a solution in $\mathcal{S}$ such that $||\varphi||_{L^1(\mathbb{R})} = \lambda$ and for $\lambda > \lambda^*$, there exists no solution in $\mathcal{S}$ satisfying $||\varphi||_{L^1(\mathbb{R})} = \lambda$.

Beside these and the previous results by Herrero, Medina and Velázquez and those by M. P. Brenner, P. Constantin, L. P. Kadanoff, A. Schenkel and S. C. Venkataramani [18, 19] I am not aware of any other results on self-similar radial solutions for Keller-Segel type models.

8.2. One dimensional traveling wave and pulse solutions. The existence of traveling wave solutions for the Keller-Segel model has been started by Keller and Segel [71] and was followed by [30, 32, 65, 73, 86] and [106].

The initiation of these studies have been experimental observations by J. Adler and his group starting in 1966. One of the typical experiments with *Escherichia coli* bacteria done by J. Adler in 1966 has been quoted in [125] and I want to give a quotation of this part from [125], to give the reader an illustration of the experimental setting:

> "*About a million motile cells of E. coli are placed at one end of a capillary tube filled with a solution containing* 2.5×10^{-4} *molar galactose as the energy source, and the ends of the tube are closed with plugs of agar and clay.... The galactose is present in excess over the oxygen, since the concentration of oxygen in water saturated with air at* 37^oC *is about* 2.0×10^{-4} *mole/l and it takes six molecules of oxygen to fully oxidize a molecule of galactose.... Soon afterward, two sharp, easily visible bands of bacteria have moved out from the origin, and some bacteria remain at the origin.*"

As a conclusion of these observations Adler stated that

> "*...the bacteria create a gradient of oxygen or of an energy source, and then they move preferentially in the direction of the higher concentration of the chemical. As a consequence, bands of bacteria ... form and move out.*"

In [71] E. F. Keller and L. A. Segel used this observation by Adler and showed that a simplified version (proposed in [70]) of their original model (2.4) can describe this phenomenon. They studied in [71] the following system:

$$\left.\begin{array}{rcl} u_t &=& \mu u_{xx} - (\delta u v^{-1} v_x)_x \\ v_t &=& -ku \end{array}\right\} \tag{8.3}$$

with homogeneous Neumann boundary data. The traveling wave ansatz yields

$$u(z) = \text{const} \cdot (v(z))^{\delta/\mu} e^{-cz/\mu} \text{ and } v(z) = \left(\text{const} \cdot kc^{-2}(\delta-\mu)e^{-cz/\mu} + v_\infty^{1-(\delta/\mu)}\right)^{-1/((\delta/\mu)-1)},$$

where $-\infty < v_\infty := v(\infty) < \infty$. If one assumes $v_\infty^{1-(\delta/\mu)} = \text{const} \cdot kc^{-2}(\delta-\mu)$ then

$$\frac{v(z)}{v_\infty} = \left(1+e^{-cz/\mu}\right)^{-\delta/(\delta-\mu)} \text{ and } \frac{u(z)}{c^2 v_\infty (\mu k)^{-1}} = \frac{1}{(\delta/\mu)-1} e^{-cz/\mu}\left(1+e^{-cz/\mu}\right)^{-\delta/(\delta/\mu-1)}. \tag{8.4}$$

Since we are looking for bounded solutions we suppose $\delta > \mu$, which implies $\lim\limits_{cz/\mu\to-\infty} v = 0$, $\lim\limits_{cz/\mu\to-\infty} u = 0$.

However the condition $\delta > \mu$ is not necessary to guarantee the existence of a traveling wave solution. In [73] E. F. Keller and G. M. Odell generalized this result also for the range $0 < \delta < \mu$. In this case the traveling wave ansatz gives a solution for which $u(z)$ and $v(z)$ reach zero at a finite z_0. Since it is possible to locate the coordinate origin anywhere one pleases by choosing the arbitrary constant generated by integrating the first equation of the Keller-Segel model in the traveling wave ansatz once, places the origin at that point. Full solutions of the system are then obtained by joining the solution of the system with the trivial solution from the point z_0. This idea leads to the definition of a "generalized traveling wave solution" for the Keller-Segel model.

Definition 8.2 (see also Keller & Odell and Ebihara, Furusho & Nagai). *We denote the set of all real valued functions $(u,v) \in C(\mathbb{R}) \times C^1(\mathbb{R})$ such that $u \in C^1(P(v))$, where $P(v) := \{z \in \mathbb{R} \mid v(z) > 0\}$, as $\mathcal{W}(\mathbb{R})$. For $(u,v) \in \mathcal{W}(\mathbb{R})$ we set*

$$\mathcal{J}(z) := \begin{cases} k_1(u,v)u' - k_2(u,v)v' + cu, & \text{for } z \in P(v) \\ 0, & \text{for } z \notin P(v) \end{cases}$$

The set of all generalized traveling wave solutions of the Keller-Segel equations is defined as the set of all $(u,v) \in \mathcal{W}(\mathbb{R})$ such that there exists a $z_0 > -\infty$ with $\mathcal{J} \in C^1(\mathbb{R})$ satisfying $\mathcal{J}' = 0$ in $\mathbb{R}$ and $v \in C^2(\mathbb{R})$ solving $k_c v'' + cv' + g(u,v) = 0$ in $\mathbb{R}$ for some $c \in \mathbb{R}$ and with $v(z) \equiv 0$ for all $z < z_0$. If $z_0 = -\infty$ we call the solution traveling wave solution of the Keller-Segel system.

Remark 1. *Other authors talk of traveling wave, normalized traveling wave and singular traveling wave solutions in this context.*

In [86], T. Nagai and T. Ikeda studied the existence and stability of traveling wave solutions of the system

$$\left.\begin{array}{rcl} u_t &=& \mu u_{xx} - (\delta u v^{-1} v_x)_x \\ v_t &=& \varepsilon v_{xx} - ku \end{array}\right\} \tag{8.5}$$

with homogeneous Neumann boundary data and $\delta > 1$. They considered both cases, $\varepsilon = 0$ and $\varepsilon > 0$. It was proved that traveling wave solutions $(u_\varepsilon, v_\varepsilon)$ for (8.5) with $\varepsilon \geq 0$ satisfy

$$||v_\varepsilon - v_0||_{L^\infty(\mathbb{R})} = O(\varepsilon) \text{ and } ||u_\varepsilon - u_0||_{L^\infty(\mathbb{R})} = O(\sqrt{\varepsilon}) \text{ as } \varepsilon \to 0. \tag{8.6}$$

Here $u_\varepsilon(z)$ is given by $e^{-cz}v_\varepsilon^\delta$. Furthermore it was shown that traveling wave solutions are linearly unstable for perturbations in the sets

$$X := \{(u,v) \mid u,v \in L^1(\mathbb{R}), \int_{-\infty}^{\infty} u(z)\,dz = 0\} \text{ and } X_w := \{(u,v) \in X \mid wu, wv \in L^1(\mathbb{R})\}$$

where the weight function $w(z)$ equals $e^{-\varrho z}$ for $z<1$ and $e^{\omega z}$ for $z>1$. Here $\omega \geq c$, which is the wave speed, and $\varrho \geq c/(\delta-1)$.

In [30] Ebihara et al. proved the existence of a traveling wave solution of (8.5) for $\delta \leq 1$. In the case of $\delta < 1$ they proved that the traveling wave solution is a generalized traveling wave solution with $z_0 > -\infty$ and $v(z) \equiv 0$ for all $z < z_0$. Other results on simplified versions of the Keller-Segel equations can also be found in [8, 9, 10, 74, 77, 107].

Let us now try to look a bit closer at the problem of the existence of traveling wave solutions for more general versions of the Keller-Segel model. Thus let us consider system (6.1) in $\mathbb{R} \times \{t \geq 0\}$ with $k(u,v) = k(u)$ and $h(u,v) = u\phi(v)$. With the traveling wave ansatz $u(t,x) = u(x-ct) = u(z)$, $v(t,x) = v(x-ct) = v(z)$ and no-flux boundary conditions for the first equation we obtain

$$-cu = k(u)u_z - u\Phi(v)_z + \text{const}, \text{ where } \Phi(v) = \int_0^v \phi(s)\, ds. \tag{8.7}$$

Caused by the conservation of mass we can only expect traveling pulse solutions for u. This results in the condition that $u(z) \to 0,\ z \to \pm\infty$. Together with Neumann boundary condition this implies that the constant in (8.7) equals zero. Dividing by u we obtain

$$-c = \frac{k(u)}{u} u_z - \Phi(v)_z.$$

If $\frac{k(u)}{u}$ is integrable with respect to u and $K(u)$ is invertible with $K'(u) = \frac{k(u)}{u}$ then $u(z) = K^{-1}\left(\Phi(v(z)) - cz\right)$. If $k(u) = 1$, respectively $k(u)/u = 1/u$, then $K(u) = \log u + c_1$ and $K^{-1}(y) = e^{y-c_1}$. Without loss of generality we can assume that the constant c_1 is equal to zero. If the chemotactical sensitivity function $\Phi(v(z)) = \log(v(z))$ then $v(z) = u(z)e^{cz}$. Therefore the traveling wave ansatz allows us to solve the first equation of the Keller-Segel model explicitly.

8.3. Some short comments on the chemotactic sensitivity function in connection with traveling wave solutions and on necessary and sufficient conditions for chemotactic bands. One major mathematical concern in connection with the existence of traveling wave solutions for the Keller-Segel model is the necessity of a singularity in the chemotactic sensitivity function as v approaches small values resp. zero. This has already been mentioned in [71]. This problem has also been discussed in [125] and some relaxations of the first statements of Keller and Segel in [70] on the necessity of the singular term had been discussed in that paper. But one fact still remains (see [125, page 198]):

> "*Although our results perhaps weaken the theoretical reasons for taking $\phi(v) = \delta/v$, experimental results have shown, that $\phi(v) = \delta/v$ is in fact the appropriate form (as we have already mentioned) so it does not appear worthwhile to explore these results further.*"

In [102] the specific chemotactic sensitivity $\phi(v) = \delta/v$ is also discussed. There it is also mentioned, that this choice seems to fit well to experiments, unless the concentrations are very large or very small, for obvious reasons.

Beside numerical simulations of traveling waves solution for some simplified versions of the Keller-Segel model one can also find the discussion of plausible functional forms of the other terms in the model in [125]. Therefore I refer the interested reader to that paper and for other proposed functional forms to [74] and [102].

E. F. Keller and G. M. Odell gave in [72] necessary and sufficient conditions for the existence of traveling wave solutions for the system

$$\left.\begin{array}{rcl} u_t &=& \nabla(k_1(v)\nabla u - u\nabla(\Phi(v))) \\ v_t &=& -g(v)u \end{array}\right\}. \tag{8.8}$$

They summarized their results in the following theorem:

Theorem 8.3 (Keller & Odell). *Let $k_1(v)$, $\Phi'(v)$ and $g(v)$ be functions which are*

1. *positive and continuous for $v > 0$;*
2. *bounded and uniformly bounded away from zero for $v \geq \varepsilon$, where ε is a positive number;*
3. *allowed to tend to zero, or become unbounded as $v \to 0^+$.*

Then, necessary and sufficient conditions for the system (8.8) *to have traveling wave solutions are:*

$$0 = \lim_{v \to 0^+} \frac{1}{g(v)\Phi'(v)}, \tag{8.9}$$

$$\infty > \int_0^{v(\infty)} \frac{dv}{g(v)}, \tag{8.10}$$

$$0 = \lim_{v \to 0^+} \int_v^{v(\infty)} \frac{exp\left(-\int_v^w \frac{\Phi'(s)}{k_1(s)} ds\right)}{k_1(w)g(w)} dw. \tag{8.11}$$

Furthermore Keller and Odell gave in [72] examples for different classes of functions that satisfy the conditions mentioned in the previous Theorem. I refer the interested reader to that paper for more details. In case of $k_1(v) = const$, $g(v) = const \cdot v^m$ $(m \geq 0)$ and $\Phi(v) = \delta \log(v)$, with a constant $\delta > 0$, G. Rosen studied in [122] the existence of propagation of bands of chemotactic bacteria and in [123] together with S. Baloga the stability of these traveling wave solutions for the case $m < 1$.

8.3.1. Reduction of the problem to one autonomous ODE. The fact that the traveling wave ansatz allows to solve the first equation of the Keller-Segel (6.1) under the assumptions made in this subsection might lead to the idea to study a single autonomous ordinary differential boundary value problem on the half line $[0, \infty)$ for the function v. This approach has been considered for some special cases in [30, 86] and in a more general version in [65]. In this subsection we only want to sketch certain ideas connected with this approach to traveling waves in chemotaxis. We consider system (6.1) where $k(u)/u$ is integrable with respect to u and its integral $K(u)$ is assumed to be invertible, on $\mathbb{R}$, so $K^{-1}(\phi(v) - cz) = u$. Then we are left to analyze

$$-cv' = k_c v'' + g\left(K^{-1}(\phi(v) - cz), v\right)$$

with the boundary conditions $v(-\infty) = 0$ and $v(\infty) = const$ where the condition $K^{-1}(\phi(v) - cz) \to 0$ for $z \to \pm\infty$ has to be fulfilled. Now we choose $v(z) = w(e^{-cz}) = w(y)$ where c is the traveling wave speed and $w(\infty) = 0, w(0) = const$. Then we obtain

$$0 = k_c c^2 y^2 w'' + (k_c - 1)c^2 y w' + g\left(K^{-1}(\phi(w) + \log y), w\right).$$

Defining now $\tilde{g}(y, w) := g\left(K^{-1}(\phi(w) + \log y), w\right)$, we have reduced the problem to a single autonomous ODE on the half-line $[0, \infty)$. In general one now would try to find suitable sub- and supersolutions for this problem to prove the existence of a solution via Perron's method. This has been done in [30] and [86] in the special case of system (8.5). In this case it was sufficient to find a supersolution, since a subsolution was immediately given by the constant solution $w_{\text{sub}} \equiv 0$. As a supersolution one can find in [86] the function w_{super} which solves the problem

$$w'' = p y^q w^{\frac{\delta}{\mu}} \text{ in } (0, \infty), \text{ with } w(y) \geq 0 \text{ and } w'(y) \leq 0 \ (0 < y < \infty)$$

with initial value $w(0) = const$. Here $p = \frac{k}{\varepsilon}\left(\frac{c}{\varepsilon}\right)^{\varepsilon - 2}$ and $q = \varepsilon - 2$. Of course this allows also to construct more examples of systems which have traveling wave solutions by assuming that

$$\tilde{g}(y, w) = \sum_{k=1}^{m} (-1)^{h_k} y^{q_k} w^{l_k}$$

for some exponents h_k, q_k and l_k. However assuming more general functional forms the following cases can be treated, too.

(a) $k_c \neq 1$: In this case we look for a solution w which satisfies: $(1 - k_c)c^2 w' = a_1(w)$, $k_c c^2 w'' = -a_2(w)$ and $a_3(w, y) = 0$. So $-a_2(w) = \frac{k_c}{1-k_c} \frac{d}{dy} a_1(w) = \frac{k_c}{(1-k_c)^2 c^2} a_1(w) a_1'(w)$. Therefore

$$\int_0^y ds = \frac{k_c}{k_c - 1} \int_{w(0)}^{w(y)} \frac{\frac{d}{dr} a_1(r)}{a_2(r)} dr =: A(w).$$

We assume that $A(w)$ is invertible. Then $w = A^{-1}(y)$ with $a_3(w, y) = 0$ solves our problem.

(b) $k_c = 1$: Here we define $\tilde g(y,w) = a_2(w)y^2 + \tilde a(y,w)$. And we assume that $c^2 w'' = -a_2(w)$ solves $\tilde a(w,y) = 0$. Then

$$c^2 \int_0^y w''(s)ds = c^2 w'(y) = \int_{w(0)}^{w(y)} -a_2(q)dq.$$

So in this case

$$y = -\int_{w(0)}^{w(y)} \frac{c^2}{\int_{r(0)}^{r(y)} a_2(q)dq} dr =: A(w)$$

has to be invertible. And $w = A^{-1}(y)$ has to solve $\tilde a(y,w) = 0$.

(c) $k_c = 0$: Here we define $\tilde g(y,w) = a_1(w)y + \tilde a(y,w)$ and we assume that $c^2 w' = a_1(w)$ solves $\tilde a(y,w) = 0$. Then

$$c^2 \int_0^y w'(s)ds = c^2(w(y) - w(0)) = \int_{w(0)}^{w(y)} a_1(q)dq.$$

This identity gives us the function w, which has to solve $\tilde a(w,y) = 0$.

Since we are looking for traveling front solutions for the chemical with a "classical shape", we can look for a function $w(s)$ that satisfies $w'(s) < 0$ on $[0,\infty)$. Now we see in case (a) that $a_1(s)$ has to be a positive function for all $s \in \mathbb{R}^+$ if $k_c > 1$. This shows that in such a case a production term is necessary to guarantee the existence of a traveling wave solution. Furthermore this implies for $k_c \neq 0$ that $a_2(w)$ does not vanish, which is obvious in case (b) if one looks for nontrivial solutions. This however implies that we need a u^2-term in the second equation to construct examples for traveling wave solutions with this approach. In case (c) we see that $a_1(s)$ has to be negative for all $s \in \mathbb{R}^+$ to guarantee the existence of a traveling front solution. Thus a degradation term is the crucial term in this case. Furthermore we see that it has to be a u-term of order one in contrast to case (a) where we need a u-term of order 2 to get traveling wave solutions.

8.4. Construction of traveling wave solutions for chemotaxis without reproduction. Of course one can also raise the question the other way around, namely, which nonlinearity in the Keller-Segel model (6.1) might favor traveling wave solutions? To be more precise one might ask: Are there functional forms for the chemotactic sensitivity, the chemical decay and the chemical production that lead to traveling wave solutions for Keller-Segel type models? Or: Does the interplay between the chemotactic sensitivity, the chemical diffusion, decay and production might cause traveling waves? To get more insights into this "inverse problem" for traveling waves in chemotaxis we turn to a constructive approach that has been demonstrated in [65]. As a general assumption in this subsection we assume that $f(v) = 0$, $k(u,v) = k(u)$ and $h(u,v) = u\phi(v)$. Furthermore we use the notation

$$\Phi(v) := \int^v \phi(s)\ ds$$

and we will assume here for simplicity that $k(u) = 1$. Using the traveling wave ansatz one concludes from the first equation that $u(z) = G(v)e^{-cz} = G(v)u(z)$. Similar to the approach in [121] let us now assume that $\frac{d}{dz}v = F(v)b(z)$ which implies $v_{zz} = F'(v)F(v)b^2(z) + F(v)b'(z)$. Substituting this in the second equation of the Keller-Segel model we get after some rearrangements

$$F'(v) = -\frac{c}{k_c b(z)} - \frac{b'(z)}{b^2(z)} - \frac{g(v, G(v)u(z))}{k_c F(v) b^2(z)}. \tag{8.12}$$

If b solves $b'(z) = -\frac{c}{k_c} b(z)$ which means $b(z) = \tilde C e^{-\frac{c}{k_c}z}$ then we see that

$$F(v) = \sqrt{-\frac{2\tilde C e^{2\frac{c}{k_c}z}}{k_c} \int^v g(s, G(s)e^{-cz})ds + const.}$$

Thus g has to be chosen such that both terms of the equation depend only on v. All equations are supposed to hold for general v. To get further information let us now proceed with our constructive ansatz introduced in [65] by assuming that

$$g(v, G(v)e^{-cz}) = \sum_{j=-\infty}^{\infty} e^{-cjz} G^j(v) \sum_{i=-\infty}^{\infty} a_{i,j} v^i \text{ where } G(v) = \sum_{l=-\infty}^{\infty} g_l v^l.$$

Furthermore let

$$b(z) = \sum_{j=0}^{\infty} b_j e^{-cjz} \text{ and } F(v) = \sum_{i=-\infty}^{\infty} f_i v^i.$$

Then, after substituting these expressions in (8.12) we have to compare the coefficients for the $v^m e^{-cnz}$. This results in

$$k_c \sum_{i=-\infty}^{\infty} i f_i f_{m-i+1} \sum_{l=0}^{\infty} b_l b_{n-l} = -c f_m b_n + c k_c n f_m b_n - A_m,$$

where A_m is the overall coefficient of v^m in $\sum_{i=-\infty}^{\infty} a_{in} v^i G^n$. Now it is possible to treat different cases as it has been done in detail in [65]. Here we only illustrate the method for an explicit example. First we consider the following u-equation

$$u_t = u_{xx} - \chi \left(u \left(\log v\right)_x\right)_x$$

with Neumann boundary conditions and $\chi \in \mathbb{R}$. After applying the traveling wave ansatz we get $u = v^{\chi} e^{-cz}$ resp. $G(v) = v^{\chi}$ in our notation. Now we want to construct suitable equations for v to get a traveling pulse for u and a traveling wave for v. Let us assume $F(v) = f_1 v$. Since $b_0 = 0$ in this case we get:

$$-k_c f_1 b_1 c = -c f_1 b_1 - k_1 \quad \text{or} \quad k_1 \ = \ v^{\chi} \sum_{i=-\infty}^{\infty} a_{i1} v^{i-1} \ = \ (k_c - 1) b_1 c f_1.$$

For simplicity we assume that $b(z) = e^{-cz}$, so $b_1 = 1$ and $b_j = 0$ for $j \neq 1$. With this we have

$$k_2 = v^{2\chi} \sum_{i=-\infty}^{\infty} a_{i2} v^{i-1} = -k_c f_1^2.$$

So $a_{1-\chi,1} = (k_c - 1) c f_1$ and $a_{1-2\chi,2} = -k_c f_1^2$. Therefore the second equation has the form

$$v_t = k_c v_{xx} + (k_c - 1) c f_1 v^{1-\chi} u - k_c f_1^2 v^{1-2\chi} u^2. \tag{8.13}$$

The general system has the traveling pulse and wave solution

$$u(x,t) = e^{-\frac{\chi f_1}{c} e^{-c(x-ct)}} e^{-c(x-ct)}, \ v(x,t) = e^{-\frac{f_1}{c} e^{-c(x-ct)}}, \tag{8.14}$$

where c is the wave speed.

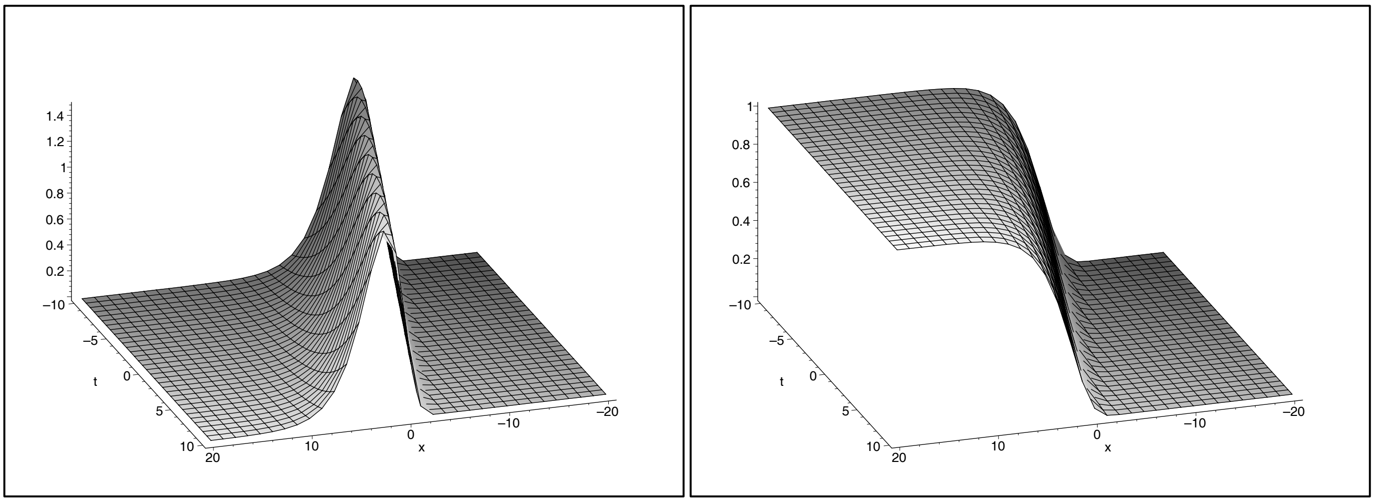

FIG. 8.1. *u and v given by (8.14) for $f_1 = 1$ and $\chi = 0.25$.*

To avoid singularities in the production and degradation terms it is of special interest to look for $0 < \chi < \frac{1}{2}$ which means $\frac{1}{2} < 1 - \chi, 1$, respectively $0 < 1 - 2\chi, 1$ for the powers of v in the v-equation. However, qualitatively the shape of the solutions remains the same, independent from the choice of $\chi \in \mathbb{R}_+$. For $k_c = 0$ respectively $k_c = 1$ one obtains pure degradation for v.

It is also possible to construct examples where multiple peak traveling pulse solutions exist by making appropriate choices for the sensitivity functional. This follows immediately from the following statement from [65]:

Assume there exists a traveling wave solution for the system

$$\left.\begin{array}{rcl} u_t &=& (u_x - u(\log(v))_x)_x \\ v_t &=& k_c v_{xx} + g(u,v) \end{array}\right\} \tag{8.15}$$

with homogeneous Neumann boundary data. Then there exists a traveling wave solution for

$$\left.\begin{array}{rcl} u_t &=& (u_x - u(\log(\Phi(v))_x)_x \\ v_t &=& k_c v_{xx} + g\left(\frac{vu}{\Phi(v)}, v\right), \end{array}\right\} \tag{8.16}$$

since in (8.15) we have $u(z) = ve^{-cz}$ and in (8.16) one obtains $u(z) = \Phi(v)e^{-cz}$.

Let us consider now a concrete example for multiple peaks traveling waves. As one can easily check the system

$$\left.\begin{array}{rcl} u_t &=& (u_x - u(\log(4v^2 - 4v^{3/2} + v))_x)_x \\ v_t &=& v_{xx} - 6\alpha^2 \frac{v^2 u^2}{4v^2 - 4v^{3/2} + v}. \end{array}\right\}$$

has the traveling wave solution

$$u(x - \alpha t) = \frac{(1 - e^{-\alpha(x-\alpha t)})^2}{(1 + e^{-\alpha(x-\alpha t)})^4} e^{-\alpha(x-\alpha t)} \text{ and } v(x - \alpha t) = (1 + e^{-\alpha(x-\alpha t)})^{-2}.$$

The second traveling peak in the population is caused by the sensitivity function. Substituting the solution into the

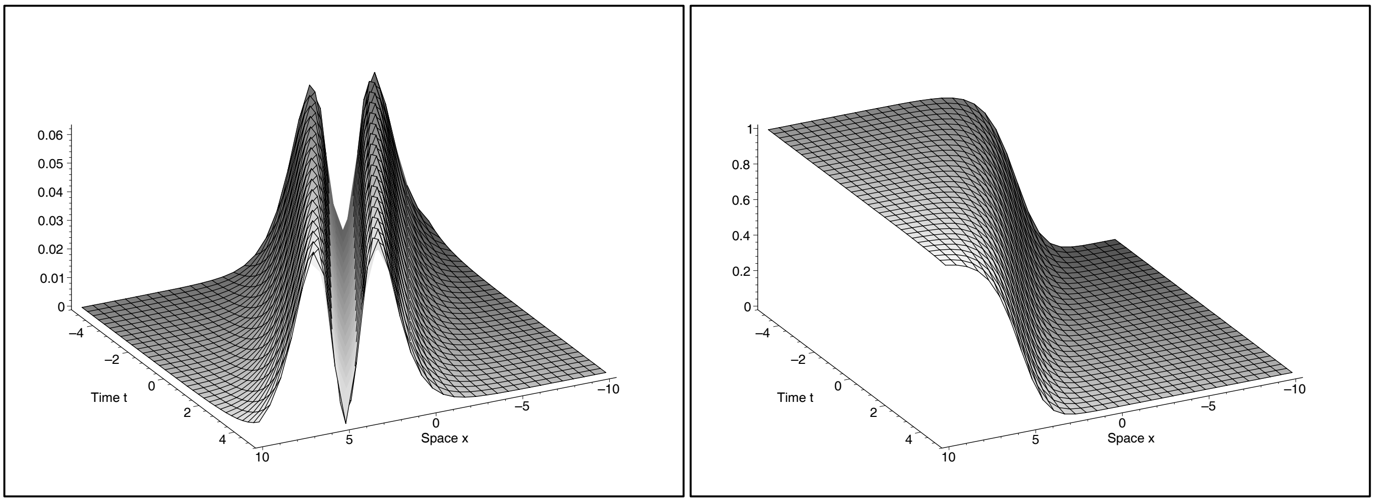

FIG. 8.2. *The function $u(x - \alpha t)$ and $v(x - \alpha t)$ given by the formula above for $\alpha = 1$.*

functional form of the sensitivity function it becomes a perturbation of the traveling front for the chemical distribution. It is also possible to construct more than two traveling peaks by appropriate choices of the sensitivity function.

I want to mention that the constructive approach sketched here also allows different assumptions on $k(u)$ and that there are also Keller-Segel type systems with $k(u) \not\equiv 1$ that also have traveling wave solutions. For more details see once more [65].

Under certain additional assumptions the presented results allow to conclude that there are also traveling wave solutions on multi-dimensional cylindrical domains. For instance the system

$$u_t = \nabla(\nabla u - u\nabla(\log(v)))$$
$$v_t = 2\Delta v + 2v + \alpha^2 u - 2\alpha^2 \frac{u^2}{v},$$

on $\Omega = \mathbb{R} \times [0, \pi]$ with no-flux boundary conditions for the first equation and where v satisfies homogeneous Dirichlet boundary data on $\partial\Gamma$ has the explicit solution

$$u(x,y,t) = sin(y)e^{-\alpha(x-\alpha t)}e^{-e^{-\alpha(x-\alpha t)}},\ v(x,y,t) = sin(y)e^{-e^{-\alpha(x-\alpha t)}}. \tag{8.17}$$

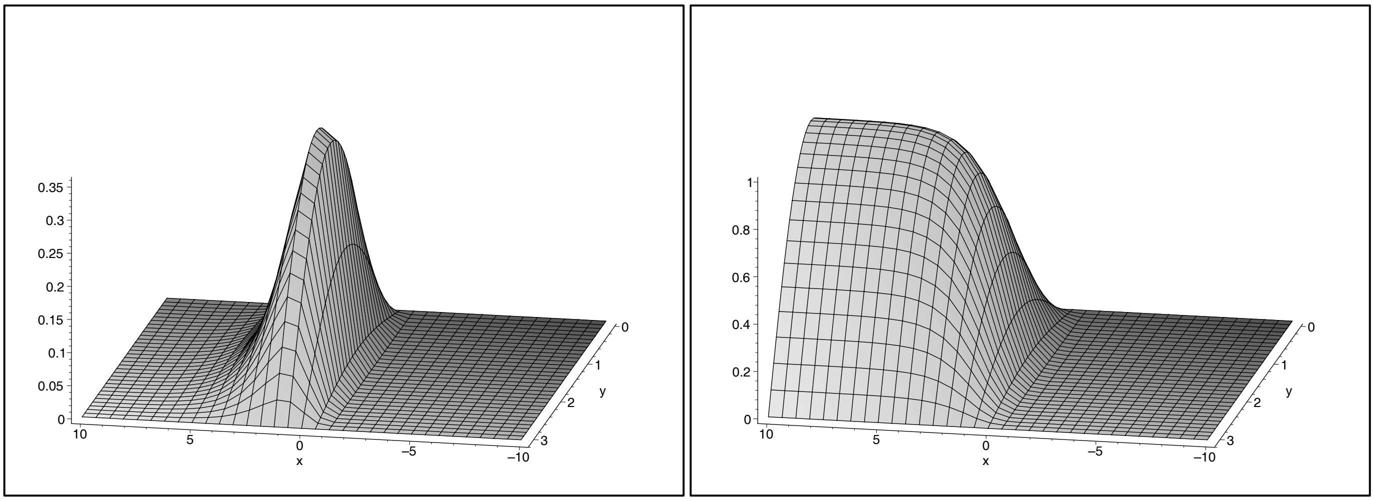

FIG. 8.3. *The function $u(x, y, t)$ on the left and $v(x, y, t)$ on the right for $\alpha = 1$ and $t = 1$ given by the formula (8.17) .*

8.5. Transversal stability of traveling wave fronts for chemotaxis with a growth term. It is possible to observe interesting patterns during the aggregation of certain species. These patterns sometimes consist of some particle streamings resp. some kind of fingering structures that change as time evolves but the wave-like movement in direction of the aggregation centers remain. In this subsection we only concentrate on the results form [41] to demonstrate that an additional growth term can generate pattern during the chemotactical movement of mobile species and might be important to explain these streaming effects. As we have seen the interplay of the chemotactical sensitivity of the species and the production and consumption terms of the chemoattractant can generate the formation of a traveling pulse for non-reproductive species. In the case of reproduction we cannot expect the existence of a traveling pulse. Therefore we shall look for traveling fronts of both, the species and the chemoattractant.

Funaki, Mimura and Tsujikawa study the system

$$\left.\begin{array}{rcl} u_t &=& \nabla(\epsilon^2 \nabla u - \epsilon k u \nabla \chi(v)) + u(1-u)(u-a), \\ v_t &=& \Delta v - \gamma v + u, \end{array}\right\} \tag{8.18}$$

on the strip domain $\Omega_l := \{(x, y) \in \mathbb{R}^2 \mid -\infty < x < \infty,\ 0 < y < l\}$ with width $l > 0$, where the boundary conditions are

$$(u, v)(t, -\infty, y) = \left(1, \frac{1}{\gamma}\right),\ t > 0,\ 0 < y < l,\ (u, v)(t, \infty, y) \ = \ (0, 0)\,,\ t > 0,\ 0 < y < l,$$
$$\left(\frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}\right)(t, x, 0) = (0, 0),\ t > 0,\ \ -\infty < x < \infty,\ \ \left(\frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}\right)(t, x, l) \ = \ (0, 0),\ t > 0,\ \ -\infty < x < \infty.$$

Introducing the new variable $z = x - \epsilon\theta t$ they first construct 1-dimensional traveling front solutions for this system in the following way. If there exists a 1-dimensional traveling front solution then it satisfies the equations

$$\left.\begin{array}{rcl} 0 &=& \epsilon^2 u_{zz} - \epsilon\theta u_z - \epsilon k \left(u\chi'(v)v_z)\right)_z + u(1-u)(u-a), \\ 0 &=& v_{zz} + \epsilon\theta v_z - \gamma v + u, \end{array}\right\} \tag{8.19}$$

with boundary conditions $(u, v)(-\infty) = \left(1, \frac{1}{\gamma}\right)$ and $(u, v)(\infty) = (0, 0)$. They then construct outer and inner approximations of the solution of this problem, taking the limits $\epsilon \to 0$. Setting $\epsilon = 0$, the lowest solution (u^0, v^0) of (8.19) satisfies

$$0 = u(1-u)(u-a) \text{ and } 0 = v_{zz} + u - \gamma v.$$

Then the first equation implies that we may take

$$u^0(z) = \begin{cases} 0, & z > 0 \\ 1, & z < 0 \end{cases}$$

Substituting this into the second equation we obtain

$$v^0(z) = \begin{cases} \frac{1}{2\gamma} e^{-\sqrt{\gamma} z}, & z > 0 \\ \frac{1}{\gamma} - \frac{1}{2\gamma} e^{-\sqrt{\gamma} z}, & z < 0 \end{cases}$$

which belongs to $C^1(\mathbb{R})$. This solution is called an outer solution in $\mathbb{R}$.

However as a discontinuous function $u^0(z)$ is not a good approximation of the solution in a neighbourhood of $z = 0$. In order to construct an approximate solution in a neighbourhood of $z = 0$, Funaki et al. introduce the stretched variable $\xi = z/\epsilon$ and rewrite the problem as

$$\left.\begin{array}{rcl} 0 &=& \tilde{u}_{\xi\xi} + (\theta - k\chi'(\tilde{v})\tilde{v}_z)\tilde{u}_\xi - \epsilon k(\chi'(\tilde{v})\tilde{v}_z)_z + u(1-u)(u-a), \\ 0 &=& \tilde{v}_{\xi\xi} + \epsilon^2(\theta\tilde{v}_\xi - \gamma\tilde{v} + \tilde{u}), \end{array}\right\} \tag{8.20}$$

where $(\tilde{u}, \tilde{v})(\xi) = (u, v)(\epsilon\xi)$. Setting now $\epsilon = 0$ and noting that $v^0(0) = \frac{1}{2\gamma}$ we see

$$\left.\begin{array}{rcl} 0 &=& \tilde{u}_{\xi\xi} + (\theta - k\chi'(\tilde{v})\tilde{v}_z)\tilde{u}_\xi + u(1-u)(u-a), \\ 0 &=& \tilde{v}_{\xi\xi}, \end{array}\right\} \tag{8.21}$$

with boundary conditions $\tilde{u}(-\infty) = 1$, $\tilde{u}(\infty) = 0$, $\tilde{v}(\pm\infty) = \frac{1}{2\gamma}$. Thus we get $\tilde{v}(\xi) \equiv \frac{1}{2\gamma}$ which implies

$$0 = \tilde{u}_{\xi\xi} + \left(\theta + \frac{k}{2\sqrt{\gamma}}\chi'\left(\frac{1}{2\gamma}\right)\right)\tilde{u}_\xi + u(1-u)(u-a).$$

The existence of an inner solution $(\tilde{u}, \tilde{v})(z/\epsilon)$ in a neighbourhood of $z = 0$ is now a consequence from the results in [33]. Matching the outer and inner solution a traveling front solution of the problem can be constructed in one spatial dimension. This is summarized in the following Theorem:

THEOREM 8.4 (M. Funaki, M. Mimura & T. Tsujikawa). *Fix $k > 0$ arbitrary. There is $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ the system* (8.19) *has a solution $(U^\epsilon, V^\epsilon)(z)$ for $\theta = (\theta(\epsilon; k)$ satisfying*

$$\lim_{\epsilon\to 0} U^\epsilon(z) = u^0(z) \text{ uniformly in } (-\infty, -\delta) \cup (\delta, \infty), \quad \lim_{\epsilon\to 0} V^\epsilon(z) \;=\; v^0(z) \text{ uniformly in } \mathbb{R}$$

with any small constant $\delta > 0$ and

$$\lim_{\epsilon\to 0} \theta(\epsilon; k) = \theta^*(k) = c - \frac{k}{2\sqrt{\gamma}}\chi'\left(\frac{1}{2\gamma}\right).$$

Moreover $\lim_{\epsilon\to 0} U^\epsilon(\epsilon\xi) \;=\; \tilde{u}(\xi)$, $\lim_{\epsilon\to 0} V^\epsilon(\epsilon\xi) \;=\; \frac{1}{2\gamma}$ uniformly on any compact subset of $\mathbb{R}$ with respect to $C^2(\mathbb{R})$.

Studying in two spatial dimensions the distribution of the eigenvalues of the eigenvalue problem of the linearized system around the front solutions with velocity $\epsilon\theta(\epsilon; k)$ in the strip domain Ω_l Funaki et al. find out that the transversal stability of the front shows the following behaviour:

THEOREM 8.5 (M. Funaki, M. Mimura & T. Tsujikawa).

1. *When $\chi''\left(\frac{1}{2\gamma}\right) \le 0$, for any fixed $l > 0$ and $k > 0$, there is $\epsilon_0 > 0$ such that all eigenvalues are in the left half of the complex plane if $0 < \epsilon < \epsilon_0$. Thus the traveling front solution ids transversally stable.*
2. *When $\chi''\left(\frac{1}{2\gamma}\right) > 0$, for any fixed $l > 0$ and $k > 0$, there is $\epsilon_0(l, k) > 0$ such that the traveling front solution is unstable for any $0 < \epsilon < \epsilon_0(l, k)$.*

Thus the tranversal stability of the wave front depends of the chemotactical sensitivity, where transversally stable is defined in [41] in the following way:

DEFINITION 8.6. *A traveling front solution of* (8.18) *is transversally stable except for translational invariance in x, if and only if zero is a simple eigenvalue of the eigenvalue problem associated with the linearized system of* (8.18) *around the traveling front solution and the remaining spectrum is contained in a closed sector lying in the left half of the complex plane. The traveling front is unstable if it is not stable.*

Further results related to this system can also be found in [13] and [14].

9. Conclusions and personal comments. There are of course still many open problems in connection with the various approaches that have been studied for the Keller-Segel model.

This paper dealt solely with the parabolic model proposed by Keller and Segel for the aggregation phase of mobile species caused by chemotaxis. However there are also different approaches to chemotaxis and hencealso numerous different models describing chemotaxis. First of all, one must always keep in mind that the model one uses is based either on a microscopic or a macroscopic approach and always depends on the species studied. Thus transport and hyperbolic models for chemotaxis have also been proposed. For example transport and hyperbolic models for chemotactical movement have been studied in [4, 5, 34, 37, 38, 55, 57, 59] and [110]. The connection between chemotaxis equations as the parabolic limit of velocity jump processes or transport models for chemotaxis has been studied in [4, 5, 22, 58, 113] and [114]. I refer to [37] for surveyes on different models for chemotactical movement and to [59] for a survey on the hyperbolic approach to chemotaxis.

Of course there are many publications presenting experimental data on chemotactic effects and the influences of changes in the motility or the chemotactic sensitivity of the given species (see for example [35, 36]). Models which take the chemotactical movement of n populations according to k sensitivity agents into account have been proposed in [3, Beispiel 2.47, page 58] and in [152] by G. Wolansky. Wolansky has studied a generalization of the Keller-Segel model for n populations in the absence of conflicts. Also the effect of multiple attractant gradients on chemotactical movement has been studied [140] and numerical solutions for the corresponding models have been calculated [39, 40]. Numerical analysis for the Keller-Segel model has been performed, for example, by Gajewski and Zacharias using a chemotaxis version of the well-established TOSCA code for solving semiconductor problems and by Nakaguchi and Yagi studying the full discrete approximation of the Keller-Segel model by Galerkin Runge-Kutta methods [100, 101]. The transport chemotaxis model is dealt with in [21] and [40], for example.

The results available for systems related to the Keller-Segel equations such as the Othmer-Stevens model are so numerous that I mention only a few [77, 112, 141] and [154]. The well-posedness of the Othmer-Stevens model follows directly from the results by Rascle [117].

Of course, the functional forms appearing in the original Keller-Segel model can vary from species to species and some explicit models for the cAMP oscillation have been proposed (see [80] for the so called Martiel and Goldbeter model, [111] for a survey on the oscillatory cAMP signaling in Dictyostelium discoideum and [120] for the description of the role of cAMP in the development of Dictyostelium discoideum). The Keller-Segel model has also been used to describe different problems. For example, in [84] the Keller-Segel equations have been proposed for strip pattern formation in alligator embryos. Angiogenesis has also been proposed as another application of Keller-Segel type models (see for example [27] including references). The large number of applications and of possible functional forms results directly in a large number of models depending on the considered problem. In some particular papers this has also resulted in the addition of a third equation to the Keller-Segel model (2.4) or the rediscovery of a more complicated version of the Keller-Segel model. This results from attempts to describe more complicated pattern formations during the aggregation phase of mobile species such as the attempt to describe spiral waves during the aggregation (see, for example, [134, 144] and [145] for such extended models).

At the conclusion of this survey I would ask the reader to allow me a personal comment. The references given in this text are far from complete. I have tried to give the interested reader a brief summary of the latest developments in the Keller-Segel model. Thus, this article is intended as continuation of Evelyn Fox Keller's article "Assessing the Keller-Segel model: How has it fared" of 1980 [74]. It is left to the reader to decide whether I have succeeded.

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Appendix A. Notation. Throughout the paper we use the following notations. Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be an open set. We use the following notation for spaces of differentiable functions on Ω. $C^k(\Omega) := \{u : \Omega \to \mathbb{R} \mid u \text{ is } k\text{−times continuous differentiable}\}$, $C^k(\overline{\Omega}) : \{u \in C^k(\Omega) \mid D^\alpha u \text{ is continuous for all } |\alpha| \leq k\}$, $C^\infty(\Omega) := \bigcap_{k=0}^{\infty} C^k(\Omega)$, and $C^\infty(\overline{\Omega}) := \bigcap_{k=0}^{\infty} C^k(\overline{\Omega})$. For the Hölder continuous functions we use the notation

$$C^{k,\beta}(\overline{\Omega}) := \{u \in C^k(\overline{\Omega}) \mid \sum_{|\alpha|\leq k} ||D^\alpha u||_{C(\overline{\Omega})} + \sum_{|\alpha|=k} [D^\alpha u]_{C^{0,\beta}(\overline{\Omega})} < \infty\},$$

$$\text{where } ||u||_{C(\overline{\Omega})} \equiv \sup_{x\in\Omega} |u(x)| \text{ and } [u]_{C^{0,\beta}(\overline{\Omega})} \equiv \sup_{x,y\in\Omega, x\neq y} \left\{ \frac{|u(x)-u(y)|}{|x-y|^\beta} \right\}.$$

We also use the following function spaces. $L^r(\Omega) := \{u : \Omega \to \mathbb{R} \mid u \text{ is measurable and } ||u||_{L^r(\Omega)} < \infty\}$, where

$$||u||_{L^r(\Omega)} := \left(\int_\Omega |u|^r dx \right)^{1/r} \quad \text{for } 1 \leq r < \infty.$$

$L^\infty(\Omega) := \{u : \Omega \to \mathbb{R} \mid u \text{ is measurable and } ||u||_{L^\infty(\Omega)} < \infty\}$, where $||u||_{L^\infty(\Omega)} := \operatorname{ess\,sup}_\Omega |u|$.

$W^{k,r}(\Omega) := \{u : \Omega \to \mathbb{R} \mid u \in L^r(\Omega)$ and $D^\alpha u$ exists weakly for all multiindices α with $|\alpha| \leq k$ and belong to $L^r(\Omega)\}$, where the norm in $W^{k,r}(\Omega)$ is given by

$$||u||_{W^{k,r}(\Omega)} := \begin{cases} \left(\sum_{|\alpha|\leq k} \int_\Omega |D^\alpha u|^r dx \right)^{1/r}, & 1 \leq r < \infty \\ \sum_{|\alpha|\leq k} \operatorname{ess\,sup}_\Omega |D^\alpha u|, & r = \infty \end{cases}.$$

We use the notation $H^1(\Omega)$ for functions belonging to $W^{1,2}(\Omega)$ and $H^2(\Omega)$ for functions belonging to $W^{2,2}(\Omega)$. We denote the dual space of $H^1(\Omega)$ by $(H^1(\Omega))^*$. Finally, the following notations are used for time dependent L^r-spaces: $L^r(\mathcal{X};(0,T)) := \{u : [0,T] \to \mathcal{X} \mid u$ is measurable and $||u||_{L^r(\mathcal{X};(0,T))} < \infty\}$,

$$||u||_{L^r(\mathcal{X};(0,T))} := \left(\int_0^T ||u(t)||^r_{\mathcal{X}} dt \right)^{1/r}$$

for $1 \leq r < \infty$ and, $||u||_{L^\infty(\mathcal{X};(0,T))} := \operatorname*{ess\,sup}_{0 \leq t \leq T} ||u||_{\mathcal{X}}$ for $r = \infty$.

$C(\mathcal{X};[0,T]) := \{u : [0,T] \to \mathcal{X} \mid u$ is continuous and $||u||_{C(\mathcal{X};[0,T])} < \infty\}$, where $||u||_{C(\mathcal{X};[0,T])} := \max_{0 \leq t \leq T} ||u(t)||_{\mathcal{X}}$.

We write $L^r(\Omega_T)$ for the function spaces $L^r(L^r(\Omega);(0,T))$ for all $1 \leq r \leq \infty$.