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MORTAR METHODS WITH CURVED INTERFACES

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Abstract. We analyze a nonoverlapping domain decomposition technique with curvilinear boundaries. The weak coupling at the curved interfaces is carried out in terms of Lagrange multiplier spaces. We use the abstract framework of mortar and blending elements to obtain a priori results for this nonconforming discretization scheme. Introducing a mesh dependent jump on the curved interfaces based on piecewise linear approximations of the interfaces, the consistency error for the piecewise linear approximation can be decomposed into a consistency error for blending elements and a variational crime. Numerical results illustrate the performance of the method.

Key words. curved interface, mortar method, domain decomposition, curvilinear elements

1. Introduction. Nonconforming domain decomposition methods provide an efficient and flexible tool for the numerical approximation of partial differential equations. Here, we consider a nonoverlapping decomposition into subdomains with curvilinear boundaries. The characteristic idea of many nonconforming methods is to replace the strong pointwise continuity across the interior interfaces by a weak integral condition or to penalize the jump in the discrete solution across the interfaces. To obtain optimal discretization schemes, the consistency error has to be small enough. The coupling of different discretization schemes or of nonmatching triangulations can be analyzed within the framework of mortar methods. These nonconforming domain decomposition techniques provide a more flexible approach than standard conforming approaches. We generalize well-known mortar formulations [4, 5] to the case of curvilinear boundaries and non-matching triangulations. The a priori analysis can be carried out in terms of blending elements [9–11] and within the abstract mortar setting. Of crucial importance are norm equivalences and uniform inf-sup conditions [2]. Recently, a lot of work has been done to analyze mortar settings and the coupling of different model equations. However, most approaches are restricted to straight or planar interfaces. In the special case of two subdomains without crosspoints and one curvilinear interface, a first theoretical result can be found in [13]. There, a piecewise constant Lagrange multiplier space is used, which does not guarantee a uniform inf-sup condition, and a priori estimates for the H^1 -norm of the discretization error are given. In the present paper, we work with discrete Lagrange multiplier spaces for which uniform inf-sup conditions hold and consider many subdomains. We focus on the analysis of the variational crime which enters by using piecewise linear approximations of the curvilinear interfaces. According to these approximations, we use piecewise linear interpolations to map the non-matching meshes on the master and slave sides onto a reference segment, in order to define the jump of a finite element function across the curved interface.

In the rest of the introduction, we consider the model problem, provide some notation and discuss the discrete saddle point problem. In Section 2, we carry out the convergence analysis and provide optimal a priori results for the discretization error in the H^1 -norm for the primal variable and in the $H^{-1/2}$ -norm for the Lagrange multiplier.

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Numerical results are given in Section 3. In a test series, we show the stability and flexibility of the approach and consider the influence of the number of subdomains on the discretization errors.

1.1. Problem formulation. We consider the classical model problem

$$-\Delta u = f \quad \text{on } \Omega \subset \mathbb{R}^2, \quad u = 0 \text{ on } \partial\Omega, \quad (1.1)$$

where $f \in L^2(\Omega)$ is piecewise Lipschitz continuous. We assume that $\Omega \subset \mathbb{R}^2$ is a polygon; this assumption is not necessary and merely done in order to concentrate on the curved interfaces. Let N_Γ smooth (C^2 is sufficient) curves Γ_i , $i = 1, \dots, N_\Gamma$, be given in terms of the parametrizations

$$\gamma_i \in C^2(\hat{I}, \mathbb{R}^2), \quad (1.2a)$$

where

$$\hat{I} := [0, 1] \quad (1.2b)$$

is the closed reference interval. We assume furthermore that the parametrizations γ_i are injective and satisfy, for some $c > 0$,

$$|\gamma_i(t) - \gamma_i(t')| \geq c|t - t'| \quad \forall t, t' \in \hat{I}. \quad (1.2c)$$

REMARK 1.1. The above setting excludes closed curves Γ_i ; however, closed curves can be incorporated into our theory by taking as the reference configuration \hat{I} the torus instead of the unit interval. ■

We assume that the interior of Γ_i is contained in Ω , i.e., no part except possibly the endpoints of Γ_i lies on $\partial\Omega$. We allow these curves to meet at $N_C \in \mathbb{N}_0$ *cross points*. Without loss of generality, we will assume that the curves Γ_i are selected such that cross points can only occur at their endpoints. These curves divide the domain Ω into N_Ω subdomains Ω_j , $j = 1, \dots, N_\Omega$, which we assume to be Lipschitz domains, see Figure 1.1. We require the number of interfaces Γ_i belonging to one subdomain Ω_j to be bounded. Each curve Γ_i is shared by two domains $\Omega_{m(i)}$, $\Omega_{n(i)}$, i.e., $\overline{\Gamma_i} \subset \partial\Omega_{m(i)} \cap \partial\Omega_{n(i)}$, where $\Omega_{m(i)}$ is called the *master side* of Γ_i , and $\Omega_{n(i)}$ is called the *slave side*. In what follows, we will frequently omit the argument i in $m(i)$ and $n(i)$. We employ usual Sobolev spaces and norms, [1], and introduce the space $X = \prod_{j=1}^{N_\Omega} H^1(\Omega_j)$ with the corresponding broken H^1 -norm given by

$$\|u\|_X^2 := \sum_{j=1}^{N_\Omega} \|u\|_{H^1(\Omega_j)}^2. \quad (1.3)$$

1.2. Formulation of the numerical method.

1.2.1. Spaces. We will restrict our attention to piecewise linear discretizations for the domain parts; the Lagrange multiplier can be discretized by means of any of the standard stable spaces, [4–6, 15, 16]. For each subdomain Ω_j , we have a quasi-uniform, shape-regular simplicial triangulation \mathcal{T}_j with mesh size h_j of a domain $\Omega_{j,h}$ that approximates Ω_j in the sense that $\partial\Omega_{j,h}$ is a piecewise linear interpolation of $\partial\Omega_j$. We insist on the endpoints of the curves Γ_i , $i = 1, \dots, N_\Gamma$, being interpolation points. For each curve Γ_i , we obtain in this way two piecewise linear approximations that we denote by $\Gamma_{i,h}^n$ and $\Gamma_{i,h}^m$. The superscripts n and m indicate that $\Gamma_{i,h}^n$ and $\Gamma_{i,h}^m$

are parts of $\partial\Omega_{n(i),h}$, $\partial\Omega_{m(i),h}$, respectively. Since $\Gamma_{i,h}^n$ and $\Gamma_{i,h}^m$ are piecewise linear interpolations of Γ_i , we can parametrize them in the standard way by continuous piecewise linear functions

$$\gamma_{i,h}^n, \gamma_{i,h}^m : \hat{I} \rightarrow \mathbb{R}^2, \quad (1.4)$$

as illustrated in the right picture of Figure 1.1.

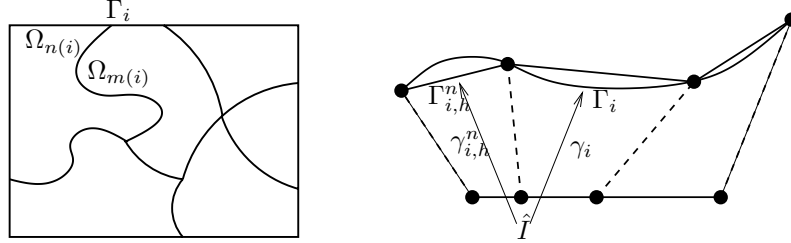


FIG. 1.1. Decomposition of Ω into six subdomains and three cross points given by eight curves (left) and interface Γ_i and its piecewise linear interpolation $\Gamma_{i,h}^n$ (right).

In particular, the endpoints of \hat{I} are mapped to the same points under these two mappings. The set \mathcal{N}_i of nodes, $\mathcal{N}_i := \{x \in \hat{I} : \gamma_{i,h}^n(x) = p, \text{ where } p \text{ is an interpolation point on } \Gamma_{i,h}^n\}$, forms a mesh on \hat{I} . Since the triangulation $\mathcal{T}_{n(i)}$ is assumed to be shape regular and the parametrizations γ_i satisfy (1.2), the nodes \mathcal{N}_i form a shape regular mesh on \hat{I} with mesh size \hat{h} . Moreover, we can find constants C_j such that $C_j^{-1}h_j \leq \hat{h} \leq C_j h_j$, $j = 1, \dots, N_\Omega$.

We work with two different dual spaces

$$M^0 := \prod_{i=1}^{N_\Gamma} H^{-1/2}(\Gamma_i), \quad H^{-1/2}(\Gamma_i) := \left(H_{00}^{1/2}(\Gamma_i) \right)',$$

$$M := \prod_{i=1}^{N_\Gamma} \left(H^{1/2}(\Gamma_i) \right)'$$

and equip the spaces $H^{1/2}(\Gamma_i)$ and $H_{00}^{1/2}(\Gamma_i)$ with the ‘‘intrinsic’’ norms, i.e., the Slobodecki norms:

$$\|v\|_{H^{1/2}(\Gamma_i)}^2 = \|u\|_{L^2(\Gamma_i)}^2 + \int_{\Gamma_i} \int_{\Gamma_i} \frac{|v(x) - v(y)|^2}{|x - y|^2} dx dy, \quad (1.5a)$$

$$\|v\|_{H_{00}^{1/2}(\Gamma_i)}^2 = \|v\|_{H^{1/2}(\Gamma_i)}^2 + \int_{\Gamma_i} \frac{1}{\text{dist}(x, \partial\Gamma_i)} |v(x)|^2 dx. \quad (1.5b)$$

The duality pairings for a curve D between $H^{1/2}(D)$ and its dual, and between $H_{00}^{1/2}(D)$ and its dual are denoted by $\langle \cdot, \cdot \rangle_D$ and $\langle \cdot, \cdot \rangle_{H^{-1/2}, D}$, respectively. We note that in case of $v \in H_{00}^{1/2}(D)$, $\mu \in (H^{1/2}(D))'$, the two duality pairings coincide. Moreover, it holds that $(H^{1/2}(\partial\Omega_j))' = H^{-1/2}(\partial\Omega_j)$, $1 \leq j \leq N_\Omega$. On the spaces M^0 and M , we define the norms $\|\cdot\|_{M^0}$ and $\|\cdot\|_M$ by

$$\|\mu\|_{M^0}^2 := \sum_{i=1}^{N_\Gamma} \|\mu\|_{H^{-1/2}(\Gamma_i)}^2, \quad \mu \in M^0, \quad \|\mu\|_M^2 := \sum_{i=1}^{N_\Gamma} \|\mu\|_{(H^{1/2}(\Gamma_i))'}^2, \quad \mu \in M,$$

respectively. It is easy to see that $M \subset M^0$ and $\|\cdot\|_M$ is the stronger norm, i.e., $\|\mu\|_M \geq \|\mu\|_{M^0}$, $\mu \in M$. By assumption f is in the dual space of $H_0^1(\Omega_j)$, $1 \leq j \leq N_\Omega$, and thus the normal flux of the weak solution $u|_{\Omega_j}$ on $\partial\Omega_j$ can be identified with a unique element $\sigma_j \in H^{-1/2}(\partial\Omega_j)$. Let the interface Γ_i be a subset of $\partial\Omega_j$. Then we can define the restriction of σ_j to Γ_i by

$$\langle v, \sigma_j|_{\Gamma_i} \rangle_{H^{-\frac{1}{2}}; \Gamma_i} := \langle v, \sigma_j \rangle_{H^{-\frac{1}{2}}; \partial\Omega_j}, \quad v \in H_{00}^{\frac{1}{2}}(\Gamma_i),$$

where the trivial extension of v onto $\partial\Omega_j$ is still denoted by v . By definition, we find $\sigma_j|_{\Gamma_i} \in H^{-1/2}(\Gamma_i)$. In what follows, we assume that $\sigma_j|_{\Gamma_i} \in \widehat{H}^{-1/2}(\Gamma_i) \subset H^{-1/2}(\Gamma_i)$ and that $\sigma_{n(i)}|_{\Gamma_i} = -\sigma_{m(i)}|_{\Gamma_i}$, $1 \leq i \leq N_\Gamma$. We then define the Lagrange multiplier $\lambda \in M \subset M^0$ by

$$\lambda|_{\Gamma_i} := -\frac{1}{g_i} \nabla u \cdot \vec{n}_{n(i)} = -\frac{1}{g_i} \sigma_{n(i)}|_{\Gamma_i} = \frac{1}{g_i} \sigma_{m(i)}|_{\Gamma_i}, \quad i = 1, \dots, N_\Gamma; \quad (1.6)$$

here $\vec{n}_{n(i)}$ denotes the outer normal vector of $\Omega_{n(i)}$ on Γ_i and the functions g_i are defined in (1.11) ahead.

The nonconforming discretization is based on a saddle point problem, and we have to discretize the spaces X and M^0 . The space of piecewise linear functions on \mathcal{T}_j is denoted by $X_{j,h}$, and we set $X_h := \prod_{j=1}^{N_\Omega} X_{j,h}$. We denote by F_K the element map from the reference element \widehat{K} associated with the element $K \in \mathcal{T}_j$. Additionally, we need to introduce discrete spaces of Lagrange multipliers associated with the curves Γ_i , $i = 1, \dots, N_\Gamma$. Here, we exploit the fact that the curves Γ_i , $\Gamma_{i,h}^n$, and $\Gamma_{i,h}^m$ can be identified with the reference interval \widehat{I} via the maps γ_i , $\gamma_{i,h}^n$, $\gamma_{i,h}^m$. On \widehat{I} , we take for each i any one of the standard Lagrange multiplier spaces [4-6, 15, 16] that are based on the mesh on \widehat{I} determined by the nodes \mathcal{N}_i , and we denote this space by $\widehat{M}_{i,h} \subset L^2(\widehat{I})$. Via γ_i , we define on Γ_i the space

$$M_{i,h} := \widehat{M}_{i,h} \circ \gamma_i^{-1}, \quad M_h := \prod_{i=1}^{N_\Gamma} M_{i,h}. \quad (1.7)$$

One of the key properties of the discrete Lagrange multiplier spaces $\widehat{M}_{i,h}$ is the existence of a projection $\widehat{\Pi}_{i,h} : H_{00}^{1/2}(\widehat{I}) \rightarrow \widehat{W}_{i,h}$ satisfying

$$(v, \mu)_{\widehat{I}} = (\widehat{\Pi}_{i,h} v, \mu)_{\widehat{I}} \quad \forall \mu \in \widehat{M}_{i,h}, \quad (1.8a)$$

$$\|\widehat{\Pi}_{i,h} v\|_{H_{00}^{1/2}(\widehat{I})} \leq C_{\widehat{\Pi}} \|v\|_{H_{00}^{1/2}(\widehat{I})}, \quad \|\widehat{\Pi}_{i,h} v\|_{L^2(\widehat{I})} \leq C_{\widehat{\Pi}} \|v\|_{L^2(\widehat{I})} \quad \forall v \in H_{00}^{1/2}(\widehat{I}), \quad (1.8b)$$

where $(\cdot, \cdot)_{\widehat{I}}$ is the L^2 -scalar product on \widehat{I} , and $\widehat{W}_{i,h}$ is the pull-back to \widehat{I} of the trace of $X_{n(i),h}$ on $\Gamma_{i,h}^n$, i.e., $\widehat{W}_{i,h} := \{u|_{\Gamma_{i,h}^n} \circ \gamma_{i,h}^n : u \in X_{n(i),h}\}$. The continuity constant $C_{\widehat{\Pi}}$ of the projections $\widehat{\Pi}_{i,h}$ is assumed to be independent of the discretization parameters. Another important property of the mortar space M_h is that it contains the characteristic functions χ_{Γ_i} of the interfaces Γ_i (cf. Lemma 2.2).

REMARK 1.2. The assumptions (1.2) imply that γ_i and γ_i^{-1} are C^1 . A direct calculation with the involved norms implies the existence of $C > 0$ such that for each $i = 1, \dots, N_\Gamma$

$$C^{-1} \|u\|_{L^2(\Gamma_i)} \leq \|u \circ \gamma_i\|_{L^2(\widehat{I})} \leq C \|u\|_{L^2(\Gamma_i)} \quad \forall u \in L^2(\Gamma_i), \quad (1.9a)$$

$$C^{-1} \|u\|_{H_{00}^{1/2}(\Gamma_i)} \leq \|u \circ \gamma_i\|_{H_{00}^{1/2}(\widehat{I})} \leq C \|u\|_{H_{00}^{1/2}(\Gamma_i)} \quad \forall u \in H_{00}^{1/2}(\Gamma_i). \quad (1.9b)$$

Moreover, for a curve Γ and a function $g \in W^{1,\infty}(\Gamma)$ that is bounded away from 0, we obtain

$$C^{-1}\|gu\|_{H^{1/2}(\Gamma)} \leq \|u\|_{H^{1/2}(\Gamma)} \leq C\|u/g\|_{H^{1/2}(\Gamma)} \quad \forall u \in H^{1/2}(\Gamma), \quad (1.9c)$$

$$C^{-1}\|gu\|_{H_{00}^{1/2}(\Gamma)} \leq \|u\|_{H_{00}^{1/2}(\Gamma)} \leq C\|u/g\|_{H_{00}^{1/2}(\Gamma)} \quad \forall u \in H_{00}^{1/2}(\Gamma). \quad (1.9d)$$

We note that the constant C in (1.9) depends on the length of the corresponding interface Γ_i . In this particular case, we will not rigorously analyze the dependence of the constant on the decomposition into subdomains. However, in all the following proofs, we will exploit the fact that the number of interfaces Γ_i belonging to one subdomain Ω_j is bounded, and demonstrate that the constants depend on this bound, and not on the total number of subdomains N_Ω . Moreover, in Section 3, we provide a numerical example illustrating this independence. ■

1.2.2. Bilinear forms and finite element method. To define our saddle point formulation, we also need to define the jump across the interface Γ_i . For a function $u \in X$, the jump across Γ_i is given in the standard way and denoted by $[u]_i$. For $u \in X_h$, we exploit the fact that $\Gamma_{i,h}^n, \Gamma_{i,h}^m$ can be identified with \hat{I} in order to define the discrete jump across Γ_i as

$$[u]_{i,h} := u|_{\Omega_{n(i),h}} \circ \gamma_{i,h}^n \circ \gamma_i^{-1} - u|_{\Omega_{m(i),h}} \circ \gamma_{i,h}^m \circ \gamma_i^{-1}.$$

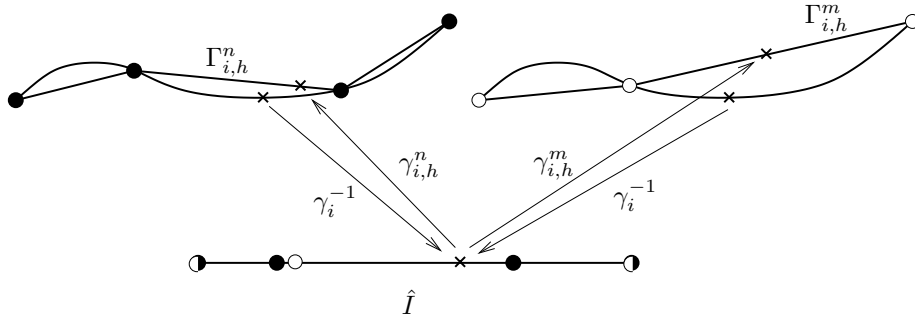


FIG. 1.2. The mappings involved in the definition of the discrete jump.

In Figure 1.2, we illustrate the mappings which are involved in the above definition of the discrete jump, and their action on an arbitrary point of the interface Γ_i . We finally define the following domain bilinear forms

$$a(u, v) := \sum_{j=1}^{N_\Omega} \int_{\Omega_j} \nabla u \cdot \nabla v \, dx, \quad (1.10a)$$

$$a_h(u, v) := \sum_{j=1}^{N_\Omega} \sum_{K \in \mathcal{T}_j} \int_K \nabla u \cdot \nabla v \, dx. \quad (1.10b)$$

Related to the jump across the interfaces Γ_i are the bilinear forms

$$b(u, \mu) := \sum_{i=1}^{N_\Gamma} \langle g_i[u]_i, \mu \rangle_{\Gamma_i}, \quad (u, \mu) \in X \times M,$$

$$b_h(u, \mu) := \sum_{i=1}^{N_\Gamma} (g_i[u]_{i,h}, \mu)_{L^2(\Gamma_i)}, \quad (u, \mu) \in X_h \times M_h,$$

where the weight functions $g_i \in C^1(\Gamma_i)$ are given by

$$g_i := \frac{1}{|\gamma_i' \circ \gamma_i^{-1}|}, \quad i = 1, \dots, N_\Gamma. \quad (1.11)$$

It is then possible to check that the pair $(u, \lambda) \in X \times M$ satisfies the following saddle point problem:

$$\begin{aligned} a(u, v) + b(v, \lambda) &= l(v) := \int_\Omega f v \, dx & \forall v \in X, \\ b(u, \mu) &= 0 & \forall \mu \in M, \end{aligned}$$

Correspondingly, the approximate solution $u_h \in X_h$ and the approximation to the Lagrange multiplier, $\lambda_h \in M_h$, are given as the solution of

$$\begin{aligned} a_h(u_h, v) + b_h(v, \lambda_h) &= l_h(v) & \forall v \in X_h, \\ b_h(u_h, \mu) &= 0 & \forall \mu \in M_h. \end{aligned} \quad (1.12)$$

Here, the approximation l_h to the linear functional l is given by

$$l_h(v) = \sum_{j=1}^{N_\Omega} \int_{\Omega_{j,h}} f_j v \, dx,$$

where we assume that f_j is a Lipschitz continuous function on $\Omega_j \cup \Omega_{j,h}$ such that

$$f_j = f \quad \text{in } \Omega_j \cap \Omega_{j,h}, \quad (1.13a)$$

$$\|f_j\|_{W^{1,\infty}(\Omega_j \cup \Omega_{j,h})} \leq C_f, \quad (1.13b)$$

and the constant C_f is independent of mesh parameters. We note that the bilinear form b on $X \times M$ is continuous but does not yield a inf-sup condition with respect to the norms $\|\cdot\|_X$ and $\|\cdot\|_M$. This also holds true for the discrete space $X_h \times M_h$ where no uniform inf-sup condition can be shown. Of crucial importance will be to establish a uniform discrete inf-sup condition for the weaker norm $\|\cdot\|_{M^0}$.

2. Convergence analysis. We do not analyze problem (1.12) directly. To obtain a priori bounds for the discretization error, we proceed in two steps. In the first step, we introduce and analyze a new discrete variational problem based on blending elements, where the curved interfaces are resolved in an exact way. In the second step, we interpret (1.12) as a perturbed blending approach, and estimate the perturbation terms obtained from the first Strang lemma.

2.1. Error analysis using blending elements.

2.1.1. Spaces of blending elements. We start with the simpler case where the subdomains Ω_j , $j = 1, \dots, N_\Omega$, are represented exactly by the triangulations. Since Ω is a polygon, only the elements that have two vertices on one of the interfaces Γ_i , $i = 1, \dots, N_\Gamma$, require a treatment as a *blending element*, [10, 11]. We assume that the blending function for elements that have an edge lying on Γ_i is taken as the parametrization γ_i of Γ_i . The triangulation of Ω_j using blending elements will be denoted by $\tilde{\mathcal{T}}_j$ with element maps $\tilde{F}_{\tilde{K}}$, $\tilde{K} \in \tilde{\mathcal{T}}_j$. The triangulations \mathcal{T}_j and $\tilde{\mathcal{T}}_j$ are illustrated in Figure 2.1.

We define the spaces

$$\tilde{X}_{j,h} := \{u \in H^1(\Omega_j) : u|_{\tilde{K}} \circ \tilde{F}_{\tilde{K}} \in \mathcal{P}_1 \quad \forall \tilde{K} \in \tilde{\mathcal{T}}_j \quad \text{and } u|_{\partial\Omega} = 0\},$$

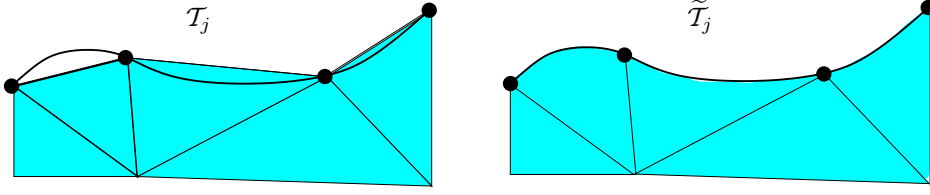


FIG. 2.1. Triangulations \mathcal{T}_j and $\tilde{\mathcal{T}}_j$.

where \mathcal{P}_1 denotes the classical space of polynomials of degree 1 on the reference element \hat{K} . We introduce the problem: Find $(\tilde{u}_h, \tilde{\lambda}_h) \in \tilde{X}_h \times M_h$ such that

$$\begin{aligned} a(\tilde{u}_h, v) + b(v, \tilde{\lambda}_h) &= l(v) & \forall v \in \tilde{X}_h, \\ b(\tilde{u}_h, \mu) &= 0 & \forall \mu \in M_h. \end{aligned} \quad (2.1)$$

In order to analyze the error $u - \tilde{u}_h$ and the error of the Lagrange multiplier $\lambda - \tilde{\lambda}_h$, we introduce the *constrained space*

$$\tilde{V}_h := \{v \in \tilde{X}_h : b(v, \mu) = 0 \quad \forall \mu \in M_h\}. \quad (2.2)$$

Additionally, we require, as is standard for blending elements (see, e.g., the more detailed construction in the Appendix A):

$$h_{\hat{K}}^{-1} \|\tilde{F}'_{\hat{K}}\|_{L^\infty(\hat{K})} + h_{\hat{K}} \left(\tilde{F}'_{\hat{K}} \right)^{-1} \|_{L^\infty(\hat{K})} \leq C_\gamma, \quad (2.3a)$$

$$\|\tilde{F}''_{\hat{K}}\|_{L^\infty(\hat{K})} \leq C_\gamma h_{\hat{K}}^2, \quad (2.3b)$$

where $\tilde{F}'_{\hat{K}}$ and $\tilde{F}''_{\hat{K}}$ denote the Jacobian and the Hessian of $\tilde{F}_{\hat{K}}$, respectively. Of essential use to us will be the following observation that stems from using blending elements based on the parametrizations γ_i :

LEMMA 2.1. *Let $X_{j,h}$ and $\widehat{W}_{i,h}$ be defined as in Section 1.2.1. Then for each interface Γ_i , $i = 1, \dots, N_\Gamma$, we have*

$$\tilde{X}_{n(i),h}|_{\Gamma_i \circ \gamma_i} = \widehat{W}_{i,h}.$$

Furthermore, there exists a constant $C > 0$, which depends solely on the parametrizations γ_i , the subdomains Ω_j , and the shape-regularity constant C_γ , and there exists a lifting operator $Z_i : \widehat{W}_{i,h} \cap H_{00}^{1/2}(\hat{I}) \rightarrow \tilde{X}_{n(i),h}$ such that

$$\begin{aligned} \|Z_i \hat{v}_h\|_{H^1(\Omega_{n(i)})} &\leq C \|\hat{v}_h\|_{H_{00}^{1/2}(\hat{I})}, \\ (Z_i \hat{v}_h)|_{\partial\Omega_{n(i)} \setminus \Gamma_i} &= 0, \\ (Z_i \hat{v}_h)|_{\Gamma_i \circ \gamma_i} &= \hat{v}_h. \end{aligned}$$

Proof. Let $\hat{v}_h \in \widehat{W}_{i,h} \cap H_{00}^{1/2}(\hat{I})$ be given. Define v_h on Γ_i via $v_h := \hat{v}_h \circ \gamma_i^{-1}$. By Remark 1.2 we then get $\|v_h\|_{H_{00}^{1/2}(\Gamma_i)} \leq C \|\hat{v}_h\|_{H_{00}^{1/2}(\hat{I})}$. Next, we extend v_h to $\partial\Omega_{n(i)}$ by zero. The thus extended function (still denoted v_h) satisfies $\|v_h\|_{H^{1/2}(\partial\Omega_{n(i)})} \leq C \|\hat{v}_h\|_{H_{00}^{1/2}(\hat{I})}$ (see, for example, [12, Thms. 1.5.1.3, 1.5.2.3]). Next, let Ev_h be the harmonic extension into $\Omega_{n(i)}$ of v_h . Finally, let $I^C : H^1(\Omega_{n(i)}) \rightarrow \tilde{X}_{n(i),h}$ be a Clément type interpolation operator on $\tilde{\mathcal{T}}_j$ that is continuous and preserves piecewise polynomial boundary conditions; such operators have been constructed in [3, 14]. We finally set $Z_i \hat{v}_h := I^C Ev_h$. \square

2.1.2. A priori results for blending elements. In this subsection, we establish optimal a priori bounds for the discretization errors $u - \tilde{u}_h$ and $\lambda - \tilde{\lambda}_h$. Of crucial importance are the best approximation property of the constrained space and a uniform inf-sup condition. Using classical compactness arguments, we have the following ellipticity result on the space

$$X_\chi := \{u \in X : b(u, \mu) = 0 \ \forall \mu \in \text{span}\{\chi_{\Gamma_i} \mid i = 1, \dots, N_\Gamma\}\}, \quad (2.4)$$

where $\chi_{\Gamma_i} \in M^0$ is the function that is identically 1 on Γ_i and vanishes on all other interfaces.

LEMMA 2.2. *There exists a constant $C > 0$ such that for all $v \in X_\chi$ there holds*

$$C\|v\|_X^2 \leq a(v, v).$$

Proof. The coercivity assertion is well-known within the framework of mortar methods with straight interfaces [4]. Moreover, in the case of straight interfaces, the ellipticity constant is independent of the number of subdomains [8]. \square

For the moment, we do not show the existence of a unique solution $(\tilde{u}_h, \tilde{\lambda}_h)$ of (2.1). Since (2.1) corresponds to a square system of linear equations, existence follows from uniqueness. For all standard Lagrange multiplier spaces $\text{span}\{\chi_{\Gamma_i} \mid i = 1, \dots, N_\Gamma\} \subset M_h$ and thus the bilinear form a is coercive on the constrained space \tilde{V}_h . To obtain uniqueness of the solution, it is sufficient to establish a suitable uniform inf-sup condition, see, e.g., [7], the proof of which will be given in Proposition 2.7.

Concerning the approximation we have the following classical result:

PROPOSITION 2.3. *Let u be the weak solution of (1.1), and assume that $\lambda \in M$ is defined by (1.6). Let \tilde{u}_h be the solution of (2.1). Then*

$$\|u - \tilde{u}_h\|_X \leq C \inf_{v_h \in \tilde{V}_h} \|u - v_h\|_X + C \sup_{w_h \in \tilde{V}_h} \frac{b(w_h, \lambda)}{\|w_h\|_X}, \quad (2.5)$$

for some $C > 0$ that is independent of h . Moreover, if the Lagrange multiplier λ is in $\prod_{i=1}^{N_\Gamma} L^2(\Gamma_i)$, we find

$$\|u - \tilde{u}_h\|_X \leq C \inf_{v_h \in \tilde{V}_h} \|u - v_h\|_X + C \left\{ \sum_{i=1}^{N_\Gamma} h_{n(i)} \inf_{\mu_h \in M_{i,h}} \|\lambda - \mu_h\|_{L^2(\Gamma_i)}^2 \right\}^{\frac{1}{2}}, \quad (2.6)$$

for some $C > 0$ that is independent of h .

Proof. The first estimate (2.5) is a standard result for nonconforming finite elements. We note that the second term measures the consistency error. For the second error bound, (2.6), we employ the following approximation property of typical Lagrange multiplier spaces, which we obtain from the case of straight interfaces via the equivalence (1.9a):

$$\|w - \Pi_i w\|_{L^2(\Gamma_i)} = \inf_{z \in M_{i,h}} \|w - z\|_{L^2(\Gamma_i)} \leq Ch_{n(i)}^{1/2} |w|_{H^{1/2}(\Gamma_i)}, \quad (2.7)$$

where Π_i denotes the L^2 -projection onto $M_{i,h}$. Given this approximation property and using the definition of \tilde{V}_h , we can bound the consistency error by

$$\begin{aligned} b(w_h, \lambda) &= \sum_{i=1}^{N_\Gamma} \int_{\Gamma_i} g_i[w_h]_i (\lambda - \Pi_i \lambda) dx = \sum_{i=1}^{N_\Gamma} \int_{\Gamma_i} (g_i[w_h]_i - \Pi_i(g_i[w_h]_i)) (\lambda - \Pi_i \lambda) dx \\ &\leq Ch_{n(i)}^{1/2} |g_i[w_h]_i|_{H^{1/2}(\Gamma_i)} \|\lambda - \Pi_i \lambda\|_{L^2(\Gamma_i)}. \end{aligned}$$

Estimate (1.9c) and the trace theorem then imply

$$\sup_{w_h \in \tilde{V}_h} \frac{b(w_h, \lambda)}{\|w_h\|_X} \leq C \left(\sum_{i=1}^{N_\Gamma} h_{n(i)} \inf_{\mu_h \in M_{i,h}} \|\lambda - \mu_h\|_{L^2(\Gamma_i)}^2 \right)^{1/2}.$$

□

Next, we estimate the infimum and the supremum in the a priori bound (2.5). This is done in two steps. In a first step, we provide an upper bound for the best approximation error in \tilde{X}_h .

LEMMA 2.4. *Let $u \in H^2(\Omega_j)$. Then there exists $C > 0$ depending only on C_γ of (2.3), and there exists a unique element $\tilde{I}_j u := v_h \in \tilde{X}_{j,h}$ such that v_h and u coincide in the nodes of $\tilde{\mathcal{T}}_j$ and*

$$\|u - v_h\|_{H^1(\Omega_j)} \leq Ch_j [\|\nabla^2 u\|_{L^2(\Omega_j)} + \|\nabla u\|_{L^2(\Omega_j)}], \quad (2.8)$$

$$\left\{ \sum_{e \in \tilde{\mathcal{E}}_j} \|u - v_h\|_{H_{00}^{1/2}(e)}^2 \right\}^{1/2} \leq Ch_j [\|\nabla^2 u\|_{L^2(\Omega_j)} + \|\nabla u\|_{L^2(\Omega_j)}], \quad (2.9)$$

where we denote by $\tilde{\mathcal{E}}_j$ the set of all edges of the triangulation $\tilde{\mathcal{T}}_j$. The $H_{00}^{1/2}(e)$ -norm is defined as in (1.5) (with Γ_i replaced with e).

Proof. Let $\tilde{K} \in \tilde{\mathcal{T}}_j$ be an element. Then for any function $w \in H^2(\tilde{K})$, we obtain from (2.3), denoting by $I_{\tilde{K}} : C(\tilde{K}) \rightarrow \mathcal{P}_1$ the nodal linear interpolant and by $\hat{w} := w \circ \tilde{F}_{\tilde{K}}$,

$$\|\hat{w} - I_{\tilde{K}} \hat{w}\|_{H^1(\tilde{K})} \leq C \|\nabla^2 \hat{w}\|_{L^2(\tilde{K})} \leq Ch_{\tilde{K}} [\|\nabla^2 w\|_{L^2(\tilde{K})} + \|\nabla w\|_{L^2(\tilde{K})}].$$

Transforming the left-hand side back to the element \tilde{K} , and indicating by $I_{\tilde{K}}$ the nodal linear interpolant on \tilde{K} , we get

$$h_{\tilde{K}}^{-1} \|w - I_{\tilde{K}} w\|_{L^2(\tilde{K})} + \|\nabla(w - I_{\tilde{K}} w)\|_{L^2(\tilde{K})} \leq Ch_{\tilde{K}} [\|\nabla^2 w\|_{L^2(\tilde{K})} + \|\nabla w\|_{L^2(\tilde{K})}].$$

This proves (2.8), and \tilde{I}_j is given elementwise by $I_{\tilde{K}}$. The bound (2.9) is obtained by similar reasoning exploiting that on the reference element \hat{K} we have for the edges e_i , $i = 1, \dots, 3$ of \hat{K} :

$$\|\hat{w} - I_{\hat{K}} \hat{w}\|_{H_{00}^{1/2}(e_i)} \leq C \|\nabla^2 \hat{w}\|_{L^2(\hat{K})}, \quad i = 1, 2, 3.$$

Transforming now to the physical element \tilde{K} allows us to infer (2.9). □

As a consequence of the upper bound (2.9), we find that

$$\|(u - v_h)|_{\Omega_k}\|_{H_{00}^{1/2}(\Gamma_i)}^2 \leq Ch_k [\|\nabla^2 u\|_{L^2(\Omega_k)} + \|\nabla u\|_{L^2(\Omega_k)}], \quad (2.10)$$

for $k \in \{n(i), m(i)\}$. Here, we have exploited the fact that the square of the $H_{00}^{1/2}(\Gamma_i)$ -norm is bounded by a constant times the sum of the square of the $H_{00}^{1/2}$ -norm restricted to the edges of Γ_i . This is not true for the $H^{1/2}(\Gamma_i)$ -norm.

PROPOSITION 2.5. *Let $u \in H_0^1(\Omega) \cap \prod_{j=1}^{N_\Omega} H^2(\Omega_j)$. Then*

$$\inf_{v_h \in \tilde{V}_h} \|u - v_h\|_X \leq C \left(\sum_{j=1}^{N_\Omega} h_j^2 \left\{ \|\nabla^2 u\|_{L^2(\Omega_j)}^2 + \|\nabla u\|_{L^2(\Omega_j)}^2 \right\} \right)^{1/2}. \quad (2.11)$$

Proof. The techniques involved in the proof are known from standard mortar element methods, [2]. We note that the assumptions on u guarantee that u is continuous. The construction is done in two steps: first, we define the function $w_h \in \tilde{X}_h$ subdomainwise by $w_h|_{\Omega_j} := \tilde{I}_j u$, where \tilde{I}_j is the piecewise linear interpolant on Ω_j introduced in Lemma 2.4. This leads to

$$\|u - w_h\|_X \leq C \left(\sum_{j=1}^{N_\Omega} h_j^2 \left\{ \|\nabla^2 u\|_{L^2(\Omega_j)}^2 + \|\nabla u\|_{L^2(\Omega_j)}^2 \right\} \right)^{1/2}.$$

Since the endpoints of the curves Γ_i are nodes, we have that the jump $\iota_i := [w_h]_i$ across the curve Γ_i satisfies $\iota_i \in H_{00}^{1/2}(\Gamma_i)$, and we obtain from (2.10)

$$\|\iota_i\|_{H_{00}^{1/2}(\Gamma_i)}^2 \leq C \sum_{j \in \{n(i), m(i)\}} h_j^2 \left\{ \|\nabla^2 u\|_{L^2(\Omega_j)}^2 + \|\nabla u\|_{L^2(\Omega_j)}^2 \right\}.$$

In a second step, we correct this jump using the mortar projection $\hat{\Pi}_{i,h}$ associated with the curve Γ_i . More precisely, we define the pull-back $\hat{\iota}_i := \iota_i \circ \gamma_i$ and set $z_i := Z_i(\hat{\Pi}_{i,h} \hat{\iota}_i)$, where the lifting operator Z_i is defined in Lemma 2.1. We may think of z_i as being extended by zero outside of $\Omega_{n(i)}$. Proceeding in this fashion for each interface Γ_i , we can construct a function $z := \sum_{i=1}^{N_\Gamma} z_i \in \tilde{X}_h$ such that

$$\|z\|_X^2 = \sum_{j=1}^{N_\Omega} \left\| \sum_{i=1}^{N_\Gamma} z_i \right\|_{H^1(\Omega_j)}^2 \leq C \sum_{j=1}^{N_\Omega} \sum_{i=1}^{N_\Gamma} \|z_i\|_{H^1(\Omega_j)}^2 = C \sum_{i=1}^{N_\Gamma} \|z_i\|_{H^1(\Omega_{n(i)})}^2,$$

with C independent of the number of subdomains. Therefore, by Lemma 2.1, together with (1.8b) and (1.9b),

$$\|z\|_X^2 \leq C \sum_{i=1}^{N_\Gamma} \|\iota_i\|_{H_{00}^{1/2}(\Gamma_i)}^2 \leq C \sum_{j=1}^{N_\Omega} \|u - w_h\|_{H^1(\Omega_j)}^2.$$

We now check that the function $v_h := w_h - z$ is an element of \tilde{V}_h . For $\mu \in M_{i,h}$ we calculate

$$b(v_h, \mu) = \sum_{i=1}^{N_\Gamma} \int_{\Gamma_i} g_i [\iota_i - z_i]_i \mu \, ds = \sum_{i=1}^{N_\Gamma} \int_{\hat{I}} (\hat{\iota}_i - \hat{\Pi}_{i,h} \hat{\iota}_i) \hat{\mu} \, d\hat{s} = 0$$

by definition of $\hat{\Pi}_{i,h}$ since $\hat{\mu} \in \hat{M}_{i,h}$. \square

REMARK 2.6. Due to the appearance of ∇u in (2.11), an affine solution u will in general not be reproduced by the numerical scheme (2.1). However, this term only stems from the blending elements, the number of which is considerably smaller than the total number of elements involved. In typical meshes, curved elements are used only near the boundary so that (2.11) can be sharpened: If $S_j \subset \Omega_j$ denotes the region where blending elements are used (and affine elements are used on $\Omega_j \setminus S_j$), then we obtain the error bound

$$\inf_{v_h \in \tilde{V}_h} \|u - v_h\|_X \leq C \left(\sum_{j=1}^{N_\Omega} h_j^2 \left\{ \|\nabla^2 u\|_{L^2(\Omega_j)}^2 + \|\nabla u\|_{L^2(S_j)}^2 \right\} \right)^{1/2}. \quad (2.12)$$

If S_j is contained in a strip of width $O(h_j)$ near $\partial\Omega_j$ and if the function u is smooth, then this improved bound yields an error bound $O(\max_{j=1,\dots,N_\Omega} h_j^{3/2})$; we will illustrate this effect in a numerical example in Section 3. \blacksquare

The error in the Lagrange multiplier is given as follows:

PROPOSITION 2.7. *Let u be the weak solution of (1.1), and assume that $\lambda \in M$ is defined by (1.6). Assume that $(\tilde{u}_h, \tilde{\lambda}_h)$ are given by (2.1). Then there exists a constant $C > 0$ depending only on the interfaces Γ_i , the stability constants of the mortar projections $\hat{\Pi}_{i,h}$, and the constant C_γ of (2.3) such that*

$$\|\lambda - \tilde{\lambda}_h\|_{M^0} \leq C \left(\inf_{\mu \in M_h} \|\lambda - \mu\|_{M^0} + \|u - \tilde{u}_h\|_X \right). \quad (2.13)$$

Proof. The key step of the proof consists in establishing continuity of the bilinear form b together with with an inf-sup condition for it on finite dimensional spaces. This is the purpose of the first two steps. The final step then uses these properties via standard arguments. Of crucial importance is the following observation: By definition the bilinear form b is continuous on $X \times M$ with respect to the norms $\|\cdot\|_X$ and $\|\cdot\|_M$ but not on $X \cap X_0 \times M^0 \cap M$ with respect to the norms $\|\cdot\|_X$ and $\|\cdot\|_{M^0}$. On the other hand, we cannot establish a uniform inf-sup condition with respect to the norms $\|\cdot\|_X$ and $\|\cdot\|_M$. A uniform inf-sup condition can be only established with respect to the norms $\|\cdot\|_X$ and $\|\cdot\|_{M^0}$.

1. *step:* We define

$$X_0 := \{u \in X : [u]_i \in H_{00}^{1/2}(\Gamma_i) \quad i = 1, \dots, N_\Gamma\}.$$

and equip the space X_0 with the stronger norm

$$\|u\|_{X_0}^2 := \|u\|_X^2 + \sum_{i=1}^{N_\Gamma} \|[u]_i\|_{H_{00}^{1/2}(\Gamma_i)}^2. \quad (2.14)$$

The bilinear form b is continuous on $X_0 \times M^0 \cap M$, i.e., there exists $C_b > 0$ such that

$$|b(v, \mu)| \leq C_b \|v\|_{X_0} \|\mu\|_{M^0} \quad \forall v \in X_0, \quad \mu \in M^0 \cap M. \quad (2.15)$$

This follows from (1.9a):

$$\begin{aligned} |b(v, \mu)| &\leq \sum_{i=1}^{N_\Gamma} |\langle g_i[v]_i, \mu \rangle_{\Gamma_i}| \leq \sum_{i=1}^{N_\Gamma} \|g_i[v]_i\|_{H_{00}^{1/2}(\Gamma_i)} \|\mu\|_{H^{-1/2}(\Gamma_i)} \\ &\leq C \left\{ \sum_{i=1}^{N_\Gamma} \|[v]_i\|_{H_{00}^{1/2}(\Gamma_i)}^2 \right\}^{1/2} \|\mu\|_{M^0} \leq C_b \|v\|_{X_0} \|\mu\|_{M^0}. \end{aligned}$$

We note that in contrast to standard mortar formulations, we have included a weighting factor g to define the bilinear form b . This weighting factor depends on the parametrization.

2. *step:* The existence of the mortar projections $\hat{\Pi}_{i,h}$ implies that the bilinear form b satisfies a discrete inf-sup condition of the following type: There exists a constant $C > 0$ (depending only on the interfaces Γ_i , the stability constants of the mortar projections $\hat{\Pi}_{i,h}$, and the constant C_γ) such that

$$C \|\mu\|_{M^0} \leq \sup_{z \in \tilde{X}_h \cap X_0} \frac{b(z, \mu)}{\|z\|_{X_0}} \quad \forall \mu \in M_h. \quad (2.16)$$

In order to see this, let $\mu \in M_h$. We set $\mu_i := \mu|_{\Gamma_i}$ and use (1.9) to obtain

$$\begin{aligned} \|\mu_i\|_{H^{-1/2}(\Gamma_i)} &= \sup_{v \in H_{00}^{1/2}(\Gamma_i)} \frac{\langle v, \mu_i \rangle_{\Gamma_i}}{\|v\|_{H_{00}^{1/2}(\Gamma_i)}} \leq C \sup_{v \in H_{00}^{1/2}(\Gamma_i)} \frac{\langle \widehat{v}|\gamma'_i|, \widehat{\mu}_i \rangle_{\widehat{I}}}{\|\widehat{v}\|_{H_{00}^{1/2}(\widehat{I})}} \\ &= C \sup_{v \in H_{00}^{1/2}(\Gamma_i)} \frac{\langle \widehat{v}, \widehat{\mu}_i \rangle_{\widehat{I}}}{\|\widehat{v}/|\gamma'_i|\|_{H_{00}^{1/2}(\widehat{I})}} \leq C \sup_{v \in H_{00}^{1/2}(\Gamma_i)} \frac{\langle \widehat{v}, \widehat{\mu}_i \rangle_{\widehat{I}}}{\|\widehat{v}\|_{H_{00}^{1/2}(\widehat{I})}}, \end{aligned}$$

where the superscript $\widehat{\cdot}$ indicates the pull-back to \widehat{I} . Employing the stability properties of mortar projection $\widehat{\Pi}_{i,h}$, we arrive at

$$\begin{aligned} \|\mu_i\|_{H^{-1/2}(\Gamma_i)} &\leq C \sup_{v \in H_{00}^{1/2}(\Gamma_i)} \frac{\langle \widehat{v}, \widehat{\mu}_i \rangle_{\widehat{I}}}{\|\widehat{v}\|_{H_{00}^{1/2}(\widehat{I})}} \leq C \sup_{v \in H_{00}^{1/2}(\Gamma_i)} \frac{\langle \widehat{\Pi}_{i,h}\widehat{v}, \widehat{\mu}_i \rangle_{\widehat{I}}}{\|\widehat{\Pi}_{i,h}\widehat{v}\|_{H_{00}^{1/2}(\widehat{I})}} \\ &\leq C \sup_{z \in \widehat{W}_{i,h} \cap H_{00}^{1/2}(\widehat{I})} \frac{\langle z, \widehat{\mu}_i \rangle_{\widehat{I}}}{\|z\|_{H_{00}^{1/2}(\widehat{I})}} \leq C \langle \widehat{z}_i, \widehat{\mu}_i \rangle_{\widehat{I}}, \end{aligned} \quad (2.17)$$

for an element $\widehat{z}_i \in \widehat{W}_{i,h} \cap H_{00}^{1/2}(\widehat{I})$ with $\|\widehat{z}_i\|_{H_{00}^{1/2}(\widehat{I})} = 1$. We extend \widehat{z}_i to $z_i \in \widetilde{X}_{n(i),h}$ by means of the lifting operator from Lemma 2.1, and define $z := \sum_{i=1}^{N_\Gamma} b(z_i, \mu)z_i$. We then obtain:

$$\langle \widehat{z}_i, \widehat{\mu}_i \rangle_{\widehat{I}} = \langle g_i z_i, \mu_i \rangle_{\Gamma_i} = b(z_i, \mu),$$

and, therefore,

$$\|\mu\|_{M^0}^2 \leq C \sum_{i=1}^{N_\Gamma} (b(z_i, \mu))^2 = b(z, \mu) \leq C_b \|z\|_{X_0} \|\mu\|_{M^0}, \quad (2.18)$$

where, in the last step, we employed the continuity of b stated in (2.15). This allows us to estimate

$$\begin{aligned} \|z\|_X^2 &= \sum_{j=1}^{N_\Omega} \left\| \sum_{i=1}^{N_\Gamma} b(z_i, \mu)z_i \right\|_{H^1(\Omega_j)}^2 \leq C \sum_{i=1}^{N_\Gamma} b(z_i, \mu)^2 = Cb(z, \mu), \\ \sum_{i=1}^{N_\Gamma} \| [z]_i \|_{H_{00}^{1/2}(\Gamma_i)}^2 &\leq C \sum_{i=1}^{N_\Gamma} b(z_i, \mu)^2 = Cb(z, \mu). \end{aligned}$$

so that we arrive, by summing these last two bounds, at

$$\|z\|_{X_0}^2 \leq Cb(z, \mu). \quad (2.19)$$

From (2.18) we infer $\|\mu\|_{M^0} \leq C\|z\|_{X_0}$; inserting this in (2.19) then allows us to conclude (2.16).

3. step: We now turn to estimating $\|\lambda - \widetilde{\lambda}_h\|_{M^0}$. To that end, let $\lambda'_h \in M_h$ be arbitrary. Then, by the continuity of b (cf. (2.15)), the Galerkin orthogonality, and (2.19), we have

$$\begin{aligned} \|\lambda'_h - \widetilde{\lambda}_h\|_{M^0} &\leq C \sup_{z \in \widetilde{X}_h \cap X_0} \frac{b(z, \lambda'_h - \widetilde{\lambda}_h)}{\|z\|_{X_0}} \leq C \sup_{z \in \widetilde{X}_h \cap X_0} \frac{b(z, \lambda'_h - \lambda)}{\|z\|_{X_0}} + \frac{b(z, \lambda - \widetilde{\lambda}_h)}{\|z\|_{X_0}} \\ &\leq C\|\lambda'_h - \lambda\|_{M^0} + C \sup_{z \in \widetilde{X}_h \cap X_0} \frac{a(u - \widetilde{u}_h, z)}{\|z\|_{X_0}} \\ &\leq C\|\lambda'_h - \lambda\|_{M^0} + C\|u - \widetilde{u}_h\|_X. \end{aligned}$$

The standard argument using the triangle inequality then gives the desired result. \square

2.2. Convergence analysis on straight elements. We now turn to the analysis of the method proposed in Section 1.2. Given the triangulations \mathcal{T}_j , $j = 1, \dots, N_\Omega$, with affine elements, we denote by $\tilde{\mathcal{T}}_j$, $j = 1, \dots, N_\Omega$, triangulations with blending elements as analyzed in the preceding section. These blending elements are chosen such that there is a one-to-one correspondence between the elements $K \in \mathcal{T}_j$ and the elements $\tilde{K} \in \tilde{\mathcal{T}}_j$, i.e., the vertices of the two triangulations coincide. The particular construction of the blending method then implies that the mappings F_K and $\tilde{F}_{\tilde{K}}$ of two elements $K \in \mathcal{T}_j$, $\tilde{K} \in \tilde{\mathcal{T}}_j$ corresponding to each other satisfy

$$\tilde{F}_{\tilde{K}} = F_K + R_K, \quad \|R_K\|_{L^\infty(\hat{K})} + \|R'_K\|_{L^\infty(\hat{K})} + \|R''_K\|_{L^\infty(\hat{K})} \leq Ch_K^2, \quad (2.20)$$

where the constant $C > 0$ depends only on the parametrizations γ_i , $i = 1, \dots, N_\Gamma$. We will give a detailed proof of this fact in Appendix A. In particular, the assumptions (2.3) follow from those for the affine element maps F_K .

Having established a one-to-one correspondence between the elements of the triangulations \mathcal{T}_j and $\tilde{\mathcal{T}}_j$, we can construct a one-to-one correspondence between the elements of X_h and \tilde{X}_h as follows: a function $v_h \in X_h$ corresponds to $\tilde{v}_h \in \tilde{X}_h$ if for every element $K \in \mathcal{T}_j$, we have $\tilde{v}_h \circ \tilde{F}_{\tilde{K}} = v_h \circ F_K$. This mapping exists and is denoted by

$$\mathcal{S} : X_h \rightarrow \tilde{X}_h, \quad v_h \mapsto \tilde{v}_h \quad (2.21)$$

with the inverse map

$$\mathcal{C} := \mathcal{S}^{-1} : \tilde{X}_h \rightarrow X_h, \quad \tilde{v}_h \mapsto v_h. \quad (2.22)$$

In particular, for every element K we have that $v_h|_K$ is the linear nodal interpolant of \tilde{v}_h on \tilde{K} . Considering the mapping G_K from \tilde{K} to K , given by

$$G_K := F_K \circ \tilde{F}_{\tilde{K}}^{-1} = \text{Id} + O(h_K^2), \quad (2.23)$$

which follows from property (2.20), we can ascertain that

$$a_h(\mathcal{C}\tilde{v}_h, \mathcal{C}\tilde{v}_h) \sim a(\tilde{v}_h, \tilde{v}_h) \quad \forall \tilde{v}_h \in \tilde{X}_h, \quad (2.24)$$

where the constants hidden in the \sim -notation are independent of h .

Using the bijection \mathcal{S} we can reformulate the problem (1.12) as a perturbation of a problem of the form analyzed in Section 2.1. On $\tilde{X}_h \times \tilde{X}_h$, we define the bilinear form

$$a'_h(w_h, v_h) := a_h(\mathcal{C}w_h, \mathcal{C}v_h). \quad (2.25)$$

In view of (2.24) and Lemma 2.2, we have ellipticity of the bilinear form a'_h on $\tilde{X}_h \cap X_\chi$, with the space X_χ defined in (2.4):

$$a'_h(v_h, v_h) \geq C \|v_h\|_{\tilde{X}}^2 \quad \forall v_h \in \tilde{X}_h \cap X_\chi. \quad (2.26)$$

We write $u'_h = \mathcal{S}u_h$ for the solution u_h of (1.12) and can rewrite the problem (1.12) as

$$a'_h(u'_h, v) + b(v, \lambda_h) = l_h(\mathcal{C}v) \quad \forall v \in \tilde{X}_h, \quad (2.27a)$$

$$b(u'_h, \mu) = 0 \quad \forall \mu \in M_h. \quad (2.27b)$$

Here, we exploited in an essential way the fact that the space \tilde{X}_h is based on blending elements associated with the parametrization of the interfaces Γ_i , which implies the important relation $[u_h]_{i,h} = [\mathcal{S}u_h]_i$, so that we have

$$b(u, \lambda) = b_h(\mathcal{C}u, \lambda) \quad \forall (u, \lambda) \in \tilde{X}_h \times M_h. \quad (2.28)$$

From the first Strang Lemma, we can now assess the error $u - u'_h$:

$$\begin{aligned} \|u - u'_h\|_X \leq C & \left\{ \inf_{v_h \in \tilde{V}_h} \left(\|u - v_h\|_X + \sup_{w \in \tilde{V}_h} \frac{a(v_h, w) - a'_h(v_h, w)}{\|w\|_X} \right) \right. \\ & \left. + \sup_{w \in \tilde{V}_h} \frac{b(w, \lambda)}{\|w\|_X} + \sup_{w \in \tilde{V}_h} \frac{l(w) - l_h(\mathcal{C}w)}{\|w\|_X} \right\}. \end{aligned}$$

The first and the third term have already been estimated in Section 2.1. For the second term, we have

LEMMA 2.8.

$$\sup_{v \in \tilde{X}_h} \sup_{w \in \tilde{X}_h} \frac{a(v, w) - a'_h(v, w)}{\|v\|_X \|w\|_X} \leq C \max_{1 \leq j \leq N_\Omega} h_j.$$

Proof. On each element \tilde{K} , the difference $a(v, w) - a'_h(v, w)$ can be written as

$$\int_{\tilde{K}} \nabla v (\text{Id} - A_K) \nabla w, \quad (2.29)$$

where the matrix A_K is given by

$$A_K := G'_K{}^{-\top} G'_K{}^{-1} \det G'_K,$$

with G_K defined in (2.23). The properties (2.20) of the functions R_K imply

$$G_K = \text{Id} + O(h_K^2) \quad \text{and} \quad \det G'_K = 1 + O(h_K). \quad (2.30)$$

Hence,

$$\|\text{Id} - A_K\|_{L^\infty(\tilde{K})} \leq Ch_K \quad \forall K \in \mathcal{T}_j.$$

Summation of (2.29) over all involved triangles gives

$$\begin{aligned} a(v, w) - a'_h(v, w) &= \sum_K \int_{\tilde{K}} \nabla v (\text{Id} - A_K) \nabla w \\ &\leq \max_{\tilde{K}} \|\text{Id} - A_K\|_{L^\infty(\tilde{K})} \sum_K \int_{\tilde{K}} |\nabla v| |\nabla w| \\ &\leq C \max_{1 \leq j \leq N_\Omega} h_j \left(\sum_{j=1}^{N_\Omega} \int_{\Omega_j} |\nabla v|^2 \right)^{1/2} \left(\sum_{j=1}^{N_\Omega} \int_{\Omega_j} |\nabla w|^2 \right)^{1/2} \\ &\leq C \max_{1 \leq j \leq N_\Omega} h_j \|v\|_X \|w\|_X, \end{aligned}$$

which yields the desired result. \square

We now turn to estimating the error introduced by the right-hand side $l(\tilde{w}_h) - l_h(\mathcal{C}\tilde{w}_h)$.
LEMMA 2.9.

$$\sup_{\tilde{w}_h \in \tilde{X}_h} \frac{l(\tilde{w}_h) - l_h(\mathcal{C}\tilde{w}_h)}{\|\tilde{w}_h\|_X} \leq C \sum_{j=1}^{N_\Omega} h_j^2.$$

Proof. For each element $\tilde{K} \in \tilde{T}_j$, we have

$$\begin{aligned} \int_{\tilde{K}} f \tilde{w}_h dx - \int_{\tilde{K}} f_j \mathcal{C}\tilde{w}_h dx &= \int_{\tilde{K}} f \tilde{w}_h dx - \int_{\tilde{K}} f_j \circ G_K \tilde{w}_h \det G'_K dx \\ &= \int_{\tilde{K}} (f - f_j \circ G_K \det G'_K) \tilde{w}_h dx. \end{aligned}$$

Our assumptions (1.13) on the choice of the functions f_j together with the fact that $G_K(x) = x$ for the vertices of the triangle \tilde{K} give

$$\begin{aligned} \|f - f_j \circ G_K\|_{L^\infty(\tilde{K})} &\leq \|f - f_j\|_{L^\infty(\tilde{K})} + \|f_j - f_j \circ G_K\|_{L^\infty(\tilde{K})} \\ &\leq Ch_{\tilde{K}} + C\|\text{Id} - G_K\|_{L^\infty(\tilde{K})} \leq Ch_K, \end{aligned} \quad (2.31)$$

where the constant C is determined by the Lipschitz constant C_f of the functions f, f_j . We therefore conclude

$$\|f - f_j \circ G_K \det G'_K\|_{L^\infty(\tilde{K})} \leq Ch_K.$$

Hence,

$$\left| \int_{\tilde{K}} f \tilde{w}_h dx - \int_{\tilde{K}} f_j \mathcal{C}\tilde{w}_h dx \right| \leq Ch_K \|\tilde{w}_h\|_{L^1(\tilde{K})} \leq Ch_K^2 \|\tilde{w}_h\|_{L^2(\tilde{K})}.$$

□

The above considerations allow us to conclude

$$\|u - u'_h\|_X \leq C \max_{j=1, \dots, N_\Omega} h_j, \quad (2.32)$$

if the exact solution u satisfies $u \in H_0^1(\Omega) \cap \prod_{j=1}^{N_\Omega} (\Omega_j)$.

It remains to estimate the error in the Lagrange multiplier. Here, we proceed as in Section 2.1.

PROPOSITION 2.10. *The error of the Lagrange multiplier is bounded by*

$$\|\lambda - \lambda_h\|_{M^0} \leq C \left[\inf_{\mu \in M_h} \|\lambda - \mu\|_{M^0} + \|u - u'_h\|_X + (1 + \|u\|_X) \max_{1 \leq j \leq N_\Omega} h_j \right]$$

Proof. The proof follows the lines of the proof of Proposition 2.7. We have the discrete inf-sup condition (2.16). The approximation $u'_h = \mathcal{C}u_h$ and the discrete Lagrange multiplier λ_h satisfy by (2.27) for arbitrary $\mu \in M_h$ and $v \in \tilde{X}_h \cap X_0$

$$\begin{aligned} b(v, \mu - \lambda_h) &= b(v, \mu - \lambda) + b(v, \lambda - \lambda_h) \\ &= b(v, \mu - \lambda) + l(v) - l_h(\mathcal{C}v) + a(u, v) - a'_h(u'_h, v) \\ &\leq C\|v\|_{X_0} \left(\max_{1 \leq j \leq N_\Omega} h_j + \|\lambda - \mu\|_{M^0} + \|u - u'_h\|_X + \frac{a(u'_h, v) - a'_h(u'_h, v)}{\|v\|_{X_0}} \right) \\ &\leq C\|v\|_{X_0} \left((1 + \|u'_h\|_X) \max_{1 \leq j \leq N_\Omega} h_j + \|\lambda - \mu\|_{M^0} + \|u - u'_h\|_X \right). \end{aligned}$$

The triangle inequality allows us then to estimate $\|u'_h\|_X \leq \|u - u'_h\|_X + \|u\|_X$. Finally, from the discrete inf-sup condition (2.16), arguments as in the proof of Proposition 2.7, Lemmata 2.8, 2.9 we get the desired result. \square

Using Proposition 2.5, Lemmata 2.8, 2.9 and Proposition 2.10 in combination with the best approximation property of M_h , we find for $u \in H^2(\Omega)$

$$\|u - u_h\|_{X_h} + \|\lambda - \lambda_h\|_{M^0} \leq C(u) \max_{1 \leq j \leq N_\Omega} h_j,$$

where the mesh dependent norm $\|\cdot\|_{X_h}$ is defined by $\|v\|_{X_h}^2 := \sum_{j=1}^{N_\Omega} \|v\|_{H^1(\Omega_{j,h})}^2$. In the last step, we have used the triangle inequality, the norm equivalence of $\|v_h\|_{X_h}$ and $\|\mathcal{S}v_h\|_X$, $v_h \in X_h$, Lemma 2.4 and the fact that $\tilde{I}_h u = \mathcal{S}I_h u$, where \tilde{I}_h and I_h are the global nodal interpolation operators on \tilde{X}_h and X_h , respectively.

3. Numerical results. We present various numerical examples. In Subsection 3.1, we consider the case of two subdomains sharing one curved interface. In Subsection 3.2, a decomposition into eight subdomains with straight as well as curved interfaces is investigated. We give numerical evidence of the independence of the constants from the number of subdomains in Subsection 3.3, where we vary the number of subdomains from 4 to 25. Finally, we consider a domain with reentrant corner in Subsection 3.4.

3.1. Two subdomains. In a first test, we consider the case of the domain $\Omega = (0, 2) \times (0, 1)$ divided into two subdomains Ω_1 and Ω_2 as illustrated in Figure 3.1. Dual basis functions with respect to the grid on Ω_2 are used to span the Lagrange multiplier space $\tilde{M}_{1,h}$. We solve the model problem (1.1) with Dirichlet boundary conditions and source term f derived from the exact solution $u(x, y) = x^2 - 2y^2$, starting with the initial triangulation displayed in Figure 3.1.

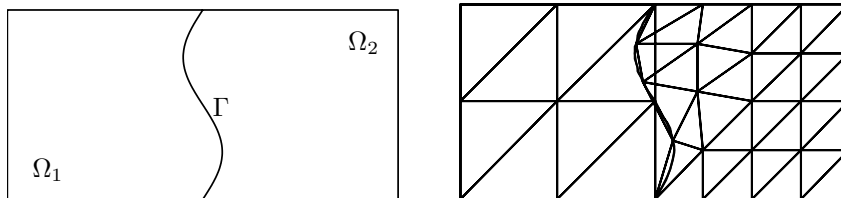


FIG. 3.1. Left: Decomposition into subdomains Ω_1, Ω_2 . Right: initial grid.

The error decay of the finite element solution u_h , measured in the H^1 - and in the L^2 -norm, as well as the behavior of the error in the Lagrange multiplier, are plotted in Figure 3.2. We note that in order to evaluate the error in the Lagrange multiplier, we mimic the $H^{-1/2}$ -norm by $(\sum_e h_e \|\lambda - \lambda_h\|_{0,e}^2)^{1/2}$, where the sum is taken over all edges e on the slave side of the interfaces, and h_e denotes the length of e . For the H^1 -norm, the results are in perfect agreement with the theory presented in Section 2. Moreover, the numerical convergence order for the Lagrange multiplier is better than predicted from the theory. This can be explained by the following observation. The weighted L^2 -norm of the error in the Lagrange multiplier can be bound by the sum of the best approximation error and the discretization error in the H^1 -norm restricted to a strip of width h around the interface. Once the solution is $H^{5/2}$ -regular, the best approximation error of the Lagrange multiplier space is of order $O(h^{3/2})$. If, moreover, the discretization error can be assumed to be equidistributed over the domain, the

overall order $O(h^{3/2})$ can be derived, as is experienced in this example, as well as in the next examples 3.2 and 3.3.

An interesting special case is the approximation of a linear solution. In contrast to straight interfaces, the finite element solution does not coincide with the exact one, as pointed out in Remark 2.6. Numerical evidence of this behavior is illustrated in the right picture of Figure 3.2, where the error decays are plotted for the same geometrical setting as before with the exact solution $u(x, y) = y$. The error measured in the H^1 -norm is of order $h^{3/2}$ as shown in Remark 2.6.

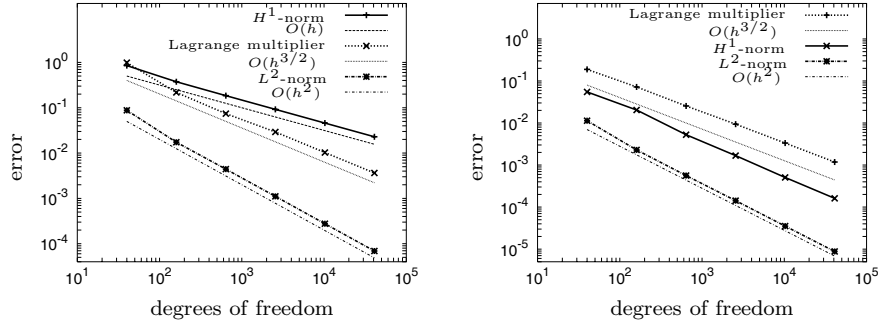


FIG. 3.2. Error decays. Quadratic solution (left), linear solution (right).

In a further test, we start from a conforming triangulation consisting of two elements on each subdomain, and refine the right subdomain Ω_2 1 (2, 3) time(s). Thus, a ratio $q_r^l = 2:1$ (4:1, 8:1) of the number of element edges on the left side of the interface to the number of edges on the right side is obtained. We test the stability of our method choosing first the Lagrange multiplier with respect to the finer (left) grid, and then with respect to the coarser (right) grid, so that the ratio q_m^n of the number of slave edges to master edges is out of $\{2:1, 4:1, 8:1, 1:2, 1:4, 1:8\}$.

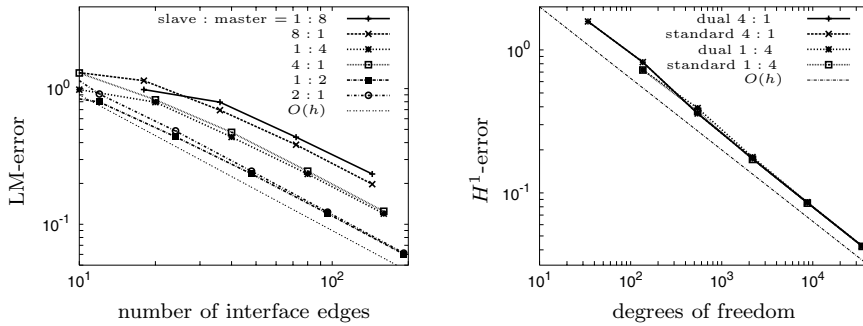


FIG. 3.3. Different ratios of the number of slave edges to the number of master edges.

In the left picture of Figure 3.3, the error $\lambda - \lambda_h$, measured in the L^2 -norm, is plotted against the total number of slave and master edges. Looking at the exact solution, it is not surprising that the error decreases as the ratio q_m^n tends to 1. What is more interesting and demonstrates the stability of our approach, is the very moderate influence of whether to choose the fine grid as master side or the coarse one. Of course, this also depends on the characteristics of the exact solution. Especially in the case of discontinuous coefficients, it might be important to choose the discrete Lagrange multipliers on the correct side. In the right picture of Figure 3.3, we provide

a comparison of the H^1 -error decays for the ratios 1 : 4 and 4 : 1, where once, dual Lagrange multipliers are taken, and in the other case, standard Lagrange multipliers are chosen. All approaches yield qualitatively the same and quantitatively almost the same results.

3.2. Eight subdomains. We investigate the behavior of our method on the domain $\Omega = (-1, 1)^2$, subdivided into eight subdomains sharing twelve interfaces, four of which are curved, and five crosspoints. The exact solution is chosen to be $u(x, y) = (\sin \pi x)(\sin 2\pi y)$. The initial triangulation is displayed in the right picture of Figure 3.4. The Lagrange multiplier spaces are defined with respect to the finer grids, again spanned by the corresponding dual linear basis functions. The various error decays shown in Figure 3.4 (right) illustrate the same qualitative behavior as in the case of two subdomains.

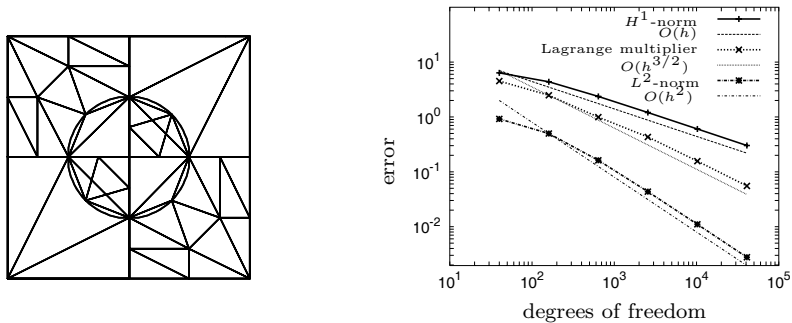


FIG. 3.4. *Eight subdomains: initial triangulation (left), error decays (right)*

3.3. Many subdomains. In this example, the global domain $\Omega = (0, 1)^2$ is divided into $N_\Omega = 4, 9, 16, 25$ subdomains. Figure 3.5 displays the domain decompositions and the initial triangulations. The discrete Lagrange multiplier spaces are always chosen with respect to the finer triangulations. As exact solution, we use $u(x, y) = \exp(-2((x - 0.5)^2 + (y - 0.5)^2))$. We run a test for each of the decom-

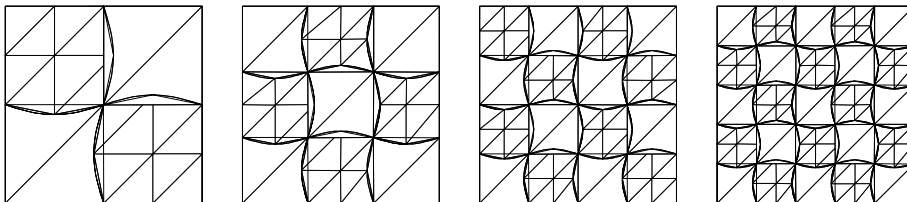


FIG. 3.5. *Many Subdomains: initial triangulations.*

positions, measuring the error decay under uniform refinement of the triangulations. In the left picture of Figure 3.6, the discretization error measured in the H^1 -norm is plotted versus the number of elements. Only very slight differences can be seen between the considered decompositions, which illustrates nicely the independence of the constants appearing in the a priori analysis from the total number of subdomains. In order to achieve comparable results for the error in the Lagrange multiplier, the weighted L^2 -norm was divided by the corresponding total length of the interfaces. Also here, it becomes hard to distinguish between the various decompositions, as illustrated in the right picture of Figure 3.6.

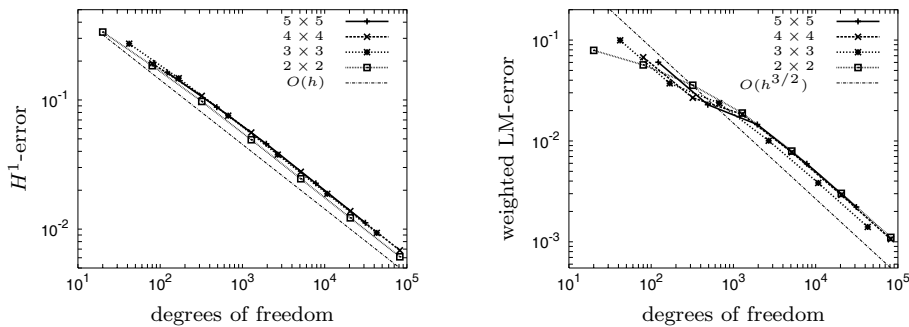


FIG. 3.6. Many subdomains: H^1 -error (left), weighted LM-error (right).

3.4. Reentrant corner. We consider the domain Ω being three quarters of the unit circle with reentrant corner $(0,0)$, and choose the harmonic exact solution to be $u(r, \phi) = r^{2/3} \sin 2/3\phi$, having $H^{5/3-\varepsilon}$ -regularity. Figure 3.7 shows that the error decays are optimal with respect to the regularity of the solution. Here, we cannot expect the behavior of the Lagrange multiplier error to be better than $O(h^{2/3})$.

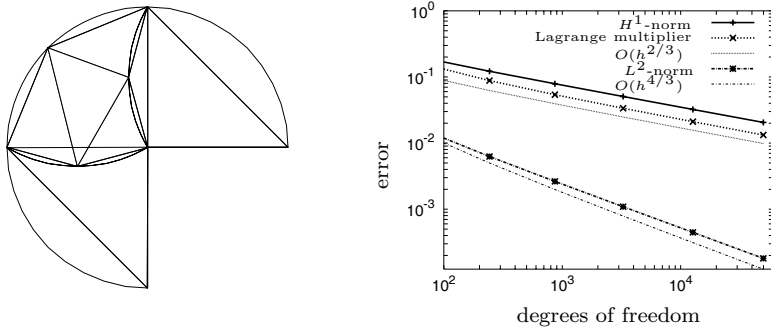


FIG. 3.7. Reentrant corner: initial triangulation (left), error decays (right).

Appendix A. Element maps of blending maps. We provide a proof of (2.20), restated in the following lemma.

LEMMA A.1. Under Assumption (1.2), the mappings F_K and $\tilde{F}_{\tilde{K}}$ of two elements $K \in \mathcal{T}_j$, $\tilde{K} \in \tilde{\mathcal{T}}_j$ corresponding to each other satisfy

$$\tilde{F}_{\tilde{K}} = F_K + R_K, \quad \|R_K\|_{L^\infty(\hat{K})} + \|R'_K\|_{L^\infty(\hat{K})} + \|R''_K\|_{L^\infty(\hat{K})} \leq Ch_K^2, \quad (\text{A.1})$$

where the constant $C > 0$ depends only on the parametrizations γ_i , $i = 1, \dots, N_\Gamma$.

Proof. Without loss of generality, we consider the situation illustrated in Figure (A.1). All other configurations involving one curved edge can be achieved by translations, rotations, and deformations of \tilde{K} with parameters independent of the diameter h . Additional edges can be taken into account by summing up the contributions from each edge. The function $F_K(\xi, \eta) = (h\xi, h\eta)^T$ maps the reference element $\hat{K} = ((0,0), (1,0), (0,1))$ onto the element $K = ((0,0), (h,0), (0,h))$. The blending element \tilde{K} has one curved edge given by $\gamma \in C^2([0, h])$, which is twice continuously differentiable as a consequence of Assumption (1.2) by transforming the subinterval of \hat{I} corresponding to the curved edge onto the interval $[0, h]$.

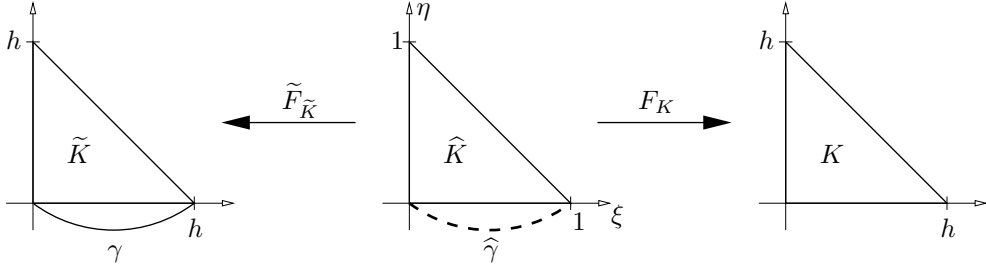


FIG. A.1. *Element mappings*

We denote the pull-back of γ onto the reference configuration by $\hat{\gamma} \in C^2([0, 1])$, defined by $\hat{\gamma}(\xi) = \gamma(h\xi)$. The element mapping $\tilde{F}_{\tilde{K}}$ is given by $\tilde{F}_{\tilde{K}} = F_K + R_K$, where

$$R_K = \left(0, \frac{1-\xi-\eta}{1-\xi} \hat{\gamma}(\xi) \right)^T.$$

We note that R_K is discontinuous in $(1, 0)$, but nevertheless well-behaved, since on K , η tends to 0 as ξ tends to 1, and $\hat{\gamma}(1) = 0$. In order to estimate the terms involved in R_K, R'_K, R''_K , we develop $\hat{\gamma}$ into its Taylor series around 1, yielding

$$\hat{\gamma}(\xi) = \hat{\gamma}(1) + (\xi - 1)\hat{\gamma}'(1) + r(\xi),$$

with

$$r(\xi) = \int_1^\xi (\xi - t)\hat{\gamma}''(t) dt.$$

By differentiating r , we obtain

$$|r(\xi)| \leq (\xi - 1)^2 \|\hat{\gamma}''\|_\infty, \quad |r'(\xi)| \leq (\xi - 1) \|\hat{\gamma}''\|_\infty, \quad |r''(\xi)| \leq \|\hat{\gamma}''\|_\infty. \quad (\text{A.2})$$

Considering the fact that $\hat{\gamma}(1) = 0$, we can give R_K as

$$R_K = \left(0, (\xi + \eta - 1)h\gamma'(h) + \frac{1-\xi-\eta}{1-\xi} r(\xi) \right)^T.$$

The term $h\gamma'(h)$ is bounded by Ch^2 , since $\hat{\gamma}(0) = \hat{\gamma}(1) = 0$, and the mean value theorem guarantees the existence of $h_* \in [0, 1]$ such that $\gamma'(h_*) = 0$. Thus, using (A.2), we see that $\|R_K\|_{L^\infty(\tilde{K})}$ is bounded by Ch_K^2 . For the Jacobian R'_K , we obtain

$$R'_K = \begin{pmatrix} 0 & 0 \\ h\gamma'(h) - \frac{\eta}{(1-\xi)^2} r(\xi) + \frac{1-\xi-\eta}{1-\xi} r'(\xi) & h\gamma'(h) - \frac{1}{1-\xi} r(\xi) \end{pmatrix},$$

which can be bounded such that $\|R_K\|_{L^\infty(\tilde{K})} \leq Ch_K^2$. The coefficients of the Hessian R''_K involve the following terms:

$$\frac{1}{(1-\xi)^2} r(\xi), \quad \frac{1}{1-\xi} r'(\xi), \quad r''(\xi),$$

which can be handled as before, and

$$\frac{2\eta}{(1-\xi)^3} r(\xi), \quad \frac{\eta}{(1-\xi)^2} r'(\xi), \quad \frac{\eta}{1-\xi} r''(\xi).$$

Regarding again the fact that η tends to 0 as ξ tends to 1, since $0 \leq \eta \leq 1 - \xi$, these terms also pose no difficulty. \square

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