Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

Equivalence of Tripartite Quantum States under Local Unitary Transformations

by

Sergio Albeverio, Laura Cattaneo, Shao-Ming Fei, and Xiao-Hong Wang

Preprint no.: 109 2005

Equivalence of Tripartite Quantum States under Local Unitary Transformations

Sergio Albeverio[a 1], Laura Cattaneo[a 2], Shao-Ming Fei[a,b 3], Xiao-Hong Wang[b 4]

[a] Institut für Angewandte Mathematik, Universität Bonn, D-53115

[b] Department of Mathematics, Capital Normal University, Beijing 100037

Abstract

The equivalence of tripartite pure states under local unitary transformations is investigated. The nonlocal properties for a class of tripartite quantum states in $\mathbb{C}^K \otimes \mathbb{C}^M \otimes \mathbb{C}^N$ composite systems are investigated and a complete set of invariants under local unitary transformations for these states is presented. It is shown that two of these states are locally equivalent if and only if all these invariants have the same values.

PACS numbers: 03.67.-a, 02.20.Hj, 03.65.-w

Key words: tripartite quantum states, local unitary transformation, entanglement, invariants

Quantum entanglement is one of the most striking features of quantum phenomena [1]. It is playing very important roles in quantum information processing such as quantum computation [2], quantum teleportation [3] (for discussions of experimental realizations see [4]), dense coding [5] and quantum cryptographic schemes [6]. As the degree of entanglement of two parts of a quantum system remains invariant under local unitary transformations of these parts, the invariants of local unitary transformations give rise to an effective description of entanglement. Two states are equivalent under local unitary transformations if and only if they have the same values of all the invariants under local unitary transformations. The method developed in [7, 8], in principle, allows one to compute all the invariants of local unitary transformations, though in general it is not operational. In [9], the invariants for general two-qubit systems are studied and a complete set of 18 polynomial invariants is presented. It is proven that two qubit mixed states are locally equivalent if and only if

[1]SFB 611; IZKS; BiBoS; CERFIM(Locarno); Acc. Arch.; USI(Mendrisio)
e-mail: albeverio@uni-bonn.de

[2]e-mail: cattaneo@wiener.iam.uni-bonn.de

[3]e-mail: fei@uni-bonn.de

[4]e-mail: wangxh@mail.cnu.edu.cn

all these 18 invariants have equal values in these states. In [10] three qubits states are also discussed in detail from a similar point of view. In [11] a complete set of invariants is presented for bipartite generic mixed states.

In this letter, we discuss the locally invariant properties of arbitrary dimensional tripartite quantum states in $\mathbb{C}^K \otimes \mathbb{C}^M \otimes \mathbb{C}^N$ composite systems. We present a complete set of invariants for a class of pure states and show that two of these states are locally equivalent if and only if all our invariants have equal values.

Let H_A resp. H_B resp. H_C be K resp. M resp. N dimensional complex Hilbert spaces. We denote by $\{|e_i\rangle\}_{i=1}^K$, $\{|f_i\rangle\}_{i=1}^M$ and $\{|h_i\rangle\}_{i=1}^N$ the orthonormal bases in H_A, H_B and H_C respectively. A general pure state on $H_A \otimes H_B \otimes H_C$ is of the form

$$|\Psi\rangle = \sum_{i=1}^{K}\sum_{j=1}^{M}\sum_{k=1}^{N} a_{ijk}|e_i\rangle \otimes |f_j\rangle \otimes |h_k\rangle, \qquad a_{ijk} \in \mathbb{C} \tag{1}$$

with the normalization $\sum_{i=1}^{K}\sum_{j=1}^{M}\sum_{k=1}^{N} a_{ijk}a_{ijk}^* = 1$ ($*$ denotes complex conjugation).

$|\Psi\rangle$ can be regarded as a state on the bipartite systems $A-BC$, $B-AC$ or $C-AB$. For each such bipartite decomposition, let us consider the matrix whose entries are the coefficients of the state $|\Psi\rangle$ with respect to the bipartite decomposition. Let A_1 be the matrix corresponding to $|\Psi\rangle$ as a bipartite state in the $A-BC$ system, with the row (resp. column) indices from the subsystem A (resp. BC). For example, if $K = M = N = 2$,

$$A_1 = \begin{pmatrix} a_{111} & a_{112} & a_{121} & a_{122} \\ a_{211} & a_{212} & a_{221} & a_{222} \end{pmatrix}.$$

Similarly, denoting A_2 resp. A_3 the matrices treating $|\Psi\rangle$ as a state in the $B-AC$ resp. $C-AB$ bipartite system, for $K = M = N = 2$ one has:

$$A_2 = \begin{pmatrix} a_{111} & a_{112} & a_{211} & a_{212} \\ a_{121} & a_{122} & a_{221} & a_{222} \end{pmatrix}, \quad A_3 = \begin{pmatrix} a_{111} & a_{121} & a_{211} & a_{221} \\ a_{112} & a_{122} & a_{212} & a_{222} \end{pmatrix}.$$

Taking partial trace of $|\Psi\rangle\langle\Psi|$ over the respective subsystems, we have $Tr_1|\Psi\rangle\langle\Psi| = A_1^t A_1^*$, $Tr_2|\Psi\rangle\langle\Psi| = A_2^t A_2^*$, $Tr_3|\Psi\rangle\langle\Psi| = A_3^t A_3^*$, where t represents the transpose of a matrix. The following quantities are invariants associated with the state $|\Psi\rangle$ given by (1):

$$I_\alpha = Tr(Tr_1|\Psi\rangle\langle\Psi|)^\alpha, \quad \alpha = 1, 2, \cdots, S, \tag{2}$$

where $S = min\{K, M, N\}$.

In fact, if $|\Psi'\rangle = U_1 \otimes U_2 \otimes U_3|\Psi\rangle$, with U_i unitary matrices acting on the space H_i, $i = 1, 2, 3$, then A_1' corresponding to $|\Psi'\rangle$ and A_1 have the following relation:

$$A_1' = U_1 A_1 (U_2 \otimes U_3)^t = U_1 A_1 V^t,$$

where $V = U_2 \otimes U_3$ is also a unitary matrix. So we have

$$Tr_1|\Psi'\rangle\langle\Psi'| = {A'}_1^t {A'}_1^* = (u_1 A_1 V^t)^t (u_1 A_1 V^t)^* = V(A_1^t A_1^*)V^\dagger$$

and we get

$$Tr(Tr_1|\Psi'\rangle\langle\Psi'|^\alpha)Tr(V(A_1^tA_1^*)^\alpha V^\dagger)=Tr(A_1^tA_1^*)^\alpha=Tr(Tr_1|\Psi\rangle\langle\Psi|)^\alpha,$$

i.e., I_α, $\alpha=1,\cdots,S$ are invariants.

Similarly, we can construct the following invariants:

$$J_\alpha=Tr(Tr_2|\Psi\rangle\langle\Psi|)^\alpha,\quad \alpha=1,2,\cdots,S. \tag{3}$$

$$K_\alpha=Tr(Tr_3|\Psi\rangle\langle\Psi|)^\alpha,\quad \alpha=1,2,\cdots,S. \tag{4}$$

There are also other invariants like

$$Tr(Tr_i(Tr_j|\Psi\rangle\langle\Psi|)^\alpha)^\beta,\quad i,j=1,2,3,\quad i\neq j,\alpha,\beta=1,2,\cdots,S. \tag{5}$$

Relevant quantities for states like the Frobenius norm, singular values and the degree of entanglement are all invariants under local unitary transformations. Generally one needs all the invariants to judge whether two tripartite states are locally equivalent. However, for some class of special states, only one kind of invariants (2), (3) or (4) is sufficient, as we are going to prove. We first recall some results on matrix realignment [12] and give some definitions.

If Z is an $m\times m$ block matrix with each block of size $n\times n$, the realigned matrix $\tilde{Z}$ is defined by

$$\tilde{Z}=[vec(Z_{11}),\cdots,vec(Z_{m1}),\cdots,vec(Z_{1m}),\cdots,vec(Z_{mm})]^t, \tag{6}$$

where

$$vec(A)=[a_{11},\cdots,a_{m1},a_{12},\cdots,a_{m2},\cdots,a_{1n},\cdots,a_{mn}]^t$$

for any $m\times n$ matrix A with entries a_{ij}.

It is straightforward to verify that a matrix U can be expressed as the tensor product of two matrices X and Y, $U=X\otimes Y$ if and only if

$$\tilde{U}=vec(X)vec(Y)^t. \tag{7}$$

(cf, e.g., [13])

[Definition]. An $mn\times mn$ unitary matrix U is called unitarily decomposable, if there exist an $m\times m$ unitary matrix U_1 and an $n\times n$ unitary matrix U_2 , such that $U=U_1\otimes U_2$.

[Lemma]. Let U be an $mn\times mn$ unitary matrix. U is a unitarily decomposable matrix if and only if the rank of $\tilde{U}$ is one, $r(\tilde{U})=1$.

[Proof]. Let U be a unitarily decomposable matrix, i.e., there exist unitary matrices U_1 and U_2 such that $U=U_1\otimes U_2$. Applying (7) and using the property that a matrix is rank one if and only if it can be written as product of a column vector and a row vector, we have $r(\tilde{U})=1$.

Conversely, if $r(\tilde{U})=1$, there are matrices X and Y such that $U=X\otimes Y$. On the other hand, due to the unitarity of U, X and Y should satisfy the following equation:

$$UU^\dagger=(X\otimes Y)(X^\dagger\otimes Y^\dagger)=XX^\dagger\otimes YY^\dagger=I_{mn}.$$

Let x_{ij} denote the entries of $XX^\dagger$. The above relation implies that $x_{ij} = 0$ if $i \neq j$ and $x_{ii} = k^{-1} \neq 0, \quad i,j = 1,\cdots,m$, and $YY^\dagger$ is a diagonal scalar matrix, i.e., $XX^\dagger = k^{-1}I_m$ and $YY^\dagger = kI_n$.

Similarly, we have $X^\dagger X = k'^{-1}I_m$, and $Y^\dagger Y = k'I_n$. It is easily proven that $k' = k$. Therefore $XX^\dagger = X^\dagger X = k^{-1}I_m$ and $YY^\dagger = Y^\dagger Y = kI_n$. Since $XX^\dagger$ and $YY^\dagger$ are positive and selfadjoint, k is real an positive. Hence $U_1 = \sqrt{k}X$ and $U_2 = \frac{Y}{\sqrt{k}}$ are unitary matrices such that $U = U_1 \otimes U_2$ is unitarily decomposable. □

Note that if $U = X \otimes Y$ is a unitary matrix, then X and Y are either both unitary or both not unitary. An $mn \times mn$ unitary matrix U is a unitarily decomposable matrix if and only if $r(\tilde{U}) = 1$.

We can judge whether an $mn \times mn$ unitary matrix U is unitarily decomposable or not in the following way: if the rank of the realigned matrix $\tilde{U}$ is not one, $r(\tilde{U}) \neq 1$, then U is not decomposable. If $r(\tilde{U}) = 1$, then it can be written as a product of a column vector and a row vector, i.e., there exist $(a_1,\cdots,a_{m^2})^t$ and $(b_1,\cdots,b_{n^2})$ such that $\tilde{U} = (a_1,\cdots,a_{m^2})^t(b_1,\cdots,b_{n^2})$. These vectors can be obtained from the realignment of certain matrices, say $vec(X) = (a_1,\cdots,a_{m^2})^t$, $vec(Y) = (b_1,\cdots,b_{n^2})^t$, so that $U = X \otimes Y$. If one of X and Y is unitary, then U is unitarily decomposable.

We consider now the state $|\Psi\rangle$ in (1) as a bipartite state $A - BC$. As shown in [11] bipartite states $|\psi\rangle = \sum_{i=1}^{M}\sum_{j=1}^{N} a_{ij}|ij\rangle$ and $|\psi'\rangle = \sum_{i=1}^{M}\sum_{j=1}^{N} a'_{ij}|ij\rangle$, are equivalent under local unitary transformations if and only if they have the same values of all the corresponding invariants: $T_\alpha = T'_\alpha$, for $\alpha = 1,\cdots,min\{N,M\}$, where $T_\alpha = Tr(AA^\dagger)^\alpha$, $T'_\alpha = Tr(A'A'^\dagger)^\alpha$, and A, A' are the $M \times N$ matrices with the entries a_{ij} and a'_{ij}. If $T_\alpha = T'_\alpha$, there exist unitary matrices U and V such that $|\psi'\rangle = U \otimes V|\psi\rangle$, which also implies $A' = UAV^t$, i.e., $AA^\dagger$ and $A'A'^\dagger$ are unitary equivalent and have the same singular values. U and V are dependent on $|\psi\rangle$ and $|\psi'\rangle$, and can be obtained by using the singular value decomposition method: $U = u'u^\dagger$ and $V = v'v^\dagger$, where $A = uDv^\dagger$ and $A' = u'Dv'^\dagger$ are the singular value decomposition of A and A' with the singular values ordered descending.

Summarizing the above discussions we have the following theorem:

[Theorem]. For two tripartite states $|\Psi\rangle$ and $|\Psi'\rangle$ on $H_A \otimes H_B \otimes H_C$, if they have the same values of the invariants given by (2), $I_\alpha = I'_\alpha$, there are unitary matrices U_1 on H_A and V_1 on $H_B \otimes H_C$ such that $|\Psi'\rangle = U_1 \otimes V_1|\Psi\rangle$. $|\Psi\rangle$ and $|\Psi'\rangle$ are equivalent under local unitary transformations if the corresponding unitary matrix U_1 satisfies $r(\tilde{U}_1) = 1$.

[Remark]. If we say that two pure tripartite states $|\Psi\rangle$ and $|\Psi'\rangle$ are a pair of D_1 states if they satisfy $|\Psi'\rangle = V_1 \otimes U_1|\Psi\rangle$ with U_1 a unitary matrix on H_A and V_1 a unitarily decomposable matrix on $H_B \otimes H_C$, we have defined an equivalence relation $|\Psi'\rangle \sim |\Psi\rangle$. Indeed, as $|\Psi\rangle = U_1^\dagger \otimes V_1^\dagger|\Psi'\rangle$, where $U_1^\dagger$ is unitary, and $V_1^\dagger$ is also unitarily decomposable with $r(\tilde{V}_1^\dagger) = 1$, one has that if $|\Psi'\rangle \sim |\Psi\rangle$ then $|\Psi\rangle \sim |\Psi'\rangle$. Transitivity also holds, namely, if $|\Psi''\rangle \sim |\Psi'\rangle$ and $|\Psi'\rangle \sim |\Psi\rangle$, then $|\Psi''\rangle \sim |\Psi\rangle$.

As an example we consider two states $|\Psi\rangle = \frac{1}{\sqrt{2}}(|001\rangle + |100\rangle)$, and $|\Psi'\rangle = \frac{1}{\sqrt{2}}(|010\rangle + |111\rangle)$ in $H_A \otimes H_B \otimes H_C$, where $K = \dim H_A = 4, M = \dim H_B = 2, N = \dim H_C = 2$. Let us denote by $\{|0\rangle,\ |1\rangle,\ |2\rangle,\ |3\rangle\}$ the orthonormal basis of H_A, and $\{|0\rangle,\ |1\rangle\}$ the orthonormal basis of H_B and H_C. We have

$$\rho = Tr_1|\Psi\rangle\langle\Psi| = \mathrm{diag}(\frac{1}{2}, \frac{1}{2}, 0, 0), \quad \rho' = \mathrm{Tr}_1|\Psi'\rangle\langle\Psi'| = \mathrm{diag}(0, 0, \frac{1}{2}, \frac{1}{2}),$$

and

$$I_\alpha = Tr(Tr_1|\Psi\rangle\langle\Psi|)^\alpha = \frac{1}{2^{\alpha-1}}, \quad I'_\alpha = Tr(Tr_1|\Psi'\rangle\langle\Psi'|)^\alpha = \frac{1}{2^{\alpha-1}}.$$

Since $I_\alpha = I'_\alpha$, we treat $|\Psi\rangle$ and $|\Psi'\rangle$ as states in the bipartite system $H_A \otimes H_{BC}$, where $H_{BC} = H_B \otimes H_C$. Then we get the corresponding 2×2 block matrices $A_1 = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}$, $A'_1 = \begin{pmatrix} 0 & T'_1 \\ 0 & 0 \end{pmatrix}$, where $T_1 = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$, $T'_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$. From the singular value decomposition of matrices we have unitary matrices U_1 in H_A and V_1 in $H_B \otimes H_C$ such that $|\Psi'\rangle = U_1 \otimes V_1|\Psi\rangle$. In this case $V_1 = I$. Therefore $|\Psi\rangle$ and $|\Psi'\rangle$ are D_1 states and they are equivalent under local unitary transformations.

Analogously we can also say that two pure tripartite states $|\Psi\rangle$ and $|\Psi'\rangle$ are a pair of D_2 (resp. D_3) states. For example, if we treat $|\Psi\rangle$ as a state in the B-AC system, then $Tr_2|\Psi\rangle\langle\Psi| = A_2^t A_2^*$. If $J_\alpha = J'_\alpha$, from the result on bipartite systems we have that $|\Psi'\rangle = U_2 \otimes V_2|\Psi\rangle$, where U_2 acts on H_B and V_2 on $H_A \otimes H_C$. If the corresponding unitary matrix V_2 satisfies $r(\tilde{V_2}) = 1$, then $|\Psi\rangle$ and $|\Psi'\rangle$ are a pair of D_2 states and they are equivalent under local unitary transformations. A pair of D_3 states can be defined in a similar way.

If $|\Psi\rangle$ and $|\Psi'\rangle$ are not a pair of D_1 states, one can check whether they are a pair of D_2 or D_3 states, by using J_α and J'_α or K_α and K'_α to check if $|\Psi\rangle$ and $|\Psi'\rangle$ are equivalent under local unitary transformations or not.

Conclusion: we have discussed the local invariants for arbitrary dimensional tripartite quantum states in $\mathbb{C}^K \otimes \mathbb{C}^M \otimes \mathbb{C}^N$ composite systems and have presented a set of invariants under local unitary transformations. The set of invariants is not necessarily independent (they could be represented by each other in some cases), but they are sufficient to judge if two states are a pair of D_i, $i = 1, 2, 3$, states, which are equivalent under local unitary transformations.

Acknowledgments The second named author gratefully acknowledges the financial support by the Stefano Franscini Fund. The fourth author gratefully acknowledges the support provided by the China-Germany Cooperation Project 446 CHV 113/231, "Quantum information and related mathematical problems".

References

[1] A. Peres, Quantum Mechanics: Concepts and Methods, Kluwer, Dordrecht (1993).

[2] See, for example, D.P. Di Vincenzo, *Science* **270**, 255 (1995);
M. Nielsen and I.L. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press (2000).

[3] C.H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W.K. Wootters, *Phys. Rev. Lett.* **70**, 1895 (1993);
S. Albeverio and S.M. Fei, *Phys. Lett. A* **276**, 8-11 (2000);
G.M. D'Ariano, P. Lo Presti, and M.F. Sacchi, *Phys. Lett. A* **272**, 32 (2000);
S. Albeverio, S.M. Fei, and W.L. Yang, *Phys. Rev. A* **66**, 012301 (2002).

[4] D. Bouwmeester, J.-W. Pan, K. Mattle, M. Elbl, H. Weinfurter, and A. Zeilinger, *Nature* **390**, 575 (1997);
D. Boschi, S. Brance, F. De Martini, L. Hardy, and S. Popescu, *Phys. Rev. Lett.* **80**, 1121 (1998);
A. Furusawa, J.L. Srensen, S.L. Braunstein, C.A. Fuchs, H.J. Kimble, and E.S. Polzik, *Science* **282**, 706 (1998);
M.A. Nielsen, E. Knill, and R. Laflamme, *Nature* **396**, 52 (1998).

[5] C.H. Bennett and S.J. Wiesner, *Phys. Rev. Lett.* **69**, 2881 (1992).

[6] A. Ekert, *Phys. Rev. Lett.* **67**, 661 (1991);
D. Deutsch, A. Ekert, P. Rozsa, C. Macchiavello, S. Popescu, and A. Sanpera, *Phys. Rev. Lett.* **77**, 2818 (1996);
C.A. Fuchs, N. Gisin, R.B. Griffiths, C-S. Niu, and A. Peres, *Phys. Rev. A* **56**, 1163 (1997).

[7] E.M. Rains, *IEEE Transactions on Information Theory* **46**, 54 (2000).

[8] M. Grassl, M. Rötteler, and T. Beth, *Phys. Rev. A* **58**, 1833 (1998).

[9] Y. Makhlin, *Quant. Info. Proc.* **1**, 243 (2002).

[10] N. Linden, S. Popescu, and A. Sudbery, *Phys. Rev. Lett.* **83**, 243 (1999).

[11] S. Albeverio, S.M. Fei, P. Parashar, and W.L. Yang, *Phys. Rev. A* **68**, 010313 (R) (2003).

[12] K. Chen and L.A. Wu, *Phys. Lett. A* **306**, 14 (2002).

[13] N.P. Pitsianis, Ph.D. thesis, *The Kronecker Product in Approximation and Fast Transform Generation*, Cornell University, New York (1997).