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Surfaces

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SUPER-LIOUVILLE EQUATIONS ON CLOSED RIEMANN SURFACES

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ABSTRACT. Motivated by the supersymmetric extension of Liouville theory in the recent physics literature, we couple the standard Liouville functional with a spinor field term. The resulting functional is conformally invariant. We study geometric and analytic aspects of the resulting Euler-Lagrange equations, culminating in a blow up analysis.

1. INTRODUCTION

The classical Liouville functional for a real-valued function u on M is

$$E(u) = \int_M \left\{ \frac{1}{2} |\nabla u|^2 + K_g u - e^{2u} \right\} dv, \quad (1)$$

where K_g is the Gaussian curvature of M . The Euler-Lagrange equation for $E(u)$ is the Liouville equation

$$-\Delta u = 2e^{2u} - K_g \quad (2)$$

where Δ is the Laplacian with respect to g . Liouville [Liou] studied this equation in the plane, that is, for $K_g = 0$. The Liouville equation arises in many contexts of complex analysis and differential geometry of Riemann surfaces, in particular in the prescribing curvature problem. The interplay between the geometric and analytic aspects makes the Liouville equation mathematically rich. It also occurs naturally in string theory as discovered by Polyakov [P2], from the gauge anomaly in quantizing the string action. There then also is a natural supersymmetric version of the Liouville functional and equation, coupling the bosonic scalar field to a fermionic spinor field. It turns out, however, that we also obtain a very interesting mathematical structure if we consider ordinary instead of fermionic (Grassmann valued) spinor fields. In particular, the fundamental conformal invariance of the Liouville action can be preserved under the coupling. This makes the resulting functional geometrically very natural and, so it seems to us, a worthy and interesting object of mathematical analysis.

Therefore, in this paper, we consider the following functional for a real-valued function u and a spinor ψ

$$E(u, \psi) = \int_M \left\{ \frac{1}{2} |\nabla u|^2 + K_g u + \langle (\not{D} + e^u)\psi, \psi \rangle - e^{2u} \right\} dv. \quad (3)$$

The Euler-Lagrange system for $E(u, \psi)$ is

$$\begin{cases} -\Delta u &= 2e^{2u} - e^u \langle \psi, \psi \rangle - K_g \\ \not{D}\psi &= -e^u \psi \end{cases} \quad \text{in } M, \quad (4)$$

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This system couples the Liouville equation and the Dirac equation in a rather natural way. We call (4) the *super-Liouville equations*. When ψ vanishes, we obtain the original Liouville equation. In other words, here we are considering a system generalizing the prescribing curvature equation. The important point is that this generalization preserves a fundamental property of the energy functional on Riemann surfaces, namely its conformal invariance.

In this paper we aim to provide an analytic foundation for system (4). We start with basic points like the regularity of weak solutions. An analytic foundation was established for the Liouville equation (2) in [BM], [LS] and for a Toda system in [JW], [JW2] and [JLW]. In those references, it was established that the key analytical points are that singularities in solutions u_n of the equations on closed surfaces, or, more generally with bounded energy $\int e^{2u_n}$, can form only at isolated points x where the limit $u_n(x)$ tends to infinity. Away from those singularities, u_n remains either uniformly bounded or converges to $-\infty$ which, in fact, is a regular situation for the field ϕ with $u = \log \phi$. At those isolated singularities, rescaling produces an entire solution of the Liouville equation of finite energy $\int_{\mathbb{R}^2} e^{2u}$ in the plane which then can be compactified to a solution on the 2-sphere. Therefore, the asymptotic behavior of such entire solutions is also an important point. In this paper, we therefore perform such an analysis for the super Liouville equations. As in the classical case, this provides a complete analytical picture, and other regularity results follow in a standard manner that is known to the experts and therefore need not be repeated here.

Assume that (u_n, ψ_n) is a sequence of solutions of (4) with

$$\int_M e^{2u_n} dv < \varepsilon_0, \text{ and } \int_M |\psi_n|^4 dv < C$$

for some positive constants ε_0 and C . If ε_0 is sufficiently small (in fact $\varepsilon_0 < \pi$ suffices), then we can show that (u_n, ψ_n) admits a subsequence, which we still denote by (u_n, ψ_n) , converging to a smooth solution (u, ψ) of (4). Note that $\int_M e^{2u_n} dv$ and $\int_M |\psi_n|^4 dv$ are conformally invariant, see Section 3.

When ε_0 is big, then the so-called “blow-up” phenomenon may occur. Let (u_n, ψ_n) be a sequence of solutions of (4) and satisfying

$$\int_M e^{2u_n} dv < C, \text{ and } \int_M |\psi_n|^4 dv < C.$$

Define

$$\begin{aligned} \Sigma_1 &= \{x \in M, \text{ there is a sequence } y_n \rightarrow x \text{ such that } u_n(y_n) \rightarrow +\infty\} \\ \Sigma_2 &= \{x \in M, \text{ there is a sequence } y_n \rightarrow x \text{ such that } |\psi_n(y_n)| \rightarrow +\infty\}. \end{aligned}$$

Then, one can show that $\Sigma_2 \subset \Sigma_1$ and (u_n, ψ_n) admits a subsequence, still denoted by (u_n, ψ_n) , satisfying one of the following cases:

- i) u_n is bounded in $L^\infty(M)$.
- ii) $u_n \rightarrow -\infty$ uniformly on M .
- iii) Σ_1 is finite, nonempty and either

$$u_n \text{ is bounded in } L_{loc}^\infty(M \setminus \Sigma_1)$$

or

$$u_n \rightarrow -\infty \text{ uniformly on compact subsets of } M \setminus \Sigma_1.$$

Furthermore, we rule out the first case in iii) if $\Sigma_1 \setminus \Sigma_2 \neq \emptyset$. Then the only case is $u_n \rightarrow -\infty$ uniformly on compact subsets of $M \setminus \Sigma_1$.

Finally, we consider entire solutions of the super Liouville equations on \mathbb{R}^2 with finite energy $\int_{\mathbb{R}^2} e^{2u} + |\psi|^4$, which can be viewed as “bubbles” or obstructions to the compactness of equation (4). We analyze the asymptotic behavior of such solutions and obtain

$$u(x) = -\frac{\alpha}{2\pi} \ln |x| + C + O(|x|^{-1}) \quad \text{for } |x| \text{ near } \infty,$$

$$\psi(x) = -\frac{1}{2\pi} \frac{x}{|x|^2} \cdot \xi_0 + o(|x|^{-1}) \quad \text{for } |x| \text{ near } \infty,$$

where \cdot is the Clifford multiplication, $C \in \mathbb{R}$ is some constant, $\alpha = \int_{\mathbb{R}^2} 2e^{2u} - e^u |\psi|^2 dx$, and $\xi_0 = \int_{\mathbb{R}^2} e^u \psi dx$ is a constant spinor.

Furthermore, by using the associated holomorphic quantity $T(z) = (\partial_z u)^2 - \partial_z^2 u + \frac{1}{4} \langle \psi, dz \cdot \partial_{\bar{z}} \psi \rangle + \frac{1}{4} \langle d\bar{z} \cdot \partial_z \psi, \psi \rangle$, we show $\alpha = 4\pi$. For the definition of T , see Section 3. Then we show such an entire solution can be extended to a smooth solution on \mathbb{S}^2 , i.e. the global singularity (the singularity at infinity) is removable.

2. SPINORS

For presenting our equations, we need to recall some background about spin structures and spinors. Let (M, g) be a closed Riemann surface and $P_{SO(2)} \rightarrow M$ its oriented orthonormal frame bundle. A *Spin*-structure is a lift of the structure group $SO(2)$ to $Spin(2)$, i.e., there exists a principal *Spin*-bundle $P_{Spin(2)} \rightarrow M$ such that there is a bundle map

$$\begin{array}{ccc} P_{Spin(2)} & \longrightarrow & P_{SO(2)} \\ \downarrow & & \downarrow \\ M & \longrightarrow & M. \end{array}$$

Let $\Sigma^+ M := P_{Spin(2)} \times_{\rho} \mathbb{C}$ be a complex line bundle over M associated to $P_{Spin(2)}$ and to the standard representation $\rho : \mathbb{S}^1 \rightarrow U(1)$. This is the bundle of positive half-spinors. Its complex conjugate $\Sigma^- M := \overline{\Sigma^+ M}$ is called the bundle of negative half-spinors. The spinor bundle is $\Sigma M := \Sigma^+ M \oplus \Sigma^- M$. There exists a Clifford multiplication

$$\begin{array}{ccc} TX \times_{\mathbb{C}} \Sigma^+ M & \rightarrow & \Sigma^- M \\ TX \times_{\mathbb{C}} \Sigma^- M & \rightarrow & \Sigma^+ M \end{array}$$

denoted by $v \otimes \psi \rightarrow v \cdot \psi$, which satisfies the Clifford relations

$$v \cdot w \cdot \psi + w \cdot v \cdot \psi = -2g(v, w)\psi,$$

for all $v, w \in TM$ and $\psi \in \Gamma(\Sigma M)$.

On the spinor bundle ΣM , the metric g induces a natural Hermitian metric $\langle \cdot, \cdot \rangle$. Let ∇ be the Levi-Civita connection on M with respect to g . Likewise, ∇ induces a connection (also denoted by ∇) on ΣM compatible with the Hermitian metric.

The Dirac operator \mathcal{D} is defined by $\mathcal{D}\psi := \sum_{\alpha=1}^2 e_{\alpha} \cdot \nabla_{e_{\alpha}} \psi$, where $\{e_1, e_2\}$ is an orthonormal basis on TM . (For more details about the spin bundle and the Dirac operator, we refer to [LM] or [J].)

3. PROPERTIES OF SUPER LIOUVILLE EQUATIONS

We start by giving some examples of solutions of the super Liouville equations (4). When $M = \mathbb{S}^2$, the standard sphere with Gaussian curvature $K = 1$, it is obvious that solutions u of (2),

$$-\Delta u + 1 - 2e^{2u} = 0 \text{ on } \mathbb{S}^2. \quad (5)$$

yield solutions of the form $(u, 0)$ of (4),

$$\begin{cases} -\Delta u &= 2e^{2u} - e^u \langle \psi, \psi \rangle - 1 \\ \not{D}\psi &= -e^u \psi \end{cases} \quad (6)$$

In fact, all solutions of (5) are of the form $u = \frac{1}{2} \log \frac{1}{2} + \frac{1}{2} \log \det |d\varphi|$, where φ is a conformal map of \mathbb{S}^2 . This can be understood in terms of the complex geometry behind the Liouville equation, but we do not go into this aspect here.

There exists another type of solution of (4). Let us recall that a *Killing spinor* is a spinor ψ satisfying

$$\nabla_X \psi = \lambda X \cdot \psi, \quad \text{for any vector field } X$$

for some constant λ . On the standard sphere, there are Killing spinors with the Killing constant $\lambda = \frac{1}{2}$, see for instance [BFGK]. Such a Killing spinor is an eigenspinor, i.e.

$$\not{D}\psi = -\psi,$$

with constant $|\psi|^2$. Choosing a Killing spinor ψ with $|\psi|^2 = 1$, $(0, \psi)$ is a solution of (4). If we identify $\mathbb{S}^2 \setminus \{\text{northpole}\}$ by the stereographic projection with the Euclidean plane \mathbb{R}^2 with the metric

$$\frac{4}{(1 + |x|^2)^2} |dx|^2,$$

then any Killing spinor has the form

$$\frac{v + x \cdot v}{\sqrt{1 + |x|^2}},$$

up to a translation or a dilation. See [BFGK].

Now we come to an important property of the functional E .

Proposition 3.1. *The functional $E(u, \psi)$ is conformally invariant. Namely, for any conformal diffeomorphism $\varphi : M \rightarrow M$, set*

$$\begin{aligned} \tilde{u} &= u \circ \varphi - \ln \lambda \\ \tilde{\psi} &= \lambda^{-\frac{1}{2}} \psi \circ \varphi \end{aligned} \quad (7)$$

where λ is the conformal factor of the conformal map φ , i.e., $\varphi^*(g) = \lambda^2 g$. Then $E(u, \psi) = E(\tilde{u}, \tilde{\psi})$. In particular, if (u, ψ) is a solution of (4), so is $(\tilde{u}, \tilde{\psi})$.

Proof. It is well-known that $\int_M \frac{1}{2} |\nabla u|^2 + K_g u$ is conformally invariant, see e.g. [H]. Since the terms

$$\int_M e^{2u} dv, \quad \int_M e^u |\psi|^2 dv,$$

are invariant under a conformal transformation, it is sufficient to show the conformality of $\int_M \langle \not{D}\psi, \psi \rangle dv$. Let $\tilde{g} = \varphi^* g$, where g is the metric on M . Let $\tilde{\not{D}}$ be the

Dirac operator with respect to the new metric \tilde{g} . By the conformality of φ , we have $\tilde{g} = \lambda^2 g$ for a positive function λ on M . We identify the new and old spin bundles as in [H]. Since the relation between the two Dirac operators \mathcal{D} and $\tilde{\mathcal{D}}$ is

$$\tilde{\mathcal{D}}\tilde{\psi} = \lambda^{-\frac{3}{2}} \mathcal{D}(\lambda^{\frac{1}{2}}\tilde{\psi}) = \lambda^{-\frac{3}{2}} \mathcal{D}\psi,$$

we can show by a direct computation that

$$\int_M \langle \mathcal{D}\psi, \psi \rangle dvol(g) = \int_M \langle \tilde{\mathcal{D}}\tilde{\psi}, \tilde{\psi} \rangle dvol(\tilde{g}).$$

The proof of the proposition is complete. \square

As before, we identify $\mathbb{S}^2 \setminus \{\text{northpole}\}$ by stereographic projection with the Euclidean plane \mathbb{R}^2 with the metric

$$\frac{4}{(1+|x|^2)^2} |dx|^2.$$

By Proposition 3.1 from any solution of equation (4) on \mathbb{S}^2 one can obtain a solution of

$$\begin{cases} -\Delta u &= 2e^{2u} - e^u \langle \psi, \psi \rangle \\ \mathcal{D}\psi &= -e^u \psi \end{cases} \text{ in } \mathbb{R}^2, \quad (8)$$

where Δ and \mathcal{D} are operators with respect to the standard metric on \mathbb{R}^2 .

Equation (8) is very interesting, since its solutions are obstructions for the compactness of equation (4), namely they are the so-called ‘‘bubbles’’ in the geometric analysis.

Let us note that on a surface the (usual) Dirac operator \mathcal{D} can be seen as the (doubled) Cauchy-Riemann operator. Consider \mathbb{R}^2 with the Euclidean metric $dx_1^2 + dx_2^2$. Let $e_1 = \frac{\partial}{\partial x_1}$ and $e_2 = \frac{\partial}{\partial x_2}$ be the standard orthonormal frame. A spinor field is simply a map $\Psi : \mathbb{R}^2 \rightarrow \Delta_2 = \mathbf{C}^2$, and e_1 and e_2 acting on spinor fields can be identified by multiplication with matrices

$$e_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

If $\Psi := \begin{pmatrix} f \\ g \end{pmatrix} : \mathbb{R}^2 \rightarrow \mathbf{C}^2$ is a spinor field, then the Dirac operator is

$$\mathcal{D}\Psi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial g}{\partial x_1} \end{pmatrix} + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial x_2} \\ \frac{\partial g}{\partial x_2} \end{pmatrix} = 2 \begin{pmatrix} \frac{\partial g}{\partial \bar{z}} \\ -\frac{\partial f}{\partial z} \end{pmatrix},$$

where

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \right).$$

Therefore, the elliptic estimates developed for (anti-) holomorphic functions can be used to study the Dirac equation.

Proposition 3.2. *Let $M = \mathbb{S}^2$ and ψ a Killing spinor with $|\psi| = 1$. Then*

$$(0, \psi)$$

is a solution of (4),

Proof. This is obvious, and we have observed it above. In order to understand the conformal invariance of the super Liouville equation better, it is instructive to carry out the proof on \mathbb{R}^2 . From the above discussion and Prop.3.1, it is sufficient to show that

$$\begin{cases} u &= -\log(1 + |x|^2) + \log 2, \\ \psi &= \left(\frac{2}{|1 + |x|^2}\right)^{\frac{1}{2}} \frac{v + x \cdot v}{\sqrt{1 + |x|^2}} \end{cases} \quad (9)$$

with $v \in \{v \in \mathbb{C}^2 \mid |v| = 1\}$ is a solution of equation (8).

We write $x \cdot v = x_1 e_1 \cdot v + x_2 e_2 \cdot v$. Recall the Clifford multiplication

$$e_i \cdot e_j \cdot \psi + e_j \cdot e_i \cdot \psi = -2\delta_{ij}\psi, \text{ for } 1 \leq i, j \leq 2$$

and

$$\langle \psi, \varphi \rangle = \langle e_i \cdot \psi, e_i \cdot \varphi \rangle$$

for any spinor fields ψ, φ . It is clear that

$$\langle x \cdot v, x \cdot v \rangle = |x|^2, \quad \text{and} \quad \langle v, x \cdot v \rangle + \langle x \cdot v, v \rangle = 0.$$

Then by a direct computation, we have

$$\begin{aligned} \langle \psi, \psi \rangle &= \frac{2}{(1 + |x|^2)^2} \langle v + x \cdot v, v + x \cdot v \rangle \\ &= \frac{2}{(1 + |x|^2)^2} (\langle v, v \rangle + \langle x \cdot v, x \cdot v \rangle + \langle v, x \cdot v \rangle + \langle x \cdot v, v \rangle) \\ &= \frac{2}{1 + |x|^2}. \end{aligned}$$

Thus we can easily check that (u, ψ) satisfies the first equation.

Next we calculate that

$$\partial_{x_1} \psi = \frac{-2\sqrt{2}x_1}{(1 + |x|^2)^2} (v + x \cdot v) + \frac{\sqrt{2}}{1 + |x|^2} e_1 \cdot v,$$

and

$$\partial_{x_2} \psi = \frac{-2\sqrt{2}x_2}{(1 + |x|^2)^2} (v + x \cdot v) + \frac{\sqrt{2}}{1 + |x|^2} e_2 \cdot v.$$

Then we have

$$\begin{aligned} \not{D}\psi &= e_1 \cdot \partial_{x_1} \psi + e_2 \cdot \partial_{x_2} \psi \\ &= -\frac{2\sqrt{2}x_1}{(1 + |x|^2)^2} (e_1 \cdot v - x_1 v + x_2 e_1 \cdot e_2 \cdot v) \\ &\quad - \frac{2\sqrt{2}x_2}{(1 + |x|^2)^2} (e_2 \cdot v - x_2 v + x_1 e_2 \cdot e_1 \cdot v) - \frac{2\sqrt{2}}{1 + |x|^2} v \\ &= -\frac{2\sqrt{2}}{(1 + |x|^2)^2} (v + x \cdot v) \\ &= -e^u \psi. \end{aligned}$$

This implies that (u, ψ) satisfy the second equation. \square

By conformal transformations, we know that

$$\left(\log \frac{\sqrt{2}}{1 + |x - x_0|^2}, 0\right) \text{ and } \left(\log \frac{2}{1 + |x - x_0|^2}, \sqrt{2} \frac{v + (x - x_0) \cdot v}{1 + |x - x_0|^2}\right)$$

are solutions of (8). It is clear that all such solutions of (8) obtained from solutions of (4) on \mathbb{S}^2 satisfy

$$I(u, \psi) := \int_{\mathbb{R}^2} \{|\nabla u|^2 + |\psi|^4\} < C. \quad (10)$$

In the last section, we will show that all solutions of (8) with bounded energy I are obtained from solutions of (4) on \mathbb{S}^2 .

Proposition 3.3. *Let (u, ψ) be a smooth solution of (4) and $z = x + iy$ a local isothermal parameter with $g = ds^2 = \rho |dz|^2$. Then the quadratic differential*

$$T(z)dz^2 = \{(\partial_z u)^2 - \partial_z^2 u + \frac{1}{4}\langle \psi, dz \cdot \partial_z \psi \rangle + \frac{1}{4}\langle d\bar{z} \cdot \partial_z \psi, \psi \rangle\}dz^2$$

is holomorphic when M is a constant curvature surface. Here $dz = dx + idy$ and $d\bar{z} = dx - idy$.

Proof. We prove this lemma by a direct computation. Let $\{e_1, e_2\}$ be a local orthonormal basis on M . It follows from the Clifford multiplication that

$$\langle e_\alpha \cdot \psi, \psi \rangle = \langle e_\alpha \cdot e_\alpha \cdot \psi, e_\alpha \cdot \psi \rangle = -\langle \psi, e_\alpha \cdot \psi \rangle.$$

Therefore we obtain the real part of $\langle e_\alpha \cdot \psi, \psi \rangle$ vanishes, i.e.

$$\operatorname{Re} \langle e_\alpha \cdot \psi, \psi \rangle = 0. \quad (11)$$

Furthermore we have

$$\langle \psi, e_\alpha \cdot \nabla_{e_\beta} \psi \rangle = -\langle e_\alpha \cdot \psi, \nabla_{e_\beta} \psi \rangle = -\langle e_\beta \cdot e_\alpha \cdot \psi, e_\beta \cdot \nabla_{e_\beta} \psi \rangle$$

and

$$\begin{aligned} \langle \psi, e_\alpha \cdot \nabla_{e_\beta} \psi \rangle - \langle \psi, e_\beta \cdot \nabla_{e_\alpha} \psi \rangle &= -\langle e_\beta \cdot e_\alpha \cdot \psi, \nabla_{e_\beta} \psi \rangle \\ &= \langle e_\beta \cdot e_\alpha \cdot \psi, e^u \psi \rangle, \end{aligned}$$

Hence from (11) we have $\operatorname{Re} \langle \psi, e_\alpha \cdot \nabla_{e_\beta} \psi \rangle$ is symmetric.

Set

$$T_1(z) = (\partial_z u)^2 - \partial_z^2 u,$$

and

$$T_2(z) = \langle \psi, dz \cdot \partial_z \psi \rangle + \langle d\bar{z} \cdot \partial_z \psi, \psi \rangle.$$

Then, we choose a local orthonormal basis $\{e_1, e_2\}$ on M such that $\nabla_{e_\alpha} e_\beta = 0$ at a considered point. By using the Ricci curvature formula we have

$$\partial_{z\bar{z}\bar{z}} u = \frac{1}{4}(\partial_z(\Delta u) + 2K_g \partial_z u).$$

Now we can compute

$$\begin{aligned} \partial_{\bar{z}} T_1(z) &= 2\partial_{z\bar{z}} u \partial_z u - \partial_{z\bar{z}\bar{z}} u \\ &= \frac{1}{2} \Delta u \partial_z u - \frac{1}{4} \partial_z(\Delta u) - \frac{1}{2} K_g \partial_z u \\ &= \frac{1}{2} (-2e^{2u} + e^u |\psi|^2 + K_g) \partial_z u + \frac{1}{4} \partial_z (2e^{2u} - e^u |\psi|^2 - K_g) - \frac{1}{2} K_g \partial_z u \\ &= \frac{1}{4} e^u |\psi|^2 \partial_z u - \frac{1}{4} e^u \partial_z |\psi|^2 - \frac{1}{4} \partial_z K_g. \end{aligned}$$

By using the symmetry of $\operatorname{Re} \langle \psi, e_\alpha \cdot \nabla_{e_\beta} \psi \rangle$, we have

$$\begin{aligned}
\partial_{\bar{z}} T_2(z) &= \frac{1}{2} \partial_{\bar{z}} (\langle (e_1 - ie_2) \cdot (\nabla_{e_1} \psi - i \nabla_{e_2} \psi), \psi \rangle + \langle \psi, (e_1 + ie_2) \cdot (\nabla_{e_1} \psi + i \nabla_{e_2} \psi) \rangle) \\
&= \partial_{\bar{z}} (\operatorname{Re} \langle \psi, e_1 \cdot \nabla_{e_1} \psi \rangle - 2i \operatorname{Re} \langle \psi, e_1 \cdot \nabla_{e_2} \psi \rangle - \operatorname{Re} \langle \psi, e_2 \cdot \nabla_{e_2} \psi \rangle) \\
&= \frac{1}{2} (\operatorname{Re} \langle \nabla_{e_1} \psi, e_1 \cdot \nabla_{e_1} \psi \rangle - 2i \operatorname{Re} \langle \nabla_{e_1} \psi, e_1 \cdot \nabla_{e_2} \psi \rangle - \operatorname{Re} \langle \nabla_{e_1} \psi, e_2 \cdot \nabla_{e_2} \psi \rangle) \\
&\quad + \frac{1}{2} (i \operatorname{Re} \langle \nabla_{e_2} \psi, e_1 \cdot \nabla_{e_1} \psi \rangle + 2 \operatorname{Re} \langle \nabla_{e_2} \psi, e_2 \cdot \nabla_{e_1} \psi \rangle - i \operatorname{Re} \langle \nabla_{e_2} \psi, e_2 \cdot \nabla_{e_2} \psi \rangle) \\
&\quad + \frac{1}{2} (\operatorname{Re} \langle \psi, e_1 \cdot \nabla_{e_1} \nabla_{e_1} \psi \rangle - 2i \operatorname{Re} \langle \psi, e_1 \cdot \nabla_{e_1} \nabla_{e_2} \psi \rangle - \operatorname{Re} \langle \psi, e_2 \cdot \nabla_{e_1} \nabla_{e_2} \psi \rangle) \\
&\quad + \frac{1}{2} (i \operatorname{Re} \langle \psi, e_1 \cdot \nabla_{e_2} \nabla_{e_1} \psi \rangle + 2 \operatorname{Re} \langle \psi, e_2 \cdot \nabla_{e_2} \nabla_{e_1} \psi \rangle - i \operatorname{Re} \langle \psi, e_2 \cdot \nabla_{e_2} \nabla_{e_2} \psi \rangle);
\end{aligned}$$

It follows from (11) that

$$\operatorname{Re} \langle \nabla_{e_i} \psi, e_j \cdot \nabla_{e_i} \psi \rangle = 0$$

for any $i, j = 1, 2$. Furthermore, by using the definition of the curvature operator $R^{\Sigma M}$ of the connection ∇ on the spinor bundle ΣM , that is

$$\nabla_{e_\alpha} \nabla_{e_\beta} \psi - \nabla_{e_\beta} \nabla_{e_\alpha} \psi = R^{\Sigma M}(e_\alpha, e_\beta) \psi,$$

and a formula for this curvature operator (see for example [J])

$$\sum_{\alpha=1}^2 e_\alpha \cdot R^{\Sigma M}(e_\alpha, X) \psi = \frac{1}{2} \operatorname{Ric}(X) \cdot \psi, \quad \text{for } \forall X \in \Gamma(TM)$$

we can obtain that

$$\begin{aligned}
\partial_{\bar{z}} T_2(z) &= \frac{1}{2} (-3 \operatorname{Re} \langle \nabla_{e_1} \psi, e_2 \cdot \nabla_{e_2} \psi \rangle + 3i \operatorname{Re} \langle \nabla_{e_2} \psi, e_1 \cdot \nabla_{e_1} \psi \rangle) \\
&\quad + \frac{1}{2} (\operatorname{Re} \langle \psi, \nabla_{e_1} (\not{D}\psi) \rangle - i \operatorname{Re} \langle \psi, \nabla_{e_2} (\not{D}\psi) \rangle) \\
&\quad + (\operatorname{Re} \langle \psi, e_2 \cdot R^{\Sigma M}(e_2, e_1) \psi \rangle - i \operatorname{Re} \langle \psi, e_1 \cdot R^{\Sigma M}(e_1, e_2) \psi \rangle).
\end{aligned}$$

By (11) we have

$$\operatorname{Re} \langle \psi, e_2 \cdot R^{\Sigma M}(e_1, e_2) \psi \rangle = \operatorname{Re} \langle \psi, \frac{1}{2} \operatorname{Ric}(e_1) \cdot \psi \rangle = 0,$$

and

$$\operatorname{Re} \langle \psi, e_1 \cdot R^{\Sigma M}(e_1, e_2) \psi \rangle = \operatorname{Re} \langle \psi, \frac{1}{2} \operatorname{Ric}(e_2) \cdot \psi \rangle = 0.$$

We also have

$$\begin{aligned}
\operatorname{Re} \langle \nabla_{e_1} \psi, e_2 \cdot \nabla_{e_2} \psi \rangle &= \operatorname{Re} \langle \nabla_{e_1} \psi, -e^u \psi - e_1 \cdot \nabla_{e_1} \psi \rangle \\
&= \operatorname{Re} \langle \nabla_{e_1} \psi, -e^u \psi \rangle - \operatorname{Re} \langle \nabla_{e_1} \psi, e_1 \cdot \nabla_{e_1} \psi \rangle \\
&= -\frac{1}{2} e^u \nabla_{e_1} |\psi|^2,
\end{aligned}$$

and in the similar way

$$\operatorname{Re} \langle \nabla_{e_2} \psi, e_1 \cdot \nabla_{e_1} \psi \rangle = -\frac{1}{2} e^u \nabla_{e_2} |\psi|^2.$$

We also compute

$$\begin{aligned}
& \operatorname{Re}\langle\psi, \nabla_{e_1}(\not{D}\psi)\rangle - i\operatorname{Re}\langle\psi, \nabla_{e_2}(\not{D}\psi)\rangle \\
&= -\operatorname{Re}\langle\psi, \nabla_{e_1}(e^u\psi)\rangle + i\operatorname{Re}\langle\psi, \nabla_{e_2}(e^u\psi)\rangle \\
&= -2e^u|\psi|^2\partial_z u - e^u\partial_z|\psi|^2
\end{aligned}$$

Therefore we get

$$\partial_{\bar{z}}T_2(z) = e^u\partial_z|\psi|^2 - e^u|\psi|^2\partial_z u.$$

Hence

$$\partial_{\bar{z}}T(z) = \partial_{\bar{z}}T_1(z) + \frac{1}{4}\partial_{\bar{z}}T_2(z) = -\frac{1}{4}\partial_{\bar{z}}K_g.$$

Therefore $\partial_{\bar{z}}T(z) = 0$ when K_g is constant and $T(z)$ is holomorphic. We finish the proof. \square

Remark 3.4. It is well-known that every holomorphic quadratic differential on \mathbb{S}^2 vanishes identically (see [J]). Therefore $T(z) = 0$ if $M = \mathbb{S}^2$.

Remark 3.5. By a similar method as in [CJLW], we can construct the holomorphic quantity in the following way. Let (u, ψ) be a solution of (4) on M . Define a tensor

$$\begin{aligned}
T_{\alpha\beta} &= 2(u_{\alpha}, u_{\beta}) - \delta_{\alpha\beta} \sum_{r=1}^2 (u_r, u_r) - 2u_{\alpha\beta} + \delta_{\alpha\beta} \sum_{r=1}^2 u_{rr} + 2\operatorname{Re}\langle\psi, e_{\alpha} \cdot \nabla_{e_{\beta}}\psi\rangle \\
&\quad + \delta_{\alpha\beta} e^u |\psi|^2
\end{aligned}$$

where $u_{\alpha} = \nabla_{e_{\alpha}} u$, and $\{e_1, e_2\}$ is a local orthonormal basis on M . Then we can check as in Proposition 3.3,

- (1) $T_{11} + T_{22} = 0$,
- (2) $T_{\alpha\beta} = T_{\beta\alpha}$, i.e., the tensor $T_{\alpha\beta}$ is symmetric.
- (3) $\sum_{\alpha=1}^2 \nabla_{e_{\alpha}} T_{\alpha\beta} = -\partial_{\beta} K_g$.

Define $T(z) = \frac{1}{4}(T_{11} - iT_{12})$. Then $T(z)dz^2$ is the holomorphic quadratic differential of Proposition 3.3.

4. COMPACTNESS THEOREM

In this section we consider the compactness of solutions of (4) under the condition that

$$I(u, \psi) := \int_M (e^{2u} + |\psi|^4) dv < C.$$

Since (4) is conformally invariant, in general the set of solutions of (4) with a uniformly bounded energy $I(u, \psi)$ is non-compact.

First, we define weak solutions of (4). We say that (u, ψ) is a weak solution of (4), if $u \in W^{1,2}(M)$ and $\psi \in W^{1, \frac{4}{3}}(\Gamma(\Sigma M))$ satisfy

$$\begin{aligned}
\int_M \nabla u \nabla \phi dv &= \int_M (2e^{2u} - e^u |\psi|^2 - K_g) \phi dv \\
\int_M \langle \psi, \not{D}\xi \rangle dv &= - \int_M e^u \langle \psi, \xi \rangle dv
\end{aligned}$$

for any smooth function ϕ and any smooth spinor ξ . It is clear that $(u, \psi) \in W^{1,2}(M) \times W^{1, \frac{4}{3}}(\Gamma(\Sigma M))$ is a weak solution if and only if (u, ψ) is a critical point

of E in $W^{1,2}(M) \times W^{1,\frac{4}{3}}(\Gamma(\Sigma M))$. A weak solution is a classical solution by the following

Proposition 4.1. *Any weak solution (u, ψ) to (4) on M with $I(u, \psi) < \infty$ is smooth.*

To prove the proposition, we first need a basic inequality in [BM].

Lemma 4.2. *Assume $\Omega \subset \mathbb{R}^2$ is a bounded domain and let u be a solution of*

$$\begin{cases} -\Delta u = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $f \in L^1(\Omega)$. Then for every $\delta \in (0, 4\pi)$ we have

$$\int_{\Omega} \exp\left\{\frac{(4\pi - \delta)|u(x)|}{\|f\|_1}\right\} dx \leq \frac{4\pi^2}{\delta} (\text{diam}\Omega)^2, \quad (12)$$

where $\|f\|_1 = \int_{\Omega} |f(x)| dx$.

Let $B_r = B_r(x)$ be a geodesic ball at a point x on M with radius r . Here r is smaller than the injective radius of M .

Lemma 4.3. *If (u, ψ) is a weak solution to (4) in B_r satisfying $\int_{B_r} e^{2u} + |\psi|^4 dx < \infty$, then we have*

$$u^+ \in L^\infty(B_{\frac{r}{4}}) \quad \text{and} \quad |\psi| \in L^\infty(B_{\frac{r}{4}}).$$

Proof. First we consider u . Set

$$f_1 = 2e^{2u} - e^u |\psi|^2 - K_g.$$

Then we have

$$-\Delta u = f_1.$$

We consider the following Dirichlet problem

$$\begin{cases} -\Delta u_1 = f_1, & \text{in } B_r \\ u_1 = 0, & \text{on } \partial B_r. \end{cases} \quad (13)$$

Since $\int_{B_r} e^{2u} dx < \infty$ and $\int_{B_r} |\psi|^4 dx < \infty$ we know that $f_1 \in L^1(B_r)$. By applying Lemma 4.2 on a smaller domain we have

$$e^{k|u_1|} \in L^1(B_r) \quad (14)$$

for some $k > 1$ and in particular $u_1 \in L^p(B_r)$ for some $p > 1$.

Let $u_2 = u - u_1$ so that $\Delta u_2 = 0$ on B_r . The mean value theorem for harmonic functions implies that

$$\|u_2^+\|_{L^\infty(B_{\frac{r}{2}})} \leq C \|u_2^+\|_{L^1(B_r)}.$$

Since $u_2^+ \leq u^+ + |u_1|$ and $2 \int_{B_r} u^+ \leq \int_{B_r} e^{2u} < \infty$, we have $u_2^+ \in L^1(B_r)$ and consequently

$$\|u_2^+\|_{L^\infty(B_{\frac{r}{2}})} < \infty. \quad (15)$$

Next we write

$$f_1 = 2e^{2u_2} e^{2u_1} - e^{u_1} e^{u_2} |\psi|^2 - K_g.$$

From (15) and (14) we have $f_1 \in L^{1+\varepsilon}(B_{\frac{r}{2}})$ for some $\varepsilon > 0$. Hence standard elliptic estimates imply that

$$\|u^+\|_{L^\infty(B_{\frac{r}{4}})} \leq C \|u^+\|_{L^1(B_r)} + C \|f_1\|_{L^{1+\varepsilon}(B_{\frac{r}{2}})} < \infty.$$

Since $u^+ \in L^\infty(B_{\frac{r}{4}})$, then the right hand of equation $\mathcal{D}\psi = -e^u\psi$ is in $L^4(\Gamma(\Sigma B_{\frac{r}{4}}))$. Hence $\psi \in C^0(\Gamma(\Sigma B_{\frac{r}{4}}))$ and especially $|\psi| \in L^\infty(B_{\frac{r}{4}})$. \square

Proof of Proposition 4.1. The standard method, together with Lemma 4.3, implies that u and ψ are smooth. \square

Next we discuss the compactness of a sequence of smooth solutions to (4). We begin with studying uniformly L^∞ boundedness of solutions for (4). Assume that (u_n, ψ_n) is a sequence of solutions of (4). Similarly as before we set

$$f_1^n = 2e^{2u_n} - e^{u_n} |\psi_n|^2 - K_g,$$

Lemma 4.4. *Let $\varepsilon_0 < \pi$ be a constant. For any sequence of solutions (u_n, ψ_n) with*

$$\int_{B_r} e^{2u_n} dx < \varepsilon_0, \quad \int_{B_r} |\psi_n|^4 dx < C$$

for some fixed constant $C > 0$ we have that $\|u_n^+\|_{L^\infty(B_{\frac{r}{4}})}$ is uniformly bounded.

Proof. Similarly as in the proof of lemma 4.3, it is sufficient to show that f_1^n is uniformly bounded in $L_{loc}^q(B_r)$ for some $q > 1$.

Let w_n be the solution of following problem:

$$\begin{cases} -\Delta w_n = 2e^{2u_n}, & \text{in } B_r(x) \\ w_n = 0, & \text{on } \partial B_r(x). \end{cases}$$

It is clear that $w_n \geq 0$ in $B_r(x)$. Since $\varepsilon_0 < \pi$, we can choose $\delta > 0$ such that $4\pi - \delta > 2\varepsilon_0(2 + \delta)$. By lemma 4.2 we get

$$\int_{B_r(x)} e^{(2+\delta)w_n} \leq C \tag{16}$$

for some constant C .

Next let z be the solution of the following equation

$$\begin{cases} -\Delta z = -K_g, & \text{in } B_r(x) \\ z = 0, & \text{on } \partial B_r(x). \end{cases}$$

It is clear that $\Delta(u_n - w_n - z) = e^u |\psi|^2 \geq 0$ on $B_r(x)$ and

$$\begin{aligned} \int_{B_r(x)} (u_n - w_n - z)^+ &\leq \int_{B_r(x)} (u_n - w_n)^+ + |z| dx \\ &\leq \int_{B_r(x)} (u_n^+ + |z|) \leq \int_{B_r(x)} e^{2u_n} + C_1 \leq C, \end{aligned}$$

for some constant $C > 0$. Here we have used $w_n \geq 0$. Therefore, by the mean value theorem for subharmonic function, for any $y \in B_{\frac{r}{2}}(x)$, we have

$$\begin{aligned} (u_n - w_n - z)(y) &\leq C \int_{B_r(x)} (u_n - w_n - z) \\ &\leq C \int_{B_r(x)} (u_n - w_n - z)^+ \leq C \end{aligned} \tag{17}$$

Thus, from (16) and (17), we deduce that

$$\int_{B_{\frac{\pi}{2}}(x)} e^{(2+\delta)u_n} \leq C. \quad (18)$$

By the Hölder inequality, for $l = \frac{4+2\delta}{4+\delta} > 1$ we have

$$\int_{B_{\frac{\pi}{2}}} (e^{u_n} |\psi_n|^2)^l dx \leq \left(\int_{B_{\frac{\pi}{2}}} e^{(2+\delta)u_n} dx \right)^{\frac{l}{2+\delta}} \left(\int_{B_{\frac{\pi}{2}}} |\psi_n|^4 dx \right)^{\frac{2+\delta-l}{2+\delta}} \leq C.$$

Let $q = \min\{l, 2 + \delta\}$. We have established that f_1^n is uniformly bounded in $L^q(B_{\frac{\pi}{2}}(x))$ with $q > 1$. \square

Since $\|u_n^+\|_{L^\infty(B_{\frac{\pi}{4}})}$ is uniformly bounded, by the standard method and the bootstrapping method of elliptic equations, we can get uniform estimates for higher derivatives of the functions u_n and ψ_n . That is,

Theorem 4.5. *Assume that (u_n, ψ_n) is a sequence of solutions for (4) with*

$$\int_M e^{2u_n} dv < \varepsilon_0, \text{ and } \int_M |\psi_n|^4 dv < C$$

for some positive constant $\varepsilon_0 < \pi$ and C . Then we have

$$\|u_n\|_{C^k(B_{\frac{1}{8}}(x))} + \|\psi_n\|_{C^k(B_{\frac{1}{8}}(x))} \leq C. \quad (19)$$

for any geodesic ball $B_{\frac{1}{8}}(x)$ of M .

From Theorem 4.5, we have the following Theorem:

Theorem 4.6. *Assume that (u_n, ψ_n) is a sequence of solutions for (4) with*

$$\int_M e^{2u_n} dv < \varepsilon_0, \text{ and } \int_M |\psi_n|^4 dv < C$$

for some positive constant $\varepsilon_0 < \pi$ and C . Then (u_n, ψ_n) admits a subsequence converging to (u, ψ) which is a smooth solution of (4).

5. BLOW UP BEHAVIOR

When the energy $\int_M e^{2u} dv$ is large, then the blow-up phenomenon may occur as in the case of the Liouville equation. In this section we will analyze the asymptotic behavior of a sequence of solutions for (4) when the blow-up phenomenon happens. Assume that (u_n, ψ_n) satisfies

$$\begin{cases} -\Delta u_n &= 2e^{2u_n} - e^{u_n} |\psi_n|^2 - K_g, \\ \not{D}\psi_n &= -e^{u_n} \psi_n, \end{cases} \quad \text{on } M \quad (20)$$

with

$$\int_M e^{2u_n} dv < C, \text{ and } \int_M |\psi_n|^4 dv < C \quad (21)$$

for some positive constant C .

We shall follow [BM], where the authors analyze the behavior of a sequence of solutions for the Liouville-type equation on a bounded domain. Similar results for the Toda system, which is another natural generalization of the Liouville equation, were obtained in [JW].

Theorem 5.1. Let (u_n, ψ_n) be a sequence of solutions to (20) satisfying (21).

Define

$$\begin{aligned}\Sigma_1 &= \{x \in M, \text{ there is a sequence } y_n \rightarrow x \text{ such that } u_n(y_n) \rightarrow +\infty\} \\ \Sigma_2 &= \{x \in M, \text{ there is a sequence } y_n \rightarrow x \text{ such that } |\psi_n(y_n)| \rightarrow +\infty\}.\end{aligned}$$

Then, we have $\Sigma_2 \subset \Sigma_1$. Moreover, (u_n, ψ_n) admits a subsequence, denoted still by (u_n, ψ_n) , satisfying that

- a) ψ_n is bounded in $L_{loc}^\infty(M \setminus \Sigma_2)$.
- b) For u_n , one of the following alternatives holds:
 - i) u_n is bounded in $L^\infty(M)$.
 - ii) $u_n \rightarrow -\infty$ uniformly on M .
 - iii) Σ_1 is finite, nonempty and either

$$u_n \text{ is bounded in } L_{loc}^\infty(M \setminus \Sigma_1) \quad (22)$$

or

$$u_n \rightarrow -\infty \text{ uniformly on compact subsets of } M \setminus \Sigma_1. \quad (23)$$

Proof. First, if $x \in M \setminus \Sigma_1$, then from the equation $\mathcal{D}\psi_n = -e^{u_n}\psi_n$ we know $x \in M \setminus \Sigma_2$. Therefore we have $\Sigma_2 \subset \Sigma_1$ and ψ_n are bounded in $L_{loc}^\infty(M \setminus \Sigma_2)$.

Next let f_1^n be as before. Since e^{2u_n} is bounded in $L^1(M)$, we may extract a subsequence from u_n (still denoted u_n) such that e^{2u_n} converges in the sense of measures on M to some nonnegative bounded measure μ i.e.

$$\int_M e^{2u_n} \varphi dv \rightarrow \int_M \varphi d\mu$$

for every $\varphi \in C(M)$. A point $x \in M$ is called an ε -regular point with respect to μ if there is a function $\varphi \in C(M)$, $\text{supp}\varphi \subset B_r(x) \subset M$, $0 \leq \varphi \leq 1$ with $\varphi = 1$ in a neighborhood of x such that

$$\int_M \varphi d\mu < \varepsilon$$

We define

$$\Omega_1(\varepsilon) = \{x \in M : x \text{ is not an } \varepsilon\text{-regular point with respect to } \mu\}.$$

By definition and (21) we see that $\Omega_1(\varepsilon)$ is finite. We divide the proof into three steps.

Step 1. $\Sigma_1 = \Omega_1(\varepsilon_0)$ provided $\varepsilon_0 < \pi$.

First we show that $\Omega_1(\varepsilon_0) \subset \Sigma_1$. Supposing that $x_0 \in \Omega_1(\varepsilon_0)$, we claim that for any $R > 0$, $\lim_{n \rightarrow +\infty} \|u_n^+\|_{L^\infty(B_R(x_0))} = +\infty$. We demonstrate the claim by a contradiction. So we assume that there would be some $R_0 > 0$ and a subsequence such that $\|u_n^+\|_{L^\infty(B_{R_0}(x_0))}$ is bounded. Especially we have $\|e^{2u_n}\|_{L^\infty(B_{R_0}(x_0))} \leq C$ and therefore $\int_{B_R(x_0)} e^{2u_n} dx \leq CR^\delta$ for all $R < R_0$ and some $\delta > 0$. This implies

$$\int_M \varphi d\mu < \varepsilon_0 \text{ for some suitable } \varphi.$$

Therefore x_0 is regular, contradicting $x_0 \in \Omega_1(\varepsilon_0)$. So the claim is proved. Now we choose $R > 0$ small enough so that $\overline{B}_R(x_0)$ does not contain any other point of $\Omega_1(\varepsilon_0)$. Let $x_n \in B_R(x_0)$ be such that

$$u_n^+(x_n) = \max_{\overline{B}_R(x_0)} u_n^+ \rightarrow +\infty.$$

We claim that $x_n \rightarrow x_0$, i.e. $x_0 \in \Sigma_1$. Otherwise there would be a subsequence

$$x_{n_k} \rightarrow \bar{x} \neq x_0 \text{ and } \bar{x} \notin \Omega_1(\varepsilon_0)$$

that is, \bar{x} is a regular point. This is a contradiction. Therefore we have proved that $\Omega_1(\varepsilon_0) \subset \Sigma_1$.

Next we show that $\Sigma_1 \subset \Omega_1(\varepsilon_0)$ by using the approach to the Toda system in [JW]. Let $x_0 \in \Sigma_1$. Assume by contradiction that $x_0 \notin \Omega_1(\varepsilon_0)$. Thus $\int_{B_\delta(x_0)} e^{2u_n} \leq \varepsilon_0$ for any small constant $\delta > 0$. Note that $-\Delta u_n = 2e^{2u_n} - e^{u_n} |\psi_n|^2 - K_g \leq 2e^{2u_n} - K_g$. Define $w : B_\delta(x_0) \rightarrow \mathbb{R}$ by

$$\begin{cases} -\Delta w = 2e^{2u_n} - K_g & \text{in } B_\delta(x_0) \\ w = u_n & \text{on } \partial B_\delta(x_0). \end{cases} \quad (24)$$

The maximum principle implies that $u_n \leq w$. Since Σ_1 is finite, we may assume that u_n is uniformly bounded in $L^\infty(\partial B_\delta(x_0))$. In view of $\int_{B_\delta(x_0)} e^{2u_n} \leq \varepsilon_0 < \pi$, and the boundedness of the curvature R of M , as in the proof of lemma 4.4 we have $w^+ \in L^\infty(B_{\frac{\delta}{2}}(x_0))$, which in turn implies that $u_n^+ \in L^\infty(B_{\frac{\delta}{2}}(x_0))$. Hence we have a contradiction. Therefore $\Sigma_1 \subset \Omega_1(\varepsilon_0)$.

So we have $\Sigma_1 = \Omega_1(\varepsilon_0)$.

Step 2. $\Sigma_1 = \emptyset$ implies (1) and (2) hold.

$\Sigma_1 = \emptyset$ means that u_n^+ is bounded in $L^\infty(M)$. Consequently ψ_n is bounded in $L^\infty(M)$. Thus, f_1^n is bounded in $L^p(M)$ for any $p > 1$. Applying the Harnack inequality as in [BM], we have (1) or (2).

Step 3. $\Sigma_1 \neq \emptyset$ implies (3).

In this case, we know that u_n^+ is bounded in $L^\infty_{loc}(M \setminus \Sigma_1)$ and therefore f_1^n is bounded in $L^p_{loc}(M \setminus \Sigma_1)$ for any $p > 1$. Then as in step (2) we know that either

$$u_n \text{ is bounded in } L^\infty_{loc}(M \setminus \Sigma_1),$$

or

$$u_n \rightarrow -\infty \quad \text{on any compact subset of } M \setminus \Sigma_1$$

Thus we complete the proof of the Theorem. \square

Actually in Theorem 5.1 the case (22) will not occur if $\Sigma_1 \setminus \Sigma_2 \neq \emptyset$. Next we will show this.

Theorem 5.2. *In Theorem 5.1, if in addition $\Sigma_1 \setminus \Sigma_2 \neq \emptyset$, then the first case of iii) does not happen, i.e. $u_n \rightarrow -\infty$ uniformly on compact subsets of $M \setminus \Sigma_1$. Moreover, setting $\Sigma_1 = \{p_1, p_2, \dots, p_l\}$, we have*

$$e^{2u_n} \rightharpoonup \mu = \sum_{i=1}^l \alpha_i \delta_{p_i}, \quad \text{with } \alpha_i \geq \pi.$$

Proof. We should show that (22) does not happen when $\Sigma_1 \setminus \Sigma_2 \neq \emptyset$. Fix some point $x_0 \in \Sigma_1 \setminus \Sigma_2$ and choose $\delta > 0$ to be so small that x_0 is the only point of $\Sigma_1 \setminus \Sigma_2$ in $\overline{B}_\delta(x_0)$. Let f_1^n be as before, i.e.

$$f_1^n = 2e^{2u_n} - e^{u_n} |\psi_n|^2 - K_g.$$

Since x_0 is a point of $\Sigma_1 \setminus \Sigma_2$, we can select δ to be sufficiently small such that

$$\begin{aligned} f_1^n &= e^{2u_n}(2 - e^{-u_n}|\psi_n|^2 - K_g e^{-2u_n}) \\ &= e^{2u_n}v_n(x), \end{aligned}$$

where $v_n(x) = 2 - e^{-u_n}|\psi_n|^2 - K_g e^{-2u_n}$ and $v_n(x) \rightarrow 2$ in $B_\delta(x_0)$. Therefore we can rewrite the first equation of (20) as

$$\begin{cases} -\Delta u_n = v_n(x)e^{2u_n}, & \text{in } B_\delta(x_0) \\ 0 \leq v_n(x) \leq b, & \text{in } B_\delta(x_0) \\ \int_{B_\delta(x_0)} e^{2u_n} dx \leq C \end{cases}$$

for b and C positive constants.

Noting that x_0 is a blow up point for u_n , we can apply the Brezis-Merle result (see [BM]) to conclude that

$$u_n \rightarrow -\infty, \quad \text{for any compact subset } K \subset B_\delta(x_0) \setminus \{x_0\}.$$

Consequently, by the alternative proved in Theorem 5.1, we have that (22) does not happen and only (23) holds.

Moreover since the case (23) is valid, then $e^{2u_n} \rightarrow 0$ in $L^p_{loc}(M \setminus \Sigma_1)$ for any $p \geq 1$. Therefore, if $e^{2u_n} \rightharpoonup \mu$, then the measure μ is supported on Σ_1 . Hence, setting $\Sigma_1 = \{p_1, p_2, \dots, p_l\}$, we have $e^{2u_n} \rightharpoonup \mu = \sum_{i=1}^l \alpha_i \delta_{p_i}$ with $\alpha_i \geq \pi$. \square

6. ASYMPTOTIC BEHAVIOR OF RESCALING EQUATIONS

It is well known that a ‘‘bubble’’, an entire solution of (4) with finite energy, will be obtained after a suitable rescaling at a blow-up point. In the rest of the paper we will analyze the asymptotic behavior of an entire solution with finite energy. We will show that an entire solution on \mathbb{R}^2 can be extended to \mathbb{S}^2 , i.e. the singularity of infinity is removable.

The considered equations are

$$\begin{cases} -\Delta u &= 2e^{2u} - e^u \langle \psi, \psi \rangle, & x \in \mathbb{R}^2 \\ \not{D}\psi &= -e^u \psi, & x \in \mathbb{R}^2. \end{cases} \quad (25)$$

The energy condition is

$$I(u, \psi) = \int_{\mathbb{R}^2} (e^{2u} + |\psi|^4) dx < \infty. \quad (26)$$

Next we start to deal with the asymptotic behavior of solutions of (25) and (26). First we have

Lemma 6.1. *Let (u, ψ) be a solution of (25) and (26) with $u \in H^1_{loc}(\mathbb{R}^2)$ and $\psi \in H^{1, \frac{4}{3}}_{loc}(\mathbb{R}^2)$. Then $u^+ \in L^\infty(\mathbb{R}^2)$.*

The proof of Lemma 6.1 follows from the idea of [CL2]. Since $u^+ \in L^\infty(\mathbb{R}^2)$, it follows from the discussion in the previous section that (u, ψ) is smooth in \mathbb{R}^2 .

Denote (v, ϕ) be the Kelvin transformation of (u, ψ) , i.e.

$$\begin{aligned} v(x) &= u\left(\frac{x}{|x|^2}\right) - 2 \ln |x| \\ \phi(x) &= |x|^{-1} \psi\left(\frac{x}{|x|^2}\right) \end{aligned}$$

Then (v, ϕ) satisfies

$$\begin{cases} -\Delta v &= 2e^{2v} - e^v \langle \phi, \phi \rangle, & x \in \mathbb{R}^2 \setminus \{0\} \\ \not{D}\phi &= -e^v \phi, & x \in \mathbb{R}^2 \setminus \{0\}. \end{cases} \quad (27)$$

And, by change of variable,

$$\begin{aligned} \int_{|x| \leq r_0} e^{2v} dx &= \int_{|x| \geq \frac{1}{r_0}} e^{2u} dx \\ \int_{|x| \leq r_0} |\phi|^4 dx &= \int_{|x| \geq \frac{1}{r_0}} |\psi|^4 dx \end{aligned}$$

could be small if r_0 is small. Therefore we obtain that there is a r_0 small enough such that (v, ϕ) is a smooth solution to (27) on $B_{r_0} \setminus \{0\}$ with energy $\int_{|x| \leq r_0} e^{2v} dx < \varepsilon_0 < \pi$ for any sufficiently small positive number ε_0 , and $\int_{|x| \leq r_0} |\phi|^4 dx < C$. Since (27) and (26) are conformally invariant, in the sequel we may assume B_{r_0} to be the unit disk B_1 .

Lemma 6.2. *There is an $0 < \varepsilon_0 < \pi$ if (v, ϕ) is a smooth solution to (27) on $B_1 \setminus \{0\}$ with energy $\int_{|x| \leq 1} e^{2v} dx < \varepsilon_0$, and $\int_{|x| \leq 1} |\phi|^4 dx < C$, then for any $x \in B_{\frac{1}{2}}$ we have*

$$|\phi(x)||x|^{\frac{1}{2}} + |\nabla \phi(x)||x|^{\frac{3}{2}} \leq C \left(\int_{B_{2|x|}} |\phi|^4 dx \right)^{\frac{1}{4}}. \quad (28)$$

Furthermore, if we assume that $e^{2v} = O(\frac{1}{|x|^{2-\varepsilon}})$, then, for any $x \in B_{\frac{1}{2}}$, we have

$$|\phi(x)||x|^{\frac{1}{2}} + |\nabla \phi(x)||x|^{\frac{3}{2}} \leq C|x|^{\frac{1}{4\varepsilon}} \left(\int_{B_1} |\phi|^4 dx \right)^{\frac{1}{4}}, \quad (29)$$

for some positive constant C . Here ε is any sufficiently small positive number.

Proof. We use a similar argument as in [CJLW] to prove the Lemma. Fix any $x_0 \in B_{\frac{1}{2}} \setminus \{0\}$, and define $(\tilde{v}, \tilde{\phi})$ by

$$\begin{aligned} \tilde{v}(x) &= v(x_0 + |x_0|x) + \log |x_0|, \\ \tilde{\phi}(x) &= |x_0|^{\frac{1}{2}} \phi(x_0 + |x_0|x). \end{aligned}$$

It is clear that $(\tilde{v}, \tilde{\phi})$ is a smooth solution to (25) on B_1 with $\int_{B_1} e^{2\tilde{v}} dx < \varepsilon_0$ and $\int_{B_1} |\tilde{\phi}|^4 dx < C$. Applying Theorem 4.5, we have

$$\|\tilde{\phi}\|_{C^1(B_{\frac{1}{2}})} \leq C \|\tilde{\phi}\|_{L^4(B_1)}.$$

Scaling back, we obtain (28).

Next recall that the spinor field $\phi(x)$ satisfies

$$\not{D}\phi = -e^v \phi \quad \text{in } B_1 \setminus \{0\}.$$

We choose a cut-off function $\eta_\varepsilon \in C_0^\infty(B_{2\varepsilon})$ such that $\eta_\varepsilon = 1$ in $B_\varepsilon(0)$ and $|\nabla \eta_\varepsilon| < \frac{C}{\varepsilon}$. Then we have

$$\not{D}((1 - \eta_\varepsilon)\phi) = -(1 - \eta_\varepsilon)e^v \phi - d\eta_\varepsilon \cdot \phi.$$

From the elliptic estimate with boundary (see [CJLW]), we have

$$\begin{aligned} & \| (1 - \eta_\varepsilon)\phi \|_{W^{1, \frac{4}{3}}(B_1)} \\ & \leq C \| e^v \|_{L^2(B_1)} \| \phi \|_{L^4(B_1)} + C \| \phi \|_{W^{1, \frac{4}{3}}(\partial B_1)} + C \| d\eta_\varepsilon \cdot \phi \|_{L^{\frac{4}{3}}(B_1)}. \end{aligned} \quad (30)$$

By (28) we have

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \|\phi\|_{L^{\frac{4}{3}}(B_{2\varepsilon})} = 0.$$

Now letting $\varepsilon \rightarrow 0$, and in virtue of the smallness of $\int_{B_1} e^{2v} dx$ and the Sobolev embedding theorem, we obtain

$$\left(\int_{B_1} |\phi|^4 dx\right)^{\frac{1}{4}} \leq C \left(\left(\int_{\partial B_1} |\nabla \phi|^{\frac{4}{3}} ds\right)^{\frac{3}{4}} + \left(\int_{\partial B_1} |\phi|^4 ds\right)^{\frac{1}{4}} \right).$$

By rescaling, we have for any $0 \leq r \leq 1$

$$\begin{aligned} \left(\int_{B_r} |\phi|^4 dx\right)^{\frac{1}{4}} &\leq C(r \int_{\partial B_r} |\nabla \phi|^{\frac{4}{3}} ds)^{\frac{3}{4}} + C(r \int_{\partial B_r} |\phi|^4 ds)^{\frac{1}{4}} \\ &\leq C(r \int_{\partial B_r} |\nabla \phi|^{\frac{4}{3}} ds)^{\frac{1}{4}} + C(r \int_{\partial B_r} |\phi|^4 ds)^{\frac{1}{4}}. \end{aligned}$$

i.e.

$$\int_{B_r} |\phi|^4 dx \leq Cr \left(\int_{\partial B_r} |\nabla \phi|^{\frac{4}{3}} ds + \int_{\partial B_r} |\phi|^4 ds \right). \quad (31)$$

Next let $\bar{\phi} := \frac{1}{|B_1|} \int_{B_1} \phi dx$. Note that

$$\mathcal{D}(\phi - \bar{\phi}) = -e^v(\phi - \bar{\phi}) - e^v \bar{\phi} \quad \text{in } B_1 \setminus \{0\}.$$

By an similar argument for obtaining (30) and using the Poincare inequality, we have

$$\|\phi - \bar{\phi}\|_{W^{1, \frac{4}{3}}(B_1)} \leq C(\|e^v\|_{L^2(B_1)} \|\phi - \bar{\phi}\|_{W^{1, \frac{4}{3}}(B_1)} + \|\nabla \phi\|_{L^{\frac{4}{3}}(\partial B_1)} + \|e^v \bar{\phi}\|_{L^{\frac{4}{3}}(B_1)}).$$

Again, in virtue of the smallness of $\int_{B_1} e^{2v} dx$ we obtain

$$\begin{aligned} \left(\int_{B_1} |\nabla \phi|^{\frac{4}{3}} dx\right)^{\frac{3}{4}} &\leq C \left(\int_{\partial B_1} |\nabla \phi|^{\frac{4}{3}} ds\right)^{\frac{3}{4}} + C |\bar{\phi}| \left(\int_{B_1} e^{2v} dx\right)^{\frac{1}{2}} \\ &\leq C \left(\int_{\partial B_1} |\nabla \phi|^{\frac{4}{3}} ds\right)^{\frac{3}{4}} + C \left(\int_{B_1} |\phi|^4 dx\right)^{\frac{1}{4}} \left(\int_{B_1} e^{2v} dx\right)^{\frac{1}{2}} \\ &\leq C \left(\int_{\partial B_1} |\nabla \phi|^{\frac{4}{3}} ds\right)^{\frac{3}{4}} + \varepsilon_1 \left(\int_{B_1} |\phi|^4 dx\right)^{\frac{3}{4}} + C(\varepsilon_1) \left(\int_{B_1} e^{2v} dx\right)^{\frac{3}{4}}, \end{aligned}$$

where ε_1 is a small constant. Hence, for $0 \leq r \leq 1$, we have

$$\int_{B_r} |\nabla \phi|^{\frac{4}{3}} dx \leq Cr \int_{\partial B_r} |\nabla \phi|^{\frac{4}{3}} ds + \varepsilon_1 \int_{B_r} |\phi|^4 dx + C(\varepsilon_1) \int_{B_r} e^{2v} dx. \quad (32)$$

Note that $e^{2v} = O\left(\frac{1}{|x|^{2-\varepsilon}}\right)$ for some $\varepsilon > 0$. We have

$$\int_{B_r} e^{2v} dx \leq Cr \int_{\partial B_r} e^{2v} ds \quad (33)$$

Form (31),(32) and (33), for any $0 \leq r \leq 1$, we obtain

$$\int_{B_r} e^{2v} dx + \int_{B_r} |\nabla \phi|^{\frac{4}{3}} dx + \int_{B_r} |\phi|^4 dx \leq Cr \left(\int_{\partial B_r} e^{2v} ds + \int_{\partial B_r} |\nabla \phi|^{\frac{4}{3}} ds + \int_{\partial B_r} |\phi|^4 ds \right),$$

for some constant $C > 0$. Denote $F(r) := \int_{B_r} e^{2v} dx + \int_{B_r} |\nabla \phi|^{\frac{4}{3}} dx + \int_{B_r} |\phi|^4 dx$. Then we get

$$F(r) \leq CrF'(r).$$

Integrating this inequality yields

$$F(r) \leq F(1)r^{\frac{1}{c}}. \quad (34)$$

From (34), we can easily get (29). Thus we complete the proof of Lemma. \square

From Lemma 6.2 and the Kelvin transformation, we obtain the asymptotic estimate of the spinor $\psi(x)$

$$|\psi(x)| \leq C|x|^{-\frac{1}{2}-\delta_0} \quad \text{for } |x| \text{ near } \infty \quad (35)$$

for some positive number δ_0 provided that $e^{2v} = O(\frac{1}{|x|^{2-\varepsilon}})$.

Now let $\alpha = \int_{\mathbb{R}^2} 2e^{2u} - e^u|\psi|^2 dx$, and a constant spinor $\xi_0 = \int_{\mathbb{R}^2} e^u \psi dx$. It will turn out that the constant spinor ξ_0 is well defined. Then we have

Proposition 6.3. *Let (u, ψ) be a solution of (25) and (26). Then u satisfies*

$$u(x) = -\frac{\alpha}{2\pi} \ln|x| + C + O(|x|^{-1}) \quad \text{for } |x| \text{ near } \infty, \quad (36)$$

$$\psi(x) = -\frac{1}{2\pi} \frac{x}{|x|^2} \cdot \xi_0 + o(|x|^{-1}) \quad \text{for } |x| \text{ near } \infty, \quad (37)$$

where \cdot is the Clifford multiplication, $C \in \mathbb{R}$ is some constant, and $\alpha = 4\pi$.

Proof. First, we analyze the asymptotic behavior of $u(x)$. To show (36), we follow essentially an argument used in [CL1]. Set

$$w_1(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} (\ln|x-y| - \ln(|y|+1)) e^{2u} dy,$$

$$w_2(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} (\ln|x-y| - \ln(|y|+1)) e^u |\psi|^2 dy.$$

Then, it is easy to check that

$$\frac{w_1(x)}{\ln|x|} \rightarrow -\frac{1}{2\pi} \int_{\mathbb{R}^2} e^{2u} dx, \quad \text{as } |x| \rightarrow +\infty, \quad \text{uniformly,}$$

$$\frac{w_2(x)}{\ln|x|} \rightarrow -\frac{1}{2\pi} \int_{\mathbb{R}^2} e^u |\psi|^2 dx, \quad \text{as } |x| \rightarrow +\infty, \quad \text{uniformly,}$$

Moreover, $-\Delta w_1(x) = e^{2u}$ and $-\Delta w_2(x) = e^u |\psi|^2$ on \mathbb{R}^2 . Therefore, if we define $v = u(x) - 2w_1(x) + w_2(x)$, we have $\Delta v(x) = 0$ on \mathbb{R}^2 . Since $u^+ \in L^\infty(\mathbb{R}^2)$ by Lemma 6.1, we get that

$$v(x) \leq C_1 + C_2 \ln|x|,$$

for $|x|$ sufficiently large, with C_1, C_2 positive constants. Therefore, by Liouville's theorem on harmonic functions, $v(x)$ has to be constant and hence we get

$$\frac{u(x)}{\ln|x|} \rightarrow -\frac{\alpha}{2\pi} \quad \text{as } |x| \rightarrow +\infty, \quad \text{uniformly.}$$

Since $\int_{\mathbb{R}^2} e^{2u} dx < +\infty$, the above result implies

$$\alpha \geq 2\pi.$$

Next we show that $\alpha > 2\pi$. Assume by contradiction that $\alpha = 2\pi$. Let (v, ϕ) be the Kelvin transformation of (u, ψ) . We know (v, ϕ) satisfy (27) in $B_1 \setminus \{0\}$. Denote $f(x) := 2e^{2v} - e^v|\phi|^2$. Then we have

$$-\Delta v = f(x) \quad \text{in } B_1 \setminus \{0\}.$$

From the asymptotic estimate (28), we know that $f(x) > 0$ in a small punctured disk $B_1 \setminus \{0\}$. Set

$$h(x) = -\frac{1}{2\pi} \int_{B_1} \log|x-y|f(y)dy$$

and $g(x) = v(x) - h(x)$. It is clear that $\Delta h = -f$ and $\Delta g = 0$.

On the other hand, we can check that

$$\lim_{|x| \rightarrow 0} \frac{v}{-\log|x|} = 0$$

which implies

$$\lim_{|x| \rightarrow 0} \frac{g(x)}{-\log|x|} = \lim_{|x| \rightarrow 0} \frac{v(x) - h(x)}{-\log|x|} = \lim_{|x| \rightarrow 0} \frac{u(\frac{x}{|x|^2}) - 2\log|x|}{-\log|x|} = 1.$$

Since $g(x)$ is harmonic in $B_1 \setminus \{0\}$, we have $g(x) = -\log|x| + g_0(x)$ with a smooth harmonic function g_0 in B_1 . By definition, we have $h(x) > 0$. Thus, we have

$$\int_{B_1} e^{2v} dx = \int_{B_1} e^{2g+h} dx \geq \int_{B_1} \frac{1}{|x|^2} e^{g_0} dx = +\infty,$$

which is a contradiction with $\int_{\mathbb{R}^2} e^{2v} dx < \infty$. Hence we have shown that $\alpha > 2\pi$.

From $\alpha > 2\pi$, we can improve the estimate for e^{2u} to

$$e^{2u} \leq C|x|^{-2-\varepsilon} \quad \text{for } |x| \text{ near } \infty. \quad (38)$$

From (38), and by using potential analysis we also get

$$-\frac{\alpha}{2\pi} \ln|x| - C \leq u(x) \leq -\frac{\alpha}{2\pi} \ln|x| + C$$

for some constant $C > 0$, see [CL2].

Then by using (38) and (35) and following the derivation of gradient estimates in [CK], we get

$$|\langle x, \nabla u \rangle + \frac{\alpha}{2\pi}| \leq C|x|^{-\varepsilon} \quad \text{for } |x| \text{ near } \infty,$$

consequently we have

$$|u_r + \frac{\alpha}{2\pi r}| \leq C|x|^{-1-\varepsilon} \quad \text{for } |x| \text{ near } \infty. \quad (39)$$

In the similar way, we can also get

$$|u_\theta| \leq C|x|^{-\varepsilon} \quad \text{for } |x| \text{ near } \infty. \quad (40)$$

Here (r, θ) is the polar coordinate system on \mathbb{R}^2 and C, ε are positive constants. From (40) and (39), we can obtain (36). The idea of proving (36) can also be seen in [WZ].

Next, we show that $\alpha = 4\pi$. Set

$$T(z) = (\partial_z u)^2 - \partial_z^2 u + \frac{1}{4} \langle \psi, dz \cdot \partial_{\bar{z}} \psi \rangle + \frac{1}{4} \langle d\bar{z} \cdot \partial_z \psi, \psi \rangle,$$

where \cdot is Clifford multiplication. From Proposition 3.3, we know that $T(z)$ is a holomorphic function. Using (35) and (36), we have the following expansion of $T(z)$ near infinity

$$\begin{aligned} & \frac{1}{4} \left(\frac{\alpha}{2\pi} \right)^2 \frac{1}{z^2} - \frac{1}{2} \frac{\alpha}{2\pi} \frac{1}{z^2} + o\left(\frac{1}{z^2}\right) + \dots \\ &= \frac{1}{2z^2} \left(\frac{1}{2} \left(\frac{\alpha}{2\pi} \right)^2 - \frac{\alpha}{2\pi} \right) + o\left(\frac{1}{z^2}\right) + \dots \end{aligned}$$

Hence, $T(z)$ is a constant and $\frac{1}{2} \left(\frac{\alpha}{2\pi} \right)^2 - \frac{\alpha}{2\pi} = 0$, i.e. $\alpha = 4\pi$.

From $\alpha = 4\pi$, we can improve the estimate for e^{2u} to

$$e^{2u} \leq C|x|^{-4} \quad \text{for } |x| \text{ near } \infty. \quad (41)$$

This implies that the constant spinor ξ_0 is well defined.

Finally, we analyze the asymptotic behavior of the spinor $\psi(x)$. We set

$$\xi(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2} \cdot e^u \psi dy,$$

where \cdot is Clifford multiplication. Since the Green function of the Dirac operator in \mathbb{R}^2 is

$$G(x, y) = \frac{1}{2\pi} \frac{x-y}{|x-y|^2},$$

for any $x, y \in \mathbb{R}^2$ and $x \neq y$, see [AHM], we have $\not{D}\xi = -e^u \psi$.

We compute

$$\begin{aligned} |x \cdot \xi(x) - \frac{1}{2\pi} \xi_0| &= \frac{1}{2\pi} \left| \int_{\mathbb{R}^2} \left(\frac{x \cdot (x-y)}{|x-y|^2} + 1 \right) \cdot e^u \psi(y) dy \right| \\ &= \frac{1}{2\pi} \left| \int_{\mathbb{R}^2} \left(\frac{x \cdot (x-y)}{|x-y|^2} - \frac{(x-y) \cdot (x-y)}{|x-y|^2} \right) \cdot e^u \psi(y) dy \right| \\ &= \frac{1}{2\pi} \left| \int_{\mathbb{R}^2} \frac{(x-y) \cdot y}{|x-y|^2} \cdot e^u \psi(y) dy \right| \\ &\leq \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{|y|}{|x-y|} e^u |\psi| dy. \end{aligned} \quad (42)$$

From (41), we also have

$$|\psi| e^u \leq C|x|^{-2-\varepsilon} \quad \text{for } |x| \text{ near } \infty, \quad (43)$$

for some positive constants C and ε . Then following the derivation of gradient estimates in [CK], we get

$$|x \cdot \xi(x) - \frac{1}{2\pi} \xi_0| \leq C|x|^{-\varepsilon} \quad \text{for } |x| \text{ near } \infty. \quad (44)$$

Set $\eta(x) = \psi(x) - \xi(x)$. Since $\not{D}\psi = -e^u \psi$, we have $\not{D}\eta(x) = 0$. By (35) and (44) we have $|\eta(x)| \leq C|x|^{-1-\delta_0}$, which implies $\eta(x) = 0$, i.e. $\psi(x) = \xi(x)$.

Furthermore,

$$\begin{aligned} \left| \psi(x) + \frac{1}{2\pi} \frac{x}{|x|^2} \cdot \xi_0 \right| &= \left| \frac{x}{|x|^2} \cdot (x \cdot \psi(x) - \frac{1}{2\pi} \xi_0) \right| \\ &\leq \frac{1}{|x|} \left| x \cdot \psi(x) - \frac{1}{2\pi} \xi_0 \right| \\ &\leq C|x|^{-1-\varepsilon}, \end{aligned}$$

for $|x|$ near ∞ . This proves (37). \square

Since the equation (25) is conformally invariant, the solutions u and ψ of (25) can be viewed as a function and a spinor on $\mathbb{S}^2 \setminus \{\text{northpole}\}$ with finite energy. In the following Theorem, we shall prove that such a singularity can be removed as in many conformal problems. Hence, at the end we obtain that the solutions are actually defined on \mathbb{S}^2 .

Theorem 6.4. *Let (u, ψ) be a smooth solution of (25) and (26). Then (u, ψ) extends to a smooth solution on \mathbb{S}^2 .*

Proof. Let (v, ϕ) be the Kelvin transformation of (u, ψ) . Then (v, ϕ) satisfies (27) on $\mathbb{R}^2 \setminus \{0\}$. To prove the Theorem, it is sufficient to show that (v, ϕ) is smooth on \mathbb{R}^2 . Applying Proposition 6.3, we have

$$v(x) = \left(\frac{\alpha}{2\pi} - 2 \right) \ln|x| + O(1) \quad \text{for } |x| \text{ near } 0. \quad (45)$$

Since $\alpha = 4\pi$, we get that v is bounded near 0. By recalling that ϕ is also bounded near 0, elliptic theory implies that (v, ϕ) is smooth. \square

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