Symbolic synchronization and the detection of global properties of coupled dynamics from local information.

by

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Preprint no.: 40 2006
Symbolic Synchronization and the Detection of Global Properties of Coupled Dynamics from Local Information

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(Dated: April 20, 2006)

Abstract

We study coupled dynamics on networks using symbolic dynamics. The symbolic dynamics is defined by dividing the state space into a small number of regions (typically 2), and considering the relative frequencies of the transitions between these regions. It turns out that the global qualitative properties of the coupled dynamics can be classified into three different phases based on the synchronization of the variables and the homogeneity of the symbolic dynamics. Of particular interest is the homogeneous unsynchronized phase where the coupled dynamics is in a chaotic unsynchronized state, but exhibits (almost) identical symbolic dynamics at all the nodes in the network. We refer to this dynamical behaviour as symbolic synchronization. In this phase, the local symbolic dynamics of any arbitrarily selected node reflects global properties of the coupled dynamics, such as qualitative behavior of the largest Lyapunov exponent, complete synchronization, and phase synchronization. This phase depends mainly on the network architecture, and only to a smaller extent on the local dynamical function. We present results for two model dynamics, iterations of the one-dimensional logistic map and the two-dimensional Hénon map, as local dynamical function.

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Nonlinear dynamical elements interacting with each other can lead to synchronization or other types of coherent behaviour at the system scale. Coupled map models are one of the most widely accepted models to understand these behaviours in systems from many diverse fields such as physics, biology, ecology etc. Their important feature is that the individual elements can already exhibit some complex behaviour, for example chaotic dynamics. The question then is how to detect coordination at larger scales beyond the simplest one, synchronization. An important tool in the analysis of dynamical systems are symbolic dynamics. We develop a new scheme of symbolic dynamics that is based on the special partitions of the phase space which prevent the occurrence of certain symbol sequences related to the characteristics of the dynamics. In particular, we report a new behaviour of coupled dynamics, which we refer to as symbolic synchronization, i.e. synchronization of the nodes at the coarse grained level, whereas microscopically all elements behave differently. Through the framework of this symbolic dynamics, we detect various global properties of coupled dynamics on networks by using a scalar time series of any randomly selected node. A decisive advantage of our method is that the global properties are inferred by using a very short time series, hence the method is computationally very fast, also it does not depend on the size of the network and is robust against external noise.

I. INTRODUCTION

In order to gain insights into the behaviour of real systems from many diverse fields ranging from chemical, physical and biological systems, it is useful to identify model systems that on one hand exhibit essential dynamical features of those real world systems, but on the other hand suppress individual details that are not really relevant for the qualitative behavior [1]. Coupled map models have emerged as one such paradigm [2, 3]. Here, we have a system of elements with identical local dynamical functions. These elements are arranged in a network that expresses their couplings so that the local dynamical iteration depends not only on the own state of an element, but also on the ones of its neighbour in the network. The inhomogeneities in the network then translate into qualitative features of the global dy-
dynamics. While such coupled map models already present an important simplification in view of the complexities of real world dynamics, their behavior can nevertheless be sufficiently complicated and difficult to analyze. Thus, it is important to identify parameters that allow for a facile and robust detection of different qualitative states. One needs a coarse grained description to analyze the complicated time evolution of a chaotic dynamical system [4, 5]. In doing so one inevitably simplifies the dynamics a lot and some of the information are lost, but the aim is that important invariants and robust properties of the dynamical systems can be kept. Such a coarse graining means that we divide the possible dynamical states of the system into finitely many discrete classes and investigate the derived symbolic dynamics [4]. Coarse graining of the time evolution of lower dimensional systems have been studied at various levels [5, 6], but symbolic dynamical studies of higher dimensional spatio-temporal chaotic systems are rare and so far limited to coupled map lattices [7]. In the present paper, continuing the approach developed in [8], we study symbolic dynamics of coupled map networks and demonstrate that they can serve the above purpose well. Thereby, we attempt to provide a general framework for coupled dynamics on networks.

In [8], we have studied symbolic dynamics of a discrete dynamical iteration of a function based on non-generating partitions. We have shown two important uses of this symbolic dynamics, namely, distinguishing a chaotic iteration and a random iteration with the same density distribution (this is related to the earlier work [10] on transition entropy and [9] on permutation entropy), and detecting synchronization in coupled dynamics on large networks. In this paper we extend these studies and propose a general method to investigate collective behavior of coupled systems. Besides the applications mentioned in [8], we show further applications for detecting various dynamical properties of coupled dynamics in relation to structural parameters of the underlying network.

We take coupled map network models as generic models to apply our method. Chaotic coupled maps show rich spatio-temporal behaviour. One phenomenon that has received a lot of attention is synchronization, where different random or chaotic units of a system behave in unison [11, 12]. (For a selection of recent references, see also [13].) One application of our symbolic dynamics is the detection of synchronization in large complex systems. Traditional methods for the detection of synchronization in coupled systems focus on the correlation analysis of the time series measured at pairs of the nodes. In [8] we have introduced a method based on symbolic dynamics, which uses a very short time series of any single
arbitrarily selected node to detect global synchrony of all the units. In this paper we show that the same type of symbolic dynamics can be used as a measure of phase synchronization, a novel phenomenon shown by coupled dynamics on networks [14].

In more detail, by our method we classify the coupled dynamics into different states, depending upon the synchronization of the nodes and the homogeneity of the symbolic dynamics of the nodes. The most interesting phase is the unsynchronized homogeneous phase, which refers to the state where the local chaotic dynamics of the individual nodes are different, but the derived symbolic dynamics of all the nodes are similar. We call this state as symbolic synchronization of the nodes. Recently the unsynchronized region of coupled maps is shown to have fractal stationary density function [15]. We show that, in this phase, the transition probabilities of any randomly selected node reflect the qualitative information of the largest Lyapunov exponent ($\lambda_1$) and the phase synchronization of the coupled dynamics. For the calculation of the largest Lyapunov exponent we utilize only a very short time series, where as traditional methods to calculate the largest Lyapunov exponent from a scalar time series require rather long time series and also involve various computational complications [16]. We point out, however, that – as to be expected from such a simplistic reduction – our symbolic dynamics gives only the qualitative behaviour of the Lyapunov exponent $\lambda_1$, but of course not its exact value.

The paper is organized as follows. After an introductory section we introduce the definitions of the different phases based on the symbolic dynamical properties in Section II. In Section III, we then present numerical examples illustrating the behaviour of nodes in different phases. Mostly we present results for homogeneous synchronized phase which is of main interest. The key point is that the derived symbolic dynamics allows for the detection of the global properties of coupled dynamics from local measurements, that is, we can infer global properties of the dynamical network by considering the symbolic dynamics at a single node. Section IV distinguishes different dynamical phases based on the network parameters. Section V describes the relation between symbolic dynamics and phase synchronization. Section VI discusses the coupled Hénon map.
II. MODEL AND DEFINITION OF SYMBOLIC DYNAMICS

We consider the dynamical system defined by the iteration rule

\[ x(t + 1) = f(x(t)) \]  

(1)

where \( t \in \mathbb{Z} \) is the discrete time and \( f : S \rightarrow S \) is a map on a subset \( S \) of \( \mathbb{R}^n \). Let \( \{ S_i : i = 1, \ldots, m \} \) be a partition of \( S \), i.e., a collection of mutually disjoint and nonempty subsets satisfying \( \bigcup_{i=1}^{m} S_i = S \). The symbolic dynamics corresponding to (1) is the sequence of symbols \( \{ \ldots, s_{t-1}, s_t, s_{t+1}, \ldots \} \) where \( s_t = i \) if \( x(t) \in S_i \). For the purposes of this paper, a useful partition is defined as follows. Let \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \), and suppose the scalar \( x_m, 1 \leq m \leq n \), is available for measurement. For a given threshold value \( x^* \in \mathbb{R} \), define the sets

\[ S_1 = \{ x \in S : x_m < x^* \} \]
\[ S_2 = \{ x \in S : x_m \geq x^* \} \]

(2)

The value \( x_m \) can be chosen to make the sets \( S_1, S_2 \) nonempty, in which case they form a non-trivial partition of \( S \). For this special partition, we use the two-symbol dynamics generated by

\[ s_t = \begin{cases} 
\alpha & \text{if } x_m(t) < x^* \\
\beta & \text{if } x_m(t) \geq x^*. 
\end{cases} \]

(3)

The symbolic dynamics depends only on the measurements \( x_m \), yielding a sequence of symbols determined by whether a measured value exceeds the threshold \( x^* \) or not. Essentially any choice of the threshold \( x^* \) will yield a non-generating partition. For practical calculations using short time series, however, a judicious choice of \( x^* \) becomes important. We will address this issue later in the paper (see section V).

We take the well known coupled map model [17],

\[ x_i(t + 1) = f(x_i(t)) + \frac{\epsilon}{k_i} \sum_j w_{ij} g(x_j(t), x_i(t)) \]

(4)

where \( x_i(t) \) is the dynamical variable of the \( i \)-th node \((1 \leq i \leq N)\) at time \( t \), \( w \) is the adjacency matrix with elements \( w_{ij} \) taking values between 0 and 1 depending upon the weight of the connection between \( i \) and \( j \), and \( k_i \) is some normalization factor depending on the node \( i \), for example its degree. The function \( f(x) \) defines the local nonlinear map and the function \( g(x) \) defines the nature of the coupling between the nodes. In the first three sections,
we present the results for the local dynamics given by the logistic map $f(x) = \mu x (1 - x)$ and coupling function $g(x_j(t), x_i(t)) = f(x_j(t)) - f(x_i(t))$. The weight $w_{ij}$ is simply one when nodes $i$ and $j$ are neighbours in the undirected network, and 0 otherwise. In particular, the matrix $w$ is symmetric; $k_i$ then is the degree of node $i$, as already indicated.

We evolve Equation (4) starting from random initial conditions and estimate the transition probabilities using time series of length $\tau = 1000$. Note that the length of the time series is independent of the size of the network. We calculate the transition probability $P(i, j)$ by the ratio $\frac{\sum_i n(s_i = i, s_{i+1} = j)}{\sum_i n(s_i = i)}$, where $n$ is a count of the number of times of occurrence [8].

**III. DIFFERENT STATES OF THE COUPLED DYNAMICS**

We classify the coupled dynamics in three different categories based on the dynamical behaviour, and we show that how one category differs from another based on some of the parameters of underlying network:

1. Unsynchronized or phase synchronized *non-homogeneous* behaviour: *phase one*,

2. Partially synchronized or phase synchronized *homogeneous* behaviour: *phase two*, and

3. Fully synchronized *homogeneous* behaviour: *phase three*.

Here, synchronization refers to the variables at different nodes having the same value $x_i(t) = x_j(t)$ for all $i, j$. The network is globally synchronized when at each time $t$, all nodes have the same value. Partial synchronization means that some of the nodes form a cluster inside which all the nodes are synchronized while nodes in different clusters are not in synchrony. We note, however, that this state usually does not occur in our coupled dynamics because the phase differences between the various clusters will interfere with the internal synchronizations. The following behavior, however, does robustly occur in suitable parameter regions. A pair of nodes is called phase synchronized [14] when they have their minima (maxima) matching for all $t > t_0$, that is, when one of them attains a minimum then so does the other. The concrete values may and can be different. In a phase synchronized cluster all nodes are phase synchronized.
Complete synchronization is indicated by the variance $\sigma^2$ of the variables over the network tending to zero, where

$$\sigma^2 = \left\langle \frac{1}{N-1} \sum_{i} [x_i(t) - x(t)]^2 \right\rangle_t,$$

$x(t) = \frac{1}{N} \sum_i x_i(t)$ denotes an average over the nodes of the network, and $\langle \ldots \rangle_t$ denotes an average over time. We define homogeneous and non-homogeneous behaviour based on the symbolic dynamics of the individual nodes. If all the nodes have exactly the same symbolic dynamics, i.e., if the transition probabilities of all the nodes are equal, then we say that the coupled dynamics is homogeneous; otherwise it is non-homogeneous. Homogeneity is indicated by the variance of the transition probability over the network being zero,

$$\varsigma^2 = \left\langle \frac{1}{N-1} \sum_{k=1}^{N} [P_k(\alpha,\alpha) - \overline{P(\alpha,\alpha)}]^2 \right\rangle,$$

where $\overline{P(\alpha,\alpha)} = \sum_{k=1}^{N} P(\alpha,\alpha)$ denotes an average over the nodes of the network.

IV. HOMOGENEOUS PHASES AND COUPLED DYNAMICS ON NETWORK

A. Homogeneous phase and network properties

We shall now connect the symbolic homogeneity with network properties. When all the nodes in a network have the same degree and the network is homogeneously connected, i.e. if the network is completely symmetric, like a nearest neighbour coupled network with periodic boundary conditions, then, unless the dynamics breaks the symmetry, each node should have qualitatively the same symbolic dynamics, i.e. all transition probabilities for all the nodes would be equal. Note that homogeneous symbolic dynamics need not correspond to synchronization. For random networks, homogeneity of the symbolic dynamics depends on the number of connections in the network. To relate different dynamical states with the network parameter we consider the quantity $r = N_c/N^2$. This ratio estimates the number of connections $N_c$ in the network with the number of possible connections $N(N - 1)/2$, and shows how far a network is from being fully connected. We use $r$ as a broad border to roughly distinguish the three phases. For a strongly connected network, i.e. when $r$ is close to one (number of connections $N_c$ of order $N^2$), we get a fully synchronized state for appropriate coupling strengths $\epsilon$. Then the transition probabilities of all nodes are obviously
equal (phase 3). For \( N_e \sim N \), we get phase two, i.e. the nodes are partially synchronized or partially phase synchronized, but the symbolic dynamics of the nodes are identical. Note here we are only roughly relating \( N_e \) and phase, later we will provide a more accurate relation between the number of connections and the phases.

B. Symbolic synchronized phase and global properties of coupled dynamics

We concentrate on the phase where the nodes are not synchronized though their symbolic dynamics are identical. This is the most interesting phase as the complexity of the coupled dynamics can be understood by observing the symbol sequence of any arbitrarily selected node. We now illustrate our results using numerical simulations. Figure 1 is plotted for the logistic map as the local map and a scale-free network [21] as the coupling network. Figure 1 \((a - a')\), \((b - b')\), \((c - c')\) and \((d - d')\) are for average degree 2, 6, 10, and 20 respectively. In the figures, the \( x \)-axis represents the coupling strength and the \( y \)-axis (for figures \((a),(b),(c),(d)\)) plots the synchronization measure for the whole network \((\sigma^2)\), and also the measure of homogeneity \((\varsigma^2)\). In the box we plot the largest Lyapunov exponent \((\lambda)\) as a function of the coupling strength. Figures \((a')\), \((b')\), \((c')\) and \((d')\) plot the transition probability \(P(\alpha,\alpha)\) for different nodes. For clarity we plot only a few arbitrarily selected nodes. We start with the example of networks having coupled dynamics in phase one (non-homogeneous unsynchronized) and we move towards the examples of networks showing homogeneous synchronized state, phase 3.

For \( \epsilon < 0.2 \), the coupled logistic map model (4) exhibits a similar behaviour for different coupling architectures, with quasiperiodic behaviour around \( \epsilon = 0.2 \). The behaviour varies with the coupling architecture for larger coupling strengths. In the periodic regions the symbolic dynamics of the nodes, given by (3), would always be similar irrespective of the underlying coupling network. So, in the periodic regions we do not get any extra information about the network by observing symbolic sequences, but the symbolic dynamics is informative when the coupled dynamics lies on the chaotic attractor. Subfigures 1(a), \((a')\), are plotted for scale free networks with average degree 2. The transition probabilities \(P(\alpha,\alpha)\) are completely different for the different nodes (except for \( \epsilon < 0.2 \)). Here, the nodes are not synchronized, \( \sigma^2 \) being nonzero for the entire coupling strength range. This state corresponds to phase one. Interesting phenomena occur when we increase the number of
connections in the networks. Subfigures 1(b) and (b') are plotted for a scale-free network with average degree 6. It can be seen that $P(\alpha, \alpha)$ for different nodes are remarkably similar. Note that we calculate $P(\alpha, \alpha)$ for coupled dynamics being in the chaotic and unsynchronized regime ($\lambda$ and $\sigma^2$ both are greater than zero). So we do not take periodic and synchronized regions into account which obviously yield similar symbolic dynamics for all the nodes. This homogeneity becomes more prominent as we increase the average degree of the network. Figure 1(c), (c') and 1 (d), (d') are plotted for average degree 10 and 20 respectively. The transition probabilities of all the nodes are the same except for a few places where some nodes have different transition probability (e.g. node number 50 in (c') having a different value of $P(\alpha, \alpha)$). Note that here the nodes are not synchronized, which is indicated by the nonzero value of $\sigma^2$ throughout the coupling range, except for $\epsilon = 1$ in (c) and for $\epsilon > 0.8$ in (d).

The second interesting feature is the qualitatively similar behaviour of the largest Lyapunov exponent, which is calculated from (4), and $P(\alpha, \alpha)$, which is calculated from a scalar time series of an arbitrarily selected node. Note that the time series used for the calculation of $P(\alpha, \alpha)$ is very short compared to the traditional methods to calculate the largest Lyapunov exponent from a scalar time series. This similar behaviour of the Lyapunov exponent and the ordering relations between the values of the state variable was first observed by Bandt and Pompe [9] in the case of isolated dynamics. Here we show that a similar relation exists for coupled dynamics on networks, depending upon the network parameters, namely the connection architecture and the connection ratio $N_c/N^2$.

Figure 2 is plotted for various networks being in the phase two (homogeneous unsynchronized phase). The figures show the similar behaviour of $\lambda$ and $P(\alpha, \alpha)$ of any arbitrarily selected node. Figure 2 (a) is for a one-dimensional nearest neighbour coupled network with size 20 and average degree two, and Figure 2 (b) is for size 50 and average degree six. For $k$-nearest neighbour coupled networks we always find the homogeneous phase, independent of the average degree or the ratio $N_c/N^2$, because of the symmetry between the nodes. Figure 2 (c) is plotted for a random network of size 100 and average degree 10, and Figure 2 (d) is plotted for a scale-free network of size 200 and average degree 10. In all the subfigures, $P(\alpha, \alpha)$ qualitatively matches the largest Lyapunov exponent of the coupled dynamics. Note that $P(\alpha, \alpha)$ gives a qualitatively similar behaviour as the Lyapunov exponent and not the exact value. For phase three any arbitrary selected node gives the trend of the largest
Lyapunov exponent. Since phase three is defined as the synchronized phase, the largest Lyapunov exponent is the Lyapunov exponent of the uncoupled map. Therefore, it is more interesting to consider examples of networks where the coupled dynamics is in phase two. Note that only the network property is responsible for the homogeneous or non-homogeneous behaviour of the coupled dynamics. Figure 2 is plotted for the coupled logistic map but a similar behaviour is shown by the Hénon maps also (see below).

V. RELATION BETWEEN DYNAMICAL PHASES AND NETWORK PROPERTIES

We can also exhibit a direct relation between network parameters and dynamical behaviour. The symmetry properties of the network and the average degree affect the homogeneity of the symbolic sequences. Figure 3 plots the deviation from the homogeneity indicated by \( \zeta^2 = \left( \frac{1}{N} \sum_{i=1}^{N} [P_i(\alpha, \alpha) - \bar{P}(\alpha, \alpha)]^2 \right)_e \), as a function of \( N_e/N^2 \). Here, \( P_i \) is the transition probability of \( i \)th node and \( \bar{P}(\alpha, \alpha) = \frac{1}{N} \sum_{i=1}^{N} P_i(\alpha, \alpha) \), and \( \langle \cdot \rangle_e \) denotes the average over all coupling strengths. We start with one-dimensional nearest neighbour coupled network (homogeneous phase) and randomly add connections. For nearest neighbour coupled networks we obtain the homogeneous phase, as already explained in the previous section. As we add the connections randomly, first the homogeneity gets perturbed, but gets reestablished as the number of connections is increased further. Note that here we always calculate the deviation in the non-synchronized regime only, because the synchronized regime obviously corresponds to the homogeneous phase. For each randomly added connection we take the average of the twenty networks. Note that in this region (phase two) \( \sigma^2 \) is not zero. For \( N_e/N^2 \) close to one, we get a synchronized state after a coupling strength [19] that corresponds to the homogeneous state (phase three).

VI. PHASE SYNCHRONIZATION : SYMBOLIC SYNCHRONIZATION

If nodes \( i \) and \( j \) have the same symbolic dynamics, \( s_t(i) = s_t(j) \), then we say nodes \( i, j \) are symbolically synchronized. Also, a cluster of nodes is symbolically synchronized if all pairs of nodes belonging to that cluster are symbolically synchronized. Note that in a symbolically synchronised cluster, the state values of the nodes may differ. The symbolic synchronization
is observed in the *phase two*, where the number of the connections in the networks is very small, in general of the order of $N$. With the increase in the number of connections we usually get a fully synchronized cluster, which is trivially symbolically synchronized. Real-world networks are in general sparsely connected ($N_c \sim N$), and complete synchronization is rare, though phase synchronization or symbolic synchronization is possible. We show that $P(\alpha, \alpha)$ can be used as a good measure of the phase synchronization in the coupled map network (4). Figure 4 shows the correlation between the phase synchronization and the transition probability $P(\alpha, \alpha)$ of an arbitrary selected node. We see that in the homogeneous region $P(\alpha, \alpha)$ matches considerably well with the phase synchronization. Note that this is a partially ordered phase region, so phase synchronized clusters vary with time. We plot the number of clusters calculated for a certain time length, and the number of clusters may change with the evolution of the coupled dynamics. Therefore at some coupling strength region(s), the transition probability $P(\alpha, \alpha)$ does not match with the phase synchronization (For example in the Figure 4 (c), at coupling strength 0.59, the value of $P(\alpha, \alpha)$ is very high although the nodes are phase synchronized).

### VII. COUPLED HÉNON MAPS

In this section we apply our method to coupled Hénon maps. The Hénon map is a two-dimensional map [20],

$$x(t + 1) = y(t) + 1 - ax(t)^2$$

$$y(t + 1) = by(t).$$

When one introduces the possibility of a time delay, the above equation can be written as the scalar equation,

$$x(t + 1) = bx(t - 1) + 1 - ax(t)^2$$

(5)

For the parameters we take the values $a = 1.4$ and $b = 0.3$, for which the Hénon map is known to have a chaotic attractor.

We define the symbolic dynamics as given in (3). The choice of the threshold $x^*$ requires some care. A judicious choice should make certain short transition probabilities very small, which may be useful for detecting network dynamics from single-node measurements. Clearly, increasing the threshold decreases the probability of occurrence of the repeated
sequence $\beta \beta$. However, it also decreases the probability of observing the single symbol $\beta$, making it difficult to work with short time series. Hence, the choice of the threshold is a compromise between these two effects. We use the natural density defined by the data to choose a threshold. Figure (5) depicts how the the probabilities of observing a single symbol $\beta$ and the repeated sequence $\beta \beta$ change depending on the value of the threshold $x^*$. It can be seen that a choice of $x^*$ roughly in the range $(0.55, 1.20)$ would be useful, since it renders the sequence $\beta \beta$ very unlikely without constraining the occurrence of the symbol $\beta$. Note that it is immediate from their definitions that the probabilities $P(\beta)$ and $P(\beta, \beta)$ will be decreasing as functions of $x^*$, and will approach zero as $x^*$ increases; furthermore, $P(\beta) > P(\beta, \beta)$. It follows that one can find a threshold $x^*$ for which $P(\beta)$ is large compared to $P(\beta, \beta)$. Figure (5) shows the ratio $P(\beta, \beta)/P(\beta)$, and the sharp decrease at about $x^* \approx 0.6$ suggests to take some value near 0.6 as the threshold, yielding a very small $P(\beta, \beta)$ and a large $P(\beta)$ at the same time. We evolve (4) starting from random initial conditions, with (5) as local dynamics, and estimate the transition probabilities $P(i, j)$ as discussed in the first section, using a time series of length $\tau = 1000$. At the globally synchronized state $x_i(t) = x_j(t); \forall i, j, t$, with all nodes evolving according to the rule (5), the symbolic sequences measured from a node will be subject to the same constraints as that generated by (5).

We now discuss some results based on numerical simulations on various networks. Figure 6 plots the transition probabilities as a function of the coupling strength. We consider the symbolic sequences of length two and three. For length two, we consider the transition probabilities $P(\alpha, \alpha)$ and $P(\beta, \beta)$. For sequences of length three we have 6 possible transitions, but some of them are very small (like $P(\beta, \beta, i)$ ), so we plot only those transition probabilities which vary with the couplings. It is clear from the figures that the synchronized state is easily detected by looking at the transition probabilities of any arbitrarily selected node. Whenever the transition probabilities are equal to the transition probabilities of the map (5), the network is globally synchronized. It is clear from subfigures (d), (e) and (f) that for the synchronized region (zero $\sigma^2$), the deviation of transition probabilities from the transition probabilities of the uncoupled map (5) is also zero. Here, the deviation of $P(i, j)$ of any node is defined as $\delta^2_{i,j} = \frac{1}{m-1} \sum_{k=1}^{m} [P_k(i, j) - \overline{P[i,j]}]^2$, where $\overline{P(i, j)} = \frac{1}{m} \sum_k P_k(i, j)$, calculated at $\epsilon = 0 \ k = 1, \ldots \ldots$ are $m$ different sets of random initial conditions taken between $-1.5$ and 1.5. In all the figures (except (f)) we get synchronization for larger coupling
strengths, so the deviation is almost zero there, i.e. all transition probabilities match completely with those of the uncoupled map. Note that there are certain regions (small coupling strength range $\epsilon < 0.2$) where the nodes do not get synchronized while the deviations are quite small. That is because for sufficiently small coupling strength, couplings do not affect the behaviour of the individual nodes very much, and so the transition probabilities also do not differ much from the uncoupled function. However, as we increase the coupling strength, the transition probabilities become dependent on the couplings. Still, if we look at the small coupling strength regions carefully we see that not all the deviations are small. For example, although the deviations of $P(\beta, \alpha, \alpha)$ (---) and $P(\alpha, \alpha, \beta)$ (-) are very small, the deviation in $P(\alpha, \alpha, \alpha)$ (---) is still large, whereas for the synchronized regime all deviations are very close to zero. Figure 7 plots the deviation from the homogeneity $\chi^2$ as a function of $N_c/N^2$ (see the caption of Figure 3, which shows a similar plot with logistic map as a local dynamical function). The only difference is that for the Hénon map we plot the transition probability of three-symbol sequences instead of two-symbol sequences for logistic and tent maps.

VIII. CONCLUSION

We have studied the symbolic dynamics of coupled maps on networks. We define our symbolic dynamics based on non-generating partitions leading to some forbidden transitions of symbols in the time evolution of the function. The optimal partitions are those which lead to the maximal difference between the permutation entropy of the dynamical iteration and corresponding random iteration. For one-dimensional systems finding these partitions is simple, whereas for higher dimensional systems it may be more difficult. However, it turns out that symbolic dynamics drawn from any non-generating partitions is usually good enough for the applications we have considered in this paper. The symbolic dynamics can be drawn when the system parameters are not known, as well as for experimental data taken in a noisy environment.

We use symbolic dynamics as a measure of dynamical state of the coupled system and show various applications of this measure. The first important application is the detection of synchronization by comparing the transition probabilities of any arbitrarily selected node with those of the uncoupled function. In the global synchronized state the coupled dynamics
collapses to the dynamics of the uncoupled function, and the symbolic dynamics of any node is subject to the same constraints as that generated by the uncoupled function.

Moreover, we define three different states of the coupled dynamics based on the synchronisation and the symbolic dynamical properties. Phase two which refers to the homogeneous unsynchronized phase or symbolic synchronized phase is of our prime interest where the nodes are not synchronized but have identical symbolic dynamics. Although these phases are detected dynamically, we find that the homogeneous unsynchronized phase is related to the ratio $N_c/N^2$ and to a smaller extent to the chaotic dynamical function used. This is the region observed for networks with $N_c \sim r \times N^2$ where $0.05 < r < 0.1$. Most of the real networks are sparsely connected and come under the category of phase two. In this phase we can deduce the global properties of the coupled dynamics such as the largest Lyapunov exponent and phase synchronization by simply observing the local symbolic dynamics of any randomly selected node. Future investigations will involve an analytical understanding of symbolic synchronization and application to detect various levels of synchronization in experimental data taken from coupled systems.

Acknowledgments

We acknowledge Dr. Wenlian Lu for useful discussions.


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[21] We have generated that scale-free network by the standard preferential attachment scheme, but one should note that large classes of scale-free networks may exhibit a qualitatively different behavior as regards other crucial network parameters besides the degree sequence, in particular concerning synchronizability, see [18].
FIG. 1: Examples of coupled networks showing all three phases. All figures are plotted for scale-free networks of size $N = 200$ and (a) average degree 2 (phase 1), (b) average degree 6, (c) average degree 10, (d) average degree 20. The $x$-axis represents the coupling strength and the $y$-axis gives the synchronization measure $\sigma^2$ (○) and the homogeneity measure $\zeta^2$ (●) for the whole network. The largest Lyapunov exponent is plotted as a function of the coupling strength (see inset). Figures (a'), (b'), (c') and (d') show exact values of the transition probability $P(\alpha, \alpha)$ for different nodes as a function of the coupling strength.
FIG. 2: The global measure of coupled dynamics from the local symbolic dynamics. We take various networks having coupled dynamics in the phase two. The x-axis gives the coupling strength $\epsilon$ and the y-axis depicts $\lambda_i$ (-) (largest Lyapunov exponent for the coupled dynamics) as well as $P(\alpha, \alpha)$ (..) (transition probability for a randomly selected node). (a) for nearest neighbour coupled network of size $N = 20$, (b) for 3-nearest neighbour coupled network, $N = 50$, (c) and (d) are for random and scale-free networks, respectively, with average degree 10 and $N = 200$.

FIG. 3: The measure of homogeneity $\varsigma^2$ as a function of connectivity ratio $N_c/N^2$. 

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FIG. 4: The ratio of the number phase synchronized clusters to the maximum possible clusters, $N_{\text{clus}}/N$ and the transition probability $P(\alpha, \alpha)$ for the coupled logistic map as a function of the coupling strength $\varepsilon$, (a) for a nearest neighbour coupled network with average degree 20 and $N = 100$, (b) for a scale-free network with average degree 10 and $N = 100$, (c) for a random network with average degree 10 and $N = 100$, (d) for a nearest neighbour coupled network with average degree 6 and $N = 50$, and the tent map ($f(x) = (1 - 2|x - \frac{1}{2}|)$ as the local chaotic function.
FIG. 5: Illustration of the choice of the threshold $x^*$ as the point where $P(\beta, \beta)/P(\beta)$ sharply drops to near zero.
FIG. 6: The transition probability measure for coupled Hénon maps. The $x$-axis displays the coupling strength and the $y$-axis shows the different transition probabilities and the measure of synchronization $\sigma^2$. (a) is for two coupled nodes and plots the transition probability $P(\alpha, \alpha)$ for the symbolic sequence of length 2. (b) and (c) are plotted for globally coupled networks with $N = 50$. (b) plots $P(\alpha, \alpha)$ for symbolic sequences of length two and (c) plots transition probabilities for symbolic sequences of length 3, namely $P(\alpha, \alpha, \alpha)$ (□), $P(\alpha, \alpha, \beta)$ (●), and $P(\beta, \alpha, \alpha)$ (○). Some of the transition probabilities are close to zero for most of the coupling strengths, so we do not plot these. The synchronized state is detected when all the transition probabilities are equal to those of the uncoupled map; i.e. the transition probabilities at the zero coupling strength. (d), (e) and (f) show the standard deviation $\sigma^2$ (solid thick line) and $\delta^2$ (vertical dashed line) for these three transition probabilities of an arbitrary selected node with respect to the transition probabilities of the uncoupled function (solid line), i.e for $\epsilon = 0$. $\sigma^2$ is calculated for 20 simulations for the dynamics with different sets of random initial conditions. (d) is plotted for a globally connected network with $N = 50$, (e) and (f) for a random network with $N = 100$, average degree 10 and 2 respectively. The last subfigure is plotted to show the behaviour of the transition probabilities when we do not get global synchronization even at large coupling strengths.
FIG. 7: The measure of homogeneity $\zeta^2$ as a function of connection ratio $N_c/N^2$, with Hénon map as the local dynamical function.