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by

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RELAXATION OF THREE SOLENOIDAL WELLS AND CHARACTERIZATION OF THREE-PHASE H-MEASURES

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Abstract. We study the problem of characterizing quasiconvex hulls for three "solenoidal" (divergence free) wells in dimension three when the wells are pairwise incompatible. A full characterization is achieved by combining certain ideas based on Šverák's example of a "nontrivial" quasiconvex function and on Müller's wavelet expansions estimates in terms of the Riesz transform. As a by-product, we obtain a new more general "geometrical" result: characterization of extremal three-point H-measures for three-phase mixtures in dimension three. We also discuss the applicability of the latter result to problems with other kinematic constrains, in particular to that of three linear elastic wells.

Key words: Differential inclusions, Relaxation, H-measures and their characterization, Three-well problem, Quasiconvex hulls.

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1. Introduction

This paper addresses two distinct but related problems. The first one is the characterization of the quasiconvex hull of a set of three 3×3 matrices in the context of divergence free fields. The second is the characterization of extremal three-point H-measures for mixtures of three characteristic functions in dimension three. The first problem is expressed as follows: given a set $K = \{A_1, A_2, A_3\} \subset \mathbb{M}^{3\times 3}$ of three real 3×3 matrices, characterize the set, denoted by K_S^{qc}, of all matrices B_0 such that there exists a sequence $\{B_h\} \subset L^2_{loc}(\mathbb{R}^3, \mathbb{M}^{3\times 3})$, L^2_{loc}-equi-integrable, Q-periodic, with $Q = (0, 2\pi)^3$, and such that

$$(1.1) \qquad \begin{cases} \mathrm{Div} B_h = 0 & \text{in } \mathcal{D}'(\mathbb{R}^3, \mathbb{R}^3)\,, \\ \mathrm{dist}(B_h, K) \to 0 & \text{in measure,} \\ \fint_Q B_h = B_0 \quad \forall\, h\,. \end{cases}$$

The set K_S^{qc} is called the S-quasiconvex hull of K. When the fields B_h are curl-free rather then divergence-free, and thus are gradients of suitable vector-fields, the analogous problem has been solved by Šverák [19] and we refer the reader to [12] for a broad introduction into this subject. More generally (1.1) falls in the framework of $\mathcal{A}$-quasiconvexity where the differential constraint on the fields B_h is replaced by more general ones related to the so-called constant rank hypothesis (see, e.g., [4]).

As recently shown by Palombaro and Ponsiglione [15], exact non-constant solution to the problem (1.1), i.e., divergence free fields taking values in K, may only exist if K contains rank-2 connections, that is $\mathrm{rank}(A_i - A_j) \leq 2$ for some $i \neq j$. In contrast, as Garroni and Nesi have shown [5], there exist sets K with no rank-2 connections, for which (1.1) admits solutions for some $B_0 \notin K$. The examples exhibited in [5] share the property that the two-dimensional vector space generated by $A_2 - A_1$ and $A_3 - A_1$ contains three distinct "rank-2 directions" and the mutual position of A_1, A_2 and A_3 is such that the corresponding rank-2 affine lines through A_1, A_2 and A_3 intersect at points inside the convex hull of K in order to form an "inner" triangle like in Figure 2-(1) in Section 5. (This is reminiscent of work done in the context of gradient fields, see, e.g., Székelyhidi [21] and in particular Fig. 1 therein.) Throughout the paper we will refer to all sets that enjoy such property as sets of *Type 1*. One of the main results of this paper (Theorem 5.24) is that if K contains no rank-2 connections, then the set K_S^{qc} is non-trivial, i.e., $K_S^{qc} \supsetneq K$, if and only if K is a set of *Type 1*. The characterization of K_S^{qc} when K is of *Type 1*, is performed in two different steps. First one seeks an inner bound for K_S^{qc} and then one proves the optimality of such bound. An explicit construction for the inner bound is provided by an "infinite-rank" sequential lamination, the idea successfully exploited earlier in a number of different settings, see for instance [17], [13], [24], [1]. All the essential ideas, in the divergence free (Div-free) context, have been introduced in the Garroni and Nesi construction [5]. Establishing the optimality of the inner bound requires an additional analysis. For the "Tartar square" example of "approximate non-rigidity" for four pairwise incompatible gradient wells, one way to establish the sharpness of the inner bound is by employing the Šverák's [20] "nontrivial" quasiconvex (but not polyconvex) function $\det^+$. Our strategy is similar in spirit. We construct a suitable modification of a function originally introduced by Tartar [22] in the study of composites in homogenization. It is a rank-2 convex function which is quadratic and therefore quasiconvex in the space of Div-free fields. Our modification resembles Šverák's function since it behaves like his $\det^+$ function on the two-dimensional plane determined by the three-wells (see Lemma 5.12). A central accompanying ingredient is in establishing that the rank-two convexity on such plane implies quasiconvexity with respect to solenoidal fields (Theorem 5.14). For proving the latter we follow Müller [11] and develop an appropriate modification of Haar wavelet estimates in terms of the Riesz transform employing deep Paley-Littlewood techniques of harmonic analysis. This allows us to fully characterize the S-quasiconvex hull for all sets of *Type 1*. In other cases when K does not contain any rank-2 connection but it is not a set of *Type 1*, we employ our analogue of Šverák's function to "disconnect" the set K_S^{qc} (Theorem 5.22) and we then use a result due to Kirchheim [7] and Matoušek [9] (see Lemma 5.21) to prove that in such a case necessarily $K_S^{qc} = K$. All the remaining cases (see Definitions 5.3 and 5.6 and the subsequent discussion for an account of all cases) can be treated without any special difficulty (see Theorem 5.23 and Proposition 8.1).

The second, related, issue addressed in this paper is the characterization of the H-measures associated with three-phase mixtures in dimension three. Such a problem arises in the context of evaluating the "relaxation" of three-well energies. To set the scene let us consider the following problem. Given the function $F(\eta) = \frac{1}{2}\min\{|\eta - A_i|^2\,, i = 1,\ldots,N\}$, $\eta \in \mathbb{M}^{3\times3}$, $N \in \mathbb{N}$, $N \geq 2$, given $\theta \in (0,1)^N$ with $\sum_i \theta_i = 1$, characterize the quasiconvexification of F, $Q^\theta F$, at fixed volume fractions θ:

$$\forall\eta \in \mathbb{M}^{3\times3} \quad Q^\theta F(\eta) := \inf_{\fint \chi_i = \theta_i} \inf_{\phi \in W^{1,2}_\sharp} \frac{1}{2} \fint_Q \left|\eta + \nabla\phi(x) - \sum_{i=1}^N \chi_i A_i\right|^2 dx\,, \tag{1.2}$$

where $W^{1,2}_\sharp(Q,\mathbb{R}^3)$ is the space of $W^{1,2}(Q,\mathbb{R}^3)$ functions which are Q-periodic, with $Q = (0,2\pi)^3$, and χ_i's are characteristic functions of measurable subsets of Q. In the framework of linearized elasticity, the function F is the minimum of N quadratic functions of the linear strain with same elastic moduli but different stress-free strains (see, e.g., [1, 8, 18]). More generally, one could consider a problem analogous to (1.2) for general differential constraints (in the above mentioned framework of $\mathcal{A}$-quasiconvexity), in particular with the divergence-free constraints. Computing $Q^\theta F$ amounts to finding energetically optimal microstructures that mix the N given phases with prescribed volume fractions θ.

One possible approach to this problem is based on the idea of using Fourier analysis and as a result reformulating (1.2) into a problem of minimization with respect to special measures on the unit sphere S^2, the H-measures, introduced by Tartar [23], and independently by Gerard [6], the idea proposed and advanced in this context by Kohn [8] and Smyshlyaev and Willis [18]. This reduces the problem of relaxation to that of characterizing the extremal points of the (closed convex) set of the H-measures. An attractive feature of the H-measures is that those are purely "geometrical" objects, i.e., they do not depend on any differential constraint but only on the microgeometry of mixing. In particular they depend on the number of component phases, N, and on the volume fractions θ. When the number of phases is two, i.e., $N = 2$, the set of the H-measures is known for every value of θ and the relaxation of a two-well energy may be explicitly computed (see Kohn [8]). In contrast, for $N > 2$, the set of the H-measures is not fully characterized. It is known that they satisfy some restrictions but these are in general not sufficient to characterize them. For $N = 3$, Smyshlyaev and Willis [18] proved that the known restrictions for the H-measures describe a bigger convex set whose extremal points are explicitly characterized (being matrix Borel measures supported in no more than three Dirac masses), and showed that among these points there is a large class of "actual" H-measures. They also provide a sufficient condition for which an extremal point of the convex superset is in fact an H-measure. In this paper we prove that such condition is also necessary, at least for all the measures supported on three independent directions. As a consequence we are able to fully characterize all the extremal three-point H-measures supported on three independent directions (Theorems 7.3, 7.8 and 7.10). Our strategy for proving this is the following. We study problem (1.2) for $N = 3$ replacing the minimization over gradient fields by that over solenoidal fields. In other words, we study the quasiconvexification of the three-well energy $F(\eta) = \frac{1}{2}\min\{|\eta - A_i|^2\,, i = 1, 2, 3\}$ with respect to divergence-free fields (see Definition (3.1)), denoted by $Q^\theta_S F$. Following the recipe of Kohn [8], we rewrite the problem as a minimization over the H-measures. Next we use an algorithm developed by Smyshlyaev and Willis [18] which allows one to compute a lower bound on $Q^\theta_S F$ by minimizing over all extremal points of the superset containing the H-measures. When the (unique) measure that delivers the bound is an H-measure, then the lower bound turns out to be optimal. Therefore, if we know that for certain matrices A_1, A_2, A_3 and a certain value of θ the lower bound is not optimal, then we may conclude that the measure that gives the bound cannot be an H-measure. What we find is that every three-point extremal measure of the superset is the extremizing measure that delivers a zero lower bound on $Q^\theta_S F$ at the point $\eta = \sum_i \theta_i A_i$, for a suitable choice of the matrices A_1, A_2, A_3 and of the volume fractions θ. Then we use the results of the first part of the paper to establish the optimality of such lower bound. Indeed, the zero lower bound is optimal if and only if $Q^\theta_S F\Big(\sum_{i=1}^{3}\theta_i A_i\Big) = 0$, equivalently, if and only if $\sum_{i=1}^{3}\theta_i A_i \in K^{qc}_S$, with $K = \{A_1, A_2, A_3\}$.

The structure of the paper is as follows. Section 2 reviews the definition and the basic properties of "multi-phase" H-measures. The reformulation of the relaxation problem in the language of minimization with respect to the H-measures is discussed in Section 3 and follows [8] and [18]. Section 4 specializes that to the three divergence-free wells problem. Section 5 reviews the results from [5] and [15], provides the main tool for proving the (sharp) outer bound for the quasiconvex hull of an arbitrary three-point set (Lemma 5.12, Theorem 5.14 and Corollary 5.16) and finally gives the characterization of the quasiconvex hull (Theorem 5.24). Section 6 is devoted to the proof of Theorem 5.14 (with the key wavelet analysis and estimates in terms of the Riesz transform) and some other technical results. The main results on the H-measures are stated and proved in Section 7. Theorem 7.3 essentially establishes that the sufficient conditions ([18], Proposition 6.1) for realizability of some extremal three-point measures of the convex superset by the H-measures are also necessary, while Theorems 7.8 and 7.10 characterize all extremal three-point measures which do not satisfy these conditions, hence ruling them out from the set of the H-measures. Section 8 completes the description of the remaining cases and summarizes the results. Section 9 discusses applications of the "generic" H-measure results, in particular to the problem of three linear elastic wells.

2. Preliminaries

In the present section we recall the definition and some basic properties of the H-measures associated with periodic micro-geometries. First we set some notation.

Let $N\,, d \in \mathbb{N}$, $N \geq 2$, $d \geq 2$, and let $\theta = (\theta_1, \ldots, \theta_N) \in [0,1]^N$, with $\sum_{i=1}^{N} \theta_i = 1$ (d is the spatial dimension, N is the number of "wells" and θ_i, $i = 1, ..., N$ are "the volume fractions"). Let further the set $Q = (0, 2\pi)^d$ be the "periodicity cell". We define $I(\theta)$ as the set of all characteristic functions $\chi(x) = (\chi_1(x), \ldots, \chi_N(x))$ of non-intersecting measurable subsets comprising Q with fixed volume fractions θ, i.e.

$$I(\theta) = \left\{ \chi : \mathbb{R}^d \to \{0,1\}^N, Q\text{–periodic and measurable } : \sum_{j=1}^{N} \chi_j = 1 \text{ a.e.}, \fint_Q \chi = \theta \right\},$$

where $\fint_Q \chi$ stands for "the volume average" $\frac{1}{|Q|}\int_Q \chi$, $|Q| := (2\pi)^d$ is the volume of Q. We denote by $\hat{\chi}_j(k)\,, k \in \mathbb{Z}^d$, the Fourier coefficients for the Q-periodic functions χ_j:

$$\hat{\chi}_j(k) := \fint_Q \chi_j(x) e^{-ik\cdot x} dx\,.$$

For every $\chi \in I(\theta)$, we call H-measure generated by χ the matrix-valued measure $\mu = (\mu_{ij})_{i,j}$ defined as follows:

$$\mu_{ij} = \mathrm{Re} \sum_{k \neq 0} \hat{\chi}_i(k) \overline{\hat{\chi}_j(k)} \delta_{k/|k|}\,, \quad 1 \leq i, j \leq N\,, \tag{2.1}$$

where $\delta_{k/|k|}$ denotes a unit Dirac mass at the point $\xi = k/|k|$ on the unit sphere S^{d-1} and k has integer components. In fact, our definition is a restrictive one, although sufficient for the purposes of the present work. For a general construction, involving functions that need not be periodic, see e.g. [6] and [23].

We introduce the notation

$$\int_{S^{d-1}} \varphi(\xi) \mu_{ij}(\xi)\, ds(\xi)$$

to denote integration of an appropriate function φ with respect to the measure μ_{ij}.
The set of all possible H-measures for a given θ, which we will denote by $Y^H(\theta)$, is characterized by including all weak-star limits of (2.1), i.e., all Borel matrix-valued measures $\mu_{ij}(\xi)$ such that there exists a sequence of measures $\mu^m_{ij}(\xi)$ of the form (2.1) for some $\chi^m \in I(\theta)$ for each $m = 1, 2, \ldots$, and $\mu^m_{ij} \overset{*}{\rightharpoonup} \mu_{ij}$, that is

$$\int_{S^{d-1}} \varphi_{ij}(\xi)\mu^m_{ij}(\xi)\, ds(\xi) \longrightarrow \int_{S^{d-1}} \varphi_{ij}(\xi)\mu_{ij}(\xi)\, ds(\xi)$$

as $m \to \infty$ for all continuous functions φ_{ij} on the unit sphere S^{d-1}. So

$$Y^H(\theta) = \{\mu_{ij} : \exists\, \mu^m_{ij} \text{ of the form (2.1) and } \mu^m_{ij} \overset{*}{\rightharpoonup} \mu_{ij} \text{ as } m \to \infty\}\,. \tag{2.2}$$

Notice that $Y^H(\theta)$ is an infinite-dimensional convex set in the space of matrix measures, compact in the sense of the above (weak) convergence, see e.g. [8, 18]. It is therefore fully characterized by its extremal points. It can be easily checked that the H-measures satisfy the following properties:

$$\mu_{ij} = \mu_{ji} \text{ and } \sum_{i=1}^{N} \mu_{ij} = 0\,, \quad 1 \le j \le N\,, \tag{2.3}$$

$$\int_{S^{d-1}} \mu_{ij} ds(\xi) = \delta_{ij}\theta_i - \theta_i\theta_j\,, \tag{2.4}$$

$$\mu_{ij}(\xi) = \mu_{ij}(-\xi)\,, \tag{2.5}$$

$$\sum_{i,j=1}^{N} \int_{S^{d-1}} \varphi_i(\xi)\varphi_j(\xi)\mu_{ij}(\xi) ds(\xi) \ge 0 \text{ for any continuous functions } \varphi_j, j = 1, ..., N. \tag{2.6}$$

We denote the set of all Borel measures on S^{d-1} subject to restrictions (2.3)-(2.6) by $Y(\theta)$:

$$Y(\theta) = \{\mu = (\mu_{ij})_{i,j} : (2.3) - (2.6) \text{ hold }\}\,.$$

The set $Y(\theta)$ is also convex and weakly compact. Kohn (see [8], Theorem 6.4) has shown that for the case of two wells ($N = 2$) the conditions (2.3)-(2.6) are necessary and sufficient to characterize the whole set $Y^H(\theta)$, i.e then the sets $Y^H(\theta)$ and $Y(\theta)$ coincide for $N = 2$. In contrast, for $N > 2$, the above restrictions are generally insufficient (Kohn, personal communications; see also discussion in [18]). The latter is a highly non-trivial fact, and one of the main results of the present work (Theorems 7.3, 7.8 and 7.10) substantially clarifies it further, providing a criterion of whether or not certain extremal points of the "bigger" (convex) set $Y(\theta)$ actually belong to the "true" set $Y^H(\theta)$, in effect introducing additional restrictions. Therefore the set $Y^H(\theta)$ is strictly contained, at least in some cases, in $Y(\theta)$: $Y^H(\theta) \subset Y(\theta)$.

3. Relaxation and H-measures

The aim of this section is to show how the H-measures arise in the relaxation of a "multi-well energy" of the form

$$F(\eta) = \tfrac{1}{2} \min\{|\eta - A_i|^2 \,, i = 1, \dots, N\} \,, \quad \eta \in \mathbb{M}^{d\times d} \,, \tag{3.1}$$

which is in turn related to characterizing the quasiconvex hull of a set of $A_1, \dots, A_N$ of given matrices in $\mathbb{M}^{d\times d}$, as discussed later. [In (3.1) for $A \in \mathbb{M}^{d\times d}$ we denote $|A| := \big(\mathrm{Tr}(A^T A)\big)^{1/2}$.] Here the "relaxation" is to be understood in the context of solenoidal (divergence free) fields. Actually we will deal with the so-called "S-quasiconvexification at fixed volume fractions":

Definition 3.1. For any $\theta = (\theta_1, \dots, \theta_N) \in [0,1]^N$, with $\sum_{i=1}^{N} \theta_i = 1$, we define the S-quasiconvexification of F at fixed "volume fractions" θ, and denote it by $Q^\theta_S F$, in the following way:

$$\forall \eta \in \mathbb{M}^{d\times d} \quad Q^\theta_S F(\eta) := \inf_{\chi \in I(\theta)} \inf_{B \in V} \tfrac{1}{2} \fint_Q \left| \eta + B(x) - \sum_{i=1}^{N} \chi_i A_i \right|^2 dx \,, \tag{3.2}$$

where V is the space of Q-periodic divergence free matrix fields with zero average on Q, that is

$$V = \left\{ B \in L^2_{loc}(\mathbb{R}^d, \mathbb{M}^{d\times d}),\ Q\text{-periodic},\ \fint_Q B(x)dx = 0 \,, \mathrm{Div} B = 0 \text{ in } \mathcal{D}'(\mathbb{R}^d, \mathbb{R}^d) \right\}. \tag{3.3}$$

Definition 3.1 is a particular case of a more general definition which falls in the framework of $\mathcal{A}$-quasiconvexity (see e.g. [4]). Indeed formula (3.2) involves matrix fields subject to differential constraints of "solenoidal" (i.e. divergence free) type, hence the label S, rather than fields which satisfy more general differential constraints.

We will use Fourier analysis to execute the "internal" minimization in (3.2) for an arbitrary number N of "wells" and we will essentially follow the same method as used by Kohn [8], with appropriate modifications for the solenoidal fields. (Further minimization over χ leads to the exact computation of $Q^\theta_S F$ in the case $N = 2$, see [14].)
Let us fix $\chi \in I(\theta)$ and compute the infimum (in fact, the minimum) over B in (3.2). Elementary manipulation exploiting the quadratic nature of F and the periodicity of B transforms the integral in (3.2) into

$$\tfrac{1}{2} \fint_Q \left| B(x) + \eta - \sum_{i=1}^{N} \chi_i A_i \right|^2 dx =$$

$$\frac{1}{2}\Bigg\{|\eta-\sum_{i=1}^{N}\theta_iA_i|^2+\sum_{i=1}^{N}\theta_i(1-\theta_i)|A_i|^2-\sum_{i\neq j=1}^{N}\theta_i\theta_j\langle A_i,A_j\rangle+$$

$$\fint_Q\left(|B(x)|^2-2\langle B(x),\sum_{i=1}^{N}\chi_iA_i\rangle\right)dx\Bigg\}, \tag{3.4}$$

with $\langle\cdot\,,\cdot\rangle$ denoting henceforth the symmetric inner product of (possibly complex) matrices: $\langle A,B\rangle := \mathrm{Tr}(A^TB)$, $A,B\in\mathbb{M}^{d\times d}$. Rewriting (3.4) in the Fourier space and using the Plancherel's formula, the last term of (3.4) can be rewritten in the form

$$\frac{1}{2}\sum_{k\in\mathbb{Z}^n}\left(|\hat{B}(k)|^2-2\langle\hat{B}(k),\sum_{i=1}^{N}\overline{\hat{\chi}_i(k)}A_i\rangle\right) \tag{3.5}$$

where $\hat{B}(k)$ and $\hat{\chi}_i(k)$ are Fourier coefficients for the Q-periodic functions B and χ_i respectively. Minimization of (3.5) can be done separately for each k, with respect to all $\hat{B}(k)$ consistent with the divergence-free trial fields (3.3). The frequency $k=0$ contributes nothing to (3.5), since $\hat{B}(0)=0$. For $k\neq 0$, the minimizing value of $\hat{B}(k)$ turns out to be

$$\hat{B}(k)=\sum_{i=1}^{N}\hat{\chi}_i(k)\Pi_{V(k)}A_i \tag{3.6}$$

(see e.g. [8] for similar straightforward linear algebra), where $\Pi_{V(k)}A_i$ denotes the orthogonal projection (in the sense of the inner product $\langle\cdot,\cdot\rangle$) of A_i onto the space

$$V(k)=\{\zeta\in\mathbb{M}^{d\times d}:\ \zeta k=0\}\,.$$

Notice that $V(k)$ is the space of Fourier transforms of divergence free fields "of frequency k" and it actually depends only on the "direction of oscillation" $k/|k|$. Moreover, the orthogonal space to $V(k)$ is given by the space $V(k)^\perp$ of Fourier transforms of gradient fields:

$$V(k)^\perp=\{\zeta\in\mathbb{M}^{d\times d}:\zeta=v\otimes k\ \text{ for some } v\in\mathbb{R}^d\}\,.$$

Therefore, for every $\zeta\in\mathbb{M}^{d\times d}$ we have

$$\Pi_{V(k)}\zeta=\zeta-(\zeta k)\otimes k/|k|^2. \tag{3.7}$$

Plugging (3.6) into (3.5) we find that the minimum value of (3.5) is given by

$$-\frac{1}{2}\sum_{k\neq 0}\left|\sum_{i=1}^{N}\hat{\chi}_i(k)\Pi_{V(k)}A_i\right|^2. \tag{3.8}$$

Now using (2.1) we rewrite (3.8) in terms of the H-measures as follows:

$$-\tfrac{1}{2}\sum_{k\neq 0}\left|\sum_{i=1}^{N}\hat{\chi}_i(k)\Pi_{V(k)}A_i\right|^2 = -\tfrac{1}{2}\sum_{i,j=1}^{N}\int_{S^{d-1}}\langle \Pi_{V(k)}A_i, \Pi_{V(k)}A_j\rangle \, d\mu_{ij}.$$

Next we set

$$\forall \xi \in S^{d-1} \quad f^{ij}(\xi) := -\tfrac{1}{2}\langle \Pi_{V(\xi)}A_i, \Pi_{V(\xi)}A_j\rangle + \tfrac{1}{2}\langle A_i, A_j\rangle \tag{3.9}$$

and by (3.7) we find that

$$f^{ij}(\xi) = \tfrac{1}{2}\langle A_i\xi, A_j\xi\rangle \, , \tag{3.10}$$

(with $\langle \cdot, \cdot \rangle$ denoting here the conventional inner product of vectors).

Then, taking in (3.2) into account (3.4)–(3.9), the minimization problem for $Q^{\theta}_{S}F$ becomes

$$Q^{S}_{\theta}F(\eta) = \tfrac{1}{2}\left|\eta - \sum_{i=1}^{N}\theta_i A_i\right|^2 + \inf_{\mu\in Y^H(\theta)}\sum_{i,j=1}^{N}\int_{S^{d-1}} f^{ij}(\xi)\, d\mu_{ij}(\xi)\, , \tag{3.11}$$

where the minimization is taken with respect to all H-measures associated with N characteristic functions in the sense of Section 2, see (2.1).

4. Description of the set $Y(\theta)$ for $N = 3$

We now focus on the case $N = 3$. As already remarked, the set $Y^H(\theta)$ is typically *not* fully characterized in this case by the conditions (2.3)-(2.6). The aim of this section is to give a description of the set $Y(\theta)$ determined by (2.3)-(2.6), following Smyshlyaev & Willis [18]. More precisely, we will focus on the extremal points of $Y(\theta)$ and their representation as given in [18]. Before going into further details, let us see how the restriction (2.3) specializes to the case $N = 3$. Since $\chi_1 = 1 - \chi_2 - \chi_3$, we have:

$$\mu_{12} = \mu_{21} = -\mu_{22} - \mu_{23}\, , \quad \mu_{13} = \mu_{31} = -\mu_{23} - \mu_{33}\, , \quad \mu_{11} = \mu_{22} + 2\mu_{32} + \mu_{33}\, .$$

We can thus restrict our analysis and equivalently consider only those measures generated by χ_2 and χ_3, for every $\chi \in I(\theta)$. We set to this end scalar measures

$$\begin{aligned} a(\xi) &:= \mu_{22}(\xi) \\ b(\xi) &:= \mu_{23}(\xi) = \mu_{32}(\xi) \\ c(\xi) &:= \mu_{33}(\xi)\, . \end{aligned} \tag{4.1}$$

Then relations (2.4) and (2.5) reduce to

$$\begin{aligned} &\int_{S^{d-1}} a(\xi)\, ds(\xi) = \theta_2(1-\theta_2)\,, \\ &\int_{S^{d-1}} b(\xi)\, ds(\xi) = -\theta_2\theta_3\,, \qquad (4.2) \\ &\int_{S^{d-1}} c(\xi)\, ds(\xi) = \theta_3(1-\theta_3)\,, \end{aligned}$$

$$a(\xi) = a(-\xi)\,,\ b(\xi) = b(-\xi)\,,\ c(\xi) = c(-\xi)\,. \qquad (4.3)$$

The condition of non-negativeness (2.6) can be rewritten as

$$\int_{S^{d-1}} \left(a(\xi)\varphi^2(\xi) + 2b(\xi)\varphi(\xi)\psi(\xi) + c(\xi)\psi^2(\xi) \right) ds(\xi) \geq 0 \qquad (4.4)$$

for any continuous functions φ and ψ on the unit sphere S^{d-1}.
Note that the restriction (4.3) requires the measures to be distributed over the sphere symmetrically. Therefore we can always identify the opposite points $\pm\xi$ on the sphere (hence, in effect dealing with the projective space $\mathbb{R}P^{d-1}$ rather than S^{d-1}). Now consider the set $Y(\theta_2, \theta_3)$ of all Borel 2×2 symmetric matrix measures μ on S^{d-1} which satisfy (4.2)-(4.4):

$$Y(\theta_2, \theta_3) = \left\{ \mu(\xi) = \begin{pmatrix} a(\xi) & b(\xi) \\ b(\xi) & c(\xi) \end{pmatrix} : \text{(4.2)-(4.4) hold} \right\}. \qquad (4.5)$$

Notice that condition (4.4) requires the matrix measures $\begin{pmatrix} a(\xi) & b(\xi) \\ b(\xi) & c(\xi) \end{pmatrix}$ to be non-negative. Moreover by (4.2) for any measure $\mu \in Y(\theta_2, \theta_3)$ its "total mass" $M := \int_{S^{d-1}} \mu(\xi) ds(\xi)$ is fixed and we have

$$M = \begin{pmatrix} \theta_2(1-\theta_2) & -\theta_2\theta_3 \\ -\theta_2\theta_3 & \theta_3(1-\theta_3) \end{pmatrix}. \qquad (4.6)$$

Smyshlyaev and Willis [18] have shown that the extremal points of the (closed convex) set $Y(\theta_2, \theta_3)$ have the form of a weighted sum of at most three Dirac masses (counting the pair $\pm\xi$ as one point):

$$\mu(\xi) = \sum_{r=1}^{3} \mu^r \delta_{\xi_r}\,, \qquad (4.7)$$

where $\mu^r = \begin{pmatrix} a_r & b_r \\ b_r & c_r \end{pmatrix} \in \mathbb{M}^{2\times 2}$ and $\xi_r \in S^{d-1}$ for $r = 1, 2, 3$. On use of conditions (4.2) and (4.4), it is easily checked that the numbers $a_r\,, b_r\,, c_r$ satisfy the following inequalities:

$$a_r \geq 0\,,\ c_r \geq 0\,,\ a_r c_r - b_r^2 \geq 0\,,\ \text{for each } r = 1, 2, 3\,, \tag{4.8}$$

$$\sum_{r=1}^{3} a_r = \theta_2(1-\theta_2)\,,\ \sum_{r=1}^{3} b_r = -\theta_2\theta_3\,,\ \sum_{r=1}^{3} c_r = \theta_3(1-\theta_3)\,. \tag{4.9}$$

We denote by $Y_3(\theta_2, \theta_3)$ the set of all measures of the form (4.7) subject to restrictions (4.8)–(4.9):

$$Y_3(\theta_2, \theta_3) = \left\{ \mu(\xi) = \sum_{r=1}^{3} \mu^r \delta_{\xi_r} : \text{(4.8)-(4.9) hold} \right\}. \tag{4.10}$$

Every matrix $\mu = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ satisfying the condition (4.8) belongs to the convex cone $\mathcal{K}$ of non-negative symmetric matrices in the (a, b, c) space:

$$\mathcal{K} = \{(a, b, c) \in \mathbb{R}^3 : a \geq 0\,,\ c \geq 0\,,\ ac - b^2 \geq 0\}\,.$$

Every matrix μ belonging to the cone $\mathcal{K}$ is uniquely characterized by its trace $\operatorname{tr}\mu = a + c$ and its "projection" μ_{cs} on the cross-section $\mathcal{K}_{cs}$ of the unit trace:

$$\mu = (\operatorname{tr}\mu)\mu_{cs}\,.$$

The cross-section $\mathcal{K}_{cs}$ is described by the relations $a + c = 1\,,\ b^2 + (c - 1/2)^2 \leq 1/4$ and so can be identified with a unit disc in the (c, b)-plane (see Figure 1).
The total mass M belongs to $\mathcal{K}$ and it can be easily checked using (4.6) that its projection M_{cs} on the cross-section $\mathcal{K}_{cs}$ always lies inside the triangle defined by the points

$$\nu_1 = (0, 0)\,,\ \nu_2 = (1, 0)\,,\ \nu_3 = (1/2, -1/2)$$

in the (c, b)-plane, by noticing that $M = \theta_1\theta_3\nu_1 + \theta_2\theta_3\nu_2 + 2\theta_1\theta_2\nu_3$. Moreover, the condition (4.7) implies that, for any $\mu \in Y_3(\theta_2, \theta_3)$, M_{cs} lies inside the triangle defined by the points $\mu^1_{cs}, \mu^2_{cs}, \mu^3_{cs}$.

The extremal measures of $Y(\theta_2, \theta_3)$ are a subset of $Y_3(\theta_2, \theta_3)$. It can be seen as an immediate consequence of methods of [18, Prop. 5.1, Lemma 5.2] that the extremal measures are either supported in three or two points and then are such that $\det \mu^r = 0$, $r = 1, 2(, 3)$, or in a single point with the total mass (4.6). Let us focus our attention on those supported in three points. We denote by $\widehat{Y}_3(\theta_2, \theta_3)$ the set of all such measures. Since $\det \mu^r = 0$, $r = 1, 2, 3$, the associated points on the (c, b)-plane belong to the circle $C := \{(b, c) : b^2 + (c - 1/2)^2 = 1/4\}$, i.e., are extremal for the set $\mathcal{K}_{cs}$. Therefore

$$\widehat{Y}_3(\theta_2, \theta_3) := \{\mu \in Y_3(\theta_2, \theta_3) :\ \mu^r_{cs} \in C \quad \forall\, r = 1, 2, 3\}\,. \tag{4.11}$$

It will be convenient to parametrize C by the angle $\phi \in [0, 2\pi)$, see Fig. 1:

$$\begin{cases} a = (1-\cos\phi)/2 \ (=\ 1-c) \\ b = \frac{1}{2}\sin\phi \\ c = (1+\cos\phi)/2\,. \end{cases} \tag{4.12}$$

In this way, for any $\mu \in \widehat{Y}_3(\theta_2, \theta_3)$, the points μ^r_{cs} can be identified by the angles ϕ_r, $r = 1, 2, 3$.

Let us now go back to the problem of computing $Q^\theta_S F$. Since for $N = 3$ we do not know the set $Y^H(\theta)$, we can instead attempt in (3.11) the minimization over the larger set $Y(\theta)$. This strategy will lead to the precise evaluation of the relaxed energy $Q^\theta_S F$ provided the minimizing measure $\mu \in Y(\theta)$ turns out to be an H-measure, i.e. $\mu \in Y^H(\theta)$. Otherwise, it will provide a lower bound on $Q^\theta_S F$. With this aim we set

$$L(\theta) := \inf_{\mu\in Y(\theta)} \sum_{i,j=1}^{3} \int_{S^2} f^{ij}(\xi)\, d\mu_{ij}(\xi)\,, \tag{4.13}$$

where f^{ij} is defined by (3.10).

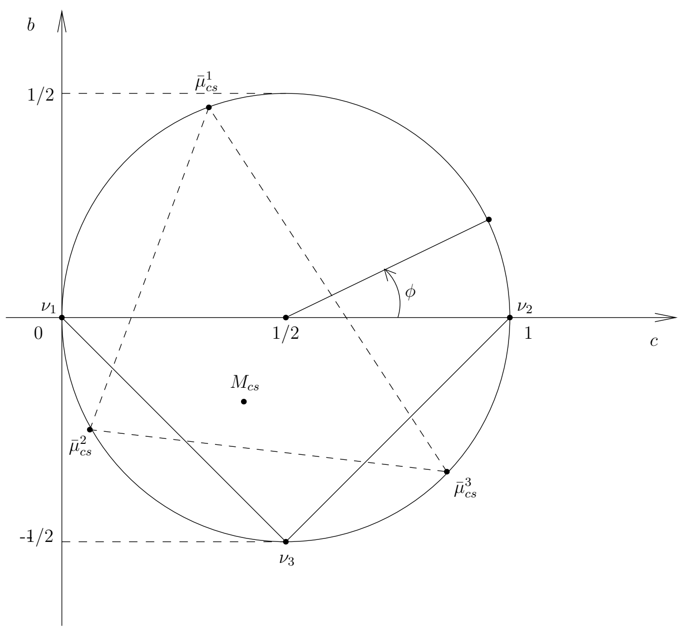

FIGURE 1. The cross-section $\mathcal{K}_{cs}$ of the cone $\mathcal{K}$ on the (c, b)-plane.

The next lemma, where we give an explicit formula for $L(\theta)$, clarifies the role of the set $\widehat{Y}_3(\theta_2, \theta_3)$ in the minimization problem defined in (4.13). It will be assumed, without loss of generality, that $A_1 = 0$.

Lemma 4.1. ([18]) *Let* $\theta \in (0,1)^3$ *be given with* $\sum_{i=1}^{3} \theta_i = 1$ *and* $A_1, A_2, A_3 \in \mathbb{M}^{d\times d}$ *with* $A_1 = 0$. *Then the infimum in* (4.13) *is attained and the minimizing measure can be chosen in* $\widehat{Y}_3(\theta_2, \theta_3)$. *Moreover we have*

$$L(\theta) = (\operatorname{tr} M)\psi_c(M_{cs}),$$

where ψ_c *denotes the convexification of the function* $\psi : \mathcal{K} \to \mathbb{R}$ *defined by*

$$\psi(a,b,c) = \inf_{\xi \in S^{d-1}} \left\{ af^{22}(\xi) + 2bf^{23}(\xi) + cf^{33}(\xi) \right\}. \tag{4.14}$$

Proof. The complete proof of Lemma 4.1 can be found in [18] (in particular see Proposition 5.1 and 5.3 and Lemma 5.2 therein). We will only give a brief sketch of the associated minimization algorithm in order to highlight the role of the set $\widehat{Y}_3(\theta_2, \theta_3)$ defined by (4.11). First we remark that in the minimization problem (4.13) the infimum is attained and the minimizing measure can always be chosen to belong to $Y_3(\theta_2, \theta_3)$, [18, Prop. 5.1]. Moreover, since $f^{ij}(\xi) = 0$ for either $i = 1$ or $j = 1$ (due to (3.10) and $A_1 = 0$), we can re-write (4.13) as

$$L(\theta) = \inf_{\mu \in Y_3(\theta_2,\theta_3)} \sum_{r=1}^{3} \{a_r f^{22}(\xi_r) + 2b_r f^{23}(\xi_r) + c_r f^{33}(\xi_r)\}. \tag{4.15}$$

Problem (4.15) may be approached using the following strategy:
(i) consider all possible splits of the total mass M into the sum of at most three "matrices" μ^r subject to condition (4.8):

$$M = \sum_{r=1}^{3} \mu^r ; \tag{4.16}$$

(ii) for any decomposition (4.16) choose ξ_r in order to generate

$$\psi(a_r, b_r, c_r) = \inf_{\xi \in S^{d-1}} \left\{ a_r f^{22}(\xi) + 2b_r f^{23}(\xi) + c_r f^{33}(\xi) \right\}, \ r = 1,2,3 ;$$

(iii) finally minimize with respect to all admissible splits of the form (4.16). As a result

$$L(\theta) = \inf_{\{a_r,b_r,c_r\}} \sum_{r=1}^{3} \psi(a_r, b_r, c_r).$$

Now remember that the total mass M is decomposed as $M = (\operatorname{tr} M)M_{cs}$. Then, for any decomposition (4.16), we have

$$M_{cs} = \sum_{r=1}^{3} \alpha_r \mu^r_{cs}$$

where $\alpha_r = \operatorname{tr}\mu^r/\operatorname{tr} M$.
Next notice that the function $\psi(\mu)$ is homogeneous of degree one, i.e. $\psi(t\mu) = t\psi(\mu)$ for any $t \geq 0$. Therefore, the problem of computing $L(\theta)$ reduces to minimizing

$$(\operatorname{tr} M)\sum_{r=1}^{3} \alpha_r \psi(\mu^r_{cs})$$

over all possible decomposition of M_{cs} into the convex combination of $\{\mu^r_{cs}\}$ on the cross-section. Moreover, since the function $\psi(\mu)$ is concave, it is enough to consider only those points μ^r_{cs} which lie on the circle C, i.e. are extremal for the set $\mathcal{K}_{cs}$. This procedure leads to finding (no more than) three critical points $\bar{\mu}^1_{cs}, \bar{\mu}^2_{cs}, \bar{\mu}^3_{cs}$ on C (see Figure 1) such that the extremizing measure $\bar{\mu}$ can be written as

$$\bar{\mu} = (\operatorname{tr} M)\sum_{i=1}^{3} \alpha_r \bar{\mu}^r_{cs}\delta_{\xi_r}, \quad \alpha_r \geq 0\,, \quad \alpha_1 + \alpha_2 + \alpha_3 = 1\,.$$

Notice finally that by construction $L(\theta) = (\operatorname{tr} M)\psi_c(M_{cs})$. □

5. Characterization of the S-quasiconvex hull of three-point sets

This section is devoted to the study of the differential inclusions problem related to the relaxation of the energy (3.1). Our strategy is to make a connection between this problem and that of characterizing the H-measures arising in (3.11) when $d = N = 3$.
In order to proceed we need to give some definitions. We give first one of several possible equivalent definitions of the approximate non-rigidity (specialized to the case of solenoidal fields) as follows:

Definition 5.1. Given a set of real 3×3 matrices $K \subset \mathbb{M}^{3\times 3}$, we say that K is non-rigid for approximate solutions of solenoidal-type, if there exists a sequence $\{B_h\} \subset L^2_{loc}(\mathbb{R}^3, \mathbb{M}^{3\times 3})$, L^2_{loc}-equi-integrable, Q-periodic and such that

$$\begin{cases} \operatorname{Div} B_h = 0 & \text{in } \mathcal{D}'(\mathbb{R}^3, \mathbb{R}^3)\,, \\ \operatorname{dist}(B_h, K) \to 0 & \text{in measure,} \end{cases} \tag{5.1}$$

and

$$\forall\, \{B_{h_j}\} \text{ subsequence of } \{B_h\}, \text{ there does not exist } A \in K \text{ such that } B_{h_j} \to A \text{ in measure.}$$

Definition 5.2. We call S-quasiconvex hull of the set K, and denote it by K^{qc}_S, the set defined in the following way

$$K^{qc}_S = K \cup \left\{ B_0 \in \mathbb{M}^{3\times 3} : \exists\, \{B_h\} \text{ solution to (5.1) and } ⨍_Q B_h = B_0 \;\forall\, h \right\}.$$

The main purpose of this section is to the characterize the S-quasiconvex hull of any three-point set $K = \{A_1, A_2, A_3\} \subset \mathbb{M}^{3\times 3}$. One can easily check that $K \subseteq K_S^{qc} \subseteq K^c$, where K^c denotes the convex hull of K.

The hard work will be to characterize K_S^{qc} when K contains at least one pair of rank-2 disconnected matrices, i.e., $\text{rank}(A_i - A_j) = 3$ for some $i \neq j$, and the plane through A_1, A_2, A_3 contains three distinct rank-2 directions. Then, depending on the intersection of the affine rank-2 lines through A_1, A_2 and A_3 on this plane, the set K may be of three different types. This suggests to give the following definition.

Definition 5.3. We say that K is of *Type 1* if the mutual position of the matrices A_1, A_2 and A_3 is such that the rank-2 lines through A_1, A_2 and A_3 form an "inner triangle" inside K^c (see Figure 2-(1)). We say that K is of *Type 2* if the mutual position of the matrices A_1, A_2 and A_3 is such that no "inner triangle" may be formed by the rank-2 lines through A_1, A_2 and A_3, but there is one point of intersection inside K^c (see Figure 2-(2)). We say that K is of *Type 3* if there is no point of intersection inside K^c (see Figure 2-(3)).

Remark 5.4. One can easily see that the situation where the plane through A_1, A_2, A_3 contains three distinct rank-2 directions occurs when, after reduction to $K = \{0, I, A\}$ (which is always possible by shifting by a constant matrix and left multiplying by an invertible matrix), the matrix A is diagonalizable with distinct real eigenvalues.

Remark 5.5. Notice that the definition of the sets of *Type 1* does not include the case when the "inner triangle" degenerates into a single point (the point S_0 in Fig. 5). This motivates the following definition.

Definition 5.6. We say that K is a set of *degenerate Type 1* if the "inner triangle" degenerates into a single point that we denote by S_0 (see Fig. 5).

We will study separately the sets of *Type 1* and the sets of *degenerate Type 1.* The latter are treated in Proposition 7.6.
We will see that if K is of *Type 1*, then $K \subsetneq K_S^{qc} \subsetneq K^c$ (Corollary 5.18), while for sets of *Type 2* the S-quasiconvex hull turns out to be trivial, i.e., $K_S^{qc} = K$ (Theorem 5.22), unless they contain rank-2 connections. The sets of *Type 3* with no rank-2 connections have trivial S-quasiconvex hull either. Their study does not present any special difficulty and will be treated in Section 8.

The study of the case when A has multiple eigenvalues follows the same approach used for the sets of *Type 1* and *Type 2* and the related results are described in Theorem 5.23. Finally, the case when A is not diagonalizable will be treated in Section 8.

For ease of reading we give in Theorem 5.24 an account of all the cases together.

Remark 5.7. If $K = \{A_1, A_2, A_3\}$ does not contain any rank-2 connection, then there exists no divergence free matrix field B such that $B \in K$ a.e., and $⨍ B = \sum_{i=1}^{3} \theta_i A_i$ with $\theta_i \in (0,1)\, \forall\, i = 1,2,3$ (see [15]).

The next lemma shows that for the purpose of characterizing the S-quasiconvex hull of a set, we can make any convenient change of variables. In particular it allows us to reduce to the diagonal case when dealing with sets of *Type 1, 2* and *3*.

Lemma 5.8. *Let $K = \{A_1, A_2, A_3\} \subset \mathbb{M}^{3\times 3}$ and $\bar{K} = \{NGA_1G^{-1} + M, NGA_2G^{-1} + M, NGA_3G^{-1} + M\}$ with $G, N \in GL(3,\mathbb{R})$, $M \in \mathbb{M}^{3\times 3}$. Then $B_0 \in K_S^{qc}$ if and only if $NGB_0G^{-1} + M \in \bar{K}_S^{qc}$.*

The proof of Lemma 5.8 is contained in Section 6.

We will now focus on the sets of *Type 1*. The following result characterizes all sets of *Type 1* which do not contain rank-2 connections.

Lemma 5.9. ([15]) *Assume that $K \subset \mathbb{M}^{3\times 3}$ does not contain any rank-2 connection. Then K is of Type 1, if and only if there exist $q_1,\ q_2,\ q_3 \in (0,1)$, $G, N \in GL(3,\mathbb{R})$, $M \in \mathbb{M}^{3\times 3}$ such that*

$$K = \{M\,, N + M\,, NA + M\}\,, \tag{5.2}$$

where

$$A = \frac{1}{q_3}\left[\left(1 - \prod_{i=1}^{3}(1-q_i)\right) G^{-1} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} G - q_2(1-q_3)I\right], \tag{5.3}$$

with $\lambda_1 = 0\,, \lambda_2 = 1/(1-q_1)\,, \lambda_3 = q_2/(q_1+q_2-q_1q_2)$.

Before stating the main results of this section let us briefly explain how the sets of *Type 1* "look like" (see [5], [14] and [15] for further details). A key "geometric" property of every set $K = \{A_1, A_2, A_3\}$ of *Type 1*, see Figure 2, is that one can find three matrices $S_1, S_2, S_3 \in \mathbb{M}^{3\times 3}$ such that

$$\begin{aligned} S_2 &= q_1 A_1 + (1-q_1) S_1\,, \\ S_3 &= q_2 A_2 + (1-q_2) S_2\,, \\ S_1 &= q_3 A_3 + (1-q_3) S_3\,, \end{aligned} \tag{5.4}$$

where $q_1, q_2, q_3 \in (0,1)$ (unless there are rank-2 connections in which case $q_i = 0$ for some i), and

$$\det(A_i - S_i) = 0 \quad \forall\, i = 1,2,3\,.$$

Therefore, for any $i = 1,2,3$ the matrices A_i and S_i are rank-2 connected. The rank-2 lines A_1S_1, A_2S_2, A_3S_3 intersect in order to form the triangle $S_1S_2S_3$ as in Figure 2.

Notation. For every $K = \{A_1, A_2, A_3\}$ of *Type 1* we set (see Figure 2):

$$\Gamma_1(K) = \{\xi \in \mathbb{M}^{3\times 3} : \xi = t_1 A_1 + t_2 S_1 + t_3 A_3\,,\ t_i \in [0,1)\,, t_1, t_3 \neq 0\,, \sum_{i=1}^{3} t_i = 1\}\,,$$

$$\Gamma_2(K) = \{\xi \in \mathbb{M}^{3\times 3} : \xi = t_1 A_2 + t_2 S_2 + t_3 A_1\,, t_i \in [0,1)\,, t_1, t_3 \neq 0\,, \sum_{i=1}^{3} t_i = 1\}\,,$$

$$\Gamma_3(K) = \{\xi \in \mathbb{M}^{3\times 3} : \xi = t_1 A_3 + t_2 S_3 + t_3 A_2\,, t_i \in [0,1)\,, t_1, t_3 \neq 0\,, \sum_{i=1}^{3} t_i = 1\}\,,$$

$$T(K) := K^c - \bigcup_{i=1}^{3} \Gamma_i(K). \tag{5.5}$$

(Hence $T(K)$ is the union of the closed triangle $S_1S_2S_3$ and the three "arms" $[A_1S_2)$, $[A_2S_3)$ and $[A_3S_1)$.) The notation introduced above also includes the case when K is of *Type 1* and contains one or two rank-2 connections. For example, if $\mathrm{rank}(A_1 - A_2) \le 2$, then $A_1 = S_2$ and the set $T(K)$ is given in this case by the union of the closed triangle $S_1S_2S_3$ and the two "arms" $[A_2S_3)$ and $[A_3S_1)$.

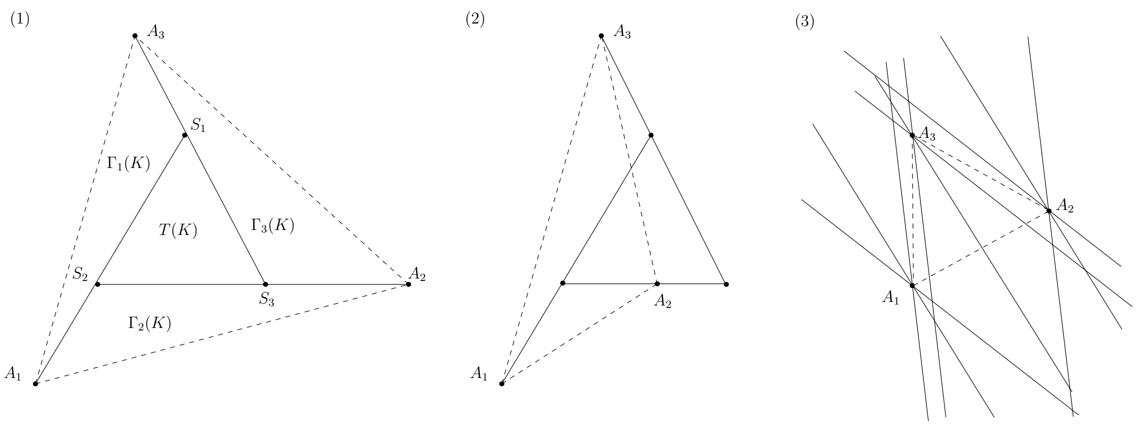

FIGURE 2. (1) The "inner triangle" $S_1S_2S_3$ formed by the rank-2 lines for the sets of *Type 1*. (2) A set of *Type 2*. (3) A set of *Type 3*. In (1)-(2)-(3) the dashed lines delimit the convex hull while the solid lines are rank-2 lines.

The next result provides an inner bound for the S-quasiconvex hulls of the sets of *Type 1*.

Lemma 5.10. *If K is of Type 1, then $T(K) \subseteq K_S^{qc}$.*

Proof. The proof of Lemma 5.10 relies on a generalization of the construction presented in [5] (in particular see Lemma 4.1 and Lemma 4.2 therein). We remark that an explicit construction realizing a point in $T(K)$ is that of infinite-rank sequential lamination, cf. [17], [13], [24], [1]. $\square$

In the sequel we will show that in fact $K_S^{qc} = T(K)$. Since by Lemma 5.10, $K_S^{qc} \supseteq T(K)$, we only need to prove that $K_S^{qc} \subseteq T(K)$. By Lemma 5.8 it suffices to prove the latter inclusion in the diagonal case, that is when K is of the form $K = \{0, I, D(q)\}$, where $D(q)$ is given by (5.3) with $G = I$ and (q_1, q_2, q_3) any arbitrary point in $(0,1)^3$.

In order to prove the desired outer bound on K_S^{qc}, we need to introduce the notion of S-quasiconvexity (see [4] for the general setting).

Definition 5.11. *A continuous function $f : \mathbb{M}^{3\times 3} \to \mathbb{R}$ with quadratic growth is said to be S-quasiconvex if for every Q-periodic divergence free matrix field $B \in L^2_{loc}(\mathbb{R}^3, \mathbb{M}^{3\times 3})$ the following inequality holds:*

$$\fint_Q f(B)\,dx \geq f\left(\fint_Q B\,dx\right). \tag{5.6}$$

S-quasiconvex functions turn out to be the natural tool in bounding the S-quasiconvex hull of a set K. Indeed if $\{B_h\}$ satisfies (5.1) for $K = \{A_1, A_2, A_3\}$ and $\fint_Q B_h = B_0$, then one has $B_0 = \sum_{i=1}^3 \theta_i A_i$ for some $\theta \in [0,1]^3$, with $\sum_{i=1}^3 \theta_i = 1$, and

$$f(B_0) \leq \sum_{i=1}^3 \theta_i f(A_i). \tag{5.7}$$

Consequently, if for some S-quasiconvex f, $f(B_0) > \sum_{i=1}^3 \theta_i f(A_i)$ then $B_0 \notin K_S^{qc}$. Unfortunately we do not know any explicit S-quasiconvex function which can provide the optimal bound on K_S^{qc} when the set K is of the type (5.2). Therefore the characterization $K_S^{qc} = T(K)$ will be performed in several steps. Let us briefly sketch our plan.

Step 1. We construct a function $\mathcal{T}^+$ defined on the plane π generated by the rank-2 matrices

$$V_1 = \mathrm{diag}(1,1,0), \quad V_2 = \mathrm{diag}(-1,0,-1). \tag{5.8}$$

Hence $\pi := \{M \in \mathbb{M}^{3\times 3} :\ M = uV_1 + vV_2 \ \text{ for some } u, v \in \mathbb{R}\}$. The function $\mathcal{T}^+$ is rank-2 convex on its domain, i.e., it is convex along the rank-2 lines contained in π (Lemma 5.12).

Step 2. We prove that inequality (5.7) holds true whenever $K \subset \pi$ and f is a rank-2 convex function on π (Corollary 5.16).

Step 3. We show that, up to a transformation, the sets K under investigation are subsets of the plane π. More precisely, for every $i = 1, 2, 3$, there exists a transformation that maps the rank-2 lines A_iS_i, $A_{i+1}S_{i+1}$ to V_1 and V_2 respectively (where $A_{i+1}S_{i+1} = A_1S_1$ for $i = 3$). This will allow us to use the function $\mathcal{T}^+$ to prove the optimal bounds on K_S^{qc} (Theorem 5.17).

In order to proceed we need to set some notation. We denote by π^+ the subset of π defined as follows

$$\pi^+ := \{M \in \pi : \, M = uV_1 + vV_2 \text{ for some } u > 0, v > 0\} .$$

We define V_π as the space of Q-periodic divergence free matrix fields which take values in π:

$$V_\pi = \left\{B \in L^2_{loc}(\mathbb{R}^3, \mathbb{M}^{3\times 3}),\ Q\text{−periodic},\ \mathrm{Div}B = 0 \text{ in } \mathcal{D}'(\mathbb{R}^3, \mathbb{R}^3),\ B \in \pi \ a.e.\right\} . \tag{5.9}$$

It will be convenient to recall that a function $f : \mathbb{M}^{3\times 3} \to \mathbb{R}$ is said to be rank-2 convex if f is convex along rank-2 lines, *i.e.* if $t \to f(M + tV)$ is convex for every $M, V \in \mathbb{M}^{3\times 3}$ with rank(V)=2.

Lemma 5.12. *(Construction of $\mathcal{T}^+$.) There exists a continuous function $\mathcal{T}^+ : \pi \to \mathbb{R}$ which satisfies the following properties:*

(5.10) $\quad \mathcal{T}^+$ *is rank-2 convex on* π, *i.e.*, $t \to f(M + tV)$ *is convex for every* $M, V \in \pi$ *with rank*$(V) = 2$;

(5.11) $\quad \mathcal{T}^+(M) > 0$ *if* $M \in \pi^+$;

(5.12) $\quad \mathcal{T}^+(M) = 0$ *if* $M \in \pi - \pi^+$.

Proof. Let us consider the function $\mathcal{T} : \mathbb{M}^{3\times 3} \to \mathbb{R}$ given by

$$\mathcal{T}(M) = 2\mathrm{tr}\,(M^T M) - (\mathrm{tr}\, M)^2 . \tag{5.13}$$

One can check that the above (S-quasiconvex) function, which is due to Tartar [23], satisfies conditions (5.10) and (5.11), but not (5.12). The idea now is to modify the restriction of $\mathcal{T}$ to the plane π in order to achieve condition (5.12). Let us first evaluate the restriction of $\mathcal{T}$ to π :

$$\forall\, u, v \in \mathbb{R} \quad \mathcal{T}(uV_1 + vV_2) = 2[(u - v)^2 + u^2 + v^2] - 4(u - v)^2 = 4uv.$$

Then we define $\mathcal{T}^+ : \pi \to \mathbb{R}$ in the following way:

$$\forall\, u, v \in \mathbb{R} \qquad \mathcal{T}^+(uV_1 + vV_2) = u^+ v^+, \tag{5.14}$$

where the symbol u^+ denotes the positive part of u: $u^+ := \max\{0, u\}$. (The function $\mathcal{T}^+$ is loosely analogous to the function $\det^+$ of Šverák [20].) The function $\mathcal{T}^+$ satisfies (5.11) and (5.12) by construction. Furthermore, $\mathcal{T}^+$ is rank-2 convex since the only rank-2 matrices in π are $V_1, V_2, V_1 + V_2$ and their multiples. □

We now claim that inequality (5.7) holds true for all rank-2 convex functions on π and $K \subset \pi$. For the sake of simplicity we will regard any function f defined on π as a function on $\mathbb{R}^2$ via the identification:

$$(y_1, y_2) \in \mathbb{R}^2 \to y_1 V_1 + y_2 V_2 \in \pi ,$$

and, when no ambiguity may arise, by abuse of notation we will write $f(y_1, y_2)$ instead of $f(y_1V_1 + y_2V_2)$. In particular, if f is a rank-2 convex function on π, then it may be viewed as a separately convex function on $\mathbb{R}^2$ which in addition is convex in the diagonal direction $(1,1)$, i.e., $t \in \mathbb{R} \to f(y_1+t, y_2+t)$ is convex for every $(y_1, y_2) \in \mathbb{R}^2$. This follows by the fact that the only rank-2 directions in π are those generated by V_1, V_2 and $V_1 + V_2$.

Before introducing the main ingredient needed in the proof of our claim, we state the following lemma.

Lemma 5.13. *Let $B \in V_\pi$. Then there exist $\eta_1, \eta_2, \eta_3 \in L^2_{loc}(\mathbb{R})$, $(0, 2\pi)$-periodic, such that*

$$B(x_1, x_2, x_3) = (\eta_3(x_3) - \eta_1(x_1))V_1 + (\eta_2(x_2) - \eta_1(x_1))V_2 \,. \tag{5.15}$$

Moreover we have

$$\fint_Q f(B) \geq f\Big(\fint_Q B \Big) , \tag{5.16}$$

for every rank-2 convex function f on π with quadratic growth.

Proof. By assumptions there exist $u, v \in L^2_{loc}(\mathbb{R}^3)$, $(0,2\pi)^3$-periodic such that

$$B(x) = u(x)V_1 + v(x)V_2 \,.$$

The equation $\mathrm{Div} B = 0$ yields

$$\partial_1(u - v) = 0 \,, \quad \partial_2 u = 0 \,, \quad \partial_3 v = 0 \,, \qquad \text{in } \mathcal{D}'(\mathbb{R}^3) \,, \tag{5.17}$$

denoting $\partial_j := \frac{\partial}{\partial x_j}$, $j = 1, 2, 3$. Then (5.15) follows from (5.17) by explicit integration.

As far as the inequality (5.16) is concerned, we find

$$\fint_Q f(B)\, dx = \fint_Q f\big(\eta_3(x_3) - \eta_1(x_1), \eta_2(x_2) - \eta_1(x_1)\big)\, dx_1 dx_2 dx_3 \geq$$
$$\geq f(\bar\eta_3 - \bar\eta_1, \bar\eta_2 - \bar\eta_1) = f\big(\fint_Q B\, dx \big) ,$$

where the symbols $\bar\eta_1, \bar\eta_2, \bar\eta_3$ denote the average of the functions η_1, η_2, η_3 over the interval $(0, 2\pi)$. In the last inequality we have used the convexity of the integrand in $\eta_2(x_2)$, $\eta_3(x_3)$ and in $\eta_1(x_1)$. □

The next result provides the key argument to prove our claim. It is a well-known result due to Müller ([11], Theorem 1) of which we give a slightly modified version suitable to our setting.

Theorem 5.14. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a separately convex function which in addition is convex in the direction $(1,1)$ and satisfies $0 \leq f(y) \leq C(1 + |y|^2)$. Suppose that*

$$u_h \rightharpoonup u_\infty \,, \quad v_h \rightharpoonup v_\infty \quad \text{ in } L^2_{loc}(\mathbb{R}^3) \text{ as } h \to \infty \,,$$

$$\partial_2 u_h \to \partial_2 u_\infty \,, \quad \partial_3 v_h \to \partial_3 v_\infty \,, \quad \partial_1(u_h - v_h) \to \partial_1(u_\infty - v_\infty) \quad \text{ in } H^{-1}_{loc}(\mathbb{R}^3) \text{ as } h \to \infty \,.$$

Then for every open set $V \subset \mathbb{R}^3$

$$\int_V f(u_\infty, v_\infty)\, dx \leq \liminf_{h\to\infty} \int_V f(u_h, v_h)\, dx\,.$$

The proof of Theorem 5.14 is postponed to Section 6.

Remark 5.15. Theorem 5.14 can be rephrased by saying that rank-2 convexity on π implies S-quasiconvexity on π.

Corollary 5.16. *Let* $K = \{A_1, A_2, A_3\} \subset \pi$ *and let* $B_0 = \sum_{i=1}^{3} \theta_i A_i \in K_S^{qc}$. *Then* (5.7) *holds true for every function* f *satisfying the assumptions of Theorem 5.14.*

Proof. By definition of K_S^{qc} there exists a sequence $\{B_h\}$ satisfying (5.1) with $⨍_Q B_h = B_0$. We want to show that the the off-plane component of B_h may be considered negligible. One can easily check that there exists a sequence $\{\chi^h\} \subset L^\infty(\mathbb{R}^3)$ of Q-periodic characteristic functions such that

$$\begin{aligned}
&A^h := \sum \chi_i^h A_i \in \pi \text{ a.e.}\,,\\
&⨍_Q A^h = B_0\,,\\
&\mathrm{Div} A^h \to 0 \quad \text{strongly in } H^{-1}_{loc}(\mathbb{R}^3)\,,\\
&A^h \overset{*}{\rightharpoonup} A^\infty \quad \text{in } L^\infty(\mathbb{R}^3)\,,\ \text{ with } \mathrm{Div} A^\infty = 0 \ \text{ in } \mathcal{D}'(\mathbb{R}^3, \mathbb{R}^3)\,.
\end{aligned}$$

Therefore we find that $A^h = u_h V_1 + v_h V_2$ for some functions u_h, v_h which satisfy

$$\begin{aligned}
&u_h \rightharpoonup u_\infty\,,\ v_h \rightharpoonup v_\infty \quad \text{weakly in } L^2_{loc}(\mathbb{R}^3)\,,\\
&\partial_2 u_h \to 0\,,\ \partial_3 v_h \to 0\,,\ \partial_1(u_h - v_h) \to 0 \quad \text{strongly in } H^{-1}_{loc}(\mathbb{R}^3)\,.
\end{aligned}$$

We can then apply Theorem 5.14 with $V = Q$ to get

$$\int_Q f(u_\infty, v_\infty)\, dx \leq \liminf_{h\to\infty} \int_Q f(u_h, v_h)\, dx\,.$$

Since $\liminf_{h\to\infty} \int_Q f(u_h, v_h)\, dx = \sum_{i=1}^{3} \theta_i f(A_i)$, inequality (5.7) follows by remarking that $A^\infty \in V_\pi$ a.e. and by applying Proposition 5.13. □

We are now ready to demonstrate that $K_S^{qc} = T(K)$ for all sets K of the form $K = \{0, I, D(q)\}$, where $D(q)$ is defined by (5.3) with $G = I$ and $q = (q_1, q_2, q_3) \in (0,1)^3$.

It will be convenient to give the explicit expression for the matrices $D(q), S_1(q), S_2(q), S_3(q)$ in this case, which follows by straightforward calculation from (5.4):

$$
\begin{aligned}
(5.18) \qquad D(q) &= \operatorname{diag}\left(-\frac{q_2}{q_3}(1-q_3),\, -\frac{(1-q_1)(1-q_3)-1}{q_3(1-q_1)},\, -\frac{q_2}{(1-q_1)(1-q_2)-1}\right),\\
S_1(q) &= \operatorname{diag}\left(0,\, \frac{1}{1-q_1},\, \frac{q_2}{q_1+q_2-q_1q_2}\right),\\
S_2(q) &= \operatorname{diag}\left(0,\, 1,\, \frac{q_2(1-q_1)}{q_1+q_2-q_1q_2}\right),\\
S_3(q) &= \operatorname{diag}\left(q_2,\, 1,\, \frac{q_2}{q_1+q_2-q_1q_2}\right).
\end{aligned}
$$

Theorem 5.17. *Let $q \in (0,1)^3$ and let $K = \{0, I, D(q)\}$. Then $K_S^{qc} = T(K)$.*

Proof. Let $B_0 \in K_S^{qc}$. By Lemma 5.10, it suffices to prove that $B_0 \notin \bigcup_{i=1}^{3} \Gamma_i(K)$. For simplicity of notation we will omit the dependence on q in the matrices (5.18).
We first show that $B_0 \notin \Gamma_3(K)$. Recall that the lines S_3S_1 and S_2S_3 are rank-2 lines. Then, for $N = \operatorname{diag}\left(-\frac{1}{q_2}, \frac{1-q_1}{q_1}, -\frac{q_1+q_2-q_1q_2}{q_1q_2}\right)$ we find

$$N(S_1 - S_3) = V_1 \quad \text{and} \quad N(S_3 - S_2) = V_2.$$

By assumption there exists a sequence $\{B_h\}$ satisfying (5.1) with $\fint_Q B_h = B_0$. We now define the new sequence $\{B'_h\}$ and the set K' in the following way:

$$\forall\, h \quad B'_h := N(B_h - S_3)\,, \quad K' := \{-NS_3, N(I - S_3), N(D - S_3)\}\,.$$

It is readily seen that $\{B'_h\}$ satisfies the following properties:

$$
(5.19) \qquad \begin{cases} \operatorname{Div} B'_h = 0 & \text{in } \mathcal{D}'(\mathbb{R}^3, \mathbb{R}^3)\,,\\ \operatorname{dist}(B'_h, K') \to 0 & \text{in measure,}\\ \fint_Q B'_h = N_1(B_0 - S_3)\,. \end{cases}
$$

Next remark that the condition $B_0 \notin \Gamma_3(K)$ is equivalent to $N(B_0 - S_3) \in \pi - \pi^+$. In order to prove the latter inclusion we use the function $\mathcal{T}^+$ of Lemma 5.12. Since $\mathcal{T}^+|_{K'} = 0$, using (5.19) and Corollary 5.16, one gets

$$\mathcal{T}^+(N(B_0 - S_3)) \le 0,$$

implying the desired via (5.11). In order to prove that $B_0 \notin \Gamma_1(K)$, one first find a matrix N' such that

$$N'(S_1 - S_3) = \operatorname{diag}(1,1,0)\,, \quad N'(S_2 - S_1) = \operatorname{diag}(0,-1,-1),$$

which is possible since S_1S_3 and S_2S_1 are rank-2 lines. Then one employs Lemma 5.8 to make a change of variable (via a permutation matrix) and reduce to the previous case. In a fully analogous way one shows that $B_0 \notin \Gamma_2(K)$. $\square$

Corollary 5.18. *If K is a set of Type 1, then $K_S^{qc} = T(K)$.*

Proof. The proof follows straightforwardly by Lemma 5.9, Lemma 5.8 and Theorem 5.17. □

We now turn our attention to the characterization of the S-quasiconvex hull of the sets of *Type 2*. In order to proceed we need to give the following definition.

Definition 5.19. We define the rank-2 convex hull K^{r2} of a set K as

$$K^{r2} = \{M \in \mathbb{M}^{3\times 3} : f(M) \leq \sup_K f \text{ , for all rank-2 convex } f\} \,.$$

Trivially, the rank-2 convex hull provides an inner approximation of the S-quasiconvex hull of a set: $K^{r2} \subseteq K_S^{qc}$.

An immediate consequence of Corollary 5.16 is the following

Corollary 5.20. *Let $K = \{A_1, A_2, A_3\} \subset \pi$. Then $K_S^{qc} = K^{r2}$.*

In order to characterize the S-quasiconvex hull of the sets of *Type 2*, we will rather deal with the rank-2 convex hull. Furthermore, we will employ the following lemma which is a particular case of a more general result first claimed in [16, 10] and later proved in [7, 9].

Lemma 5.21. *Let $C_1, \ldots, C_k$ be disjoint compact sets and suppose that $K^{r2} \subset \cup_i C_i$. Then $K^{r2} = \cup_i (K \cap C_i)^{r2}$.*

Theorem 5.22. *If K is a set of Type 2 and does not contain any rank-2 connection, then $K_S^{qc} = K$.*

Proof. Using Lemma 5.8 we can make a suitable change of variables and reduce as before to the case where $K \subset \pi$. Then, employing the same arguments used in the study of the sets of *Type 1*, we can rule out from K_S^{qc} the three triangles $A_1P_1A_2$, $A_1P_3A_3$ ($\subset A_1P_4A_3$), $A_2A_3P_5$ ($\subset A_2A_3P_6$) (see Figure 3). Therefore, it only remains to eliminate the triangle $P_1P_2A_2$ and the "arm" $(A_1, P_1]$. Let $C_1 = \{A_3\}$ and let C_2 be $P_1P_2A_2 \cup [A_1, P_1]$. Then, by Lemma 5.21, $K^{r2} = A_3 \cup (K \cap C_2)^{r2}$. Clearly $(K \cap C_2)^{r2} = \{A_1, A_2\}^{r2} = \{A_1, A_2\}$ since A_1 and A_2 are rank-two disconnected. Hence $K^{r2} = \{A_1, A_2, A_3\} = K$. Finally, by Corollary 5.20 $K^{qc} = K$ as required. □

We conclude this section with treating the case when $K = \{0, I, A\}$ and A is diagonalizable (on $\mathbb{R}$) with multiple eigenvalues.

Theorem 5.23. *Assume that the set $K = \{0, I, A\}$ does not contain any rank-2 connection and that the matrix A is diagonalizable with a real eigenvalue of multiplicity two or three. Then $K_S^{qc} = K$.*

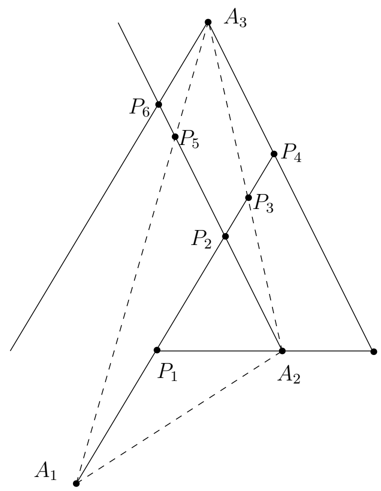

FIGURE 3

Proof. If A has an eigenvalue of multiplicity three, then, up to a transformation we may assume that A is diagonal, i.e., $A = aI$ for some $a \in \mathbb{R}$. Then the original Tartar's function $\mathcal{T}$ defined by (5.13) provide the desired bound. Indeed inequality (5.7) yields

$$-3(\theta_2 + a\theta_3)^2 \leq -3\theta_2 - 3a^2\theta_3\,,$$

which is never satisfied for any value of $\theta_2, \theta_3 \in (0,1)$, with $\theta_2 + \theta_3 < 1$.

Now assume that A has two distinct eigenvalues, one of multiplicity two. In this case the two-dimensional subspace generated by I and A contains only two rank-2 lines. The case when the corresponding affine rank-2 lines through 0, I and A do not intersect inside K^c does not present any special difficulty and is treated in Section 8 together with the sets of *Type 3*. Here we assume that there is one point of intersection inside K^c. Then the proof is very similar to that of Theorem 5.17 and Theorem 5.22 and therefore it will not be detailed. Up to a transformation we may reduce to the case where A is diagonal and $K \subset \tilde{\pi}$, with $\tilde{\pi}$ the plane generated by the matrices $W_1 := \mathrm{diag}(0,1,1)$ and $W_2 := \mathrm{diag}(1,0,0)$:

$$\tilde{\pi} := \{M \in \mathbb{M}^{3\times3} :\, M = uW_1 + vW_2 \text{ for some } u, v \in \mathbb{R}\}\,.$$

Then we follow the same strategy as before. Namely, we define the function $\widetilde{\mathcal{T}}^+ : \tilde{\pi} \to \mathbb{R}$ as $\widetilde{\mathcal{T}}^+(uW_1 + vW_2) = u^+v^+$ and we observe that $\widetilde{\mathcal{T}}^+$ is rank-2 convex on $\tilde{\pi}$ since the only rank-2 directions contained in $\tilde{\pi}$ are those generated by W_1 and W_2. Next, up to an obvious modification of Theorem 5.14 and Corollary 5.16, we show that rank-2 convexity on $\tilde{\pi}$ implies S-quasiconvexity on $\tilde{\pi}$. Then, arguing as for the sets of *Type 2*, we first show that K^{r2} is the union of two disjoint sets and then we apply Lemma 5.21. $\square$

The following theorem sums up all the possible cases including those discussed so far.

Theorem 5.24. *Let $K = \{A_1, A_2, A_3\}$. Then we have:*

(i) if $\operatorname{rank}(A_i - A_j) \le 2 \quad \forall\, i, j = 1, 2, 3$, *then* $K_S^{qc} = K^c$*;*

(ii) if K is a set of Type 1 then $K_S^{qc} = T(K)$*;*

(iii) if K is a set of degenerate Type 1 then $K_S^{qc} = [A_1, S_0] \cup [A_2, S_0] \cup [A_3, S_0]$*;*

(iv) if K contains at least one pair of rank-2 disconnected matrices, say A_1 and A_2, and is not a set of Type 1, then $K_S^{qc} = K$, unless we have one of the following cases:

(iv-1) if $\det(A_1 - A_3) = 0$ *and* $\det\big(A_2 - (tA_1 + (1-t)A_3)\big) = 0$ *for some* $t \in [0, 1)$*, then* $K_S^{qc} = [A_1 A_3] \cup [A_2, tA_1 + (1-t)A_3]$*;*

(iv-2) if $\det(A_2 - A_3) = 0$ *and* $\det\big(A_1 - (tA_2 + (1-t)A_3)\big) = 0$ *for some* $t \in [0, 1)$*, then* $K_S^{qc} = [A_2 A_3] \cup [A_1, tA_2 + (1-t)A_3]$*;*

(iv-3) if $\det(A_1 - A_3) = 0$ *and* $\det\big(A_2 - (tA_1 + (1-t)A_3)\big) \neq 0$ *for all* $t \in [0, 1]$*, then* $K_S^{qc} = [A_1 A_3] \cup \{A_2\}$*;*

(iv-4) if $\det(A_2 - A_3) = 0$ *and* $\det\big(A_1 - (tA_2 + (1-t)A_3)\big) \neq 0$ *for all* $t \in [0, 1]$*, then* $K_S^{qc} = [A_2 A_3] \cup \{A_1\}$ *.*

Proof. The cases *(i),(ii),(iv-1),(iv-2)* either are trivial or are covered by the arguments used in the previous part of this section. In the cases *(iv-3),(iv-4)* either K is a set of *Type 2*, which has already been studied, or, after reduction to $K = \{0, I, A\}$, either we have that A has multiple eigenvalues, in which case we can still apply the arguments in the proof of Theorem 5.23, or that A is not diagonalizable. The latter case is treated in Proposition 8.1 when K does not contain any rank-2 connection. However the proof extends as well to the present case.

The case *(iii)* is treated in Proposition 7.6. The cases which are left to complete the proof of Theorem 5.24 are first when K is a set of *Type 3* and second when $K = \{0, I, A\}$ contains no rank-2 connection and A is not diagonalizable. The treatment of these cases is postponed to Section 8 (see Proposition 8.1). □

6. Proofs of Lemma 5.8 and Theorem 5.14

The present section is devoted to the proofs of Lemma 5.8 and Theorem 5.14.

Proof of Lemma 5.8. We may assume that $M = 0$ and $N = I$ since, as already remarked, shift by a matrix and left-multiplication by an invertible matrix do not play any role (with the latter e.g. keeping the divergence-free property). Let $B_0 \in K_S^{qc}$ and $G \in GL(3, \mathbb{R})$. By definition there exists a sequence of Q-periodic L^2 equi-integrable divergence free matrix fields $\{B_h\}$ which satisfy

$$\operatorname{dist}(B_h, K) \to 0 \quad \text{in measure} \quad \text{and} \quad \fint_Q B_h = B_0.$$

We introduce the new variable y in $\mathbb{R}^3$ given by

$$y = G^{-T}x$$

and define the sequence $\{\bar{B}_h\}$ in the following way:

$$\bar{B}_h(y) := GB_h(G^T y)G^{-1}\,. \tag{6.1}$$

Then one can check that B_h is still divergence free in $\mathbb{R}^3$, and it satisfies

$$\operatorname{dist}(B_h, \bar{K}) \to 0 \quad \text{in measure.} \tag{6.2}$$

If $G \in GL(3, \mathbb{Q})$, then there exists a positive integer l such that $\bar{B}_h$ is periodic with periodicity cube $(0, 2l\pi)^3$, which can be re-scaled back to a 2π-periodic field $\bar{B}_h(y/l)$ and thus the proof is concluded. If G has irrational entries, decompose $\bar{B}_h$ in the following way:

$$\bar{B}_h(y) = GB_0G^{-1} + G(B_h - B_0)(G^T y)G^{-1}\,.$$

Notice then that $G(B_h - B_0)(G^T y)G^{-1}$ is divergence free and periodic with periodicity cell $G^{-T}Q$, therefore there exists a sequence of matrix fields $\{V_h\} \subset H^1_{loc}(\mathbb{R}^3)$ bounded in $L^2(G^{-T}Q)$ and with the same periodicity as $\{B_h\}$ such that

$$G(B_h - B_0)(G^T y)G^{-1} = \operatorname{Curl} V_h(y)\,.$$

Now let $\{L_h\}$ be an increasing sequence of positive numbers such that $L_h \to \infty$ as $h \to \infty$ and define

$$\varphi_h(y) := \min\{1, \operatorname{dist}(y, \partial Q_h)\} \text{ for } y \in Q_h$$

where $Q_h = (0, 2\pi L_h)^3$ and extend φ_h periodically to the whole $\mathbb{R}^3$. Next set

$$\widehat{B}_h(y) = GB_0G^{-1} + \frac{1}{L_h}\operatorname{Curl}\Big(\varphi_h(L_h y)V_h(L_h y)\Big)$$

and observe that the sequence $\{\widehat{B}_h\}$ is Q-periodic, L^2_{loc}-equi-integrable and satisfies (6.2) since

$$\int_Q |V_h(L_h y) \wedge \nabla\varphi_h(L_h y)|^2\, dy \le \frac{1}{L_h^3}\int_{Q_h \cap \{\nabla\varphi_h \neq 0\}} |V_h(y)|^2\, dy \approx \frac{|\det G|}{L_h}\|V_h\|^2_{L^2(G^{-T}Q)} \to 0\,.$$

□

The proof of Theorem 5.14 requires the introduction of some new objects. Let $h : \mathbb{R} \to \mathbb{R}$ be defined as $h = 1$ on $(0, 1/2]$, $h = -1$ on $(1/2, 1]$ and $h = 0$ elsewhere. For $j \in \mathbb{Z}, k \in \mathbb{Z}^3, \varepsilon \in \{0,1\}^3 \setminus (0,0,0)$ we define the three-dimensional Haar basis $\{h^\varepsilon_{j,k}(x)\}_{k,j,\varepsilon}$ as

$$h^\varepsilon_{j,k}(x) = h^\varepsilon(2^j x - k)$$

where $h^{(\varepsilon_1,\varepsilon_2,\varepsilon_3)}(x) = (h(x_1))^{\varepsilon_1}(h(x_2))^{\varepsilon_2}(h(x_3))^{\varepsilon_3}$ (with the convention $(-1)^0 = 1, 0^0 = 0$). For every $u \in L^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ with $\int_{\mathbb{R}^2} u\,dx = 0$ we can consider the expansion of u into Haar wavelets

$$u = \sum_{j,k,\varepsilon} a^{\varepsilon}_{j,k} h^{\varepsilon}_{j,k}$$

and define the projection operator P^{ε} by

$$P^{\varepsilon} u := \sum_{j,k} a^{\varepsilon}_{j,k} h^{\varepsilon}_{j,k}\,.$$

The following theorem plays a central role and provides a key estimate of the wavelet coefficients in terms of the Riesz transform $R_k = -i\partial_k(-\Delta)^{-1/2}$.

Theorem 6.1. ([11]) *The operator P^{ε} can be extended to a bounded operator on L^2 and $\forall\, k = 1, 2, 3\,, \forall\, \varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$ with $\varepsilon_k \neq 0$ one has*

$$\|P^{(\varepsilon)} u\|_2 \leq C \|u\|_2^{1/2} \|R_k u\|_2^{1/2}\,.$$

The proof of Theorem 6.1 is a direct line-by-line adaptation of the Müller's proof [11, Thm 5] employing the deep Littlewood-Paley decomposition and the "almost orthogonality" properties to the three-dimensional case, and is not reproduced here.

We will need the following lemma which is a straightforward modification of Lemma 6 in [11].

Lemma 6.2. *Let f satisfy the assumptions of Theorem* (5.14). *Assume that $u, v \in L^2(\mathbb{R}^3)$ have the finite expansions*

$$u = \sum_{j=J}^{K} \sum_{k\in\mathbb{Z}^3} a^{(0,0,1)}_{j,k} h^{(0,0,1)}_{j,k} + c^{(1,0,0)}_{j,k} h^{(1,0,0)}_{j,k}\,,$$

$$v = \sum_{j=J}^{K} \sum_{k\in\mathbb{Z}^3} b^{(0,1,0)}_{j,k} h^{(0,1,0)}_{j,k} + c^{(1,0,0)}_{j,k} h^{(1,0,0)}_{j,k}\,.$$

Then

$$\int_{\mathbb{R}^3} (f(u, v) - f(0, 0))\,dx \geq 0\,.$$

We are now in a position to prove Theorem 5.14.

Proof of Theorem 5.14. By Theorem 6.1 it follows that

$$P^{\varepsilon} u_h \to 0 \ \text{ in } L^2 \quad \forall\, \varepsilon \text{ such that } \varepsilon_2 \neq 0\,, \tag{6.3}$$

$$P^{\varepsilon} v_h \to 0 \ \text{ in } L^2 \quad \forall\, \varepsilon \text{ such that } \varepsilon_3 \neq 0\,, \tag{6.4}$$

$$P^{\varepsilon} (u_h - v_h) \to 0 \ \text{ in } L^2 \quad \forall\, \varepsilon \text{ such that } \varepsilon_1 \neq 0\,. \tag{6.5}$$

Additionally we have

$$(6.4) - (6.5) \Rightarrow P^{(1,0,1)} u_h \to 0 \;\text{ in } L^2_{loc}(\mathbb{R}^3)\,,$$
$$(6.3) - (6.5) \Rightarrow P^{(1,1,0)} v_h \to 0 \;\text{ in } L^2_{loc}(\mathbb{R}^3)\,,$$

which allow us to reduce to Lemma 6.2 adapting Müller's techniques in a straightforward way. □

7. Extremal three-point H-measures

In the present section we go back to the problem of characterizing the H-measures arising in (3.11) for $d = N = 3$. Our goal is to characterize the set $\widehat{Y}_3(\theta_2, \theta_3) \cap Y^H(\theta)$, that is the set of the "true" H-measures among the extremal points of the convex superset $Y(\theta)$. We will show that the solution to the problem discussed in Section 5, that is finding the optimal bound on the set K_S^{qc}, also provides the solution to this problem. More precisely, we will consider three-point measures of the form $\mu(\xi) = \sum_{r=1}^{3} \mu^r \delta_{\xi_r}$ when ξ_1, ξ_2, ξ_3 are linearly independent vectors and we will distinguish two cases. The first case is when the associated normalized masses, i.e., the points μ^r_{cs} on the (c, b)-plane, lie on the circular segments $\nu_2\nu_1$, $\nu_1\nu_3$, $\nu_3\nu_2$, *one* on each segment. Theorem 7.1 shows that characterizing the H-measures in this case is equivalent to characterizing the S-quasiconvex hull of the sets of *Type 1*. As a consequence we will obtain a criteria (Theorem 7.3) which allows us to "recognize" the H-measures among the three-point measures having normalized masses on different arches.

The second case is when the normalized masses are distinct but *two* of them lie on the *same* circular segment. Theorem 7.8 asserts that such measures are not H-measures. This result is in turn related to the characterization of the S-quasiconvex hull of the sets of *Type 2* (as shown in the proof of Theorem 7.8).

Finally, the case when two of the normalized masses merge is ruled out by Theorem 7.10.

The general strategy is as follows: to each measure $\mu \in \widehat{Y}_3(\theta_2, \theta_3)$ we associate a set $K = \{A_1, A_2, A_3\}$ for which μ turns out to be the only extremizing measure in (4.13) and is such that the delivered lower bound $L(\theta)$ is zero. Then we use the results of Section 5, namely the knowledge of the S-quasiconvex hull of K, to establish the attainability of the lower bound. If the lower bound turns out to be attained (optimal), then $\mu \in Y^H(\theta)$, otherwise $\mu \notin Y^H(\theta)$.

Let us introduce some notation needed in the presentation of our results. For the three-point measure $\mu = \sum_{r=1}^{3} \mu^r \delta_{e_r} \in \widehat{Y}_3(\theta_2, \theta_3)$, let ϕ_r be the angle associated with the mass μ^r

via (4.7)-(4.12), and let t_r be defined as follows:

$$t_r := \tan\frac{\phi_r}{2}\,, \quad r = 1, 2, 3\,,$$

(assuming $\phi_r \neq \pi$).

Theorem 7.1. *Let $\theta \in (0,1)^3$, $d = 3$ and let $\bar\mu \in \widehat{Y}_3(\theta_2, \theta_3)$ be supported on three linearly independent vectors $\xi_1, \xi_2, \xi_3 \in S^2$:*

$$\bar\mu(\xi) = \sum_{r=1}^{3} \bar\mu^r \delta_{\xi_r}\,.$$

Suppose that

$$\bar\phi_1 \in (0,\pi)\,, \quad \bar\phi_2 \in (\pi, \tfrac{3}{2}\pi)\,, \quad \bar\phi_3 \in (\tfrac{3}{2}\pi, 2\pi)\,, \quad \text{and} \quad t_1(1+t_3) \neq t_3(1+t_2)\,.$$

Then there exists a set $K = \{A_1, A_2, A_3\}$ of the form (5.2) such that

$$\bar\mu \in Y^H(\theta) \iff \sum_{i=1}^{3} \theta_i A_i \in K_S^{qc}\,. \tag{7.1}$$

Proof. We first assume that $\bar\mu$ is supported on the canonical basis of $\mathbb{R}^3$, (e_1, e_2, e_3). Since by assumption $t_1(1+t_3) \neq t_3(1+t_2)$, we may have either $t_1(1+t_3) < t_3(1+t_2)$ or $t_1(1+t_3) > t_3(1+t_2)$. We study the two cases separately.
Case (*i*). Assume that $t_1(1+t_3) < t_3(1+t_2)$. Let $q = (q_1, q_2, q_3)$ be defined as follows:

$$q_1 = \frac{t_1(1+t_3) - t_3(1+t_2)}{t_3(t_1 - t_2)}\,, \quad q_2 = \frac{t_1(1+t_3) - t_3(1+t_2)}{t_2 - t_3}\,, \quad q_3 = \frac{t_1(1+t_3) - t_3(1+t_2)}{(1+t_2)(t_1 - t_3)}\,. \tag{7.2}$$

Since by assumption $t_1 \in (0, +\infty)$, $t_2 \in (-\infty, -1)$, $t_3 \in (-1, 0)$, we find that $q \in (0,1)^3$ and it can be easily checked that $D(q) = \mathrm{diag}(t_1, t_2, t_3)$, where $D(q)$ is defined by (5.18) with q as in (7.2). Then set

$$A_1 = 0\,, \quad A_2 = I\,, \quad A_3 = D(q)\,, \quad K = \{A_1, A_2, A_3\}\,, \quad B_0 = \theta_1 A_1 + \theta_2 A_2 + \theta_3 A_3\,. \tag{7.3}$$

By construction the set K is of the form (5.2). Next consider the problem (3.2) corresponding to the above choice of matrices:

$$\forall \eta \in \mathbb{M}^{3\times 3} \quad Q_\theta^S F(\eta) = \inf_{\chi \in I(\theta)} \inf_{B \in V} \tfrac{1}{2} \fint_Q \left| B(x) + \eta - \sum_{i=1}^{3} \chi_i A_i \right|^2 dx\,, \tag{7.4}$$

where V is defined in (3.3). Using (3.9), (3.11), (4.1), (7.3) we specialize $Q_\theta^S F$ at the point $\eta = B_0$ in terms of the H-measures as follows:

$$Q_\theta^S F(B_0) = \inf_{\mu \in Y^H(\theta)} \int_{S^2} \big(a(\xi) f^{22}(\xi) + 2b(\xi) f^{23}(\xi) + c(\xi) f^{33}\big)\, ds(\xi)\,. \tag{7.5}$$

Next notice that the condition $B_0 \in K_S^{qc}$ is equivalent to $Q_\theta^S F(B_0) = 0$. In order to prove (7.1) we will show that $Q_\theta^S F(B_0) = 0$ if and only if the extremizing measure in (7.5) is the given measure $\bar{\mu}$. To this end note that by condition (4.4) and by definition of f^{ij} it follows that

$$a(\xi)f^{22}(\xi) + 2b(\xi)f^{23}(\xi) + c(\xi)f^{33}(\xi) \geq 0$$

anywhere on the sphere S^2. Therefore it is enough to prove that the function $\psi(\mu)$ defined in (4.14) vanishes only at the points $\mu = \bar{\mu}^r$, $r = 1, 2, 3$. Parametrize C by the angle ϕ as in (4.12) and set $e(\phi) = \sin(\phi/2)A_2 + \cos(\phi/2)A_3$. Evaluation of the function $\psi(\mu)$ for μ belonging to the circle C gives, cf. (4.14), (3.10) and (4.12):

$$\psi(a, b, c) = \tfrac{1}{2} \inf_{k \in S^2} \{a|A_2 k|^2 + 2b\langle A_2 k, A_3 k\rangle + c|A_3 k|^2\} = \tfrac{1}{2} \inf_{|k|=1} \langle e(\phi)^T e(\phi) k, k\rangle. \tag{7.6}$$

Therefore the value of $\psi(a, b, c)$ is the smallest eigenvalue of the symmetric non-negative matrix

$$e(\phi)^T e(\phi) = \left(\sin\frac{\phi}{2}A_2 + \cos\frac{\phi}{2}A_3\right)^T \left(\sin\frac{\phi}{2}A_2 + \cos\frac{\phi}{2}A_3\right).$$

Recalling the definition of the matrices A_2 and A_3 given in (7.3), we find that

$$e(\phi)^T e(\phi) = \left(\sin\frac{\phi}{2}I + \cos\frac{\phi}{2}D(q)\right)^2.$$

In other words ψ is the infimum of three linear functions whose graphs are planes intersecting the cylinder $\{(b, c, \psi) : (b, c) \in C\}$ over ellipses. It is easily seen that there are precisely three critical points ϕ_1, ϕ_2 and ϕ_3, that can be obtained by equating to zero the eigenvalues of $e(\phi)$:

$$\begin{aligned}
&\tan\frac{\phi_1}{2} - \frac{q_2}{q_3}(1 - q_3) = 0\,, \\
&\tan\frac{\phi_2}{2} - \frac{1}{q_3}\left[1 - q_3 - \frac{1}{1 - q_1}\right] = 0\,, \\
&\tan\frac{\phi_3}{2} - \frac{q_2}{(1 - q_1)(1 - q_2) - 1} = 0.
\end{aligned} \tag{7.7}$$

Using the definition of q given in (7.2), one can check that $\tan\dfrac{\phi_r}{2} = t_r$ in (7.7) and therefore $\phi_r = \bar{\phi}_r$ for every $r = 1, 2, 3$. Moreover one can immediately see that for every $r = 1, 2, 3$, the extremizing point in (7.6) for $\phi = \bar{\phi}_r$ is the eigenvector of $e(\bar{\phi}_r)^T e(\bar{\phi}_r)$ corresponding to the zero eigenvalue, that is the vector e_r. Therefore, $Q_\theta^S F(B_0) = 0$ if and only if $\bar{\mu} \in Y^H(\theta)$, and (7.1) follows.

Case (ii). Now assume that $t_1(1 + t_3) > t_3(1 + t_2)$. Then choose $q = (q_1, q_2, q_3)$ in the following way:

(7.8)
$$q_1 = \frac{t_1(1+t_3) - t_3(1+t_2)}{t_1 - t_3}\,, \quad q_2 = \frac{t_1(1+t_3) - t_3(1+t_2)}{t_1(t_3 - t_2)}\,, \quad q_3 = \frac{t_1(1+t_3) - t_3(1+t_2)}{(1+t_3)(t_1 - t_2)}\,,$$

and set

(7.9)
$$A_1 = 0\,, \quad A_2 = PD(q)P\,, \quad A_3 = I\,, \quad K = \{A_1, A_2, A_3\}\,, \quad B_0 = \theta_1 A_1 + \theta_2 A_2 + \theta_3 A_3\,,$$

where P is the permutation matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ and $D(q)$ is given by (5.18) with q as in (7.8). In particular $PD(q)P = \mathrm{diag}(\frac{1}{t_1}, \frac{1}{t_2}, \frac{1}{t_3})$. Then one defines problem (7.4) corresponding to the choice (7.9) and proceeds as in the previous case.

To conclude the proof we observe that if the measure $\bar{\mu}$ is supported on any three linearly independent vectors $\xi_1, \xi_2, \xi_3 \in S^2$, then it is enough to replace the matrix $D(q)$ in (7.3) and (7.9) by $G^{-1}D(q)G$, where G^{-1} is the matrix with columns ξ_1, ξ_2, ξ_3:

$$G^{-1} := (\xi_1, \xi_2, \xi_3)\,.$$

$\square$

Remark 7.2. Theorem 7.1 establishes the equivalence between two problems: the one of understanding whether a three-point measure in $\widehat{Y}_3(\theta_2, \theta_3)$ is an H-measure and the one of characterizing the S-quasiconvex hull of the set K defined via (7.3) or (7.9). The nature of this correspondence can be visualized as follows, see Figure 4. On the cross-section $\mathcal{K}_{cs}$ in the (c, b)-plane consider the triangle specified by the points $\nu_1 = (0, 0)$, $\nu_2 = (1, 0)$, $\nu_3 = (\frac{1}{2}, -\frac{1}{2})$. Every point in the interior of K^c can be identified with a point inside the triangle $\nu_1\nu_2\nu_3$ via the correspondence

(7.10)
$$\sum_i^3 \theta_i A_i \in \mathrm{Int}(K^c) \xrightarrow{\rho} \left(\frac{\theta_3(1-\theta_3)}{\theta_2(1-\theta_2) + \theta_3(1-\theta_3)}, \frac{-\theta_2\theta_3}{\theta_2(1-\theta_2) + \theta_3(1-\theta_3)} \right) \in \mathcal{K}_{cs}\,.$$

Now let $\bar{\mu} \in \widehat{Y}_3(\bar{\theta}_2, \bar{\theta}_3)$ satisfy the assumptions of Theorem 7.1 and let K be the set associated with $\bar{\mu}$ via (7.3) or (7.9). Then set

$$T_{cs}(K) := \rho(T(K) \cap \mathrm{Int}(K^c))\,,$$

where $T(K)$ is the set defined by (5.5). Since by Corollary 5.18 $K_S^{qc} = T(K)$, Theorem 7.1 can be re-phrased by saying that $\bar{\mu}$ is an H-measure *if and only if* the projection M_{cs} on $\mathcal{K}_{cs}$ of the total mass of $\bar{\mu}$ belongs to $T_{cs}(K)$. Moreover it can be checked that the set $T_{cs}(K)$ is the region delimited by the lines $\nu_1\bar{\mu}^3_{cs}$, $\nu_2\bar{\mu}^2_{cs}$ and $\nu_3\bar{\mu}^1_{cs}$ on the (c, b)-plane (see Figure 4). We briefly illustrate its construction. On the (c, b)-plane draw the three segments $\nu_3\bar{\mu}^1_{cs}$, $\nu_2\bar{\mu}^2_{cs}$, $\nu_1\bar{\mu}^3_{cs}$. Consider the intersections of each of the segments with the two others and with the segments $\nu_1\nu_2$, $\nu_1\nu_3$, $\nu_2\nu_3$:

$$\{R_1\} = \nu_3\bar{\mu}^1_{cs} \cap \nu_1\bar{\mu}^3_{cs}\,, \quad \{R_2\} = \nu_3\bar{\mu}^1_{cs} \cap \nu_2\bar{\mu}^2_{cs}\,, \quad \{R_3\} = \nu_2\bar{\mu}^2_{cs} \cap \nu_1\bar{\mu}^3_{cs}\,,$$

$$\{R_4\} = \nu_3\bar{\mu}^1_{cs} \cap \nu_1\nu_2\,, \quad \{R_5\} = \nu_2\bar{\mu}^2_{cs} \cap \nu_1\nu_3\,, \quad \{R_6\} = \nu_1\bar{\mu}^3_{cs} \cap \nu_3\nu_2\,.$$

Then the set $T_{cs}(K)$ is given by the union of the closed triangle $R_1R_2R_3$ and the segments $[R_1, R_6)$, $[R_2, R_4)$, $[R_3, R_5)$.

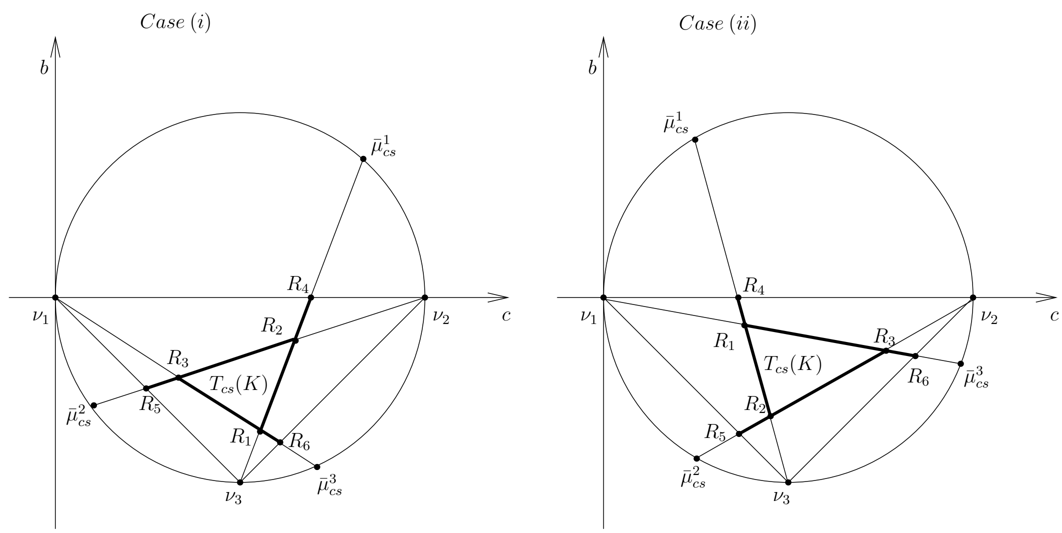

FIGURE 4. The set $T_{cs}(K)$ in cases (i) and (ii) of the proof of Theorem 7.1.

Keeping the notation introduced in Remark 7.2, we can then state the following

Theorem 7.3. *Let $\bar{\mu} \in \widehat{Y}_3(\theta_2, \theta_3)$ satisfy the assumptions of Theorem 7.1. Then*

$$\bar{\mu} \in Y^H(\theta) \Longleftrightarrow M_{cs} \in R_1R_2R_3 \cup [R_1, R_6) \cup [R_2, R_4) \cup [R_3, R_5)\,.$$

Remark 7.4. The results of Theorem 7.3 can be extended to the case when the measure $\bar{\mu}$ is such that one or more of the points $\bar{\mu}^r_{cs}$ coincide with some of the basic points ν_r. In this case some of the points R_1, R_2, R_3 in Figure 4 would merge with some of the points ν_r. The set associated with μ in the sense of Theorem 7.1 is still a set of *Type 1*, but with rank-2 connections.

Conversely, if we study problem (3.2) when the set K contains one or more rank-2 connections, then the resulting extremizing measure will have one or more of the normalized masses coinciding with some of the basic points ν_r. In particular, if the matrices A_1, A_2, A_3 are pairwise rank-2 connected, then $\{A_1, A_2, A_3\}^{qc}_S = \{A_1, A_2, A_3\}^c$ and the minimizing measure μ will have normalized masses equal to ν_1, ν_2, ν_3.

Remark 7.5. Theorem 7.3 essentially establishes that the sufficient conditions ([18], Proposition 6.1) for realizability of some extremal three-point measures of $Y(\theta)$ by the H-measures are also necessary. This result cannot apparently be derived from polyconvexity/ quadratic translation - type arguments (cf. e.g. [3], [2]).

We wish to discuss now what happens if the triangle $R_1R_2R_3$ on the (c,b)-plane degenerates into one single point, which we denote by R_0 (see Figure 5-(2)). In this case the associated measure μ satisfies

$$t_1(1+t_3) = t_3(1+t_2)$$

and therefore there is no set K of the type (5.2) for which (7.1) may hold. More precisely, the set associated with such measure μ is of *degenerate Type 1*. Proposition 7.6 makes some of this precise.

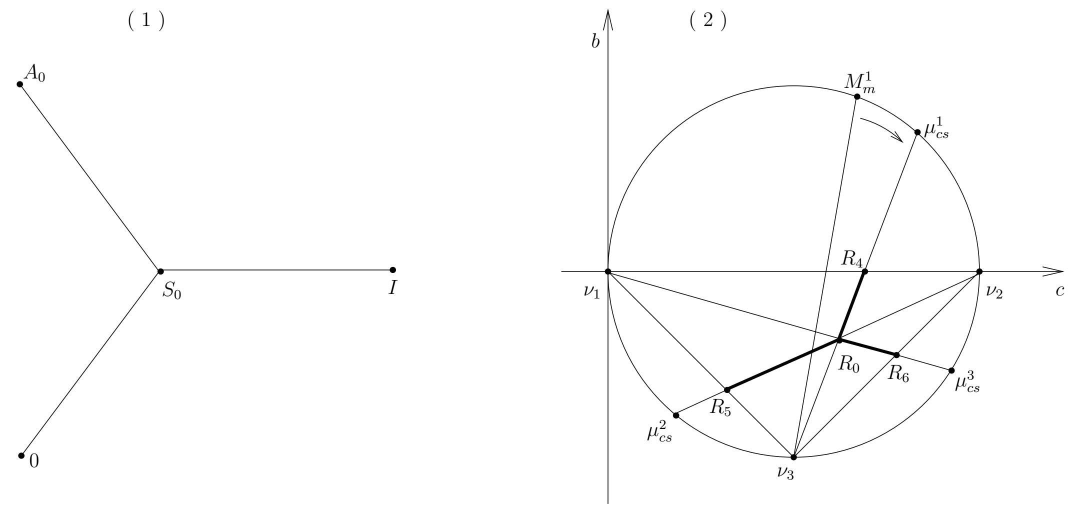

FIGURE 5. (1) The set K_0. (2) The set $T_{cs}(K_0)$ on the (c,b)-plane.

Proposition 7.6. *Let $\theta \in (0,1)^3$, $d=3$ and let $\mu \in \widehat{Y}_3(\theta_2,\theta_3)$ be supported on three linearly independent vectors $\xi_1,\xi_2,\xi_3 \in S^2$:*

$$\mu(\xi) = \sum_{r=1}^{3} \mu^r \delta_{\xi_r}\,.$$

Suppose that

$$\phi_1 \in (0,\pi)\,, \quad \phi_2 \in (\pi, \tfrac{3}{2}\pi)\,, \quad \phi_3 \in (\tfrac{3}{2}\pi, 2\pi)\,, \quad \textit{and} \quad t_1(1+t_3) = t_3(1+t_2)\,.$$

Then

$$\mu \in Y^H(\theta) \Longleftrightarrow M_{cs} \in [R_0,R_4) \cup [R_0,R_5) \cup [R_0,R_6) \Longleftrightarrow \theta_2 I + \theta_3 A_0 \in \{0, I, A_0\}^{qc}_S\,,$$

where the matrix A_0 is defined as follows:

$$A_0 = G^{-1}\mathrm{diag}(-t_1, -t_2, -t_3)G\,, \quad G = (\xi_1, \xi_2, \xi_3)^{-1}\,.$$

Proof. Set $K_0 = \{0, I, A_0\}$ and $S_0 = \mathrm{diag}(0, 1, -t_3)$. The first observation we make is that the matrix S_0 is rank-2 connected with each of the three matrices $0, I, A_0$ and that the set $T(K_0)$ defined by (5.5) is given in this case by the union of three segments:

$$T(K_0) = [0, S_0] \cup [I, S_0] \cup [A_0, S_0]$$

(see Figure 5-(1)). According to Definition 5.6, K_0 is a set of *degenerate Type 1*. Moreover it is easily checked that

$$\rho((0, S_0] \cup (I, S_0] \cup (A_0, S_0]) = [R_0, R_4) \cup [R_0, R_5) \cup [R_0, R_6)\,, \quad \rho(S_0) = R_0,$$

with the mapping ρ defined by (7.10). Using the function $\mathcal{T}^+$ introduced in Section 5 and arguing as for the sets of *Type 1*, one can show that every point outside $T(K_0)$ does not belong to $(K_0)^{qc}_S$. Then, using the algorithm illustrated in Lemma 4.1 and proceeding as in the proof of Theorem 7.1, one checks that the lower bound for $Q^\theta_S F(B_0)$, with $K = K_0$ and $B_0 = \theta_2 I + \theta_3 A_0$, is zero and is delivered by the given measure μ. Therefore if $\mu \in Y^H(\theta)$ then $M_{cs} \in [R_0, R_4) \cup [R_0, R_5) \cup [R_0, R_6)$.

Now let $M_{cs} \in [R_0, R_5) \cup [R_0, R_6)$. A way to prove that $\mu \in Y^H(\theta)$ is to use an approximation argument. We consider a sequence of points M^1_m on the circle C such that $M^1_m \to \mu^1_{cs}$ as $m \to \infty$ (see Figure 5-(2)). By Theorem 7.3 it follows that for every m the measure μ^m corresponding to the split $M^1_m, \mu^2_{cs}, \mu^3_{cs}$ is an H-measure. By construction $\mu^m \rightharpoonup \mu$ and therefore $\mu \in Y^H(\theta)$ by the closedness property of the H-measures, see (2.2). If $M_{cs} \in [R_0, R_4)$ then one introduces a perturbation around the point μ^2_{cs} or μ^3_{cs} and proceeds as before. We have thus proved that

$$\mu \in Y^H(\theta) \Longleftrightarrow M_{cs} \in [R_0, R_4) \cup [R_0, R_5) \cup [R_0, R_6)$$

and

$$(K_0)^{qc}_S = [0, S_0] \cup [I, S_0] \cup [A_0, S_0]\,.$$

□

Remark 7.7. Observe that the case when *all* the points μ^r_{cs} lie on the *same* circular segment (i.e. either $\nu_2\nu_1$, or $\nu_1\nu_3$ or $\nu_3\nu_2$) is clearly *not* associated with an H-measure: the projection on the cross-section of the total mass of the measure is then outside the triangle $\nu_1\nu_2\nu_3$, which must not be the case.

The next result describes the case when two of the normalized masses lie on the same arch. Figure 6 represents a measure with one normalized mass on the arch $\nu_1\nu_2$ and the other two masses on the same arch $\nu_1\nu_3$.

Theorem 7.8. *Let $\theta \in (0,1)^3$, $d = 3$ and let $\mu \in \widehat{Y}_3(\theta_2, \theta_3)$ be supported on three linearly independent vectors $\xi_1, \xi_2, \xi_3 \in S^2$:*

$$\mu(\xi) = \sum_{r=1}^{3} \mu^r \delta_{\xi_r} .$$

Assume that the points $\mu^1_{cs}, \mu^2_{cs}, \mu^3_{cs}$ are pairwise distinct and that $\mu^r_{cs} \neq \nu_i$ for all $r, i = 1, 2, 3$. If two and only two of the normalized masses $\mu^1_{cs}, \mu^2_{cs}, \mu^3_{cs}$ lie on the same circular segment, then $\mu \notin Y^H(\theta)$.

Proof. Let $A = G^{-1}\mathrm{diag}(-t_1, -t_2, -t_3)G$, with $G = (\xi_1, \xi_2, \xi_3)^{-1}$. It can be easily checked that the set $K = \{0, I, A\}$ is of *Type 2* and does not contain any rank-2 connection. We now proceed as in the previous cases. We study problem (3.2) for $A_1 = 0, A_2 = I, A_3 = A$ and, again using the algorithm of Lemma 4.1, we find that the lower bound $L(\theta)$ for $Q^\theta_S F(\theta_2 I + \theta_3 A)$ is zero and is delivered by the given measure μ. Since by Theorem 5.22 $K^{qc}_S = K$, it must be $Q^\theta_S F(\theta_2 I + \theta_3 A) > 0$ and therefore μ is not an H-measure. □

Remark 7.9. Observe that the results of Theorem 7.8 extend to the case when one of the normalized masses coincides with one of the basic points (for example in Figure 6 the point μ^1_{cs} may merge with ν_2 or we may as well have $\mu^2_{cs} = \nu_1$ or $\mu^3_{cs} = \nu_3$). In this case the set K associated with μ is still of *Type 2* but it contains a rank-2 connection.

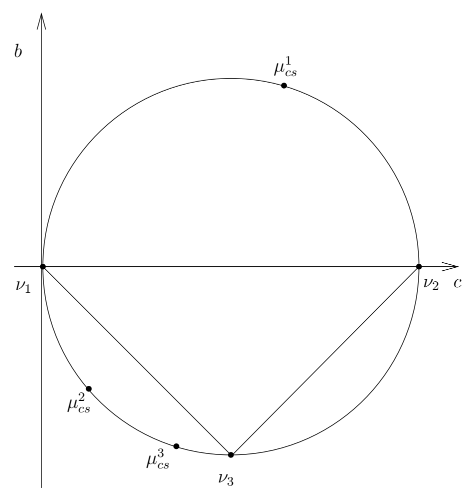

FIGURE 6

Theorem 7.10. *Let $\theta \in (0,1)^3$, $d = 3$ and let $\mu \in \widehat{Y}_3(\theta_2, \theta_3)$ be supported on three linearly independent vectors $\xi_1, \xi_2, \xi_3 \in S^2$:*

$$\mu(\xi) = \sum_{r=1}^{3} \mu^r \delta_{\xi_r} .$$

Assume that $\mu_{cs}^r \neq \nu_i$ *for all* $r, i = 1, 2, 3$. *If* μ_{cs}^1 *and* μ_{cs}^2 *lie on different arches and* $\mu_{cs}^2 = \mu_{cs}^3$, *then* $\mu \notin Y^H(\theta)$.

Proof. We use the same strategy as before. Namely, we associate to μ a set K for which μ is the extremizing measure in (4.13) and delivers a zero lower bound. Such set is given by $K = \{0, I, A\}$, where $A = G^{-1}\mathrm{diag}(-t_1, -t_2, -t_3)G$. By assumption we have that $t_2 = t_3$ and therefore the set K satisfies the assumptions of Theorem 5.23. Then $K_S^{qc} = K$ and μ is not an H-measure. □

Remark 7.11. Observe that the results of Theorem 7.10 may not extend to the case when one normalized mass coincides with a basic point. Indeed it is easy to check that if μ_{cs}^1 coincides with a basic point and $\mu_{cs}^2 = \mu_{cs}^3$ lie on the opposite arch (e.g., $\mu_{cs}^1 = \nu_1$ and $\mu_{cs}^2 = \mu_{cs}^3 \in \nu_2\nu_3$), or if $\mu_{cs}^2 = \mu_{cs}^3$ merge with a basic point and μ_{cs}^1 lies on the opposite arch (e.g., $\mu_{cs}^2 = \mu_{cs}^3 = \nu_1$ and $\mu_{cs}^1 \in \nu_2\nu_3$), then the corresponding measure is an H-measure for all M_{cs} on the segment $\mu_{cs}^1\mu_{cs}^2 \cap \nu_1\nu_2\nu_3$. Moreover, the sets K associated with such measures would contain one rank-2 connection and would have the form $K = \{0, A_2, A_3\}$ and the subspace generated by A_2 and A_3 would contain only two rank-2 directions. The S-quasiconvex hull of these sets is indeed non-trivial, as follows from Theorem 5.24, *(iv-1)-(iv-2)*.

Remark 7.12. The results in Theorems 7.3, 7.8 and 7.10 provide full characterization of the extremal H-measures supported in (no more than) three linearly independent directions. This characterization appears to be sufficient for purposes of full resolution of the problem of characterizing the quasiconvex hulls for three solenoidal wells. However the above results *do not* imply a full characterization of the three-phase H-measures $Y^H(\theta)$ themselves: while the latter are fully characterized by their extremal points, there remains a possibility that there are extremal points of $Y^H(\theta)$ supported in more than three points, therefore *not* being extremal points of the (fully characterized) superset $Y(\theta)$. Indeed, we sketch below an argument establishing the existence of extremal H-measures supported in *four* points.

Consider H-measures supported in three Dirac masses according to Theorem 7.3, i.e., associated with sets of *Type 1*, with fixed $\bar{\mu}_{cs}^r$, and linearly independent ξ_r (e.g. $\xi_r = e_r$), $r = 1, 2, 3$. This could be achieved for a range of volume fractions θ, in particular such that $M_{cs}(\theta)$ is well inside the triangle $R_1R_2R_3$, see Fig. 4. Select such $\theta = \theta^{(0)}$ and let the corresponding extremal H-measures be $\bar{\mu}^{(0)} \in Y^H(\theta^{(0)})$. By continuity, there exists $\Delta > 0$ such that the above property is held for all $|\theta - \theta^{(0)}| < \Delta$. Select on the circle C one more "cross-sectional mass" $\bar{\mu}_{cs}^4$ such that for $\theta = \theta^{(0)}$ there *does not* exist an H-measure corresponding to $\{\bar{\mu}_{cs}^r, r = 1, 2, 4\}$, which is clearly possible by Theorem 7.3 and let $\xi_4 \neq \xi_r$, $r = 1, 2, 3$ such that any three vectors out of $\xi_1, ..., \xi_4$ are linearly independent.

Hence for the corresponding three-point (ξ_1, ξ_2, ξ_4) Borel measure $\bar{\mu}$, extremal for $Y(\theta^{(0)})$, $\bar{\mu} \notin Y^H(\theta^{(0)})$. For $0 < t < 1$, the Borel measures $\bar{\mu}(t) := (1-t)\bar{\mu}^{(0)} + t\bar{\mu}$ are hence supported in *four* points. We argue that at least for small enough positive t those *are* H-measures, therefore the one corresponding to the maximal value of such t ($t = t_0$ $0 < t_0 < 1$) can only be an extremal H-measure, supported in four points. To establish this, notice that since $\bar{\mu}^4_{cs} \in C$ is extremal, $\bar{\mu}^4_{cs} = m \otimes m$ for some $m \in \mathbb{R}^2$, $|m| = 1$. Let $\theta^{(1)} := \theta^{(0)} + \Delta m/2$, $\theta^{(2)} := \theta^{(0)} - \Delta m/2$ and let the corresponding extremal H-measures be $\bar{\mu}^{(1)} \in Y^H(\theta^{(1)})$, $\bar{\mu}^{(2)} \in Y^H(\theta^{(2)})$, respectively (hence all supported in the same ξ_r with the same $\bar{\mu}^r_{cs}$, $r = 1, 2, 3$). Then "mix" these two H-measures in equal volume fractions via a lamination in layers perpendicular to ξ_4. The "mixing formula" for H-measures (see e.g. [23], [8], [18, §6(a) (6.4)]) produces the following new H-measure $\bar{\mu}^{(12)} \in Y^H(\theta^{(0)})$:

$$\bar{\mu}^{(12)} = \frac{1}{2}\bar{\mu}^{(1)} + \frac{1}{2}\bar{\mu}^{(2)} + \frac{\Delta^2}{4}\bar{\mu}^4_{cs}\delta_{\xi_4}.$$

This is clearly an H-measure supported in the four points. Such a measure can only be a convex combination of $\bar{\mu}^{(0)}$ and $\bar{\mu}$ and hence $\bar{\mu}^{(12)} = \bar{\mu}(t^*)$, for some $0 < t^* < 1$. By convexity and closedness, there exists "maximal" t_0, $t_0 \geq t^* > 0$ such that $\bar{\mu}(t) \in Y^H(\theta^{(0)})$ if and only if $t \in [0, t_0]$. (Since $\bar{\mu}(1) = \bar{\mu} \notin Y^H(\theta^{(0)})$, $t_0 < 1$.) □

8. Last part of Theorem 5.24 and a brief summary

In the present section we complete the proof of Theorem 5.24 and give a summary of the main findings of the paper.

Proposition 8.1. *Let $K = \{0, I, A\}$ where* $\det(A) \neq 0$ *and* $\det(A - I) \neq 0$*. Assume that one of the following conditions is satisfied:*
(i) K is a set of Type 3;
(ii) A is diagonalizable and the plane formed by K contains only two distinct rank-2 directions, and the corresponding affine rank-2 lines through 0*, I and A do not intersect at points inside K^c;*
(iii) A is not diagonalizable.
Then $K^{qc}_S = K$.

Proof. We consider problem (3.2) for the given set K and show that the lower bound $L(\theta)$ defined by (4.13) is strictly positive for all values of the volume fractions θ, implying that the quasiconvex hull is trivial.

Assume (i). Then the matrix A is diagonalizable and has three distinct real eigenvalues. Using the algorithm of Lemma 4.1, one can see that $L(\theta)$ could be zero only if the extremizing measure in (4.13) had all the three normalized masses on the same circular segment, either $\nu_1\nu_2$, or $\nu_1\nu_3$ or $\nu_3\nu_2$. Since this cannot be the case (see Remark 7.7), the lower bound is strictly positive.

Assume (ii). Then the matrix A is diagonalizable and has two distinct real eigenvalues, one of multiplicity two. As in case (i), one can see that $L(\theta)$ could be zero only if the extremizing measure had normalized masses on the same circular segment, except that in this case two of them would merge. Again, this cannot be the case.

Now assume (iii). If A has one real eigenvalue and two complex (hence complex conjugate), then the function ψ defined by (4.14) never vanishes and therefore the lower bound is strictly positive. If A has two distinct eigenvalues, one of which has algebraic multiplicity two but geometric multiplicity one, or if A has one eigenvalue of algebraic multiplicity three but geometric multiplicity two, then the lower bound may be zero but is delivered by a two-point supported measure. On the other hand, two-point measures are not H-measures (this can be shown e.g. via the Šverák's incompatibility result for three gradient wells [20], see [18], §7-(a)).

Finally, if A has one eigenvalue of algebraic multiplicity three but geometric multiplicity one, then the lower bound is strictly positive. $\square$

Summary. The results presented in Section 7 provide the characterization of all three-point H-measures of the form $\mu(\xi) = \sum_{r=1}^{3} \mu^r \delta_{\xi_r}$ when ξ_1, ξ_2, ξ_3 are linearly independent vectors and the associated points μ^r_{cs} on the (c, b)-plane lie on the circular segments $\nu_2\nu_1$, $\nu_1\nu_3$, $\nu_3\nu_2$, *one* on each segment including possibly the endpoints, Figure 4. The only extremal H-measures in this class are those described by Theorem 7.3 including the limit cases as discussed in Remark 7.4 and Proposition 7.6.

Theorems 7.8 and 7.10 complete full characterization of the extremal three point H-measures supported on three arbitrary linearly independent directions: all other measures in the set $\widehat{Y}_3(\theta_2, \theta_3)$ are *not* H-measures.

On the other hand, the presented analysis allows us to fully solve the problem of S-quasiconvexification for three arbitrary solenoidal wells. The conclusion is that a non-trivial quasiconvex hull can only emerge in the situation as in Figure 2-(1), i.e. when there are three separate rank-two directions in the plane formed by $K = \{A_1, A_2, A_3\}$ *and* the mutual position of A_1, A_2 and A_3 on this plane is such that an inner triangle is formed, including the limit cases. Then, according to Corollary 5.18, $K^{qc}_S = T(K)$. In all other cases $K^{qc}_S = K$, unless K contains rank-2 connections, in which case K^{qc}_S also contains the segment(s) joining the pair(s) of rank-2 connected matrices (Theorem 5.24).

9. On applications of the H-measure results. The three well problem for linear elasticity

An attractive feature of the H-measure is that it is a purely geometric object, i.e., independent from the kinematic constraints. Hence the same H-measures are involved in characterizing the relaxation of problems with different kinematic constraints, in particular of associated quasiconvex hulls. Therefore, any progress in characterizing the H-measures can be potentially transferred from problems with one type of kinematic constraints to those with another. In this section we discuss the application of the results on the H-measures to the problem of characterizing the quasiconvex hulls for three linear elastic wells.

The problem is formulated similarly to that in Sections 3 and 5 with $K = \{A_1, A_2, A_3\}$ and A_1, A_2 and A_3 being now three *symmetric* matrices in $\mathbb{M}^{d\times d}$ of given linearized "transformation strains". The divergence-free kinematic constraint for a field B, cf. (3.3), is in turn replaced by the requirement that B is a symmetrized gradient of a periodic displacement field u:

$$B(x) = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right), \quad u \in W^{1,2}_{\sharp}(Q, \mathbb{R}^d). \tag{9.1}$$

The multi-well energy is analogous to (3.1), being characterized more generally by a quadratic form generated by a positive definite elastic tensor $\mathbf{C}$ which would formally coincide with (3.1) for the special case of an isotropic tensor with Lamé constants $\lambda = 0$ and $\mu = 1/2$ (cf. [18, §7(b)]), resulting in $\mathbf{C} = I$ with I being the identity tensor. Notice that the exact choice of $\mathbf{C}$ does not affect the issue of characterizing the (linear elastic) quasiconvex hull K^{qc}_{le}, cf. [1], so there is no loss of generality in choosing $\mathbf{C} = I$ for this purpose. As before, $\eta = \sum_{i=1}^3 \theta_i A_i$, $\theta \in [0,1]^3$, $\sum_{i=1}^3 \theta_i = 1$, is in K^{qc}_{le} if and only if $Q^\theta_{le}F(\eta) = 0$. Here the relaxed energy $Q^\theta_{le}F$ is defined by (3.2) where in the definition (3.3) for V the divergence-free constraint is replaced by (9.1).

The relaxed energy $Q^\theta_{le}F(\eta)$ can in turn be equivalently expressed in terms of minimization with respect to H-measures [8, 18], namely (3.11) still holds with the same set of H-measures $Y^H(\theta)$ as before but $f^{ij}(\xi)$ requiring re-evaluation for the linear elasticity context. Specializing to the three-dimensional elasticity ($d = 3$), $f^{ij}(\xi)$ is as follows (cf. [18, §7(b)]):

$$f^{ij}(\xi) = \frac{1}{2} A_i^{kl} \Delta_{klpq}(\xi) A_j^{pq}. \tag{9.2}$$

Here A_i^{kl} denotes the (kl) components of the matrix A_i and summation is implied with respect to repeated indices, and

$$\Delta_{klpq}(\xi) := \frac{1}{2}\{T_{kp}(\xi)T_{lq}(\xi) + T_{kq}(\xi)T_{lp}(\xi)\}, \quad T_{kl}(\xi) := \delta_{kl} - \xi_k \xi_l.$$

Hence (9.2) can be equivalently re-written as follows:

$$f^{ij}(\xi) = \mathrm{Tr}\left[A_i T(\xi) A_j T(\xi)\right], \tag{9.3}$$

where $T(\xi) = I - \xi \otimes \xi$.

Further, the lower bound $L(\theta)$ for $Q^{\theta}_{le}F$, as computed in [18] is given by (4.13) with the same "universal" superset $Y(\theta)$.

Assuming further without loss of generality $A_1 = 0$, the linear elastic analog of (4.14) when restricted to the circle C parametrized as before by $\phi \in [0, 2\pi)$, see (4.12), can be computed, as in (7.6), and in the present case reads

$$\psi(a,b,c) = \psi(\phi) = \inf_{\xi \in S^2} \mathrm{Tr}\left[\left(e(\phi) T(\xi) \right)^2 \right], \tag{9.4}$$

where $e(\phi) := \sin(\phi/2)A_2 + \cos(\phi/2)A_3$. Denoting by $\nu_j(a,b,c) = \nu_j(\phi)$, $j = 1,2,3$, the eigenvalues of $e(\phi)$ and by k_1, k_2 and k_3 the components of ξ with respect to the (orthonormal) basis of the eigenvectors diagonalizing $e(\phi)$, (9.4) via a straightforward calculation reads:

$$\begin{aligned} \psi(a,b,c) = \inf_{k \in S^2} \ \Big\{ & \left(\nu_1 k_2^2 + \nu_2 k_1^2\right)^2 + \left(\nu_2 k_3^2 + \nu_3 k_2^2\right)^2 + \left(\nu_3 k_1^2 + \nu_1 k_3^2\right)^2 \\ & + 2\nu_1^2 k_2^2 k_3^2 + 2\nu_2^2 k_3^2 k_1^2 + 2\nu_3^2 k_1^2 k_2^2 \Big\}. \end{aligned} \tag{9.5}$$

It is easy to see that $\psi(\phi) = 0$ if and only if at least one of the eigenvalues ν_j, $j = 1,2,3$ is zero and the two others are not of the same sign, i.e.

$$\nu_1 \le \nu_2 = 0 \le \nu_3. \tag{9.6}$$

In particular, for the case of strict inequalities ($\nu_1 < \nu_2 = 0 < \nu_3$) the zero minimum in (9.5) is achieved at exactly *two* different directions $k \in S^2$: $k_2 = 0$, $k_3 = \pm|\nu_3/\nu_1|^{1/2} k_1$ giving rise to two different locations on the sphere for the component extremal mass corresponding to such ϕ.

The condition of compatibility of two linear elastic matrices A_i and A_j is known to be of similar type: one of the eigenvalues of $(A_i - A_j)$ must be zero and the two others must not be of the same sign, see e.g. [1]. Hence, for pairwise compatible wells $\psi(0) = \psi(\pi) = \psi(3\pi/2) = 0$, in which case $K^{qc}_{le} = K^c$, e.g. [1]. Therefore it remains to consider the cases when the wells are *not* pairwise compatible. We assume without loss of generality that A_2 and $A_1 = 0$ are incompatible, i.e., upon diagonalization, $A_2 = \mathrm{diag}\,(\alpha_1, \alpha_2, \alpha_3)$ with $\alpha_1\alpha_2 > 0$. We then argue that the equation $\psi(\phi) = 0$ does not have more than three solutions for ϕ (within the range $[0, 2\pi)$) unless, in the chosen basis, $\alpha_3 = 0$ and $A_3^{k3} = A_3^{3k} = 0$, $k = 1,2,3$. The latter corresponds to two-dimensional linear elasticity, for which the quasiconvex hull is known and is in particular trivial in the case of pairwise incompatible wells, see e.g. [18, §7(b)]. For the former assertion, a necessary condition for $\psi(\phi) = 0$ is $\det e(\phi) = 0$. The latter equation does not have more than three solutions: if $\det e(\phi) = 0$ then either $\phi = \pi$ implying $\det A_2 = 0$ or $\det(A_3 + t\,A_2) = 0$ which is (a nontrivial) at most cubic equation in $t := \tan(\phi/2)$. From these values of ϕ those failing (9.6) should be excluded further, and as a result we end up with no more than three values of t such that $\psi(\phi) = 0$.

Further, $\eta \in K_{le}^{qc}$ if and only if $L(\theta) = 0$ *and* the minimizing measure in $Y(\theta)$ is an H-measure. The previous reasoning assures that, as in the divergence-free case, the total mass could generically only be split in a no more than a single triple of extremal masses corresponding to $\psi(\phi_r) = 0$, $r = 1, 2, 3$. Since for each of such ϕ_r there are generically two corresponding "minimizing" directions on the sphere, $\xi_r^{(1)}$ and $\xi_r^{(2)}$, the minimizing measures could be supported in *up to six* Dirac masses.

On the other hand, the results of this paper, in particular of Section 7, provide full characterization of H-measures supported in no more than three points. The most interesting case of three solutions $\psi(\phi_r) = 0$, $r = 1, 2, 3$ corresponds to the situation when the plane (A_1, A_2, A_3) contains three "linear elastically compatible" directions, see [1] and [18, §7(b)] for such examples. The results of Section 7 are directly applicable to characterize the "inner bound" for K_{le}^{qc}, namely in the Type 1 case when the "internal triangle" is formed, Fig. 2, $T(K) \subset K_{le}^{qc}$, cf. [1].

However, for establishing the outer bounds, some further developments are required to eliminate (or otherwise) the possibility of the minimizing H-measure being supported in four to six points. For example, our results in Section 7 ensure that in the *Type 1* case the exterior to $T(K)$ is *not* realized by any extremal point of the superset $Y(\theta)$, i.e. by (generically) any extremal measure supported in a triple of points $\xi_1^{(k)}$, $\xi_2^{(l)}$ and $\xi_3^{(m)}$, where $(k, l, m) \in \{1, 2\}^3$. This argument does not however eliminate the possibility of an H-measure being a convex combination of those points. This poses an interesting open problem, whose resolution would possibly require further developments of the ideas of harmonic analysis akin to [11] and/or other ideas.

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