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by

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ERROR ESTIMATES FOR A MIXED FINITE ELEMENT DISCRETIZATION OF SOME DEGENERATE PARABOLIC EQUATIONS

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Abstract. We consider a numerical scheme for a class of degenerate parabolic equations, including both slow and fast diffusion cases. A particular example in this sense is the Richards' equation modeling the flow in porous media. The numerical scheme is based on the mixed finite element method (MFEM) in space, and is of first order implicit in time. The lowest order Raviart-Thomas elements are used. We derive error estimates in terms of the discretization parameters and show the convergence of the scheme. The paper is concluded by numerical examples.

1. Introduction. In this paper we analyze a mixed finite element scheme for the nonlinear, possibly degenerate, parabolic equation

$$\partial_t b(u) - \nabla \cdot (\nabla u + k(b(u))\ \mathbf{e}_z) = 0, \tag{1.1}$$

where $\mathbf{e}_z$ denotes the vertical unit vector.

Many porous media models can be brought to the form above. In this sense we mention the equation

$$\partial_t \Theta(\psi) - \nabla \cdot (K(\Theta(\psi))\nabla(\psi + z)) = 0, \tag{1.2}$$

which has been proposed by L.A Richards in 1930 to model the water flow in a porous medium (see e.g. [6]). In (1.2), ψ denotes the pressure head, Θ the saturation reduced to the standard interval $[0, 1]$, K stands for the hydraulic conductivity of the medium and z for the height against the gravitational direction. Based on experimental results, different curves have been proposed for describing the dependency between K, Θ and ψ (see e. g. [6]), yielding the nonlinear model (1.2). In this sense we mention the van Genuchten - Mualem framework, where

$$\Theta(\psi) = \left(1 + (c|\psi|)^{\frac{1}{1-m}}\right)^{-m}, \qquad K(\Theta) = K_s \Theta^{\frac{1}{2}} \left[1 - \left(1 - \Theta^{\frac{1}{m}}\right)^m\right]^2. \tag{1.3}$$

whenever the flow is unsaturated ($\psi < 0$). Here $K_s > 0$, $c > 0$ and $m \in (0, 1)$ are medium dependent parameters. For the fully saturated regime ($\psi \geq 0$) we have $\Theta = 1$ and $K = K_s$. Notice that in the present setting Richards' equation degenerates whenever ψ goes to $-\infty$, implying that both $\Theta'(\psi)$ and $K(\Theta(\psi))$ are approaching 0, or in the fully saturated regime ($\psi \geq 0$), when $\Theta'(\psi) = 0$. The regions of degeneracy depend on the saturation of the medium; therefore these regions are not known a-priori and may vary in time and space.

The Richards' equation is written in the pressure based formulation. In this way all flow regimes up to the case of completely dry soils can be considered: unsaturated, partially saturated and fully (water) saturated. As proven in [2], the Kirchhoff

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transformation

$$\begin{aligned}\mathcal{K} : \mathbb{R} &\longrightarrow \mathbb{R} \\ \psi &\longmapsto \int_0^{\psi} K(\Theta(s))\, ds\end{aligned} \tag{1.4}$$

allows writing the model in the more regular unknown $u := \mathcal{K}(\psi)$. Notice that the hydraulic conductivity K can only vanish in the completely unsaturated case, when $\Theta = 0$. Hence $K(\Theta(s)) > 0$ whenever $\Theta > 0$, so the transformation is bijective. With

$$\begin{array}{rcl} b(u) & := & \Theta \circ \mathcal{K}^{-1}(u), \\ k(b(u)) & := & K \circ \Theta \circ \mathcal{K}^{-1}(u), \end{array} \tag{1.5}$$

we can bring equation (1.2) to the form stated in (1.1). Further, for the nonlinearities in (1.3), $b'(\cdot)$ vanishes in the fully saturated regime, when $u \geq 0$. In this case the equation (1.1) becomes elliptic. This type of degeneracy is commonly called as "fast diffusion". Moreover, in the above setting u approaches a negative value M as the soil is drying, and in this case $b'(\cdot)$ is blowing up. This corresponds to a vanishing diffusion in the original form (1.2), and the equation becomes hyperbolic or ordinary in such a case (the "slow diffusion case").

Another example is the porous medium equation (see [4])

$$\partial_t v = \Delta v^m, \tag{1.6}$$

with $m \geq 1$. Here we seek for nonnegative solutions v. Taking $u = v^m$ and $b(u) = u^{1/m}$, as well as $k \equiv 0$, we end up with the equation (1.1). In this case the degeneracy appears at $u = 0$, for which $b'(\cdot)$ becomes unbounded. Again we can speak about a "slow diffusion" degeneracy.

Having in mind the above examples, we are interested in solving equation (1.1) endowed with initial and boundary conditions

$$\begin{array}{rclll} \partial_t b(u) - \nabla \cdot (\nabla u + k(b(u))\mathbf{e}_z) & = & 0 & \text{in} & (0, T] \times \Omega, \\ u & = & u_I & \text{in} & 0 \times \Omega, \\ u & = & 0 & \text{on} & (0, T] \times \Gamma. \end{array} \tag{1.7}$$

In the above problem Ω is a d-dimensional domain, where $d = 1$, 2 or 3. Its boundary is denoted by Γ. Further, $T > 0$ is a given finite time.

Throughout this paper we make use of the following assumptions:

(A1) $\Omega \subset \mathbb{R}^d$ is open, bounded and has a Lipschitz continuous boundary.

(A2) $b(\cdot) \in C^{0,\alpha}$ is nondecreasing and Hölder continuous: there exists an $\alpha \in (0, 1]$ and $C_b > 0$ so that $|b(u_1) - b(u_2)| \leq C_b |u_1 - u_2|^{\alpha}$ for all $u_1, u_2 \in \mathbb{R}$. For simplicity we assume $b(\cdot)$ continuously differentiable almost everywhere.

(A3) $k(b(\cdot))$ is continuous and bounded and satisfies for all $u_1, u_2 \in \mathbb{R}$, $| k(b(u_2)) - k(b(u_1)) |^2 \leq C_k (b(u_2) - b(u_1))(u_2 - u_1)$.

(A4) The initial data satisfies $u_I \in L^2(\Omega)$ and $b(u_I) \in L^{\infty}(\Omega)$.

Remark 1.1. *For the Richards' equation in the Mualem - van Genuchten setting, the assumption (A2) holds with $\alpha = 2m/(3m + 2)$, whereas (A3) is satisfied whenever $m \in [2/3, 1)$. For the porous medium equation (A2) is satisfied with $\alpha = 1/m$.*

Remark 1.2. *In (1.7) we have considered only a vertical convection. The results in this paper can be straightforwardly extended to the more general case, where the convection term is a vector satisfying (A3).*

Remark 1.3. *For the ease of presentation we have only considered homogeneous Dirichlet boundary conditions. The results can be extended to more general ones, as well as for problems involving a reaction term satisfying a condition that is similar to (A3).*

We mention [3, 15, 18, 24] for a mixed finite element discretization of (1.1). Specifically, the lowest order Raviart-Thomas finite elements are used, whereas the time discretization is achieved by an Euler implicit scheme. For the spatial discretization, optimal error estimates are obtained in [3, 18]. For proving the convergence of the fully discrete scheme, the solution is assumed sufficiently regular.

Similar results are obtained in [23] for an expanded MFEM. In [15], the ideas in [3, 18] are combined with the techniques for degenerate parabolic equations that are developed in [10]. The convergence order in [15] is optimal and obtained for a Lipschitz continuous nonlinearity $b(\cdot)$. An essential point of the proof is the equivalence between the mixed and conformal formulations, for both the continuous and the time discrete problems.

All the results mentioned above are obtained for either the slow diffusion case, or the fast diffusion one. As mentioned in the introduction, the Richards' equation (1.2) features both type of degeneracies. This is allowed in [12], where a conformal scheme is analyzed but only for the saturation based formulation. Therefore the results there do not apply to the fully saturated flow regime.

In the present paper we prove the convergence of the Raviart-Thomas MFEM, Euler implicit discretization in a general framework. By assuming only the Hölder continuity of $b(\cdot)$, both degeneracies mentioned before are allowed. This applies in particular to the Richards' equation (1.2). Furthermore, the error estimates are obtained now directly, in a transparent manner. The equivalence with a conformal formulation remains valid also in this general framework but plays just a secondary role in the proof, precisely to establish the regularity of the solution. Another advantage of the new approach is that the convergence in the nondegenerate case, as well as in the fast diffusion case where $b(\cdot)$ is Lipschitz continuous, can be obtained directly as particular cases of the current results.

The paper is organized as follows. In Section 2 the mixed continuous variational formulation is stated and the regularity of the solution is discussed. The error estimates for the time discrete scheme are obtained in the next section. The fully discrete scheme is considered in Section 4, where error estimates are derived in terms of the discretization parameters. In Section 5 we present some numerical simulations and the conclusions.

2. The mixed formulation.

In what follows we seek for weak solutions for the problem (1.7), written in the mixed form. We start with the weak conformal formulation, and investigate the equivalence between the mixed and conformal solutions. To define a solution in the weak sense, we make use of common notations in the functional analysis. By $\langle \cdot, \cdot \rangle$ we mean the inner product on $L^2(\Omega)$, or the duality pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. Further, $\| \cdot \|$, $\| \cdot \|_1$ and $\| \cdot \|_{-1}$ stand for the norms in $L^2(\Omega)$, $H^1(\Omega)$, respectively $H^{-1}(\Omega)$. The functions in $H(\mathrm{div};\Omega)$ are vector valued, having a L^2 divergence. By C we mean a positive constant, not depending on the unknowns or the discretization parameters.

A weak, conformal solution of (1.7) is solving the following problem:

Problem P_C. Find $u \in L^2(0,T;H_0^1(\Omega))$ such that $b(u) \in H^1(0,T;H^{-1}(\Omega))$,

$u(0) = u_I \in L^2(\Omega)$, and

$$\int_0^T \langle \partial_t b(u(t)), \varphi(t)\rangle \ + \ \langle \nabla u(t) + k(b(u(t)))\mathbf{e}_z, \nabla\varphi(t)\rangle dt = 0, \tag{2.1}$$

for all $\varphi \in L^2(0,T;H_0^1(\Omega))$.

Existence, uniqueness and essential boundedness for a weak solution of (1.7) is studied in several papers (see, for example, [2], [11] and the references therein). We also mention [20] for the analysis of an outflow problem in unsaturated media that is based on regularization. In particular, the following regularity is proven in [2]

$$b(u) \in L^\infty(0,T;L^1(\Omega)), \tag{2.2}$$

$$\mathbf{q} := -\left(\nabla u + k(b(u))\mathbf{e}_z\right) \in L^2(0,T;(L^2(\Omega))^d). \tag{2.3}$$

In the present paper we have in mind the applications mentioned in the beginning, where $b(u)$ models the water content or the air density. Thus it is physically reasonable to assume

(A5) $\quad b(u) \in L^\infty(0,T;L^\infty(\Omega))$.

In this way, by Lemma 9 and Corollary 4 of [21], we have $b(u) \in C(0,T;H^{-1}(\Omega))$.

As stated in the introduction, our aim is to prove the convergence of a mixed finite element discretization of (1.1). Due to the degeneracy of this equation, its solution lacks regularity. In particular, $\partial_t b(u)$ is only in $L^2(0,T;H^{-1}(\Omega))$, so in the variational formulation of (1.1) the spatial regularity of the test functions should be H^1. However, the mixed formulation requires test functions that are only L^2 in space. To overcome this difficulty we follow [3] (see also [23]) and integrate (1.1) in time from 0 to any $t \in (0,T]$. With $\mathbf{q}$ defined in (2.3) this gives

$$b(u(t)) + \nabla \cdot \int_0^t \mathbf{q}(s)\,ds = b(u_I), \tag{2.4}$$

for all t, in the sense of H^{-1}.

By (A4) and (A5), from (2.4) and (2.3) we can conclude that

$$\int_0^t \mathbf{q}ds \in X := H^1(0,T;(L^2(\Omega))^d) \cap L^2(0,T;H(\mathrm{div};\Omega)). \tag{2.5}$$

In fact we also have

$$\nabla \cdot \int_0^t \mathbf{q}ds \in L^\infty(\Omega)$$

for almost every t. In this way, for all $\varphi \in H_0^1(\Omega)$ and $t \in (0,T]$ we obtain

$$\langle b(u(t),\varphi\rangle + \left\langle \nabla \cdot \int_0^t \mathbf{q}(s)ds, \varphi \right\rangle = \langle b(u_I),\varphi\rangle. \tag{2.6}$$

Moreover, $b(u(t))$ and $\nabla \cdot \int_0^t \mathbf{q}(\tau)d\tau$ are L^2 for almost every t. Since these are defined for all t, by density arguments we conclude that equation (2.6) holds for all $\varphi \in L^2(\Omega)$ and all t.

Having in mind the above we can now define the mixed, time integrated variational form of (1.7):

Problem P_M. Find $(p, \mathbf{q}) \in L^2(0,T; L^2(\Omega)) \times L^2(0,T; (L^2(\Omega))^d)$ such that $b(p) \in L^\infty(0,T; L^\infty(\Omega))$ and $\int_0^t \mathbf{q}(s)\, ds \in X$, and

$$\langle b(p(t)) - b(p^0), w\rangle + \langle \nabla \cdot \int_0^t \mathbf{q}(s) ds, w\rangle = 0, \tag{2.7}$$

$$\langle \int_0^t \mathbf{q}(s) ds, \mathbf{v}\rangle - \langle \int_0^t p(s) ds, \nabla \cdot \mathbf{v}\rangle + \langle \int_0^t k(b(p(s)))\mathbf{e}_z ds, \mathbf{v}\rangle = 0, \tag{2.8}$$

for all $t \in (0,T]$, $w \in L^2(\Omega)$ and $\mathbf{v} \in H(\mathrm{div}; \Omega)$, with $p(0) = u_I \in L^2(\Omega)$.

The problems P_C and P_M are equivalent, as follows from Proposition 2.2 in [15]:

PROPOSITION 2.1. *A function u solves Problem P_C if and only if $(p, \mathbf{q})$ defined as*

$$(p, \mathbf{q}) = (u, -(\nabla u + k(b(u))\mathbf{e}_z)) \tag{2.9}$$

solves Problem P_M. Moreover, in this case we have $p \in L^2(0,T; H_0^1(\Omega))$.

Proof. In [15], $b(\cdot)$ is assumed Lipschitz continuous, which is more restrictive than (A2). However, the regularity of u and $\mathbf{q}$ stated above allows us to prove the equivalence in the present setting as well. To do so, we simply have to follow the steps in [15]. The details are omitted here. ∎

3. The time discretization. We now proceed with the time discretization Problem P_M, which is achieved by the Euler implicit scheme. Let $N > 1$ be an integer giving the time step $\tau = T/N$. For a given $n \in \{1, 2, \ldots, N\}$, with $t_n = n\tau$ we define the time discrete mixed variational problem:

Problem P_M^n. Let p^{n-1} be given. Find $(p^n, \mathbf{q}^n) \in L^2(\Omega) \times H(\mathrm{div}; \Omega)$ such that

$$\langle b(p^n) - b(p^{n-1}), w\rangle + \tau\langle \nabla \cdot \mathbf{q}^n, w\rangle = 0, \tag{3.1}$$

$$\langle \mathbf{q}^n, \mathbf{v}\rangle - \langle p^n, \nabla \cdot \mathbf{v}\rangle + \langle k(b(p^n))\mathbf{e}_z, \mathbf{v}\rangle = 0, \tag{3.2}$$

for all $w \in L^2(\Omega)$, and $\mathbf{v} \in H(\mathrm{div}; \Omega)$.
Initially we take $p^0 = u_I \in L^2(\Omega)$.

Assuming $b(\cdot)$ Lipschitz continuous, the convergence of the time discrete numerical scheme is proven in [15] by showing the equivalence between the conformal and mixed forms of the temporal discretization. Then the proof is done for the conformal method. In this section we give a simplified convergence proof, which applies directly to the mixed time discretization given in (3.1) and (3.2). Before doing so, it is worth noticing that the equivalence between the semidiscrete mixed and conformal schemes holds in the present generalized setting as well. To prove this, we simply have to proceed as in the proof of Proposition 2.3 in [15]. Furthermore, this equivalence also provides the existence of a solution for Problem P_M^n, as well as its uniqueness at least under a mild restriction on τ: $\tau < 4/C_k$. This is due to the existence and uniqueness for the conformal problem that is equivalent to Problem P_M^n. In the case of a Lipschitz continuous nonlinearity $b(\cdot)$, these results are proven for example in [9], Chapter 4. If $b(\cdot)$ is only Hölder continuous, the existence can be obtained by approximating it in the $C^{0,\alpha}$ norm by a family of Lipschitz continuous functions $b_\delta(\cdot)$. The resulting regularized problems have unique solutions that are uniformly bounded in $H_0^1(\Omega)$. The weak limit u of a sequence $\{u_{\delta_n}\}_{n\in\mathbb{N}}$ with $\delta_n \searrow 0$ is solving Problem P_M^n, and uniqueness follows by standard energy arguments. We omit the details here and only mention that, as a result of this equivalence, p^n has a better regularity:

$$p^n \in H_0^1(\Omega). \tag{3.3}$$

In what follows we will make use of the elementary result below.

PROPOSITION 3.1. *For any vectors* $\mathbf{a}_k \in \mathbb{R}^d$ $(k \in \{1, \dots, N\}, d \geq 1)$ *we have*

$$2\sum_{n=1}^{N} \mathbf{a}_n \sum_{k=1}^{n} \mathbf{a}_k = \left(\sum_{n=1}^{N} \mathbf{a}_n\right)^2 + \sum_{n=1}^{N} (\mathbf{a}_n)^2. \tag{3.4}$$

To prove the convergence of the semidiscrete scheme (3.1)-(3.2), we use the following stability estimates:

LEMMA 3.2. *Assuming (A1)-(A5), we have*

$$\tau \sum_{n=1}^{N} \|p^n\|_1^2 + \tau \sum_{n=1}^{N} \|\mathbf{q}^n\|^2 \leq C, \tag{3.5}$$

$$\sum_{n=1}^{N} \langle b(p^n) - b(p^{n-1}), p^n - p^{n-1} \rangle + \tau \max_{n=1,\dots,N} \|\mathbf{q}^n\|^2$$

$$+\tau \sum_{n=1}^{N} \|\mathbf{q}^n - \mathbf{q}^{n-1}\|^2 \leq C\tau, \tag{3.6}$$

$$\tau \sum_{n=1}^{N} \|\nabla \cdot \mathbf{q}^n\|^2 \leq C\tau^{\dfrac{2(\alpha-1)}{1+\alpha}}. \tag{3.7}$$

Proof. We test (3.1) with p^n and (3.2) with $\tau \mathbf{q}^n$, add the equalities and sum the resulting up for $n = 1, \dots, N$. This gives

$$\sum_{n=1}^{N} \langle b(p^n) - b(p^{n-1}), p^n \rangle + \tau \sum_{n=1}^{N} \|\mathbf{q}^n\|^2 + \tau \sum_{n=1}^{N} \langle k(b(p^n))\mathbf{e}_z, \mathbf{q}^n \rangle = 0.$$

The three terms in the above are denoted by T_1, T_2 and T_3. To estimate T_1 we notice that if $b(\cdot)$ satisfies the assumption (A2), for any reals x and y we have

$$(b(x) - b(y))x \geq \int_y^x s b'(s) ds, \quad \text{and} \quad \int_0^x s b'(s) ds \geq 0.$$

Together with the Hölder continuity of $b(\cdot)$ this gives

$$\begin{aligned} T_1 &\geq \sum_{n=1}^{N} \int_\Omega \int_{p^{n-1}}^{p^n} s b'(s) ds dx = \int_\Omega \int_0^{p^N} s b'(s) ds dx - \int_\Omega \int_0^{p^0} s b'(s) ds dx \\ &\geq - \int_\Omega \int_0^{p^0} s b'(s) ds dx = \int_\Omega \int_0^{p^0} b(s) - b(p^0) ds dx \geq - \int_\Omega C_b |p^0|^{\alpha+1} dx. \end{aligned}$$

Since $p^0 = u_I \in L^2(\Omega)$, for $\alpha = 1$ we immediately obtain $T_1 \geq -C$. The case $\alpha \in (0, 1)$ is solved by applying the inequality of Hölder

$$|ab| \leq \frac{|a|^p}{p} + \frac{|b|^q}{q} \tag{3.8}$$

with $p = \dfrac{2}{1+\alpha}$ and $q = \dfrac{2}{1-\alpha}$. We obtain again $T_1 \geq -C$ for some constant $C > 0$.

T_2 needs no further treatment, while for T_3 we apply the Cauchy inequality, as well as the inequality of means to prove the estimates for $\mathbf{q}$ in (3.5). For the H^1 estimates for p we first notice that, by (3.3), (3.2) becomes:

$$\langle \nabla p^n, \mathbf{v}\rangle = -\langle \mathbf{q}^n, \mathbf{v}\rangle - \langle k(b(p^n))\mathbf{e}_z, \mathbf{v}\rangle,$$

for all $\mathbf{v} \in H(\mathrm{div};\Omega)$. Since both $\mathbf{q}^n$ and $k(b(p^n))$ are actually L^2, the above inequality holds for any $\mathbf{v} \in (L^2(\Omega))^d$. This, together with the boundedness of $k(\cdot)$, the estimates for $\mathbf{q}^n$ and the inequality of Poincaré completes the proof of (3.5).

To prove (3.6), one simply has to follow the proof of Proposition 3.5 in [15]. Finally, for obtaining the estimate (3.7), we test (3.1) by $\nabla \cdot \mathbf{q}^n$. Applying the Cauchy inequality gives

$$\tau \|\nabla \cdot \mathbf{q}^n\| \leq \left\| b(p^n) - b(p^{n-1})\right\|,$$

yielding straightforwardly

$$\tau \sum_{n=1}^{N} \|\nabla \cdot \mathbf{q}^n\|^2 \leq \frac{1}{\tau} \sum_{n=1}^{N} \left\| b(p^n) - b(p^{n-1})\right\|^2. \tag{3.9}$$

If the Hölder exponent α in (A2) is 1 (thus if $b(\cdot)$ is Lipschitz continuous), the proof is concluded by the estimate in (3.6). For the case $\alpha \in (0,1)$ we first notice that (A2) and (3.6) immediately imply

$$\sum_{j=1}^{n} \left\| b(p^j) - b(p^{j-1})\right\|_{L^{1+1/\alpha}(\Omega)}^{1+1/\alpha} \leq C\tau. \tag{3.10}$$

Further, with $r = \dfrac{2(1-\alpha)}{1+\alpha}$ we use the inequality (3.8) to estimate the sum on the right in (3.9) by

$$\sum_{n=1}^{N} \left\| b(p^n) - b(p^{n-1})\right\|^2 \leq \frac{1}{\tau^r} \sum_{n=1}^{N} \left(\int_\Omega \frac{\tau^{rp}}{p} dx + \frac{1}{q} \|b(p^n) - b(p^{n-1})\|_{L^{2q}(\Omega)}^{2q} \right).$$

With $p = \dfrac{\alpha+1}{1-\alpha}$ and $q = \dfrac{\alpha+1}{2\alpha}$ and recalling (3.10) this gives

$$\sum_{n=1}^{N} \left\| b(p^n) - b(p^{n-1})\right\|^2 \leq C\tau^{1-r},$$

and the rest of the proof is straightforward. ☐

In what follows we prove the convergence of the mixed time discrete scheme (3.1)-(3.2). This extends the result stated in Theorem 4.6 of [15] to the case of Hölder continuous nonlinearities $b(\cdot)$. As following from below, the present demonstration is simplified by giving up the convergence proof for the conformal scheme, as proceeded in [15]. The error estimates are obtained now in a direct manner, which applies to mixed variational formulations with test functions in $L^2(\Omega)$.

For any time dependent function f defined on the interval $[0,T]$, we first introduce the notations:

$$\overline{f}^n = \frac{1}{\tau} \int_{t_{n-1}}^{t_n} f(t)\, dt,$$

$$f_\Delta(t) = f^n, \quad \text{for} \quad t \in (t_{n-1}, t_n],$$

whenever $n \in \{1, \ldots, N\}$. For $n = 0$ we take $\overline{f}^0 = f(0)$. Recalling that $p \in L^2(0, T; H^1(\Omega))$ by the equivalence between the Problems P_C and P_M, and by (3.5), we have

$$\sum_{n=1}^{N} \tau \|\overline{p}^n - p^n\|_1^2 \le \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \|p(t) - p^n\|_1^2 dt \le C. \tag{3.11}$$

This further implies

$$\sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \|\mathbf{q}(t) - \mathbf{q}^n\|^2 dt \le C, \tag{3.12}$$

yielding

$$\sum_{n=1}^{N} \left\| \int_{t_{n-1}}^{t_n} \mathbf{q}(t) - \mathbf{q}^n dt \right\|^2 \le \tau \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \|\mathbf{q}(t) - \mathbf{q}^n\|^2 dt \le C\tau. \tag{3.13}$$

Now we can proceed by estimating the error for the mixed time discrete scheme (3.1)-(3.2). In what follows we assume that τ is sufficiently small, so that the discrete Gronwall lemma can be applied.

Lemma 3.3. *Assuming (A1)-(A5), for any $K = 1, \ldots, N$ we have the estimates*

$$\sum_{n=1}^{K} \int_{t_{n-1}}^{t_n} \langle b(p(t)) - b(p^n), p(t) - p^n \rangle dt + \left\| \sum_{n=1}^{K} \int_{t_{n-1}}^{t_n} (\mathbf{q}(t) - \mathbf{q}^n)\, dt \right\|^2 \le C\tau.$$

Proof. For any $n \in \{1, \ldots, N\}$, (3.1) immediately implies

$$\langle b(p^n) - b(p^0), w \rangle + \tau \langle \nabla \cdot \sum_{k=1}^{n} \mathbf{q}^k, w \rangle = 0, \tag{3.14}$$

for all $w \in L^2(\Omega)$. Further, (2.7) and (2.8) can be rewritten as

$$\langle b(p(t_n)) - b(p(0)), w \rangle + \langle \tau \sum_{k=1}^{n} \nabla \cdot \mathbf{q}^k, w \rangle = 0, \tag{3.15}$$

$$\langle \overline{\mathbf{q}}^n, \mathbf{v} \rangle - \langle \overline{p}^n, \nabla \cdot \mathbf{v} \rangle + \langle \overline{k(b(p))}^n \mathbf{e}_z, \mathbf{v} \rangle = 0, \tag{3.16}$$

for all $w \in L^2(\Omega)$, respectively $\mathbf{v} \in H(\mathrm{div}; \Omega)$. Subtracting now (3.14) from (3.15) and (3.2) from (3.16) and recalling that $p(0) = p^0$ gives

$$\langle b(p(t_n)) - b(p^n), w \rangle + \langle \tau \sum_{k=1}^{n} \nabla \cdot \langle \overline{\mathbf{q}}^k - \mathbf{q}^k), w \rangle = 0, \tag{3.17}$$

$$\langle \overline{\mathbf{q}}^n - \mathbf{q}^n, \mathbf{v} \rangle - \langle \overline{p}^n - p^n, \nabla \cdot \mathbf{v} \rangle + \langle (\overline{k(b(p))}^n - k(b(p^n)))\mathbf{e}_z, \mathbf{v} \rangle = 0, \tag{3.18}$$

for all $w \in L^2(\Omega)$ and $\mathbf{v} \in H(\mathrm{div}; \Omega)$. Taking now $w = \overline{p}^n - p^n$ in (3.17) and

$\mathbf{v} = \tau \sum_{k=1}^{n} (\overline{\mathbf{q}}^k - \mathbf{q}^k) \in H(\mathrm{div};\Omega)$ in (3.18), and adding the resulting yields

$$\begin{aligned} \langle b(p(t_n)) - b(p^n), \overline{p}^n - p^n\rangle + \langle \overline{\mathbf{q}}^n - \mathbf{q}^n, \tau \sum_{k=1}^{n} (\overline{\mathbf{q}}^k - \mathbf{q}^k)\rangle & \\ + \langle (\overline{k(b(p))}^n - k(b(p^n)))\mathbf{e}_z, \tau \sum_{k=1}^{n} (\overline{\mathbf{q}}^k - \mathbf{q}^k)\rangle &= 0. \end{aligned}$$

Summing the above for $n = 1, \ldots, K$ leads to

$$\begin{aligned} \sum_{n=1}^{K} \langle b(p(t_n)) - b(p^n), \overline{p}^n - p^n\rangle + \tau \sum_{n=1}^{K} \langle \overline{\mathbf{q}}^n - \mathbf{q}^n, \sum_{k=1}^{n} (\overline{\mathbf{q}}^k - \mathbf{q}^k)\rangle & \\ + \tau \sum_{n=1}^{K} \langle (\overline{k(b(p))}^n - k(b(p^n)))\mathbf{e}_z, \sum_{k=1}^{n} (\overline{\mathbf{q}}^k - \mathbf{q}^k)\rangle &= 0. \end{aligned} \tag{3.19}$$

We denote by T_1, T_2 and T_3 the terms in the above and estimate them separately. By (3.4), we have

$$T_2 = \frac{\tau}{2} \| \sum_{n=1}^{K} (\overline{\mathbf{q}}^n - \mathbf{q}^n)\|^2 + \frac{\tau}{2} \sum_{n=1}^{K} \|(\overline{\mathbf{q}}^n - \mathbf{q}^n)\|^2. \tag{3.20}$$

To estimate T_1, we first split it as

$$\begin{aligned} T_1 &= \frac{1}{\tau} \sum_{n=1}^{K} \int_{t_{n-1}}^{t_n} \langle b(p(t_n)) - b(p(t)), p(t) - p^n\rangle dt \\ &\quad + \frac{1}{\tau} \sum_{n=1}^{K} \int_{t_{n-1}}^{t_n} \langle b(p(t)) - b(p^n), p(t) - p^n\rangle dt. \end{aligned} \tag{3.21}$$

Denoting the terms above by T_{11} and T_{12}, since $b(\cdot)$ is nondecreasing we immediately get $T_{12} > 0$. For T_{11} we use (3.11) and the regularity of $\partial_t b(p)$ and p, and obtain

$$\begin{aligned} |T_{11}| &= \frac{1}{\tau} \left| \sum_{n=1}^{K} \int_{t_{n-1}}^{t_n} \langle \int_{t}^{t_n} \partial_s b(p)(s) ds, p(t) - p^n\rangle dt \right| \\ &\leq \frac{1}{\tau} \sum_{n=1}^{K} \int_{t_{n-1}}^{t_n} \int_{t}^{t_n} \|\partial_s b(p)\|_{-1} \|p(t) - p^n\|_1 \, ds dt \\ &\leq \frac{1}{2\tau} \sum_{n=1}^{K} \left(\int_{t_{n-1}}^{t_n} \|\partial_s b(p)\|_{-1} \, ds \right)^2 + \frac{1}{2\tau} \sum_{n=1}^{K} \left(\int_{t_{n-1}}^{t_n} \|p(t) - p^n\|_1 dt \right)^2 \\ &\leq \frac{1}{2} \left\{ \int_0^T \|\partial_t b(p)\|_{-1}^2 dt + \sum_{n=1}^{K} \int_{t_{n-1}}^{t_n} \|p(t) - p^n\|_1^2 dt \right\} \quad \leq C. \end{aligned} \tag{3.22}$$

Finally, the convection term T_3 gives

$$\begin{aligned} |T_3| &\leq \sum_{n=1}^{K} \left| \langle (\overline{k(b(p))}^n - k(b(p^n)))\mathbf{e}_z, \tau \sum_{k=1}^{n} (\overline{\mathbf{q}}^k - \mathbf{q}^k)\rangle \right| \\ &\leq \frac{1}{2C_k} \sum_{n=1}^{K} \|\overline{k(b(p))}^n - k(b(p^n))\|^2 + \frac{\tau^2 C_k}{2} \sum_{n=1}^{K} \left\| \sum_{k=1}^{n} (\overline{\mathbf{q}}^k - \mathbf{q}^k) \right\|^2 \\ &=: T_{31} + T_{32}. \end{aligned} \tag{3.23}$$

For T_{31} we use (A3) and obtain

$$
\begin{aligned}
T_{31} &= \frac{1}{2\tau^2 C_k}\sum_{n=1}^{K}\int_\Omega\left[\int_{t_{n-1}}^{t_n} k(b(p(t))) - k(b(p^n))dt\right]^2 dx \\
&\le \frac{1}{2\tau C_k}\sum_{n=1}^{K}\int_\Omega\int_{t_{n-1}}^{t_n}(k(b(p(t))) - k(b(p^n)))^2 dtdx \\
&\le \frac{1}{2\tau}\sum_{n=1}^{K}\int_{t_{n-1}}^{t_n}\langle b(p(t)) - b(p^n), p(t) - p^n\rangle dt.
\end{aligned}
\tag{3.24}
$$

Using (3.20)-(3.24) into (3.20) gives

$$
\begin{aligned}
\frac{1}{2\tau}\sum_{n=1}^{K}\int_{t_{n-1}}^{t_n}\langle b(p(t)) - b(p^n), p(t) - p^n\rangle dt + \frac{\tau}{2}\left\|\sum_{n=1}^{K}(\overline{\mathbf{q}}^n - \mathbf{q}^n)\right\|^2 \\
+\frac{\tau}{2}\sum_{n=1}^{K}\|(\overline{\mathbf{q}}^n - \mathbf{q}^n)\|^2 \le C + \frac{\tau^2 C_k}{2}\sum_{n=1}^{K}\left\|\sum_{k=1}^{n}(\overline{\mathbf{q}}^k - \mathbf{q}^k)\right\|^2,
\end{aligned}
$$

and the result follows by applying the discrete Gronwall lemma. ∎

Remark 3.4. *As following from the Gronwall lemma, the constant C appearing in the estimates proven above is depending exponentially on T. In the absence of convection there is no need for the Gronwall lemma, and the constant C does not depend on T anymore. In particular we obtain:*

$$
\left\|\int_0^t \mathbf{q}(s) - \mathbf{q}_\Delta ds\right\|^2_{L^\infty(0,T;L^2(\Omega))} \le C\tau. \tag{3.25}
$$

Remark 3.5. *By (A2), the estimates in Lemma 3.3 immediately imply*

$$
\int_0^T \|b(p(t)) - b(p_\Delta(t))\|^{1+1/\alpha}_{L^{1+1/\alpha}(\Omega)} dt \le C\tau. \tag{3.26}
$$

The estimates in Lemma 3.3 can be improved under stronger assumptions on $b(u)$, and by ruling out the fast diffusion case. Specifically, we have the following:

Corollary 3.6. *Assuming*

$$
\partial_t b(u) \in L^2(0,T;L^2(\Omega)) \quad \text{and} \quad b'(\cdot) \ge C_{inf} > 0, \tag{3.27}
$$

the estimates in Lemma 3.3 become optimal:

$$
\sum_{n=1}^{K}\int_{t_{n-1}}^{t_n}\langle b(p(t)) - b(p^n), p(t) - p^n\rangle dt + \left\|\sum_{n=1}^{K}\int_{t_{n-1}}^{t_n}(\mathbf{q}(t) - \mathbf{q}^n)\, dt\right\|^2 \le C\tau^2,
$$

for any $K = 1, \ldots, N$.

Proof. The proof follows the ideas in the demonstration of Lemma 3.3. The regularity of $\partial_t b(u)$ allows estimating T_{11} as

$$
|T_{11}| = \frac{1}{\tau}\left|\sum_{n=1}^{K}\int_{t_{n-1}}^{t_n}\langle\int_t^{t_n}\partial_s b(p)(s)ds, p(t) - p^n\rangle dt\right|
$$

$$\leq \sum_{n=1}^{K} \int_{t_{n-1}}^{t_n} \int_t^{t_n} \frac{1}{2\delta} \|\partial_s b(p(s))\|^2 + \frac{\delta}{2\tau^2} \|p(t) - p^n\|^2 \, ds dt$$

$$\leq \frac{\tau}{2C_{inf}} \|\partial_s b(p)\|^2_{L^2(0,T;L^2(\Omega))} + \sum_{n=1}^{K} \frac{C_{inf}}{2\tau} \int_{t_{n-1}}^{t_n} \|p(t) - p^n\|^2 dt.$$

By (3.27), the first term on the right in the above is bounded by τC for some $C > 0$. Further, for dealing with the remaining term we estimate from below the last term in (3.21) as follows

$$\frac{1}{\tau} \sum_{n=1}^{K} \int_{t_{n-1}}^{t_n} \langle b(p(t)) - b(p^n), p(t) - p^n \rangle dt \geq$$
$$\frac{1}{2\tau} \sum_{n=1}^{K} \int_{t_{n-1}}^{t_n} \langle b(p(t)) - b(p^n), p(t) - p^n \rangle dt + \frac{C_{inf}}{2\tau} \sum_{n=1}^{K} \int_{t_{n-1}}^{t_n} \|p(t) - p^n\|^2 dt.$$

Now the proof can be completed exactly as proceeded for Lemma 3.3. ☐

In the following we will make use of the following technical lemma:

LEMMA 3.7. *Given a $w \in L^2(\Omega)$, a $\mathbf{v} \in H(\mathrm{div}; \Omega)$ exists such that*

$$\nabla \cdot \mathbf{v} = w \text{ and } \|\mathbf{v}\| \leq \mathrm{C}\|\mathrm{w}\|, \tag{3.28}$$

with $C > 0$ not depending on w.

Proof. Let u be the (weak) solution of the Poisson equation

$$-\Delta u = w, \text{ in } \Omega,$$

and having a vanishing trace on Γ. Testing the above equation by u and recalling the Poincaré inequality, we immediately obtain $\|\nabla u\| \leq C\|w\|$. The result follows now by taking $\mathbf{v} = -\nabla u$. ☐

The estimates in Lemma 3.3 can now be enriched.

LEMMA 3.8. *Assuming (A1)-(A5), for any $K = 1, \ldots, N$ we have*

$$\begin{aligned} \left\| \sum_{n=1}^{K} \int_{t_{n-1}}^{t_n} p(t) - p^n dt \right\|^2 \quad &\leq \quad C \left\| \sum_{n=1}^{K} \int_{t_{n-1}}^{t_n} \mathbf{q}(t) - \mathbf{q}^n dt \right\|^2 \\ &\quad + C \sum_{n=1}^{K} \int_{t_{n-1}}^{t_n} \langle b(p(t)) - b(p^n), p(t) - p^n \rangle dt. \end{aligned}$$

Proof. Subtracting (3.2) from (3.16) and adding the result for $n = 1, \ldots, K$, we end up with

$$\langle \sum_{n=1}^{K} (\overline{\mathbf{q}}^n - \mathbf{q}^n), \mathbf{v} \rangle - \langle \sum_{n=1}^{K} (\overline{p}^n - p^n), \nabla \cdot \mathbf{v} \rangle + \langle \sum_{n=1}^{K} (\overline{k(b(p))}^n - k(b(p^n))) \mathbf{e}_z, \mathbf{v} \rangle = 0,$$

for all $\mathbf{v} \in H(\mathrm{div}; \Omega)$. Further, by Lemma 3.7, a $\mathbf{v} \in H(\mathrm{div}; \Omega)$ exists such that (3.28) holds for $w = \sum_{n=1}^{K} (\overline{p}^n - p^n)$. Taking this $\mathbf{v}$ in the above yields

$$\left\| \sum_{n=1}^{K} (\overline{p}^n - p^n) \right\|^2 = \langle \sum_{n=1}^{K} (\overline{\mathbf{q}}^n - \mathbf{q}^n), \mathbf{v} \rangle + \langle \sum_{n=1}^{K} (\overline{k(b(p))}^n - k(b(p^n))) \mathbf{e}_z, \mathbf{v} \rangle.$$

Since $\|\mathbf{v}\| \leq C\|w\|$, it follows that

$$\left\|\sum_{n=1}^{K}(\overline{p}^n - p^n)\right\|^2 \leq 2C\left\{\left\|\sum_{n=1}^{K}(\overline{\mathbf{q}}^n - \mathbf{q}^n)\right\|^2 + \left\|\sum_{n=1}^{K}(\overline{k(b(p))}^n - k(b(p^n)))\mathbf{e}_z\right\|^2\right\}$$

The last term in the above is estimated by

$$\begin{aligned} T &\leq \frac{1}{\tau^2}\int_\Omega \left(\sum_{n=1}^{K}\int_{t_{n-1}}^{t_n} k(b(p(t))) - k(b(p^n)dt\right)^2 dx \\ &\leq \frac{K}{\tau}\int_\Omega \sum_{n=1}^{K}\int_{t_{n-1}}^{t_n} (k(b(p(t))) - k(b(p^n)dt)^2\, dx \\ &\overset{A3}{\leq} \frac{KC_k}{\tau}\sum_{n=1}^{K}\int_{t_{n-1}}^{t_n} \langle b(p(t)) - b(p^n), p(t) - p^n\rangle dt. \end{aligned}$$

In this way we obtain

$$\begin{aligned} \left\|\sum_{n=1}^{K}(\overline{p}^n - p^n)\right\|^2 \quad &\leq \quad C\frac{1}{\tau^2}\sum_{n=1}^{K}\int_{t_{n-1}}^{t_n} \langle b(p(t)) - b(p^n), p(t) - p^n\rangle dt \\ &\quad +C\left\|\sum_{n=1}^{K}(\overline{\mathbf{q}}^n - \mathbf{q}^n)\right\|^2. \end{aligned} \tag{3.29}$$

The result follows straightforwardly, multiplying the above by τ^2. □

Summarizing the estimates in the lemmas 3.3 and 3.8, we obtain the following theorem:

THEOREM 3.9. *Assuming (A1)-(A5), with $(p, \mathbf{q})$ and $(p^n, \mathbf{q}^n)$ solving Problem P_M, respectively Problem P_M^n with $n = 1, \ldots, N$, for any $K = 1, \ldots, N$ we have*

$$\begin{aligned} &\sum_{n=1}^{K}\int_{t_{n-1}}^{t_n} \langle b(p(t)) - b(p^n), p(t) - p^n\rangle dt \\ &\qquad + \left\|\sum_{n=1}^{K}\int_{t_{n-1}}^{t_n}(p(t) - p^n)dt\right\|^2 + \left\|\sum_{n=1}^{K}\int_{t_{n-1}}^{t_n}(\mathbf{q}(t) - \mathbf{q}^n)dt\right\|^2 \leq C\tau. \end{aligned}$$

4. The fully discrete mixed discretization. In this section we proceed by estimating the error for the fully discrete approximation. This is done first for the flux variable $\mathbf{q}$, and then for the p unknown. In doing so we let $\mathcal{T}_h$ be a regular decomposition of $\Omega \subset \mathbb{R}^d$ into closed d-simplices; h stands for the mesh-size (see [8]). Here we assume $\overline{\Omega} = \cup_{T\in\mathcal{T}_h} T$, hence Ω is polygonal. Thus we neglect the errors caused by an approximation of a nonpolygonal domain and avoid an excess of technicalities (a complete analysis in this sense can be found in [10]).

The discrete subspaces $W_h \times V_h \subset L^2(\Omega) \times H(\mathrm{div}; \Omega)$ are defined as

$$\begin{aligned} W_h &:= \{p \in L^2(\Omega) |\ p \text{ is constant on each element } T \in \mathcal{T}_h\}, \\ V_h &:= \{\mathbf{q} \in H(\mathrm{div}; \Omega) |\ \mathbf{q}_{|T} = \mathbf{a} + b\mathbf{x} \ \text{ for all } T \in \mathcal{T}_h\}. \end{aligned} \tag{4.1}$$

So W_h denotes the space of piecewise constant functions, while V_h is the RT_0 space (see [7]). Notice that $\nabla \cdot \mathbf{q} \in W_h$ for any $\mathbf{q} \in V_h$.

In what follows we make use of the usual L^2 projector:

$$P_h : L^2(\Omega) \to W_h, \quad \langle P_h w - w, w_h \rangle = 0, \tag{4.2}$$

for all $w_h \in W_h$. Furthermore, a projector Π_h can be defined on $(H^1(\Omega))^d$ (see [7, p. 131]) such that

$$\Pi_h : (H^1(\Omega))^d \to V_h, \quad \langle \nabla \cdot (\Pi_h \mathbf{v} - \mathbf{v}), w_h \rangle = 0, \tag{4.3}$$

for all $w_h \in W_h$. Following [14], p. 237, this operator can be extended to $H(\mathrm{div}, \Omega)$. For the above operators there holds

$$\begin{aligned} \|w - P_h w\| &\le Ch\|w\|_1, \\ \|\mathbf{v} - \Pi_h \mathbf{v}\| &\le Ch\|\mathbf{v}\|_1 \end{aligned} \tag{4.4}$$

for any $w \in H^1(\Omega)$ and $\mathbf{v} \in (H^1(\Omega))^d$.

The following technical lemma is proven in [22] (see also [19, p. 38]).

Lemma 4.1. *Assuming (A1) and given a $f_h \in W_h$, a $\mathbf{v}_h \in V_h$ exists such that*

$$\nabla \cdot \mathbf{v}_h = f_h \quad \text{and} \quad \|\mathbf{v}_\mathrm{h}\| \le \mathrm{C}\,\|\nabla \cdot \mathbf{v}_\mathrm{h}\|,$$

with $C > 0$ being a constant not depending on h, f_h, or v_h. Further we will make use also of the following stability estimates.

Lemma 4.2. *Assuming (A1)-(A5), and p^n being the first component in the the solution of Problem P_M^n, with $n = 1, \ldots, N$ we have the following estimates*

$$\tau \sum_{n=1}^{N} \|p^n - P_h p^n\|_{L^{1+\alpha}(\Omega)}^{1+\alpha} \le Ch^{1+\alpha}, \tag{4.5}$$

$$\tau \sum_{n=1}^{N} \|p^n - P_h p^n\|_{L^{1+\alpha}(\Omega)}^{1+\alpha} \le C(\tau^2 + h^2 \tau^{\frac{2(\alpha-1)}{1+\alpha}}). \tag{4.6}$$

Proof. For $\alpha = 1$ the proof is straightforward, by using the estimates in (4.4) and (3.5). For $\alpha \in (0, 1)$, using the imbedding Theorem 2.8 in [1], p. 25, the Hölder inequality and (4.4) we obtain

$$\begin{aligned} \tau \sum_{n=1}^{N} \|p^n - P_h p^n\|_{L^{1+\alpha}(\Omega)}^{1+\alpha} &\le C\tau h^{1+\alpha} \sum_{n=1}^{N} \|p^n\|_1^{1+\alpha} \\ &\le Ch^{1+\alpha} \sum_{n=1}^{N} \left(\frac{\tau^{(1-r)s}}{s} + \frac{\tau^{rl}}{l} \|p^n\|_1^{(1+\alpha)l} \right). \end{aligned}$$

With $l = \dfrac{2}{1+\alpha}$, $s = \dfrac{2}{1-\alpha}$ and $r = \dfrac{1}{l}$, the first estimate follows by (3.5).

The second estimate can be proven similarly. We omit the details here. ∎

Let again $n = 1, \ldots, N$. We can now define the fully discrete problems:

Problem $P_M^{n,h}$. Let p_h^{n-1} be given. Find $(p_h^n, \mathbf{q}_h^n) \in W_h \times V_h$ such that

$$\langle b(p_h^n) - b(p_h^{n-1}), w_h \rangle + \tau \langle \nabla \cdot \mathbf{q}_h^n, w \rangle = 0, \tag{4.7}$$

$$\langle \mathbf{q}_h^n, \mathbf{v}_h \rangle - \langle p_h^n, \nabla \cdot \mathbf{v}_h \rangle + \langle k(b(p_h^n))\mathbf{e}_z, \mathbf{v}_h \rangle = 0, \tag{4.8}$$

for all $w_h \in W_h$ and $\mathbf{v}_h \in V_h$.
Initially we take $p_h^0 \in W_h$. Since $P_h b(u_I)$ is constant in any $T \in \mathcal{T}_h$, p_h^0 can be taken piecewise constant. Its restriction to any $T \in \mathcal{T}_h$ should satisfy the condition $b(p_h^0) = P_h b(u_I)$. Moreover, with this choice, for all $w_h \in W_h$, we obtain

$$\langle b(p_h^0), w_h\rangle = \langle b(u_I), w_h\rangle = \langle b(p^0), w_h\rangle.$$

LEMMA 4.3. *Assuming (A1)-(A5), with* $(p^n, \mathbf{q}^n)$ *and* $(p_h^n, \mathbf{q}_h^n)$ *solving Problem* P_M^n*, respectively Problem* $P_M^{n,h}$ *with* $n = 1, \dots, N$*, for any* $K = 1, \dots, N$ *we have*

$$\begin{aligned}&\sum_{n=1}^{K}\left\{\langle b(p^n) - b(p_h^n), p^n - p_h^n\rangle + \|b(p^n) - b(p_h^n)\|_{L^{1+1/\alpha}(\Omega)}^{1+1/\alpha}\right\}\\ &\quad+\tau\left\|\sum_{n=1}^{K}(\Pi_h\mathbf{q}^n - \mathbf{q}_h^n)\right\|^2 \le C\sum_{n=1}^{N}\left\{\|\mathbf{q}^n - \Pi_h\mathbf{q}^n\|^2 + \|P_h p^n - p^n\|_{L^{1+\alpha}(\Omega)}^{1+\alpha}\right\}.\end{aligned} \tag{4.9}$$

Proof. From (4.7) we immediately obtain

$$\langle b(p_h^n) - b(p_h^0), w_h\rangle + \tau\langle \nabla\cdot\mathbf{q}_h^n, w\rangle = 0, \tag{4.10}$$

for all $w_h \in W_h$ and for any $n = 1, \dots, N$. Subtracting (4.10) from (3.14), respectively (4.8) from (3.2), and recalling the definition of the projectors in (4.2) and (4.3), we end up with

$$\langle b(p^n) - b(p_h^n), w_h\rangle + \tau\sum_{j=1}^{n}\langle\nabla\cdot\Pi_h(\mathbf{q}^j - \mathbf{q}_h^j), w_h\rangle = 0 \tag{4.11}$$

$$\langle \mathbf{q}^n - \mathbf{q}_h^n, \mathbf{v}_h\rangle - \langle P_h p^n - p_h^n, \nabla\cdot\mathbf{v}_h\rangle + \langle (k(b(p^n)) - k(b(p_h^n)))\mathbf{e}_z, \mathbf{v}_h\rangle = 0 \tag{4.12}$$

for all $w_h \in W_h$ and $\mathbf{v}_h \in V_h$. With $w_h = P_h p^n - p_h^n \in W_h$, and $\mathbf{v}_h = \tau\sum_{j=1}^{n}(\Pi_h\mathbf{q}^j - \mathbf{q}_h^j) \in V_h$ into (4.11), respectively (4.12), adding the resulting and summing up for $n = 1, \dots, K$ with $K \le N$ gives

$$\begin{aligned}&\sum_{n=1}^{K}\langle b(p^n) - b(p_h^n), P_h p^n - p_h^n\rangle + \tau\sum_{n=1}^{K}\langle\mathbf{q}^n - \mathbf{q}_h^n, \sum_{j=1}^{n}(\Pi_h\mathbf{q}^j - \mathbf{q}_h^j)\rangle\\ &\quad+\tau\sum_{n=1}^{K}\langle (k(b(p^n)) - k(b(p_h^n)))\mathbf{e}_z, \sum_{j=1}^{n}\Pi_h\mathbf{q}^j - \mathbf{q}_h^j\rangle = 0.\end{aligned} \tag{4.13}$$

Now we proceed by estimating separately the terms in the above, denoted T_1, T_2 and T_3. It is worth mentioning that the estimates for $T2$ and $T3$ are obtained as in Proposition 4.10, p. 1470 in [15]. However, since $b(\cdot)$ is only Hölder continuous here, and not necessarily Lipschitz, T_1 requires a special attention. We start by writing

$$\begin{aligned}T_1 &= \sum_{n=1}^{K}\langle b(p^n) - b(p_h^n), p^n - p_h^n\rangle + \sum_{n=1}^{K}\langle b(p^n) - b(p_h^n), P_h p^n - p^n\rangle\\ &=: T_{11} + T_{12}.\end{aligned} \tag{4.14}$$

The first term above is positive. Moreover, by (A2) we have

$$T_{11} \geq \frac{1}{2}\sum_{n=1}^{K}\langle b(p^n)-b(p_h^n),p^n-p_h^n\rangle + \frac{C_b^{-\frac{1}{\alpha}}}{2}\sum_{n=1}^{K}\|b(p^n)-b(p_h^n)\|_{L^{1+\frac{1}{\alpha}}(\Omega)}^{1+\frac{1}{\alpha}}. \tag{4.15}$$

For T_{12} we use the Hölder inequality and obtain with $\delta = (4^\alpha C_b)^{-\frac{1}{1+\alpha}}\left(1+\frac{1}{\alpha}\right)^{\frac{\alpha}{1+\alpha}} > 0$:

$$T_{12} \leq \frac{\delta^{1+\frac{1}{\alpha}}}{1+\frac{1}{\alpha}}\sum_{n=1}^{K}\|b(p^n)-b(p_h^n)\|_{L^{1+\frac{1}{\alpha}}(\Omega)}^{1+\frac{1}{\alpha}} + \frac{\delta^{-(1+\alpha)}}{1+\alpha}\sum_{n=1}^{K}\|P_hp^n-p^n\|_{L^{1+\alpha}(\Omega)}^{1+\alpha}. \tag{4.16}$$

As in Proposition 4.10, p. 1470 in [15], we use (3.4) and rewrite T_2 as

$$\begin{aligned} T_2 = {} & \sum_{n=1}^{K}\langle \mathbf{q}^n - \Pi_h\mathbf{q}^n, \tau\sum_{j=1}^{n}\Pi_h\mathbf{q}^j-\mathbf{q}_h^j\rangle \\ & + \frac{\tau}{2}\left\|\sum_{n=1}^{K}(\Pi_h\mathbf{q}^n-\mathbf{q}_h^n)\right\|^2 + \frac{\tau}{2}\sum_{n=1}^{K}\|\Pi_h\mathbf{q}^n-\mathbf{q}_h^n\|^2. \end{aligned} \tag{4.17}$$

The first term on the right, denoted T_{21}, is estimated by

$$|T_{21}| \leq \frac{1}{2}\sum_{n=1}^{K}\|\mathbf{q}^n-\Pi_h\mathbf{q}^n\|^2 + \frac{\tau^2}{2}\sum_{n=1}^{K}\left\|\sum_{j=1}^{n}(\Pi_h\mathbf{q}^j-\mathbf{q}_h^j)\right\|^2. \tag{4.18}$$

Using (A3), T_3 gives for any $\delta > 0$

$$|T_3| \leq \frac{C_k\delta}{2}\sum_{n=1}^{K}\langle b(p^n)-b(p_h^n),p^n-p_h^n\rangle + \frac{\tau^2}{2\delta}\sum_{n=1}^{K}\left\|\sum_{j=1}^{n}(\Pi_h\mathbf{q}^j-\mathbf{q}_h^j)\right\|^2. \tag{4.19}$$

Inserting (4.14)-(4.19) into (4.13) and choosing δ properly leads to

$$\begin{aligned} & \sum_{n=1}^{K}\langle b(p^n)-b(p_h^n),p^n-p_h^n\rangle + \tau\left\|\sum_{n=1}^{K}(\Pi_h\mathbf{q}^n-\mathbf{q}_h^n)\right\|^2 + \|b(p^n)-b(p_h^n)\|_{L^{1+\frac{1}{\alpha}}(\Omega)}^{1+\frac{1}{\alpha}} \\ & \leq C\sum_{n=1}^{K}\left\{\|\mathbf{q}^n-\Pi_h\mathbf{q}^n\|^2 + \|P_hp^n-p^n\|_{L^{1+\alpha}(\Omega)}^{1+\alpha} + \tau^2\left\|\sum_{j=1}^{n}(\Pi_h\mathbf{q}^j-\mathbf{q}_h^j)\right\|^2\right\}. \end{aligned}$$

Finally, (4.9) follows applying the discrete Gronwall lemma. ☐

The above estimates for the flux error can be completed by estimates in p. To this aim we can proceed as in Proposition 4.12, p. 1472 in [15]. We omit the details here.

LEMMA 4.4. *Under the assumptions of Lemma 4.3, for any $K = 1, \ldots, N$ we have*

$$\begin{aligned} \tau\left\|\sum_{n=1}^{K}(P_hp^n-p_h^n)\right\|^2 \leq C\Bigg\{ & \sum_{n=1}^{K}\langle b(p^n)-b(p_h^n),p^n-p_h^n\rangle \\ & + \tau\left\|\sum_{n=1}^{K}(\Pi_h\mathbf{q}^n-\mathbf{q}_h^n)\right\|^2 + \sum_{n=1}^{K}\|\mathbf{q}^n-\Pi_h\mathbf{q}^n\|^2\Bigg\}. \end{aligned} \tag{4.20}$$

The error estimates between the time discrete and the fully discrete solution provided in Lemma 4.3 and 4.4 can be comprised in the following theorem.

THEOREM 4.5. *Assuming (A1)-(A5), let* $(p^n, \mathbf{q}^n) \in L^2(\Omega) \times H(\mathrm{div}, \Omega)$ *and* $(p_h^n, \mathbf{q}_h^n) \in W_h \times V_h$ *solve the problems* P_M^n, *respectively* $P_M^{n,h}$, *where* $n = 1, \dots, N$. *For any* $K = 1, \dots, N$ *we have*

$$
\begin{aligned}
&\sum_{n=1}^{K} \left\{ \langle b(p^n) - b(p_h^n), p^n - p_h^n \rangle + \|b(p^n) - b(p_h^n)\|_{L^{1+\frac{1}{\alpha}}(\Omega)}^{1+\frac{1}{\alpha}} \right\} \\
&\quad + \tau \left\| \sum_{n=1}^{K} (P_h p^n - p_h^n) \right\|^2 + \tau \left\| \sum_{n=1}^{K} (\Pi_h \mathbf{q}^n - \mathbf{q}_h^n) \right\|^2 \\
&\quad \le C \sum_{n=1}^{K} \left\{ \|\mathbf{q}^n - \Pi_h \mathbf{q}^n\|^2 + \|P_h p^n - p^n\|_{L^{1+\alpha}(\Omega)}^{1+\alpha} \right\}.
\end{aligned} \tag{4.21}
$$

The last term in the above is bounded, as follows from Lemma 4.2. For the first term on the right we make the following additional assumption

(A6) $q^n \in H^1(\Omega)^d$ for all $n = 1, \dots, N$ and $\displaystyle\sum_{n=1}^{N} \tau \|\mathbf{q}^n\|_1^2 \le C\tau^{\frac{-2(1-\alpha)}{1+\alpha}}$.

The assumption above is not too restrictive, since it involves a negative exponent of the time step τ. It is suggested by the estimate (3.7), obtained for $\nabla \cdot \mathbf{q}^n$. Here we assume a similar bound for all partial derivatives of $\mathbf{q}^n$. Notice that (A6) is fulfilled in the case of one spatial dimension, when the spaces $H(\mathrm{div}, \Omega)$ and $H^1(\Omega)$ coincide. Furthermore, the H^1 regularity for $\mathbf{q}^n$ in the multi dimensional case is ensured at least for domains Ω with sufficiently smooth boundaries, and whenever $k(\cdot)$ is differentiable and $b(\cdot)$ is Lipschitz (see for example [9], Chapter 4).

Using now Theorems 3.9 and 4.5, the projection estimates (4.4), as well as the inequality (3.26) and the stability estimates we end up with the error estimates for the fully discrete mixed finite element scheme:

THEOREM 4.6. *Assuming (A1)-(A6), for any* $K = 1, \dots, N$ *we have*

$$
\begin{aligned}
&\sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \|b(p(t)) - b(p_h^n))\|_{L^{1+\frac{1}{\alpha}}(\Omega)}^{1+\frac{1}{\alpha}} dt + \left\| \sum_{n=1}^{K} \int_{t_{n-1}}^{t_n} (p(t) - p_h^n) dt \right\|^2 \\
&\quad + \left\| \sum_{n=1}^{K} \int_{t_{n-1}}^{t_n} (\mathbf{q}(t) - \mathbf{q}_h^n) dt \right\|^2 \le C \left(\tau + h^2 \tau^{\frac{2(\alpha-1)}{1+\alpha}} \right).
\end{aligned} \tag{4.22}
$$

REMARK 4.7. *As in Corollary 3.6, allowing only the slow diffusion case and assuming (A6), we obtain*

$$
\begin{aligned}
&\sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \|b(p(t)) - b(p_h^n))\|_{L^{1+\frac{1}{\alpha}}(\Omega)}^{1+\frac{1}{\alpha}} dt + \left\| \sum_{n=1}^{K} \int_{t_{n-1}}^{t_n} (p(t) - p_h^n) dt \right\|^2 \\
&\quad + \left\| \sum_{n=1}^{K} \int_{t_{n-1}}^{t_n} (\mathbf{q}(t) - \mathbf{q}_h^n) dt \right\|^2 \le C \left(\tau^2 + h^2 \tau^{\frac{2(\alpha-1)}{1+\alpha}} \right).
\end{aligned}
$$

For $\tau = h^{\frac{1+\alpha}{2}}$ *this gives a convergence of order* $h^{1+\alpha}$.

Table 5.1
Numerical results (final time $T = 2$, $m = 2$).

N	τ	h	Error	$\tau^2 + h^{1.5}$	Convergence Order
1	0.333	0.2	1.173610e-04	0.20033172	—
2	0.181818	0.1	3.739292e-05	0.06468056	3.1386
3	0.10526316	0.05	1.305061e-05	0.02226067	2.8652
4	0.0625	0.025	4.713953e-06	0.00785910	2.7685
5	0.037	0.0125	1.675419e-06	0.00276654	2.8136

5. Numerical results. We test the considered numerical scheme on the following equation:

$$\partial_t u^{1/m} - \Delta u = 0, \qquad \text{in } \Omega \times (0, T], \tag{5.1}$$

where $m > 1$ is a given parameter, $\Omega = [0, L] \times [0, L]$ and $T > 0$ the final time. All the computations have been performed in the software package *UG* (see [5]). In this setting, assumption (A2) is fulfilled with $\alpha = 1/m$. The above equation is derived in a straightforward manner from the porous medium equation, a typical slow diffusion model. With appropriate initial and boundary conditions, the equation above admits the similarity solution (see [4]):

$$u(t, x, y) = \frac{1}{t+1}\left[1 - \frac{m-1}{4m^2}\frac{x^2+y^2}{(t+1)^{1/m}}\right]_+^{1/(m-1)}. \tag{5.2}$$

The computations are performed for $m = 2$, $m = 4$, $L = 1$, $T = 2$ and $T = 200$, and the errors are given in the tables 5.1 - 5.4. The initial grid is uniform, with $h = 0.2$. This grid is then refined successively by halving h. Correspondingly, the time step is taken $\tau = h^{(m+1)/2m}$, and we compute the errors as given in (4.22). As revealed from the error tables 5.1 - 5.4, the numerical results are confirming the theoretically estimated convergence order of $\tau^2 + h^{(m+1)/m}$, since the reduction factor is close to $2^{(m+1)/m}$ at each refinement.

In the second numerical example we add a source term to the equation 5.1:

$$\partial_t u^{1/m} - \Delta u = f \quad \text{in} \quad \Omega \times (0, T]. \tag{5.3}$$

with Ω being the unit square. The source term is given by

$$f(t, x, y) = \frac{1}{m}[tx(1-x)y(1-y) + \epsilon]^{1/m-1}x(1-x)y(1-y) + 2tx(1-x) + 2ty(1-y),$$

with ϵ denoting a small regularization parameter. This parameter has been introduced to overcome the difficulties that are due to the degeneracy. In particular, this choice guarantees the convergence of the Newton method, when applied to the emerging nonlinear fully discrete problems (see [16]).

With appropriate initial and boundary conditions, an explicit solution of the equation (5.3) is:

$$u(t, x, y) = tx(1-x)y(1-y) + \epsilon. \tag{5.4}$$

Notice that by introducing the regularization parameter ε, the solution u is bounded away from 0. In his way, the problem is not degenerate anymore. In this case one

TABLE 5.2
Numerical results (final time $T = 200$, $m = 2$).

N	τ	h	Error	$\tau^2 + h^{1.5}$	Convergence Order
1	0.333	0.2	1.915036e-04	0.20033172	—
2	0.181818	0.1	5.991603e-05	0.06468056	3.1962
3	0.10526316	0.05	2.074323e-05	0.02226067	2.8885
4	0.0625	0.025	7.469926e-06	0.00785910	2.7769
5	0.037	0.0125	2.660661e-06	0.00276654	2.8075

TABLE 5.3
Numerical results (final time $T = 2$, $m = 4$).

N	τ	h	Error	$\tau^2 + h^{1.25}$	Convergence Order
1	0.4	0.2	4.075752e-05	0.29374806	—
2	0.25	0.1	1.684891e-05	0.11873413	2.4190
3	0.1538	0.05	6.672659e-06	0.04729798	2.5251
4	0.1	0.025	2.899402e-06	0.01994088	2.3014
5	0.0645	0.0125	1.228340e-06	0.00833988	2.3604

TABLE 5.4
Numerical results (final time $T = 200$, $m = 4$).

N	τ	h	Error	$\tau^2 + h^{1.25}$	Convergence Order
1	0.4	0.2	7.741041e-05	0.29374806	—
2	0.25	0.1	3.131379e-05	0.11873413	2.4721
3	0.1538	0.05	1.225694e-05	0.04729798	2.5548
4	0.1	0.025	5.298637e-06	0.01994088	2.3132
5	0.0645	0.0125	2.244762e-06	0.00833988	2.3604

would expect better estimates. However, these estimates are altered by ε. A numerical scheme exploiting this idea of perturbing the data for preventing the solution to reach the degeneracy values is analyzed in [13]. We mention here that the elliptic case was also treated in [17]. For the above problem, the final computational time is $T = 1$. We took $\varepsilon = 10^{-14}$. Table 5.5 presents the results obtained for $\tau = h$, and both being halved successively. Clearly, the computations are confirming the expected convergence order of h. Further, we also performed computations for $\tau = h^{1.2}$. According to the estimates of Theorem 4.6, the convergence should be of order $h^{1.2}$. This means a reduction of the error by the factor $2^{1.2} \approx 2.297$ at each refinement. The results presented in Table 5.6 are confirming the expectations again.

Conclusions. We have analyzed a numerical scheme for a class of degenerate parabolic equations, including both slow and fast diffusion cases. In particular, the results apply to the Richards' equation modeling the flow in unsaturated porous media. The spatial discretization is mixed and based on the lowest order Raviart - Thomas finite elements, whereas the time stepping is performed by the Euler implicit method. We have proven the convergence of the scheme by estimating the error in terms of the discretization parameters. The numerical experiments agree with the theoretically derived estimates.

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TABLE 5.5
Numerical results (final time $T = 1$, $m = 2$).

N	τ	h	Error	Reduction
1	0.1	0.1	1.035385e-04	–
2	0.05	0.05	4.098669e-05	2.5261
3	0.025	0.025	1.855009e-05	2.2095
4	0.0125	0.0125	8.957746e-06	2.0708
5	0.00625	0.00625	4.443746e-06	2.0158

TABLE 5.6
Numerical results (final time $T = 1$, $m = 2$).

N	τ	h	Error	Reduction
1	0.0625	0.1	8.042695e-05	–
2	0.0278	0.05	2.747263e-05	2.9275
3	0.012	0.025	1.025398e-05	2.6792
4	0.0052	0.0125	4.125775e-06	2.4853
5	0.0023	0.00625	1.750638e-06	2.3567

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