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Approximation of $W^{2,2}$ isometric immersions by smooth isometric immersions

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by

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Approximation of flat $W^{2,2}$ isometric immersions by smooth ones

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Abstract

Let $S \subset \mathbb{R}^2$ be a bounded Lipschitz domain and denote by $W^{2,2}_{\mathrm{iso}}(S;\mathbb{R}^3)$ the set of mappings $u \in W^{2,2}(S;\mathbb{R}^3)$ which satisfy $(\nabla u)^T(\nabla u) = Id$ almost everywhere. Under an additional regularity condition on the boundary ∂S (which is satisfied if ∂S is piecewise continuously differentiable) we prove that the strong $W^{2,2}$ closure of $W^{2,2}_{\mathrm{iso}}(S;\mathbb{R}^3) \cap C^\infty(\overline{S};\mathbb{R}^3)$ agrees with $W^{2,2}_{\mathrm{iso}}(S;\mathbb{R}^3)$.

Contents

1 Introduction

Let $S \subset \mathbb{R}^2$ be a bounded Lipschitz domain. A smooth mapping $u : S \to \mathbb{R}^3$ is called an isometric immersion of S (endowed with the flat metric) if

$$\partial_i u \cdot \partial_j u = \delta_{ij} \text{ for all } i, j = 1, 2 \tag{1}$$

pointwise on S. In this paper we study the Sobolev space

$$W^{2,2}_{\mathrm{iso}}(S;\mathbb{R}^3) = \{u \in W^{2,2}(S;\mathbb{R}^3) : \text{ (1) is satisfied almost everywhere on } S\ \}. \tag{2}$$

In order to state our main result, let us say that a bounded Lipschitz domain $S \subset \mathbb{R}^2$ satisfies condition $(*)$ if there is a closed subset $\Sigma \subset \partial S$ with $\mathcal{H}^1(\Sigma) = 0$ such that the outer unit normal $\hat{\nu}$ to S exists and is continuous on $\partial S \setminus \Sigma$. Condition $(*)$ is satisfied, e.g., if ∂S is piecewise continuously differentiable. Our main result is this:

Theorem 1 *Let $S \subset \mathbb{R}^2$ be a bounded Lipschitz domain satisfying condition* $(*)$. *Then the strong $W^{2,2}$ closure of*

$$W^{2,2}_{iso}(S;\mathbb{R}^3) \cap C^\infty(\overline{S};\mathbb{R}^3)$$

agrees with $W^{2,2}_{iso}(S;\mathbb{R}^3)$.

A similar result was obtained in [13] for convex domains S. The main contribution of the present paper therefore is to extend the density result to quite general domains S. There are two motivations for this: The first motivation comes from applications of Theorem 1 in nonlinear elasticity. In fact, the space (2) arises naturally in nonlinear elasticity; more precisely, $W^{2,2}$ isometric immersions form the natural class of admissible deformations in Kirchhoff's plate theory as derived in [3]. Density results like Theorem 1 are used in the derivation of related thin-film theories from nonlinear three dimensional elasticity, cf. [1] and [16], as well as [14]. The second motivation arises from the observation that the construction in [13] for convex domains S heavily depends on the convexity of S; it breaks down completely even when S is an annulus, but also for generic simply connected domains. This is because it uses a global construction. It is a natural question whether or not one can construct smooth *local* approximations satisfying appropriate local boundary conditions, and then glue together these local approximations.
This is the approach taken in the present paper. In order to outline the proof of Theorem 1, we need to recall some facts about the space (2). As in [7], we introduce the directed distance function ν by setting

$$\nu(x,\mu) = \inf\{\theta > 0 : x + \theta\mu \notin S\} \tag{3}$$

for all $(x,\mu) \in S \times (\mathbb{R}^2 \setminus \{0\})$. Observe that the open line segment with endpoints $x \pm \nu(x, \pm\mu)\mu$ is just the maximal subinterval of the line $x + \mathbb{R}\mu$ contained in S and which itself contains the point x. For $u \in C^1(S;\mathbb{R}^3)$ define

$$C_{\nabla u} = \{x \in S : \nabla u \text{ is constant in a neighbourhood of } x\};$$

this is the set of points on which the surface u is locally a plane. The following theorem is the basis of all results in the present paper.

Theorem 2 *If $u \in W^{2,2}_{iso}(S;\mathbb{R}^3)$ then $u \in C^1(S;\mathbb{R}^3)$, and ∇u is countably developable in the sense of [7], i.e., there exists a vector field*

$$q_{\nabla u} : S \setminus C_{\nabla u} \to \mathbb{S}^1$$

such that the following is true: For all $x \in S \setminus C_{\nabla u}$ the gradient ∇u is constant on the open line segment

$$[x] := (x - \nu(x, -q_{\nabla u}(x))q_{\nabla u}(x), x + \nu(x, q_{\nabla u}(x))q_{\nabla u}(x)),$$

and we have

$$[x] \cap [y] \neq \emptyset \implies [x] = [y]$$

whenever $x, y \in S \setminus C_{\nabla u}$.

Figure 1 illustrates the level set structure of ∇u according to Theorem 2. Theorem 2 generalizes the classical results of [4], where the same conclusion was obtained under the stronger hypothesis that $u \in C^2$. In above form, Theorem 2 was proven in [13], using ideas from [9]. Recently, their results were generalized in [8]; a generalization in another direction can be found in [15], where it is only assumed that $u \in C^1$ and that the spherical image of $u(S)$ have zero area. The proof of the first

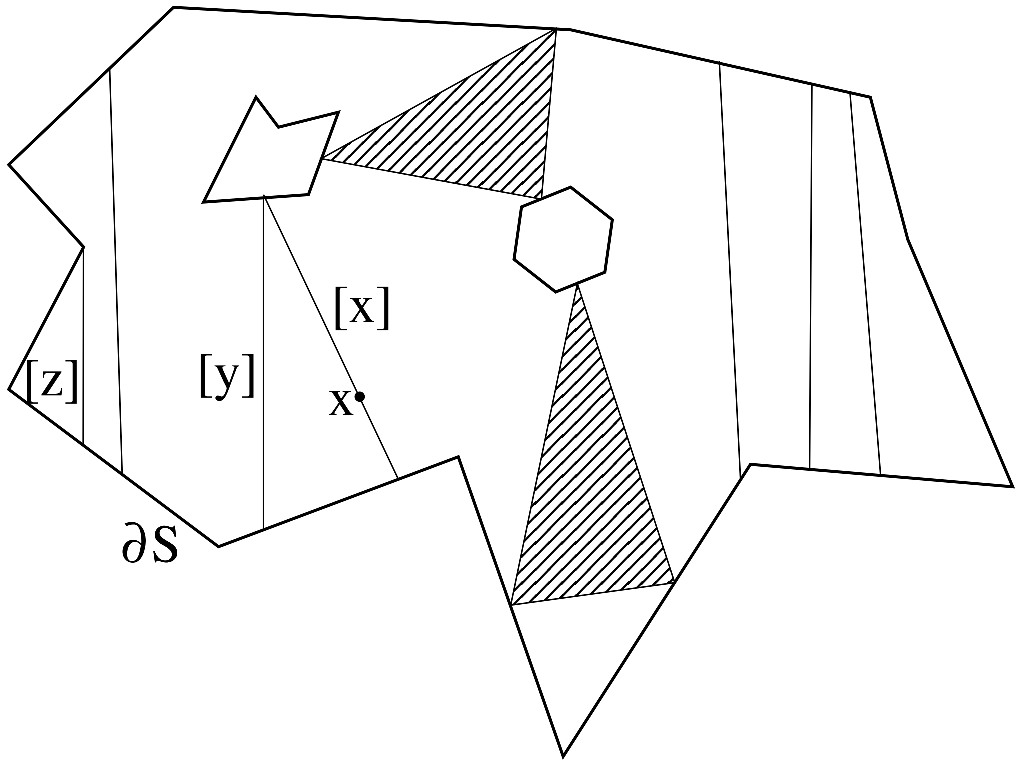

Figure 1: Level set structure of ∇u. The segment of constancy of ∇u passing through x is denoted $[x]$. The shaded regions belong to $C_{\nabla u}$. The segment $[z]$ intersects ∂S tangentially; the segments $[x]$ and $[y]$ intersect each other on ∂S.

part of Theorem 2 (that ∇u is continuous in the interior of S) can be found in [12]; the authors attribute it to Kirchheim. A mapping is countably developable in the sense of [7] precisely if it is continuous and satisfies condition (L) from [12]. Such mappings are studied in [12] and, in some more detail, in [7].

We will now outline the strategy to prove Theorem 1 and explain the differences to the case when the domain is convex. There are three main difficulties which arise in passing from convex domains to domains S as in the hypothesis of Theorem 1:

(a) The domain S need not be simply connected; this creates the need for some topological arguments in order to control the set $C_{\nabla u}$.

(b) There exist $x \in S \setminus C_{\nabla u}$ such that the line segment $[x]$ intersects ∂S tangentially at one endpoint. This has nothing to do with the topology of S and also occurs on generic simply connected domains.

(c) As mentioned earlier, one needs a local construction; this involves finding an appropriate notion of local boundary conditions and being able to construct smooth approximations which satisfy these boundary conditions.

Item (a) is about the set $C_{\nabla u}$. In the convex case, each connected component of this set is a convex polygon, cf. [4], [9]; some facts about this set for general S were proven in [4]. Here we will need the more precise results from [7]. However, we will use them only implicitly because we directly invoke the theorems in [7]. This takes care of the topological difficulties; hence in the present paper we only need to address (b) and (c).

Item (b), i.e., the fact that (on nonconvex domains S) level set segments $[x]$ of ∇u can intersect ∂S tangentially, has two consequences which are relevant in this paper. The first one is that the mapping

$$S \to 2^S,\ x \mapsto [x] \tag{4}$$

is not continuous with respect to Hausdorff distance: it can be discontinuous at those $x \in S \setminus C_{\nabla u}$ for which $[x]$ intersects ∂S tangentially. This leads to some phenomena which are not present at all in the convex case. For instance, it leads to a failure of regularity in [6]. Its relevance for the present paper will be discussed later in this introduction.

Now we discuss the second consequence of (b). It is related to a question which is of some independent interest and which we also address in this paper:

Does $u \in W^{2,2}_{\text{iso}}(S;\mathbb{R}^3)$ admit an *isometric* extension $\tilde{u}$?

More precisely: Does there exist a domain $\tilde{S}$ containing the closure of S and an *isometric immersion* $\tilde{u} \in W^{2,2}_{\text{iso}}(\tilde{S};\mathbb{R}^3)$ which agrees with u on S? In general, the answer to this question is negative; even after suitably localizing it. However, a suitably localized and modified version of it will be answered positively in Section 3, using results from [7].

The nonexistence of isometric extensions has exactly two causes. The first cause can occur even on convex domains. Roughly speaking, it occurs when two different segments $[x]$ and $[y]$ intersect on the boundary ∂S (recall that they never intersect inside S); cf. Figures 1 and 2. If this happens, then it is impossible to extend u as an isometric immersion to larger domain $\tilde{S}$ containing the closure of S, because this would require to extend the segments $[x]$ and $[y]$ as straight line segments which do not intersect inside $\tilde{S}$. That is impossible because they already intersect on ∂S.

The second cause for nonexistence of isometric extensions can only occur if S is nonconvex: Also segments of the kind mentioned in (b) which intersect ∂S tangentially, e.g. the segment denoted by $[z]$ in Figures 1 and 2, can be an obstacle to isometric extension of u. This is not as obvious as in the case of segments intersecting on ∂S, but it should be clear from Figure 2.

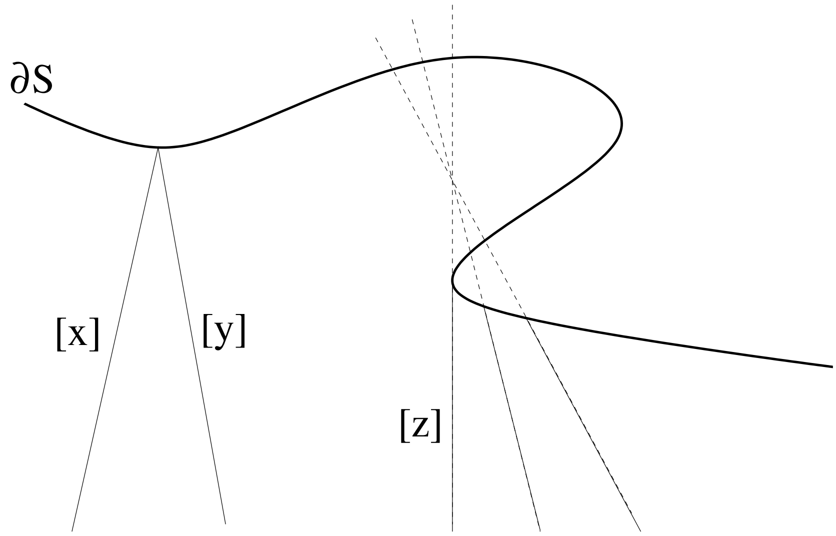

Figure 2: The two reasons why isometric extension can fail: On the left two segments $[x]$ and $[y]$ intersect on the boundary. On the right the segment $[z]$ intersects ∂S tangentially. Enlarging the domain would lead to an inadmissible level set geometry where segments intersect inside the new domain.

Isometric extensions are relevant in this paper on one hand because we wish to construct approximants which are C^∞ up to the boundary (although this would not require *isometric* extensions). On the other hand, isometric extensions are relevant because u which do admit an isometric extension enjoy a 'stable' gradient level set geometry. By this we mean that the segments $[x]$ on which ∇u is constant can be slightly perturbed (i.e. rotated), and this perturbed family of segments still gives rise to (or stems from) a well-defined (modified) isometric immersion. In particular, no two segments in the perturbed family intersect inside S.

Exactly this kind of stability is needed in the smoothing process, because the smoothing will perturb the level set geometry of ∇u. From the previous discussion we therefore see that, before attempting to smoothen u, we must first modify it in a preparatory 'stabilizing' step in such a way that, for instance, no two segments intersect on ∂S. In the convex case addressed in [13] this is achieved by dilating the original level set geometry. This is possible for convex (or strictly star-shaped) domains because then S is contained in dilated versions of itself. Unfortunately, generic domains as in Theorem 1 lack this dilation property; moreover, such a dilation process spoils any kind of local boundary conditions. Hence a completely different approach to this preparatory 'stabilizing' step is needed here.

Finally, we address problem (c), namely the construction of local smooth approximants satisfying appropriate boundary conditions. The corresponding sudomains of S are obtained in Theorem 4 in [7]. It asserts that S can be decomposed, up to a remainder that is irrelevant in this discussion, into finitely many mutually disjoint subdomains $V_1, ..., V_N$, each of which admits a global (on V_k) line of curvature parametrization. We will smoothen u on each V_k separately, thus obtaining smooth local approximants $u^{(k)}$. Naively, in order to glue together these local approximants $u^{(k)}$, one would require

$$(u^{(k)}, \nabla u^{(k)}) = (u, \nabla u) \tag{5}$$

on $S \cap \partial V_k$. The set $S \cap \partial V_k$ is a union of straight (maximal) segments of the form $[x]$. However, a problematic consequence of (4) is that, on nonconvex domains S, the set $S \cap \partial V_k$ can consist of infinitely many such segments, cf. [7] and the arguments in this paper for more details on this. If u is nonsmooth then, in general, there can be no smooth $u^{(k)}$ satisfying (5) on each of these infinitely many line segments. Nevertheless, one can achieve (5) on all *long* segments $[x] \subset S \cap \partial V_k$. Roughly speaking, this is because each V_k has finite perimeter, so there are only finitely many long segments. And we will see that the boundary conditions can be ignored on the infinitely many short segments.

However, there is yet another point in the construction where the junctures $S \cap \partial V_k$ play a role. Namely, we need *smooth* transitions between the local approximants $u^{(k)}$, but the boundary conditions (5) only involve derivatives up to the first order. In order to achieve C^∞-transitions we will require

$$u^{(k)} \text{ is locally affine in a neighbourhood of } S \cap \partial V_k. \tag{6}$$

Here it will not be possible to ignore the short subsegments in $S \cap \partial V_k$, but we will see that (6) can indeed be achieved on all of $S \cap \partial V_k$.

The actual smoothing step involves mollification of normal curvature κ_n and geodesic curvature κ of a line of curvature which determines u on V_k; this was also done in [13]. But clearly a generic mollification spoils the boundary conditions (5) on the long segments. And so does the process leading to (6). Therefore one needs a method to correct such errors. And this 'correction' process should be smooth because otherwise it undoes the mollification step.

We close this introduction by emphasizing that the arguments and results of the present paper rely heavily on the special level set structure of ∇u described in Theorem 2. They also apply to other problems leading to a similar level set geometry (e.g. solutions to the homogeneous Monge-Ampère equation), but they are unlikely to be of much use in problems where this kind of structure is not present. The techniques in this paper are quite elementary because the nice level set structure allows to reduce the central analytic difficulties to problems involving ordinary differential equations. The main results of this paper were announced in [5].

Now follows an overview of the paper. In Section 2 we study local line of curvature patches of u; as mentioned in the introduction, a global approximant of u will be obtained by gluing together local approximants, each of which is defined on such a patch. In Section 3 we derive a positive result regarding the existence of isometric extensions. Section 4 constitutes the analytic core of the argument; it consists of two slightly technical lemmas. The first one, Lemma 3, contains the preliminary 'stabilizing' step mentioned earlier. The second one, Lemma 4 performs the smooth 'correction' procedure. In Section 5 we construct local approximants living on a single line of curvature patch. These are glued together in Section 6 to give the proof of Theorem 1. Section 6 also contains Proposition 5, which is an approximation result of some independent interest. It yields Lipschitz continuous approximants (thus recovering Theorem 2 from [12]). More importantly, the gradient of each approximant enjoys a special level set structure: Each approximant consists of finitely many developable pieces and finitely many planar pieces. In Section 7 we give an example of an irregular $W^{2,2}$ isometric immersion. This example shows that the preliminary 'stabilizing' step carried out in Lemma 3 cannot be replaced by a simple localization. It also shows that naive variations are not possible in [6], and it clarifies some misunderstandings in the literature.

Notation. The letters Γ, γ, $\tilde{\Gamma}$, $\bar{\Gamma}$ etc. will always denote arclength parametrized curves. When making pointwise statements about $f \in L^1_{loc}$ we always refer to the precise representative. We write $f(\Phi)$ to denote the composition $f \circ \Phi$. If $X \subset \mathbb{R}^n$ then $B_r(X)$ is the union of all open balls with radius r centered at some $x \in X$.

2 Line of curvature parametrizations

Throughout this paper $S \subset \mathbb{R}^2$ denotes bounded Lipschitz domain; when assuming $(*)$ this will be stated explicitly. Let $u \in W^{2,2}_{\text{iso}}(S;\mathbb{R}^3)$. In [7], countably developable mappings were studied in some detail. By Theorem 2 all results derived there apply to ∇u when $u \in W^{2,2}_{\text{iso}}(S;\mathbb{R}^3)$. But of course more can be said in the present case. Moreover, certain notions used in [7] have a natural interpretation in the present context. In [7] the set

$$\hat{C}_{\nabla u}$$

was introduced, which by definition is the union of all connected components U of $C_{\nabla u}$ whose relative boundary $S \cap \partial U$ consists of at least three connected components. It was also shown that each connected component of $S \cap \partial U$ is of the form $[x]$ for some $x \in S \setminus C_{\nabla u}$. Clearly the set $\hat{C}_{\nabla u}$ is contained in $C_{\nabla u}$, and this inclusion is generally strict. An important observation is that the vector field $q_{\nabla u}$ provided by Theorem 2 can be extended to the set

$$S \setminus \hat{C}_{\nabla u}$$

in such a way that the conclusion of Theorem 2 remains true when $C_{\nabla u}$ is replaced throughout by $\hat{C}_{\nabla u}$. This is proven in Proposition 9 in [7].
It is shown in [7] that, regarded as a mapping into the projective space $\mathbb{P}^1$, the mapping $q_{\nabla u}$ is unique on $S \setminus C_{\nabla u}$ (cf. Lemma 4 in [7]), but its extension to $S \setminus \hat{C}_{\nabla u}$ is (in general) no longer unique. This non-uniqueness does not create any difficulties for the arguments in this paper. From now on, whenever $u \in W^{2,2}_{\text{iso}}(S;\mathbb{R}^3)$ is given, we will fix one possible extension $q_{\nabla u}$ to $S \setminus \hat{C}_{\nabla u}$.
In [7] we introduced curves $\Gamma \in W^{2,\infty}([0,T];S)$ satisfying $\Gamma'(t) = -q^{\perp}_{\nabla u}(\Gamma(t))$ for all $t \in [0,T]$. Since the ∇u is constant in the direction of $q_{\nabla u}(x)$, it follows that $q_{\nabla u}(x)$ is a principal curvature direction for the surface u corresponding to the principal curvature zero. Thus $\Gamma'(t)$ is a principal curvature direction that corresponds to the nontrivial principal curvature at $\Gamma(t)$. Hence a curve satisfying (7) below is a line of curvature of u. This leads us to the following definition:

Definition 1 *If $u \in W^{2,2}_{iso}(S;\mathbb{R}^3)$ then any curve*

$$\Gamma \in W^{2,\infty}([0,T]; S \setminus \hat{C}_{\nabla u})$$

satisfying (for some extension of $q_{\nabla u}$ to $S \setminus \hat{C}_{\nabla u}$)

$$\Gamma'(t) = -q^{\perp}_{\nabla u}(\Gamma(t)) \text{ for all } t \in [0,T], \text{ and} \tag{7}$$

$$\Gamma'(t) \cdot \Gamma'(t') > 0 \text{ for all } t, t' \in [0,T] \tag{8}$$

is called a line of curvature of u.

Associated with any (arclength parametrized) curve $\Gamma \in W^{2,\infty}((0,T);\mathbb{R}^2)$ is its normal $N = (\Gamma')^{\perp}$ and its curvature $\kappa = \Gamma'' \cdot N$. Its Frénet frame $R = (\Gamma', N)^T$ satisfies the Frénet equation

$$R' = \begin{pmatrix} \Gamma' \\ N \end{pmatrix}' = \begin{pmatrix} 0 & \kappa \\ -\kappa & 0 \end{pmatrix} R. \tag{9}$$

Now let $s^{\pm} : [0,T] \to \mathbb{R}$ be a pair of functions such that $\pm s^{\pm}$ are positive, bounded and lower semicontinuous. We introduce the bounded domain

$$M_{s^{\pm}} = \{(s,t) : t \in (0,T) \text{ and } s \in (s^-(t), s^+(t))\},$$

the mapping $\Phi_\Gamma : M_{s^\pm} \to \mathbb{R}^2$ given by

$$\Phi_\Gamma(s,t) = \Gamma(t) + sN(t),$$

the mappings $\beta_\Gamma^\pm : [0,T] \to \mathbb{R}^2$ given by

$$\beta_\Gamma^\pm(t) = \Gamma(t) + s^\pm(t)N(t),$$

the open line segments

$$\begin{aligned}[]
[\Gamma(t)] &= (\beta_\Gamma^-(t), \beta_\Gamma^+(t)) \\
&= (\Gamma(t) + s^-(t)N(t), \Gamma(t) + s^+(t)N(t)),
\end{aligned}$$

and the bounded domain

$$[\Gamma(0,T)] = \Phi_\Gamma(M_{s^\pm}) = \bigcup_{t\in(0,T)} [\Gamma(t)].$$

The curve Γ is called locally $s^\pm$-admissible if $1 - s^\pm(t)\kappa(t) \geq 0$ for almost every $t \in (0,T)$; it is called $s^\pm$-admissible if

$$[\Gamma(t_1)] \cap [\Gamma(t_2)] = \emptyset \text{ for all unequal } t_1, t_2 \in [0,T],$$

it is called uniformly locally $s^\pm$-admissible if there is $c > 0$ such that

$$1 - s^\pm(t)\kappa(t) \geq c \text{ almost everywhere on } (0,T),$$

and it is called uniformly $s^\pm$-admissible if it is $s^\pm$-admissible and uniformly locally $s^\pm$-admissible.
If Γ takes values in S then we define

$$s_\Gamma^\pm(t) = \pm\nu(\Gamma(t), \pm N(t)) \text{ for all } t \in [0,T],$$

and the above definitions are always understood with $s^\pm = s_\Gamma^\pm$. We will omit the prefix $s^\pm$ if it is clear from the context.

Proposition 15 in [7] implies:

Remark 1 *If $\Gamma \in W^{2,\infty}([0,T]; S \setminus \hat{C}_{\nabla u})$ is a line of curvature of $u \in W^{2,2}_{iso}(S;\mathbb{R}^3)$ then Γ is admissible on $[0,T]$.*

Proposition 1 *Let $S \subset \mathbb{R}^2$ be a bounded Lipschitz domain, let $u \in W^{2,2}_{iso}(S;\mathbb{R}^3)$ and let $\Gamma \in W^{2,\infty}([0,T]; S \setminus \hat{C}_{\nabla u})$ be a line of curvature for u. Set*

$$\gamma = u(\Gamma) \text{ and } v = \nabla u(\Gamma)N \text{ and } n = \gamma' \wedge v$$

and define

$$\kappa := \Gamma'' \cdot N \text{ and } \kappa_n := \gamma'' \cdot n.$$

Then $\kappa \in L^\infty(0,T)$, $\kappa_n \in L^2(0,T)$, $\gamma \in W^{2,2}((0,T);\mathbb{R}^3)$, and $r := (\gamma', v, n)^T \in W^{1,2}((0,T); SO(3))$ solves the system

$$r' = \begin{pmatrix} 0 & \kappa & \kappa_n \\ -\kappa & 0 & 0 \\ -\kappa_n & 0 & 0 \end{pmatrix} r. \tag{10}$$

Moreover,

$$u(\Phi_\Gamma(s,t)) = \gamma(t) + sv(t), \tag{11}$$

$$\nabla u(\Phi_\Gamma(s,t)) = \gamma'(t) \otimes \Gamma'(t) + v(t) \otimes N(t) \tag{12}$$

for all $(s,t) \in \bar{M}_{s_\Gamma^\pm}$.
The functions

$$A_{ij} := (\partial_1 u \wedge \partial_2 u) \cdot \partial_i \partial_j u$$

satisfy

$$\partial_i \partial_j u = A_{ij}(\partial_1 u \wedge \partial_2 u) \tag{13}$$

for $i,j = 1,2$, *and*

$$A(\Phi_\Gamma(s,t)) = \frac{\kappa_n(t)}{1 - s\kappa(t)}(\Gamma'(t) \otimes \Gamma'(t)) \text{ for almost every } (s,t) \in M_{s_\Gamma^\pm}. \tag{14}$$

Moreover,

$$(s,t) \mapsto \frac{\kappa_n^2(t)}{1 - s\kappa(t)} \in L^1(M_{s_\Gamma^\pm}), \tag{15}$$

and

$$\int_{[\Gamma(0,T)]} |\nabla^2 u(x)|^2 \, dx = \int_0^T \Big(\int_{s_\Gamma^-(t)}^{s_\Gamma^+(t)} \frac{\kappa_n^2(t)}{1 - s\kappa(t)} \, ds \Big) \, dt. \tag{16}$$

Remarks.

1. The mapping r is the Darboux-frame of γ (see e.g. [18]). Due to isometry the geodesic curvature $\gamma'' \cdot v$ agrees with the curvature κ of the preimage Γ. The curve γ is a line of curvature of the surface $u(S)$, which is reflected by the vanishing of the geodesic torsion $v' \cdot n$. Of course, A is the second fundamental form of u (up to a sign).

2. The mapping $(s,t) \mapsto \gamma(t) + sv(t)$ is a line of curvature parametrization of $u([\Gamma(0,T)])$. Identifying S and $u(S)$, one can therefore regard Φ_Γ as a (local) line of curvature parametrization.

3. Observe that (15) implies that $\kappa_n = 0$ almost everywhere on the set

$$I_0 = \big\{ t \in (0,T) : \kappa(t) \in \{ \frac{1}{s_\Gamma^-(t)}, \frac{1}{s_\Gamma^+(t)} \} \big\}.$$

For $t \in (0,T) \setminus I_0$ we have

$$\int_{s_\Gamma^-(t)}^{s_\Gamma^+(t)} \frac{1}{1 - s\kappa(t)} \, ds = \begin{cases} -\frac{1}{\kappa(t)} \log \frac{1 - s_\Gamma^+(t)\kappa(t)}{1 - s_\Gamma^-(t)\kappa(t)} & \text{if } \kappa(t) \neq 0 \\ s_\Gamma^+(t) - s_\Gamma^-(t) & \text{otherwise.} \end{cases} \tag{17}$$

Observe that (17) can be estimated from below by

$$\inf \operatorname{dist}_{\partial S}(\Gamma([0,T])) > 0.$$

Therefore, (15) implies that $\kappa_n \in L^2(0,T)$.

Proof. The relation (13) is proven in [2] Proposition 6. Since $u \in C^1(S;\mathbb{R}^3)$ by Theorem 2, and since $\Gamma \in C^1((0,T);S)$, we have $\gamma \in C^1$ with $\gamma' = (\nabla u)(\Gamma)\Gamma'$. Since Φ_Γ is locally Bilipschitz on $M_{s_\Gamma^\pm}$ (cf. Proposition 10 in [7]) we can apply [19] Theorem 2.2.2 to conclude that $(\nabla u)(\Phi_\Gamma) \in W^{1,2}_{loc}(M_{s_\Gamma^\pm};\mathbb{R}^{3\times 2})$ with

$$\partial_t(\partial_i u(\Phi_\Gamma(s,t))) = (1 - s\kappa(t))\partial_i \partial_k u(\Phi_\Gamma(s,t))\Gamma_k'(t) \tag{18}$$

$$\partial_s(\nabla u(\Phi_\Gamma(s,t))) = 0 \tag{19}$$

for almost every $(s,t) \in M_{s_\Gamma^\pm}$ and for $i = 1, 2$. Here we used that

$$\partial_t \Phi_\Gamma(s,t) = (1 - s\kappa(t))\Gamma'(t)$$

and that ∇u is constant on $[\Gamma(t)]$ because Γ is line of curvature in the sense of Definition 1. We sum over repeated indices. A change of variables shows that the right-hand side of (18) is in $L^2(M_{s_\Gamma^\pm})$ (since clearly $(1 - s\kappa) \in L^\infty$). Thus $\nabla u(\Phi_\Gamma) \in W^{1,2}(M_{s_\Gamma^\pm}; \mathbb{R}^{3\times 2})$ by (18, 19). Hence $t \mapsto \nabla u(\Phi_\Gamma(s,t)) \in W^{1,2}((0,T); \mathbb{R}^{3\times 2})$ for almost every $s \in \mathbb{R}$ with $|s| \leq \frac{1}{2}\operatorname{dist}_{\partial S}(\Gamma([0,T]))$. Since $\nabla u(\Phi_\Gamma(s,t))$ does not depend on s and $\Gamma = \Phi_\Gamma(0,\cdot)$, this implies that

$$\nabla u(\Gamma) \in W^{1,2}((0,T); \mathbb{R}^{3\times 2}). \tag{20}$$

Hence by the product rule $\gamma' \in W^{1,2}((0,T); \mathbb{R}^3)$ with

$$\begin{aligned} \gamma'' &= \partial_t(\nabla u(\Gamma)\Gamma') = \nabla^2 u(\Gamma)(\Gamma', \Gamma') + \kappa \nabla u(\Gamma) N \\ &= \Big(A(\Gamma)(\Gamma', \Gamma')\Big) n + \kappa v. \end{aligned} \tag{21}$$

Here we used (13) and that

$$(\partial_1 u \wedge \partial_2 u)(\Phi_\Gamma(s,t)) = \nabla u(\Gamma(t))\Gamma'(t) \wedge \nabla u(\Gamma(t))N(t) = n(t)$$

since $\nabla u(\Phi_\Gamma(s,t))$ does not depend on s. From (21) we deduce $\kappa_n := \gamma'' \cdot n = A(\Gamma)(\Gamma', \Gamma')$. Combining this with (18) and again using (13) and symmetry of A we deduce (14).
By (20) and the product rule we have $v = \nabla u(\Gamma)N \in W^{1,2}$, and thus, again by the product rule, $n \in W^{1,2}$. Equations (11, 12) follow from the definitions. Equation (16) follows from (14), from $|\nabla^2 u|^2 = |A|^2$ and from a change of variables. □

Notation. In what follows, if $r : (0,T) \to SO(3)$ then we define $v = r^T e_2$ and $n = r^T e_3$, where $e_2 = (0,1,0)^T$ and $e_3 = (0,0,1)^T$.

The next proposition will be the basic tool in constructing new isometric immersions from a given one; in the present paper it will be used to construct smooth local approximants, and in [6] it is used to produce local variations of a putative minimizer of a certain functional.

Proposition 2 *Let $\Gamma \in W^{2,\infty}((0,T); \mathbb{R}^2)$, $\kappa_n \in L^2(0,T)$ and let $s^\pm \in L^\infty(0,T)$ be given such that $\pm s^\pm$ are lower semicontinuous and uniformly bounded from below by a positive constant. Denote by $r \in W^{1,2}((0,T); SO(3))$ the solution of (10) with initial value $r(0) = Id$ and set $\gamma(t) := \int_0^t \gamma'$. Define the mapping*

$$(\Gamma, \kappa_n) : [\Gamma(0,T)] \to \mathbb{R}^3$$

by setting

$$(\Gamma, \kappa_n)(\Phi_\Gamma(s,t)) := \gamma(t) + s v(t) \tag{22}$$

for all $(s,t) \in M_{s^\pm}$.

(i) If Γ is $s^\pm$-admissible then (Γ, κ_n) is well defined on $[\Gamma(0,T)]$, and

$$(\Gamma, \kappa_n) \in W^{2,2}_{\text{loc,iso}}([\Gamma(0,T)]; \mathbb{R}^3).$$

Moreover, $\nabla(\Gamma, \kappa_n)(\Phi_\Gamma) \in C^0(\bar{M}_{s^\pm}; \mathbb{R}^{3\times 2})$ with

$$\nabla(\Gamma, \kappa_n)(\Phi_\Gamma(s,t)) = \gamma'(t) \otimes \Gamma'(t) + v(t) \otimes N(t) \tag{23}$$

for all $(s,t) \in \bar{M}_{s^\pm}$.

(ii) If Γ is $s^{\pm}$-admissible and

$$\int_0^T \left(\int_{s^-(t)}^{s^+(t)} \frac{\kappa_n^2(t)}{1 - s\kappa(t)} ds \right) dt < \infty \tag{24}$$

(this is in particular the case if Γ is uniformly admissible) then $(\Gamma, \kappa_n) \in W^{2,2}_{iso}([\Gamma(0,T)]; \mathbb{R}^3)$.

(iii) If $\Gamma \in W^{2,\infty}([0,T]; S)$ is admissible and satisfies (8) then

$$(\Gamma, \kappa_n) \in C^1(S \cap \overline{[\Gamma(0,T)]}; \mathbb{R}^3)$$

and

$$\nabla(\Gamma, \kappa_n)(\Gamma(t) + sN(t)) = \gamma'(t) \otimes \Gamma'(t) + v(t) \otimes N(t)$$

for all $t \in [0,T]$ and all

$$s \in (\underline{s}_\Gamma^-(t), \overline{s}_\Gamma^+(t)),$$

where

$$\overline{s}_\Gamma^+(t) := \limsup_{[0,T] \ni t' \to t} s_\Gamma^+(t') \text{ and } \underline{s}_\Gamma^-(t) := \liminf_{[0,T] \ni t' \to t} s_\Gamma^-(t'). \tag{25}$$

Remarks.

(i) Regarding Proposition 2 (i) notice that, while $\nabla(\Gamma, \kappa_n)(\Phi_\Gamma)$ is continuous up to the boundary of $M_{s^\pm}$, without the hypothesis (24) it is not true in general that $\nabla(\Gamma, \kappa_n)$ is continuous up to the boundary of $[\Gamma(0,T)] = \Phi_\Gamma(M_{s^\pm})$.

(ii) Regarding Proposition 2 (iii), the proof shows that $\nabla(\Gamma, \kappa_n)(\Phi_\Gamma)$ is continuous on the set

$$\hat{M}_{s^\pm} := \bigcup_{t \in [0,T]} (\underline{s}^-(t), \overline{s}^+(t)) \times \{t\}. \tag{26}$$

An extra hypothesis like (8) is needed because otherwise $\nabla(\Gamma, \kappa_n)$ might not have well-defined traces on parts of $S \cap \partial[\Gamma(0,T)]$.

Proof. We omit most indices. Define

$$U : M \to \mathbb{R}^3 \text{ by } U(s,t) := \gamma(t) + sv(t),$$

so $(\Gamma, \kappa_n)(\Phi) = U$. Clearly $U \in W^{1,2}(M; \mathbb{R}^3)$ because of the regularity of γ and v. Let us prove (i). By Proposition 10 in [7], the mapping Φ^{-1} is locally Bilipschitz on $[\Gamma(0,T)]$. So we can apply Theorem 2.2.2 in [19] to conclude that

$$(\Gamma, \kappa_n) := U(\Phi^{-1}) \in W^{1,2}_{loc}([\Gamma(0,T)]; \mathbb{R}^3)$$

with

$$\nabla(\Gamma, \kappa_n) = \nabla U(\Phi^{-1}) \nabla(\Phi^{-1})$$

almost everywhere. Hence (23) holds for almost every (s,t) in M. This follows from an easy calculation using the fact that

$$\nabla(\Phi^{-1})(x) = \Big(\nabla\Phi(\Phi^{-1}(x))\Big)^{-1} \text{ for almost every } x \in [\Gamma(0,T)],$$

(cf. [7]) and the fact that Φ maps $\mathcal{L}^2$-null sets onto $\mathcal{L}^2$-null sets because it is locally Bilipschitz; it also uses

$$\nabla U(s,t) = (v | (1 - s\kappa)\gamma') \text{ and } \Big(\nabla\Phi(s,t)\Big)^{-1} = (N | \frac{1}{1 - s\kappa} \Gamma')^T.$$

So

$$\nabla(\Gamma, \kappa_n)(\Phi) \in W^{1,2}(M;\mathbb{R}^{3\times 2}) \cap C^0(\overline{M};\mathbb{R}^{3\times 2})$$

by the regularity of the right-hand side in (23). We apply Theorem 2.2.2 in [19] again to conclude that

$$\nabla(\Gamma, \kappa_n) \in W^{1,2}_{loc}([\Gamma(0,T)];\mathbb{R}^{3\times 2})$$

with the usual expression for the derivative. So we can take derivatives in (23) and use a change of variables to find that (16) holds with $u = (\Gamma, \kappa_n)$ and $s^{\pm}$ instead of $s^{\pm}_{\Gamma}$. Since Γ is admissible, by Proposition 10 in [7] it is locally admissible. It is easy to check that then

$$\frac{1}{1 - s\kappa} \in L^{\infty}_{loc}(M)$$

Thus f as defined in part (ii) of the statement lies in $L^1_{loc}(M)$. Hence (16) implies that

$$|\nabla^2(\Gamma, \kappa_n)| \in L^2_{loc}(\Phi(M)).$$

If $f \in L^1(M)$ then of course $|\nabla^2(\Gamma, \kappa_n)| \in L^2(\Phi(M))$. And if Γ is uniformly admissible then $1/(1 - s\kappa) \in L^{\infty}(M)$, so then $f \in L^1(M)$ for any choice of $\kappa_n \in L^2(0,T)$.

To prove (iii) note that by Proposition 10 in [7] we have

$$S \cap \overline{[\Gamma(0,T)]} = S \cap \overline{\Phi(M)} = S \cap \Phi(\bar{M}).$$

By the definition of $\hat{M}$ in (26) and since

$$\bar{M}_{s^{\pm}} = \bigcup_{t\in[0,T]} [\underline{s}^-(t), \overline{s}^+(t)] \times \{t\},$$

we have $\Phi(\bar{M} \setminus \hat{M}) \subset \partial S$. In fact, since ∂S is closed and contains $\beta^{\pm}([0,T])$, it also contains $\Gamma(t) + \overline{s}^+(t)N(t)$ and $\Gamma(t) + \underline{s}^-(t)N(t)$ for all $t \in [0,T]$. We conclude that

$$S \cap \overline{[\Gamma(0,T)]} \subset \Phi(\hat{M}).$$

Thus the claim follows from Proposition 15 in [7]. □

2.1 Transversality, regularity of the directed distance and consequences

Recall the definition of ν in (3). A pair $(x, \mu) \in S \times (\mathbb{R}^2 \setminus \{0\})$ is said to be transversal if the line $x + \mathbb{R}\mu$ intersects ∂S transversally in the point $x + \nu(x,\mu)\mu$, i.e. the direction μ is not contained in the tangent cone to ∂S at $x + \nu(x,\mu)\mu$; cf. Definition 5 in [7] for details. The set of all transversal pairs is open, and ν is locally Lipschitz on it, cf. Lemma 12 in [7]. A curve $\Gamma \in W^{2,\infty}([0,T];S)$ is said to be transversal on $[t_0, t_1] \subset [0,T]$ if both $(\Gamma(t), \pm N(t))$ are transversal for every $t \in [t_0, t_1]$. Proposition 12 in [7] implies the following remark:

Remark 2 *If $\Gamma \in W^{2,\infty}([0,T]; S \setminus \hat{C}_{\nabla u})$ is a line of curvature of $u \in W^{2,2}_{iso}(S;\mathbb{R}^3)$ and Γ is transversal, then the set $[\Gamma(0,T)]$ agrees with the connected component of $S \setminus ([\Gamma(0)] \cup [\Gamma(T)])$ containing $\Gamma(0,T)$, and so*

$$S \cap \partial[\Gamma(0,T)] = [\Gamma(0)] \cup [\Gamma(T)]. \tag{27}$$

Lemma 1 *Let $(x,\mu) \in S \times \mathbb{S}^1$ be such that $(x, \pm\mu)$ are transversal. Then there exists $T_{x,\mu} > 0$, depending only on S, x and μ, such that the following holds:*
If $T \in (0, T_{x,\mu}]$ and if $\Gamma : [-T,T] \to S$ is a locally admissible curve satisfying

$$(\Gamma(0), N(0)) = (x, \mu),$$

then Γ is transversal and admissible on $[-T,T]$. Moreover, the following are satisfied:

$$\Gamma'(a) \cdot \Gamma'(b) \geq \frac{1}{2} \text{ for all } a, b \in [-T,T], \tag{28}$$

$$(\beta_\Gamma^\pm)' \cdot \Gamma'(a) \geq \frac{1}{2}(1 - s^\pm \kappa) \tag{29}$$

for all $a \in [-T,T]$ and almost everywhere on $(-T,T)$, and there exist disjoint Lipschitz graphs $G^\pm \subset \partial S$ such that

$$\beta_\Gamma^\pm(t) \in G^\pm \text{ for all } t \in [-T,T].$$

Proof. Since ν is Lipschitz in a neighbourhood of $(x, \pm\mu)$, the proof of Lemma 2.4 in [6] goes through without changes. □

Remark 3 *The proof of Lemma 2.4 in [6] shows the following:*

(i) The following stronger statement than admissibility is satisfied: If $a, b \in [-T,T]$ with $a \neq b$ then $[\Gamma(b)]$ does not intersect the line $\Gamma(a) + \mathrm{span}\{N(a)\}$.

(ii) For all $\varepsilon > 0$ there is $T_\varepsilon > 0$ such that the following is true: If ν is Lipschitz on $B_\varepsilon(x, \pm\mu)$ with

$$\|\nabla \nu\|_{L^\infty(B_\varepsilon(x,\pm\mu);\mathbb{R}^2)} \leq \frac{1}{\varepsilon}$$

and if

$$\mathrm{dist}_{\partial S}(x) \geq \varepsilon$$

then $T_{x,\mu}$ in the conclusion of Lemma 1 can be chosen to be greater than T_ε.

Notation. From now on the symbol $T_{x,\mu}$ (sometimes in the form $T_{\Gamma(t),N(t)}$) always refers to the number in the conclusion of Lemma 1.

3 Isometric extensions and smoothness up to the boundary

A natural question is whether or not a given isometric immersion $u : S \to \mathbb{R}^3$ can be extended *as an isometric immersion* to a larger domain. This question is directly related to the possibility of appropriately extending the vector field $q_{\nabla u}$ to a larger domain. On convex domains, there is only one obstacle to this, namely when $[x]$ and $[y]$ intersect on ∂S. On nonconvex domains there is an additional obstacle, namely the presence of segments $[z]$ which intersect ∂S tangentially. These possibilities are illustrated in Figure 2.
In Proposition 3, the former problem is ruled out by the uniform admissibility assumption, and the latter one by assuming transversality; this is implicit in the condition $T < T_{x,\mu}$. The second part of the proposition links higher regularity of Γ and κ_n to that of the induced isometric immersion (Γ, κ_n).

Proposition 3 *Let $(x, \pm\mu) \in S \times \mathbb{S}^1$ be transversal and let $T \in (0, T_{x,\mu})$. Then the following is true: If $\kappa_n \in L^2(0,T)$ and if $\Gamma \in W^{2,\infty}([0,T]; S)$ is uniformly locally admissible and satisfies $(\Gamma(0), N(0)) = (x, \mu)$, then $(\Gamma, \kappa_n) \in W^{2,2}_{iso}([\Gamma(0,T)]; \mathbb{R}^3)$, and there is a domain $\tilde{S} \subset \mathbb{R}^2$ and a map $\tilde{u} \in W^{2,2}_{iso}(\tilde{S}; \mathbb{R}^3)$ such that*

$$\overline{[\Gamma(0,T)]} \subset \tilde{S} \text{ and } \tilde{u}|_{[\Gamma(0,T)]} = (\Gamma, \kappa_n).$$

Moreover,

(i) If $\kappa_n = 0$ in a neighbourhood of $\{0, T\}$ then there is $\varepsilon > 0$ such that $\tilde{u}$ is affine on $B_\varepsilon([\Gamma(0)] \cup [\Gamma(T)])$.

(ii) If Γ and κ_n are C^∞ and $\kappa_n = 0$ in a neighbourhood of $\{0, T\}$, then $\tilde{u} \in C^\infty(\tilde{S}; \mathbb{R}^3)$. In particular, $(\Gamma, \kappa_n) \in C^\infty(\overline{[\Gamma(0,T)]}; \mathbb{R}^3)$.

Proof. Proposition 2 (ii) implies that $(\Gamma, \kappa_n) \in W^{2,2}_{\text{iso}}([\Gamma(0,T)]; S)$. Denote by $\tilde{\Gamma} \in W^{2,\infty}_{loc}(\mathbb{R}; \mathbb{R}^2)$ the arclength parametrized curve that agrees with Γ on $(0,T)$ and has curvature zero elsewhere. Define $\tilde{\kappa}_n \in L^2(\mathbb{R})$ by setting $\tilde{\kappa}_n := \kappa_n$ on $(0,T)$ and $\tilde{\kappa}_n := 0$ elsewhere. For $\delta > 0$ define $\tilde{s}^\pm_\delta : \mathbb{R} \to \mathbb{R}$ by

$$\pm\tilde{s}^\pm_\delta(t) := \begin{cases} \pm s^\pm_\Gamma(t) + \delta & \text{if } t \in [0,T], \\ \pm s^\pm_\Gamma(0) + \delta & \text{if } t < 0, \\ \pm s^\pm_\Gamma(T) + \delta & \text{if } t > T. \end{cases}$$

Observe that $\tilde{s}^\pm_\delta$ are continuous because so are $s^\pm_\Gamma$ by the transversality of Γ, cf. Lemma 1. By Lemma 1 we also have

$$(\beta^+_\Gamma)' \cdot \Gamma'(0) \geq \frac{1}{2}(1 - s^+_\Gamma \kappa) \text{ on } (0,T).$$

By uniform local admissibility, the right-hand side is bounded from below on $(0,T)$ by a positive constant. Hence $\beta^+_\Gamma \cdot \Gamma'(0)$ is strictly increasing. This and a similar argument for β^-_Γ shows that

$$\beta^\pm_\Gamma : [0,T] \to \mathbb{R}^2 \text{ are injective.}$$

Lemma 1 also implies that $\beta^\pm_\Gamma([0,T])$ are disjoint. Thus we can apply Proposition 11 in [7] to conclude that there is $\delta > 0$ such that $\Phi_{\tilde{\Gamma}}$ is globally Bilipschitz on the set

$$\tilde{M}_\delta := \bigcup_{t \in (-\delta, T+\delta)} \left(\tilde{s}^-_\delta(t), \tilde{s}^+_\delta(t)\right) \times \{t\},$$

that $\tilde{\Gamma}$ is uniformly $\tilde{s}^\pm_\delta$-admissible on $[-\delta, T+\delta]$, and that $\tilde{S} := \Phi_{\tilde{\Gamma}}(\tilde{M}_\delta)$ is open and contains

$$\Phi_{\tilde{\Gamma}}(\bar{M}) = \overline{\Phi_\Gamma(M)}.$$

Since $\tilde{\Gamma}$ is uniformly $\tilde{s}^\pm_\delta$-admissible, Proposition 2 implies that

$$\tilde{u} := (\tilde{\Gamma}, \tilde{\kappa}_n) \in W^{2,2}_{\text{iso}}\Big(\Phi_{\tilde{\Gamma}}(\tilde{M}_\delta); \mathbb{R}^3\Big).$$

And clearly $(\tilde{\Gamma}, \tilde{\kappa}_n)$ is an extension of (Γ, κ_n).

To prove (i) notice that since $\Phi^{-1}_{\tilde{\Gamma}}$ is uniformly continuous (even Lipschitz) in a neighbourhood of $\overline{[\Gamma(0,T)]}$, for small $\eta > 0$ there is $\varepsilon > 0$ such that the image under $\Phi_{\tilde{\Gamma}}$ of

$$B_\eta\Big((s^-(0), s^+(0)) \times \{0\}\Big)$$

contains $B_\varepsilon([\Gamma(0)])$. But $\tilde\kappa_n = 0$ on $(-\rho,\rho)$ for some $\rho > 0$. So $(\tilde\Gamma,\tilde\kappa_n)(\Phi_{\tilde\Gamma})$ is affine on

$$\bigcup_{t\in(-\rho,\rho)} (\tilde s_\delta^-(t), \tilde s_\delta^+(t)) \times \{t\}.$$

For small η this set contains $B_\eta\Big((s^-(0), s^+(0)) \times \{0\}\Big)$ by continuity of $s^\pm$. Thus $(\tilde\Gamma,\tilde\kappa_n)$ is affine on $B_\varepsilon([\Gamma(0)])$. An analogous argument at T concludes the proof of (i).

To prove (ii) notice that by smoothness of $\tilde\Gamma = \Gamma$ and $\tilde\kappa_n = \kappa_n$ on $\mathbb{R}\setminus\{0,T\}$, we have

$$(\tilde\Gamma,\tilde\kappa_n)(\Phi_{\tilde\Gamma}) \in C^\infty\Big(\bigcup_{t\in(0,T)} (\tilde M_\delta \setminus (\mathbb{R}\times\{0,T\}); \mathbb{R}^3\Big).$$

Since $\Phi_{\tilde\Gamma}$ is a C^∞-diffeomorphism on $\tilde M_\delta \setminus (\mathbb{R}\times\{0,T\})$ (see Proposition 11 in [7]), we conclude

$$(\tilde\Gamma,\tilde\kappa_n) \in C^\infty\Big(\Phi_{\tilde\Gamma}\big(\bigcup_{t\in(0,T)} (\tilde M_\delta \setminus (\mathbb{R}\times\{0,T\})\big); \mathbb{R}^3\Big). \tag{30}$$

But by part (i) we know that $(\tilde\Gamma,\tilde\kappa_n)$ is affine in a neighbourhood of

$$\Phi_{\tilde\Gamma}\Big(\tilde M_\delta \cap (\mathbb{R}\times\{0,T\})\Big).$$

Thus indeed $(\tilde\Gamma,\tilde\kappa_n) \in C^\infty\Big(\Phi_{\tilde\Gamma}(\tilde M_\delta; \mathbb{R}^3)\Big)$. □

4 Local modification of $W^{2,2}$ isometric immersions

The main results of this section are Lemma 3 and Lemma 4. Lemma 3 provides the 'stabilization' step mentioned in the Introduction, and Lemma 4 provides the 'correcting' step.

4.1 Local boundary conditions

In this section $\bar T$ and T will always be positive numbers, and Γ, $\tilde\Gamma$, $\bar\Gamma$, γ etc. are arclength parametrized curves. The curves and frames $\tilde\Gamma$, $\tilde\gamma$, $\tilde R$, $\tilde r$, $\bar\Gamma$, etc. always denote the ones solving (9) and (10) for appropriate initial conditions with given $\tilde\kappa$, $\tilde\kappa_n$ or $\bar\kappa$, $\bar\kappa_n$ replacing κ and κ_n. Conversely, $\tilde\kappa$, $\tilde\kappa_n$ or $\bar\kappa$, $\bar\kappa_n$ are the curvatures found by differentiating given $\tilde\Gamma$, $\tilde\gamma$, $\tilde R$, $\tilde r$, $\bar\Gamma$, etc. Unless stated otherwise, we assume that coordinates are chosen such that $\Gamma(0) = \gamma(0) = 0$ and $R(0)$ and $r(0)$ agree with the 2×2 and with the 3×3 unit matrix, respectively. Unless stated otherwise, we assume that $\tilde\Gamma(0) = \bar\Gamma(0) = \Gamma(0)$ and similarly for γ and for the frames r and R.

Let $\Gamma \in W^{2,\infty}([0,T]; S)$, $\bar\Gamma \in W^{2,\infty}([0,\bar T]; S)$ be admissible and such that (8) and the analogue for $\bar\Gamma$ hold, and let $\kappa_n \in L^2(0,T)$ and $\bar\kappa_n \in L^2(0,\bar T)$. We write

$$(\Gamma,\kappa_n) \sim (\bar\Gamma,\bar\kappa_n)$$

if the following hold:

$$[\bar\Gamma(0)] = [\Gamma(0)] \text{ and } [\bar\Gamma(\bar T)] = [\Gamma(T)] \tag{31}$$
$$[\Gamma(0,T)] = [\bar\Gamma(0,\bar T)] \tag{32}$$
$$(\Gamma,\kappa_n) = (\bar\Gamma,\bar\kappa_n) \text{ on } [\Gamma(0)] \cup [\Gamma(T)] \tag{33}$$
$$\nabla(\Gamma,\kappa_n) = \nabla(\bar\Gamma,\bar\kappa_n) \text{ on } [\Gamma(0)] \cup [\Gamma(T)]. \tag{34}$$

Observe that by Proposition 2 we have $(\Gamma,\kappa_n)\in W^{2,2}_{loc}([\Gamma(0,T)];\mathbb{R}^3)$ and $(\bar\Gamma,\bar\kappa_n)\in W^{2,2}_{loc}([\bar\Gamma(0,\bar T)];\mathbb{R}^3)$. By Proposition 2 (iii) and (8), both (Γ,κ_n) and $(\bar\Gamma,\bar\kappa_n)$ are C^1 on $[\Gamma(0,T)]\cup[\Gamma(0)]\cup[\Gamma(T)]$. Hence (33, 34) make sense pointwise.

Lemma 2 *Let $(x,\pm\mu)\in S\times\mathbb{S}^1$ be transversal, let $\bar T,T\in(0,T_{x,\mu})$ and let $\Gamma\in W^{2,\infty}([0,T];S)$, and $\bar\Gamma\in W^{2,\infty}([0,\bar T];S)$ be locally admissible and such that*

$$(\Gamma(0),N(0))=(\bar\Gamma(0),\bar N(0))=(x,\mu), \tag{35}$$

and assume that

$$\bar\Gamma(\bar T)-\Gamma(T)\parallel N(T) \tag{36}$$

$$\bar R(\bar T)=R(T). \tag{37}$$

Then

$$[\bar\Gamma(0,\bar T)]=[\Gamma(0,T)].$$

Assume, in addition, that $\kappa_n\in L^2(0,T)$ and $\bar\kappa_n\in L^2(0,\bar T)$ and

$$(\bar\gamma(0),\bar r(0))=(\gamma(0),r(0))$$

and

$$\bar\gamma(\bar T)-\gamma(T)=(v(T)\otimes N(T))(\bar\Gamma(\bar T)-\Gamma(T)) \tag{38}$$

$$\bar r(\bar T)=r(T). \tag{39}$$

Then

$$(\Gamma,\kappa_n),(\bar\Gamma,\bar\kappa_n)\in W^{2,2}_{loc,iso}([\Gamma(0,T)];\mathbb{R}^3)$$

and

$$(\bar\Gamma,\bar\kappa_n)\sim(\Gamma,\kappa_n).$$

Proof. By Lemma 1 both Γ and $\bar\Gamma$ are admissible and transversal. Hence Proposition 2 implies that

$$(\Gamma,\kappa_n)\in W^{2,2}_{\mathrm{loc,iso}}([\Gamma(0,T)];\mathbb{R}^3)\cap C^1(S\cap\overline{[\Gamma(0,T)]};\mathbb{R}^3)$$

and a similar fact for $(\bar\Gamma,\bar\kappa_n)$. It is easy to see that (36, 37) imply that

$$[\bar\Gamma(\bar T)]=[\Gamma(T)]\text{ and }[\bar\Gamma(0)]=[\Gamma(0)] \tag{40}$$

by the initial data. Remark 2 implies that the set $[\Gamma(0,T)]$ agrees with the connected component of $S\setminus([\Gamma(0)]\cup[\Gamma(T)])$ containing $\Gamma(0,T)$ and the analogous fact for $\bar\Gamma$. Hence (40) implies that

$$[\Gamma(0,T)]=[\bar\Gamma(0,\bar T)].$$

In fact, $\Gamma(0,T)$ and $\bar\Gamma(0,\bar T)$ intersect the same connected component of $S\setminus([\Gamma(0)]\cup[\Gamma(T)])$ because $\bar\Gamma(0)=\Gamma(0)$ and $\bar\Gamma'(0)=\Gamma'(0)$ by virtue of (35).
Proposition 2 implies that the gradient of (Γ,κ_n) satisfies (23), and that a similar formula is true for the gradient of $(\bar\Gamma,\bar\kappa_n)$. The rest of the proof is the same as that of Lemma 2.9 in [6]. □

4.2 Equivalence of plane curves

For the following it will be useful to introduce the notion of equivalent curves: Let $\Gamma^{(i)} \in W^{2,\infty}((t_0^{(i)}, t_1^{(i)}); \mathbb{R}^2)$, $i = 1, 2$, be parametrized by arclength and let $\tau : [t_0^{(1)}, t_1^{(1)}] \to [t_0^{(2)}, t_1^{(2)}]$ with $\tau' \geq 1/8$ be a Bilipschitz homeomorphism. We say $\Gamma^{(2)}(\tau)$ is equivalent to $\Gamma^{(1)}$ on $[t_0^{(1)}, t_1^{(1)}]$ if

$$(\Gamma^{(2)})'(\tau(t)) = (\Gamma^{(1)})'(t) \tag{41}$$

$$\Gamma^{(2)}(\tau(t)) - \Gamma^{(1)}(t) \parallel N^{(1)}(t) \tag{42}$$

for all $t \in [t_0^{(1)}, t_1^{(1)}]$. This is equivalent to

$$[\Gamma^{(2)}(\tau(t))]^{\mathbb{R}^2} = [\Gamma^{(1)}(t)]^{\mathbb{R}^2} \text{ for all } t \in [t_0^{(1)}, t_1^{(1)}]. \tag{43}$$

In fact, (41, 42) clearly imply (43). Conversely, (43) clearly implies $\Gamma^{(2)}(\tau(t)) \in [\Gamma^{(1)}(t)]^{\mathbb{R}^2}$, which is equivalent to (42). Moreover (43) implies that $N^{(2)}(\tau(t)) \parallel N^{(1)}(t)$, so $(\Gamma^{(2)})'(\tau(t)) \parallel (\Gamma^{(1)})'(t)$ for all t. By continuity of $\Gamma^{(i)}$ this implies that either (41) holds or $(\Gamma^{(2)})'(\tau(t)) = -(\Gamma^{(1)})'(t)$ for all t. But in the latter case one could clearly not have $\Gamma^{(2)}(\tau(t)) \in [\Gamma^{(1)}(t)]^{\mathbb{R}^2}$ for all t.
If $\Gamma^{(2)}(\tau)$ is equivalent to $\Gamma^{(1)}$ then we have

$$\Gamma^{(2)}(\tau(t)) - \Gamma^{(1)}(t) = \theta N^{(1)}(t) \tag{44}$$

for all $t \in [t_0^{(1)}, t_1^{(1)}]$, where $\theta := (\Gamma^{(2)}(t_0^{(2)}) - \Gamma^{(1)}(t_0^{(1)})) \cdot N(t_0^{(1)})$ is constant. Notice that τ is uniquely determined by θ. It is given by

$$\tau(t) = t_0^{(2)} + \int_{t_0^{(1)}}^{t} (1 - \theta\kappa^{(1)}(s))\, ds, \tag{45}$$

and the curvature $\kappa^{(2)}$ of $\Gamma^{(2)}$ satisfies

$$\kappa^{(2)}(\tau) = \frac{\kappa^{(1)}}{\tau'}.$$

In fact, equation (44) is seen by differentiating its left-hand side and using (41, 42) together with the fact that N' is always parallel to Γ'. Differentiation of (44) gives

$$\tau'(\Gamma^{(2)})'(\tau) - (\Gamma^{(1)})' = -\theta\kappa^{(1)}(\Gamma^{(1)})',$$

which due to (41) gives $\tau' = 1 - \theta\kappa^{(1)}$, and (45) follows. The relation between the curvatures follows from their definition.
If $\Gamma^{(1)}([0,T]) \subset S$ (and τ' is uniformly close to 1) then

$$\int_{s^-_{\Gamma^{(2)}}(\tau)}^{s^+_{\Gamma^{(2)}}(\tau)} \frac{1}{1 - s\kappa^{(2)}(\tau)}\, ds = \tau' \cdot \int_{s^-_{\Gamma^{(1)}}}^{s^+_{\Gamma^{(1)}}} \frac{1}{1 - s\kappa^{(1)}}\, ds \text{ a.e. on } (t_0^{(1)}, t_1^{(1)}). \tag{46}$$

In fact, by (44) we have

$$\nu(\Gamma^{(2)}(\tau), \pm N^{(2)}(\tau)) = \mp\theta + \nu(\Gamma^{(1)}, \pm N^{(1)}). \tag{47}$$

A short calculation shows:

$$1 - s^{\pm}_{\Gamma^{(2)}}(\tau)\kappa^{(2)}(\tau) = \frac{1 - s^{\pm}_{\Gamma^{(1)}}\kappa^{(1)}}{1 - \theta\kappa^{(1)}}. \tag{48}$$

Thus $\frac{1 - s^{+}_{\Gamma^{(2)}}(\tau)\kappa^{(2)}(\tau)}{1 - s^{-}_{\Gamma^{(2)}}(\tau)\kappa^{(2)}(\tau)} = \frac{1 - s^{+}_{\Gamma^{(1)}}\kappa^{(1)}}{1 - s^{-}_{\Gamma^{(1)}}\kappa^{(1)}}$. This implies (46).
Finally, if $\Gamma^{(1)}$ is admissible, uniformly admissible or transversal, then the same will clearly be true for $\Gamma^{(2)}$.

4.3 Main technical results

Given $\Gamma \in W^{2,\infty}([0,T];S)$, we introduce the following sets:

$$I_\eta := \{t \in (0,T) : 1 - s_\Gamma^+(t)\kappa(t) > \eta \text{ and } 1 - s_\Gamma^-(t)\kappa(t) > \eta\},$$

and

$$I_0 := \{t \in (0,T) : 1 - s_\Gamma^+(t)\kappa(t) = 0 \text{ or } 1 - s_\Gamma^-(t)\kappa(t) = 0\}. \tag{49}$$

As mentioned in the remark to Proposition 1, we have $\kappa_n = 0$ almost everywhere on I_0 if $(\Gamma, \kappa_n) \in W^{2,2}$.

The next Lemma 3 allows to make Γ *uniformly* admissible on a subinterval on which it is transversal. The issue here is to obtain uniform admissibility on an open interval. The example in Section 7 shows that, in general, no such interval exists a priori; hence one must really modify Γ to achieve uniform admissibility on an interval. In contrast, there always exists some small interval on which Γ is transversal (except in a degenerate case).

Recall that condition $(*)$ ensures the existence of a closed $\mathcal{H}^1$ null set $\Sigma \subset \partial S$ outside which ∂S is C^1; in the rest of this paper, the symbol Σ refers to this set.

Lemma 3 *Assume that S satisfies condition $(*)$, let $(x, \pm\mu) \in S \times \mathbb{S}^1$ be transversal, let*

$$T \in (0, \frac{T_{x,\mu}}{2}) \tag{50}$$

and let $\Gamma \in W^{2,\infty}([0,T];S)$ be locally admissible and such that $(\Gamma(0), N(0)) = (x, \mu)$ and such that

$$\beta_\Gamma^\pm([0,T]) \cap \Sigma = \emptyset \tag{51}$$

and

$$\mathcal{L}^1\big((0,T) \setminus I_0\big) > 0.$$

Let $\kappa_n \in L^2(0,T)$ be such that

$$\int_0^T \left(\int_{s_\Gamma^-(t)}^{s_\Gamma^+(t)} \frac{\kappa_n^2(t)}{1 - s\kappa(t)} ds \right) dt < \infty. \tag{52}$$

Then, for all $\delta \in (0, T/2)$, there are $\bar{T} \in (T-\delta, T+\delta)$ and $\bar{\Gamma} \in W^{2,\infty}([0,\bar{T}];S)$ with

$$(\bar{\Gamma}(0), \bar{R}(0)) = (\Gamma(0), R(0))$$

and $\bar{\kappa}_n \in L^2(0,\bar{T})$ such that $\bar{\Gamma}$ is uniformly locally admissible on $(0,\bar{T})$, and such that

$$(\bar{\Gamma}, \bar{\kappa}_n) \sim (\Gamma, \kappa_n) \text{ and } \|(\bar{\Gamma}, \bar{\kappa}_n) - (\Gamma, \kappa_n)\|_{W^{2,2}([\Gamma(0,T)];\mathbb{R}^3)} < \delta. \tag{53}$$

In particular, $\bar{T} \in (0, T_{x,\mu})$ and $\bar{\Gamma}$ is uniformly admissible and transversal on $[0,\bar{T}]$.

Proof. Lemma 1 implies that Γ is admissible and transversal on $[0,T]$. Hence by (51), Proposition 14 (iii) in [7] implies that

$$\nu \text{ is } C^1 \text{ in a neighbourhood of } \bigcup_{t\in[0,T]} (\Gamma(t), N(t)). \tag{54}$$

Denote by ν_1 the gradient of $\nu(x,v)$ with respect to $x \in \mathbb{R}^2$. Set

$$h(t) := \kappa(t)\chi_{\{\kappa>0\}}(t)\nu_1(\Gamma(t), N(t)) - \kappa(t)\chi_{\{\kappa<0\}}(t)\nu_1(\Gamma(t), -N(t)).$$

For $f \in L^\infty(0,T)$ (to be specified later) let $y : (0,T) \to \mathbb{R}^2$ be the unique Lipschitz continuous solution of

$$y' = (\varepsilon f + h \cdot y)\Gamma' \text{ with } y(0) = 0. \tag{55}$$

Denoting by $X \in W^{1,\infty}((0,T);\mathbb{R}^2)$ the fundamental solution of the homogeneous system $X' = (\Gamma' \otimes h)X$ with initial value $X(0) = I$, the variation of constants formula yields

$$y(t) = \varepsilon X(t) \int_0^t X^{-1}(s)\Gamma'(s)f(s)\, ds. \tag{56}$$

By (56),

$$\|y\|_{L^\infty(0,T)} \leq C\varepsilon \|f\|_{L^1(0,T)} \tag{57}$$

for some constant C independent of f and ε. Define $\tau : [0,T] \to \mathbb{R}$ by setting

$$\tau(t) = t + \int_0^t y' \cdot \Gamma'.$$

If ε is small enough then (57) implies that $\tau' \geq 1/2$ on $(0,T)$. Hence we can define

$$\bar{R} : \tau([0,T]) \to SO(2) \text{ and } \bar{r} : \tau([0,T]) \to SO(3)$$

by setting

$$\bar{R}(\tau(t)) = R(t) \text{ and } \bar{r}(\tau(t)) = r(t) \text{ for all } t \in [0,T]. \tag{58}$$

For all $t \in \tau([0,T])$ define

$$\bar{\Gamma}(t) = \Gamma(0) + \int_0^t \bar{R}_1 \text{ and } \bar{\gamma}(t) = \gamma(0) + \int_0^t \bar{r}_1,$$

where the subindex 1 denotes the first row of the corresponding matrix. By a change of variables one finds

$$\bar{\Gamma}(\tau) = \Gamma + y \text{ and } \bar{\gamma}(\tau(t)) = \gamma(t) + \int_0^t (\gamma' \otimes \Gamma')y' \tag{59}$$

as well as

$$\bar{\kappa}(\tau) = \frac{\kappa}{\tau'} \text{ and } \bar{\kappa}_n(\tau) = \frac{\kappa_n}{\tau'}. \tag{60}$$

Using this and (54) we have

$$s^{\pm}_{\bar{\Gamma}}(\tau(t)) - s^{\pm}_{\Gamma}(t) = \pm\nu_1(\Gamma(t), \pm N(t)) \cdot y(t) + \omega^{\pm}(t) \text{ for all } t \in [0,T] \tag{61}$$

for some $\omega^{\pm} \in C^0(0,T)$. As y, τ, $\bar{\Gamma}$, $\bar{\gamma}$, also $\omega^{\pm}$ depends implicitly on ε. By (54), since $\bar{N}(\tau) = N$ and since $\|y\|_{L^\infty(0,T)} \leq C\varepsilon$, we have

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \sup_{t \in [0,T]} |\omega^{\pm}(t)| = 0.$$

By (60, 61) and the definition of τ',

$$|\tau' - 1| + |\bar{\kappa}(\tau) - \kappa| + |s^+_{\Gamma} - s^+_{\bar{\Gamma}}(\tau)| + |s^-_{\Gamma} - s^-_{\bar{\Gamma}}(\tau)| \leq C\varepsilon \text{ a.e. on } (0,T). \tag{62}$$

Fix a small $\eta \in (0, 1/2)$. Then (62) implies that, for ε small enough (depending on η),

$$1 - s^{\pm}_{\bar{\Gamma}}(\tau)\bar{\kappa}(\tau) \geq \frac{\eta}{2} \text{ on } I_\eta. \tag{63}$$

If $t \in (0,T)$ is such that

$$1 - s^+_{\Gamma}(t)\kappa(t) \leq \eta \tag{64}$$

then $\kappa(t) > 0$ and so

$$(h \cdot y)(t) = \kappa(t)\nu_1(\Gamma(t), N(t)) \cdot y(t).$$

Hence by (61, 55) we have

$$\begin{aligned}\tau'(t)\cdot\Big(1-s^+_{\bar\Gamma}(\tau(t))\bar\kappa(\tau(t))\Big)&-\Big(1-s^+_{\Gamma}(t)\kappa(t)\Big)\\ &=(\tau'(t)-1)-h(t)\cdot y(t)-\omega^+(t)\kappa(t)\\ &=\varepsilon f(t)-\omega^+(t)\kappa(t). \end{aligned}\tag{65}$$

for all t satisfying (64). We will choose f to be identically one on $(0,T)\setminus I_\eta$. Then the right-hand side of (65) is greater than $\frac{\varepsilon}{2}$ for small ε (depending on η) and all t satisfying (64). Since $1-s^+_\Gamma\kappa\geq 0$, this implies that, for small ε, $1-s^+_{\bar\Gamma}(\tau)\bar\kappa(\tau)\geq\frac{\varepsilon}{4}$ on $(0,T)\setminus I_\eta$. From this, from (63) and from an analogous argument with $-$ instead of $+$ we conclude that

$$1-s^\pm_{\bar\Gamma}\bar\kappa\geq\frac{\varepsilon}{4}\text{ on }\tau(0,T)\tag{66}$$

for ε small enough. Thus $\bar\Gamma$ is uniformly locally admissible on $\tau(0,T)$, provided that $f=1$ on $(0,T)\setminus I_\eta$.

Using (59) we see that the constraints (38, 36) — with $\bar T=\tau(T)$ — are equivalent to

$$\int_0^T(\gamma'(t)-v(T)\otimes N(T)\Gamma'(t))\;(\Gamma'(t)\cdot y'(t))\;dt=0\tag{67}$$

$$\Gamma'(T)\cdot y(T)=0.\tag{68}$$

Setting

$$Q:=(\nabla u(\Gamma))^T=\Gamma'\otimes\gamma'+N\otimes v,\tag{69}$$

we have $Q'=\kappa_n\;(\Gamma'\otimes n)$. By definition we have $(\gamma')^T=Q^T\Gamma'$ and $v^T=Q^TN$. Using this together with $y'\parallel\Gamma'$, constraint (67) can be written as

$$\int_0^T\big(Q^T(t)-v(T)\otimes N(T)\big)\;y'(t)\;dt=0,$$

which after partial integration yields

$$\int_0^T(Q'(t))^Ty(t)\;dt=0.\tag{70}$$

In fact, since $y(0)=0$ the boundary term equals $\big(Q^T(T)-v(T)\otimes N(T)\big)\;y(T)=0$ because $y(T)\cdot\Gamma'(T)=0$ by the constraint (68) and by the definition of Q.
Define the linear operator $A:L^2(0,T)\to\mathbb{R}^4$ by setting

$$A(f):=\begin{pmatrix}\int_0^T f(t)\cdot\big[X^{-T}(t)X^T(T)\Gamma'(T)\cdot\Gamma'(t)\big]\;dt\\ \int_0^T f(t)\cdot\big[X^{-T}(t)\int_t^T X^T(s)Q'(s)e_1\;ds\;\cdot\Gamma'(t)\big]\;dt\\ \int_0^T f(t)\cdot\big[X^{-T}(t)\int_t^T X^T(s)Q'(s)e_2\;ds\;\cdot\Gamma'(t)\big]\;dt\\ \int_0^T f(t)\cdot\big[X^{-T}(t)\int_t^T X^T(s)Q'(s)e_3\;ds\;\cdot\Gamma'(t)\big]\;dt\end{pmatrix}.$$

Claim #1. There is $\eta>0$ and $f\in L^\infty(0,T)$ such that $f=1$ on $(0,T)\setminus I_\eta$ and $A(f)=0$.

For the moment, let us take Claim #1 for granted and let f as in the conclusion of Claim #1. By (56), the condition $A(f)=0$ implies that the constraints (68, 70) are satisfied. So (67) is satisfied. Then by (67, 68) and the definitions of $\bar\Gamma$, $\bar\gamma$ and of y, the constraints (36, 38) are satisfied. Moreover, the constraints (37, 39) are satisfied automatically by (58). Since $(\bar\Gamma(0),\bar N(0))=(x,\mu)$, from (50, 62, 66) we deduce that $\bar\Gamma$ is transversal and uniformly admissible on $\tau([0,T])$ by virtue of Lemma 1. By Lemma 2 we have

$$[\bar\Gamma(\tau(0,T))]=[\Gamma(0,T)].$$

In particular, $(\bar{\Gamma}, \bar{\kappa}_n) \in W^{2,2}_{\text{iso}}([\Gamma(0,T)];\mathbb{R}^3)$ by Proposition 2 and by (66). Lemma 2 shows that $(\bar{\Gamma}, \bar{\kappa}_n) \sim (\Gamma, \kappa_n)$.

We claim that there exists a constant C, independent of ε and t, such that

$$\int_{s_{\bar{\Gamma}}^-(\tau(t))}^{s_{\bar{\Gamma}}^+(\tau(t))} \frac{1}{1 - s\bar{\kappa}(\tau(t))}\, ds \leq C\left(1 + \int_{s_{\Gamma}^-(t)}^{s_{\Gamma}^+(t)} \frac{1}{1 - s\kappa(t)}\right) \tag{71}$$

for almost every $t \in (0,T) \setminus I_0$. In fact, by (66) the left-hand side of (71) is finite for all $t \in (0,T)$, and the right-hand side is finite for $t \in (0,T) \setminus I_0$. For all $\eta > 0$ small enough, the estimate (63) implies that on I_η the left-hand side of (71) is bounded by a constant C depending on η, but independent of ε and t; hence (71) is true for all $t \in I_\eta$. Now let $t \in (0,T)$ be such that $1 - s_\Gamma^+(t)\kappa(t) \leq \eta$. Then $f(t) = 1$, so (65) implies

$$1 - s_{\bar{\Gamma}}^+(\tau(t))\bar{\kappa}(\tau(t)) \geq \frac{1}{2}(1 - s_\Gamma^+(t)\kappa(t)). \tag{72}$$

Since η is small we have $\kappa(t) > 0$, so also $\bar{\kappa}(\tau(t)) > 0$. Hence the left-hand side of (72) is less than 1. Thus there exists a universal constant C such that

$$\left|\log\left(1 - s_{\bar{\Gamma}}^+(\tau(t))\bar{\kappa}(\tau(t))\right)\right| \leq C\left(1 + \left|\log\left(1 - s_\Gamma^+(t)\kappa(t)\right)\right|\right).$$

Since $\bar{\kappa}(\tau(t)) > 0$, we trivially have

$$\left|\log\left(1 - s_{\bar{\Gamma}}^-(\tau(t))\bar{\kappa}(\tau(t))\right)\right| \leq C$$

for some constant C independent of ε and t. We conclude that (71) is satisfied for this t. The argument in the case $1 - s_\Gamma^-(t)\kappa(t) \leq \eta$ is similar. Hence (71) is true for all $t \in (0,T) \setminus I_\eta$, and the claim follows.
By (60, 62), for small $\varepsilon > 0$ we clearly have

$$|\bar{\kappa}_n(\tau)| \leq 2|\kappa_n|$$

almost everywhere on $(0,T)$. Hence (71) implies that

$$\int_{s_{\bar{\Gamma}}^-(\tau)}^{s_{\bar{\Gamma}}^+(\tau)} \frac{(\bar{\kappa}_n(\tau))^2}{1 - s\bar{\kappa}(\tau)}\, ds \leq C\left(\kappa_n^2 + \int_{s_\Gamma^-}^{s_\Gamma^+} \frac{\kappa_n^2}{1 - s\kappa} ds\right) \tag{73}$$

almost everywhere on $(0,T)$, because both sides are zero on I_0. On the other hand, one readily checks that

$$\int_{s_{\bar{\Gamma}}^-(\tau)}^{s_{\bar{\Gamma}}^+(\tau)} \frac{(\bar{\kappa}_n(\tau))^2}{1 - s\bar{\kappa}(\tau)}\, ds \to \int_{s_\Gamma^-}^{s_\Gamma^+} \frac{\kappa_n^2}{1 - s\kappa} ds \tag{74}$$

pointwise almost everywhere on $(0,T)$. This is trivial on I_0, and elsewhere it follows from (60, 62). But the right-hand side of (73) belongs to $L^1(0,T)$. Hence Lebesgue's dominated convergence theorem together with (16) and the analogous expression for $(\bar{\Gamma}, \bar{\kappa}_n)$ implies that

$$\int_{[\Gamma(0,T)]} |\nabla^2(\bar{\Gamma}, \bar{\kappa}_n)|^2 \to \int_{[\Gamma(0,T)]} |\nabla^2(\Gamma, \kappa_n)|^2 \text{ as } \varepsilon \to 0 \tag{75}$$

after applying a change of variables involving τ to the left-hand side. From this and using $(\Gamma, \kappa_n) \sim (\bar{\Gamma}, \bar{\kappa}_n)$ together with a Poincaré inequality, we deduce that there is a sequence $\varepsilon_k \to 0$ and there is $\hat{u} \in W^{2,2}([\Gamma(0,T)];\mathbb{R}^3)$ such that

$$(\bar{\Gamma}, \bar{\kappa}_n) \rightharpoonup \hat{u} \text{ weakly in } W^{2,2}([\Gamma(0,T)];\mathbb{R}^3)$$

along this sequence. Hence by Sobolev embedding $(\bar{\Gamma}, \bar{\kappa}_n) \to \hat{u}$ uniformly on $[\Gamma(0,T)]$. By (22) we have $(\bar{\Gamma}, \bar{\kappa}_n)(\Phi_{\bar{\Gamma}}(s, \tau(t))) = \bar{\gamma}(\tau(t)) + s\bar{v}(\tau(t))$. This converges to $\gamma(t) + sv(t) = (\Gamma, \kappa_n)(\Phi_\Gamma(s,t))$ for all $(s,t) \in M_{s_\Gamma^\pm}$. Since, moreover, $\Phi_{\bar{\Gamma}}(s, \tau(t)) \to \Phi_\Gamma(s,t)$ pointwise, by uniform convergence we deduce that also

$$(\bar{\Gamma}, \bar{\kappa}_n)(\Phi_{\bar{\Gamma}}) \to \hat{u}(\Phi_\Gamma) \text{ pointwise on } M_{s_\Gamma^\pm}.$$

Hence

$$\hat{u}(\Phi_\Gamma) = (\Gamma, \kappa_n)(\Phi_\Gamma) \text{ on } M_{s_\Gamma^\pm}.$$

Since Γ is admissible, Φ_Γ is invertible on $M_{s_\Gamma^\pm}$. We conclude that $\hat{u} = (\Gamma, \kappa_n)$ on $[\Gamma(0,T)]$. So

$$(\bar{\Gamma}, \bar{\kappa}_n) \rightharpoonup (\Gamma, \kappa_n) \text{ weakly in } W^{2,2}([\Gamma(0,T)]; \mathbb{R}^3).$$

Together with (75) this proves strong convergence in $W^{2,2}([\Gamma(0,T)]; \mathbb{R}^3)$. This concludes the proof of (53).

Let us finally prove Claim #1. Set

$$\mathcal{L} := \bigcup_{\eta > 0} \{\chi_{I_\eta} f : f \in L^\infty(0,T)\}.$$

This is a dense linear subspace of

$$\{(1 - \chi_{I_0}) f : f \in L^2(0,T)\}.$$

Therefore,

$$A(\mathcal{L}) = A\left(\{(1 - \chi_{I_0}) f : f \in L^2(0,T)\}\right), \tag{76}$$

because the target space is finite dimensional. We claim that

$$A\left(L^2(0,T)\right) \subset A\left(\{(1 - \chi_{I_0}) f : f \in L^2(0,T)\}\right). \tag{77}$$

In fact, let $\tilde{\lambda} := (\lambda_0, \lambda) \in \mathbb{R} \times \mathbb{R}^3$ lie in the orthogonal complement of the right-hand side of (77). That is, $\tilde{\lambda}$ is perpendicular to $A((1 - \chi_{I_0}) f)$ for all $f \in L^2(0,T)$. Then

$$H(t) \cdot \Gamma'(t) = 0 \text{ for all } t \in (0,T) \setminus I_0. \tag{78}$$

Here we have introduced

$$H(t) = X^{-T}(t)\Big(\int_t^T X^T(s) Q'(s) \lambda \, ds + \lambda_0 \; X^T(T) \Gamma'(T)\Big).$$

From the definition of X we see that H satisfies the inhomogeneous linear ODE

$$H' = -(h \otimes \Gamma') \; H - Q'\lambda \tag{79}$$

with the terminal condition $H(T) = \lambda_0 \Gamma'(T)$. Since $\kappa_n = 0$ on I_0, also $Q' = 0$ on I_0. Thus

$$H' = -(h \otimes \Gamma') \; H \text{ almost everywhere on } I_0. \tag{80}$$

One readily checks from the definition of ν that

$$\pm \nu_1(x, \pm N) \cdot N = -1.$$

Hence

$$N \cdot Q'\lambda = 0 \text{ and } h \cdot N = -\kappa.$$

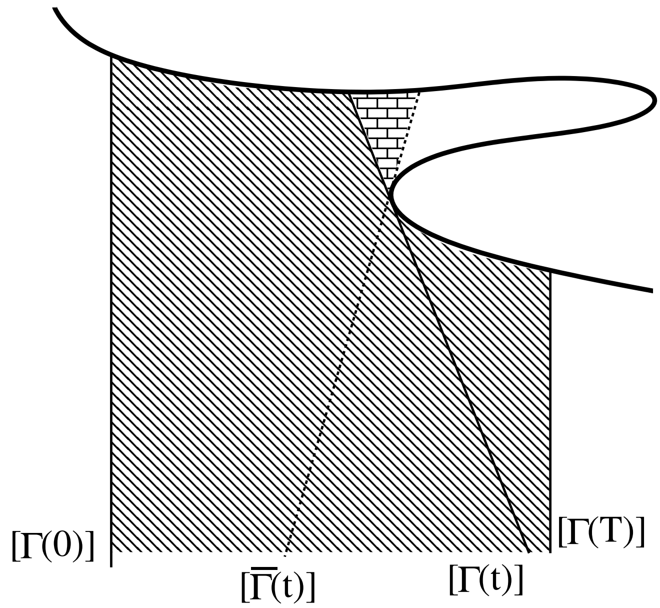

Figure 3: The dashed region is $[\Gamma(0,T)]$. The brick pattern region belongs to $[\bar\Gamma(0,\bar T)]$ but not to $[\Gamma(0,T)]$. This situation must be avoided by prescribing $\bar\Gamma$ on a larger set than $\bar\kappa_n$.

Hence $H'\cdot N = (-N\cdot h)(H\cdot\Gamma') - N\cdot Q'\lambda = \kappa H\cdot\Gamma' = -H\cdot N'$. We conclude that

$$(H\cdot N)' = 0 \text{ on all of } (0,T).$$

Hence $H\cdot N \equiv H(T)\cdot N(T) = 0$ everywhere. Hence by (78) on $(0,T)\setminus I_0$ we have $H=0$. In particular, the open set $\{H\neq 0\}$ must be contained in I_0, so $\kappa_n = 0$ on $\{H\neq 0\}$. And if $H\neq 0$ everywhere, then $(0,T) = I_0$, contradicting the hypotheses. Hence there exists a maximal interval J of $\{H\neq 0\}$ such that at at least one of the two boundary points of J we have $H=0$. But on J the function H satisfies the homogeneous ODE (80), so $H=0$ on J, a contradiction. We conclude that $H=0$ on all of $(0,T)$. Thus $\tilde\lambda$ is perpendicular to the left-hand side of (77). This concludes the proof of (77).
Now by (76) and (77) there are $\eta>0$ and $\tilde f\in L^\infty(0,T)$ such that $A(\chi_{I_\eta}\tilde f) = -A(1)$. Setting $f := 1+\chi_{I_\eta}\tilde f$ we have $f\in L^\infty(0,T)$ and $A(f)=0$. Moreover, $f=1$ on $(0,T)\setminus I_\eta$. This concludes the proof of Claim #1. □

The next lemma states that if Γ is uniformly admissible and transversal on some open interval then L^2-small perturbations of κ and κ_n can be "corrected" by adding compactly supported smooth corrections in such a way that the resulting curve pair $\bar\Gamma$, $\bar\gamma$ is again admissible and satisfies the boundary conditions of the original mapping.

Lemma 4 *Let $\Gamma\in W^{2,\infty}([0,T];S)$ be admissible and such that (8) holds, and let $\kappa_n\in L^2(0,T)$ be such that (52) is satisfied. Assume, moreover, that there exists $t_0\in[0,T]$ such that*

$$(\Gamma(t_0), \pm N(t_0)) \text{ are transversal}$$

and that there is

$$t_1 \in (t_0, t_0 + \frac{T_{\Gamma(t_0),N(t_0)}}{2}) \cap [0,T]$$

such that

$$\mathcal{L}^1\left(\{t\in(t_0,t_1) : \kappa_n(t)\neq 0\right) > 0$$

and

$$\Gamma \text{ is uniformly admissible on } [t_0, t_1].$$

Then there exists $\iota > 0$ *such that the following is true:*
Let $c > 0$ *and let*

$$\{\hat\kappa^\varepsilon\}_{\varepsilon>0} \subset L^\infty(0,T) \text{ and } \{\hat\kappa_n^\varepsilon\}_{\varepsilon>0} \subset L^2(0,T) \tag{81}$$

be one-parameter families with

$$\|\hat\kappa^\varepsilon - \kappa\|_{L^2(0,T)} + \|\hat\kappa_n^\varepsilon - \kappa_n\|_{L^2(0,T)} \to 0 \text{ as } \varepsilon \downarrow 0$$

and which, for all $\varepsilon > 0$*, satisfies*

$$\hat\kappa^\varepsilon \in \Big[\frac{1}{s_\Gamma^-} + c, \frac{1}{s_\Gamma^+} - c\Big] \text{ almost everywhere on } (t_0, t_1), \tag{82}$$

$$\hat\kappa^\varepsilon = \kappa \text{ almost everywhere on } (0,T) \setminus (t_0, t_1), \tag{83}$$

$$|\hat\kappa_n^\varepsilon| \le |\kappa_n| \text{ almost everywhere on } (0,T) \setminus (t_0, t_1). \tag{84}$$

Then for every $\varepsilon > 0$ *there exists* $\tau_\varepsilon \in W^{1,\infty}(0,T)$ *with*

$$\tau_\varepsilon \begin{cases} = id & \text{on } (0, t_0) \\ \text{is affine} & \text{on } (t_0, t_1), \end{cases}$$

and there exist

$$\varphi^\varepsilon,\ \psi^\varepsilon \in C_0^\infty(t_0 + \iota, t_1 - \iota)$$

such that

$$\|\tau_\varepsilon' - 1\|_{L^\infty(0,T)} + \|\varphi^\varepsilon\|_{L^\infty(0,T)} + \|\psi^\varepsilon\|_{L^\infty(0,T)} \to 0 \text{ as } \varepsilon \downarrow 0 \tag{85}$$

and such that with

$$\bar\kappa^\varepsilon(\tau^\varepsilon) := \frac{\hat\kappa^\varepsilon + \varphi^\varepsilon}{\tau'} \text{ and } \bar\kappa_n^\varepsilon(\tau^\varepsilon) := \frac{\hat\kappa_n^\varepsilon + \psi^\varepsilon}{\tau'}, \tag{86}$$

and denoting by $(\bar\Gamma^\varepsilon, \bar R^\varepsilon)$ *the corresponding solution of (9) with initial values* $(\bar\Gamma^\varepsilon(0), \bar R^\varepsilon(0)) = (\Gamma(0), R(0))$*, we have*

$$(\Gamma, \kappa_n),\ (\bar\Gamma^\varepsilon, \bar\kappa_n^\varepsilon) \in W^{2,2}_{iso}([\Gamma(0,T)]; \mathbb{R}^3)$$

and

$$(\bar\Gamma^\varepsilon, \bar\kappa_n^\varepsilon) \sim (\Gamma, \kappa_n)$$

and

$$(\bar\Gamma^\varepsilon, \bar\kappa_n^\varepsilon) \to (\Gamma, \kappa_n) \text{ strongly in } W^{2,2}([\Gamma(0,T)]; \mathbb{R}^3)$$

as $\varepsilon \downarrow 0$*. Moreover,*

$$\bar\Gamma^\varepsilon(\tau^\varepsilon) \begin{cases} = \Gamma & \text{on } (0, t_0) \\ \text{is equivalent to } \Gamma & \text{on } (t_1, T), \end{cases}$$

and $\bar\Gamma^\varepsilon$ *is uniformly admissible and transversal on* $\tau^\varepsilon([t_0, t_1])$*, and*

$$(\bar\Gamma^\varepsilon)'(t) \cdot (\bar\Gamma^\varepsilon)'(t') > 0 \text{ for all } t, t' \in \tau_\varepsilon([0,T]). \tag{87}$$

Remarks.

(i) The boundary values of $(\bar{\Gamma}, \bar{\kappa}_n)$ in general do not agree with those of (Γ, κ_n) on the set

$$\Big(S \cap \partial[\Gamma(0,T)]\Big) \setminus \Big([\Gamma(0)] \cup [\Gamma(T)]\Big). \tag{88}$$

This is because $\hat{\kappa}_n$ need not agree with κ_n outside $[t_0, t_1]$. Observe that the set (88) is in general nonempty because Γ is not assumed to be transversal on all of $[0,T]$.

(ii) Observe that the corrections φ^ε, ψ^ε are not of lower order. Their key property is that they are smooth and have small support. It is important that their support is small even though the perturbation $\hat{\kappa}_n$ of κ_n can be spread over all of $(0,T)$.

(iii) The (a priori nonlocal) effect that a local modification of κ has on Γ is corrected to become also local on the level of Γ (up to equivalence). This is crucial since we only have knowledge about transversality and uniform admissibility of Γ on (t_0, t_1), so an uncontrolled modification outside this interval could spoil admissibility. Due to the lack of transversality outside (t_0, t_1), it could even happen that $[\bar{\Gamma}^\varepsilon(\tau_\varepsilon(0,T))] \neq [\Gamma(0,T)]$, cf. Figure 3.

(iv) Condition (82) cannot be omitted, since uniform admissibility is not stable under general perturbations of κ which are small in L^2 but not small in L^∞. Take e.g. $\Gamma(t) = te_1$, $t \in [0,1]$ on $S = [-2,2]^2$. Then $\kappa = 0$ and Γ is clearly uniformly admissible. Setting $\hat{\kappa}^\varepsilon := 10\chi_{(\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon)}$, it is easy to see that the corresponding curve $\hat{\Gamma}^\varepsilon$ is not admissible for small ε, nor is the curve corresponding to (86).

Proof. We assume without loss of generality that $\Gamma(0) = 0$ and $\gamma(0) = 0$ and that $R(0) = I$ and $r(0) = I$ (where I is the identity matrix in the appropriate dimension). For arbitrary $\tilde{\kappa}, \tilde{\kappa}_n \in L^2(0,T)$ define $(\tilde{\Gamma}, \tilde{R})$ and $(\tilde{\gamma}, \tilde{r})$ in the usual way by (9, 10) with initial values

$$(\tilde{\Gamma}(0), \tilde{R}(0)) = (\Gamma(0), R(0)) \text{ and } (\tilde{\gamma}(0), \tilde{r}(0)) = (\gamma(0), r(0)),$$

and set

$$\tilde{\Gamma}(t) = \int_0^t \tilde{R}^T e_1 \text{ and } \tilde{\gamma}(t) = \int_0^t \tilde{r}^T e_1.$$

In this proof, $U(\kappa)$ denotes a generic $L^2(0,T)$-neighbourhood of κ whose size may change from line to line.

Define the functional

$$L : U(\kappa) \to \mathbb{R}$$

by setting

$$L[\tilde{\kappa}] = \frac{\big(\Gamma(t_1) - \Gamma(t_0)\big) \cdot \Gamma'(t_1)}{\big(\tilde{\Gamma}(t_1) - \Gamma(t_0)\big) \cdot \Gamma'(t_1)} \tag{89}$$

for all $\tilde{\kappa} \in U(\kappa)$. Here, $U(\kappa)$ is an L^2-neighbourhood of κ such that the right-hand side of (89) is well defined for all $\tilde{\kappa} \in U(\kappa)$; such a neighbourhood exists by virtue of (8).

Next define the function $\tau \in W^{1,\infty}(0, t_1)$ by setting

$$\tau(t) = \begin{cases} t & \text{if } t \in (0, t_0) \\ t_0 + (t - t_0)L[\tilde{\kappa}] & \text{if } t \in [t_0, t_1]. \end{cases} \tag{90}$$

If $\tilde{\kappa} \to \kappa$ in $L^2(0,T)$ then $\tilde{\Gamma}(t_1) \to \Gamma(t_1)$, so

$$L[\tilde{\kappa}] \to 1 \text{ as } \tilde{\kappa} \to \kappa \text{ in } L^2(0,T).$$

This implies that

$$\tau' \to 1 \text{ uniformly on } (0,t_1) \text{ as } \tilde{\kappa} \to \kappa.$$

Hence by choosing $U(\kappa)$ small enough, we obtain a well- defined curve $\bar{\Gamma} : \tau([0,t_1]) \to S$ and frame $\bar{R} : \tau([0,t_1]) \to SO(2)$ by setting

$$\overline{R}(\tau(t)) := \tilde{R}(t) \text{ and } \bar{\Gamma}(\tau(t)) := \int_0^{\tau(t)} \bar{R}e_1 \tag{91}$$

for all $t \in [0,t_1]$. Now we define $\theta : U(\kappa) \to \mathbb{R}$ by setting

$$\theta[\tilde{\kappa}] = (\bar{\Gamma}(\tau(t_1)) - \Gamma(t_1)) \cdot N(t_1). \tag{92}$$

By the above we have

$$\theta[\tilde{\kappa}] \to 0 \text{ as } \tilde{\kappa} \to \kappa \text{ in } L^2(0,T).$$

We extend τ to a Lipschitz function on $(0,T)$ by defining

$$\tau(t) = \tau(t_1) + \int_{t_1}^{t} \big(1 - \theta[\tilde{\kappa}]\kappa(s)\big)\, ds \text{ if } t \in (t_1,T).$$

Now we extend $(\bar{\Gamma}, \bar{R})$ to $\tau([0,T])$ via (91). The function τ clearly satisfies the bound

$$\|\tau' - 1\|_{L^\infty(0,T)} \leq \chi_{(t_0,t_1)}|L[\tilde{\kappa}] - 1| + \chi_{(t_1,T)}\|\kappa\|_\infty|\theta[\tilde{\kappa}]|, \tag{93}$$

In analogy to $(\bar{\Gamma}, \bar{R})$ define $(\bar{\gamma}, \bar{r})$ by setting

$$\bar{r}(\tau(t)) = \tilde{r}(t) \text{ and } \bar{\gamma}(\tau(t)) = \int_0^{\tau(t)} \bar{r}^T e_1 \text{ for all } t \in [0,T].$$

A short calculation shows that

$$\bar{\kappa}(\tau) = \frac{\tilde{\kappa}}{\tau'} \text{ and } \bar{\kappa}_n(\tau) = \frac{\tilde{\kappa}_n}{\tau'} \text{ on } (0,T). \tag{94}$$

Next, for $i = 1,2,3$ and $m_1 = m_2 = 3$ and $m_3 = 1$ and for $U(\kappa)$ as in the definition of L, $\bar{\Gamma}$ and θ, we introduce the functionals

$$\hat{\mathcal{G}}_i : U(\kappa) \times L^2(0,T) \to \mathbb{R}^{m_i}$$

by setting

$$\hat{\mathcal{G}}_1(\tilde{\kappa}, \tilde{\kappa}_n) = \bar{\gamma}(\tau(T)) - \gamma(T) - \theta[\tilde{\kappa}]v(T) \tag{95}$$

$$\hat{\mathcal{G}}_2(\tilde{\kappa}, \tilde{\kappa}_n) = \begin{pmatrix} \tilde{\gamma}'(T) \cdot v(T) \\ -\tilde{\gamma}'(T) \cdot n(T) \\ \tilde{n}(T) \cdot v(T) \end{pmatrix} \tag{96}$$

$$\hat{\mathcal{G}}_3(\tilde{\kappa}, \tilde{\kappa}_n) = -\tilde{N}(t_1) \cdot \Gamma'(t_1). \tag{97}$$

Consider the conditions

$$\bar{\gamma}(\tau(T)) - \gamma(T) = (v(T) \otimes N(t_1))(\bar{\Gamma}(\tau(t_1)) - \Gamma(t_1)) \tag{98}$$

$$\bar{r}(\tau(T)) = r(T) \tag{99}$$

$$\bar{\Gamma}(\tau(t_1)) - \Gamma(t_1) \parallel N(t_1) \tag{100}$$

$$\bar{R}(\tau(t_1)) = R(t_1). \tag{101}$$

One can easily show that if $\|\tilde{\kappa}-\kappa\|_{L^2(0,T)}$ and $\|\tilde{\kappa}_n-\kappa_n\|_{L^2(0,T)}$ are small, then the implication

$$\hat{\mathcal{G}}_i(\tilde{\kappa},\tilde{\kappa}_n)=0 \text{ for } i=1,2,3 \implies (98,\ 99,\ 100,\ 101) \text{ are satisfied} \tag{102}$$

holds. Notice that the constraint (100) is satisfied automatically: Since $\Gamma(t_0)=\tilde{\Gamma}(t_0)=\bar{\Gamma}(t_0)$, by definition of τ we have

$$\bar{\Gamma}(\tau(t_1))=\Gamma(t_0)+L[\tilde{\kappa}](\tilde{\Gamma}(t_1)-\Gamma(t_0)).$$

Thus (100) follows from (89).

Claim 1. Assume that $\|\tilde{\kappa}-\kappa\|_{L^2(0,T)}$ and $\|\tilde{\kappa}_n-\kappa_n\|_{L^2(0,T)}$ are small, that $\hat{\mathcal{G}}_i(\tilde{\kappa},\tilde{\kappa}_n)=0$ for $i=1,2,3$, that

$$\tilde{\kappa}=\kappa \text{ almost everywhere on } (0,T)\setminus(t_0,t_1)$$

and that $\bar{\Gamma}$ is locally admissible on (t_0,t_1). Then $\bar{\Gamma}$ satisfies (87) and is admissible on $\tau([0,T])$ and transversal on $\tau([t_0,t_1])$. Moreover,

$$\bar{\Gamma}(\tau)\begin{cases}=\Gamma & \text{on } (0,t_0)\\ \text{is equivalent to } \Gamma & \text{on } (t_1,T),\end{cases}$$

and (98, 99, 100, 101) as well as

$$\bar{\gamma}(\tau(T))-\gamma(T)=(v(T)\otimes N(T))(\bar{\Gamma}(\tau(T))-\Gamma(T)) \tag{103}$$

$$\bar{\Gamma}(\tau(T))-\Gamma(T)\parallel N(T) \tag{104}$$

$$\bar{R}(\tau(T))=R(T) \tag{105}$$

are satisfied. In particular, $(\bar{\Gamma},\bar{\kappa}_n)\sim(\Gamma,\kappa_n)$.

To prove Claim 1, first observe that local admissibility of $\bar{\Gamma}$ implies that $\bar{\kappa}\in L^\infty(\tau(t_0,t_1))$. Since $\bar{\Gamma}'(\tau)$ is uniformly close to Γ', it is clear that $\bar{\Gamma}$ satisfies (87) provided $\tilde{\kappa}$ is L^2-close to κ. By virtue of (93) we have

$$|\tau(t_1)-\tau(t_0)|<T_{\Gamma(t_0),N(t_0)} \tag{106}$$

for small $U(\kappa)$. Since $\tilde{\kappa}=\kappa$ on $(0,t_0)$, we have

$$(\bar{\Gamma}(\tau(t_0)),\pm\overline{N}(\tau(t_0)))=(\Gamma(t_0),\pm N(t_0)).$$

Hence Lemma 1 implies that $\bar{\Gamma}$ is transversal and admissible on $\tau([t_0,t_1])$.
Next observe that (98, 99, 100, 101) are satisfied by virtue of (102). By (101) and since $\tilde{\kappa}=\kappa$ on (t_1,T) we have $\tilde{\Gamma}'=\Gamma'$ on (t_1,T), so by definition of $\bar{R}$ we have $\bar{N}(\tau)=N$ on (t_1,T). Now it is easy to deduce from (100) that $\bar{\Gamma}(\tau)$ is equivalent to Γ on (t_1,T). By (44) we therefore have

$$\bar{\Gamma}(\tau)-\Gamma=\theta[\tilde{\kappa}]N$$

on (t_1,T), so (98, 100, 101) imply (103, 104, 105). And from the definition of τ we have $\bar{\Gamma}=\Gamma$ on $(0,t_0)$.
By (100, 101) and since $\bar{R}(\tau(t_0))=R(t_0)$ and $\bar{\Gamma}(\tau(t_0))=\Gamma(t_0)$, Lemma 2 implies that

$$[\bar{\Gamma}(\tau(t_0,t_1))]=[\Gamma(t_0,t_1)] \tag{107}$$

because $\bar{\Gamma}$ is admissible on $[t_0,t_1]$. Since $\bar{\Gamma}(\tau)$ is equivalent to Γ outside (t_0,t_1), we have

$$[\bar{\Gamma}(\tau(t))]=[\Gamma(t)] \text{ for all } t\notin(t_0,t_1). \tag{108}$$

From this, from (107) and from admissibility of Γ on $[0,T]$, we conclude that $\bar{\Gamma}$ is admissible on $\tau([0,T])$ and that

$$[\bar{\Gamma}(\tau(0,T))] = [\Gamma(0,T)].$$

Combining this with (99, 103, 104, 105), using Proposition 2 (iii) it is easy to see that indeed $(\bar{\Gamma}, \bar{\kappa}_n) \sim (\Gamma, \kappa_n)$ (despite the fact that Γ is not transversal on $[0,T]$). This concludes the proof of Claim 1.

Now let $\{\hat{\kappa}^\varepsilon\}$, $\{\hat{\kappa}_n^\varepsilon\}$ be one-parameter families as in the hypothesis.

Claim 2. There exists $\iota > 0$ (independent of $\hat{\kappa}^\varepsilon$, $\hat{\kappa}_n^\varepsilon$) such that the following is true: For all $\varepsilon > 0$ small enough there exist

$$\varphi^\varepsilon,\ \psi^\varepsilon \in C_0^\infty(t_0 + \iota, t_1 - \iota) \tag{109}$$

such that the curvatures

$$\tilde{\kappa}^\varepsilon := \hat{\kappa}^\varepsilon + \varphi^\varepsilon \text{ and } \tilde{\kappa}_n^\varepsilon := \hat{\kappa}_n^\varepsilon + \psi^\varepsilon \tag{110}$$

satisfy $\mathcal{G}_1(\tilde{\kappa}^\varepsilon, \tilde{\kappa}_n^\varepsilon) = \mathcal{G}_2(\tilde{\kappa}^\varepsilon, \tilde{\kappa}_n^\varepsilon) = 0$ and $\mathcal{G}_3(\tilde{\kappa}^\varepsilon, \tilde{\kappa}_n^\varepsilon) = 0$. Moreover,

$$\|\varphi^\varepsilon\|_{L^\infty(0,T)} + \|\psi^\varepsilon\|_{L^\infty(0,T)} \to 0 \text{ as } \varepsilon \downarrow 0.$$

To prove this claim, define the functional

$$\mathcal{G} : U(\kappa) \times L^2(0,T) \to \mathbb{R}^7$$

by setting $\mathcal{G} := (\hat{\mathcal{G}}_1, \hat{\mathcal{G}}_2, \hat{\mathcal{G}}_3)^T$. The functionals $\hat{\mathcal{G}}_i$ are continuously Fréchet differentiable in an $\left(L^2(0,T)\right)^2$-neighbourhood of (κ, κ_n) by Lemma 5. By Lemma 5, and with notation as in that lemma (in particular, if $D\mathcal{G}$ denotes the Fréchet derivative of $\mathcal{G}$ then $\dot{\mathcal{G}} := D\mathcal{G}(0,0)$), the restriction of the linearized constraint functional $\dot{\mathcal{G}}$ to $\left(C_0^\infty(t_0,t_1)\right)^2$ is surjective onto $\mathbb{R}^7$. Hence there exist $\hat{\varphi}_k, \hat{\psi}_k \in C_0^\infty(t_0,t_1)$, $k = 1, ..., 7$, such that the 7×7 matrix with entries $\dot{\hat{\mathcal{G}}}_j[\hat{\varphi}_i, \hat{\psi}_i]$ is invertible.
Let $r > 0$ and let $\tilde{U}$ be a neighbourhood of zero (specified below) in $L^2(t_0,t_1)$, and define the function

$$F : \tilde{U} \times \tilde{U} \times B_r(0) \to \mathbb{R}^7,$$

where $B_r(0) \subset \mathbb{R}^7$, by setting

$$F(\mu_1, \mu_2, \eta) = \mathcal{G}\big(\kappa + \mu_1 + \sum_{k=1}^7 \eta_k \hat{\varphi}_k, \kappa_n + \mu_2 + \sum_{k=1}^7 \eta_k \hat{\psi}_k\big).$$

We choose $\tilde{U}$ and r above small enough to ensure that for all $(\mu_1, \mu_2, \eta) \in \tilde{U} \times \tilde{U} \times B_r(0)$ the argument on the right-hand side is contained in a neighbourhood of (κ, κ_n) on which $\mathcal{G}$ is Fréchet differentiable.
For $k = 1, ..., 7$ the partial derivatives $\partial F / \partial \eta_k$ exist and are continuous in a neighbourhood of zero by continuity of $D\mathcal{G}(\cdot)[\hat{\varphi}_i, \hat{\psi}_i]$ near (κ, κ_n). And for $i, j = 1, ..., 7$ we have

$$\frac{\partial F_j}{\partial \eta_i}(\mu_1, \mu_2, \eta) = D\hat{\mathcal{G}}_j(\kappa + \mu_1 + \sum_{k=1}^7 \eta_k \hat{\varphi}_k, \kappa_n + \mu_2 + \sum_{k=1}^7 \eta_k \hat{\psi}_k)[\hat{\varphi}_i, \hat{\psi}_i].$$

By a standard implicit function theorem (see e.g. Theorem 3.4.10 in [11]) there is an $L^2(t_0,t_1)$-neighbourhood $U \subset \tilde{U}$ of 0 and

$$\hat{\eta} \in C^0(U \times U; B_r(0))$$

with $\hat{\eta}(0,0) = 0$ and such that

$$F(\mu_1, \mu_2, \hat{\eta}(\mu_1, \mu_2)) = 0 \text{ for all } \mu_1,\ \mu_2 \in U.$$

Setting

$$\varphi^\varepsilon := \sum_{k=1}^{7} \hat{\eta}_k(\hat{\kappa}^\varepsilon - \kappa, \hat{\kappa}_n^\varepsilon - \kappa_n)\hat{\varphi}_k \text{ and } \psi^\varepsilon := \sum_{k=1}^{7} \hat{\eta}_k(\hat{\kappa}^\varepsilon - \kappa, \hat{\kappa}_n^\varepsilon - \kappa_n)\hat{\psi}_k,$$

we obtain the desired functions. Notice that $\hat{\varphi}_k$ and $\hat{\psi}_k$ only depend on the original framed curves (γ, r) and (Γ, R) (via the functional $\mathcal{G}$). This shows that the supports of φ^ε and ψ^ε are contained in a compact subinterval of (t_0, t_1) which does not depend on ε or on $\hat{\kappa}^\varepsilon$, $\hat{\kappa}_n^\varepsilon$. This concludes the proof of Claim 2.

Now define τ^ε, $\bar{\Gamma}^\varepsilon$ etc. analogously to τ, $\bar{\Gamma}$ etc., but with $\tilde{\kappa}^\varepsilon$ and $\tilde{\kappa}_n^\varepsilon$ as defined in (110) instead of $\tilde{\kappa}$ and $\tilde{\kappa}_n$. Observe that $\tilde{\kappa}^\varepsilon$ and $\tilde{\kappa}_n^\varepsilon$ satisfy the hypotheses of Claim 1 for all $\varepsilon > 0$ small enough. We claim that

$$\bar{\Gamma}^\varepsilon \text{ is uniformly admissible on } \tau^\varepsilon([t_0, t_1]). \tag{111}$$

In fact, by transversality and by smallness of the perturbations we have

$$\|s^+_{\bar{\Gamma}^\varepsilon}(\tau^\varepsilon) - s^+_{\Gamma}\|_{L^\infty(t_0,t_1)} + \|s^-_{\bar{\Gamma}^\varepsilon}(\tau^\varepsilon) - s^-_{\Gamma}\|_{L^\infty(t_0,t_1)} \to 0 \text{ as } \varepsilon \downarrow 0, \tag{112}$$

compare e.g. the analogous argument in Lemma 3. Since

$$\bar{\kappa}^\varepsilon(\tau^\varepsilon) = \frac{\tilde{\kappa}^\varepsilon}{(\tau^\varepsilon)'} = \frac{\hat{\kappa}^\varepsilon + \varphi^\varepsilon}{(\tau^\varepsilon)'}$$

is close to $\hat{\kappa}^\varepsilon$ in $L^\infty(0,T)$, from (82) we deduce that

$$\bar{\kappa}(\tau^\varepsilon) \in \left[\frac{1}{s^-_{\bar{\Gamma}^\varepsilon}(\tau^\varepsilon)} + \frac{c}{2}, \frac{1}{s^+_{\bar{\Gamma}^\varepsilon}(\tau^\varepsilon)} - \frac{c}{2}\right] \text{ almost everywhere on } (t_0, t_1). \tag{113}$$

In particular, $\bar{\Gamma}$ is locally admissible on (t_0, t_1). Hence (111) follows from Claim 1 and from (113).

It remains to prove that

$$(\bar{\Gamma}^\varepsilon, \bar{\kappa}_n^\varepsilon) \to (\Gamma, \kappa_n) \text{ strongly in } W^{2,2}([\Gamma(0,T)]; \mathbb{R}^3) \text{ as } \varepsilon \downarrow 0. \tag{114}$$

In the rest of the proof we omit the index ε in order to avoid cumbersome expressions. The definitions (94) imply that

$$\tau' \cdot \int_{s^-_{\bar{\Gamma}}(\tau)}^{s^+_{\bar{\Gamma}}(\tau)} \frac{\bar{\kappa}_n^2(\tau)}{1 - s\bar{\kappa}(\tau)}\, ds = \int_{s^-_{\bar{\Gamma}}(\tau)}^{s^+_{\bar{\Gamma}}(\tau)} \frac{\tilde{\kappa}_n^2}{\tau' - s\tilde{\kappa}}\, ds \text{ on } (0,T). \tag{115}$$

So by (109, 83) and by (46) we have

$$\tau' \cdot \int_{s^-_{\bar{\Gamma}}(\tau)}^{s^+_{\bar{\Gamma}}(\tau)} \frac{\bar{\kappa}_n^2(\tau)}{1 - s\bar{\kappa}(\tau)}\, ds = \int_{s^-_{\Gamma}}^{s^+_{\Gamma}} \frac{\hat{\kappa}_n^2}{1 - s\kappa}\, ds \text{ on } (0,T) \setminus (t_0, t_1) \tag{116}$$

because $\bar{\Gamma}(\tau)$ is equivalent to Γ on this set. By (84) the integrand on the right-hand side of (116) is dominated by $\frac{\kappa_n^2}{1-s\kappa} \in L^1(M_{s^\pm_\Gamma})$. Since (after possibly passing to a subsequence) $\hat{\kappa}_n \to \kappa_n$ pointwise almost everywhere, Lebesgue's dominated convergence theorem together with (116) and a change of variables imply that

$$\int_{\tau((0,T)\setminus(t_0,t_1))} \int_{s^-_{\bar{\Gamma}}}^{s^+_{\bar{\Gamma}}} \frac{\bar{\kappa}_n^2}{1 - s\bar{\kappa}}\, ds\, dt \to \int_{(0,T)\setminus(t_0,t_1)} \int_{s^-_{\Gamma}}^{s^+_{\Gamma}} \frac{\kappa_n^2}{1 - s\kappa}\, ds\, dt \text{ as } \varepsilon \downarrow 0. \tag{117}$$

The inclusion (113) and smallness of $\|\tau' - 1\|_{L^\infty(t_0,t_1)}$ imply that

$$\int_{s_{\bar\Gamma}^-(\tau)}^{s_{\bar\Gamma}^+(\tau)} \frac{1}{\tau' - s\tilde\kappa}\, ds \leq C_1 \text{ on } (t_0, t_1) \tag{118}$$

for some constant C_1 independent of ε. On the other hand (112, 85) imply that the left-hand side of (118) converges pointwise to

$$\int_{s_\Gamma^-}^{s_\Gamma^+} \frac{1}{1 - s\kappa}\, ds \tag{119}$$

almost everywhere on (t_0, t_1). Hence the left-hand side of (118) converges weakly-$*$ in $L^\infty(t_0, t_1)$ to (119). But $\tilde\kappa_n^2 \to \kappa_n^2$ in $L^1(t_0, t_1)$. We conclude that

$$\int_{t_0}^{t_1} \int_{s_\Gamma^-}^{s_\Gamma^+} \frac{\tilde\kappa_n^2}{\tau' - s\tilde\kappa}\, ds\, dt \to \int_{t_0}^{t_1} \int_{s_\Gamma^-}^{s_\Gamma^+} \frac{\kappa_n^2}{1 - s\kappa}\, ds\, dt.$$

Together with (115, 117) and a change of variables we conclude:

$$\int_{\tau(0,T)} \int_{s_{\bar\Gamma}^-}^{s_{\bar\Gamma}^+} \frac{\bar\kappa_n^2}{1 - s\bar\kappa}\, ds\, dt \to \int_0^T \int_{s_\Gamma^-}^{s_\Gamma^+} \frac{\kappa_n^2}{1 - s\kappa}\, ds\, dt \text{ as } \varepsilon \downarrow 0.$$

Thus from (16) we deduce that

$$\int_{[\Gamma(0,T)]} |\nabla^2(\bar\Gamma, \bar\kappa_n)|^2 \to \int_{[\Gamma(0,T)]} |\nabla^2(\Gamma, \kappa_n)|^2 \text{ as } \varepsilon \downarrow 0.$$

Arguing as in Lemma 3 we deduce (114). This concludes the proof. □

4.4 Nondegeneracy of lateral boundary conditions

A key ingredient in the proof of Lemma 4 was Lemma 5 below. It shows that the constraint functionals introduced in the proof of Lemma 4 are linearly independent.

Lemma 5 *Let $\kappa \in L^\infty(0,T)$ and $\kappa_n \in L^2(0,T)$ induce the framed curves (Γ, R) and (γ, r) in the usual way. Let $(t_0, t_1) \subset (0,T)$, assume that*

$$\mathcal{L}^1\left(\{t \in (t_0, t_1) : \kappa_n(t) \neq 0\}\right) > 0$$

and let $\mathcal{G}$ be as defined in the proof of Lemma 4. Then $\mathcal{G}$ is continuously Fréchet-differentiable in an $(L^2(0,T))^2$-neighbourhood of (κ, κ_n). Moreover, denoting the Fréchet derivative of $\mathcal{G}$ at the point (κ, κ_n) by $\dot{\mathcal{G}}$, we have

$$\dot{\mathcal{G}}\left[\left(C_0^\infty(t_0, t_1)\right)^2\right] = \mathbb{R}^7;$$

here we regard $C_0^\infty(t_0, t_1)$ as a subspace of $L^2(0,T)$.

Proof. For given $(\tilde\kappa, \tilde\kappa_n)$ in an $(L^2(0,T))^2$-neighbourhood $U(\kappa, \kappa_n)$ of (κ, κ_n) define

$$(\tilde\Gamma[\tilde\kappa], \tilde R[\tilde\kappa]) \text{ and } (\tilde\gamma[\tilde\kappa, \tilde\kappa_n], \tilde r[\tilde\kappa, \tilde\kappa_n])$$

as well as $L[\tilde\kappa]$ and $\tau[\tilde\kappa]$ and $\theta[\tilde\kappa]$ and

$$(\bar\Gamma[\tilde\kappa], \bar R[\tilde\kappa]) \text{ and } (\bar\gamma[\tilde\kappa, \tilde\kappa_n], \bar r[\tilde\kappa, \tilde\kappa_n])$$

as in the proof of Lemma 4 — in the present proof it is convenient to display the dependence on $(\tilde\kappa, \tilde\kappa_n)$. Observe that

$$\bar\kappa(\tau[\tilde\kappa]) := \frac{\tilde\kappa}{(\tau[\tilde\kappa])'} \text{ and } \bar\kappa_n(\tau[\tilde\kappa]) := \frac{\tilde\kappa_n}{(\tau[\tilde\kappa])'}$$

are the curvatures corresponding to $(\bar\Gamma[\tilde\kappa], \bar R[\tilde\kappa])$ and to $(\bar\gamma[\tilde\kappa, \tilde\kappa_n], \bar r_[\tilde\kappa, \tilde\kappa_n])$.
As in [6], using results from [17], [10], one shows that — choosing $U(\kappa, \kappa_n)$ small enough — the functional

$$(\tilde\kappa, \tilde\kappa_n) \mapsto \tilde r[\tilde\kappa, \tilde\kappa_n](t)$$

is continuosly Fréchet differentiable on $U(\kappa, \kappa_n)$ for all $t \in [0, T]$. Similar facts are true for the other quantities introduced above.
Let $\varphi, \psi \in L^2(0, T)$ and set

$$\xi := \varphi n - \psi v,$$

where, as usual, $n := r_3 := r^T e_3$ and $v := r_2 := r^T e_2$ (we will use a similar notation for $\tilde r$). Then for all $t \in [0, T]$ we have, for $i = 1, 2, 3$,

$$\begin{aligned}
\dot r_i[\varphi, \psi](t) &:= \frac{d}{d\varepsilon}|_{\varepsilon=0}\tilde r_i[\kappa + \varepsilon\varphi, \kappa_n + \varepsilon\psi](t) = \left(\int_0^t \xi(s)\ ds\right) \wedge r_i(t),\\
\dot\gamma[\varphi, \psi](t) &:= \frac{d}{d\varepsilon}|_{\varepsilon=0}\tilde\gamma[\kappa + \varepsilon\varphi, \kappa_n + \varepsilon\psi](t) = \int_0^t \xi(s) \wedge (\gamma(t) - \gamma(s))ds,\\
\dot\Gamma'[\varphi](t) &:= \frac{d}{d\varepsilon}|_{\varepsilon=0}\tilde\Gamma'[\kappa + \varepsilon\varphi](t) = \Big(\int_0^t \varphi(s)\ ds\Big)N(t),\\
\dot N[\varphi](t) &:= \frac{d}{d\varepsilon}|_{\varepsilon=0}\tilde N[\kappa + \varepsilon\varphi](t) = -\Big(\int_0^t \varphi(s)\ ds\Big)\Gamma'(t),\\
\dot\Gamma[\varphi](t) &:= \frac{d}{d\varepsilon}|_{\varepsilon=0}\tilde\Gamma[\kappa + \varepsilon\varphi](t) = \int_0^t \varphi(s)(\Gamma^\perp(t) - \Gamma^\perp(s))\ ds.
\end{aligned}$$

These equations follow by taking derivatives with respect to ε in the defining ODEs for $\tilde r[\kappa + \varepsilon\varphi, \kappa_n + \varepsilon\psi]$ and $\tilde R[\kappa + \varepsilon\varphi]$, and then applying the variation of constants formula. We omit the details.
From now on we assume

$$\varphi, \psi \in C_0^\infty(t_0, t_1).$$

From (89) we deduce

$$\dot L[\varphi] := \frac{d}{d\varepsilon}|_{\varepsilon=0}L[\kappa + \varepsilon\varphi] = -\frac{\dot\Gamma[\varphi](t_1) \cdot \Gamma'(t_1)}{(\Gamma(t_1) - \Gamma(t_0)) \cdot \Gamma'(t_1)}.$$

From (92) we find with $\dot\Gamma[\varphi](t_0) = 0$ and using $a \cdot b = a^\perp \cdot b^\perp$ and $\Gamma' \otimes \Gamma' + N \otimes N = id$ that

$$\dot\theta[\varphi] := \frac{d}{d\varepsilon}|_{\varepsilon=0}\theta[\kappa + \varepsilon\varphi] = \frac{(\Gamma(t_1) - \Gamma(t_0))^\perp \cdot \dot\Gamma[\varphi](t_1)}{(\Gamma(t_1) - \Gamma(t_0)) \cdot \Gamma'(t_1)}.$$

Let us next simplify $\hat{\mathcal{G}}_1$ as defined in (95). If $\tilde\kappa = \kappa$ outside (t_0, t_1), then we have $-\kappa\tilde\gamma'[\tilde\kappa, \tilde\kappa_n] = (\tilde v[\tilde\kappa, \tilde\kappa_n])'$, so using the definitions of $\bar\Gamma$ and τ', we have

$$\begin{aligned}
\hat{\mathcal{G}}_1(\tilde\kappa, \tilde\kappa_n) &= \int_0^T (\bar\gamma[\tilde\kappa, \tilde\kappa_n])'(\tau[\tilde\kappa])(\tau[\tilde\kappa])'\ dt - \gamma(T) - \theta[\tilde\kappa]v(T)\\
&= \tilde\gamma[\tilde\kappa, \tilde\kappa_n](T) - \gamma(T) - \theta[\tilde\kappa]\tilde v[\tilde\kappa, \tilde\kappa_n](t_1)\\
&+ (L[\tilde\kappa] - 1)\int_{t_0}^{t_1} (\tilde\gamma[\tilde\kappa, \tilde\kappa_n])'\ dt + \theta[\tilde\kappa](\tilde v[\tilde\kappa, \tilde\kappa_n](T) - v(T)). \qquad (120)
\end{aligned}$$

For $i = 1, 2, 3$ define

$$\dot{\hat{\mathcal{G}}}_i[\varphi, \psi] := \frac{d}{d\varepsilon}|_{\varepsilon=0}\hat{\mathcal{G}}_i(\kappa + \varepsilon\varphi, \kappa_n + \varepsilon\psi).$$

From (120) we deduce:

$$\dot{\hat{\mathcal{G}}}_1[\varphi, \psi] = \dot{\gamma}[\varphi, \psi](T) - \dot{\theta}[\varphi]v(t_1) + \dot{L}[\varphi] \int_{t_0}^{t_1} \gamma' \, dt. \tag{121}$$

Define the matrix

$$E = -\frac{\big(\gamma(t_1) - \gamma(t_0)\big) \otimes \Gamma'(t_1) + v(t_1) \otimes \big(\Gamma(t_1) - \Gamma(t_0)\big)^{\perp}}{\big(\Gamma(t_1) - \Gamma(t_0)\big) \cdot \Gamma'(t_1)}. \tag{122}$$

Inserting the above equations into (121) and simplifying the resulting expression yields

$$\dot{\hat{\mathcal{G}}}_1[\varphi, \psi] = E\dot{\Gamma}[\varphi](t_1) + \dot{\gamma}[\varphi, \psi](T),$$

Using the above expressions for $\dot{r}$ and so on, we obtain

$$\begin{aligned}
\dot{\hat{\mathcal{G}}}_1[\varphi, \psi] &= E \int_{t_0}^{t_1} \varphi(t)(\Gamma(t_1) - \Gamma(t))^{\perp} \, dt + \int_{t_0}^{t_1} \xi(t) \wedge (\gamma(T) - \gamma(t)) \, dt \\
\dot{\hat{\mathcal{G}}}_2[\varphi, \psi] &= \begin{pmatrix} \Xi(t_1) \cdot n(t_1) \\ -\Xi(t_1) \cdot v(t_1) \\ -\Xi(t_1) \cdot \gamma'(t_1) \end{pmatrix} \\
\dot{\hat{\mathcal{G}}}_3[\varphi, \psi] &= \int_{t_0}^{t_1} \xi(t) \cdot n(t) \, dt,
\end{aligned}$$

where we have set

$$\Xi(t) = \int_0^t \xi(s) \, ds.$$

Now suppose that the image of $\big(C_0^\infty(t_0, t_1)\big)^2$ under the linear operator $\dot{\mathcal{G}} := (\dot{\hat{\mathcal{G}}}_1, \dot{\hat{\mathcal{G}}}_2, \dot{\hat{\mathcal{G}}}_3)$ were not all of $\mathbb{R}^7$. By linearity this would imply the existence of

$$(\lambda_1, \tilde{\lambda}_2, \tilde{\lambda}_4) \in (\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}) \setminus \{0\} \tag{123}$$

such that

$$\lambda_1 \cdot \dot{\hat{\mathcal{G}}}_1[\varphi, \psi] + \tilde{\lambda}_2 \cdot \dot{\hat{\mathcal{G}}}_2[\varphi, \psi] + \tilde{\lambda}_4 \dot{\hat{\mathcal{G}}}_4[\varphi, \psi] = 0$$

for all $\varphi, \psi \in C_0^\infty(t_0, t_1)$. Introduce

$$\lambda_3 = (E^T \lambda_1)^{\perp} \tag{124}$$

and notice that

$$\operatorname{spt} \xi \subset (t_0, t_1).$$

Using $\varphi = \xi \cdot n$, we obtain

$$\begin{aligned}
&\lambda_3 \cdot \int_{t_0}^{t_1} \xi(t) \cdot n(t) \, (\Gamma(t) - \Gamma(t_1)) \, dt + \int_{t_0}^{t_1} \xi(t) \cdot \lambda_1 \wedge (\gamma(t) - \gamma(T)) \, dt \\
&+ (\tilde{\lambda}_{21} n(t_1) - \tilde{\lambda}_{22} v(t_1) - \tilde{\lambda}_{23} \gamma'(t_1)) \cdot \int_{t_0}^{t_1} \xi(t) \, dt \\
&+ \tilde{\lambda}_4 \int_{t_0}^{t_1} \xi(t) \cdot n(t) \, dt = 0.
\end{aligned}$$

This holds for all choices of $\varphi, \psi \in C_0^\infty(t_0, t_1)$. But by appropriately choosing φ and ψ, the function ξ runs through all functions in $C_0^\infty((t_0, t_1); \mathbb{R}^3)$ satisfying $\xi(t) \cdot \gamma'(t) = 0$ for all $t \in (t_0, t_1)$. Hence with

$$\lambda_2 = \tilde{\lambda}_{21} n(t_1) - \tilde{\lambda}_{22} v(t_1) - \tilde{\lambda}_{23} \gamma'(t_1) - \lambda_1 \wedge \gamma(T),$$
$$\lambda_4 = \tilde{\lambda}_4 - \lambda_3 \cdot \Gamma(t_1)$$

we obtain

$$\lambda_2 + \lambda_1 \wedge \gamma(t) + (\lambda_3 \cdot \Gamma(t) + \lambda_4)\, n(t) \;\Big\|\; \gamma'(t) \tag{125}$$

for all $t \in (t_0, t_1)$. Observe that by virtue of (123), not all multipliers λ_1, λ_2, λ_4 are zero.
Let us first consider the case $\lambda_1 = 0$. In this case by (124) also $\lambda_3 = 0$, so (125) reduces to

$$\lambda_4 n + \lambda_2 = a\gamma', \tag{126}$$

where $a = \lambda_2 \cdot \gamma' \in W^{1,2}(t_0, t_1)$. By scalar multiplication with v and n we obtain that $\lambda_2 \cdot v = 0$ and $\lambda_2 \cdot n + \lambda_4 = 0$. Differentiating the second equation we find $\kappa_n a = 0$ almost everywhere on (t_0, t_1). Thus on the open set $\{a \neq 0\}$ we have $\kappa_n = 0$, so n is constant on each connected component of $\{a \neq 0\}$. On the set $\{a = 0\}$ we have that n is parallel to λ_2 by (126). Hence by continuity n must be constant, so κ_n vanishes almost everywhere on (t_0, t_1), a contradiction. Notice that we have assumed $\lambda_2 \neq 0$ because otherwise by $\lambda_2 \cdot n + \lambda_4 = 0$ also $\lambda_4 = 0$, so all $\lambda_i = 0$, a contradiction.

Now consider the case $\lambda_1 \neq 0$. By (125) we have $(\lambda_2 + \lambda_1 \wedge \gamma) \cdot v = 0$. So

$$\lambda_2 + \lambda_1 \wedge \gamma = a\gamma' + bn, \tag{127}$$

where now $a = (\lambda_2 + \lambda_1 \wedge \gamma) \cdot \gamma'$ and $b = (\lambda_2 + \lambda_1 \wedge \gamma) \cdot n$. Straightforward differentiation gives

$$\begin{pmatrix} a \\ b \end{pmatrix}' = \begin{pmatrix} 0 & \kappa_n \\ -\kappa_n & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 0 \\ -\lambda_1 \cdot v \end{pmatrix}. \tag{128}$$

Comparing (125) with (127) gives

$$b = -\lambda_3 \cdot \Gamma - \lambda_4. \tag{129}$$

Differentiation of the equation $(\lambda_2 + \lambda_1 \wedge \gamma) \cdot v = 0$ gives

$$\kappa a = \lambda_1 \cdot n \tag{130}$$

Multiplying (130) by κ_n and multiplying the ODE (128) satisfied by b by κ, and adding the resulting equations, we obtain $-\lambda_3 \cdot N' = \lambda_1 \cdot \gamma''$; we have used that $b' = -\lambda_3 \cdot \Gamma'$ by (129) and that $N' = -\kappa \Gamma'$. Integration yields

$$-\lambda_3 \cdot N = \lambda_1 \cdot \gamma' + c_1 \text{ on } (t_0, t_1) \tag{131}$$

for some constant c_1. We integrate (131) over (t_0, t_1) to find

$$-\lambda_3 \cdot \big(\Gamma(t_1) - \Gamma(t_0)\big)^\perp = \lambda_1 \cdot \big(\gamma(t_1) - \gamma(t_0)\big) + c_1(t_1 - t_0).$$

Together with the explicit expression (124) of λ_3 this yields $c_1 = 0$. Keeping this in mind we multiply (131) by κ and use $v' = -\kappa\gamma'$ and $\kappa N = \Gamma''$ to deduce $\lambda_3 \cdot \Gamma'' = \lambda_1 \cdot v'$. Integration yields

$$\lambda_3 \cdot \Gamma' = \lambda_1 \cdot v + c_2$$

for some constant c_2. Evaluating this at $t = t_1$ and using (124) gives $c_2 = 0$. We conclude that

$$\lambda_3 = (-\lambda_1 \cdot \gamma')N + (\lambda_1 \cdot v)\Gamma',$$

i.e. $\lambda_3^\perp = Q\lambda_1$, where Q is as in (69). So

$$0 = Q'\lambda_1 = \kappa_n(\lambda_1 \cdot n).$$

As in the proof of Lemma 3 this implies $\lambda_1 \cdot n = 0$ on (t_0, t_1), so $\kappa_n = 0$ on $\{\lambda_1 \cdot \gamma' \neq 0\}$. And $\kappa a = 0$ by (130). On the other hand by (129) and (128) we have

$$-\kappa_n a - \lambda_1 \cdot v = b' = -\lambda_3 \cdot \Gamma'.$$

Hence $\kappa_n a = 0$ because $\lambda_3 \cdot \Gamma' = \lambda_1 \cdot v$, as we saw above. Hence $\kappa_n = 0$ on $\{a \neq 0\}$. But on $\{b \neq 0\} \cup \{a = 0\}$ we have $0 = a' = \kappa_n b$, so $\kappa_n = 0$ there as well. Thus

$$\kappa_n = 0 \text{ on } \{a \neq 0\} \cup \{b \neq 0\} \cup \{\lambda_1 \cdot \gamma' \neq 0\}. \tag{132}$$

But on $\{\lambda_1 \cdot \gamma' = 0\}$ we have $\lambda_1 \cdot v \neq 0$ because $\lambda_1 \neq 0$ yet $\lambda_1 \cdot n = 0$. On the other hand, on $\{b = 0\}$ we have $0 = b' = -\lambda_1 \cdot v$, so $\{b = 0\} \subset \{\lambda_1 \cdot \gamma' \neq 0\}$. Hence (132) implies that $\kappa_n = 0$ almost everywhere on (t_0, t_1), a contradiction. □

5 Smooth local approximants of $W^{2,2}$ isometric immersions

Throughout this section, $S \subset \mathbb{R}^2$ denotes a bounded Lipschitz domain that satisfies condition $(*)$, and $u \in W^{2,2}_{\text{iso}}(S; \mathbb{R}^3)$. The purpose of this section is to construct smooth approximants to the restriction of u to a subdomain of the form $[\Gamma(0, T)]$. For the proof of Theorem 1 such local approximants will be glued together using the decomposition theorem (Theorem 4) from [7]. Hence each local approximant must satisfy boundary conditions; ideally on $S \cap \partial[\Gamma(0, T)]$. As explained in the Introduction, that is not possible in general because this set can consist of infinitely many line segments, but a closer look at Theorem 4 in [7] reveals that it is enough to satisfy the boundary conditions on the set $[\Gamma(0)] \cup [\Gamma(T)]$, which for non-transversal curves is (in general) a strict subset of $S \cap \partial[\Gamma(0, T)]$.
In order to ensure smooth transitions between the local approximants we must require, moreover, that each local approximant be affine in a neighbourhood of the set $S \cap \partial[\Gamma(0, T)]$. It is not enough to have them affine near $[\Gamma(0)]$ and $[\Gamma(T)]$ because otherwise the transition to the small boundary domains $W_1, \ldots$ in the proof of Theorem 1 would not be smooth.
Let $\Gamma \in W^{2,\infty}([0, T]; S \backslash \hat{C}_{\nabla u})$ be a line of curvature for u in the sense of Definition 1. In particular, Γ is admissible on $[0, T]$. By Proposition 1 there exists $\kappa_n \in L^2(0, T)$ such that

$$(\Gamma, \kappa_n) = u \text{ on } [\Gamma(0, T)],$$

and (15) is satisfied.

Lemma 6 *Assume that Γ is transversal on $[0, T]$ and that*

$$\beta_\Gamma^\pm([0, T]) \cap \Sigma = \emptyset, \tag{133}$$

and let $\delta > 0$. Then there is

$$\hat{u} \in C^\infty_{iso}(\overline{[\Gamma(0, T)]}; \mathbb{R}^3)$$

such that the following are satisfied:

(i) Boundary conditions:

$$(\hat{u}, \nabla\hat{u}) = \Big((\Gamma, \kappa_n), \nabla(\Gamma, \kappa_n)\Big) \text{ on } [\Gamma(0)] \cup [\Gamma(T)].$$

(ii) Approximation:

$$\|\hat{u} - (\Gamma, \kappa_n)\|_{W^{2,2}([\Gamma(0,T)];\mathbb{R}^3)} < \delta$$

(iii) Affinity near the relative boundary: There is $\varepsilon > 0$ such that $\hat{u}$ is affine on each connected component of

$$[\Gamma(0,T)] \cap B_\varepsilon(S \cap \partial[\Gamma(0,T)]) = [\Gamma(0,T)] \cap B_\varepsilon([\Gamma(0)] \cup [\Gamma(T)]).$$

Proof. We may assume without loss of generality that

$$T < \frac{T_{\Gamma(0),N(0)}}{2} \tag{134}$$

where $T_{\Gamma(0),N(0)}$ is as in the conclusion of Lemma 1. In fact, since Γ is transversal, there is $\varepsilon > 0$ such that ν is Lipschitz on an ε-neighbourhood U of the compact set

$$\bigcup_{*=+,-} \bigcup_{t\in[0,T]} (\Gamma(t), *N(t)).$$

Hence there is $\varepsilon > 0$ such that

$$\inf \operatorname{dist}_{\partial S}(\Gamma([0,T])) > \varepsilon$$
$$\|\nabla\nu\|_{L^\infty(U;\mathbb{R}^2)} < \frac{1}{\varepsilon}.$$

Hence by Remark 3 there exists $N \in \mathbb{N}$ such that $T_{\Gamma(t),N(t)}$ can be chosen to be larger than $2T/N$ for all $t \in [0,T]$. Setting $T_k = kT/N$ we can apply the lemma to each curve $\Gamma|_{[T_k,T_{k+1}]}$, for $k = 0, ..., N-1$. This yields $\hat{u}_k$ satisfying (i), (ii), (iii) from the conclusion with T_k instead of 0 and T_{k+1} instead of T, and with δ/N instead of δ. Since Γ is admissible, the sets $[\Gamma(T_k, T_{k+1})]$ are pairwise disjoint for different $k = 0, ..., N-1$, and so the mapping $\hat{u} : [\Gamma(0,T)] \to \mathbb{R}^3$ defined by setting

$$\hat{u}(x) = \hat{u}_k(x) \text{ if } x \in [\Gamma(T_k, T_{k+1})]$$

for all $k = 0, ..., N-1$ is well-defined and satisfies the conclusion as stated in the lemma. This shows that we may indeed assume (134) without loss of generality.

So suppose that (134) is satisfied. Hence (27) holds. If $\kappa_n = 0$ almost everywhere on $(0,T)$ then the lemma follows with $\hat{u} = (\Gamma, \kappa_n)$, so we assume that

$$\mathcal{L}^1\Big(\{t \in (0,T) : \kappa_n(t) \neq 0\}\Big) > 0. \tag{135}$$

In the following three steps we subsequently drop some extra hypotheses.

Step 1. Make the following extra assumptions:

$$\Gamma \text{ is uniformly admissible on } [0,T] \tag{136}$$

and there is $\varepsilon_1 > 0$ such that

$$\kappa_n = 0 \text{ almost everywhere on } (0, \varepsilon_1) \cup (T - \varepsilon_1, T).$$

Fix some $\delta \in (0, 1/4)$. We apply Lemma 4 with $t_0 = 0$ and $t_1 = T$ to obtain $\iota > 0$ with the properties stated there. Set $\varepsilon_2 := \frac{1}{4}\min\{\iota, \varepsilon_1\}$ and extend κ and κ_n

by zero to $\mathbb{R}$. Let $m : \mathbb{R} \to \mathbb{R}_+$ be a standard mollifier supported on $[-1, 1]$ and set $m_\rho(t) = \frac{1}{\rho} m(\frac{t}{\rho})$. For $\rho < \varepsilon_2$ set

$$\hat{\kappa}^\rho := m_\rho * \kappa \text{ and } \hat{\kappa}_n^\rho := m_\rho * \kappa_n. \tag{137}$$

Clearly

$$\hat{\kappa}_n^\rho \to \kappa_n \text{ and } \hat{\kappa}^\rho \to \kappa \text{ strongly in } L^2(0, T)$$

as $\rho \downarrow 0$. Moreover, (82) holds because uniform continuity of $s_\Gamma^\pm$ implies that

$$m_\rho * \frac{1}{s_\Gamma^\pm} \to \frac{1}{s_\Gamma^\pm} \text{ uniformly on } [0, T],$$

and because Γ is uniformly admissible, and by the specific form (137) of $\hat{\kappa}^\rho$.
Since $\rho < \frac{\varepsilon_1}{4}$ we have

$$\hat{\kappa}_n^\rho = 0 \text{ almost everywhere on } (0, \varepsilon_2) \cup (T - \varepsilon_2, T). \tag{138}$$

By Lemma 4 with t_0, t_1, ι as above, there is $\rho \in (0, \delta)$ and a linear function $\tau : (0, T) \to \mathbb{R}$ with $\tau' \in (1 - \delta, 1 + \delta)$ and there are $\varphi, \psi \in C_0^\infty(\iota, T - \iota)$ and curvatures

$$\bar{\kappa}(\tau) := \frac{\hat{\kappa}^\rho + \varphi}{\tau'} \text{ and } \bar{\kappa}_n(\tau) := \frac{\hat{\kappa}_n^\rho + \psi}{\tau'} \tag{139}$$

such that — imposing the usual initial conditions — the curvature $\bar{\kappa}$ induces a curve $\bar{\Gamma} \in W^{2,\infty}(\tau([0, T]); S)$. Moreover, $\bar{\kappa}, \bar{\kappa}_n \in C^\infty(\tau(0, T))$ and by (138)

$$\bar{\kappa}_n(\tau) = 0 \text{ on } (0, \varepsilon_2) \cup (T - \varepsilon_2, T)$$

because $\varepsilon_2 < \frac{\iota}{4}$. Thus

$$\bar{\kappa}_n = 0 \text{ on } B_{\varepsilon_2/2}(\{0, \tau(T)\}) \tag{140}$$

because $\tau' > 1/2$. In addition, $(\bar{\Gamma}, \bar{\kappa}_n) \sim (\Gamma, \kappa_n)$ and

$$\|(\bar{\Gamma}, \bar{\kappa}_n) - (\Gamma, \kappa_n)\|_{W^{2,2}([\Gamma(0,T)];\mathbb{R}^3)} < \delta,$$

so Lemma 6 (i) follows. Moreover, $\bar{\Gamma}$ is uniformly admissible and transversal on $\tau([0, T])$. By (140) and Proposition 3 (ii) we conclude that

$$\hat{u} := (\bar{\Gamma}, \bar{\kappa}_n) \in C_{\text{iso}}^\infty(\overline{[\Gamma(0, T)]}; \mathbb{R}^3).$$

By Proposition 3 (i) and by (27), moreover, item (iii) of the conclusion is satisfied.

Step 2. Make only the extra assumption (136).
By (135) there exists $[t_0, t_1] \subset (0, T)$ such that

$$\mathcal{L}^1\big(\{t \in (t_0, t_1) : \kappa_n(t) \neq 0\}\big) > 0.$$

For

$$\rho \in (0, \min\{|t_0|, |T - t_1|\}/4).$$

define

$$\hat{\kappa}_n^\rho := (1 - \chi_{B_{2\rho}(\{0,T\})})\kappa_n \text{ and } \hat{\kappa}^\rho := \kappa.$$

Apply Lemma 4 with $t_0, t_1, \hat{\kappa}^\rho, \hat{\kappa}_n^\rho$ as just defined. We obtain τ Bilipschitz and

$$\bar{\Gamma} \in W^{2,\infty}(\tau([0, T]); S)$$

which is uniformly admissible and transversal, and we obtain $\bar{\kappa}_n \in L^2(\tau(0, T)$ such that $(\bar{\Gamma}, \bar{\kappa}_n) \sim (\Gamma, \kappa_n)$ and the other conclusions of Lemma 4 are satisfied, too.
By (86), there is $\varepsilon_1 > 0$ such that

$$\bar{\kappa}_n(\tau) = 0 \text{ on } B_{2\varepsilon_1}(\{0, T\}),$$

since spt $\psi^{(\rho)} \subset (t_0, t_1)$, where $\psi^{(\rho)}$ is as in the conclusion of Lemma 4. Thus

$$\bar{\kappa}_n = 0 \text{ on } B_{\varepsilon_1}(\{0, \tau(T)\})$$

because $\tau' > \frac{1}{2}$ for small ρ. Hence $(\bar{\Gamma}, \bar{\kappa}_n)$ satisfies (135) and the hypotheses of Step 1. Moreover, $(\bar{\Gamma}, \bar{\kappa}_n)$ is arbitrarily close to (Γ, κ_n) in $W^{2,2}([\Gamma(0,T)]; \mathbb{R}^3)$, cf. Lemma 4.

Step 3. Let us finally consider the general case.
By (133, 134,135) and since $\kappa_n = 0$ almost everywhere on I_0, we can apply Lemma 3 to obtain $\bar{T} \in (0, T_{\Gamma(0),N(0)}/2)$ and a curve $\bar{\Gamma} \in W^{2,\infty}([0,\bar{T}]; S)$ that is transversal and uniformly admissible on $[0,\bar{T}]$, and we obtain $\bar{\kappa}_n \in L^2(0,\bar{T})$ such that $(\bar{\Gamma}, \bar{\kappa}_n) \sim (\Gamma, \kappa_n)$ and $(\bar{\Gamma}, \bar{\kappa}_n)$ is arbitrarily close to (Γ, κ_n) in $W^{2,2}([\Gamma(0,T)]; \mathbb{R}^3)$. As in Steps 1 and 2 it satisfies the required boundary conditions. The proof is completed by applying Step 2 to $(\bar{\Gamma}, \bar{\kappa}_n)$. □

In the next proposition we drop the transversality hypothesis as well as the assumption (133) made in Lemma 6.

Proposition 4 *For all $\delta > 0$ and S, u, κ_n and Γ as stated at the beginning of this section, there exists $\hat{u} \in C^\infty_{iso}(\overline{\Gamma([0,T)]}; \mathbb{R}^3)$ such that the following are satisfied:*

(i) Boundary conditions:

$$(\hat{u}, \nabla\hat{u}) = \Big((\Gamma, \kappa_n), \nabla(\Gamma, \kappa_n)\Big) \tag{141}$$

on

$$\bigcup_{t \in \{0,T\}} \left(\Gamma(t) + \underline{s}_\Gamma^-(t)N(t), (\Gamma(t) + \overline{s}_\Gamma^+(t)N(t)\right),$$

where $\underline{s}_\Gamma^-, \overline{s}_\Gamma^+$ are as in (25).

(ii) Approximation:

$$\|\hat{u} - (\Gamma, \kappa_n)\|_{W^{2,2}([\Gamma(0,T)];\mathbb{R}^3)} < \delta$$

(iii) Affinity near the relative boundary: There is $\varepsilon > 0$ such that $\hat{u}$ is affine on each connected component of

$$[\Gamma(0,T)] \cap B_\varepsilon(S \cap \partial[\Gamma(0,T)]).$$

Remarks.

(i) In general $\hat{u}$ does not agree with (Γ, κ_n) on the (nonempty) set

$$S \cap \partial[\Gamma(0,T)] \setminus ([\Gamma(0)] \cup [\Gamma(T)]);$$

these are the dashed segments in Figure 4. As mentioned earlier, segments in this set can accumulate inside S.

(ii) In order to enforce Proposition 4 (iii), the features of Lemma 4 mentioned in the remark to that lemma will be important.

(iii) Statement and proof are sketched in Figure 4: The approximant $\hat{u}$ is affine on the regions with diagonal pattern.

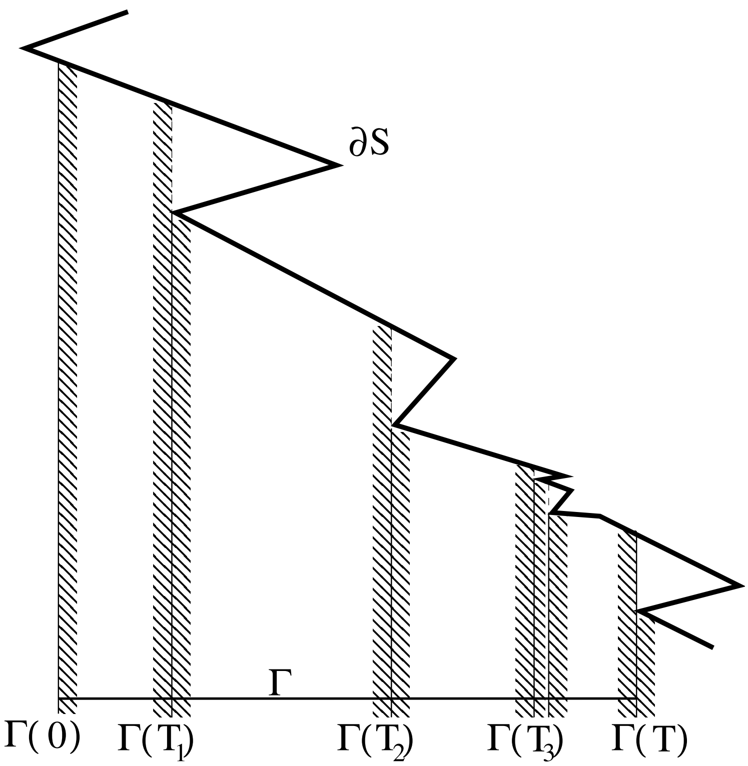

Figure 4: Proposition 4. In general, $(\hat{u}, \nabla\hat{u}) \neq ((\Gamma, \kappa_n), \nabla(\Gamma, \kappa_n))$ on the dashed segments. The intervals I_1, I_2, I_3, I_4 from the proof correspond to the subintervals $\Gamma(I_1)$, ... of Γ which do not intersect the dashed regions; in the figure there are four such subintervals.

Proof. Observe that in order to prove Proposition 4 (i) it suffices to verify that (141) is satisfied on $[\Gamma(0)] \cup [\Gamma(T)]$. The seemingly more general statement then follows from Proposition 2 (iii).
If $\kappa_n = 0$ almost everywhere on $(0, T)$ then (Γ, κ_n) is affine and the lemma is trivial. Therefore we assume that

$$\mathcal{L}^1 \left(\{t \in (0, T) : \kappa_n(t) \neq 0\}\right) > 0. \tag{142}$$

As in [7] we introduce the sets

$$D_\Gamma^\pm = \left\{t \in [0, T] : [\Gamma(t)]_{N(t)} \text{ intersects } \partial S \text{ tangentially at } \Gamma(t) + s_\Gamma^\pm(t)N(t)\right\}$$

and the set

$$E_\Gamma := \{0, T\} \cup D_\Gamma^+ \cup D_\Gamma^- \cup (\beta_\Gamma^+)^{-1}(\Sigma) \cup (\beta_\Gamma^-)^{-1}(\Sigma). \tag{143}$$

In the following steps we subsequently drop extra hypotheses.

Step 1. Make the extra assumption that there is $\varepsilon > 0$ such that

$$\kappa_n = 0 \text{ almost everywhere on } B_{2\varepsilon}(E_\Gamma). \tag{144}$$

By (142, 144), there is $\varepsilon > 0$ such that

$$[0, T] \setminus \bar{B}_\varepsilon(E_\Gamma) \neq \emptyset.$$

This set consists of finitely many nonempty disjoint open intervals $I_1, ..., I_N \subset (0, T)$. There are only finitely many because the complement $[0, T] \cap \bar{B}_\varepsilon(E_\Gamma)$ consists of finitely many connected components; in fact, each maximal interval in $[0, T] \cap \bar{B}_\varepsilon(E_\Gamma)$ has length at least ε. The relatively open set

$$J := [0, T] \setminus (\bar{I}_1 \cup ... \cup \bar{I}_N) \subset B_{2\varepsilon}(E_\Gamma)$$

contains $[0, T] \cap B_\varepsilon(E_\Gamma)$, and $\kappa_n = 0$ almost everywhere on J. So (Γ, κ_n) is affine on every connected component of $[\Gamma(J)]$. Each such component corresponds to one shaded region in Figure 4.
By definition of E_Γ

$$\Gamma \text{ is transversal on } \bar{I}_k, \text{ and } \bar{I}_k \cap (\beta_\Gamma^\pm)^{-1}(\Sigma) = \emptyset.$$

For each $k = 1, ..., N$ we can therefore apply Lemma 6 to $(\Gamma|_{I_k}, \kappa_n|_{I_k})$. It yields $u_k \in C^\infty_{\text{iso}}(\overline{[\Gamma(I_k)]}; \mathbb{R}^3)$ which on $S \cap \partial[\Gamma(I_k)]$ agrees with (Γ, κ_n) up to the first derivatives. Moreover, u_k is arbitrarily close to $(\Gamma, \kappa_n)|_{[\Gamma(I_k)]}$ in $W^{2,2}([\Gamma(I_k)]; \mathbb{R}^3)$. And there is $\varepsilon > 0$ such that

$$u_k \text{ is affine on each component of } [\Gamma(I_k)] \cap B_\varepsilon(S \cap \partial[\Gamma(I_k)])$$

for all k. Thus the composite mapping $\hat{u}$ defined by

$$\hat{u}(x) := \begin{cases} u_k(x) & \text{if } x \in [\Gamma(I_k)] \text{ with } k \in \{1, ..., N\} \\ (\Gamma, \kappa_n)(x) & \text{if } x \in [\Gamma(0, T)] \setminus \bigcup_{k=1}^N [\Gamma(I_k)] \end{cases}$$

has the desired properties, since $\hat{u}$ agrees with (Γ, κ_n) up to the first derivatives on $[\Gamma(0)] \cup [\Gamma(T)]$.

Step 2. General case.
By Proposition 14 in [7] the set E_Γ is closed. By Proposition 14 (iii) in [7] and by condition $(*)$, we have

$$\mathcal{L}^1(E_\Gamma \setminus I_0) = 0,$$

where I_0 is as in (49). Hence $\kappa_n = 0$ almost everywhere on E_Γ. Thus by (142) there exists a Lebesgue point $t_0 \in (0,T) \setminus E_\Gamma$ of κ_n such that $\kappa_n(t_0) \neq 0$. Since $t_0 \notin D_\Gamma^\pm$, the pairs $(\Gamma(t_0), \pm N(t_0))$ are transversal. Hence by Lemma 1 and openness of $(0,T) \setminus E_\Gamma$ there exists $t_1 \in (t_0, T) \setminus E_\Gamma$ such that

$$t_1 - t_0 < \frac{T_{\Gamma(t_0),N(t_0)}}{2}$$

and

$$[t_0, t_1] \subset (0,T) \setminus E_\Gamma.$$

Since t_0 is a Lebesgue point of κ_n with $\kappa_n(t_0) \neq 0$, in particular we have

$$\mathcal{L}^1\left(\{t \in (t_0, t_1) : \kappa_n(t) \neq 0\}\right) > 0.$$

After applying Lemma 3 to the restrictions of Γ and κ_n to $[t_0, t_1]$, we may assume without loss of generality that Γ is uniformly admissible on $[t_0, t_1]$.
Let $\varepsilon, \delta > 0$. Set

$$\hat{\kappa}_n^\varepsilon := (1 - \chi_{B_{4\varepsilon}(E_\Gamma)})\kappa_n.$$

Since E_Γ is closed, we have

$$E_\Gamma = \bigcap_{\varepsilon > 0} B_{4\varepsilon}(E_\Gamma).$$

Since $\kappa_n = 0$ almost everywhere on E_Γ, we conclude that

$$\hat{\kappa}_n^\varepsilon \to \kappa_n \text{ strongly in } L^2(0,T)$$

as $\varepsilon \to 0$. Clearly $|\hat{\kappa}_n^\varepsilon| \leq |\kappa_n|$. Hence we can apply Lemma 4 with t_0, t_1 and $\hat{\kappa}_n^\varepsilon$ as just defined and with $\hat{\kappa}^\varepsilon := \kappa$. It yields (omitting the parameter ε) $\tau : (0,T) \to \mathbb{R}$ with $\tau(0) = 0$ and $\|\tau' - 1\|_{L^\infty(0,T)}$ arbitrarily small, and we obtain a curve $\bar{\Gamma} \in W^{2,\infty}(\tau([0,T]); S)$ such that

$$[\bar{\Gamma}(\tau(t))] = [\Gamma(t)] \text{ for all } t \in [0,T] \setminus (t_0, t_1),$$

and $[\bar{\Gamma}(\tau(0,T))] = [\Gamma(0,T)]$. Moreover, there is $\bar{\kappa}_n \in L^2(\tau(0,T))$ such that $(\bar{\Gamma}, \bar{\kappa}_n) \in W^{2,2}_{\text{iso}}([\Gamma(0,T)]; \mathbb{R}^3)$ is arbitrarily close to (Γ, κ_n) in $W^{2,2}([\Gamma(0,T)]; \mathbb{R}^3)$ and such that $(\bar{\Gamma}, \bar{\kappa}_n) \sim (\Gamma, \kappa_n)$. In particular, $(\bar{\Gamma}, \bar{\kappa}_n)$ and (Γ, κ_n) agree up to the first derivatives on $[\Gamma(0)] \cup [\Gamma(T)]$. Moreover, $(\bar{\Gamma}, \bar{\kappa}_n)$ satisfies the hypotheses of Step 1. In fact,

$$\bar{\kappa}_n = 0 \text{ on } B_{2\varepsilon}(\tau(E_\Gamma)) \subset \tau(B_{4\varepsilon}(E_\Gamma))$$

because τ' is close to 1 in $L^\infty(0,T)$, and clearly $\tau(E_\Gamma) = E_{\bar{\Gamma}}$. Applying Step 1 to $(\bar{\Gamma}, \bar{\kappa}_n)$ concludes the proof of the proposition. □

6 Proof of Theorem 1

Define

$$E_\delta = \{x \in S : \text{dist}_{\partial S}(x) > \delta\}$$

and denote by $\mathcal{U}_\delta$ the set of all connected components U of $\hat{C}_{\nabla u}$ with $U \cap E_\delta \neq \emptyset$. A byproduct of the following proposition is the existence of Lipschitz continuous approximants; this recovers Theorem 2 from [12] (but not their Remark 12). The conclusion of Proposition 5 which is essential here, however, is that there are approximants u_δ whose gradient is finitely developable in the sense of [7]. This means that $\hat{C}_{\nabla u_\delta}$ consists of finitely many connected components U and that for each of them the relative boundary $S \cap \partial U$ consists of finitely many connected components (i.e. maximal line segments).

Proposition 5 *Let $S \subset \mathbb{R}^2$ be a bounded Lipschitz domain and let $u \in W^{2,2}_{iso}(S;\mathbb{R}^3)$. Then, for every $\delta > 0$ there exists $u_\delta \in W^{2,2}_{iso}(S;\mathbb{R}^3)$ with the following properties:*

(i) The gradient ∇u_δ is finitely developable, and $u_\delta \in W^{1,\infty}(S;\mathbb{R}^3)$.

(ii) $u_\delta = u$ on the set

$$S_\delta = \bigcup_{x\in E_\delta\setminus\hat{C}_{\nabla u}} [x] \cup \bigcup_{U\in\mathcal{U}_\delta} U,$$

and u_δ is affine on every connected component of $S \setminus \overline{S}_\delta$.

(iii) $u_\delta \to u$ strongly in $W^{2,2}(S;\mathbb{R}^3)$ as $\delta \downarrow 0$.

Proof. By Theorem 3 in [7] the set $S \setminus \overline{S}_\delta$ consists of countably many connected components $W_1, W_2, ...$ which satisfy $\bar{W}_i \cap \bar{W}_j \cap S = \emptyset$, whenever $i \neq j$, and for all i there is $x_i \in S \setminus \hat{C}_{\nabla u}$ such that $\bar{S}_\delta \cap \bar{W}_j \cap S = [x_i]$. Define

$$u_\delta = \begin{cases} u & \text{on } S \cap \bar{S}_\delta \\ \tilde{u}_{x_i} & \text{on } W_i \text{ (i = 1, 2, ...)}; \end{cases}$$

here, $\tilde{u}_{x_i} : \mathbb{R}^2 \to \mathbb{R}^3$ is the affine mapping which agrees, up to the first derivatives, with u on $[x_i]$.
The mapping u is well-defined on S and C^1, and it is easily seen to belong to $W^{2,2}_{\text{iso}}$; this follows from the fact that it is piecewise $W^{2,2}_{\text{iso}}$ and globally C^1. And clearly $u_\delta \to u$ strongly in $W^{2,2}(S;\mathbb{R}^3)$ as $\delta \downarrow 0$ because S_δ contains E_δ, and clearly the area of $S \setminus E_\delta$ converges to zero as $\delta \downarrow 0$. Theorem 3 in [7] shows that ∇u_δ is finitely developable and belongs to L^∞. □

Proofof Theorem 1. Let S be a bounded Lipschitz domain satisfying condition $(*)$, let $u \in W^{2,2}_{\text{iso}}(S;\mathbb{R}^3)$ and let $\delta > 0$. By Proposition 5 and a diagonal sequence argument, we may assume without loss of generality that ∇u is finitely developable.

By Theorem 4 in [7] there exists $N \in \mathbb{N}$ and subdomains

$$V_1, ..., V_N \subset S$$

as well as subdomains

$$W_1, W_2, ... \subset S \cap B_\delta(\partial S)$$

such that the set

$$W_0 := \text{int}\left(\bigcup_{k=1}^N \bar{V}_k\right)$$

is a subdomain of S and such that

$$W_j \cap W_k = \emptyset \text{ and } V_j \cap V_k = \emptyset \text{ whenever } j \neq k. \tag{145}$$

Moreover, for every $k \geq 1$ there exists $x_k \in S \setminus \hat{C}_{\nabla u}$ such that

$$S \cap \bar{W}_j \cap \bar{W}_k = \begin{cases} \emptyset & \text{if } 0 < j < k \\ [x_k] & \text{if } 0 = j < k. \end{cases} \tag{146}$$

In addition,

$$S = W_0 \cup \bigcup_{k\geq 1} (S \cap \bar{W}_k). \tag{147}$$

More precisely, there is $M \in \mathbb{N}$ with $M \leq N$ such that $V_1, ..., V_M$, are the connected components of $\hat{C}_{\nabla u}$ and for all $k = M + 1, ..., N$ there exist $T_k > 0$ and lines of curvature

$$\Gamma^{(k)} \in W^{2,\infty}([0, T_k]; S)$$

for u such that $V_k = [\Gamma^{(k)}(0, T_k)]$ and $V_k \cap \hat{C}_{\nabla u} = \emptyset$. Furthermore, if $j < k$ and $S \cap \bar{V}_j \cap \bar{V}_k \neq \emptyset$ then $k \geq M + 1$, and

$$S \cap \bar{V}_j \cap \bar{V}_k \subset \bigcup_{t \in \{0, T_k\}} \left(\Gamma(t) + \underline{s}_\Gamma^-(t)N(t), (\Gamma(t) + \overline{s}_\Gamma^+(t)N(t)\right), \tag{148}$$

where $\overline{s}_\Gamma^+$ and $\underline{s}_\Gamma^-$ are as in (25). We apply Proposition 4 on each $V_{M+1}, ..., V_N$. For every $k = M + 1, ..., N$ it yields

$$u^{(k)} \in C^\infty_{\text{iso}}(\bar{V}_k; \mathbb{R}^3)$$

with

$$\int_{V_k} |\nabla^2 u^{(k)} - \nabla^2 u|^2 \leq \frac{\delta}{N} \tag{149}$$

satisfying

$$(u^{(k)}, \nabla u^{(k)}) = (u, \nabla u) \text{ on } \bigcup_{t \in \{0, T_k\}} \left(\Gamma(t) + \underline{s}_\Gamma^-(t)N(t), (\Gamma(t) + \overline{s}_\Gamma^+(t)N(t)\right). \tag{150}$$

Moreover, there is $\varepsilon > 0$ such that

$$\nabla^2 u^{(k)} = 0 \text{ on } V_k \cap B_\varepsilon(S \cap \partial V_k). \tag{151}$$

In order to obtain a consistent notation we set

$$u^{(k)} = u \text{ on } V_k \text{ for } k = 1, ..., M. \tag{152}$$

Observe that then (149, 151) are trivially satisfied for $k \leq M$, too.

Now define the mapping $\hat{u} : W_0 \to \mathbb{R}^3$ by setting

$$\hat{u}(x) := u^{(k)}(x) \text{ if } x \in S \cap \bar{V}_k \text{ for some } k = 1, ..., N. \tag{153}$$

Observe that we are taking the closures here, so one must check that $\hat{u}$ is well-defined on W_0. But this follows from the definition of W_0 and by virtue of (145, 148, 150). And (148, 150) also imply that $\hat{u}_\delta \in C^1(W_0; \mathbb{R}^3)$. Now define $u_\delta : S \to \mathbb{R}^3$ by setting

$$u_\delta(x) = \begin{cases} \hat{u}(x) & \text{if } x \in W_0 \\ \tilde{u}_{x_i}(x) & \text{if } x \in S \cap \bar{W}_i \text{ for some } i = 1, 2, ...; \end{cases}$$

here, $\tilde{u}_y : \mathbb{R}^2 \to \mathbb{R}^3$ is the affine mapping satisfying

$$(\tilde{u}_y, \nabla \tilde{u}_y) = (\hat{u}, \nabla \hat{u}) \text{ on } [y].$$

It follows from (147, 146) that u_δ is well-defined on S. By definition of the $\tilde{u}_{x_i}$ it is obviously C^1 on S because it is C^1 on W_0. Since u_δ is piecewise $W^{2,2}_{\text{iso}}$ and globally C^1, we conclude $u_\delta \in W^{2,2}_{\text{iso}}(S; \mathbb{R}^3)$. Moreover, u_δ is arbitrarily close in $W^{2,2}(S; \mathbb{R}^3)$ to u. In fact, by (147),

$$\int_S |\nabla^2 u_\delta - \nabla^2 u|^2 = \sum_{k=1}^N \int_{V_k} |\nabla^2 u^{(k)} - \nabla^2 u|^2 + \sum_{i \geq 1} \int_{W_i} |\nabla^2 u|^2 \tag{154}$$

because u_δ is affine on each $W_1, W_2, ...$ Since $|\nabla^2 u| \in L^1(S)$, since $W_1, W_2, ...$ are pairwise disjoint and since each of them is contained in $B_\delta(\partial S)$, we conclude that the second sum on the right-hand side of (154) converges to zero as $\delta \downarrow 0$. The first sum on the right-hand side of (154) is bounded by δ because of (149, 152). Hence Poincaré's inequality implies that

$$u_\delta \to u \text{ strongly in } W^{2,2}(S; \mathbb{R}^3) \text{ as } \delta \downarrow 0.$$

Finally, it is clear that $u_\delta \in C^\infty(\bar{S}; \mathbb{R}^3)$ because $u^{(k)} \in C^\infty(\bar{V}_k; \mathbb{R}^3)$ for all $k = 1, ..., N$, because of (151) and because each $\tilde{u}_{x_i}$ is affine. □

7 An irregular $W^{2,2}$ isometric immersion

In this section we provide an example showing that Lemma 3 is really needed: There is a convex domain $S \subset \mathbb{R}^2$ and $\Gamma \in W^{2,\infty}([0,1];S)$ and $\kappa_n \in L^\infty(0,1)$ such that $(\Gamma, \kappa_n) \in W^{2,2}_{\text{iso}}([\Gamma(0,1)];\mathbb{R}^3)$ and such that there is no interval on which Γ is uniformly admissible. This example also clarifies some misunderstandings in [13].

Let $S := (-2,2)^2$, let $\theta \in (0, \frac{1}{2})$ and set $r_n := 2^{-(2n+1)}\theta$. Obviously, any arclength parametrized curve $\Gamma \in W^{2,\infty}([0,1];\mathbb{R}^2)$ with $\Gamma(0) = 0$ satisfies $\Gamma([0,1]) \subset S$. We construct a Cantor set as follows: Let $t^{(0)} = \frac{1}{2}$ and set $A_1^{(0)} := (t^{(0)} - \frac{r_0}{2}, t^{(1)} + \frac{r_0}{2})$. Denoting by $t_1^{(1)}$ and $t_2^{(1)}$ the middle points of the two connected components of $(0,1) \setminus A_1^{(0)}$ we define the sets $A_i^{(1)} = (t_i^{(1)} - \frac{r_1}{2}, t_i^{(1)} + \frac{r_1}{2})$, $i = 1, 2$. As usual, we define the sets $A_i^{(n)}$, $i = 1, ..., 2^n$, inductively for all $n \in \mathbb{N}$ and set $E_k = \bigcup_{n=0}^k \bigcup_{i=1}^{2^n} A_i^{(n)}$. As usual, the set $E := \bigcup_{k=1}^\infty E_k$ satisfies $|E| = \theta$, and the Cantor set $[0,1] \setminus E$ is totally disconnected, while E is open and dense in $[0,1]$.
For each k, denote by $0 = a_0^{(k)} < a_1^{(k)} < ... < a_k^{(k)}$ the minima of the connected components of $[0,1] \setminus E_k$, and by $1 = b_k^{(k)} > ... > b_0^{(k)}$ their maxima. Now we define the functions $\kappa^{(k)}$ as follows. Set $\kappa^{(k)} = 0$ on $[0,1] \setminus E_k$, so in particular $\kappa^{(k)}$ is defined on the interval $[a_0^{(k)}, b_0^{(k)}]$. Thus the unique arc-length parametrized curve $\Gamma^{(k)}$ given via the Frénet equations by the initial values $\Gamma^{(k)}(0) = 0$ and $(\Gamma^{(k)})'(0) = e_1$ and the curvature $\kappa^{(k)}$ is well defined on this interval (it is a straight line). Hence $s_k^+(t) := s_{\Gamma^{(k)}}(t) = \nu^S(\Gamma^{(k)}(t), N^{(k)}(t))$ is also defined on this interval. Next we set $\kappa^{(k)}|_{(b_0^{(k)}, a_1^{(k)})} = \frac{1}{s_k^+(b_0)}$, so $\Gamma^{(k)}$ and s_k^+ are well defined up to b_1, and then we set $\kappa^{(k)}|_{(b_1^{(k)}, a_2^{(k)})} = \frac{1}{s_k^+(b_1)}$ and so on. Proceeding inductively we obtain

$$\kappa^{(k)} = \frac{1}{s_k^+} \chi_{E_k} \tag{155}$$

and $\Gamma^{(k)}$ and s_k^+ are defined on $[0,1]$ and $s_k^+(0) = \frac{1}{2}$ for all $k \in \mathbb{N}$. Moreover, since $\Gamma^{(k)}([0,1]) \subset \bar{B}_1(0)$, we have $s_k^+ \geq 1$ on $[0,1]$ for all k. Thus $\|\kappa^{(k)}\|_{L^\infty(0,1)} \leq 1$ for all k. So after passing to subsequences $\kappa^{(k)} \overset{*}{\rightharpoonup} \kappa$ for some $\kappa \in L^\infty(0,1)$. Define the arc-length parametrized curve Γ by $\Gamma(0) = 0$, $\Gamma'(0) = e_1$ and the curvature κ, and as usual set $s_\Gamma^+(t) := \nu(\Gamma(t), N(t))$. We claim that

$$\kappa = \frac{1}{s_\Gamma^+} \chi_E. \tag{156}$$

In fact, since $E_k \subset E$ for all $k \in \mathbb{N}$, by (155) and weak-$*$ convergence we conclude $\int_{(0,1)\setminus E} \kappa = 0$. But $\kappa^{(k)} \geq 0$ for all k, so $\kappa \geq 0$. Hence

$$\kappa = 0 \text{ almost everywhere on } (0,1) \setminus E. \tag{157}$$

Moreover, $\kappa^{(k)} \overset{*}{\rightharpoonup} \kappa$ implies that $(\Gamma^{(k)})' \to \Gamma'$ pointwise because

$$(\Gamma^{(k)})' = (\cos \int_0^t \kappa, \sin \int_0^t \kappa)^T.$$

Hence $\Gamma^{(k)} \to \Gamma$ pointwise. Hence by continuity of ν^S — which follows from Proposition 14 in [7] and convexity of S — we also have that $s_k^+ \to s_\Gamma^+$ pointwise. Since also $\chi_{E_k} \to \chi_E$ pointwise, by (155) we conclude that $\kappa^{(k)} \to \chi_E \frac{1}{s_\Gamma^+}$ pointwise, and (156) follows. Notice that Γ is admissible and transversal on $[0,1]$. In fact, admissibility is clear from the construction and transversality follows from convexity of S. Moreover, we have

$$\Gamma'(t) \cdot \Gamma'(t') > 0 \text{ for all } t, t' \in [0,1]. \tag{158}$$

This is because the length of Γ is $1 = \|\kappa\|_{L^\infty(0,1)}^{-1}$. Finally, set e.g. $\kappa_n := 1 - \chi_E$. Then $(\Gamma, \kappa_n) \subset W^{2,2}_{\text{iso}}([\Gamma(0,1)]; \mathbb{R}^3)$ by Proposition 2 because by construction $I_0 = E$.

In Section 3.1 of [13] the set I_γ is introduced, which in our notation would be given by

$$I_\gamma := \{t \in [0,1] : \overline{[\Gamma(t)]}^{\mathbb{R}^2} \cap \overline{[\Gamma(t')]}^{\mathbb{R}^2} \cap \bar{S} = \emptyset \text{ for all } t' \neq t\}.$$

The omission of the condition $t \neq t'$ in [13] is clearly not intended, since it would imply that always $I_\gamma = \emptyset$. In [13] it is claimed that I_γ is relatively open in $[0,1]$. However, this is not the case e.g. for the curve Γ constructed above.
In fact, since $\kappa \geq 0$ clearly

$$\frac{1}{\sigma_\Gamma} \geq 0 \text{ on } [0,1] \times [0,1], \tag{159}$$

where σ_Γ is as defined in [7]. To see this rigorously, one can use (71) in [7]: Integrating (158) gives

$$(\Gamma(t) - \Gamma(t')) \cdot \Gamma'(t) < 0 \text{ for } t' > t.$$

But also

$$N(t') \cdot \Gamma'(t) = -\int_t^{t'} \kappa(s)\Gamma'(s) \cdot \Gamma'(t)\, ds \leq 0,$$

since $\kappa \geq 0$ and by (158). This proves (159). Thus

$$I_\gamma = \{t' \in [0,1] : \beta_\Gamma^+(t') \neq \beta_\Gamma^+(t) \text{ for all } t \in [0,1] \setminus \{t'\}\}. \tag{160}$$

Since S is a Lipschitz domain, transversality and Proposition 14 in [7] imply that ν is Lipschitz in a neighbourhood of $\bigcup_{t\in[0,1]}(\Gamma(t), N(t))$. Hence s_Γ^+ is Lipschitz with

$$(s_\Gamma^+)' = (1 - s_\Gamma^+\kappa)\nu_1(\Gamma, N) \cdot \Gamma',$$

where ν_1 denotes the gradient of ν with respect to the first argument, cf. e.g. Lemma 2.1 in [6]. Hence

$$(\beta_\Gamma^+)' = (1 - s_\Gamma^+)\Big(\Gamma' + (\nu_1(\Gamma, N) \cdot \Gamma')N\Big), \tag{161}$$

where ν_1 is the gradient with respect to the first entry. In particular, β_Γ^+ is constant on every connected component of $\{1 - s^+\kappa = 0\}$. Hence β_Γ^+ is constant on every connected component of E. Therefore,

$$I_\gamma \cap E = \emptyset. \tag{162}$$

We claim that

$$I_\gamma = \{t' \in [0,1] : \exists\varepsilon > 0 : \beta_\Gamma^+(t') \neq \beta_\Gamma^+(t) \text{ for all } t \in [0,1] \cap B_\varepsilon(t') \setminus \{t'\}\}, \tag{163}$$

i.e., the set I_γ consists of those points near which β_Γ^+ is strictly monotone. Equation (163) follows from Lemma 14 in [7], but since S is convex, here we can use the following simple analytic argument: Clearly

$$1 - s^+\kappa \geq 0 \text{ almost everywhere on } (0,1); \tag{164}$$

this is clear from the construction. In view of (161, 164) we conclude that

$$\beta_\Gamma^+ \cdot e_1 \text{ is monotone.}$$

Since $\beta_\Gamma^+ \cdot e_2 \equiv 2$, equality (163) follows.

Now let $t' \in (0,1) \setminus E$ be a Lebesgue point of κ. By (157), Lemma 8 in [7] implies that there is $\varepsilon > 0$ such that $\sigma_\Gamma(t, t') > 20$ and $\sigma_\Gamma(t', t) > 20$ for all $t \in B_\varepsilon(t')$. Hence $\beta_\Gamma^+(t) \neq \beta_\Gamma^+(t')$. Thus from (163) we deduce that almost every $t' \in (0,1) \setminus E$ is contained in I_γ. Thus

$$\mathcal{L}^1\big((0,1) \setminus E \setminus I_\gamma\big) = 0.$$

By (162), if I_γ were open then it would have to be empty. By (163), however,

$$\mathcal{L}^1(I_\gamma) \geq 1 - \theta > 0.$$

We conclude that I_γ cannot be open. Instead, combining (162) and (163), we see that, up to an $\mathcal{L}^1$-null set,

$$I_\gamma = [0,1] \setminus E,$$

and the set on the right is a Cantor set.
The reader can easily deduce from the above that the quantities S_+^γ and S_-^γ introduced in Section 3.2 of [13] are not continuous in general.

Finally, observe that Γ as above is not uniformly admissible on any subinterval. This explains the need for a result like Lemma 3: One has to modify Γ in order to obtain a curve that is uniformly admissible on some interval. This, in turn, is clearly necessary if one wants to modify that curve without spoiling local admissibility.

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