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and Geodesic Flow : A Novel Approach

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It is a pleasure to dedicate this paper to Professor Jerry Marsden on his 65th birthday.

Abstract

We use the logarithmic 2-cocycle and the action of $Vect(S^1)$ on the space of Pseudo-differential symbols to derive one particular type of supersymmetric KdV equation, known as Kuper-KdV equation. This equation was formulated by Kupershmidt and it is different from the Manin-Radul-Mathieu type equation. The two Super KdV equations behave differently under a supersymmetric transformation and Kupershmidt version does not preserve SUSY transformation. In this paper we study the second type of supersymmetric generalization of the Camassa-Holm equation corresponding to Kuper-KdV equation via standard embedding of super vector fields into the Lie algebra of graded pseudodifferential symbols. The natural lift of the action of superconformal group $SDiff$ yields $SDiff$ module. This method is particularly useful to construct Moyal quantized systems.

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1 Introduction

Supersymmetric integrable systems play a very interesting role in modern mathematical physics and consequently a number of well known integrable equations have been supersymmetrized. The supersymmetric extension of a nonlinear evolution equation characterizes to a system of coupled equations for a bosonic field and a fermionic field, which reduces to original equation in the limit where the fermionic field is zero. Here a fermionic field is described by an anticommuting function with values in an infinitely generated Grassmann algebra. There are many generalizations related to the number N of fermionic independent variables. In this paper we will focus on $N = 1$ superspace.

It is known to most of us that the Korteweg-de Vries (KdV) is the best studied equation in integrable systems. So it is not surprising that the first steps in the field of supersymmetric integrable systems consisted in constructing supersymmetric extensions of the KdV equation. The $N = 1$ supersymmetry (SUSY) means that we are dealing with only one Grassmann variable θ . It is known that $N = 1$ superconformal (or super Virasoro) algebra can be related to fermionic extensions of the KdV equation. Using the superspace formalism one can construct *two* different fermionic extensions of the KdV equation. The first extension was proposed by Manin and Radul [17]. This $N = 1$ SUSY KdV is given by

$$\Phi_t + D^6\Phi + 3D^2(\Phi D\Phi) = 0, \quad (1)$$

where $\Phi(t, x, \theta) = \phi(t, x) + \theta u(t, x)$ stands for superfield. Thus equation (1) can be expressed as

$$\begin{aligned} u_t &= -u_{xxx} + 6uu_x - 3\phi\phi_x, \\ \phi_t &= -\phi_{xxx} + 3(u\phi)_x. \end{aligned} \quad (2)$$

Mathieu [18] showed that this equation is associated to the (super) Lax operator

$$L = D^4 - \Phi D \quad \text{or} \quad L = D^4 - (D\Phi) + \Phi D,$$

where $D = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial x}$ denotes superderivative. After a suitable scaling Mathieu showed that the Eqn. (2) is equivalent to the sKdV equation obtained by Manin and Radul from the reduction of the super Kadomtsev-Petviashvili (sKP) hierarchy. In fact, the second Hamiltonian structure of this system was shown by Mathieu to correspond to the superconformal algebra of the superstring theories.

The *second class* of super KdV was proposed by Kupershmidt in 1984 [16], given by

$$\begin{aligned} u_t &= -u_{xxx} + 6uu_x - 12\phi\phi_x, \\ \phi_t &= -\phi_{xxx} + 3u_x\phi + 6u\phi_x. \end{aligned} \quad (3)$$

This equation is associated to the Lax operator

$$L = \partial^2 - u + \phi \partial^{-1} \phi.$$

In fact, Kupershmidt [16] demonstrated the bi-hamiltonian property of this equation. Ovsienko and Khesin [20] showed that the super KdV equation is the Euler equation corresponding to the inertia operator for the Neveu-Schwarz and Ramond superalgebras [cf.14], which are the simplest superanalogues of the Virasoro algebra.

In [18] Mathieu pointed out that this super equation is indeed invariant under a space supersymmetry transformation, while Kupershmidt version does not preserve SUSY transformation. So, sometimes it is appropriate call Eqn. (3) the *Kuper-KdV* equation. Moreover a Painlevé analysis by Mathieu of possible supersymmetric extensions of the KdV equation seems to suggest that the only integrable extensions are the Manin-Radul KdV and the one of Kupershmidt.

It is known that the KdV equation is the geodesic flow with respect to right invariant L^2 metric on the Bott-Virasoro group. Using superconformal group with an L^2 -metric Ovsienko and Khesin obtained the Kuper-KdV system of Kupershmidt.

In this paper we study the super analogue of the Camassa-Holm equation [4,5]

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx} \quad t \geq 0, \quad x \in \mathbb{R}. \quad (4)$$

This equation is a model for the unidirectional propagation of shallow water waves, with $u(t, x)$ representing the free surface of the water above a flat bed [4,5]. Introducing new variable $m = u - u_{xx}$, i.e., the Helmholtz operator action on u , it may be rewritten

$$m_t + um_x + 2u_x m = 0. \quad (5)$$

Misiolek [18] showed that if the metric is changed to the Sobolev H^1 -metric, $\int_{S^1} (u^2 + u_x^2) dx$, the Camassa-Holm equation is a re-expression of geodesic flow on the Bott-Virasoro group.

This had drawn considerable interest. Devchand and Schiff [6] formulated the supersymmetrisation of the Camassa-Holm equation as a geodesic flow on the superconformal group. Most recently author and Olver [8] have shown that the multicomponent Camassa-Holm equation and their supersymmetrisation are the geodesic flows on the extension of the Bott-Virasoro and superconformal groups respectively.

The Fermionic extension of Camassa-Holm equation that Devchand and Schiff have studied is related to Mathieu and Manin-Radul type KdV systems. Our goal is to give an alternative description of Kuper-KdV and to give the Euler-Poincaré framework of much awaited Kupershmidt-Camassa-Holm equation. We take a different route. We embed the vector field and its dual to pseudodifferential symbols on S^1 . These are functions of cotangent bundle T^*S^1 . The Lie algebra $Vect(M)$ of vector fields on a

manifold M has a natural embedding π into the Poisson algebra of functions on the cotangent bundle T^*M , given in the canonical coordinates (q, p) by

$$\pi \left(\sum_{i=1}^n X^i(q) \frac{\partial}{\partial q^i} \right) = \sum_{i=1}^n X^i(q) p_i.$$

In a nice piece of work Ovsienko and Roger [21,22] study – (a) formal deformations of the embeddings of the Lie algebra $Vect(S^1)$ into the Poisson algebra of functions which are Laurent polynomials on fibers and (b) polynomial deformations of the embedding of $Vect(S^1)$ into the Poisson algebra of formal Laurent series on T^*S^1 (or, rather, on T^*S^1 with the zero-section removed). In this paper we lift the action of $Vect(S^1)$ on the space of pseudodifferential symbols to derive dispersionless class of integrable systems. The full integrable system can be manufactured via logarithmic 2-cocycle [13,15]. In fact by deforming this action one could derive Moyal deformed integrable systems [10].

The *objective* of this paper is to give a new derivation of the Kupershmidt-Camassa-Holm (or Kuper Camassa-Holm) equation using algebra of pseudodifferential symbols. This method is particularly useful to derive the noncommutative (or Moyal deformed) version of the Kuper Camassa-Holm equation.

2 Pseudodifferential Symbols on S^1

The ring of pseudodifferential symbols on S^1 , $\Psi D(S^1)$, is defined to be the ring of formal Laurent series with finite number of positive powers $\sum_{k < k_0} f_k(x) \xi^k$ over $C^\infty(S^1)$ with finite number of positive powers. There are two differentiations defined in ring $\Psi D(S^1)$

$$\partial_\xi : \sum_k f_k \xi^k \mapsto \sum_k k f_k \xi^{k-1} \quad \partial : \sum_k f_k \xi^k \mapsto \sum_k f'_k \xi^k. \quad (6)$$

Differentiations (6) are differentiations of $\Psi D(S^1)$ as a $\mathbb{R}$ - ring. These differentiations may be used to define the symbolic multiplication on the ring by setting

$$(F \circ G) = \sum_{k=0}^{\infty} \frac{1}{k!} : \frac{\partial^k F}{\partial \xi^k}(x, \xi) \frac{\partial^k G}{\partial x^k}(x, \xi) :.$$

Here $: \cdot :$ is called the Normal Ordering; it is defined by

$$: f(x) \xi^k g(x) \xi^l := f(x) g(x) \xi^{k+l}.$$

This normal ordering $: \cdot :$ is a usual product of monomials.

The multiplication rules yields an associative and and a Lie algebra operation on the ring. The commutator and the residue map are defined, respectively, by setting $[F, G] = F \circ G - G \circ F$ and $\text{res} : \Psi D(S^1) \rightarrow C^\infty(S^1)$. The “trace operation” is defined by [1]

$$\text{Tr}(F) = \int_{S^1} \text{res } F dx = \int_{S^1} f_{-1} dx, \quad (7)$$

In otherwords,

$$res : \Psi D(S^1) \ni F \rightarrow f_{-1} \in C^\infty(S^1).$$

This formula is known as Adler trace formula. The main property [1] of the residue (or trace) is for $F, G \in \psi D(S^1)$,

$$Tr([F, G]) = \int res [F, G] dx = 0.$$

It is easy to see this since the residue of a commutator is a total derivative.

2.1 Construction of dispersionless KdV equation

Suppose $L_{f(x)\frac{d}{dx}}^{(\lambda)}$ is considered to be the action of Lie derivative on $\mathcal{F}_\lambda$, tensor densities of degree $-\lambda$, i.e. $a = a(x)dx^{-\lambda} \in \mathcal{F}_\lambda$. A 1-parameter family of $Vect(S^1)$ actions on $C^\infty(S^1)$

$$L_{f(x)\frac{d}{dx}}^{(\lambda)} a(x) = f(x)a' - \lambda f'(x)a(x) \quad \text{for } \lambda \in \mathbf{R}. \quad (8)$$

can be interpreted as an action of $Vect(S^1)$ on $\mathcal{F}_\lambda$. We show that this action can be realized from the Poisson action. We consider the subspace of $\Psi D(S^1)$ consisting of functions of degree λ in ξ , given by $a(x)\xi^\lambda$. This subspace is $Vect(S^1)$ module isomorphic to $\mathcal{F}_{-\lambda}$. Set

$$L_{f(x)\frac{d}{dx}}^{(\lambda)}(a)\xi^\lambda = (fa' - \lambda f'a)\xi^\lambda \quad (9)$$

where the canonical embedding $\pi : Vect(S^1) \rightarrow \psi D(S^1)$ identifies $\pi(f(x)\partial) = f(x)\xi$. This action can be realized in terms of Poisson action

$$L_{f(x)\frac{d}{dx}}^{(\lambda)}(a)\xi^\lambda = \{f(x)\xi, a(x)\xi^\lambda\},$$

where $\{.,.\}$ is the Poisson bracket defined with respect to ξ and x . For $\lambda = 1$ this coincides with the usual commutation relation between two vector fields.

Remark This idea can be extended to multivector fields on a manifold M . Consider multivector field $F \in \mathcal{X}^p(M)$ such that

$$F = \sum_{i_1 < \dots < i_p} F_{i_1 \dots i_p} \partial_{x_{i_1}} \wedge \dots \wedge \partial_{x_{i_p}} = \sum_{i_1 < \dots < i_p} F_{i_1 \dots i_p} \xi_{i_1} \dots \xi_{i_p},$$

where $\xi_{i_1} = \partial_{x_{i_1}}$ and treat it as an odd formal variable

$$\xi_i \xi_j = -\xi_j \xi_i \quad (\partial_{x_i} \wedge \partial_{x_j} = -\partial_{x_j} \wedge \partial_{x_i}).$$

Then the Schouten-Nijenhuis bracket between two multivector fields satisfies

$$[F, G] = \sum_i \partial_{\xi_i} F \partial_{x_i} G - (-1)^{(p-1)(q-1)} \partial_{\xi_i} G \partial_{x_i} F,$$

where we use following differentiation rule

$$\partial_{\xi_{i_k}} (\xi_{i_1} \dots \xi_{i_p}) = (-1)^{p-k} \xi_{i_1} \dots \hat{\xi}_{i_k} \dots \xi_{i_p}.$$

We return back to the embedding. It enables one to pass from the Virasoro algebra to the algebra of groups of area preserving diffeomorphisms. Following Kirillov [14] it is known that geometrically the regular dual space of the vector field on S^1 is the space of quadratic differentials. Note that as a vector space the space of quadratic differentials $\mathcal{F}_2 \simeq C^\infty(S^1)$. Hence the vector field on S^1 is identified with $u(x)\xi^{-2} \in \mathcal{F}_2$.

It is clear now that the space $\Psi D(S^1)$ is a $Vect(S^1)$ module. Thus the coadjoint action of $Vect(S^1)$ on its dual is given by

$$\begin{aligned} ad_{f(x)\xi}^* u(x)\xi^{-2} &= \{f(x)\xi, u(x)\xi^{-2}\} \\ &= -\left(\frac{\partial}{\partial \xi}(f(x)\xi)\frac{\partial}{\partial x}(u\xi^{-2}) - \frac{\partial}{\partial x}(f(x)\xi)\frac{\partial}{\partial \xi}(u\xi^{-2})\right) \\ &= -(fu' + 2f'u)\xi^{-2}. \end{aligned}$$

Thus we obtain the Hamiltonian operator from this expression.

The regular dual $Vect(S^1)^*$ is the subspace of $Vect(S^1)^*$ defined by linear functionals of the form

$$f \longmapsto \int_{S^1} u f dx$$

for some function $u \in C^\infty(S^1)$. The regular dual is therefore isomorphic to $C^\infty(S^1)$ by means of inner product. Let F be a smooth real valued function on $C^\infty(S^1)$. Its Fréchet derivative $dF(u)$ is a linear functional on $C^\infty(S^1)$ and it belongs to the regular dual of Lie algebra such that

$$dF(u)\mu = \left\langle \frac{\delta F}{\delta u}, \mu \right\rangle = \int_{S^1} \frac{\delta F}{\delta u} \cdot \mu dx.$$

Definition 1 *Let dH be the gradient of the Hamiltonian function $H(u)$. The Euler-Poincaré equation induced by the action of $Vect(S^1)$ on its dual is:*

$$u_t = ad_{(dH)\xi}^* u\xi^{-2} \equiv \{(dH)\xi, u\xi^{-2}\}. \quad (10)$$

As the following Proposition is a straightforward calculation from the definition given above, we omit details in the interests of brevity:

Proposition 1 *Let $H = \frac{1}{2} \int_{S^1} u^2 dx$. The Euler-Poincaré flow on $\mathcal{F}_2$, dual space of $Vect(S^1)$ yields the dispersionless KdV equation $u_t + 3uu_x = 0$.*

3 The Neveu-Schwarz super algebra and construction of the Super/Kuper KdV equation

Let us introduce the Neveu-Schwarz superalgebra. Consider the space of $-1/2$ -tensor densities $\mathcal{F}_{1/2}$ on S^1 . There exists a natural Lie superalgebra structure on the space of $Vect(S^1) \oplus \mathcal{F}_{1/2}$. The anticommutator

$$[,]_+ : \mathcal{F}_{1/2} \otimes \mathcal{F}_{1/2} \longrightarrow Vect(S^1) \in \mathcal{F}_1$$

is just the product of tensor densities:

$$[\psi(x)(dx)^{-1/2}, \phi(x)(dx)^{-1/2}]_+ := \psi(x) \phi(x) \frac{d}{dx}.$$

Consider the space of *anti-periodic* $-1/2$ densities on S^1 :

$$\phi(x)(dx)^{-1/2}, \quad \phi(x + 2\pi) = -\phi(x).$$

The space $\mathcal{F}_{1/2}^-$ is a $Vect(S^1)$ -module, known as Neveu-Schwarz space.

Definition 2 Let $\mathcal{F}_{-3/2}^-$ be the space of antiperiodic $3/2$ densities. The space $Vect(S^1) \oplus \mathcal{F}_{1/2}^-$ defines a Lie superalgebra structure, it is known as Neveu-Schwarz algebra $\mathfrak{g}$. Geometrically, the (regular) dual space of the Neveu-Schwarz algebra is given as $\mathfrak{g}^* = \mathcal{F}_{-2} \oplus \mathcal{F}_{-3/2}^-$.

Let us now embed the Neveu-Schwarz algebra into space of pseudodifferential symbols. We obtain following mappings for algebra

$$(f(x) \frac{d}{dx} + \phi(x)(dx)^{-1/2}) \longmapsto (f(x)\xi + \phi(x)\xi^{1/2}),$$

and the corresponding dual is given by

$$(u(x)dx^2 + \eta(x)(dx)^{3/2}) \longmapsto (u(x)\xi^{-2} + \eta(x)\xi^{-3/2}).$$

We will now study the action of the Neveu-Schwarz algebra on $u(x)\xi^{-2} + \eta(x)\xi^{-3/2}$.

Lemma 1 The classification of invariant bilinear differential operators on tensor fields is due to P. Grozman [7]. Let us recall the zeroth-order and the first order cases:

1. There exists a suitable constant so that a zeroth-order operator $\mathcal{F}_\nu \otimes \mathcal{F}_\mu \longrightarrow \mathcal{F}_{\nu+\mu}$ has the form

$$\phi(x)(dx)^\nu \otimes u(x)(dx)^\mu \longmapsto k\phi(x)u(x)(dx)^{\nu+\mu}. \quad (11)$$

2. Every first order operator $\mathcal{F}_\nu \otimes \mathcal{F}_\mu \longrightarrow \mathcal{F}_{\nu+\mu+1}$ is given by

$$\{\phi(x)(dx)^\nu, u(x)(dx)^\mu\} = (\nu\phi(x)u'(x) - \lambda\phi'(x)u(x))(dx)^{\nu+\mu+1}. \quad (12)$$

Lemma 2 The Hamiltonian operator $\mathcal{O}$ corresponding to the action of the Neveu-Schwarz algebra on $u(x)\xi^{-2} + \eta(x)\xi^{-3/2} \in \Psi D(S^1)$ yields

$$\mathcal{O} = \begin{pmatrix} \partial u + u\partial & \frac{1}{2}\partial\eta + \eta\partial \\ \partial\eta + \frac{1}{2}\eta\partial & \frac{1}{2}u \end{pmatrix} \quad (13)$$

Proof: Let $\mathcal{F}_\lambda$ be the space of $-\lambda$ -densities on S^1 . Any zeroth-order differential operator is the operator of multiplication by a $(\mu - \lambda)$ density:

$$\phi(x)(dx)^{\mu-\lambda} : u(x)(dx)^\lambda \longmapsto \phi(x)u(x)(dx)^\mu.$$

In standard Darboux coordinates this can be written as:

$$\phi(x)\xi^{-\mu+\lambda} : u(x))\xi^{-\lambda} \longmapsto \phi(x)u(x)\xi^{-\mu}.$$

Hence, the action of $f(x)\xi + \phi(x)\xi^{1/2} \in sVect(S^1)$ on its dual is:

$$\begin{aligned} & ad_{f(x)\xi + \phi(x)\xi^{1/2}}^*(u(x)\xi^{-2} + \eta(x)\xi^{-3/2}) \\ = & -\{f(x)\xi, u(x)\xi^{-2} + \eta(x)\xi^{-3/2}\} - \{\phi(x)\xi^{1/2}, \eta(x)\xi^{-3/2}\} \\ & + \phi(x)u(x)\xi^{-3/2}, \end{aligned}$$

where the last expression follows from the definition of zeroth operator and we have chosen $k = \frac{1}{2}$. Therefore, we obtain the Hamiltonian operator from this expression, thereby completing the proof:

$$\begin{aligned} & ad_{f(x)\xi + \phi(x)\xi^{1/2}}^*(u(x)\xi^{-2} + \eta(x)\xi^{-3/2}) \\ = & -(fu' + 2f'u + \frac{1}{2}\phi\eta' + \frac{1}{2}\phi'\eta)\xi^{-2} + (f(x)\eta' \\ & + \frac{3}{2}f'\eta(x) + \frac{1}{2}\phi(x)u(x))\xi^{-3/2}. \end{aligned}$$

Thus from this expression we obtain the desired Hamiltonian operator.

□

We use the definition of the Euler-Poincaré equation to obtain the dispersionless super-KdV equation.

Definition 3 Let $dH_{(u,\eta)} = (\delta H/\delta u, \delta H/\delta \eta)$ be a derivative of Hamiltonian at the point $(u, \eta) \in \mathcal{F}_{-2} + \mathcal{F}_{-3/2}$. The Euler-Poincaré equation induced by the action of supersymmetric $Vect(S^1)$ on its dual $\mathcal{F}_{-2} + \mathcal{F}_{-3/2}$ is defined as

$$\begin{aligned} & (u\xi^{-2} + \eta\xi^{-3/2})_t = ad_{(\frac{\delta H}{\delta u}\xi + \frac{\delta H}{\delta \eta}\xi^{1/2})}^*(u\xi^{-2} + \eta\xi^{-3/2}) \\ = & \left\{ \frac{\delta H}{\delta u}\xi, u(x)\xi^{-2} + \eta\xi^{-3/2} \right\} + \left\{ \frac{\delta H}{\delta \eta}\xi^{1/2}, \eta\xi^{-3/2} \right\} + \frac{\delta H}{\delta \eta}u(x)\xi^{-3/2} \end{aligned} \quad (14)$$

The following is a straightforward calculation:

Proposition 2 Let $\frac{\delta H}{\delta u} = 2u$ and $\frac{\delta H}{\delta \eta} = 4\eta_x$. The Euler-Poincaré flow on $\mathcal{F}_{-2} + \mathcal{F}_{-3/2}$, dual space of Neveu-Schwarz algebra, yields the dispersionless Kuper-KdV equation

$$u_t + 6uu_x + 12\eta_{xx}\eta = 0 \quad \text{and} \quad \eta_t + 6u\eta_x + 3u_x\eta = 0.$$

3.1 Generalization of Souriau cocycle to supersymmetric system

We recall that for any pseudo-differential operator P the formal product $P \circ \ln D$, where $\ln \xi$ is the symbol of $\ln D$, is not contained in ring. However, the formal commutator $[\ln D, P] = \ln D \circ P - P \circ \ln D$ belongs to the ring. In coordinate form it is given by

$$[\ln D, P] = \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} P_x^k D^{-k}.$$

For the ring $\Psi D(S^1)$ of pseudodifferential symbols one can easily verify that

$$[\ln \xi, P] = \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} P_x^k \xi^{-k} \in \Psi D(S^1).$$

We introduce logarithmic cocycle to our case. Let Φ and Ψ be the super part of the super $\Psi D(S^1)$. The nontrivial central extension of the Fermionic part of the super $\psi D(S^1)$ is:

$$c_{fermionic}(\Phi, \Psi) = \int \text{res}([\ln \xi, \Phi(x)]\Psi) dx. \quad (15)$$

The restriction to this on the subalgebra of supervector fields yields Neveu-Schwarz cocycle of the super-Virasoro algebra. This follows by checking:

$$\begin{aligned} c_{fermionic}(\phi(x)\xi^{1/2}, \psi(x)\xi^{1/2}) &= \int \text{res}([\ln \xi, \phi(x)\xi^{1/2}]\psi\xi^{1/2}) dx \\ &= \int \text{res}((\phi'\xi^{-1/2} - \frac{1}{2}\phi''\xi^{-3/2} + \dots)\psi\xi^{1/2}) \\ &= \int \text{res}(\dots - \frac{1}{2}\phi''\psi\xi^{-1} + \dots) = \frac{1}{2} \int \phi''\psi dx. \end{aligned}$$

Thus once again using the extended EP equation we obtain the full super/Kuper KdV equation

$$u_t + u_{xxx} + 6u_x u + 12\eta_{xx}\eta = 0 \quad \text{and} \quad \eta_t + \eta_{xxx} + 6u\eta_x + 3u_x\eta = 0 \quad (16)$$

3.2 Construction of dispersionless Kuper-KdV equation via scaling

We have already seen that the Kuper-KdV is not a supersymmetric system, nevertheless it gives a nontrivial coupled boson-fermion system, i.e., reducing to the KdV equation when $\phi = 0$.

As shown in [2] that finding the dispersionless limit of the Kupershmidt equation is bit tricky. If one tries scaling $\partial \rightarrow \epsilon \partial$ then one runs automatically in trouble since $\partial^{-1} \epsilon \partial$ would diverge in the limit $\epsilon \rightarrow 0$. In another attempt, if we put $\partial \rightarrow p$, then the Fermionic term in the Lax operator would vanish. So at one stage finding a Lax description for the dispersionless limit became an enigma. Anyway, it was resolved

by Barcelos-Neto et al.[2] in two ways: (A) Unlike the bosonic variables, fermionic variables need to scale for a consistent “classical” limit. Hence it can be checked that the “classical” limit involves the scaling $\partial \rightarrow \epsilon \partial$ and $\phi \rightarrow \epsilon^{-1/2} \phi$ without which the fermion terms would not be present in the boson equation. Secondly, It can be solved in [!] with a simple construction. Barcelos-Neto et al. have chosen Lax function

$$L = p^2 + u - p^{-2} \phi \phi_x.$$

This Lax function yields the dispersionless Kuper-KdV equation

$$\begin{aligned} u_t &= 6uu_x - 12\phi\phi_x, \\ \phi_t &= 3u_x\phi + 6u\phi_x. \end{aligned} \tag{17}$$

from the Lax operator equation

$$\frac{\partial L}{\partial t} = -4\{(L^{3/2})_+, L\}.$$

4 Geodesic Flows on group of area preserving diffeomorphisms on cyclider

The Lie algebra $Vect(S^1)$ of vector fields on S^1 has a natural embedding into the Poisson Lie algebra of functions on T^*S^1 . It is defined by the standard action of the Lie algebra of vector fields on the cotangent bundle. Using the local Darboux coordinates $(x, \xi) \in T^*S^1$ the explicit action is given as

$$\pi(f(x) \frac{d}{dx}) = f(x)\xi.$$

We realize cylinder C as a (trivial) cotangent bundle on S^1 , i.e. $T^*S^1 \simeq \mathbf{R} \times S^1$, locally $T^*S^1 = \{(x, \xi) \mid x \in S^1\}$ and ξ is the fibre coordinate of T^*S^1 . The base is S^1 , the fibres are homeomorphic to $\mathbf{R}$, and since the cotangent space can be represented by any nonzero real number, the structure group is ratios of nonzero real numbers and may be identified with $\mathbf{R}/\{0\}$.

The Lie algebra $Vect(S^1)$ of vector fields on a circle S^1 has a natural embedding into the Poisson-Lie algebra of functions on T^*S^1 . It is defined by the standard action of the Lie algebra of vector fields on the cotangent bundle. The Poisson-Lie algebra of smooth functions on T^*S^1 is simply a Poisson algebra with respect to the canonical symplectic structure on the cotangent bundle.

Consider the Poisson-Lie algebra of smooth functions on T^*S^1 for S^1 . This is identified as a cylinder C and the Poisson bracket is defined as

$$\begin{aligned} \{f, g\} &= \frac{\partial f}{\partial \xi} \frac{\partial g}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial g}{\partial \xi} \\ \implies &= (\partial_\xi \wedge \partial_x)(f \otimes g) \end{aligned}$$

for any pair of functions f and g on C .

4.1 Vector fields, Lie algebras and geodesic flow on area-preserving diffeomorphism group

Let μ be the volume element. Let the area-preserving diffeomorphisms group $SDiff(2)$ of a cylinder C acts on C from right. The elements of $SDiff(2)$ induce vector fields $\mathcal{X}$ on C . This set of vector fields are divergence free. For $X \in \mathcal{X}(C)$, we have

$$0 = \text{div}(X)\mu = L_X\mu = di_x\mu, \quad (18)$$

where L_X is the Lie derivative with respect to the vector field X and μ is a standard volume element $dx \wedge d\xi$ on a cylinder T^*S^1 . Thus there exists a unique function f on C for each $X \in \mathcal{X}$ satisfying $df = i_X\mu$. Here f is known as flow function corresponding to the vector field X on C . The vector field generated by $sdiff(2)$ is defined as

$$X_f = \frac{\partial f}{\partial x} \frac{\partial}{\partial \xi} - \frac{\partial f}{\partial \xi} \frac{\partial}{\partial x}, \quad (19)$$

this is typical element of the Lie algebra $sdiff(2)$ corresponding to the group of area-preserving diffeomorphism $SDiff(2)$.

The Lie algebra of vector fields X_f is then identified with C^∞ functions on C equipped with the Lie bracket defined by

$$[X_f, X_g] = X_{\{f,g\}}. \quad (20)$$

This can be easily checked by direct computation.

Remark 1 (*Physical interpretation*) The generator L_f transforms (x, ξ) to $(x - \frac{\partial f}{\partial \xi}, \xi + \frac{\partial f}{\partial x})$, infinitesimally, this is a canonical transformation generated by f , which preserve the phase-space area element $dx \wedge d\xi$.

4.1.1 Geodesic flows on $SDiff(2)$

Groups of area-preserving diffeomorphisms and their Lie algebras play an important role in modern physics literature. Hoppe [12] showed that in a suitable basis, the Lie algebra of the group $SDiff(2)$ tends to that of $SU(N)$ as $N \rightarrow \infty$. The action of the vector field X_H on $C^\infty(C)$ is given by

$$L_{X_H}F = \langle df, X_H \rangle = \{f, H\} := \frac{\partial f}{\partial \xi} \frac{\partial H}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial H}{\partial \xi},$$

where $f, H \in C^\infty(C)$.

We fix the ad-invariant quantity

$$\text{Tr}L = \int_C L dx d\xi. \quad (21)$$

Thus, we can define a weakly nondegenerate invariant inner product [3] on $sdiff(2)$ by

$$\langle L, M \rangle = \text{Tr}(LM) = \int_C LM dx d\xi \quad L, M \in sdiff(2). \quad (22)$$

This inner product is invariant under the adjoint action. Let $L \circ g$ be the (adjoint) action of $g \in SDiff(\mathcal{A})$. We

$$\begin{aligned} \langle L \circ g, M \circ g \rangle &= \int_C (L \circ g)(M \circ g) dx d\xi \\ &= \int_C ((LM) \circ g) dx d\theta = \langle L, M \rangle. \end{aligned}$$

The Lie-Poisson bracket on $sdiff(2)$ is given by

$$\{\{f, g\}\}(\alpha) = \langle \alpha, \left\{ \frac{\delta f}{\delta \alpha}, \frac{\delta g}{\delta \alpha} \right\} \rangle, \quad (23)$$

where $\frac{\delta f}{\delta \alpha} \in sdiff(2)$ denotes the Frechét derivative. Here we have used the double curly bracket notations from Bloch et al. [for details, 3].

The infinitesimal version of invariance is given by

$$\begin{aligned} \langle ad_N^* L, M \rangle &= \langle L, ad_N M \rangle \\ \implies \langle \{L, N\}, M \rangle &= \langle L, \{N, M\} \rangle, \end{aligned}$$

where $ad_f(g) = \{f, g\}$.

Hence the coadjoint action is

Lemma 3

$$ad_f^*(g) = -\{f, g\} \equiv -\left(\frac{\partial f}{\partial \xi} \frac{\partial g}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial g}{\partial \xi}\right). \quad (24)$$

It turns out that the Hamiltonian equation on the dual space of the Lie algebra $\mathfrak{g}$ can be viewed as a geodesic flow in a right invariant metric on the Lie group G in the following way. Let $I : \mathfrak{g} \rightarrow \mathfrak{g}^*$ be an inertia operator such that $(v, w)_{id} = \langle I(v), v \rangle$, where the metric $(\cdot, \cdot)_{id}$ at the identity $id \in G$ defines a nondegenerate bilinear form on the Lie algebra. Writing $u = I(v)$ allows one to relate the geodesic flow and the Hamiltonian equation on the dual $\mathfrak{g}^*$ with respect to the Hamiltonian function $H(u) = \frac{1}{2} \langle u, I^{-1}u \rangle$.

Lemma 4 *Let G be an infinite dimensional Lie group equipped with a right invariant metric. A curve $t \rightarrow c(t)$ in ΩG is a geodesic of this metric iff $u(t) = d_{c_t} R_{c_t^{-1}} \dot{c}(t)$ satisfies*

$$\frac{d}{dt} u(t) = -ad_{I^{-1}u(t)}^* u(x, t). \quad (25)$$

Example of geodesic flow: 2D Euler equation The best example of the above flow is the 2D Euler equation can be written as

$$\partial_t \Omega + \{\Omega, \Psi\} = 0, \quad (26)$$

where the bracket is given by $\{f, g\} = \partial_\xi f \partial_x g - \partial_x f \partial_\xi g$, and $H = \frac{1}{2} \int \Omega \Psi \, dx d\xi$. Here Ψ is the stream function given by,

$$\Omega = \Delta \Psi.$$

Thus using above these results we say that the 2D Euler equation is a geodesic flow on the area preserving diffeomorphic group for $H = \frac{1}{2} \langle \Omega, \Psi \rangle = \langle |\nabla \Psi|^2 \rangle$.

Proposition 3 *The dispersionless KdV and Kuper-KdV are Euler-Poincaré flows on the dual space of the Lie algebra of area preserving diffeomorphism group.*

5 H^1 geodesic flows and Kuper Camassa-Holm equation

Our goal is to give an Euler-Poincaré formalism of the Camassa-Holm analogue of Kuper-KdV equation (or Kupershmidt-Camassa-Holm equation). We will study the H^1 geodesics flow on the area preserving group of diffeomorphism.

Historically Holm, Marsden and Ratiu [11] developed the geometric (or Euler-Poincaré) view of fluid mechanics gave a formal derivation of the average Euler equations and applied this viewpoint it to many other types of fluid equations, such as those in geophysical fluid dynamics. These equations may be described in two mathematically equivalent ways - the first derivation is by means of averaging the Euler equation over rapid fluctuation and second one is interpreted in terms of H^1 geodesics flow on the volume preserving diffeomorphic group.

5.1 Geodesic flow with respect to H^1 -norm

We study geodesic flow on the area preserving diffeomorphism group with respect to H^1 -Sobolev norm on the $sdiff(\mathcal{A})$ algebra. It is defined by

$$\langle L, M \rangle_{H^1} = \int_C LM \, dx d\xi + \int_C L' M' \, dx d\xi \quad L, M \in sdiff(2), \quad (27)$$

where L' denotes derivatives with respect to both x and ξ , i.e., $L' M' = L_x M_x + L_\xi M_{\xi}$. But since all L and M are polynomial ξ . Hence for all practical purposes it boils down to

$$\langle L, M \rangle_{H^1} = \int_C LM \, dx d\xi + \nu \int_C L_x M_x \, dx d\xi \quad \nu \in \mathbb{R}. \quad (28)$$

Thus from now onwards $'$ always means the derivative with respect to x .

Let us compute again the coadjoint action:

Lemma 5 *The coadjoint action with respect to H^1 metric is given by*

$$ad_F^*(G)|_{H^1} = (1 - \nu\partial^2)^{-1}\{(F, (1 - \nu\partial^2)G)\}. \quad (29)$$

Proof: We start from

$$\begin{aligned} \langle F, \{G, H\} \rangle_{H^1} &= \int_C F' \{G, H\}' dx d\xi + \int_C F \{G, H\} dx d\xi \\ &= \int_C \{F', G'\} H dx d\xi + \int_C \{F', G\} H' dx d\xi \\ &= \int_C \{F, (1 - \nu\partial^2)G\} H dx d\xi \\ &\equiv \langle ad_G^* F, H \rangle. \end{aligned}$$

Let us compute now the left hand side (LHS) of equation (29)

$$\begin{aligned} L.H.S. &= \int_C (ad_G^* F) H dx d\xi + \int_{\mathcal{A}} (ad_G^* F)' H' dx d\xi \\ &= \int_C [(1 - \nu\partial^2) ad_G^* F] H dx d\xi. \end{aligned}$$

Thus by equating the R.H.S. and L.H.S. we obtain the above formula.

□

Proposition 4 *The Euler-Poincaré equation with respect to right invariant H^1 metric on the dual space of $sdiff(2)^*$ yields*

$$\frac{\partial \tilde{\Omega}}{\partial t} = -ad_{\frac{\delta H}{\delta \tilde{\Omega}}}^* \tilde{\Omega}, \quad (30)$$

where H is the corresponding Hamiltonian.

Proposition 5 *The flow $\eta_t(\cdot) := \eta(\cdot, t)$ of the time dependent vector field Ω is a geodesic with respect to the right invariant metric by H^1 norm iff the vector field Ω satisfies the average Euler-Poincaré flow.*

5.2 Computation of Kuperschmidt-Camassa-Holm equation

We come to the final Section of the paper. Here we finally derive the Kuperschmidt-Camassa-Holm from the knowledge of the previous Sections.

Lemma 6 *The Hamiltonian operator $\mathcal{O}$ corresponding to the H^1 -action of $f(x)\xi + \phi(x)\xi^{-1/2} \in sVect(S^1)$ on the dual space of the Neveu-Schwarz algebra on $u(x)\xi^{-2} + \eta(x)\xi^{-3/2}$ yields*

$$\mathcal{O}_{KCH} = -(1 - \partial_x^2)^{-1} \left(\frac{\partial m + m\partial}{\partial \tilde{\eta} + \frac{1}{2}\tilde{\eta}\partial} \middle| \frac{\frac{1}{2}\partial\tilde{\eta} + \tilde{\eta}\partial}{\frac{1}{2}m} \right) \quad (31)$$

Proof: This follows directly from our previous results

$$\begin{aligned} & ad_{f(x)\xi + \phi(x)\xi^{1/2}}^* (u(x)\xi^{-2} + \eta(x)\xi^{-3/2})|_{H^1} \\ &= (1 - \nu\partial^2)^{-1} \{f(x)\xi + \phi(x)\xi^{1/2}, (1 - \nu\partial^2)(u(x)\xi^{-2} + \eta(x)\xi^{-3/2})\} \end{aligned}$$

and rest of the calculation is similar to Lemma2.

□

Theorem 1 *The Euler-Poincaré flow with respect to H^1 -metric on the dual space of Neveu-Schwarz algebra yields*

$$\begin{aligned} m_t + 2mu_x + \frac{1}{2}\tilde{\eta}_x\eta_x + \frac{3}{2}\tilde{\eta}\eta_{xx} &= 0 \\ \tilde{\eta}_t + \tilde{\eta}_{xx}u + \frac{3}{2}\tilde{\eta}u_x + \frac{1}{2}m\eta_x &= 0 \end{aligned} \quad (32)$$

for (super) Hamiltonian $H = \frac{1}{2} \int (u^2 + \eta\eta_x) dx$.

6 Conclusion and Outlook

In this paper we have studied the supersymmetric KdV equation of Kupershmidt type or kuper-KdV equation and its Camassa-Holm extension. We have given the Euler-Poincaré framework of the Kupershmidt-Camassa-Holm equation. Our method is based on the action of $Vect(S^1)$ on pseudodifferential symbols on S^1 . We have used the result of logarithmic 2-cocycle to construct the dispersion terms of the Kuper-KdV equation.

It is not hard to formulate the noncommutative analogue of these equations from our approach [9,10]. This can be achieved by the followings methods:

- (A) All Poisson brackets should be replaced by Moyal brackets.
- (B) The derivatives act on the $\star$ -deformed space in a usual way

Moreover, another form of Moyal star product is very useful to study discretization of integrable system is given by the shift formula

$$(f \star_M g)(q, p) = f(q + \frac{i\hbar}{2}\partial_p, p - \frac{i\hbar}{2}\partial_q)g(q, p). \quad (33)$$

Therefore we have given a new insight to the derivation of the supersymmetric KdV equation.

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