Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

Floer homology of cotangent bundles and the loop product

by

Alberto Abbondandolo, and Matthias Schwarz

Preprint no.: 41 2008

Floer homology of cotangent bundles and the loop product

Alberto Abbondandolo and Matthias Schwarz

May 5, 2008

Contents

Introduction

Let M be a compact manifold without boundary, and let H be a time-dependent smooth Hamiltonian on T^*M, the cotangent bundle of M. We assume that H is 1-periodic in time and grows asymptotically quadratically on each fiber. Generically, the corresponding Hamiltonian system

$$x'(t) = X_H(t, x(t)), \quad x : \mathbb{T} := \mathbb{R}/\mathbb{Z} \to T^*M, \tag{1}$$

has a discrete set $\mathscr{P}(H)$ of 1-periodic orbits. The free Abelian group $F_*(H)$ generated by the elements in $\mathscr{P}(H)$, graded by their Conley-Zehnder index, supports a chain complex, the *Floer complex* $(F_*(H), \partial)$. The boundary operator ∂ is defined by an algebraic count of the maps u from the cylinder $\mathbb{R} \times \mathbb{T}$ to T^*M, solving the Cauchy-Riemann type equation

$$\partial_s u(s,t) + J(u(s,t))\big(\partial_t u(s,t) - X_H(t, u(s,t))\big) = 0, \quad \forall (s,t) \in \mathbb{R} \times \mathbb{T}, \tag{2}$$

and converging to two 1-periodic orbits of (1) for $s \to -\infty$ and $s \to +\infty$. Here J is the almost-complex structure on T^*M induced by a Riemannian metric on M, and (2) can be seen as the negative L^2-gradient equation for the Hamiltonian action functional.

This construction is due to Floer (see e.g. [Flo88a, Flo88b, Flo89a, Flo89b]) in the case of a compact symplectic manifold P, in order to prove a conjecture of Arnold on the number of periodic Hamiltonian orbits. The extension to non-compact symplectic manifolds, such as the cotangent bundles we consider here, requires suitable growth conditions on the Hamiltonian, such as the convexity assumption used in [Vit96] or the asymptotic quadratic-growth assumption used in [AS06b]. The Floer complex obviously depends on the Hamiltonian H, but its homology often does not, so it makes sense to call this homology the *Floer homology* of the underlying sympletic manifold P, which is denoted by $HF_*(P)$. The Floer homology of a compact symplectic manifold P without boundary is isomorphic to the singular homology of P, as proved by Floer for special classes of symplectic manifolds, and later extended to larger and larger classes by several authors (the general case requiring special coefficient rings, see [HS95, LT98, FO99]). Unlike the compact case, the Floer homology of a cotangent bundle T^*M is a truly infinite dimensional homology theory, being isomorphic to the singular homology of the free loop space $\Lambda(M)$ of M. This fact was first proved by Viterbo (see [Vit96]) using a generating functions approach, later by Salamon and Weber using the heat flow for curves on a Riemannian manifold (see [SW06]), and then by the authors in [AS06b]. In particular, our proof reduces the general case to the case of Hamiltonians which are fiber-wise convex, and for such a Hamiltonians it constructs an explicit isomorphism from the Floer complex of H to the *Morse complex* of the action functional

$$\mathbb{S}_L(\gamma) = \int_{\mathbb{T}} L(t, \gamma(t), \gamma'(t))\, dt, \quad \gamma \in W^{1,2}(\mathbb{T}, M),$$

associated to the Lagrangian L which is the Fenchel dual of H. The latter complex is just the standard chain complex associated to a gradient flow on a manifold. Here actually, the manifold is the infinite dimensional Hilbert manifold $W^{1,2}(\mathbb{T}, M)$ consisting of closed loops of Sobolev class $W^{1,2}$ on M, and an important fact is that the functional $\mathbb{S}_L$ is of class C^2, it is bounded from below, it satisfies the Palais-Smale condition, and its critical points have finite Morse index. The construction of the Morse complex in this infinite dimensional setting and the proof that its homology is isomorphic to the singular homology of the ambient manifold are described in [AM06]. The isomorphism between the Floer and the Morse complex is obtained by coupling the Cauchy-Riemann type equation on half-cylinders with the gradient flow equation for the Lagrangian action. We call this the *hybrid method*.

Since the space $W^{1,2}(\mathbb{T}, M)$ is homotopy equivalent to $\Lambda(M)$, we get the required isomorphism

$$\Phi : H_*(\Lambda(M)) \xrightarrow{\cong} HF_*(T^*M), \tag{3}$$

from the singular homology of the free loop space of M to the Floer homology of T^*M.

Additional interesting algebraic structures on the Floer homology of a symplectic manifold are obtained by considering other Riemann surfaces than the cylinder as domain for the Cauchy-Riemann type equation (2). By considering the pair-of-pants surface, a non-compact Riemann surface with three cylindrical ends, one obtains the *pair-of-pants product* in Floer homology (see [Sch95] and [MS04]). When the symplectic manifold P is compact without boundary and symplectically aspherical, this product corresponds to the standard cup product from topology, after identifying the Floer homology of P with its singular cohomology by Poincaré duality, while when the manifold P can carry J-holomorphic spheres the pair-of-pants product corresponds to the *quantum cup product* of P (see [PSS96] and [LT99]).

The main result of this paper is that in the case of cotangent bundles, the pair-of-pants product is also equivalent to a product on $H_*(\Lambda(M))$ coming from topology, but a more interesting one than the simple cup product:

THEOREM. *Let M be a compact oriented manifold without boundary. Then the isomorphism Φ in (3) is a ring isomorphism when the Floer homology of T^*M is endowed with its pair-of-pants product, and the homology of the space of free parametrized loops of M is endowed with its Chas-Sullivan loop product.*

The latter is an algebraic structure which was recently discovered by Chas and Sullivan [ChS99], and which is currently having a strong impact in string topology (see e.g. [CHV06] and [Sul07]). In some sense, it is the free loop space version of the classical Pontrjagin product on the singular homology of the based loop space, and it can be described in the following way. Let $\Theta(M)$ be the subspace of $\Lambda(M) \times \Lambda(M)$ consisting of pairs of parametrized loops with identical initial point. If M is oriented and n-dimensional, $\Theta(M)$ is both a co-oriented n-co-dimensional submanifold of the Banach manifold $\Lambda(M) \times \Lambda(M)$ as well as of $\Lambda(M)$ itself via the concatenation map $\Gamma \colon \Theta(M) \to \Lambda(M)$,

$$\Lambda(M) \times \Lambda(M) \overset{e}{\hookleftarrow} \Theta(M) \overset{\Gamma}{\hookrightarrow} \Lambda(M)\,. \tag{4}$$

Seen as continuous maps, e and Γ induce homomorphisms e_*, Γ_* in homology. Seen as n-co-dimensional co-oriented embeddings, they induce Umkehr maps

$$\begin{aligned} e_! \colon H_j(\Lambda(M) \times \Lambda(M)) &\to H_{j-n}(\Theta(M)), \\ \Gamma_! \colon H_j(\Lambda(M)) &\to H_{j-n}(\Theta(M)), \end{aligned} \qquad \forall j \in \mathbb{N}.$$

The loop product is the degree $-n$ product on the homology of the free loop space of M,

$$\circ : H_j(\Lambda(M)) \otimes H_k(\Lambda(M)) \to H_{j+k-n}(\Lambda(M)),$$

defined as the composition

$$H_j(\Lambda(M)) \otimes H_k(\Lambda(M)) \xrightarrow{\times} H_{j+k}(\Lambda(M) \times \Lambda(M)) \xrightarrow{e_!} H_{j+k-n}(\Theta(M)) \xrightarrow{\Gamma_*} H_{j+k-n}(\Lambda(M)),$$

where $\times$ is the exterior homology product. The loop product turns out to be associative, commutative, and to have a unit, namely the image of the fundamental class of M by the embedding of

M into $\Lambda(M)$ as the space of constant loops. More information about the loop product and about its relationship with the Pontrjagin and the intersection product are recalled in section 1. Note that, we also obtain immediately in an anologous way a coproduct of degree $-n$ by composing $e_* \circ \Gamma_!$ which corresponds to the pair-of-pants coproduct on Floer homology. It is easy to see, that this coproduct is almost entirely trivial except for homology class of dimension n, see [AS08]

Actually, the analogy between the pair-of-pants product and the loop product is even deeper. Indeed, we may look at the solutions $(x_1, x_2) : [0,1] \to T^*M \times T^*M$ of the following pair of Hamiltonian systems

$$x_1'(t) = X_{H_1}(t, x_1(t)), \quad x_2'(t) = X_{H_2}(t, x_1(2)), \tag{5}$$

coupled by the non-local boundary condition

$$\begin{aligned} q_1(0) = q_1(1) = q_2(0) = q_2(1), \\ p_1(1) - p_1(0) = p_2(0) - p_2(1). \end{aligned} \tag{6}$$

Here we are using the notation $x_j(t) = (q_j(t), p_j(t))$, with $q_j(t) \in M$ and $p_j(t) \in T^*_{q_j(t)}M$, for $j = 1, 2$. By studying the corresponding Lagrangian boundary value Cauchy-Riemann type problem on the strip $\mathbb{R} \times [0,1]$, we obtain a chain complex, the *Floer complex for figure-8 loops* $(F^\Theta(H), \partial)$, on the graded free Abelian group generated by solutions of (5)-(6). Then we can show that:

(i) The homology of the chain complex $(F^\Theta(H), \partial)$ is isomorphic to the singular homology of $\Theta(M)$.

(ii) The pair of pants product factors through the homology of this chain complex.

(iii) The first homomorphism in this factorization corresponds to the homomorphism $e_! \circ \times$, while the second one corresponds to homomorphism Γ_*.

We also show that similar results hold for the space of based loops. The Hamiltonian problem in this case is the equation (1) for $x = (q, p) : [0,1] \to T^*M$ with boundary conditions

$$q(0) = q(1) = q_0,$$

for a fixed $q_0 \in M$. The corresponding Floer homology $HF_*^\Omega(T^*M)$ is isomorphic to the singular homology of the based loop space $\Omega(M)$, and there is a product on such a Floer homology, the *triangle product*, which corresponds to the classical Pontrjagin product # on $H_*(\Omega(M))$. Actually, every arrow in the commutative diagram from topology

$$\begin{array}{ccc} H_j(M) \otimes H_k(M) & \xrightarrow{\bullet} & H_{j+k-n}(M) \\ \downarrow{\scriptstyle c_* \otimes c_*} & & \downarrow{\scriptstyle c_*} \\ H_j(\Lambda(M)) \otimes H_k(\Lambda(M)) & \xrightarrow{\circ} & H_{j+k-n}(\Lambda(M)) \\ \downarrow{\scriptstyle i_! \otimes i_!} & & \downarrow{\scriptstyle i_!} \\ H_{j-n}(\Omega(M)) \otimes H_{k-n}(\Omega(M)) & \xrightarrow{\#} & H_{j+k-2n}(\Omega(M, q_0)), \end{array} \tag{7}$$

has an equivalent homomorphism in Floer homology. Here $\bullet$ is the intersection product in singular homology, c is the embedding of M into $\Lambda(M)$ by constant loops, and $i_!$ denotes the Umkehr map induced by the n-co-dimensional co-oriented embedding $\Omega(M) \hookrightarrow \Lambda(M)$.

All the Floer homologies on cotangent bundles we consider here - for free loops, figure eight loops, or based loops - are special cases of Floer homology for *non-local conormal* boundary conditions: One fixes a closed submanifold Q of $M \times M$ and considers the Hamiltonian orbits $x : [0,1] \to T^*M$ such that $(x(0), -x(1))$ belongs to the conormal bundle N^*Q of Q, that is to the set of covectors in $T^*(M \times M)$ which are based at Q and annihilate every vector which is tangent to Q. See [APS08], where we show that the isomorphism (3) generalizes to

$$\Phi \colon H_*(\Omega^Q(M)) \xrightarrow{\cong} HF_*^Q(T^*M)$$

where $\Omega^Q(M)$ is the space of paths $\gamma\colon [0,1] \to M$ such that $(\gamma(0), \gamma(1)) \in Q$, and HF_*^Q is the Floer homology associated to N^*Q.

The first step in the proof of the main statements of this paper is to describe objects and morphisms from algebraic topology in a Morse theoretical way. The way this translation is performed is well known in the case of finite dimensional manifolds (see e.g. [Fuk93], [Sch93], [BC94], [Vit95], [Fuk97]). In section 2 we outline how these results extend to infinite dimensional Hilbert manifolds. We pay particular attention to the transversality conditions required for each construction, and we particularize the analysis to the action functional associated to a fiberwise convex Lagrangian having quadratic growth in the velocities. See also [Coh06, CHV06, CS08].

The core of the paper consists of sections 3 and 4. In the former we define the Floer complexes we are dealing with and the products on their homology. In the latter we establish the equivalence with algebraic topology, thus proving the above theorem and the other main results of this paper. The linear Fredholm theory used in these sections is described in section 5, whereas section 6 contains compactness and removal of singularities results, together with the proofs of three cobordism statements from sections 3 and 4.

Some of the proofs are based on standard techniques in Floer homology, and in this case we just refer to the literature. However, there are a few key points where we need to introduce some new ideas. We conclude this introduction by briefly describing these ideas.

Riemann surfaces as quotients of strips with slits. The definition of the pair-of-pants product requires extending the Cauchy-Riemann type equation (2) to the pair-of-pants surface. The Cauchy-Riemann operator $\partial_s + J\partial_t$ naturally extends to any Riemann surface, by letting it act on anti-linear one-forms. The zero-order term $-JX_H(t,u)$ instead does not have a natural extension when the Riemann surface does not have a global coordinate $z = s+it$. The standard way to overcome this difficulty is to make this zero-order term act only on the cylindrical ends of the pair-of-pants surface - which do have a global coordinate $z = s+it$ - by multiplying the Hamiltonian by a cut-off functions making it vanish far from the cylindrical ends (see [Sch95], [MS04]). This construction does not cause problems when dealing with compact symplectic manifolds as in the above mentioned reference, but in the case of the cotangent bundle it would create problems with compactness of the spaces of solutions. In fact, on one hand cutting off the Hamiltonian destroys the identity relating the energy of the solution with the oscillation of the action functional, on the other hand our C^0-estimate for the solutions requires coercive Hamiltonians.

We overcome this difficulty by a different - and we believe more natural - way of extending the zero-order term. We describe the pair-of-pants surface - as well as the other Riemann surfaces we need to deal with - as the quotient of an infinite strip with a slit - or more slits in the case of more general Riemann surfaces. At the end of the slit we use a chart given by the square root map. In this way, the Riemann surface is still seen as a smooth object, but it carries a global coordinate $z = s+it$ with singularities. This global coordinate allows to extend the zero-order term without cutting off the Hamoltonian, and preserving the energy identity. See subsection 3.2 below.

Cauchy-Riemann operators on strips with jumping boundary conditions. When using the above description for the Riemann surfaces, the problems we are looking at can be described in a unified way as Cauchy-Riemann type equations on a strip, with Lagrangian boundary conditions presenting a finite number of jumps. In section 5 we develop a complete linear theory for such problems, in the case of Lagrangian boundary conditions of co-normal type. This is the kind of conditions which occur naturally on cotangent bundles. Once the proper Sobolev setting has been chosen, the proof of the Fredholm property for such operators is standard. The computation of the index instead is reduced to a Liouville type statement, proved in subsection 5.5

These linear results have the following consequence. Let $Q_0, \dots, Q_k$ be submanifolds of $M \times M$, such that Q_{j-1} and Q_j intersect cleanly, for every $j = 1, \dots, k$. Let $-\infty = s_0 < s_1 < \cdots < s_k < s_{k+1} = +\infty$, and consider the space $\mathscr{M}$ consisting of the maps $u : \mathbb{R} \times [0,1] \to T^*M$ solving the Cauchy-Riemann type equation (2), satisfying the boundary conditions

$$(u(s,0), -u(s,1)) \in N^*Q_j \quad \forall s \in [s_j, s_{j+1}],\ \forall j = 0, \dots, k,$$

and converging to Hamiltonian orbits x^- and x^+ for $s \to -\infty$ and $s \to +\infty$. The results of section 5 imply that for a generic choice of the Hamiltonian H the space $\mathscr{M}$ is a manifold of dimension

$$\dim \mathscr{M} = \mu^{Q_0}(x^-) - \mu^{Q_k}(x^+) - \sum_{j=1}^{k} (\dim Q_{j-1} - \dim Q_{j-1} \cap Q_j).$$

Here $\mu^{Q_0}(x^-)$ and $\mu^{Q_k}(x^+)$ are the Maslov indices of the Hamiltonian orbits x^- and x^+, with boundary conditions $(x^-(0), -x^-(1)) \in N^*Q_0$, $(x^+(0), -x^+(1)) \in N^*Q_k$, suitably shifted so that in the case of a fiberwise convex Hamiltonian they coincide with the Morse indices of the corresponding critical points γ^- and γ^+ of the Lagrangian action functional on the spaces of paths satisfying $(\gamma^-(0), \gamma^-(1)) \in Q_0$ and $(\gamma^+(0), \gamma^+(1)) \in Q_k$, respectively. Similar formulas hold for problems on the half-strip.

Cobordism arguments. The main results of this paper always reduce to the fact that certain diagrams involving homomorphism defined either in a Floer or in a Morse theoretical way should commute. The proof of such a commutativity is based on cobordism arguments, saying that a given solution of a certain Problem 1 can be "continued" by a unique one-parameter family of solutions of a certain Problem 2, and that this family of solutions converges to a solution of a certain Problem 3. In many situations such a statement can be proved by the classical gluing argument in Floer theory: One finds the one-parameter family of solutions of Problem 2 by using the given solution of Problem 1 to construct an approximate solution, to be used as the starting point of a Newton iteration scheme which converges to a true solution. When this is the case, we just refer to the literature. However, we encounter three situations in which the standard arguments do not apply, one reason being that we face a Problem 2 involving a Riemann surface whose conformal structure is varying with the parameter: this occurs when proving that the pair-of-pants product factorizes through the figure-8 Floer homology (subsection 3.5), that the Pontrjagin product corresponds to the triangle product (subsection 4.2), and that the homomorphism $e_! \circ \times$ corresponds to its Floer homological counterpart (subsection 4.4). We manage to reduce the former two statements to the standard implicit function theorem (see subsections 6.3 and 6.4). The proof of the latter statement is more involved, because in this case the solution of Problem 2 we are looking for cannot be expected to be even C^0-close to the solution of Problem 1 we start with. We overcome this difficulty by the following algebraic observation: In order to prove that two chain maps $\varphi, \psi : C \to C'$ are chain homotopic, it suffices to find a chain homotopy between the chain maps $\varphi \otimes \psi$ and $\psi \otimes \varphi$, and to find an element $\epsilon \in C_0$ and a chain map δ from the complex C' to the trivial complex $(\mathbb{Z}, 0)$ such that $\delta(\varphi(\epsilon)) = \delta(\psi(\epsilon))$ (see Lemma 4.6 below). In our situation, the chain homotopy between $\varphi \otimes \psi$ and $\psi \otimes \varphi$ is easier to find, by using a localization argument and the implicit function theorem (see subsection 6.5). This argument is somehow reminiscent of an alternative way suggested by Hofer to prove standard gluing results in Floer homology. The construction of the element ϵ and of the chain map δ is presented in subsection 4.4, together with the proof of the required algebraic identity. This is done by considering special Hamiltonian systems, having a hyperbolic equilibrium point.

The main results of this paper were announced in [AS06a]. Related results concerning the equivariant loop product and its interpretation in the symplectic field theory of unitary cotangent bundles have been announced in [CL07].

Outlook. An immediate question raised by the main result of this paper whether other product structures in classical homoloy theories for path and loop spaces can also be constructed naturally on chain level in Floer theory. The answer appears to be affirmative for all so far considered structures. In a following paper [AS08] we give the explicit construction of the cup-product in all path and loop space cases, a direct proof of the Hopf algebra structure on $HF^{\Omega}(T^*M)$ based on Floer chain complex morphisms, and we also construct the counterpart of the very recently introduced product structure on relative cohomology

$$H^k(\Lambda(M), M) \times H^l(\Lambda(M), M) \to H^{k+l+n-1}(\Lambda(M), M)$$

from [GH07]. Also in view of the uniformization of path and loop space homology $H_*(\Omega^Q(M))$ for $Q \subset M \times M$ we show in [APS08] that the bilinear operation for cleanly intersecting submanifolds $Q_1 \cap Q_2 \neq \emptyset$, viewed as composable correspondences, which one obtains analogously to the loop product from

$$\Omega^{Q_1}(M) \times \Omega^{Q_2}(M) \hookleftarrow \Omega^{Q_1 \cap Q_2}(M) \hookrightarrow \Omega^{Q_1 \circ Q_2}(M),$$

is isomorphic to the operation

$$HF^{Q_1}(T^*M) \otimes HF^{Q_2}(T^*M) \to HF^{Q_1 \circ Q_2}(T^*M)$$

which follows from the above moduli problem $\mathscr{M}$ with conormal boundary condition jump from $Q_1 \times Q_2$ to $Q_1 \circ Q_2$. The latter is the composition as correspondences

$$Q_1 \circ Q_2 = \pi_{14}\big((Q_1 \times Q_2) \cap (M \times \triangle \times M)\big)$$

where $\pi_{14} \colon M^4 \to M^2$ is the projection onto the first and fourth factor. The special case $Q_1 = Q_2 = \triangle$ describes the Chas-Sullivan and the pair-of-pants product.

Acknowledgements. We wish to thank the Max Planck Institute for Mathematics in the Sciences of Leipzig and the Department of Mathematics at Stanford University, and in particular Yasha Eliashberg, for their kind hospitality. We are also indebted with Ralph Cohen and Helmut Hofer for many fruitful discussions. The second author thanks the Deutsche Forschungsgemeinschaft for the support by the grant DFG SCHW 892/2-3.

1 The Pontrjagin and the loop products

1.1 The Pontrjagin product

Given a topological space M and a point $q_0 \in M$, we denote by $\Omega(M, q_0)$ the space of loops on M based at q_0, that is

$$\Omega(M, q_0) := \left\{ \gamma \in C^0(\mathbb{T}, M) \mid \gamma(0) = q_0 \right\},$$

endowed with the compact-open topology. Here $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the circle parameterized by the interval $[0, 1]$. The concatenation

$$\Gamma(\gamma_1, \gamma_2)(t) := \begin{cases} \gamma_1(2t) & \text{for } 0 \leq t \leq 1/2, \\ \gamma_2(2t-1) & \text{for } 1/2 \leq t \leq 1, \end{cases}$$

maps $\Omega(M, q_0) \times \Omega(M, q_0)$ continuously into $\Omega(M, q_0)$. The constant loop q_0 is a homotopy unit for Γ, meaning that the maps $\gamma \mapsto \Gamma(q_0, \gamma)$ and $\gamma \mapsto \Gamma(\gamma, q_0)$ are homotopic to the identity map. Moreover, Γ is homotopy associative, meaning that $\Gamma \circ (\Gamma \times \mathrm{id})$ and $\Gamma \circ (\mathrm{id} \times \Gamma)$ are homotopic. Therefore, Γ defines the structure of an H-space on $\Omega(M, q_0)$.

We denote by H_* the singular homology functor with integer coefficients. The composition

$$H_j(\Omega(M, q_0)) \otimes H_k(\Omega(M, q_0)) \xrightarrow{\times} H_{j+k}(\Omega(M, q_0) \times \Omega(M, q_0)) \xrightarrow{\Gamma_*} H_{j+k}(\Omega(M, q_0))$$

where the first arrow is the exterior homology product, is by definition the *Pontrjagin product*

$$\# : H_j(\Omega(M, q_0)) \otimes H_k(\Omega(M, q_0)) \to H_{j+k}(\Omega(M, q_0)).$$

The fact that q_0 is a homotopy unit for Γ implies that $[q_0] \in H_0(\Omega(M, q_0))$ is the identity element for the Pontrjagin product. The fact that Γ is homotopy associative implies that the Pontrjagin product is associative. Therefore, the product $\#$ makes the singular homology of $\Omega(M, q_0)$ a graded ring. In general, it is a non-commutative graded ring. See for instance [tDKP70] for more information on H-spaces and the Pontrjagin product.

1.2 The Chas-Sullivan loop product

We denote by $\Lambda(M) := C^0(\mathbb{T}, M)$ the space of free loops on M. Under the assumption that M is an oriented n-dimensional manifold, it is possible to use the concatenation map Γ to define a product of degree $-n$ on $H_*(\Lambda(M))$. In order to describe the construction, we need to recall the definition of the Umkehr map.

Let $\mathcal{M}$ be a (possibly infinite-dimensional) smooth Banach manifold, and let $e : \mathcal{M}_0 \hookrightarrow \mathcal{M}$ be a smooth closed embedding, which we assume to be n-codimensional and co-oriented. In other words, $\mathcal{M}_0$ is a closed submanifold of $\mathcal{M}$ whose normal bundle $N\mathcal{M}_0 := T\mathcal{M}|_{\mathcal{M}_0}/T\mathcal{M}_0$ has codimension n and is oriented. The tubular neighborhood theorem provides us with a homeomorphism[1] $u : \mathcal{U} \to N\mathcal{M}_0$, uniquely determined up to isotopy, of an open neighborhood of $\mathcal{M}_0$ onto $N\mathcal{M}_0$, mapping $\mathcal{M}_0$ identically onto the zero section of $N\mathcal{M}_0$, that we also denote by $\mathcal{M}_0$ (see [Lan99], IV.§5-6). The *Umkehr map* is defined to be the composition

$$H_j(\mathcal{M}) \longrightarrow H_j(\mathcal{M}, \mathcal{M} \setminus \mathcal{M}_0) \xrightarrow{\cong} H_j(\mathcal{U}, \mathcal{U} \setminus \mathcal{M}_0) \xrightarrow{u_*} H_j(N\mathcal{M}_0, N\mathcal{M}_0 \setminus \mathcal{M}_0) \xrightarrow{\tau} H_{j-n}(\mathcal{M}_0),$$

where the first arrow is induced by the inclusion, the second one is the isomorphism given by excision, and the last one is the Thom isomorphism associated to the n-dimensional oriented vector bundle $N\mathcal{M}_0$, that is, the cap product with the Thom class $\tau_{N\mathcal{M}_0} \in H^n(N\mathcal{M}_0, N\mathcal{M}_0 \setminus \mathcal{M}_0)$. The Umkehr map associated to the embedding e is denoted by

$$e_! : H_j(\mathcal{M}) \longrightarrow H_{j-n}(\mathcal{M}_0).$$

We recall that if M is an n-dimensional manifold, $\Lambda(M)$ is an infinite dimensional smooth manifold modeled on the Banach space $C^0(\mathbb{T}, \mathbb{R}^n)$. The set $\Theta(M)$ of pairs of loops with the same initial point (figure-8 loops),

$$\Theta(M) := \{(\gamma_1, \gamma_2) \in \Lambda(M) \times \Lambda(M) \mid \gamma_1(0) = \gamma_2(0)\},$$

is the inverse image of the diagonal Δ_M of $M \times M$ by the smooth submersion

$$\mathrm{ev} \times \mathrm{ev} : \Lambda(M) \times \Lambda(M) \to M \times M, \quad (\gamma_1, \gamma_2) \mapsto (\gamma_1(0), \gamma_2(0)).$$

Therefore, $\Theta(M)$ is a closed smooth submanifold of $\Lambda(M) \times \Lambda(M)$, and its normal bundle[2] $N\Theta(M)$ is n-dimensional, being isomorphic to to the pull-back of the normal bundle $N\Delta_M$ of Δ_M in $M \times M$ by the map $\mathrm{ev} \times \mathrm{ev}$. If moreover M is oriented, so is $N\Delta_M$ and thus also $N\Theta(M)$. Notice also that the concatenation map Γ is well-defined and smooth from $\Theta(M)$ into $\Lambda(M)$. If we denote by e the inclusion of $\Theta(M)$ into $\Lambda(M) \times \Lambda(M)$, the Chas-Sullivan *loop product* (see [ChS99]) is defined by the composition

$$H_j(\Lambda(M)) \otimes H_k(\Lambda(M)) \xrightarrow{\times} H_{j+k}(\Lambda(M) \times \Lambda(M)) \xrightarrow{e_!} H_{j+k-n}(\Theta(M)) \xrightarrow{\Gamma_*} H_{j+k-n}(\Lambda(M)),$$

and it is denoted by

$$\circ : H_j(\Lambda(M)) \otimes H_k(\Lambda(M)) \to H_{j+k-n}(\Lambda(M)).$$

We denote by $c : M \to \Lambda(M)$ the map which associates to every $q \in M$ the constant loop q in $\Lambda(M)$. A simple homotopy argument shows that the image of the fundamental class $[M] \in H_n(M)$ under the homomorphism c_* is a unit for the loop product: $\alpha \circ c_*[M] = c_*[M] \circ \alpha = \alpha$ for every

[1] If $\mathcal{M}$ admits smooth partitions of unity (for instance, if it is a Hilbert manifold) then u can be chosen to be a smooth diffeomorphism.

[2] The Banach manifold $\Lambda(M)$ does not admit smooth partitions of unity (actually, the Banach space $C^0(\mathbb{T}, \mathbb{R}^n)$ does not admit non-zero functions of class C^1 with bounded support). So in general a closed submanifold of $\Lambda(M)$, or of $\Lambda(M) \times \Lambda(M)$, will not have a smooth tubular neighborhood. However, it would not be difficult to show that the submanifold $\Theta(M)$ and all the submanifolds we consider in this paper do have a smooth tubular neighborhood, which can be constructed explicitly by using the exponential map and the tubular neighborhood theorem on finite-dimensional manifolds.

$\alpha \in H_*(\Lambda(M))$. Since $\Gamma \circ (\mathrm{id} \times \Gamma)$ and $\Gamma \circ (\Gamma \times \mathrm{id})$ are homotopic on the space of triplets of loops with the same initial points, the loop product turns out to be associative. Finally, notice that the maps $(\gamma_1, \gamma_2) \mapsto \Gamma(\gamma_1, \gamma_2)$ and $(\gamma_1, \gamma_2) \mapsto \Gamma(\gamma_2, \gamma_1)$ are homotopic on $\Theta(M)$, by the homotopy

$$\Gamma_s(\gamma_1, \gamma_2)(t) := \begin{cases} \gamma_2(2t - s) & \text{if } 0 \leq t \leq s/2, \text{ or } (s+1)/2 \leq t \leq 1, \\ \gamma_1(2t - s) & \text{if } s/2 \leq t \leq (s+1)/2. \end{cases}$$

This fact implies the following commutation rule

$$\beta \circ \alpha = (-1)^{(|\alpha|-n)(|\beta|-n)} \alpha \circ \beta,$$

for every $\alpha, \beta \in H_*(\Lambda(M))$.

In order to get a product of degree zero, it is convenient to shift the grading by n, obtaining the graded group

$$\mathbb{H}_j(\Lambda(M)) := H_{j+n}(\Lambda(M)),$$

which becomes a graded commutative ring with respect to the loop product (commutativity has to be understood in the graded sense, that is $\beta \circ \alpha = (-1)^{|\alpha||\beta|} \alpha \circ \beta$).

1.3 Relationship between the two products

If M is an oriented n-dimensional manifold, we denote by

$$\bullet : H_j(M) \otimes H_k(M) \to H_{j+k-n}(M),$$

the intersection product on the singular homology of M (which is obtained by composing the exterior homology product with the Umkehr map associated to the embedding of the diagonal into $M \times M$). Shifting again the grading by n, we see that the product $\bullet$ makes

$$\mathbb{H}_j(M) := H_{j+n}(M)$$

a commutative graded ring.

Since $\Omega(M, q_0)$ is the inverse image of q_0 by the submersion $\mathrm{ev} : \Lambda(M) \to M$, $\mathrm{ev}(\gamma) = \gamma(0)$, $\Omega(M, q_0)$ is a closed submanifold of $\Lambda(M)$, and its normal bundle is n-dimensional and oriented. If $i : \Omega(M, q_0) \hookrightarrow \Lambda(M)$ is the inclusion map, we find that the following diagram

$$\begin{array}{ccc} H_j(M) \otimes H_k(M) & \xrightarrow{\bullet} & H_{j+k-n}(M) \\ \downarrow{\scriptstyle c_* \otimes c_*} & & \downarrow{\scriptstyle c_*} \\ H_j(\Lambda(M)) \otimes H_k(\Lambda(M)) & \xrightarrow{\circ} & H_{j+k-n}(\Lambda(M)) \\ \downarrow{\scriptstyle i_! \otimes i_!} & & \downarrow{\scriptstyle i_!} \\ H_{j-n}(\Omega(M, q_0)) \otimes H_{k-n}(\Omega(M, q_0)) & \xrightarrow{\#} & H_{j+k-2n}(\Omega(M, q_0)) \end{array} \tag{8}$$

commutes. In other words, the maps

$$\{\mathbb{H}_*(M), \bullet\} \xrightarrow{c_*} \{\mathbb{H}_*(\Lambda(M)), \circ\} \xrightarrow{i_!} \{H_*(\Omega(M, q_0), \#\}$$

are graded ring homomorphisms. Notice that the homomorphism c_* is always injective onto a direct summand, the map ev being a left inverse of c. Using the spectral sequence associated to the Serre fibration

$$\Omega(M) \hookrightarrow \Lambda(M) \to M,$$

it is possible to compute the ring $\{\mathbb{H}_*(\Lambda(M)), \circ\}$ from the intersection product on M and the Pontrjagin product on $\Omega(M)$, see [CJY03].

The aim of this paper is to show how the homomorphisms appearing in the diagram above can be described symplectically, in terms of the Floer homology of the cotangent bundle of M. The first step is to describe them in a Morse theoretical way, using suitable Morse functions on some infinite dimensional Hilbert manifolds having the homotopy type of $\Omega(M, q_0)$ and $\Lambda(M)$.

1.1. Remark. *The loop product was defined by Chas and Sullivan in [ChS99], by using intersection theory for transversal chains. The definition we use here is due to Cohen and Jones [CJ02]. See also [ChS04, Sul03, BCR06, Coh06, CHV06, Ram06, Sul07] for more information and for other interpretations of this product.*

2 Morse theoretical descriptions

2.1 The Morse complex

Let us recall the construction of the Morse complex for functions defined on an infinite-dimensional Hilbert manifold. See [AM06] for detailed proofs. Let $\mathcal{M}$ be a (possibly infinite-dimensional) Hilbert manifold, and let g be a complete Riemannian metric on $\mathcal{M}$. Let $\mathcal{F}(M, g)$ be the set of C^2 functions $f : \mathcal{M} \to \mathbb{R}$ such that:

(f1) f is bounded below;

(f2) each critical point of f is non-degenerate and has finite Morse index;

(f3) f satisfies the Palais-Smale condition with respect to g (that is, every sequence (p_n) such that $(f(p_n))$ is bounded and $\|Df(p_n)\| \to 0$ has a converging subsequence).

Here $\|\cdot\|$ is the norm on $T^*\mathcal{M}$ induced by the metric g.

Let $f \in \mathcal{F}(\mathcal{M}, g)$. We denote by $\mathrm{crit}(f)$ the set of critical points of f, and by $\mathrm{crit}_k(f)$ the set of critical points of Morse index $m(x) = k$. Let ϕ be the local flow determined by the vector field $-\mathrm{grad}\, f$ (our assumptions imply that the domain of such a flow contains $[0, +\infty[\times\mathcal{M}$, and that each orbit converges to a critical point of f for $t \to +\infty$). The stable (respectively unstable) manifold of a critical point x,

$$W^s(x) := \left\{p \in \mathcal{M} \mid \lim_{t\to+\infty} \phi(t,p) = x\right\} \quad \left(\text{resp. } W^u(x) := \left\{p \in \mathcal{M} \mid \lim_{t\to-\infty} \phi(t,p) = x\right\}\right)$$

is a submanifold of codimension (resp. dimension) $m(x)$. Actually, it is the image of an embedding of $V^+(\mathrm{Hess}\, f(x))$ (resp. $V^-(\mathrm{Hess}\, f(x))$), the positive (resp. negative) eigenspace of the Hessian of f at x. Our assumptions imply that the closure of $W^u(x)$ in $\mathcal{M}$ is compact.

The vector field $-\mathrm{grad}\, f$ is said to satisfy the *Morse-Smale condition* if for every $x, y \in \mathrm{crit}(f)$ the intersection $W^u(x) \cap W^s(y)$ is transverse. The Morse-Smale condition can be achieved by perturbing the metric g. Such perturbations can be chosen to be arbitrarily small in many reasonable senses, in particular the perturbed metric can be chosen to be equivalent to the original one, so that f satisfies the Palais-Smale condition also with respect to the new metric.

Let us assume that $-\mathrm{grad}\, f$ satisfies the Morse-Smale condition. Then we can choose an open neighborhood $\mathcal{U}(x)$ of each critical point x so small that the increasing sequence of positively invariant open sets

$$\mathcal{U}^k := \bigcup_{\substack{x\in\mathrm{crit}(f)\\ m(x)\leq k}} \phi([0, +\infty[\times\mathcal{U}(x)),$$

is a cellular filtration[3] of $\mathcal{U}^\infty := \bigcup_{k\in\mathbb{N}} \mathcal{U}^k$, meaning that $H_j(\mathcal{U}^k, \mathcal{U}^{k-1}) = 0$ if $j \neq k$. Actually, $H_k(\mathcal{U}^k, \mathcal{U}^{k-1})$ is isomorphic to the free Abelian group generated by the critical points of Morse index k, which we denote by $M_k(f)$. Indeed, $H_k(\mathcal{U}^k, \mathcal{U}^{k-1})$ is generated by the relative homology

[3] Actually here one needs a stronger transversality condition, namely that every critical point x is not a cluster point for the union of all the unstable manifolds of critical points of Morse index not exceeding $m(x)$, other than x. This assumption follows from the standard Morse-Smale condition provided that there are finitely many critical point with any given Morse index. Since the latter condition automatically holds on sublevels of f, the Morse complex can be defined under the standard Morse-Smale assumption by a direct limit argument on sublevels. See [AM06] for full details.

classes of balls in $W^u(x)$, so large that their boundaries lie in $\mathcal{U}^{k-1}$, for every $x \in \mathrm{crit}_k(f)$. Hence, the isomorphism $M_k(f) \cong H_k(\mathcal{U}^k,\mathcal{U}^{k-1})$ is determined by the choice of an orientation for the unstable manifold of each $x \in \mathrm{crit}_k(f)$. Furthermore, the inclusion $\mathcal{U}^\infty \hookrightarrow \mathcal{M}$ is a homotopy equivalence.

It is well known that the cellular complex associated to the cellular filtration $\mathcal{U} := \{\mathcal{U}^k\}_{k\in\mathbb{N}}$, namely

$$\begin{aligned} W_k\mathcal{U} &:= H_k(\mathcal{U}^k,\mathcal{U}^{k-1}),\\ \partial_k : W_k\mathcal{U} &= H_k(\mathcal{U}^k,\mathcal{U}^{k-1}) \to H_{k-1}(\mathcal{U}^{k-1}) \to H_{k-1}(\mathcal{U}^{k-1},\mathcal{U}^{k-2}) = W_{k-1}\mathcal{U}, \end{aligned}$$

is a chain complex of Abelian groups, whose homology is naturally isomorphic to the singular homology of $\mathcal{U}^\infty$, hence to the singular homology of $\mathcal{M}$ (see for instance [Dol80], V.1). Therefore, a Morse-Smale metric g and an arbitrary choice of an orientation for the unstable manifold of each critical point determine a boundary operator $\partial_*(f,g)$ on the graded group $M_*(f)$, making it a chain complex called the *Morse complex of* (f,g), which we denote by $M(f,g)$, whose homology is isomorphic to the singular homology of $\mathcal{M}$,

$$H_k M(f,g) \cong H_k(\mathcal{M}).$$

A change of the metric g and of the orientation data produces an isomorphic chain complex.

The boundary operator $\partial_*(f,g)$ can be easily interpreted in terms of intersection numbers. Indeed, since $T_xW^u(x) \oplus T_xW^s(x) = T_x\mathcal{M}$ for each $x \in \mathrm{crit}(f)$, the orientation of $W^u(x)$ determines an orientation of the normal bundle of $W^s(x)$. Therefore each (transverse) intersection $W^u(x)\cap W^s(y)$ is canonically oriented. If $m(x)-m(y) = 1$, compactness and transversality imply that $W^u(x)\cap W^s(y)$ consists of finitely many orbits $\mathbb{R}\cdot p := \phi(\mathbb{R}\times\{p\})$ of ϕ, each of which can be given a sign $\epsilon(\mathbb{R}\cdot p) = \pm 1$, depending on whether the orientation of $\mathbb{R}\cdot p$ defined above agrees or not with the direction of the flow. If we then set $n_\partial(x,y) := \sum \epsilon(\mathbb{R}\cdot p) \in \mathbb{Z}$, where the sum ranges over all the orbits of ϕ in $W^u(x)\cap W^s(y)$, there holds

$$\partial_k(f,g)x = \sum_{y\in \mathrm{crit}_{k-1}(f)} n_\partial(x,y)\, y,$$

for every $x \in \mathrm{crit}_k(f)$.

It is also useful to consider the following relative version of the Morse complex. Let $\mathcal{A}$ be an open subset of the Hilbert manifold $\mathcal{M}$, and assume that the function $f \in C^\infty(\mathcal{M})$ and the metric g on $\mathcal{M}$ satisfy:

(f1') f is bounded below on $\mathcal{M}\setminus\mathcal{A}$;

(f2') each critical point of f in $\mathcal{M}\setminus\mathcal{A}$ is non-degenerate and has finite Morse index;

(f3') f satisfies the Palais-Smale condition with respect to g on $\mathcal{M}\setminus\overline{\mathcal{A}}$;

(f4') $\mathcal{A}$ is positively invariant for the flow of $-\mathrm{grad}\, f$, and this flow is positively complete with respect to $\mathcal{A}$ (meaning that the orbits that never enter $\mathcal{A}$ are defined for every $t \geq 0$).

In particular, (f3') implies that there are no critical points on the boundary of $\mathcal{A}$ (such a critical point would be the limit of a Palais-Smale sequence in $\mathcal{M}\setminus\mathcal{A}$ which does not converge in this set). Assume that $-\mathrm{grad}\, f$ satisfies the Morse-Smale condition[4] in $\mathcal{M}\setminus\mathcal{A}$. Then one constructs the relative Morse complex $M(f,g)$, taking into account only the critical points of f in $\mathcal{M}\setminus\mathcal{A}$, and finds that its homology is isomorphic to the singular homology of the pair $(\mathcal{M},\mathcal{A})$,

$$H_k M(f,g) \cong H_k(\mathcal{M},\mathcal{A}).$$

It is well known that many operations in singular homology have their Morse theoretical interpretation, in the sense that they can be read on the Morse complex (see for instance [Sch93, Fuk93, BC94, Vit95, Fuk97]). Here we are interested only in functoriality, in the exterior homology product, and in the Umkehr map. The corresponding constructions - still in our infinite dimensional setting - are outlined in the following sections.

[4]Notice that transversality issues are more delicate here because one wants $\mathcal{A}$ to remain positively invariant for the perturbed flow. We will not state a general result, but we will discuss the transversality question case by case.

2.2 Functoriality

Let $(\mathcal{M}_1, g_1)$ and $(\mathcal{M}_2, g_2)$ be complete Riemannian Hilbert manifolds, and let $f_1 \in \mathcal{F}(\mathcal{M}_1, g_1)$, $f_2 \in \mathcal{F}(\mathcal{M}_2, g_2)$ be such that $-\mathrm{grad}\, f_1$ and $-\mathrm{grad}\, f_2$ satisfy the Morse-Smale condition. Denote by ϕ^1 and ϕ^2 the corresponding negative gradient flows.

Let $\varphi : \mathcal{M}_1 \to \mathcal{M}_2$ be a smooth map. We assume that

$$\text{each } y \in \mathrm{crit}(f_2) \text{ is a regular value of } \varphi; \tag{9}$$

$$x \in \mathrm{crit}(f_1),\ \varphi(x) \in \mathrm{crit}(f_2) \quad \Longrightarrow \quad m(x; f_1) \geq m(\varphi(x); f_2). \tag{10}$$

The set of critical points of f_2 is discrete, and in many cases (for instance, if φ is a Fredholm map) the set of regular values of φ is generic (i.e. a countable intersection of open dense sets, by Sard-Smale theorem [Sma65]). In such a situation, condition (9) can be achieved by arbitrary small (in several senses) perturbations of φ or of f_2. Also condition (10) can be achieved by an arbitrary small perturbation of φ or of f_2, simply by requiring that the image of the set of critical points of f_1 by φ does not meet the set of critical point of f_2.

By (9) and (10), up to perturbing the metrics g_1 and g_2, we may assume that

$$\forall x \in \mathrm{crit}(f_1),\ \forall y \in \mathrm{crit}(f_2), \quad \varphi|_{W^u(x; -\mathrm{grad}\, f_1)} \text{ is transverse to } W^s(y; -\mathrm{grad}\, f_2). \tag{11}$$

Indeed, by (9) and (10) one can perturb g_1 in such a way that if $p \in W^u(x; -\mathrm{grad}\, f_1)$ and $\varphi(p)$ is a critical point of f_2 then $\mathrm{rank}\, D\varphi(p)|_{T_p W^u(x)} \geq m(\varphi(p); f_2)$. The possibility of perturbing g_2 so that (11) holds is now a consequence of the following fact: if W is a finite dimensional manifold and $\psi : W \to \mathcal{M}_2$ is a smooth map such that for every $p \in W$ with $\psi(p) \in \mathrm{crit}(f_2)$ there holds $\mathrm{rank}\, D\psi(p) \geq m(\varphi(p); f_2)$, then the set of metrics g_2 on $\mathcal{M}_2$ such that the map ψ is transverse to the stable manifold of every critical point of f_2 is generic in the set of all metrics, with many reasonable topologies[5].

The transversality condition (11) ensures that if $x \in \mathrm{crit}(f_1)$ and $y \in \mathrm{crit}(f_2)$, then

$$W(x, y) := W^u(x; -\mathrm{grad}\, f_1) \cap \varphi^{-1}(W^s(y; -\mathrm{grad}\, f_2))$$

is a submanifold of dimension $m(x; f_1) - m(y; f_2)$. If $W^u(x; -\mathrm{grad}\, f_1)$ is oriented and the normal bundle of $W^s(y; -\mathrm{grad}\, f_2)$ in $\mathcal{M}_2$ is oriented, the manifold $W(x, y)$ carries a canonical orientation. In particular, if $m(x; f_1) = m(y; f_2)$, $W(x, y)$ is a discrete set, each of whose point carries an orientation sign ± 1. The transversality condition (11) and the fact that $W^u(x; -\mathrm{grad}\, f_1)$ has compact closure in $\mathcal{M}_1$ imply that the discrete set $W(x, y)$ is also compact, so it is a finite set and we denote by $n_\varphi(x, y) \in \mathbb{Z}$ the algebraic sum of the corresponding orientation signs. We can then define the homomorphism

$$M_k\varphi : M_k(f_1, g_1) \to M_k(f_2, g_2), \quad M_k\varphi\, x = \sum_{y \in \mathrm{crit}_k(f_2)} n_\varphi(x, y)\, y,$$

for every $x \in \mathrm{crit}_k(f_1)$.

We claim that $M_*\varphi$ is a chain map from the Morse complex of (f_1, g_1) to the Morse complex of (f_2, g_2), and that the corresponding homomorphism in homology coincides - via the isomorphism described in section 2.1 - with the homomorphism $\varphi_* : H_*(\mathcal{M}_1) \to H_*(\mathcal{M}_2)$.

Indeed, let us fix small open neighborhoods $\mathcal{U}_i(x)$, $i = 1, 2$, for each critical point $x \in \mathrm{crit}(f_i)$, such that the sequence of open sets

$$\mathcal{U}_i^k = \bigcup_{\substack{x \in \mathrm{crit}(f_i) \\ m(x; f_i) \leq k}} \phi^i([0, +\infty[\times \mathcal{U}_i(x)), \quad k \in \mathbb{N},\ i = 1, 2,$$

[5]Here one is interested in finding perturbations of g_1 and g_2 which are so small that $-\mathrm{grad}\, f_1$ and $-\mathrm{grad}\, f_2$ still satisfy the Morse-Smale condition and the corresponding Morse complexes are unaffected (for instance, because the perturbed flows are topologically conjugated to the original ones). This can be done by considering Banach spaces of C^k perturbations of g_1 and g_2 endowed with a Whitney norm, that is something like $\|h\| = \sum_{1 \leq j \leq k} \sup_{p \in \mathcal{M}_i} \epsilon(p) |D^j h(p)|$, for a suitable positive function $\epsilon : \mathcal{M}_i \to]0, +\infty[$. We shall not specify these topologies any further, and we shall always assume that the perturbations are so small that the good properties of the original metrics are preserved.

is a cellular filtration of $\mathcal{U}_i^\infty = \bigcup_k \mathcal{U}_i^k$. By the transversality assumption (11), if p belongs to $W^u(x; -\mathrm{grad}\, f_1)$ then $\varphi(p)$ belongs to the stable manifold of some critical point $y \in \mathrm{crit}(f_2)$ with $m(y; f_2) \leq m(x; f_1)$. A standard compactness-transversality argument shows that, up to replacing the neighborhoods $\mathcal{U}_1(x)$, $x \in \mathrm{crit}(f_1)$, by smaller ones, we may assume that

$$p \in \mathcal{U}_1^k \quad \Longrightarrow \quad \varphi(p) \in W^s(y; -\mathrm{grad}\, f_2) \text{ with } m(y; f_2) \leq k.$$

Since the set $\mathcal{U}_2^k$ is a ϕ^2-positively invariant open neighborhood of the set of the critical points of f_2 whose Morse index does not exceed k, it is easy to find a continuous function $t_0 : \mathcal{U}_1^k \to [0, +\infty[$ such that

$$p \in \mathcal{U}_1^k \quad \Longrightarrow \quad \phi^2(t_0(p), \varphi(p)) \in \mathcal{U}_2^k.$$

Therefore, $\psi(p) := \phi^2(t_0(p), \varphi(p))$ is a cellular map from $(\mathcal{U}_1^\infty, \{\mathcal{U}_1^k\}_{k\in\mathbb{N}})$ to $(\mathcal{U}_2^\infty, \{\mathcal{U}_2^k\}_{k\in\mathbb{N}})$, and it is easy to check that the induced cellular homomorphism

$$W_*\psi : W_*\{\mathcal{U}_1^k\}_{k\in\mathbb{N}} \to W_*\{\mathcal{U}_2^k\}_{k\in\mathbb{N}}$$

coincides with $M_*\varphi$, once we identify $H_k(\mathcal{U}_i^k, \mathcal{U}_i^{k-1})$ with $M_k(f_i)$, by taking the orientations of the unstable manifolds into account. Then, everything follows from the naturality of cellular homology, from the fact that the inclusions $j_i : \mathcal{U}_i^\infty \hookrightarrow \mathcal{M}_i$ are homotopy equivalences, and from the fact that $j_2 \circ \psi$ is homotopic to $\varphi \circ j_1$.

The above construction has an obvious extension to the case of a smooth map φ between two pairs $(\mathcal{M}_1, \mathcal{A}_1)$ and $(\mathcal{M}_2, \mathcal{A}_2)$, $\mathcal{A}_i$ open subset of $\mathcal{M}_i$, $i = 1, 2$ (but some care is needed to deal with transversality issues, see footnote 4).

2.1. Remark. *We recall that if two chain maps between* free *chain complexes induce the same homomorphism in homology, they are chain homotopic. So from the functoriality of singular homology, we deduce that $M_*\varphi \circ M_*\psi$ and $M_*\varphi \circ \psi$ are chain homotopic. Actually, a chain homotopy between these two chain maps could be constructed in a direct way.*

2.2. Remark. *Consider the following particular but important case: $\mathcal{M}_1 = \mathcal{M}_2$ and $\varphi = \mathrm{id}$. Then (9) holds automatically, and (10) means asking that every common critical point x of f_1 and f_2 must satisfy $m(x; f_1) \geq m(x; f_2)$. In this case, the above construction produces a chain map from $M_*(f_1, g_1)$ to $M_*(f_2, g_2)$ which induces the identity map in homology (after the identification with singular homology).*

2.3. Remark. *For future reference, let us stress the fact that if it is already known that $p \in W^u(x; -\mathrm{grad}\, f_1)$ and $\varphi(p) \in \mathrm{crit}(f_2)$ imply $\mathrm{rank}\, D\varphi(p)|_{T_pW^u(x; -\mathrm{grad}\, f_1)} \geq m(\varphi(p); f_2)$, then condition (9) is useless, condition (10) holds automatically, and there is no need of perturbing the metric g_1 on $\mathcal{M}_1$.*

2.3 The exterior homology product

Let $(\mathcal{M}_1, g_1)$ and $(\mathcal{M}_2, g_2)$ be complete Riemannian Hilbert manifolds, and let $f_1 \in \mathcal{F}(\mathcal{M}_1, g_1)$, $f_2 \in \mathcal{F}(\mathcal{M}_2, g_2)$ be such that $-\mathrm{grad}\, f_1$ and $-\mathrm{grad}\, f_2$ satisfy the Morse-Smale condition. If we denote by $f_1 \oplus f_2$ the function on $\mathcal{M}_1 \times \mathcal{M}_2$,

$$f_1 \oplus f_2\, (p_1, p_2) := f_1(p_1) + f_2(p_2),$$

we see that $f_1 \oplus f_2$ belongs to $\mathcal{F}(\mathcal{M}_1 \times \mathcal{M}_2, g_1 \times g_2)$. Moreover,

$$\mathrm{crit}_\ell(f_1 \oplus f_2) = \bigcup_{j+k=\ell} \mathrm{crit}_j(f_1) \times \mathrm{crit}_k(f_2),$$

hence

$$M_\ell(f_1 \oplus f_2) = \bigoplus_{j+k=\ell} M_j(f_1) \otimes M_k(f_2).$$

The orbits of $-\mathrm{grad}\,(f_1 \oplus f_2)$ are just pairs of orbits of $-\mathrm{grad}\,f_1$ and $-\mathrm{grad}\,f_2$, so the Morse-Smale condition holds. If we fix orientations for the unstable manifold of each critical point of f_1, f_2, and we endow the unstable manifold of each $(x_1, x_2) \in \mathrm{crit}(f_1 \oplus f_2)$,

$$W^u((x_1, x_2)) = W^u(x_1) \times W^u(x_2),$$

with the product orientation, we see that the boundary operator in the Morse complex of $(f_1 \oplus f_2, g_1 \times g_2)$ is given by

$$\partial(x_1, x_2) = (\partial x_1, x_2) + (-1)^{m(x_1)}(x_1, \partial x_2), \quad \forall x_i \in \mathrm{crit}(f_i),\ i = 1, 2.$$

We conclude that the Morse complex of $(f_1 \oplus f_2, g_1 \times g_2)$ is the tensor product of the Morse complexes of (f_1, g_1) and (f_2, g_2). So, using the natural homomorphism from the tensor product of the homology of two chain complexes to the homology of the tensor product of the two complexes, we obtain the homomorphism

$$H_j M(f_1, g_1) \otimes H_k M(f_2, g_2) \to H_{j+k} M(f_1 \oplus f_2, g_1 \times g_2). \tag{12}$$

We claim that this homomorphism corresponds to the exterior product homomorphism

$$H_j(\mathcal{M}_1) \otimes H_k(\mathcal{M}_2) \xrightarrow{\times} H_{j+k}(\mathcal{M}_1 \times \mathcal{M}_2), \tag{13}$$

via the isomorphism between Morse homology and singular homology described in section 2.1.

Indeed, the cellular filtration in $\mathcal{M}_1 \times \mathcal{M}_2$ can be chosen to be generated by small product neighborhoods of the critical points,

$$\mathcal{W}^\ell = \bigcup_{\substack{(x_1,x_2)\in\mathrm{crit}(f_1\oplus f_2)\\ m(x_1)+m(x_2)=\ell}} \phi^1([0, +\infty[\times \mathcal{U}_1(x_1)) \times \phi^2([0, +\infty[\times \mathcal{U}_2(x_2)) = \bigcup_{j+k=\ell} \mathcal{U}_1^j \times \mathcal{U}_2^k.$$

By excision and by the Kunneth theorem, together with the fact that we are dealing with free Abelian groups, one easily obtains that

$$W_\ell \mathcal{W} = H_\ell(\mathcal{W}^\ell, \mathcal{W}^{\ell-1}) \cong \bigoplus_{j+k=\ell} H_j(\mathcal{U}_1^j, \mathcal{U}_1^{j-1}) \otimes H_k(\mathcal{U}_2^k, \mathcal{U}_2^{k-1}),$$

and that the boundary homomorphism of the cellular filtration $\mathcal{W}$ is the tensor product of the boundary homomorphisms of the cellular filtrations $\mathcal{U}_1$ and $\mathcal{U}_2$. Passing to homology, we find that (12) corresponds to the exterior homology product

$$H_j(\mathcal{U}_1^\infty) \otimes H_k(\mathcal{U}_2^\infty) \xrightarrow{\times} H_{j+k}(\mathcal{U}_1^\infty \times \mathcal{U}_2^\infty) = H_{j+k}(\mathcal{W}^\infty),$$

by the usual identification of the cellular complex to the Morse complex induced by a choice of orientations for the unstable manifolds. But since the inclusion $\mathcal{U}_1^\infty \hookrightarrow \mathcal{M}_1$ and $\mathcal{U}_2^\infty \hookrightarrow \mathcal{M}_2$ are homotopy equivalences, we conclude that (12) corresponds to (13).

2.4 Intersection products

Let $\mathcal{M}_0$ be a Hilbert manifold, and let $\pi : \mathcal{E} \to \mathcal{M}_0$ be a smooth n-dimensional oriented real vector bundle over $\mathcal{M}_0$. It is easy to describe the Thom isomorphism

$$\tau : H_k(\mathcal{E}, \mathcal{E} \setminus \mathcal{M}_0) \xrightarrow{\cong} H_{k-n}(\mathcal{M}_0), \quad \alpha \mapsto \tau_{\mathcal{E}} \cap \alpha,$$

in a Morse theoretical way ($\tau_{\mathcal{E}} \in H^n(\mathcal{E}, \mathcal{E} \setminus \mathcal{M}_0)$ denotes the Thom class of the vector bundle $\mathcal{E}$).

Indeed, let g_0 be a complete metric on $\mathcal{M}_0$ and let $f_0 \in \mathcal{F}(\mathcal{M}_0, g_0)$ be such that $-\mathrm{grad}\,f_0$ satisfies the Morse-Smale condition. The choice of a Riemannian structure on the vector bundle $\mathcal{E}$ determines a class of product-type metrics g_1 on the manifold $\mathcal{E}$, and a smooth function

$$f_1(\xi) := f_0(\pi(\xi)) - |\xi|^2, \quad \forall \xi \in \mathcal{E}.$$

It is readily seen that (f_1, g_1) satisfies conditions (f1')-(f4') of section 2.1 on the pair $(\mathcal{E}, \mathcal{E} \setminus \overline{\mathcal{U}_1})$, $\mathcal{U}_1$ being the set of vectors ξ in the total space $\mathcal{E}$ with $|\xi| < 1$, and that $-\mathrm{grad}\, f_1$ satisfies the Morse-Smale condition. Therefore, the homology of the relative Morse complex of (f_1, g_1) on $(\mathcal{E}, \mathcal{E} \setminus \overline{\mathcal{U}_1})$ is isomorphic to the singular homology of the pair $(\mathcal{E}, \mathcal{E} \setminus \overline{\mathcal{U}_1})$, that is to the singular homology of $(\mathcal{E}, \mathcal{E} \setminus \mathcal{M}_0)$. Actually,

$$\mathrm{crit}_k(f_1) = \mathrm{crit}_{k-n}(f_0), \quad T_x W^u(x; -\mathrm{grad}\, f_1) = T_x W^u(x; -\mathrm{grad}\, f_0) \oplus \mathcal{E}_x,$$

so the orientation of the vector bundle $\mathcal{E}$ allows to associate an orientation of $W^u(x; -\mathrm{grad}\, f_1)$ to each orientation of $W^u(x; -\mathrm{grad}\, f_0)$. Then the relative Morse complex of (f_1, g_1) on $(\mathcal{E}, \mathcal{E} \setminus \overline{\mathcal{U}_1})$ is obtained from the Morse complex of (f_0, g_0) on $\mathcal{M}_0$ by a $-n$-shift in the grading:

$$M_k(f_1, g_1) = M_{k-n}(f_0, g_0),$$

and it is easily seen that the isomorphism τ - read on the Morse complexes by the isomorphisms described in section 2.1 - is induced by the identity mapping

$$M_k(f_1, g_1) \xrightarrow{\mathrm{id}} M_{k-n}(f_0, g_0).$$

Consider now the general case of a closed embedding $e : \mathcal{M}_0 \hookrightarrow \mathcal{M}$, assumed to be of codimension n and co-oriented. The above description of the Thom isomorphism associated to the normal bundle $N\mathcal{M}_0$ of $\mathcal{M}_0$ and the discussion about functoriality of section 2.2 yield the following Morse theoretical description for the Umkehr map

$$e_! : H_k(\mathcal{M}) \to H_{k-n}(\mathcal{M}_0).$$

It is actually useful to identify an open neighborhood of $\mathcal{M}_0$ to $N\mathcal{M}_0$ by the tubular neighborhood theorem, to consider again the open unit ball $\mathcal{U}_1$ around the zero section of $N\mathcal{M}_0$, and to see the Umkehr map as the composition

$$H_j(\mathcal{M}) \xrightarrow{i_*} H_j(\mathcal{M}, \mathcal{M} \setminus \overline{\mathcal{U}_1}) \cong H_j(N\mathcal{M}_0, N\mathcal{M}_0 \setminus \mathcal{M}_0) \xrightarrow{\tau} H_{j-n}(\mathcal{M}_0),$$

the map $i : \mathcal{M} \hookrightarrow (\mathcal{M}, \mathcal{M} \setminus \overline{\mathcal{U}_1})$ being the inclusion. Let f_0, g_0, f_1, g_1 be as above. We use the symbols f_1 and g_1 also to denote arbitrary extensions of f_1 and g_1 to the whole $\mathcal{M}$. Let g be a complete metric on $\mathcal{M}$, and let $f \in \mathcal{F}(\mathcal{M}, g)$ be such that $-\mathrm{grad}\, f$ satisfies the Morse-Smale condition. Since we would like to achieve transversality by perturbing g and g_0, but keeping g_1 of product-type near $\mathcal{M}_0$, we need the condition

$$x \in \mathrm{crit}(f) \cap \mathcal{M}_0 \quad \Longrightarrow \quad m(x; f) \geq n, \tag{14}$$

which implies that up to perturbing g we may assume that the unstable manifold of each critical point of f is transversal to $\mathcal{M}_0$. Assumption (9) is automatically satisfied by the triplet (i, f, f_1), while (10) is equivalent to asking that

$$x \in \mathrm{crit}(f) \cap \mathrm{crit}(f_0) \quad \Longrightarrow \quad m(x; f) \geq m(x; f_0) + n. \tag{15}$$

Conditions (14) and (15) are implied by the generic assumption $\mathrm{crit}(f) \cap \mathcal{M}_0 = \emptyset$. By the arguments of section 2.2 applied to the map i (in particular, condition (11)), we see that up to perturbing g and g_0 (keeping g_1 of product-type near $\mathcal{M}_0$), we may assume that for every $x \in \mathrm{crit}(f)$, $y \in \mathrm{crit}(f_0)$, the intersection

$$W^u(x; -\mathrm{grad}\, f) \cap W^s(y; -\mathrm{grad}\, f_1) = W^u(x; -\mathrm{grad}\, f) \cap W^s(y; -\mathrm{grad}\, f_0)$$

is transverse in $\mathcal{M}$, hence it is a submanifold of dimension $m(x; f) - m(y; f_0) - n$. If we fix an orientation for the unstable manifold of each critical point of f and f_0, these intersections are canonically oriented. Compactness and transversality imply that when $m(y; f_0) = m(x; f) - n$, this intersection is a finite set of points, each of which comes with an orientation sign ± 1. Denoting by $n_{e_!}(x,y)$ the algebraic sum of these signs, we conclude that the homomorphism

$$M_k(f,g) \to M_{k-n}(f_0,g_0), \quad x \mapsto \sum_{\substack{y\in \mathrm{crit}(f_0)\\ m(y;f_0)=k-n}} n_{e_!}(x,y)\, y, \quad \forall x \in \mathrm{crit}_k(f),$$

is a chain map of degree $-n$, and that it induces the Umkehr map $e_!$ in homology (by the identification of the homology of the Morse complex with singular homology described in section 2.1).

Let us conclude this section by describing the Morse theoretical interpretation of the intersection product in homology. Let M be a finite-dimensional oriented manifold, and consider the diagonal embedding $e : \Delta_M \hookrightarrow M\times M$, which is n-codimensional and co-oriented. The intersection product is defined by the composition

$$H_j(M) \otimes H_k(M) \xrightarrow{\times} H_{j+k}(M\times M) \xrightarrow{e_!} H_{j+k-n}(\Delta_M) \cong H_{j+k-n}(M),$$

and it is denoted by

$$\bullet : H_j(M)\otimes H_k(M) \longrightarrow H_{j+k-n}(M).$$

The above description of $e_!$ and the description of the exterior homology product $\times$ given in section 2.3 immediately yield the following description of $\bullet$. Let g_1, g_2, g_3 be complete metrics on M, and let $f_i \in \mathcal{F}(M,g_i)$, $i=1,2,3$, be such that $-\mathrm{grad}\, f_i$ satisfies the Morse-Smale condition. The non-degeneracy conditions (14) and (15), necessary to represent $e_!$, are now

$$x \in \mathrm{crit}(f_1)\cap \mathrm{crit}(f_2) \quad \Longrightarrow \quad m(x;f_1)+m(x;f_2) \geq n, \tag{16}$$

$$x \in \mathrm{crit}(f_1)\cap \mathrm{crit}(f_2)\cap \mathrm{crit}(f_3) \quad \Longrightarrow \quad m(x;f_1)+m(x;f_2) \geq m(x;f_3)+n. \tag{17}$$

These conditions are implied for instance by the generic assumption that f_1 and f_2 do not have common critical points. We can now perturb the metrics g_1, g_2, and g_3 on M in such a way that for every triplet $x_i \in \mathrm{crit}(f_i)$, $i=1,2,3$, the intersection

$$W^u((x_1,x_2);-\mathrm{grad}_{g_1\times g_2} f_1 \oplus f_2) \cap a(W^s(x_3;-\mathrm{grad}_{g_3} f_3)),$$

$a : M \to M\times M$ being the map $a(p)=(p,p)$, is transverse in $M\times M$, hence it is an oriented submanifold of Δ_M of dimension $m(x_1;f_1)+m(x_2;f_2)-m(x_3;f_3)-n$. By compactness and transversality, when $m(x_3;f_3) = m(x_1;f_1)+m(x_2;f_2)-n$, this intersection, which can also be written as

$$\left\{(p,p) \in W^u(x_1;-\mathrm{grad}\, f_1)\times W^u(x_2;-\mathrm{grad}\, f_2) \mid p \in W^s(x_3;-\mathrm{grad}\, f_3)\right\},$$

is a finite set of points, each of which comes with an orientation sign ± 1. Denoting by $n_\bullet(x_1,x_2;x_3)$ the algebraic sum of these signs, we conclude that the homomorphism

$$M_j(f_1,g_1)\otimes M_k(f_2,g_2) \to M_{j+k-n}(f_3,g_3), \quad x_1\otimes x_2 \mapsto \sum_{\substack{x_3\in \mathrm{crit}(f_3)\\ m(x_3;f_3)=j+k-n}} n_\bullet(x_1,x_2;x_3)\, x_3,$$

where $x_1 \in \mathrm{crit}_j(f_1)$, $x_2 \in \mathrm{crit}_k(f_2)$, is a chain map of degree $-n$ from the complex $M(f_1,g_1)\otimes M(f_2,g_2)$ to $M(f_3,g_3)$, and that it induces the intersection product $\bullet$ in homology (by the identification of the homology of the Morse complex with singular homology described in section 2.1).

2.5 Lagrangian action functionals, and Morse theoretical interpretation of the homomorphisms $\mathbf{c}_*$, $\mathbf{ev}_*$, and $\mathbf{i}_!$

We are now in a good position to describe the homomorphisms appearing in diagram (8) in a Morse theoretical way. The first thing to do is to replace the Banach manifolds $\Omega(M,q_0)$, $\Lambda(M)$, and $\Theta(M)$ by the Hilbert manifolds

$$\Omega^1(M,q_0) := \left\{\gamma \in \Lambda^1(M) \mid \gamma(0)=q_0\right\}, \quad \Lambda^1(M) := W^{1,2}(\mathbb{T},M),$$
$$\Theta^1(M) := \left\{(\gamma_1,\gamma_2) \in \Lambda^1(M)\times\Lambda^1(M) \mid \gamma_1(0)=\gamma_2(0)\right\},$$

$W^{1,2}$ denoting the class of absolutely continuous curves whose derivative is square integrable. The inclusions

$$\Omega^1(M,q_0) \hookrightarrow \Omega(M,q_0), \quad \Lambda^1(M) \hookrightarrow \Lambda(M), \quad \Theta^1(M) \hookrightarrow \Theta(M),$$

are homotopy equivalences. Therefore, we can replace $\Omega(M,q_0)$, $\Lambda(M)$, and $\Theta(M)$ by $\Omega^1(M,q_0)$, $\Lambda^1(M)$, and $\Theta^1(M)$ in the constructions of section 1.

Consider as before the smooth maps

$$c : M \to \Lambda^1(M),\ c(q)(t) \equiv q, \quad \mathrm{ev} : \Lambda^1(M) \to M,\ \mathrm{ev}(\gamma) = \gamma(0),$$
$$i : \Omega^1(M,q_0) \hookrightarrow \Lambda^1(M), \quad e : \Theta^1(M) \hookrightarrow \Lambda^1(M) \times \Lambda^1(M).$$

Applying the results of sections 2.2 and 2.4, one can describe the homomorphisms c_*, ev_*, $i_!$, and $e_!$, in terms of the Morse complexes of a quite general class of functions on M, $\Omega^1(M,q_0)$, $\Lambda^1(M)$, and $\Theta^1(M)$. However, in order to find a link with symplectic geometry, we wish to consider a special class of functions on the three latter manifolds, namely the action functionals associated to (possibly time-dependent) Lagrangian functions on TM.

For this purpose, let us assume the oriented n-dimensional manifold M to be compact. Let $L : \mathbb{T} \times TM \to \mathbb{R}$ or $L : [0,1] \times TM \to \mathbb{R}$ be a smooth Lagrangian, such that

(L1) there exists $\ell_0 > 0$ such that $\partial_{vv} L(t,q,v) \geq \ell_0 I$, for every (t,q,v);

(L2) there exists $\ell_1 \geq 0$ such that

$$|\partial_{vv} L(t,q,v)| \leq \ell_1, \quad |\partial_{vq} L(t,q,v)| \leq \ell_1(1+|v|), \quad |\partial_{qq} L(t,q,v)| \leq \ell_1(1+|v|^2),$$

for every (t,q,v).

These assumptions are stated by means of local coordinates and of a Riemannian structure on the vector bundle TM, but they actually do not depend on these objects. If $|\cdot|.$ is a Riemannian metric on M, the geodesic Lagrangian $L(t,q,v) = |v|_q^2/2$, and more generally the electro-magnetic Lagrangian

$$L(t,q,v) = \frac{1}{2}|v|_q^2 + \langle A(t,q), v\rangle_q - V(t,q).$$

satisfy conditions (L1) and (L2), for every scalar potential V and every vector potential A. The Lagrangian L defines a second order ODE on M, which in local coordinates can be written as

$$\frac{d}{dt}\partial_v L(t,\gamma(t),\gamma'(t)) = \partial_q L(t,\gamma(t),\gamma(t)), \tag{18}$$

and (L1) guarantees that the corresponding Cauchy problem is well-posed. The action functional

$$\mathbb{S}_L(\gamma) = \int_0^1 L(t,\gamma(t),\gamma'(t))\, dt$$

is of class C^2 on $\Omega^1(M,q_0)$ and on $\Lambda^1(M)$ (in the second case we assume L to be defined on $\mathbb{T} \times TM$, in the first case $[0,1] \times TM$ suffices). The restriction of $\mathbb{S}_L$ to $\Omega^1(M,q_0)$ is denoted by $\mathbb{S}_L^\Omega$, whereas its restriction to $\Lambda^1(M)$ is denoted by $\mathbb{S}_L^\Lambda$. The critical points of $\mathbb{S}_L^\Lambda$ are precisely the 1-periodic solutions of (18), while the critical points of $\mathbb{S}_L^\Omega$ are the solutions $\gamma : [0,1] \to M$ of (18) such that $\gamma(0) = \gamma(1) = q_0$ (but in general $\gamma'(0) \neq \gamma'(1)$, so these solutions do not extend to period solutions, even if L is 1-periodic in time). Denote by

$$\mathscr{P}^\Omega(L) = \mathrm{crit}(\mathbb{S}_L^\Omega), \quad \mathscr{P}^\Lambda(L) = \mathrm{crit}(\mathbb{S}_L^\Lambda),$$

these sets of solutions. Assumption (L1) implies that each critical point γ of $\mathbb{S}_L^\Omega$ and of $\mathbb{S}_L^\Lambda$ has finite Morse index, which we denote by $m^\Omega(\gamma)$ and $m^\Lambda(\gamma)$. The requirement that each critical point γ should be non-degenerate is translated into the following assumptions on the Jacobi vector fields along γ (i.e. solutions of the second order linear ODE obtained by linearizing (18) along γ):

(L0)$^\Omega$ every solution $\gamma \in \mathscr{P}^\Omega(L)$ is non-degenerate, meaning that there are no non-zero Jacobi vector fields along γ which vanish for $t=0$ and for $t=1$.

(L0)$^\Lambda$ every solution $\gamma \in \mathscr{P}^\Lambda(L)$ is non-degenerate, meaning that there are no non-zero periodic Jacobi vector fields along γ.

These conditions hold for a generic choice of L, in several reasonable senses. Notice that (L0)$^\Lambda$ forces L to be explicitly time-dependent, otherwise γ' would be a non-zero 1-periodic Jacobi vector field along γ, for every non-constant periodic solution γ. In the fixed ends case instead, L is allowed to be autonomous.

We need also to consider the following Lagrangian problem for figure-8 loops. Let $L_1, L_2 \in C^\infty([0,1]\times TM)$ be Lagrangians satisfying (L1) and (L2). Let $\mathscr{P}^\Theta(L_1 \oplus L_2)$ be the set of all pairs (γ_1,γ_2), with $\gamma_j : [0,1] \to M$ solution of the Lagrangian system given by L_j, such that

$$\gamma_1(0)=\gamma_1(1)=\gamma_2(0)=\gamma_2(1), \quad \sum_{i=0}^{1}\sum_{j=1}^{2}(-1)^i \partial_v L_j(i,\gamma_j(i),\gamma_j'(i)) = 0. \tag{19}$$

The elements of $\mathscr{P}^\Theta(L_1\oplus L_2)$ are precisely the critical points of the functional $\mathbb{S}_{L_1\oplus L_2} = \mathbb{S}_{L_1}\oplus \mathbb{S}_{L_2}$ restricted to the submanifold $\Theta^1(M)$,

$$\mathbb{S}_{L_1\oplus L_2}(\gamma_1,\gamma_2) = \int_0^1 L_1(t,\gamma_1(t),\gamma_1'(t))\,dt + \int_0^1 L_2(t,\gamma_2(t),\gamma_2'(t))\,dt.$$

Such a restriction is denoted by $\mathbb{S}^\Theta_{L_1\oplus L_2}$. If we denote by $m^\Theta(\gamma_1,\gamma_2)$ the Morse index of $(\gamma_1,\gamma_2) \in \mathscr{P}^\Theta(L_1\oplus L_2)$, we clearly have

$$\begin{aligned}\max\{m^\Omega(\gamma_1;L_1)+m^\Omega(\gamma_2;L_2), m^\Lambda(\gamma_1;L_1)+m^\Lambda(\gamma_2;L_2)-n\} &\le m^\Theta(\gamma_1,\gamma_2)\\ \le \min\{m^\Omega(\gamma_1;L_1)+m^\Omega(\gamma_2;L_2)+n, m^\Lambda(\gamma_1;L_1)&+m^\Lambda(\gamma_2;L_2)\}.\end{aligned}$$

The non-degeneracy of every critical point of $\mathbb{S}^\Theta_{L_1\oplus L_2}$ is equivalent to the condition:

(L0)$^\Theta$ every solution $(\gamma_1,\gamma_2) \in \mathscr{P}^\Theta(L_1\oplus L_2)$ is non-degenerate, meaning that there are no non-zero pairs of Jacobi vector fields ξ_1,ξ_2 along γ_1,γ_2 such that

$$\xi_1(0)=\xi_1(1)=\xi_2(0)=\xi_2(1),$$
$$\sum_{i=0}^{1}\sum_{j=1}^{2}\big(\partial_{qv}L_j(i,\gamma_j(i),\gamma_j'(i))\xi_j(i) + \partial_{vv}L_j(i,\gamma_j(i),\gamma_j'(i))\xi_j'(i)\big) = 0.$$

This condition allows both L_1 and L_2 to be autonomous. It also allows $L_1 = L_2$, but this excludes the autonomous case (otherwise pairs (γ,γ) with $\gamma \in \mathscr{P}^\Lambda(L_1) = \mathscr{P}^\Lambda(L_2)$ non-constant would violate (L0)$^\Theta$).

Assumption (L1) implies that L is bounded below, and it can be shown that (L1) and (L2) imply that $\mathbb{S}_L$ satisfies the Palais-Smale condition on $\Omega^1(M,q_0)$ and on $\Lambda^1(M)$, with respect to the standard $W^{1,2}$-metric

$$\langle\langle \xi,\eta\rangle\rangle_\gamma := \int_0^1 \big(\langle \xi(t),\eta(t)\rangle_{\gamma(t)} + \langle \nabla_t\xi(t),\nabla_t\eta(t)\rangle_{\gamma(t)}\big)\,dt,$$

where $\langle\cdot,\cdot\rangle$ is a metric on M, and ∇_t is the corresponding Levi-Civita covariant derivation along γ. The same is true for the functional $\mathbb{S}_{L_1\oplus L_2}$ on $\Theta^1(M)$. See for instance the appendix in [AF07] for a proof of the Palais-Smale condition under general non-local conormal boundary conditions, including all the case treated here. We conclude that under the assumptions (L0)$^\Omega$ (resp. (L0)$^\Lambda$, resp. (L0)$^\Theta$), (L1), (L2), the function $\mathbb{S}^\Omega_L$ (resp. $\mathbb{S}^\Lambda_L$, resp. $\mathbb{S}^\Theta_{L_1\oplus L_2}$) belongs to $\mathcal{F}(\Omega^1(M,q_0),\langle\langle\cdot,\cdot\rangle\rangle)$ (resp. $\mathcal{F}(\Lambda^1(M),\langle\langle\cdot,\cdot\rangle\rangle)$, resp. $\mathcal{F}(\Theta^1(M),\langle\langle\cdot,\cdot\rangle\rangle)$).

It is now straightforward to apply the results of section 2.2 to describe the homomorphisms

$$c_* : H_k(M) \to H_k(\Lambda^1(M)), \quad \text{and} \quad \mathrm{ev}_* : H_k(\Lambda^1(M)) \to H_k(M),$$

in terms of the Morse complexes of $\mathbb{S}_L^\Lambda$ and of a Morse function f on M. Indeed, in the case of c_* one imposes the condition

$$q \in \mathrm{crit}(f) \quad \Longrightarrow \quad c(q) \notin \mathscr{P}^\Lambda(L), \tag{20}$$

which guarantees (9) and (10) (given any Lagrangian L satisfying $(\mathrm{L0})^\Lambda$, one can always find a Morse function f on M such that (20) holds, simply because $(\mathrm{L0})^\Lambda$ implies that the Lagrangian system has finitely many constant solutions). Then one can find a metric g_M on M and a small perturbation g^Λ of the metric $\langle\langle \cdot, \cdot \rangle\rangle$ on $\Lambda^1(M)$ such that the vector fields $-\mathrm{grad}\, \mathbb{S}_L^\Lambda$ and $-\mathrm{grad}\, f$ satisfy the Morse-Smale condition, and the restriction of c to the unstable manifold of each $q \in \mathrm{crit}(f)$ is transverse to the stable manifold of each $\gamma \in \mathrm{crit}(\mathbb{S}_L)$. When $m(q; f) = m^\Lambda(\gamma)$, the intersection

$$W^u(q; -\mathrm{grad}\, f) \cap c^{-1}\left(W^s(\gamma; -\mathrm{grad}\, \mathbb{S}_L^\Lambda)\right)$$

consists of finitely many points, each of which comes with an orientation sign ± 1. The algebraic sums $n_c(q, \gamma)$ of these signs provide us with the coefficients of a chain map

$$M_k c : M_k(f, g_M) \to M_k(\mathbb{S}_L^\Lambda, g^\Lambda),$$

which in homology induces the homomorphism c_*.

The map $\mathrm{ev} : \Lambda^1(M) \to M$ is a submersion, so (9) holds automatically. Condition (10) instead is implied by

$$\gamma \in \mathscr{P}^\Lambda(L) \quad \Longrightarrow \quad \gamma(0) \notin \mathrm{crit}(f), \tag{21}$$

which again holds for a generic f, given L. Then one finds suitable metrics g_M on M and g^Λ on $\Lambda^1(M)$, and when $m^\Lambda(\gamma) = m(q; f)$ the intersection

$$W^u(\gamma; -\mathrm{grad}\, \mathbb{S}_L^\Lambda) \cap \mathrm{ev}^{-1}\left(W^s(q; -\mathrm{grad}\, f)\right)$$

consists of finitely many oriented points, which add up to the integer $n_{\mathrm{ev}}(\gamma, q)$. These integers are the coefficients of a chain map

$$M_k \mathrm{ev} : M_k(\mathbb{S}_L^\Lambda, g^\Lambda) \to M_k(f, g_M),$$

which in homology induces the homomorphism ev_*.

2.4. Remark. *The fact that* $\mathrm{ev}_* \circ c_* = \mathrm{id}_{H_*(M)}$*, together with the fact that the Morse complex is free, implies that* $M_* \mathrm{ev} \circ M_* c$ *is chain homotopic to the identity on* $M_*(f, g_M)$.

We conclude this section by using the results of section 2.4 to describe the homomorphism

$$i_! : H_k(\Lambda^1(M)) \to H_{k-n}(\Omega^1(M, q_0)),$$

working with the same action functional $\mathbb{S}_L$ both on $\Lambda^1(M)$ and on $\Omega^1(M, q_0)$. Conditions (14) and (15) are implied by the following assumption,

$$\gamma \in \mathscr{P}^\Lambda(L) \quad \Longrightarrow \quad \gamma(0) \neq q_0, \tag{22}$$

a condition holding for all but a countable set of q_0's. Under this assumption, we can perturb the standard metrics on $\Lambda^1(M)$ and $\Omega^1(M, q_0)$ to produce metrics g^Λ and g^Ω such that $-\mathrm{grad}\, \mathbb{S}_L^\Lambda$ and $-\mathrm{grad}\, \mathbb{S}_L^\Omega$ satisfy the Morse-Smale condition, and the unstable manifold of each $\gamma_1 \in \mathscr{P}^\Lambda(L)$ is transverse to the stable manifold of each $\gamma_2 \in \mathscr{P}^\Omega(L)$ in $\Lambda^1(M)$. When $m^\Omega(\gamma_2) = m^\Lambda(\gamma_1) - n$, the set

$$W^u(\gamma_1; -\mathrm{grad}\, \mathbb{S}_L^\Lambda) \cap W^s(\gamma_2; -\mathrm{grad}\, \mathbb{S}_L^\Omega)$$

consists of finitely many oriented points, which determine the integer $n_{i_!}(\gamma_1, \gamma_2)$. These integers are the coefficients of a chain map

$$M_{i_!} : M_*(\mathbb{S}_L^\Lambda, g^\Lambda) \to M_{*-n}(\mathbb{S}_L^\Omega, g^\Omega),$$

which in homology induces the homomorphism $i_!$.

2.6 Morse theoretical interpretation of the Pontrjagin product

In the last section we have described the vertical arrows of diagram (8), as well as a preferred left inverse of the top-right vertical arrow. The top horizontal arrow has already been described at the end of section 2.4. There remains to describe the middle and the bottom horizontal arrows, that is the loop product and the Pontrjagin product. This section is devoted to the description of the latter product. The following Propositions 2.5, 2.7 and 2.8 are consequences of the general statements in Sections 2.1–2.4

Given two Lagrangians $L_1, L_2 \in C^\infty([0,1] \times TM)$ such that $L(1,\cdot) = L_2(0,\cdot)$ with all the time derivatives, we define the Lagrangian $L_1 \# L_2 \in C^\infty([0,1] \times TM)$ as

$$L_1 \# L_2(t,q,v) = \begin{cases} 2L_1(2t,q,v/2) & \text{if } 0 \le t \le 1/2, \\ 2L_2(2t-1,q,v/2) & \text{if } 1/2 \le t \le 1. \end{cases} \tag{23}$$

The curve $\gamma : [0,1] \to M$ is a solution of the Lagrangian equation (18) with $L = L_1 \# L_2$ if and only if the rescaled curves $t \mapsto \gamma(t/2)$ and $t \mapsto \gamma((t+1)/2)$ solve the corresponding equation given by the Lagrangians L_1 and L_2, on $[0,1]$.

In view of the results of section 2.3, we wish to consider the functional $\mathbb{S}_{L_1}^\Omega \oplus \mathbb{S}_{L_2}^\Omega$ on $\Omega^1(M,q_0) \times \Omega^1(M,q_0)$,

$$\mathbb{S}_{L_1}^\Omega \oplus \mathbb{S}_{L_2}^\Omega(\gamma_1,\gamma_2) = \mathbb{S}_{L_1}^\Omega(\gamma_1) + \mathbb{S}_{L_2}^\Omega(\gamma_2),$$

and the functional $\mathbb{S}_{L_1\#L_2}^\Omega$ on $\Omega^1(M,q_0)$. The concatenation map

$$\Gamma : \Omega^1(M,q_0) \times \Omega^1(M,q_0) \to \Omega^1(M,q_0)$$

is nowhere a submersion, so condition (9) for the triplet $(\Gamma, \mathbb{S}_{L_1}^\Omega \oplus \mathbb{S}_{L_2}^\Omega, \mathbb{S}_{L_1\#L_2}^\Omega)$ requires that the image of Γ does not meet the critical set of $\mathbb{S}_{L_1\#L_2}^\Omega$, that is

$$\gamma \in \mathscr{P}^\Omega(L_1 \# L_2) \quad \Longrightarrow \quad \gamma(1/2) \neq q_0. \tag{24}$$

Notice that (24) allows L_1 and L_2 to be equal, and actually it allows them to be also autonomous (however, it implies that q_0 is not a stationary solution, so they cannot be the Lagrangian associated to a geodesic flow).

Assuming (24), condition (10) is automatically fulfilled. Moreover, if g_1, g_2 are metrics on $\Omega^1(M,q_0)$, we have that for every $\gamma_1 \in \mathscr{P}^\Omega(L_1)$, $\gamma_2 \in \mathscr{P}^\Omega(L_2)$,

$$\Gamma(W^u((\gamma_1,\gamma_2); -\mathrm{grad}_{g_1\times g_2} \mathbb{S}_{L_1}^\Omega \oplus \mathbb{S}_{L_2}^\Omega)) \cap \mathrm{crit}(\mathbb{S}_{L_1\#L_2}^\Omega) = \emptyset.$$

By Remark 2.3, there is no need to perturb the metric $g_1 \times g_2$ on $\Omega^1(M,q_0) \times \Omega^1(M,q_0)$ to achieve transversality, and we arrive at the following description of the Pontrjagin product.

Let L_1, L_2 be Lagrangians such that $L(1,\cdot) = L_2(0,\cdot)$ with all the time derivatives, satisfying $(\mathrm{L0})^\Omega$, (L1), (L2), and (24), such that also $L_1 \# L_2$ satisfies $(\mathrm{L0})^\Omega$. Let g_1, g_2, g be complete metrics on $\Omega^1(M,q_0)$ such that $-\mathrm{grad}_{g_1} \mathbb{S}_{L_1}^\Omega$, $-\mathrm{grad}_{g_2} \mathbb{S}_{L_2}^\Omega$, $-\mathrm{grad}\,_g \mathbb{S}_{L_1\#L_2}^\Omega$ satisfy the Palais-Smale and the Morse-Smale condition. Fix an arbitrary orientation for the unstable manifolds of each critical point of $\mathbb{S}^\Omega_{L_1}$, $\mathbb{S}^\Omega_{L_2}$, $\mathbb{S}^\Omega_{L_1\#L_2}$. Up to perturbing the metric g, we get that the restriction of Γ to the unstable manifold

$$W^u((\gamma_1,\gamma_2);-\mathrm{grad}_{g_1\times g_2}\mathbb{S}^\Omega_{L_1}\oplus\mathbb{S}^\Omega_{L_2}) = W^u(\gamma_1;-\mathrm{grad}_{g_1}\mathbb{S}^\Omega_{L_1})\times W^u(\gamma_2;-\mathrm{grad}_{g_2}\mathbb{S}^\Omega_{L_2})$$

of every critical point $(\gamma_1,\gamma_2)\in\mathscr{P}^\Omega(L_1)\times\mathscr{P}^\Omega(L_2)$ is transverse to the stable manifold

$$W^s(\gamma;-\mathrm{grad}_g\mathbb{S}^\Omega_{L_1\#L_2})$$

of each critical point of $\mathbb{S}^\Omega_{L_1\#L_2}$. When $m^\Omega(\gamma)=m^\Omega(\gamma_1)+m^\Omega(\gamma_2)$, the corresponding intersections

$$\left\{(\alpha_1,\alpha_2)\in W^u(\gamma_1;-\mathrm{grad}\,\mathbb{S}^\Omega_{L_1})\times W^u(\gamma_2;-\mathrm{grad}\,\mathbb{S}^\Omega_{L_2}) \mid \Gamma(\alpha_1,\alpha_2)\in W^s(\gamma;-\mathrm{grad}\,\mathbb{S}^\Omega_{L_1\#L_2})\right\},$$

is a finite set of oriented points. Let $n_\#(\gamma_1,\gamma_2;\gamma)$ be the algebraic sum of these orientation signs.

2.5. Proposition. *The homomorphism*

$$M_\# : M_j(\mathbb{S}^\Omega_{L_1},g_1)\otimes M_k(\mathbb{S}^\Omega_{L_2},g_2)\to M_{j+k}(\mathbb{S}^\Omega_{L_1\#L_2},g), \quad \gamma_1\otimes\gamma_2\mapsto \sum_{\substack{\gamma\in\mathscr{P}^\Omega(L_1\#L_2)\\ m^\Omega(\gamma)=j+k}} n_\#(\gamma_1,\gamma_2;\gamma)\,\gamma,$$

is a chain map, and it induces the Pontrjagin product $\#$ in homology.

2.6. Remark. *It is not necessary to consider the Lagrangian $L_1\#L_2$ on the target space of this homomorphism. One could actually work with any three Lagrangians (with the suitable non-degeneracy condition replacing (24)). The choice of dealing with two Lagrangians L_1, L_2 and their concatenation $L_1\#L_2$ will be important to get energy estimates in Floer homology. We have made this choice also here mainly to see which kind of non-degeneracy condition one needs.*

2.7 Morse theoretical interpretation of the loop product

The loop product is slightly more complicated than the other homomorphisms considered so far, because it consists of a composition where two homomorphisms are non-trivial (that is, not just identifications) when read on the Morse homology groups, namely the Umkehr map associated to the submanifold $\Theta^1(M)$ of figure-8 loops, and the homomorphism induced by the concatenation map $\Gamma:\Theta^1(M)\to\Lambda^1(M)$. We shall describe these homomorphisms separately, and then we will show a compact description of their composition.

Let us start by describing the Umkehr map

$$e_! : H_k(\Lambda^1(M)\times\Lambda^1(M))\to H_{k-n}(\Theta^1(M)).$$

Let $L_1, L_2\in C^\infty(\mathbb{T}\times TM)$ be Lagrangians satisfying $(\mathrm{L0})^\Lambda$, (L1), (L2), and such that the pair (L_1,L_2) satisfies $(\mathrm{L0})^\Theta$. Assume also

$$\gamma_1\in\mathscr{P}^\Lambda(L_1),\ \gamma_2\in\mathscr{P}^\Lambda(L_2) \quad\Longrightarrow\quad \gamma_1(0)\neq\gamma_2(0). \tag{25}$$

Notice that this condition prevents L_1 from coinciding with L_2. We shall consider the functional $\mathbb{S}^\Lambda_{L_1}\oplus\mathbb{S}^\Lambda_{L_2}$ on $\Lambda^1(M)\times\Lambda^1(M)$, and the functional $\mathbb{S}^\Theta_{L_1\oplus L_2}$ on $\Theta^1(M)$. Condition (25) implies that the unconstrained functional has no critical points on $\Theta^1(M)$, so conditions (14) and (15) hold.

By the discussion of section 2.4, we can find complete metrics $g_1\times g_2$ and g^Θ on $\Lambda^1(M)\times\Lambda^1(M)$ and on $\Theta^1(M)$ such that $-\mathrm{grad}_{g_1\times g_2}\mathbb{S}^\Lambda_{L_1}\oplus\mathbb{S}^\Lambda_{L_2}$ and $-\mathrm{grad}_{g^\Theta}\mathbb{S}^\Theta_{L_1\oplus L_2}$ satisfy the Palais-Smale condition and the Morse-Smale condition, and such that the unstable manifold $W^u(\gamma^-;-\mathrm{grad}_{g_1\times g_2}\mathbb{S}^\Lambda_{L_1}\oplus\mathbb{S}^\Lambda_{L_2})$ of every $\gamma^-=(\gamma_1^-,\gamma_2^-)\in\mathscr{P}^\Lambda(L_1)\times\mathscr{P}^\Lambda(L_2)$ is transverse to $\Theta^1(M)$ and to the stable manifold $W^s(\gamma^+;-\mathrm{grad}_{g^\Theta}\mathbb{S}^\Theta_{L_1\oplus L_2})$ of every $\gamma^+=(\gamma_1^+,\gamma_2^+)\in\mathscr{P}^\Theta(L_1\oplus L_2)$. Fix an arbitrary orientation for the unstable manifold of every critical point of $\mathbb{S}^\Lambda_{L_1}\oplus\mathbb{S}^\Lambda_{L_2}$ and $\mathbb{S}^\Theta_{L_1\oplus L_2}$. When $m^\Theta(\gamma^+)=m^\Lambda(\gamma_1^-)+m^\Lambda(\gamma_2^-)-n$, the intersection

$$W^u(\gamma^-;-\mathrm{grad}_{g_1\times g_2}\mathbb{S}^\Lambda_{L_1}\oplus\mathbb{S}^\Lambda_{L_2})\cap W^s(\gamma^+;-\mathrm{grad}_{g^\Theta}\mathbb{S}^\Theta_{L_1\oplus L_2})$$

is a finite set of oriented points. If we denote by $n_{e_!}(\gamma^-,\gamma^+)$ the algebraic sum of these orientation signs, we have the following:

2.7. Proposition. *The homomorphism*

$$M_k(\mathbb{S}^\Lambda_{L_1} \oplus \mathbb{S}^\Lambda_{L_2}, g_1 \times g_2) \to M_{k-n}(\mathbb{S}^\Theta_{L_1\oplus L_2}, g^\Theta), \quad \gamma^- \mapsto \sum_{\substack{\gamma^+ \in \mathscr{P}^\Theta(L_1\oplus L_2)\\ m^\Theta(\gamma^+)=k-n}} n_{e_!}(\gamma^-,\gamma^+)\gamma^+,$$

is a chain map, and it induces the Umkehr map $e_!$ *in homology.*

By composing this homomorphism with the Morse theoretical version of the exterior homology product described in section 2.3, that is the isomorphism

$$M_j(\mathbb{S}^\Lambda_{L_1}, g_1) \otimes M_h(\mathbb{S}^\Lambda_{L_2}, g_2) \to M_{j+h}(\mathbb{S}^\Lambda_{L_1} \oplus \mathbb{S}^\Lambda_{L_2}, g_1 \times g_2),$$

we obtain the homomorphism

$$M_! : M_j(\mathbb{S}^\Lambda_{L_1}, g_1) \otimes M_h(\mathbb{S}^\Lambda_{L_2}, g_2) \to M_{j+h-n}(\mathbb{S}^\Theta_{L_1\oplus L_2}, g^\Theta).$$

Let us describe the homomorphism

$$\Gamma_* : H_k(\Theta^1(M)) \to H_k(\Lambda^1(M)),$$

induced by the concatenation map Γ. Let L_1, L_2 be Lagrangians such that $L(1,\cdot) = L_2(0,\cdot)$ with all the time derivatives, satisfying (L1), (L2). We assume that (L_1, L_2) satisfies $(\mathrm{L0})^\Theta$ and $L_1\#L_2$ satisfies $(\mathrm{L0})^\Lambda$. We would like to apply the results of section 2.2 to the functionals $\mathbb{S}^\Theta_{L_1\oplus L_2}$ on $\Theta^1(M)$ and $\mathbb{S}^\Lambda_{L_1\#L_2}$ on $\Lambda^1(M)$. The map $\Gamma : \Theta^1(M) \to \Lambda^1(M)$ is nowhere a submersion, so condition (9) for the triplet $(\Gamma, \mathbb{S}^\Theta_{L_1\oplus L_2}, \mathbb{S}^\Lambda_{L_1\#L_2})$ requires that $\Gamma(\Theta^1(M))$ does not contain critical points of $\mathbb{S}^\Lambda_{L_1\#L_2}$, that is

$$\gamma \in \mathscr{P}^\Lambda(L_1\#L_2) \quad \Longrightarrow \quad \gamma(1/2) \neq \gamma(0). \tag{26}$$

Assuming (26), conditions (9) and (10) are automatically fulfilled. Therefore, the discussion of section 2.2 implies that we can find complete metrics g^Θ and g^Λ on on $\Theta^1(M)$ and $\Lambda^1(M)$ such that $-\mathrm{grad}_{g^\Theta} \mathbb{S}^\Theta_{L_1\oplus L_2}$ and $-\mathrm{grad}_{g^\Lambda} \mathbb{S}^\Lambda_{L_1\#L_2}$ satisfy the Palais-Smale and the Morse-Smale condition, and that the restriction of Γ to the unstable manifold

$$W^u(\gamma^-; -\mathrm{grad}_{g^\Theta} \mathbb{S}^{\Theta^1}_{L_1\oplus L_2})$$

of every critical point $\gamma^- = (\gamma_1^-, \gamma_2^-) \in \mathscr{P}^\Theta(L_1 \oplus L_2)$ is transverse to the stable manifold

$$W^s(\gamma^+; -\mathrm{grad}_{g^\Lambda} \mathbb{S}^\Lambda_{L_1\#L_2})$$

of every critical point $\gamma^+ \in \mathscr{P}^\Lambda(L_1\#L_2)$. Fix arbitrary orientations for the unstable manifolds of every critical point of $\mathbb{S}^\Theta_{L_1\oplus L_2}$ and $\mathbb{S}^\Lambda_{L_1\#L_2}$. When $m^\Lambda(\gamma^+) = m^\Theta(\gamma^-)$, the intersection

$$\left\{(\alpha_1,\alpha_2) \in W^u(\gamma^-; -\mathrm{grad}\, \mathbb{S}^\Theta_{L_1\oplus L_2}) \mid \Gamma(\alpha_1,\alpha_2) \in W^s(\gamma^+; -\mathrm{grad}\, \mathbb{S}^\Lambda_{L_1\#L_2})\right\},$$

is a finite set of oriented points. If we denote by $n_\Gamma(\gamma^-,\gamma^+)$ the algebraic sum of these orientation signs, we have the following:

2.8. Proposition. *The homomorphism*

$$M_\Gamma : M_k(\mathbb{S}^\Theta_{L_1\oplus L_2}, g^\Theta) \to M_k(\mathbb{S}^\Lambda_{L_1\#L_2}, g^\Lambda), \quad \gamma^- \mapsto \sum_{\substack{\gamma^+ \in \mathscr{P}^\Lambda(L_1\#L_2)\\ m^\Lambda(\gamma)=k}} n_\Gamma(\gamma^-,\gamma^+)\,\gamma^+,$$

is a chain map, and it induces the homomorphism $\Gamma_* : H_k(\Theta^1(M)) \to H_k(\Lambda^1(M))$ *in homology.*

Therefore, the composition $M_\Gamma \circ M_!$ induces the loop product in homology.

We conclude this section by exhibiting a compact description of the loop product

$$\circ : H_j(\Lambda^1(M)) \otimes H_k(\Lambda^1(M)) \to H_{j+k-n}(\Lambda^1(M)).$$

Since we are not going to use this description, we omit the proof. Let $L_1, L_2 \in C^\infty(\mathbb{T} \times TM)$ be Lagrangians satisfying $(\mathrm{L0})^\Lambda$, (L1), (L2), such that $L_1(0,\cdot) = L_2(0,\cdot)$ with all time derivatives, and such that the concatenated Lagrangian $L_1 \# L_2$ defined by (23) satisfies $(\mathrm{L0})^\Lambda$. We also assume (25), noticing that this condition prevents L_1 from coinciding with L_2. Let g_1, g_2, g be complete metrics on $\Lambda^1(M)$ such that $-\mathrm{grad}_{g_1} \mathbb{S}^\Lambda_{L_1}$, $-\mathrm{grad}_{g_2} \mathbb{S}^\Lambda_{L_2}$, and $-\mathrm{grad}_g \mathbb{S}^\Lambda_{L_1 \# L_2}$ satisfy the Palais-Smale and the Morse-Smale condition on $\Lambda^1(M)$. By (25), the functional $\mathbb{S}^\Theta_{L_1 \oplus L_2}$ has no critical points on $\Theta^1(M)$, so up to perturbing g_1 and g_2 we can assume that for every $\gamma_1 \in \mathscr{P}^\Lambda(L_1)$, $\gamma_2 \in \mathscr{P}^\Lambda(L_2)$, the unstable manifold

$$W^u((\gamma_1,\gamma_2); -\mathrm{grad}_{g_1 \times g_2} \mathbb{S}^\Lambda_{L_1} \oplus \mathbb{S}^\Lambda_{L_2}) = W^u(\gamma_1; -\mathrm{grad}_{g_1} \mathbb{S}^\Lambda_{L_1}) \times W^u(\gamma_2; -\mathrm{grad}_{g_2} \mathbb{S}^\Lambda_{L_2})$$

is transverse to $\Theta^1(M)$. Moreover, assumption (25) implies that the image of $\Theta^1(M)$ by the concatenation map Γ does not contain any critical point of $\mathbb{S}^\Lambda_{L_1 \# L_2}$. Therefore, up to perturbing g we can assume that for every $\gamma_1 \in \mathscr{P}^\Lambda(L_1)$, $\gamma_2 \in \mathscr{P}^\Lambda(L_2)$, the restriction of Γ to the submanifold

$$W^u((\gamma_1,\gamma_2); -\mathrm{grad}_{g_1 \times g_2} \mathbb{S}^\Lambda_{L_1} \oplus \mathbb{S}^\Lambda_{L_2}) \cap \Theta^1(M)$$

is transverse to the stable manifold $W^s(\gamma; -\mathrm{grad}_g \mathbb{S}^\Lambda_{L_1 \# L_2})$ of each $\gamma \in \mathscr{P}^\Lambda(L)$. In particular, when $m^\Lambda(\gamma) = m^\Lambda(\gamma_1) + m^\Lambda(\gamma_2) - n$, the submanifold

$$\big\{(\alpha_1,\alpha_2) \in (W^u(\gamma_1; -\mathrm{grad}\, \mathbb{S}^\Lambda_{L_1}) \times W^u(\gamma_2; -\mathrm{grad}\, \mathbb{S}^\Lambda_{L_2})) \cap \Theta^1(M) \mid \Gamma(\alpha_1,\alpha_2) \in W^s(\gamma; -\mathrm{grad}\, \mathbb{S}^\Lambda_{L_1 \# L_2})\big\}$$

is a finite set of oriented points. We contend that if $n_\circ(\gamma_1,\gamma_2;\gamma)$ denotes the algebraic sum of the corresponding orientation signs, the following holds:

2.9. PROPOSITION. *The homomorphism*

$$M_j(\mathbb{S}^\Lambda_{L_1}, g_1) \otimes M_k(\mathbb{S}^\Lambda_{L_2}, g_2) \to M_{j+k-n}(\mathbb{S}^\Lambda_{L_1 \# L_2}, g), \quad \gamma_1 \otimes \gamma_2 \mapsto \sum_{\substack{\gamma \in \mathscr{P}^\Lambda(L_1 \# L_2) \\ m^\Lambda(\gamma) = j+k-n}} n_\circ(\gamma_1,\gamma_2;\gamma)\, \gamma, \tag{27}$$

is a chain map, and it induces the loop product in homology.

3 Floer homologies on cotangent bundles and their ring structures

3.1 Floer homology for the periodic and the fixed ends orbits

In this section we recall the construction of Floer homology for periodic and for fixed-ends orbits on the cotangent bundle of a compact oriented manifold. See [AS06b] for detailed proofs.

Let M be a compact oriented manifold. We shall often use standard coordinates (q,p) on T^*M, the cotangent bundle of M. Denote by ω the standard symplectic form $\omega := dp \wedge dq$ on the manifold T^*M, that is the differential of the Liouville form $\eta := p\, dq$. Equivalently, the Liouville form η can be defined by

$$\eta(\zeta) = x(D\pi(x)[\zeta]), \quad \text{for } \zeta \in T_x T^*M,\ x \in T^*M,$$

where $\pi : T^*M \to M$ is the bundle projection. Let Y be the Liouville vector field on T^*M, defined by the identity

$$\omega(Y(x),\cdot) = \eta. \tag{28}$$

A submanifold $L \subset T^*M$ is called a co-normal iff the Liouville form η vanishes on L. Equivalently, L is the conormal bundle $N^*\pi(L)$ of its projection onto M. They are a special class of Lagrangian submanifolds.

Consider the class of Hamiltonians H on $\mathbb{T} \times T^*M$ or on $[0,1] \times T^*M$ such that:

(H1) $DH(t,q,p)[Y] - H(t,q,p) \geq h_0|p|^2 - h_1$, for every (t,q,p);

(H2) $|\partial_q H(t,q,p)| \leq h_2(1+|p|^2)$, $|\partial_p H(t,q,p)| \leq h_2(1+|p|)$, for every (t,q,p);

for some constants $h_0 > 0$, $h_1 \geq 0$, $h_2 \geq 0$. As seen for conditions (L1) and (L2), these assumptions do not depend on the Riemannian structure and on the local coordinates used to state them. Condition (H1) essentially says that H grows at least quadratically in p on each fiber of T^*M, and that it is radially convex for $|p|$ large. Condition (H2) implies that H grows at most quadratically in p on each fiber. Notice also that if H is the Fenchel transform of a convex Lagrangian L in $C^\infty([0,1]\times TM)$ (see section 4.1), then the term $DH(t,q,p)[Y(q,p)] - H(t,q,p)$ appearing in (H1) coincides with $L(t,q,\partial_p H(t,q,p))$.

Let X_H be the time-dependent Hamiltonian vector field associated to H by the formula $\omega(X_H,\cdot) = -D_x H$. Condition (H2) implies the quadratic bound

$$|X_H(t,q,p)| \leq h_3(1+|p|^2), \tag{29}$$

for some $h_3 \geq 0$. Let $(t,x) \mapsto \phi^H(t,x)$ be the non-autonomous flow associated to the vector field X_H. We are interested in the set $\mathscr{P}^\Lambda(H)$ of one-periodic orbits of ϕ^H (in this case we assume H to be defined on $\mathbb{T} \times T^*M$), and in the set $\mathscr{P}^\Omega(H)$ of orbits x of ϕ^H such that $x(0), x(1) \in T^*_{q_0}M$ (in this case H may be defined only on $[0,1] \times T^*M$). The superscripts Λ and Ω will appear often to distinguish the periodic from the fixed-ends problem. We shall omit them when we wish to consider both situations at the same time.

The non-degeneracy assumptions for the elements of $\mathscr{P}^\Lambda(H)$ and $\mathscr{P}^\Omega(H)$ are:

(H0)$^\Lambda$ for every $x \in \mathscr{P}^\Lambda(H)$, the number 1 is not an eigenvalue of $D_x\phi^H(1,x(0)) : T_{x(0)}T^*M \to T_{x(0)}T^*M$;

(H0)$^\Omega$ for every $x \in \mathscr{P}^\Omega(H)$, the linear mapping $D_x\phi^H(1,x(0)) : T_{x(0)}T^*M \to T_{x(1)}T^*M$ maps the vertical subspace $T^v_{x(0)}T^*M$ at $x(0)$ into a subspace having intersection (0) with the vertical subspace $T^v_{x(1)}T^*M$ at $x(1)$.

As in the Lagrangian case, these conditions hold for a generic choice of H in several reasonable topologies, and (H0)$^\Lambda$ forces H to be explicitly time-dependent.

Each $x \in \mathscr{P}^\Lambda(H)$ has a well-defined Conley-Zehnder index $\mu^\Lambda(x) \in \mathbb{Z}$. Indeed, the fact that T^*M has a Lagrangian fibration consisting of the fibers T^*_qM singles out a class of preferred symplectic trivializations for the vector bundle $x^*(TT^*M)$, namely those which map the Lagrangian subspace

$$\lambda_0 := (0) \times \mathbb{R}^n \subset \mathbb{R}^{2n}$$

into the vertical space $T^v_{x(t)}T^*M$. Here $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ is endowed with the symplectic form

$$\omega_0 = dp \wedge dq,$$

with respect to coordinates $(q,p) \in \mathbb{R}^n \times \mathbb{R}^n$. Any two such trivializations are isotopic in the space of all symplectic trivializations, so the Conley-Zendher index $\mu_{CZ}(\Phi)$ of the path Φ of symplectic matrices obtained by conjugating the path $t \mapsto D_x\phi^H(t,x(0))$ by such a trivialization is well-defined, and we set

$$\mu^\Lambda(x) := \mu_{CZ}(\Phi).$$

See also section 5.1.

Similarly, each $x \in \mathscr{P}^\Omega(H)$ has a well defined Maslov index $\mu^\Omega(x)$, obtained from the relative Maslov index of the path of Lagrangian subspaces $t \mapsto D_x\phi^H(t, x(0))T^v_{x(0)}T^*M$ with respect to the path of Lagrangian subspaces $t \mapsto T^v_{x(t)}T^*M$. Indeed, using a symplectic trivialization of $x^*(TT^*M)$ mapping λ_0 into the vertical subbundle, and defining Φ as above, we set

$$\mu^\Omega(x) := \mu(\Phi\lambda_0, \lambda_0) - \frac{n}{2},$$

where μ denotes the relative Maslov index (see section 5.1 for the sign conventions). When x in $\mathscr{P}^\Lambda(H)$ (respectively in $\mathscr{P}^\Omega(H)$) is non-degenerate, then $\mu^\Lambda(x)$ (respectively $\mu^\Omega(x)$) is an integer.

The elements of $\mathscr{P}^\Lambda(H)$ and $\mathscr{P}^\Omega(H)$ are the critical points of the Hamiltonian action functional

$$\mathbb{A}_H(x) = \int_{[0,1]} x^*(\eta - H dt) = \int_0^1 \big(p(t)[q'(t)] - H(t, q(t), p(t))\big)\, dt,$$

on the space of one-periodic loops in T^*M, or on the space of curves $x : [0,1] \to T^*M$ such that $x(0), x(1) \in T^*_{q_0}M$. Indeed, the differential of $\mathbb{A}_H$ on the space of free paths on T^*M is

$$D\mathbb{A}_H(x)[\zeta] = \int_0^1 \big(\omega(\zeta, x') - D_xH(t,x)[\zeta]\big)\, dt + \eta(x(1))[\zeta(1)] - \eta(x(0))[\zeta(0)], \tag{30}$$

and the boundary terms vanish if either x and ζ are 1-periodic, or $\zeta(0)$ and $\zeta(1)$ belong to the vertical subbundle.

Conditions (H0) and (H1) imply that the set of $x \in \mathscr{P}(H)$ with $\mathbb{A}_H(x) \leq A$ is finite, for every $A \in \mathbb{R}$. Indeed, this follows immediately from the following general:

3.1. Lemma. *Let $H \in C^\infty([0,1] \times T^*M)$ be a Hamiltonian satisfying (H1) and (H2). For every $A \in \mathbb{R}$ there exists a compact subset $K \subset T^*M$ such that each orbit $x : [0,1] \to T^*M$ of X_H with $\mathbb{A}_H(x) \leq A$ lies in K.*

Proof. Let $x = (q,p)$ be an orbit of X_H such that $\mathbb{A}_H(x) \leq A$. Since x is an orbit of X_H, by (28),

$$\eta(x)[x'] - H(t,x) = \omega(Y(x), X_H(t,x)) - H(t,x) = DH(t,x)[Y(x)] - H(t,x).$$

Therefore (H1) implies that $|p|$ is uniformly bounded in $L^2(]0,1[)$. By (29), $|x'|$ in uniformly bounded in $L^1(]0,1[)$. Therefore x is uniformly bounded in $W^{1,1}$, hence in L^∞. □

3.2. Remark. *Assume that the flow generated by a Hamiltonian $H \in C^\infty([0,1]\times T^*M)$ is globally defined (for instance, this holds if H is coercive and $|\partial_t H| \leq c(|H|+1)$). Then the conclusion of Lemma 3.1 holds assuming just that the function $DH[Y] - H$ is coercive (a much weaker assumption than (H1), still implying that H is coercive), without any upper bound such as (H2).*

Let us fix a Riemannian metric $\langle \cdot, \cdot \rangle$ on M. This metric induces metrics on TM and on T^*M, both denoted by $\langle \cdot, \cdot \rangle$. It induces also an identification $T^*M \cong TM$, horizontal-vertical splittings of both TTM and TT^*M, and a particular almost complex structure J on T^*M, namely the one which in the horizontal-vertical splitting takes the form

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}. \tag{31}$$

This almost complex structure is ω-compatible, meaning that[6]

$$\langle \xi, \eta \rangle = \omega(J\xi, \eta), \quad \forall \xi, \eta \in T_xT^*M,\ \forall x \in T^*M.$$

[6]Notice that our sign convention here differs from the one used in [AS06b]. The reason is that here we prefer to see the leading term in the Floer equation as a Cauchy-Riemann operator, and not as an anti-Cauchy-Riemann operator.

The L^2-negative gradient equation for the Hamiltonian action functional $\mathbb{A}_H$ is the Floer equation

$$\overline{\partial}_{J,H}(u) := \partial_s u + J(u)[\partial_t u - X_H(t,u)] = 0, \tag{32}$$

for $u = u(s,t)$, $s \in \mathbb{R}$, t in $\mathbb{T}$ or in $[0,1]$. A generic choice of the Hamiltonian $H \in C^\infty(\mathbb{T} \times T^*M)$ makes the space of solutions of the Floer equation on the cylinder,

$$\mathscr{M}_\partial^\Lambda(x,y) = \Big\{ u \in C^\infty(\mathbb{R} \times \mathbb{T}, T^*M) \mid \overline{\partial}_{J,H}(u) = 0 \text{ and } \lim_{s\to-\infty} u(s,t) = x(t), \ \lim_{s\to+\infty} u(s,t) = y(t) \Big\}$$

a manifold of dimension $\mu^\Lambda(x) - \mu^\Lambda(y)$, for every $x, y \in \mathscr{P}^\Lambda(H)$. Similarly, a generic choice of $H \in C^\infty([0,1] \times T^*M)$ makes the space of solutions of the Floer equation on the strip,

$$\mathscr{M}_\partial^\Omega(x,y) = \Big\{ u \in C^\infty(\mathbb{R} \times [0,1], T^*M) \,|\, \overline{\partial}_{J,H}(u) = 0, \ u(s,0), \ u(s,1) \in T^*_{q_0}M \ \forall s \in \mathbb{R}, \text{ and } \lim_{s\to-\infty} u(s,t) = x(t), \ \lim_{s\to+\infty} u(s,t) = y(t) \Big\}$$

a manifold of dimension $\mu^\Omega(x) - \mu^\Omega(y)$, for every $x, y \in \mathscr{P}^\Omega(H)$. Here *generic* means for a countable intersection of open and dense subsets of the space of smooth time-dependent Hamiltonians satisfying (H0), (H1), and (H2), with respect to suitable topologies (see [FHS96] for transversality issues). In particular, the perturbation of a given Hamiltonian H satisfying (H0), (H1), (H2) can be chosen in such a way that the discrete set $\mathscr{P}(H)$ is unaffected.

3.3. Remark. *As it is well-known, transversality can also be achieved for a fixed Hamiltonian by perturbing the almost complex structure J in a time-dependent way. In order to have good compactness properties for the spaces $\mathscr{M}_\partial$ one needs the perturbed almost complex structure J_1 to be C^0-close enough to the metric one J defined by (31) (see [AS06b, Theorem 1.14]). Other compactness issues in this paper would impose further restrictions on the distance between J_1 and J. For this reason here we prefer to work with the fixed almost complex structure J, and to achieve transversality by perturbing the Hamiltonian.*

The manifolds $\mathscr{M}_\partial(x,y)$ can be oriented in a coherent way. Assumptions (H1) and (H2) imply that these manifolds have nice compactifications. In particular, when $\mu(x) - \mu(y) = 1$, $\mathscr{M}_\partial(x,y)$ consists of finitely many one-parameter families of solutions $\sigma \mapsto u(\cdot + \sigma, \cdot)$, each of which comes with a sign ± 1, depending whether its orientation agrees or not with the orientation determined by letting σ increase. The algebraic sum of these numbers is an integer $n_\partial^\Lambda(x,y)$, or $n_\partial^\Omega(x,y)$. If we let $F_k(H)$ denote the free Abelian group generated by the elements $x \in \mathscr{P}(H)$ of index $\mu(x) = k$, the above coefficients define the homomorphism

$$\partial : F_k(H) \to F_{k-1}(H), \quad x \mapsto \sum_{\substack{y \in \mathscr{P}(H) \\ \mu(x)=k-1}} n_\partial(x,y)\, y,$$

which turns out to be a boundary operator. The resulting chain complexes $F^\Lambda(H,J)$ and $F^\Omega(H,J)$ are the *Floer complexes* associated to the periodic orbits problem, and to the fixed-ends problem. If we change the metric on M - hence the almost complex structure J - and the orientation data, the Floer complex $F(H,J)$ changes by an isomorphism. If we change the Hamiltonian H, the new Floer complex is homotopically equivalent to the old one. In particular, the homology of the Floer complex does not depend on the metric, on H, and on the orientation data. This fact allows us to denote this graded Abelian group as $HF_*^\Lambda(T^*M)$, in the periodic case, and $HF_*^\Omega(T^*M)$, in the fixed-ends case. Actually, the homology of $F^\Lambda(H,J)$ is isomorphic to the singular homology of the free loop space $\Lambda(M)$, while the homology of $F^\Omega(H,J)$ is isomorphic to the singular homology of the based loop space $\Omega(M,q_0)$,

$$HF_k^\Lambda(T^*M) \cong H_k(\Lambda(M)), \quad HF_k^\Omega(T^*M) \cong H_k(\Omega(M,q_0)).$$

The first fact was first proved by Viterbo in [Vit96] (see also [SW06] for a different proof). In section 4.1 we outline a third definition of both isomorphisms, which is fully described in [AS06b], and in the subsequent sections we prove that these isomorphisms are actually ring isomorphisms intertwining the pair-of-pants product and the triangle product in Floer homology, with the loop product and the Pontrjagin product in the singular homology of the free and based loop spaces.

3.2 The Floer equation on triangles and pair-of-pants

Additional algebraic structures on Floer homology are defined by extending the Floer equation to more general Riemann surfaces than the strip $\mathbb{R} \times [0, 1]$ and the cylinder $\mathbb{R} \times \mathbb{T}$.

Let (Σ, j) be a Riemann surface, possibly with boundary. For $u \in C^\infty(\Sigma, T^*M)$ consider the *nonlinear Cauchy-Riemann operator*

$$\overline{D}_J u = \frac{1}{2}(Du + J(u) \circ Du \circ j),$$

that is the complex anti-linear part of Du with respect to the almost-complex structure J. The operator $\overline{D}_J$ is a section of the bundle over $C^\infty(\Sigma, T^*M)$ whose fiber at u is $\Omega^{0,1}(\Sigma, u^*(TT^*M))$, the space of anti-linear one-forms on Σ taking values in the vector bundle $u^*(TT^*M)$. In some holomorphic coordinate $z = s + it$ on Σ, the operator $\overline{D}_J$ takes the form

$$\overline{D}_J u = \frac{1}{2}(\partial_s u + J(u)\partial_t u)\, ds - \frac{1}{2}J(u)(\partial_s u + J(u)\partial_t u)\, dt. \tag{33}$$

This expression shows that the leading term $\overline{\partial}_J := \partial_s + J(\cdot)\partial_t$ in the Floer equation (32) can be extended to arbitrary Riemann surfaces, at the only cost of considering an equation which does not take values on a space of tangent vector fields, but on a space of anti-linear one-forms.

When Σ has a global coordinate $z = s+it$, as in the case of the strip $\mathbb{R}\times[0, 1]$ or of the cylinder $\mathbb{R} \times \mathbb{T}$, we can associate to the Hamiltonian term in the Floer equation the complex anti-linear one-form

$$F_{J,H}(u) = -\frac{1}{2}(J(u)X_H(t, u)\, ds + X_H(t, u)\, dt) \in \Omega^{0,1}(\Sigma, u^*(TT^*M)). \tag{34}$$

Formula (33) shows that the Floer equation (32) is equivalent to

$$\overline{D}_J u + F_{J,H}(u) = 0. \tag{35}$$

If we wish to use the formulation (35) to extend the Floer equation to more general Riemann surfaces, we encounter the difficulty that - unlike $\overline{D}_J$ - the Hamiltonian term $F_{J,H}$ is defined in terms of coordinates.

One way to get around this difficulty is to consider Riemann surfaces with cylindrical or strip-like ends, each of which is endowed with some fixed holomorphic coordinate $z = s + it$, to define the operator $F_{J,H}$ on such ends, and then to extend it to the whole Σ by considering a Hamiltonian H which also depends on s and vanishes far from the ends. In this way, only the Cauchy-Riemann operator acts in the region far from the ends. This approach is adopted in [Sch95, PSS96, MS04].

A drawback of this method is that one looses sharp energy identities relating some norm of u to the jump of the Hamiltonian action functional. Moreover, an s-dependent Hamiltonian which vanishes for some values of s cannot satisfy assumptions (H1) and (H2). These facts lead to problems with compactness when dealing - as we are here - with a non-compact symplectic manifold.

Therefore, we shall use a different method to extend the Hamiltonian term $F_{J,H}$. We shall describe this construction in the case of the triangle and the pair-of-pants surface, although the same idea could be generalized to any Riemann surface.

Let Σ^Ω_Υ be the *holomorphic triangle* that is the Riemann surface consisting of a closed triangle with the three vertices removed (equivalently, a closed disk with three boundary points removed). Let Σ^Λ_Υ be the *pair-of-pants* Riemann surface, that is the sphere with three points removed.

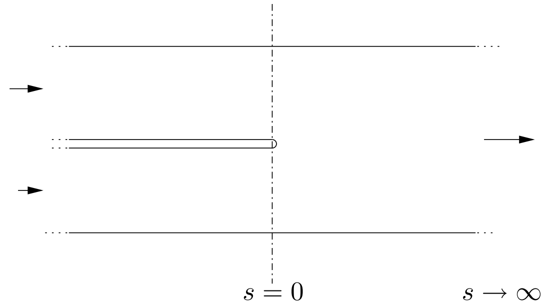

Figure 1: The strip with a slit Σ_Υ^Ω.

The Riemann surface Σ_Υ^Ω can be described as a strip with a slit: one takes the disjoint union

$$\mathbb{R} \times [-1,0] \sqcup \mathbb{R} \times [0,1]$$

and identifies $(s,0^-)$ with $(s,0^+)$ for every $s \geq 0$. See Figure 1. The resulting object is indeed a Riemann surface with interior

$$\mathrm{Int}(\Sigma_\Upsilon) = (\mathbb{R}\times]-1,1[) \setminus (]-\infty,0] \times \{0\})$$

endowed with the complex structure of a subset of $\mathbb{R}^2 \cong \mathbb{C}$, $(s,t) \mapsto s+it$, and three boundary components

$$\mathbb{R} \times \{-1\}, \quad \mathbb{R} \times \{1\}, \quad]-\infty,0] \times \{0^-,0^+\}.$$

The complex structure at each boundary point other than $0=(0,0)$ is induced by the inclusion in $\mathbb{C}$, whereas a holomorphic coordinate at 0 is given by the map

$$\{\zeta \in \mathbb{C} \mid \mathrm{Re}\,\zeta \geq 0,\ |\zeta|<1\} \to \Sigma_\Upsilon^\Omega, \quad \zeta \mapsto \zeta^2, \tag{36}$$

which maps the boundary line $\{\mathrm{Re}\,\zeta = 0,\ |\zeta|<1\}$ into the portion of the boundary $]-1,0] \times \{0^-,0^+\}$.

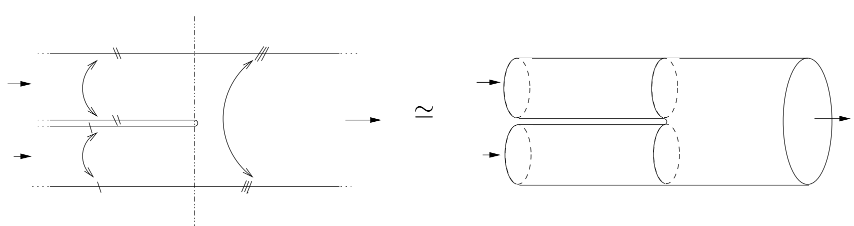

Figure 2: The pair-of-pants Σ_Υ^Λ.

Similarly, the pair-of-pants Σ_Υ^Λ can be described as the following quotient of a strip with a slit: in the disjoint union $\mathbb{R}\times[-1,0]\sqcup\mathbb{R}\times[0,1]$ we consider the identifications

$$\begin{matrix} (s,-1)\sim(s,0^-) \\ (s,0^+)\sim(s,1) \end{matrix} \quad \text{for } s \leq 0, \qquad \begin{matrix} (s,0^-)\sim(s,0^+) \\ (s,-1)\sim(s,1) \end{matrix} \quad \text{for } s \geq 0.$$

See figure 2. This object is a Riemann surface without boundary, by considering the standard complex structure at every point other than $(0,0)\sim(0,-1)\sim(0,1)$, and by choosing the holomorphic coordinate

$$\left\{\zeta \in \mathbb{C} \mid |\zeta| < 1/\sqrt{2}\right\} \to \Sigma_\Upsilon^\Lambda, \quad \zeta \mapsto \begin{cases} \zeta^2 & \text{if } \mathrm{Re}\,\zeta \geq 0, \\ \zeta^2+i & \text{if } \mathrm{Re}\,\zeta \leq 0,\ \mathrm{Im}\,\zeta \geq 0, \\ \zeta^2-i & \text{if } \mathrm{Re}\,\zeta \leq 0,\ \mathrm{Im}\,\zeta \leq 0, \end{cases} \tag{37}$$

at this point.

The advantage of these representations is that now Σ^Ω_Υ and Σ^Λ_Υ are endowed with a global coordinate $z = s + it$, which is holomorphic everywhere except at the point $(0,0)$ (identified with $(0,-1)$ and $(0,1)$ in the Λ case). We refer to such a point as the *singular* point: it is a regular point for the complex structure of Σ^Ω_Υ or Σ^Λ_Υ, but it is singular for the global coordinate $z = s+it$. In fact, the canonical map

$$\Sigma^\Lambda_\Upsilon \to \mathbb{R} \times \mathbb{T}, \quad (s,t) \mapsto (s,t),$$

is a $2:1$ branched covering of the cylinder.

Let $H \in C^\infty([-1,1] \times T^*M)$. If $u \in C^\infty(\Sigma^\Omega_\Upsilon, T^*M)$, the complex anti-linear one-form $F_{J,H}(u)$ is everywhere defined by equation (34). We just need to check the regularity of $F_{J,H}(u)$ at the singular point. Writing $F_{J,H}(u)$ in terms of the holomorphic coordinate $\zeta = \sigma + i\tau$ by means of (36), we find

$$F_{J,H}(u) = (\tau I - \sigma J(u)) X_H(2\sigma\tau, u)\, d\sigma + (\sigma I + \tau J(u)) X_H(2\sigma\tau, u)\, d\tau.$$

Therefore $F_{J,H}(u)$ is smooth, and actually it vanishes at the singular point.

Assume now that $H \in C^\infty(\mathbb{R}/2\mathbb{Z} \times T^*M)$ is such that $H(-1,\cdot) = H(0,\cdot) = H(1,\cdot)$ with all the time derivatives. If $u \in C^\infty(\Sigma^\Lambda_\Upsilon, T^*M)$, (34) defines a smooth complex anti-linear one-form $F_{J,H}(u) \in \Omega^{0,1}(\Sigma^\Lambda_\Upsilon, u^*(TT^*M))$.

A map u in $C^\infty(\Sigma^\Omega_\Upsilon, T^*M)$ or in $C^\infty(\Sigma^\Lambda_\Upsilon, T^*M)$ solves equation (35) if and only if it solves the equation

$$\overline{\partial}_{J,H}(u) = \partial_s u + J(u)(\partial_t u - X_H(t,u)) = 0$$

on $\mathrm{Int}(\Sigma_\Upsilon)$. If u solves the above equation on $[s_0, s_1] \times [t_0, t_1]$, formula (30) together with an integration by parts leads to the identity

$$\begin{aligned}\int_{s_0}^{s_1} \int_{t_0}^{t_1} |\partial_s u(s,t)|^2 \, dt\, ds &= \mathbb{A}_H^{[t_0,t_1]}(u(s_0,\cdot)) - \mathbb{A}_H^{[t_0,t_1]}(u(s_1,\cdot)) \\ &+ \int_{s_0}^{s_1} \left(\eta(u(s,t_1))[\partial_s u(s,t_1)] - \eta(u(s,t_0))[\partial_s u(s,t_0)]\right) ds,\end{aligned}$$

where $\mathbb{A}_H^I(x)$ denotes the Hamiltonian action of the path x on the interval I. We conclude that a solution u of (35) on Σ^Λ_Υ or on Σ^Ω_Υ - in the latter case with values in $T^*_{q_0}M$ on the boundary - satisfies the sharp energy identity

$$\begin{aligned}&\int\int_{(]-s_0,s_0[\times]-1,1[)\setminus(]-s_0,0]\times\{0\}} |\partial_s u(s,t)|^2 \, ds\, dt \\ &= \mathbb{A}_H^{[-1,0]}(u(-s_0,\cdot)) + \mathbb{A}_H^{[0,1]}(u(-s_0,\cdot)) - \mathbb{A}_H^{[-1,1]}(u(s_0,\cdot)).\end{aligned} \tag{38}$$

3.3 The triangle and the pair-of-pants products

Given $H_1, H_2 \in C^\infty([0,1] \times T^*M)$ such that $H_1(1,\cdot) = H_2(0,\cdot)$ with all time derivatives, we define $H_1 \# H_2 \in C^\infty([0,1] \times T^*M)$ by

$$H_1 \# H_2(t,x) = \begin{cases} 2H_1(2t,x) & \text{for } 0 \le t \le 1/2, \\ 2H_2(2t-1,x) & \text{for } 1/2 \le t \le 1. \end{cases} \tag{39}$$

Let us assume that H_1, H_2, and $H_1 \# H_2$ satisfy $(\mathrm{H0})^\Omega$. The *triangle product* on $HF^\Omega(T^*M)$ will be induced by a chain map

$$\Upsilon^\Omega : F_h^\Omega(H_1, J_1) \otimes F_k^\Omega(H_2, J_2) \to F_{h+k}^\Omega(H_1 \# H_2, J_1 \# J_2).$$

In the periodic case, we consider Hamiltonians $H_1, H_2 \in C^\infty(\mathbb{T} \times T^*M)$ such that $H_1(0,\cdot) = H_2(0,\cdot)$ with all time derivatives. Assuming that H_1, H_2, and $H_1 \# H_2$ satisfy (H0)$^\Lambda$, the *pair-of-pants product* on $HF^\Lambda(T^*M)$ will be induced by a chain map

$$\Upsilon^\Lambda : F_h^\Lambda(H_1, J_1) \otimes F_k^\Lambda(H_2, J_2) \to F_{h+k-n}^\Lambda(H_1 \# H_2, J_1 \# J_2),$$

where n is the dimension of M.

Let $H \in C^\infty([-1,1] \times T^*M)$, respectively $H \in C^\infty(\mathbb{R}/2\mathbb{Z} \times T^*M)$, be defined by

$$H(t,x) = \frac{1}{2} H_1 \# H_2((t+1)/2, x) = \begin{cases} H_1(t+1,x) & \text{if } -1 \le t \le 0, \\ H_2(t,x) & \text{if } 0 \le t \le 1. \end{cases} \tag{40}$$

Notice that $x : [-1,1] \to T^*M$ is an orbit of X_H if and only if the curve $t \mapsto x((t+1)/2)$ is an orbit of $X_{H_1 \# H_2}$.

Given $x_1 \in \mathscr{P}^\Omega(H_1)$, $x_2 \in \mathscr{P}^\Omega(H_2)$, and $y \in \mathscr{P}^\Omega(H_1 \# H_2)$, consider the space of solutions of the Floer equation
$delbar_{J,H}(u) = 0$ on the holomorphic triangle

$$\mathscr{M}_\Upsilon^\Omega(x_1, x_2; y) := \Big\{ u \in C^\infty(\Sigma_\Upsilon^\Omega, T^*M) \,\Big|\, \overline{\partial}_{J,H}(u) = 0,\ u(z) \in T_{q_0}^*M \ \forall z \in \partial \Sigma_\Upsilon^\Omega,$$
$$\lim_{s\to-\infty} u(s,t-1) = x_1(t),\ \lim_{s\to-\infty} u(s,t) = x_2(t),\ \lim_{s\to+\infty} u(s,2t-1) = y(t),\ \text{uniformly in } t \in [0,1] \Big\}.$$

Similarly, for $x_1 \in \mathscr{P}^\Lambda(H_1)$, $x_2 \in \mathscr{P}^\Lambda(H_2)$, and $y \in \mathscr{P}^\Lambda(H_1 \# H_2)$, we consider the space of solutions of the Floer equation on the pair-of-pants surface

$$\mathscr{M}_\Upsilon^\Lambda(x_1, x_2; y) := \Big\{ u \in C^\infty(\Sigma_\Upsilon^\Lambda, T^*M) \,\Big|\, \overline{\partial}_{J,H}(u) = 0,\ \lim_{s\to-\infty} u(s,t-1) = x_1(t),$$
$$\lim_{s\to-\infty} u(s,t) = x_2(t),\ \lim_{s\to+\infty} u(s,2t-1) = y(t),\ \text{uniformly in } t \in [0,1] \Big\}.$$

The following result is proved in section 5.10.

3.4. Proposition. *For a generic choice of H_1 and H_2 as above, the sets $\mathscr{M}_\Upsilon^\Omega(x_1, x_2; y)$ and $\mathscr{M}_\Upsilon^\Lambda(x_1, x_2; y)$ - if non-empty - are manifolds of dimension*

$$\dim \mathscr{M}_\Upsilon^\Omega(x_1,x_2;y) = \mu^\Omega(x_1) + \mu^\Omega(x_2) - \mu^\Omega(y), \quad \dim \mathscr{M}_\Upsilon^\Lambda(x_1,x_2;y) = \mu^\Lambda(x_1) + \mu^\Lambda(x_2) - \mu^\Lambda(y) - n.$$

These manifolds carry coherent orientations.

The energy identity (38) implies that every map u in $\mathscr{M}_\Upsilon^\Omega(x_1, x_2; y)$ or in $\mathscr{M}_\Upsilon^\Lambda(x_1, x_2; y)$ satisfies

$$\int\int_{(\mathbb{R}\times]-1,1[)\setminus]-\infty,0]\times\{0\})} |\partial_s u(s,t)|^2 \, ds\, dt = \mathbb{A}_{H_1}(x_1) + \mathbb{A}_{H_2}(x_2) - \mathbb{A}_{H_1\#H_2}(y). \tag{41}$$

As a consequence, we obtain the following compactness result, which is proved in section 6.1.

3.5. Proposition. *Assume that the Hamiltonians H_1 and H_2 satisfy (H1), (H2). Then the spaces $\mathscr{M}_\Upsilon^\Omega(x_1, x_2; y)$ and $\mathscr{M}_\Upsilon^\Lambda(x_1, x_2; y)$ are pre-compact in C_{loc}^∞.*

When $\mu^\Omega(y) = \mu^\Omega(x_1) + \mu^\Omega(x_2)$, $\mathscr{M}_\Upsilon^\Omega(x_1, x_2; y)$ is a finite set of oriented points, and we denote by $n_\Upsilon^\Omega(x_1, x_2; y)$ the algebraic sum of the corresponding orientation signs. Similarly, when $\mu^\Lambda(y) = \mu^\Lambda(x_1) + \mu^\Lambda(x_2) - n$, $\mathscr{M}_\Upsilon^\Lambda(x_1, x_2; y)$ is a finite set of oriented points, and we denote by $n_\Upsilon^\Lambda(x_1, x_2; y)$ the algebraic sum of the corresponding orientation signs. These integers are the coefficients of the homomorphisms

$$\Upsilon^\Omega : F_h^\Omega(H_1) \otimes F_k^\Omega(H_2) \to F_{h+k}^\Omega(H_1 \# H_2), \quad x_1 \otimes x_2 \mapsto \sum_{\substack{y \in \mathscr{P}^\Omega(H_1\#H_2) \\ \mu^\Omega(y) = h+k}} n_\Upsilon^\Omega(x_1, x_2; y)\, y,$$

$$\Upsilon^\Lambda : F_h^\Lambda(H_1) \otimes F_k^\Lambda(H_2) \to F_{h+k-n}^\Lambda(H_1 \# H_2), \quad x_1 \otimes x_2 \mapsto \sum_{\substack{y \in \mathscr{P}^\Lambda(H_1\#H_2) \\ \mu^\Lambda(y) = h+k-n}} n_\Upsilon^\Lambda(x_1, x_2; y)\, y.$$

A standard gluing argument shows that the homomorphisms Υ^{Ω} and Υ^{Λ} are chain maps. Therefore, they define products

$$\Upsilon^{\Omega} : HF_h^{\Omega}(T^*M) \otimes HF_k^{\Omega}(T^*M) \to HF_{h+k}^{\Omega}(T^*M),$$
$$\Upsilon^{\Lambda} : HF_h^{\Lambda}(T^*M) \otimes HF_k^{\Lambda}(T^*M) \to HF_{h+k-n}^{\Lambda}(T^*M),$$

in homology. By standard gluing arguments, it could be shown that these products have a unit element, are associative, and the second one is commutative. These facts will actually follow from the fact that these products correspond to the Pontrjagin and the loop products on $H_*(\Omega(M, q_0))$ and $H_*(\Lambda(M))$.

3.4 Floer homology for figure-8 loops

The pair-of-pants product on cotangent bundles - unlike on an arbitrary symplectic manifold - has a natural factorization. Indeed, we will show that it factors through the *Floer homology of figure-8 loops*. The aim of this sections is to define this Floer homology.

Let $H_1, H_2 \in C^{\infty}([0,1] \times T^*M)$ be two Hamiltonians satisfying (H1) and (H2). We are now interested in the set $\mathscr{P}^{\Theta}(H_1 \oplus H_2)$ of pairs of orbits (x_1, x_2) of X_{H_1} and X_{H_2} such that

$$\pi \circ x_1(0) = \pi \circ x_2(0) = \pi \circ x_1(1) = \pi \circ x_2(1), \quad x_1(1) - x_1(0) + x_2(1) - x_2(0) = 0.$$

This is the Hamiltonian version of the problem $\mathscr{P}^{\Theta}(L_1 \oplus L_2)$, introduced in section 2.5. It is a non-local Lagrangian boundary value problem on the symplectic manifold $(T^*M \times T^*M, \omega \times \omega)$, that is on the cotangent bundle of $M^2 = M \times M$ endowed with its standard symplectic structure. Indeed, consider the following n-dimensional submanifold of $M^4 = M \times M \times M \times M$,

$$\Delta_M^{\Theta} := \{(q,q,q,q) \mid q \in M\},$$

and denote by $N^*\Delta_M^{\Theta}$ its conormal bundle, that is

$$N^*\Delta_M^{\Theta} := \left\{ \zeta \in T^*M^4|_{\Delta_M^{\Theta}} \mid \zeta|_{T\Delta_M^{\Theta}} = 0 \right\}$$
$$= \left\{ (\zeta_1, \zeta_2, \zeta_3, \zeta_4) \in (T^*M)^4 \mid \pi(\zeta_1) = \pi(\zeta_2) = \pi(\zeta_3) = \pi(\zeta_4),\ \zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 = 0 \right\}.$$

We are looking at orbits $x = (x_1, x_2)$ of the Hamiltonian vector field $X_{H_1 \oplus H_2}$ on T^*M^2 such that $(x(0), -x(1))$ belongs to $N^*\Delta_M^{\Theta}$. In other words, we are looking at the intersection of the graph of $-\phi^{H_1 \oplus H_2}(1, \cdot)$ - a Lagrangian[7] submanifold of $T^*M^2 \times T^*M^2 = T^*M^4$ - with the Lagrangian submanifold $N^*\Delta_M^{\Theta}$. The corresponding non-degeneracy condition is then the following:

(H0)$^{\Theta}$ every solution $x = (x_1, x_2) \in \mathscr{P}^{\Theta}(H_1 \oplus H_2)$ is non-degenerate, meaning that the graph of the map $-\phi^{H_1 \oplus H_2}(1, \cdot)$ is transverse to the submanifold $N^*\Delta_M^{\Theta}$ at the point $(x(0), -x(1))$.

Let $x = (x_1, x_2) \in \mathscr{P}^{\Theta}(H_1 \oplus H_2)$. By means of a unitary trivialization

$$[0,1] \times \mathbb{R}^{4n} \to x^*(TT^*M^2),$$

mapping $(0) \times \mathbb{R}^{2n}$ into the vertical subbundle $T^vT^*M^2$, we can transform the differential of $\phi^{H_1 \oplus H_2}(t, \cdot)$ at $x(0)$ into a path $\Phi : [0,1] \to \mathrm{Sp}(4n)$ into the symplectic group of $\mathbb{R}^{4n}$, such that $\Phi(0) = I$. Denoting by $C \in L(\mathbb{R}^{4n}, \mathbb{R}^{4n})$ the anti-symplectic involution $C(q,p) = (q, -p)$, condition (H0)$^{\Theta}$ states that the graph of $C\Phi(1)$ has intersection (0) with the Lagrangian subspace of $\mathbb{R}^{4n} \times \mathbb{R}^{4n}$,

$$N^*\Delta_{\mathbb{R}^n}^{\Theta} := \left\{ (z_1, z_1, z_3, z_4) \in (\mathbb{R}^{2n})^4 \mid \pi z_1 = \pi z_2 = \pi z_3 = \pi z_4,\ (I - \pi)(z_1 + z_2 + z_3 + z_4) = 0 \right\},$$

$\pi : \mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ denoting the projection onto the first factor.

[7] Usually a non-local Lagrangian boundary value problem on the symplectic manifold (P, ω) is given by fixing a Lagrangian submanifold $\mathscr{L}$ of $(P \times P, \omega \oplus (-\omega))$ and considering its intersection with the graph of a Hamiltonian diffeomorphism of P, which is a Lagrangian submanifold of $(P \times P, \omega \oplus (-\omega))$. Here P is the cotangent bundle of a manifold, and so is $P \times P$. Therefore we prefer to consider always the standard symplectic structure $\omega \oplus \omega$ on $P \times P$, and not the flipped one, $\omega \oplus (-\omega)$. With this choice, ϕ is a symplectic diffeomorphism of P if and only if the graph of $-\phi$ is a Lagrangian submanifold of $P \times P$. See also section 5.1 for the consequences of adopting this sign convention.

3.6. Definition. *The Maslov index $\mu^\Theta(x) \in \mathbb{Z}$ of $x = (x_1, x_2) \in \mathscr{P}^\Theta(H_1 \oplus H_2)$ is the integer*

$$\mu^\Theta(x) := \mu(N^*\Delta^\Theta_{\mathbb{R}^n}, \operatorname{graph} C\Phi) - \frac{n}{2}.$$

Since the intersection of the Lagrangian subspaces $N^*\Delta^\Theta_{\mathbb{R}^n}$ and $\operatorname{graph} C\Phi(0) = \operatorname{graph} C$ has dimension $3n$, the relative Maslov index $\mu(N^*\Delta^\Theta_{\mathbb{R}^n}, \operatorname{graph} C\Phi)$ differs from $3n/2$ by an integer (see [RS93], Corollary 4.12), so $\mu^\Theta(x)$ is an integer. The fact that this definition does not depend on the choice of the trivialization is proved in [APS08], in the more general setting of arbitrary non-local conormal boundary conditions.

By (30), the elements of $\mathscr{P}^\Theta(H_1 \oplus H_2)$ are critical points of the action functional $\mathbb{A}_{H_1 \oplus H_2} = \mathbb{A}_{H_1} \oplus \mathbb{A}_{H_2}$ on the space of pairs of curves $x_1, x_2 : [0,1] \to T^*M$ whose four end points have the same projection on M. We endow $M \times M$ with the product metric, and T^*M^2 with the corresponding almost complex structure, still denoted by J. Let $x^- = (x_1^-, x_2^-)$ and $x^+ = (x_1^+, x_2^+)$ be two elements of $\mathscr{P}^\Theta(H_1 \oplus H_2)$, and let $\mathscr{M}^\Theta_\partial(x^-, x^+)$ be the space of maps $u \in C^\infty(\mathbb{R} \times [0,1], T^*M^2)$ which solve the Floer equation

$$\overline{\partial}_{J, H_1 \oplus H_2}(u) = 0,$$

together with the boundary and asymptotic conditions

$$(u(s,0), -u(s,1)) \in N^*\Delta^\Theta_M, \quad \forall s \in \mathbb{R},$$
$$\lim_{s \to \pm\infty} u_j(s, \cdot) = x_j^\pm, \quad \forall j = 1,2.$$

The following result is proved in section 5.10.

3.7. Proposition. *For a generic choice of H_1 and H_2 the set $\mathscr{M}^\Theta_\partial(x^-, x^+)$ is a smooth manifold of dimension $\mu^\Theta(x^-) - \mu^\Theta(x^+)$. These manifolds can be oriented in a coherent way.*

Let us deal with compactness issues. Lemma 3.1 together with assumption $(\mathrm{H0})^\Theta$ implies that for every $A \in \mathbb{R}$ the set of $(x_1, x_2) \in \mathscr{P}^\Theta(H_1 \oplus H_2)$ with $\mathbb{A}_{H_1}(x_1) + \mathbb{A}_{H_2}(x_2) \leq A$ is finite.

Assumptions (H1) and (H2) allow to prove the following compactness result for solutions of the Floer equation (see section 6.1).

3.8. Proposition. *Assume that H_1 and H_2 satisfy (H1), (H2). Then for every $x^-, x^+ \in \mathscr{P}^\Theta(H_1 \oplus H_2)$, the space $\mathscr{M}^\Theta_\partial(x^-, x^+)$ is pre-compact in $C^\infty_{\mathrm{loc}}(\mathbb{R} \times [0,1], T^*M^2)$.*

If we now assume that H_1 and H_2 satisfy $(\mathrm{H0})^\Theta$, (H1), and (H2), we can define the Floer complex in the usual way. Indeed, for $x^-, x^+ \in \mathscr{P}^\Theta(H_1 \oplus H_2)$ such that $\mu^\Theta(x^-) - \mu^\Theta(x^+) = 1$, we define $n^\Theta_\partial(x^-, x^+) \in \mathbb{Z}$ to be the algebraic sum of the orientation sign associated to the elements of $\mathscr{M}^\Theta_\partial(x^-.x^+)$, and we consider the boundary operator

$$\partial : F^\Theta_k(H_1 \oplus H_2) \to F^\Theta_{k-1}(H_1 \oplus H_2), \quad x^- \mapsto \sum_{\substack{x^+ \in \mathscr{P}^\Theta(H_1 \oplus H_2) \\ \mu^\Theta(x^+) = k-1}} n^\Theta_\partial(x^-, x^+)\, x^+,$$

where $F^\Theta_k(H_1 \oplus H_2)$ denotes the free Abelian group generated by the elements $x \in \mathscr{P}^\Theta(H_1 \oplus H_2)$ with $\mu^\Theta(x) = k$.

The resulting chain complex $F^\Theta(H_1 \oplus H_2, J)$ is the *Floer complex* associated to figure-8 loops. If we change the metric on M - hence the almost complex structure J on T^*M^2 - and the orientation data, the Floer complex $F^\Theta(H_1 \oplus H_2, J)$ changes by an isomorphism. If we change the Hamiltonians H_1 and H_2, the Floer complex changes by a chain homotopy. In particular, the homology of the Floer complex does not depend on the metric, on H_1, on H_2, and on the orientation data. This fact allows us to denote this graded Abelian group as $HF^\Theta_*(T^*M)$. We will show in section 4.1 that the Floer homology for figure-8 loops is isomorphic to the singular homology of the space of figure-8 loops $\Theta(M)$,

$$HF^\Theta_k(T^*M) \cong H_k(\Theta(M)).$$

3.5 Factorization of the pair-of-pants product

Let $H_1, H_2 \in C^\infty(\mathbb{T} \times T^*M)$ be two Hamiltonians satisfying $(\mathrm{H0})^\Lambda$, (H1), and (H2). We assume that $H_1(0,\cdot) = H_2(0,\cdot)$ with all time derivatives, so that the Hamiltonian $H_1 \# H_2$ defined in (39) also belongs to $C^\infty(\mathbb{T} \times T^*M)$. We assume that $H_1 \# H_2$ satisfies $(\mathrm{H0})^\Lambda$, while $H_1 \oplus H_2$ satisfies $(\mathrm{H0})^\Theta$. The aim of this section is to construct two chain maps

$$\begin{aligned} E : F_h^\Lambda(H_1, J) \otimes F_k^\Lambda(H_2, J) &\to F_{h+k-n}^\Theta(H_1 \oplus H_2, J), \\ G : F_k^\Theta(H_1 \oplus H_2, J) &\to F_k^\Lambda(H_1 \# H_2, J), \end{aligned}$$

such that the composition $G \circ E$ is chain homotopic to the pair-of-pants chain map Υ^Λ.

The homomorphisms E is defined by counting solutions of the Floer equation on the Riemann surface Σ_E which is the disjoint union of two closed disks with an inner and a boundary point removed. The homomorphism G is defined by counting solutions of the Floer equation on the Riemann surface Σ_G obtained by removing one inner point and two boundary points from the closed disk. Again, we find it useful to represent these Riemann surfaces as suitable quotients of strips with slits.

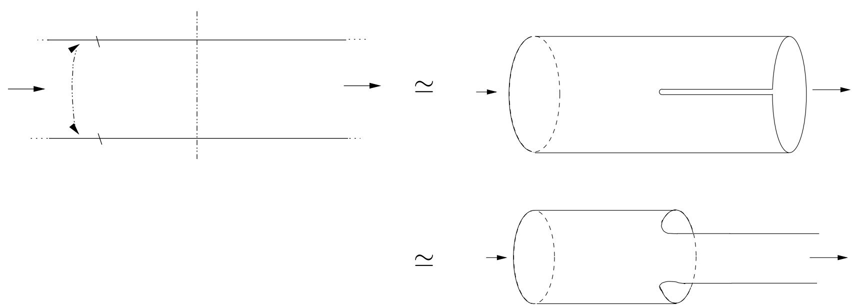

Figure 3: A component of Σ_E: the cylinder with a slit.

The surface Σ_E can be described starting from the disjoint union of two strips,

$$(\mathbb{R} \times [-1, 0]) \sqcup (\mathbb{R} \times [0, 1]),$$

by making the following identifications:

$$(s, -1) \sim (s, 0^-), \quad (s, 0^+) \sim (s, 1) \quad \text{for } s \leq 0.$$

The complex structure of Σ_E is constructed by considering the holomorphic coordinate

$$\left\{\zeta \in \mathbb{C} \mid \mathrm{Im}\,\zeta \geq 0,\ |\zeta| < 1/\sqrt{2}\right\} \to \Sigma_E, \quad \zeta \mapsto \begin{cases} \zeta^2 - i & \text{if } \mathrm{Re}\,\zeta \geq 0, \\ \zeta^2 & \text{if } \mathrm{Re}\,\zeta \leq 0, \end{cases} \tag{42}$$

at $(0,-1) \sim (0, 0^-)$, and the holomorphic coordinate

$$\left\{\zeta \in \mathbb{C} \mid \mathrm{Im}\,\zeta \geq 0,\ |\zeta| < 1/\sqrt{2}\right\} \to \Sigma_E, \quad \zeta \mapsto \begin{cases} \zeta^2 + i & \text{if } \mathrm{Re}\,\zeta \geq 0, \\ \zeta^2 & \text{if } \mathrm{Re}\,\zeta \leq 0, \end{cases} \tag{43}$$

at $(0, 0^+) \sim (0, 1)$. The resulting object is a Riemann surface consisting of two disjoint components, each of which is a cylinder with a slit: each component has one cylindrical end (on the left-hand side), one strip-like end and one boundary line (on the right-hand side). See figure 3. The global holomorphic coordinate $z = s + it$ has two singular points, at $(0, 0^-) \sim (0, -1)$, and at $(0, 0^+) \sim (0, 1)$.

The Riemann surface Σ_G is obtained from the disjoint union of two strips $(\mathbb{R} \times [-1,0]) \sqcup (\mathbb{R} \times [0,1])$ by making the identifications:

$$\begin{cases} (s,0^-) \sim (s,0^+) \\ (s,-1) \sim (s,1) \end{cases} \quad \text{for } s \geq 0.$$

A holomorphic coordinate at $(0,0)$ is the one given by (36), and a holomorphic coordinate at $(0,-1) \sim (0,1)$ is:

$$\{\zeta \in \mathbb{C} \mid \operatorname{Re}\zeta \geq 0,\ |\zeta| < 1\} \to \Sigma_G, \quad \zeta \mapsto \begin{cases} \zeta^2 - i & \text{if } \operatorname{Im}\zeta \geq 0, \\ \zeta^2 + i & \text{if } \operatorname{Im}\zeta \leq 0, \end{cases} \tag{44}$$

We obtain a Riemann surface with two boundary lines and two strip-like ends (on the left-hand side), and a cylindrical end (on the right-hand side). The global holomorphic coordinate $z = s + it$ has two singular points, at $(0,0)$, and at $(0,-1) \sim (0,1)$.

Let $H \in C^\infty(\mathbb{R}/2\mathbb{Z} \times T^*M)$ be defined by (40). Given $x_1 \in \mathscr{P}^\Lambda(H_1)$, $x_2 \in \mathscr{P}^\Lambda(H_2)$, $y = (y_1, y_2) \in \mathscr{P}^\Theta(H_1 \oplus H_2)$, and $z \in \mathscr{P}^\Lambda(H_1 \# H_2)$, we consider the following spaces of maps. The set $\mathscr{M}_E(x_1, x_2; y)$ is the space of solutions $u \in C^\infty(\Sigma_E, T^*M)$ of the Floer equation

$$\overline{\partial}_{J,H}(u) = 0,$$

satisfying the boundary conditions

$$\begin{cases} \pi u(s,-1) = \pi u(s,0^-) = \pi u(s,0^+) = \pi u(s,1), \\ u(s,0^-) - u(s,-1) + u(s,1) - u(s,0^+) = 0, \end{cases} \quad \forall s \geq 0,$$

and the asymptotic conditions

$$\lim_{s\to-\infty} u(s,t-1) = x_1(t),\ \lim_{s\to-\infty} u(s,t) = x_2(t),\ \lim_{s\to+\infty} u(s,t-1) = y_1(t),\ \lim_{s\to+\infty} u(s,t) = y_2(t),$$

uniformly in $t \in [0,1]$. The set $\mathscr{M}_G(y,z)$ is the set of solutions $u \in C^\infty(\Sigma_G, T^*M)$ of the same equation, the same boundary but for $s \leq 0$, and the asymptotic conditions

$$\lim_{s\to-\infty} u(s,t-1) = y_1(t),\ \lim_{s\to-\infty} u(s,t) = y_2(t),\ \lim_{s\to+\infty} u(s,2t-1) = z(t),$$

uniformly in $t \in [0,1]$. The following result is proved in section 5.10.

3.9. Proposition. *For a generic choice of H_1 and H_2, the spaces $\mathscr{M}_E(x_1, x_2; y)$ and $\mathscr{M}_G(y,z)$ - if non-empty - are manifolds of dimension*

$$\dim \mathscr{M}_E(x_1, x_2; y) = \mu^\Lambda(x_1) + \mu^\Lambda(x_2) - \mu^\Theta(y) - n, \quad \dim \mathscr{M}_G(y,z) = \mu^\Theta(y) - \mu^\Lambda(z).$$

These manifolds carry coherent orientations.

The energy identities are now

$$\int\int_{\mathbb{R}\times]-1,1[} |\partial_s u(s,t)|^2\, ds\, dt = \mathbb{A}_{H_1}(x_1) + \mathbb{A}_{H_2}(x_2) - \mathbb{A}_{H_1 \oplus H_2}(y), \tag{45}$$

for every $u \in \mathscr{M}_E(x_1, x_2; y)$, and

$$\int\int_{\mathbb{R}\times]-1,1[} |\partial_s u(s,t)|^2\, ds\, dt = \mathbb{A}_{H_1 \oplus H_2}(y) - \mathbb{A}_{H_1 \# H_2}(z), \tag{46}$$

for every $u \in \mathscr{M}_G(y,z)$. As usual, they imply the following compactness result (proved in section 6.1).

3.10. Proposition. *Assume that H_1, H_2 satisfy (H1), (H2). Then the spaces $\mathscr{M}_E(x_1, x_2; y)$ and $\mathscr{M}_G(y,z)$ are pre-compact in C^∞_{loc}.*

When $\mu^\Theta(y) = \mu^\Lambda(x_1) + \mu^\Lambda(x_2) - n$, $\mathscr{M}_E(x_1, x_2; y)$ is a finite set of oriented points, and we denote by $n_E(x_1, x_2; y)$ the algebraic sum of the corresponding orientation signs. Similarly, when $\mu^\Lambda(z) = \mu^\Theta(y)$, $\mathscr{M}_G(y, z)$ is a finite set of oriented points, and we denote by $n_G(y, z)$ the algebraic sum of the corresponding orientation signs. These integers are the coefficients of the homomorphisms

$$E : F_h^\Lambda(H_1) \otimes F_k^\Lambda(H_2) \to F_{h+k-n}^\Theta(H_1 \oplus H_2), \quad x_1 \otimes x_2 \mapsto \sum_{\substack{y \in \mathscr{P}^\Theta(H_1 \oplus H_2) \\ \mu^\Theta(y) = h+k-n}} n_E(x_1, x_2; y)\, y,$$

$$G : F_k^\Theta(H_1 \oplus H_2) \to F_k^\Lambda(H_1 \# H_2), \quad y \mapsto \sum_{\substack{z \in \mathscr{P}^\Lambda(H_1 \# H_2) \\ \mu^\Lambda(z) = k}} n_G(y, z)\, z.$$

A standard gluing argument shows that these homomorphisms are chain maps. The main result of this section states that the pair-of-pants product on T^*M factors through the Floer homology of figure-8 loops:

3.11. Theorem. *The chain maps*

$$\Upsilon^\Lambda,\ G \circ E : \big(F^\Lambda(H_1, J) \otimes F^\Lambda(H_2, J)\big)_k = \bigoplus_{j+h=k} F_j^\Lambda(H_1, J) \otimes\ F_h^\Lambda(H_2, J) \to F_{k-n}^\Lambda(H_1 \# H_2, J)$$

are chain homotopic.

In order to prove the above theorem, we must construct a homomorphism

$$P_{GE}^\Upsilon : \big(F^\Lambda(H_1, J) \otimes F^\Lambda(H_2, J)\big)_k \to F_{k-n+1}^\Lambda(H_1 \# H_2, J),$$

such that

$$(\Upsilon^\Lambda - G \circ E)(\alpha \otimes \beta) = \partial_{J, H_1 \# H_2}^\Lambda \circ P_{GE}^\Upsilon(\alpha \otimes \beta) + P_{GE}^\Upsilon\big(\partial_{J,H_1}^\Lambda \alpha \otimes \beta + (-1)^h \alpha \otimes \partial_{J,H_2}^\Lambda \beta\big), \tag{47}$$

for every $\alpha \in F_h^\Lambda(H_1)$ and $\beta \in F_j^\Lambda(H_2)$. The chain homotopy P_{GE}^Υ is defined by counting solutions of the Floer equation on a one-parameter family of Riemann surfaces with boundary $\Sigma_{GE}^\Upsilon(\alpha)$, $\alpha \in]0, +\infty[$, obtained by removing two open disks from the pair-of-pants.

More precisely, given $\alpha \in]0, +\infty[$, we define $\Sigma_{GE}^\Upsilon(\alpha)$ as the quotient of the disjoint union $(\mathbb{R} \times [-1, 0]) \sqcup (\mathbb{R} \times [0, 1]$ under the identifications

$$\begin{cases} (s, -1) \sim (s, 0^-) \\ (s, 0^+) \sim (s, 1) \end{cases} \quad \text{if } s \leq 0, \quad \begin{cases} (s, -1) \sim (s, 1) \\ (s, 0^-) \sim (s, 0^+) \end{cases} \quad \text{if } s \geq \alpha.$$

This object is a Riemann surface with boundary, with the holomorphic coordinates (42) and (43) at $(0, -1) \sim (0, 0^-)$ and at $(0, 0^+) \sim (0, 1)$, with the holomorphic coordinates (36) and (44) (translated by α) at $(\alpha, 0)$ and at $(\alpha, -1) \sim (\alpha, 1)$. The resulting object is a Riemann surface with three cylindrical ends, and two boundary circles.

Given $x_1 \in \mathscr{P}^\Lambda(H_1)$, $x_2 \in \mathscr{P}^\Lambda(H_2)$, and $z \in \mathscr{P}^\Lambda(H_1 \# H_2)$, we define $\mathscr{M}_{GE}^\Upsilon(x_1, x_2; z)$ to be the space of pairs (α, u), with $\alpha > 0$ and $u \in C^\infty(\Sigma_{GE}^\Upsilon(\alpha), T^*M)$ solution of

$$\overline{\partial}_{J,H}(u) = 0,$$

with boundary conditions

$$\begin{cases} \pi u(s, -1) = \pi u(s, 0^-) = \pi u(s, 0^+) = \pi u(s, 1), \\ u(s, 0^-) - u(s, -1) + u(s, 1) - u(s, 0^+) = 0, \end{cases} \quad \forall s \in [0, \alpha],$$

and asymptotic conditions

$$\lim_{s \to -\infty} u(s, t-1) = x_1(t), \quad \lim_{s \to -\infty} u(s, t) = x_2(t), \quad \lim_{s \to +\infty} u(s, 2t-1) = z(t),$$

uniformly in $t \in [0, 1]$. The following result is proved in section 5.10.

3.12. PROPOSITION. *For a generic choice of H_1 and H_2, $\mathscr{M}^{\Upsilon}_{GE}(x_1, x_2; z)$ - if non-empty - is a manifold of dimension*

$$\dim \mathscr{M}^{\Upsilon}_{GE}(x_1, x_2; z) = \mu^{\Lambda}(x_1) + \mu^{\Lambda}(x_2) - \mu^{\Lambda}(z) - n + 1.$$

The projection $(\alpha, u) \mapsto \alpha$ is smooth on $\mathscr{M}^{\Upsilon}_{GE}(x_1, x_2; z)$. These manifolds carry coherent orientations.

Energy estimates together with (H1) and (H2) again imply compactness. When $\mu^{\Lambda}(z) = \mu^{\Lambda}(x_1)+\mu^{\Lambda}(x_2)-n+1$, $\mathscr{M}^{\Upsilon}_{GE}(x_1, x_2; z)$ is a finite set of oriented points. Denoting by $n^{\Upsilon}_{GE}(x_1, x_2; z)$ the algebraic sum of the corresponding orientation signs, we define the homomorphism

$$P^{\Upsilon}_{GE} : F^{\Lambda}_h(H_1) \otimes F^{\Lambda}_k(H_2) \to F^{\Lambda}_{h+k-n+1}(H_1 \# H_2), \quad x_1 \otimes x_2 \mapsto \sum_{\substack{z \in \mathscr{P}^{\Lambda}(H_1 \# H_2) \\ \mu^{\Lambda}(z) = h+k-n+1}} n^{\Upsilon}_{GE}(x_1, x_2; z)\, z.$$

Then Theorem 3.11 follows from the following:

3.13. PROPOSITION. *The homomorphism P^{Υ}_{GE} is a chain homotopy between Υ^{Λ} and $G \circ E$.*

The proof of the above result is contained in section 6.3.

3.6 The homomorphisms C, Ev, and $\mathbf{I}_!$

Let us define the Floer homological counterparts of the the homomorphisms

$$c_* : H_k(M) \to H_k(\Lambda(M)), \quad \mathrm{ev}_* : H_k(\Lambda(M)) \to H_k(M).$$

Let f be a smooth Morse function on M, and assume that the vector field $-\mathrm{grad}\, f$ satisfies the Morse-Smale condition. Let $H \in C^{\infty}(\mathbb{T} \times T^*M)$ be a Hamiltonian satisfying $(\mathrm{H0})^{\Lambda}$, (H1), (H2). We shall define two chain maps

$$C : M_k(f, \langle\cdot,\cdot\rangle) \to F_k(H, J), \quad \mathrm{Ev} : F_k(H, J) \to M_k(f, \langle\cdot,\cdot\rangle).$$

Given $x \in \mathrm{crit}(f)$ and $y \in \mathscr{P}^{\Lambda}(H)$, consider the following spaces of maps

$$\mathscr{M}_C(x, y) = \Big\{ u \in C^{\infty}([0, +\infty[\times \mathbb{T}, T^*M) \,\Big|\, \overline{\partial}_{J,H}(u) = 0,\ \pi \circ u(0, t) \equiv q \in W^u(x)\ \forall t \in \mathbb{T}, \\ \lim_{s \to +\infty} u(s, t) = y(t) \text{ uniformly in } t \in \mathbb{T} \Big\},$$

and

$$\mathscr{M}_{\mathrm{Ev}}(y, x) = \Big\{ u \in C^{\infty}(]-\infty, 0] \times \mathbb{T}, T^*M) \,\Big|\, \overline{\partial}_{J,H}(u) = 0,\ u(0, t) \in \mathbb{O}_M\ \forall t \in \mathbb{T}, \\ u(0, 0) \in W^s(x),\ \lim_{s \to -\infty} u(s, t) = y(t) \text{ uniformly in } t \in \mathbb{T} \Big\},$$

where $\mathbb{O}_M$ denotes the zero section in T^*M.

The following result is proved in section 5.10.

3.14. PROPOSITION. *For a generic choice of H, $\mathscr{M}_C(x, y)$ and $\mathscr{M}_{\mathrm{Ev}}(y, x)$ are manifolds with*

$$\dim \mathscr{M}_C(x, y) = m(x) - \mu^{\Lambda}(y), \quad \mathscr{M}_{\mathrm{Ev}}(y, x) = \mu^{\Lambda}(y) - m(x).$$

These manifolds carry coherent orientations.

If u belongs to $\mathscr{M}_C(x, y)$ or $\mathscr{M}_{\mathrm{Ev}}(y, x)$, the fact that $u(0, \cdot)$ takes value either on the fiber of some point $q \in M$ or on the zero section of T^*M implies that

$$\mathbb{A}_H(u(0, \cdot)) = -\int_0^1 H(t, u(0, t))\, dt.$$

Therefore, we have the energy estimates

$$\int\int_{]0,+\infty[\times]0,1[} |\partial_s u(s,t)|^2 \, ds \, dt \le - \min_{(t,q)\in\mathbb{T}\times M} H(t,q,0) - \mathbb{A}_H(y),$$

for every $u \in \mathscr{M}_C(x,y)$, and

$$\int\int_{]-\infty,0[\times]0,1[} |\partial_s u(s,t)|^2 \, ds \, dt \le \mathbb{A}_H(y) + \max_{(t,q)\in\mathbb{T}\times M} H(t,q,0),$$

for every $u \in \mathscr{M}_{\mathrm{Ev}}(y,x)$. These energy estimates allow to prove the following compactness result:

3.15. Proposition. *The spaces $\mathscr{M}_C(x,y)$ and $\mathscr{M}_{\mathrm{Ev}}(y,x)$ are pre-compact in $C^\infty_{\mathrm{loc}}([0,+\infty[\times\mathbb{T}, T^*M)$ and $C^\infty_{\mathrm{loc}}(]-\infty,0]\times\mathbb{T}, T^*M)$.*

When $\mu^\Lambda(y) - m(x)$, $\mathscr{M}_C(x,y)$ and $\mathscr{M}_{\mathrm{Ev}}(y,x)$ consist of finitely many oriented points. The algebraic sums of these orientation signs, denoted by $n_C(x,y)$ and $n_{\mathrm{Ev}}(y,x)$, define the homomorphisms

$$C : M_k(f, \langle\cdot,\cdot\rangle) \to F_k^\Lambda(H,J), \quad x \mapsto \sum_{\substack{y\in\mathscr{P}^\Lambda(H)\\ \mu^\Lambda(y)=k}} n_C(x,y)\, y,$$

$$\mathrm{Ev} : F_k^\Lambda(H,J) \to M_k(f, \langle\cdot,\cdot\rangle), \quad y \mapsto \sum_{\substack{x\in\mathrm{crit}(f)\\ m(x)=k}} n_{\mathrm{Ev}}(y,x)\, x.$$

A standard gluing argument shows that C and Ev are chain maps.

We conclude this section by defining the Floer homological counterpart of the homomorphism

$$i_! : H_k(\Lambda(M)) \to H_{k-n}(\Omega(M,q_0)).$$

Let $\Sigma_{I_!}$ be a cylinder with a slit. More precisely, $\Sigma_{I_!}$ is obtained from the strip $\mathbb{R}\times[0,1]$ by the identifications $(s,0)\sim(s,1)$ for every $s\le 0$. At the point $(0,0)\sim(0,1)$ we have the holomorphic coordinate

$$\left\{\zeta\in\mathbb{C} \mid \mathrm{Re}\,\zeta \ge 0,\ |\zeta| < 1/\sqrt{2}\right\} \to \Sigma_{I_!}, \quad \zeta \mapsto \begin{cases} \zeta^2 & \text{if } \mathrm{Im}\,\zeta \ge 0, \\ \zeta^2 + i & \text{if } \mathrm{Im}\,\zeta \le 0. \end{cases}$$

It is a Riemann surface with one cylindrical end (on the right-hand side), one strip-like end and one boundary line (on the left-hand side). It is the copy of one component of Σ_E, see Figure 3.

Consider now a Hamiltonian $H \in C^\infty(\mathbb{T}\times T^*M)$ satisfying (H0)$^\Lambda$, (H0)$^\Omega$, (H1), (H2). We also assume

$$x \in \mathscr{P}^\Lambda(H) \quad \Longrightarrow \quad x(0) \notin T^*_{q_0}M. \tag{48}$$

Given $x\in\mathscr{P}^\Lambda(H)$ and $y\in\mathscr{P}^\Omega(H)$, we introduce the space of maps

$$\mathscr{M}_{I_!}(x,y) = \Big\{u \in C^\infty(\Sigma_{I_!}, T^*M) \,\Big|\, u \text{ solves (35)}, \ u(s,0)\in T^*_{q_0}M,\ u(s,1)\in T^*_{q_0}M \ \forall s\ge 0, \\ \lim_{s\to-\infty} u(s,t) = x(t),\ \lim_{s\to+\infty} u(s,t) = y(t) \text{ uniformly in } t\in[0,1]\Big\}.$$

The following result is proved in section 5.10):

3.16. Proposition. *For a generic H satisfying (48), the space $\mathscr{M}_{I_!}(x,y)$ is a manifold, with*

$$\dim \mathscr{M}_{I_!}(x,y) = \mu^\Lambda(x) - \mu^\Omega(y) - n.$$

These manifolds carry coherent orientations.

The following compactness statement follows from the general discussion of section 6.1.

3.17. PROPOSITION. *The space* $\mathscr{M}_{I_!}(x,y)$ *is pre-compact in* $C^\infty_{\mathrm{loc}}(\Sigma_{I_!}, T^*M)$.

When $\mu^\Omega(y) = \mu^\Lambda(x) - n$, the space $\mathscr{M}_{I_!}(x,y)$ consists of finitely many oriented points. The algebraic sum of these orientations is denoted by $n_{I_!}(x,y)$, and defines the homomorphism

$$I_! : F_k^\Lambda(H,J) \to F_{k-n}^\Omega(H,J), \quad x \mapsto \sum_{\substack{y \in \mathscr{P}^\Omega(H) \\ \mu^\Omega(y)=k-n}} n_{I_!}(x,y)\, y.$$

A standard gluing argument shows that $I_!$ is a chain map.

4 Isomorphisms between Morse and Floer complexes

4.1 The chain complex isomorphisms

Let $L \in C^\infty([0,1] \times T^*M)$ or $L \in C^\infty(\mathbb{T} \times T^*M)$ be a Lagrangian satisfying (L1) and (L2). Let H be the *Fenchel transform* of L, that is

$$H(t,q,p) := \max_{v \in T_qM} \Big(p(v) - L(t,q,v) \Big).$$

It is easy to see that H satisfies (H1) and (H2). If $v(t,q,p) \in T_qM$ is the (unique) vector where the above maximum is achieved, the map

$$[0,1] \times T^*M \to [0,1] \times TM, \quad (t,q,p) \mapsto (t,q,v(t,q,p)),$$

is a diffeomorphism, called the *Legendre transform* associated to the Lagrangian L. The Legendre transform induces a one-to-one correspondence $x \mapsto \pi \circ x$ between the orbits of the Hamiltonian vector field X_H and the solutions of the second order Lagrangian equation given by L. When H is the Fenchel transform of L, we have the fundamental inequality between the Hamiltonian and the Lagrangian action functionals:

$$\mathbb{A}_H(x) \le \mathbb{S}_L(\pi \circ x), \quad \forall x : [0,1] \to T^*M, \tag{49}$$

with the equality holding if and only if x is related to $(\pi \circ x, (\pi \circ x)')$ by the Legendre transform. In particular, the equality holds if x is an orbit of the Hamiltonian vector field X_H. In this section we recall the definition of the isomorphisms

$$\Phi_L^\Omega : M_k(\mathbb{S}_L^\Omega, g^\Omega) \to F_k^\Omega(H,J), \quad \Phi_L^\Lambda : M_k(\mathbb{S}_L^\Lambda, g^\Lambda) \to F_k^\Lambda(H,J),$$

between the Morse complex of the Lagrangian action functional and the Floer complex of the corresponding Hamiltonian system. See [AS06b] for detailed proofs.

We assume that L satisfies $(\mathrm{L0})^\Omega$, resp. $(\mathrm{L0})^\Lambda$, equivalently that H satisfies $(\mathrm{H0})^\Omega$, resp. $(\mathrm{H0})^\Lambda$. Consider a Riemannian metric g^Ω, resp. g^Λ, on $\Omega^1(M,q_0)$, resp. $\Lambda^1(M)$, such that the Lagrangian action functional $\mathbb{S}_L$ satisfies the Palais-Smale and the Morse-Smale conditions. Given $\gamma \in \mathscr{P}^\Omega(L)$ and $x \in \mathscr{P}^\Omega(H)$, we denote by $\mathscr{M}_\Phi^\Omega(\gamma, x)$ the space of maps $u \in C^\infty([0,+\infty[\times [0,1], T^*M)$ which solve the Floer equation

$$\overline{\partial}_{J,H}(u) = 0, \tag{50}$$

and satisfy the boundary conditions

$$u(s,0) \in T^*_{q_0}M,\ u(s,1) \in T^*_{q_0}M, \quad \forall s \ge 0,$$
$$\pi \circ u(0,\cdot) \in W^u(\gamma, -\mathrm{grad}\, \mathbb{S}_L^\Omega),$$

and the asymptotic condition

$$\lim_{s\to+\infty} u(s,t) = x(t), \tag{51}$$

uniformly in $t \in [0,1]$. Similarly, for $\gamma \in \mathscr{P}^\Lambda(L)$ and $x \in \mathscr{P}^\Lambda(H)$, we denote by $\mathscr{M}_\Phi^\Lambda(\gamma,x)$ the space of maps $u \in C^\infty([0,+\infty[\times\mathbb{T}, T^*M)$ solving the Floer equation (50) with the boundary condition

$$\pi \circ u(0,\cdot) \in W^u(\gamma, -\mathrm{grad}\, \mathbb{S}_L^\Lambda),$$

and the asymptotic conditon (51) uniformly in $t \in \mathbb{T}$. For a generic choice of H, these spaces of maps are manifolds of dimension

$$\dim \mathscr{M}_\Phi^\Omega(\gamma,x) = m^\Omega(\gamma) - \mu^\Omega(x), \quad \dim \mathscr{M}_\Phi^\Lambda(\gamma,x) = m^\Lambda(\gamma) - \mu^\Lambda(x).$$

The inequality (49) provides us with the energy estimates which allow to prove suitable compactness properties for the spaces $\mathscr{M}_\Phi^\Omega(\gamma,x)$ and $\mathscr{M}_\Phi^\Lambda(\gamma,x)$. When $\mu^\Omega(x) = m^\Omega(\gamma)$, resp. $\mu^\Lambda(x) = m^\Lambda(\gamma)$, the space $\mathscr{M}_\Phi^\Omega(\gamma,x)$, resp. $\mathscr{M}_\Phi^\Lambda(\gamma,x)$, consists of finitely many oriented points, which add up to the integers $n_\Phi^\Omega(\gamma,x)$, resp. $n_\Phi^\Lambda(\gamma,x)$. These integers are the coefficients of the homomorphisms

$$\Phi_L^\Omega : M_k(\mathbb{S}_L^\Omega, g^\Omega) \to F_k^\Omega(H,J), \quad \gamma \mapsto \sum_{\substack{x\in\mathscr{P}^\Omega(H)\\ \mu^\Omega(x)=k}} n_\Phi^\Omega(\gamma,x)\, x,$$

$$\Phi_L^\Lambda : M_k(\mathbb{S}_L^\Lambda, g^\Lambda) \to F_k^\Lambda(H,J), \quad \gamma \mapsto \sum_{\substack{x\in\mathscr{P}^\Lambda(H)\\ \mu^\Lambda(x)=k}} n_\Phi^\Lambda(\gamma,x)\, x,$$

which are shown to be chain maps. The inequality (49) together with its differential version implies that $n_\Phi(\gamma,x) = 0$ if $\mathbb{A}_H(x) \geq \mathbb{S}_L(\gamma)$ and $\gamma \neq \pi\circ x$, while $n_\Phi(\gamma,x) = \pm 1$ if $\gamma = \pi \circ x$. These facts imply that Φ^Ω and Φ^Λ are isomorphisms. We summarize the above facts into the following:

4.1. Theorem. *For a generic metric g^Ω, resp. g^Λ, on the based loop space $\Omega^1(M,q_0)$, resp. on the free loop space $\Lambda^1(M)$, and for a generic Lagrangian $L \in C^\infty([0,1]\times TM)$, resp. $L \in C^\infty(\mathbb{T}\times TM)$, satisfying (L0)$^\Omega$, resp. (L0)$^\Lambda$, (L1), (L2), the above construction produces an isomorphism*

$$\Phi_L^\Omega : M_k(\mathbb{S}_L^\Omega, g^\Omega) \to F_k^\Omega(H,J), \quad \textit{resp. } \Phi_L^\Lambda : M_k(\mathbb{S}_L^\Lambda, g^\Lambda) \to F_k^\Lambda(H,J),$$

from the Morse complex of the Lagrangian action functional to the Floer complex of (H,J), where H is the Fenchel transform of L.

The same idea produces an isomorphism

$$\Phi_{L_1\oplus L_2}^\Theta : M_k(\mathbb{S}_{L_1\oplus L_2}^\Theta, g^\Theta) \to F_k^\Theta(H_1\oplus H_2, J),$$

between the Morse and the Floer complex associated to the figure-8 problem (see sections 2.5 and 3.4). Indeed, given $\gamma \in \mathscr{P}^\Theta(L_1 \oplus L_2)$ and $x \in \mathscr{P}^\Theta(H_1\oplus H_2)$, we consider the space $\mathscr{M}_\Phi^\Theta(\gamma,x)$ of maps $u \in C^\infty([0,+\infty[\times[0,1], T^*M^2)$ solving the Floer equation

$$\overline{\partial}_{J,H_1\oplus H_2}(u) = 0,$$

with non-local boundary conditions

$$\begin{aligned} (u(s,0), -u(s,1)) &\in N^*\Delta_M^\Theta, \quad \forall s \geq 0, \\ \pi\circ u(0,\cdot) &\in W^u(\gamma, -\mathrm{grad}\, \mathbb{S}_{L_1\oplus L_2}^\Theta), \end{aligned}$$

and asymptotic condition

$$\lim_{s\to+\infty} u(s,t) = x(t),$$

uniformly in $t \in [0,1]$. The following fact is proved in section 5.10:

4.2. Proposition. *For a generic choice of g^Θ, H_1, H_2, the space $\mathscr{M}^\Theta_\Phi(\gamma, x)$ - if non-empty - is a manifold of dimension*

$$\dim \mathscr{M}^\Theta_\Phi(\gamma, x) = m^\Theta(\gamma) - \mu^\Theta(x).$$

These manifolds carry coherent orientations.

The inequality (49) implies the energy estimate which can be used to prove C^∞_{loc} compactness of the space $\mathscr{M}^\Theta_\Phi(\gamma, x)$. When $\mu^\Theta(x) = m^\Theta(\gamma)$, the space $\mathscr{M}^\Theta_\Phi(\gamma, x)$ consists of finitely many oriented points, which define an integer $n^\Theta_\Phi(\gamma, x)$. These integers are the coefficients of a homomorphism

$$\Phi^\Theta_{L_1 \oplus L_2} : M_k(\mathbb{S}^\Theta_{L_1 \oplus L_2}, g^\Theta) \to F_k(H_1 \oplus H_2, J), \quad \gamma \mapsto \sum_{\substack{x \in \mathscr{P}^\Theta(H_1 \oplus H_2) \\ \mu^\Theta(x) = k}} n^\Theta_\Phi(\gamma, x)\, x.$$

This is a chain complex isomorphism. See [APS08] for the construction of this isomorphism for arbitrary non-local conormal boundary conditions.

4.2 The Ω ring isomorphism

Let $L_1, L_2 \in C^\infty([0,1] \times TM)$ be two Lagrangians such that $L_1(1, \cdot) = L_2(0, \cdot)$ with all the time derivatives, and such that L_1 and L_2 satisfy (L0)$^\Omega$, (L1), (L2), and (24). Assume also that the Lagrangian $L_1 \# L_2$ defined by (23) satisfies (L0)$^\Omega$. Let H_1 and H_2 be the Fenchel transforms of L_1 and L_2, so that $H_1 \# H_2$ is the Fenchel transform of $L_1 \# L_2$, and the three Hamiltonians H_1, H_2, and $H_1 \# H_2$ satisfy (H0)$^\Omega$, (H1), (H2).

In section 2.6 we have shown how the Pontrjagin product can be expressed in a Morse theoretical way. In other words, we have constructed a homomorphism

$$M_\# : M_h(\mathbb{S}_{L_1}, g_1) \otimes M_j(\mathbb{S}_{L_2}, g_2) \longrightarrow M_{h+j}(\mathbb{S}_{L_1 \# L_2}, g)$$

such that the upper square in the following diagram commutes

$$\begin{tikzcd}
H_h(\Omega(M, q_0)) \otimes H_j(\Omega(M, q_0)) \arrow[r, "\#"] \arrow[d, "\cong"'] & H_{h+j}(\Omega(M, q_0)) \arrow[d, "\cong"] \\
HM_h(\mathbb{S}^\Omega_{L_1}, g_1) \otimes HM_j(\mathbb{S}^\Omega_{L_2}, g_2) \arrow[r, "HM_\#"] \arrow[d, "H\Phi^\Omega_{L_1} \otimes H\Phi^\Omega_{L_2}"'] & HM_{h+j}(\mathbb{S}^\Omega_{L_1 \# L_2}, g) \arrow[d, "H\Phi^\Omega_{L_1 \# L_2}"] \\
HF^\Omega_h(H_1, J) \otimes HF^\Omega_j(H_2, J) \arrow[r, "H\Upsilon^\Omega"] & HF^\Omega_{h+j}(H_1 \# H_2, J)
\end{tikzcd}$$

The aim of this section is to show that also the lower square commutes. Actually, we will show more, namely that the diagram

$$\begin{tikzcd}
\big(M(\mathbb{S}^\Omega_{L_1}, g_1) \otimes M(\mathbb{S}^\Omega_{L_2}, g_2)\big)_k \arrow[r, "M_\#"] \arrow[d, "\Phi^\Omega_{L_1} \otimes \Phi^\Omega_{L_2}"'] & M_k(\mathbb{S}^\Omega_{L_1 \# L_2}, g) \arrow[d, "\Phi^\Omega_{L_1 \# L_2}"] \\
\big(F^\Omega(H_1, J) \otimes F^\Omega(H_2, J)\big)_k \arrow[r, "\Upsilon^\Omega"] & F^\Omega_k(H_1 \# H_2, J)
\end{tikzcd}$$

is chain-homotopy commutative. Instead than constructing a direct homotopy between $\Phi^\Omega_{L_1 \# L_2} \circ M_\#$ and $\Upsilon^\Omega \circ \Phi^\Omega_{L_1} \otimes \Phi^\Omega_{L_2}$, we shall prove that both chain maps are homotopic to a third one, that we name K^Ω, see Figure 4.

The definition of K^Ω is based on the following space of solutions of the Floer equation for the Hamiltonian H defined in (40): given $\gamma_1 \in \mathscr{P}^\Omega(L_1)$, $\gamma_2 \in \mathscr{P}^\Omega(L_2)$, and $x \in \mathscr{P}^\Omega(H_1 \# H_2)$, let $\mathscr{M}^\Omega_K(\gamma_1, \gamma_2; x)$ be the space of solutions of the Floer equation

$$\overline{\partial}_{J,H}(u) = 0,$$

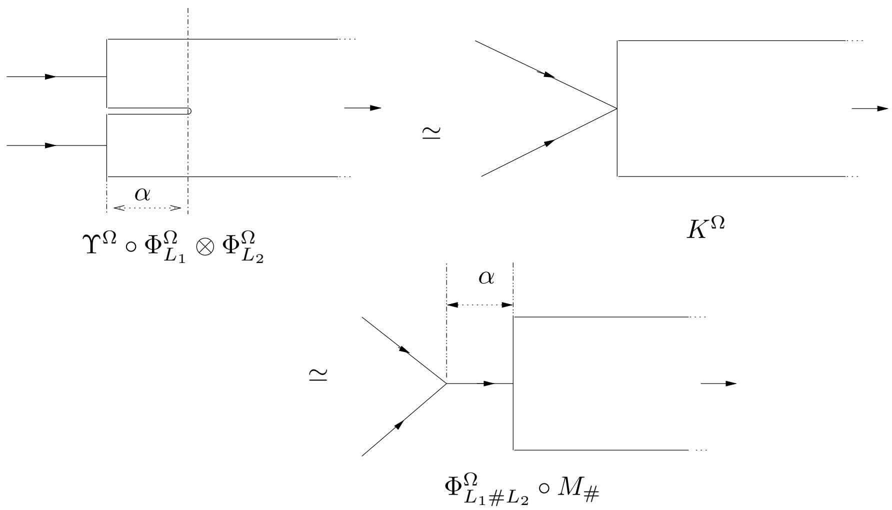

Figure 4: The homotopy through K^Ω.

with boundary conditions

$$\pi \circ u(s,-1) = \pi \circ u(s,1) = q_0, \quad \forall s \geq 0,$$
$$\pi \circ u(0,\cdot-1) \in W^u(\gamma_1, -\mathrm{grad}\, \mathbb{S}^\Omega_{L_1}),\ \pi \circ u(0,\cdot) \in W^u(\gamma_2, -\mathrm{grad}\, \mathbb{S}^\Omega_{L_2}),$$

and the asymptotic behavior

$$\lim_{s\to+\infty} u(s,2t-1) = x(t),$$

uniformly in $t \in [0,1]$. Theorem 3.2 in [AS06b] (or the arguments of section 5.10) implies that for a generic choice of g_1, g_2, H_1, and H_2, $\mathscr{M}^\Omega_K(\gamma_1,\gamma_2;x)$ - if non-empty - is a smooth manifold of dimension

$$\dim \mathscr{M}^\Omega_K(\gamma_1,\gamma_2;x) = m^\Omega(\gamma_1;L_1) + m^\Omega(\gamma_2;L_2) - \mu^\Omega(x;H_1\#H_2).$$

These manifolds carry coherent orientations. The energy identity is now

$$\int_{]0,+\infty[\times]-1,1[} |\partial_s u(s,t)|^2\, ds\, dt = \mathbb{A}_{H_1}(x_1) + \mathbb{A}_{H_2}(x_2) - \mathbb{A}_{H_1\#H_2}(x),$$

where $x_1(t) = u(0,t-1)$ and $x_2(t) = u(0,t)$. Since $\pi \circ x_1$ is in the unstable manifold of γ_1 and $\pi \circ x_2$ is in the unstable manifold of γ_2, the inequality (49) implies that

$$\mathbb{A}_{H_1}(x_1) \leq \mathbb{S}_{L_1}(\gamma_1), \quad \mathbb{A}_{H_2}(x_2) \leq \mathbb{S}_{L_2}(\gamma_2),$$

so the elements u of $\mathscr{M}^\Omega_K(\gamma_1,\gamma_2;x)$ satisfy the energy estimate

$$\int_{]0,+\infty[\times]-1,1[} |\partial_s u(s,t)|^2\, ds\, dt \leq \mathbb{S}_{L_1}(\gamma_1) + \mathbb{S}_{L_2}(\gamma_2) - \mathbb{A}_{H_1\#H_2}(x). \tag{52}$$

When $m^\Omega(\gamma_1;L_1) + m^\Omega(\gamma_2;L_2) = \mu^\Omega(x;H_1\oplus H_2)$, the space $\mathscr{M}^\Omega_K(\gamma_1,\gamma_2;x)$ is a compact zero-dimensional oriented manifold. If $n^\Omega_K(\gamma_1,\gamma_2;x)$ is the algebraic sum of its points, we can define the homomorphism

$$K^\Omega : \big(M(\mathbb{S}^\Omega_{L_1},g_1)\otimes M(\mathbb{S}^\Omega_{L_2},g_2)\big)_k \to F^\Omega_k(H_1\#H_2,J), \quad \gamma_1\otimes\gamma_2 \mapsto \sum_{\substack{x\in\mathscr{P}^\Omega(H_1\#H_2)\\ \mu^\Omega(x)=k}} n^\Omega_K(\gamma_1,\gamma_2;x)\, x.$$

A standard gluing argument shows that K^Ω is a chain map.

It is easy to construct a homotopy $P_K^\#$ between $\Phi^\Omega_{L_1\#L_2}\circ M_\#$ and K^Ω. In fact, it is enough to consider the space of pairs (α,u), where α is a positive number and u is a solution of the Floer equation on $[0,+\infty[\times[-1,1]$ converging to x for $s\to+\infty$, and such that the curve $t\mapsto \pi\circ u(0,2t-1)$ belongs to the evolution at time α of

$$\Gamma\big(W^u(\gamma_1;-\mathrm{grad}_{g_1}\mathbb{S}^\Omega_{L_1})\times W^u(\gamma_2;-\mathrm{grad}_{g_2}\mathbb{S}^\Omega_{L_2})\big),$$

by flow of $-\mathrm{grad}\,\mathbb{S}^\Omega_{L_1\#L_2}$. Here Γ is the concatenation map defined in section 2.6. More precisely, set

$$\begin{aligned}\mathscr{M}_K^\#(\gamma_1,\gamma_2;x) := \Big\{(\alpha,u)\,\Big|\,\alpha>0,\ u\in C^\infty([0,+\infty[\times[-1,1],T^*M)\text{ solves (32)},\\ \pi\circ u(s,-1)=\pi\circ u(s,1)=q_0\ \forall s\geq 0,\ \lim_{s\to+\infty}u(s,2t-1)=x(t),\text{ uniformly in } t\in[0,1],\\ \pi\circ u(0,2\cdot-1)\in\phi^\Omega_\alpha\big(\Gamma(W^u(\gamma_1,-\mathrm{grad}\,\mathbb{S}^\Omega_{L_1})\times W^u(\gamma_2,-\mathrm{grad}\,\mathbb{S}^\Omega_{L_2}))\big)\Big\},\end{aligned}$$

where ϕ^Ω_s denotes the flow of $-\mathrm{grad}\,\mathbb{S}^\Omega_{L_1\#L_2}$. For a generic choice of g_1, g_2, H_1, and H_2, $\mathscr{M}_K^\#(\gamma_1,\gamma_2;x)$ - if non-empty - is a smooth manifold of dimension

$$\dim\mathscr{M}_K^\#(\gamma_1,\gamma_2;x)=m^\Omega(\gamma_1;L_1)+m^\Omega(\gamma_2;L_2)-\mu^\Omega(x;H_1\#H_2)+1,$$

and these manifolds carry coherent orientations. The energy estimate is again (52). By counting the elements of the zero-dimensional manifolds, we obtain a homomorphism

$$P_K^\#:\big(M(\mathbb{S}^\Omega_{L_1},g_1)\otimes M(\mathbb{S}^\Omega_{L_2},g_2)\big)_k\to F^\Omega_{k+1}(H_1\#H_2,J).$$

A standard gluing argument shows that $P_K^\#$ is a chain homotopy between $\Phi^\Omega_{L_1\#L_2}\circ M_\#$ and K^Ω.

The homotopy P_Υ^K between K^Ω and $\Upsilon^\Omega\circ(\Phi^\Omega_{L_1}\otimes\Phi^\Omega_{L_2})$ is defined by counting solutions of the Floer equation on a one-parameter family of Riemann surfaces $\Sigma_\Upsilon^K(\alpha)$, obtained by removing a point from the closed disk. More precisely, given $\alpha>0$ we define $\Sigma^K_\Upsilon(\alpha)$ as the quotient of the disjoint union $[0,+\infty[\times[-1,0]\sqcup[0,+\infty[\times[0,1]$ under the identification

$$(s,0^-)\sim(s,0^+)\quad\text{for } s\geq\alpha.$$

This object is a Riemann surface with boundary: its complex structure at each interior point and at each boundary point other than $(\alpha,0)$ is induced by the inclusion, whereas the holomorphic coordinate at $(\alpha,0)$ is given by the map

$$\{\zeta\in\mathbb{C}\mid \mathrm{Re}\,\zeta\geq 0,\ |\zeta|<\epsilon\}\to\Sigma^K_\Upsilon(\alpha),\quad \zeta\mapsto\alpha+\zeta^2,$$

where the positive number ϵ is smaller than 1 and $\sqrt{\alpha}$. Given $\gamma_1\in\mathscr{P}(L_1)$, $\gamma_2\in\mathscr{P}(L_2)$, and $x\in\mathscr{P}(H_1\#H_2)$, we consider the space of pairs (α,u) where α is a positive number, and $u(s,t)$ is a solution of the Floer equation on $\Sigma^K_\Upsilon(\alpha)$ which converges to x for $s\to+\infty$, lies above some element in the unstable manifold of γ_1 for $s=0$ and $-1\leq t\leq 0$, lies above some element in the unstable manifold of γ_2 for $s=0$ and $0\leq t\leq 1$, and lies above q_0 at all the other boundary points. More precisely, $\mathscr{M}^K_\Upsilon(\gamma_1,\gamma_2;x)$ is the set of pairs (α,u) where α is a positive number and $u\in C^\infty(\Sigma^K_\Upsilon(\alpha),T^*M)$ is a solution of

$$\overline{\partial}_{J,H}(u)=0,$$

satisfying the boundary conditions

$$\begin{aligned}\pi\circ u(s,-1)=\pi\circ u(s,1)=q_0,\quad \forall s\geq 0,\\ \pi\circ u(s,0^-)=\pi\circ u(s,0^+)=q_0,\quad \forall s\in[0,\alpha],\\ \pi\circ u(0,\cdot-1)\in W^u(\gamma_1,-\mathrm{grad}\,\mathbb{S}^\Omega_{L_1}),\quad \pi\circ u(0,\cdot)\in W^u(\gamma_2,-\mathrm{grad}\,\mathbb{S}^\Omega_{L_2}),\end{aligned}$$

and the asymptotic condition

$$\lim_{s\to+\infty} u(s,2t-1) = x(t),$$

uniformly in $t \in [0,1]$. The following result is proved in section 5.10.

4.3. Proposition. *For a generic choice of H_1, H_2, g_1 and g_2, $\mathscr{M}^K_\Upsilon(\gamma_1,\gamma_2;x)$ - if non-empty - is a smooth manifold of dimension*

$$\dim \mathscr{M}^K_\Upsilon(\gamma_1,\gamma_2;x) = m^\Omega(\gamma_1;L_1) + m^\Omega(\gamma_2;L_2) - \mu^\Omega(x;H_1\#H_2) + 1.$$

These manifolds carry coherent orientations.

As before, the elements (α,u) of $\mathscr{M}^K_\Upsilon(\gamma_1,\gamma_2;x)$ satisfy the energy estimate

$$\int_{\mathbb{R}\times]-1,1[\setminus\{0\}\times[0,\alpha]} |\partial_s u(s,t)|^2\, ds\, dt \leq \mathbb{S}_{L_1}(\gamma_1) + \mathbb{S}_{L_2}(\gamma_2) - \mathbb{A}_{H_1\#H_2}(x),$$

which allows to prove compactness. By counting the zero-dimensional components, we define a homomorphism

$$P^K_\Upsilon : \big(M(\mathbb{S}^\Omega_{L_1},g_1)\otimes M(\mathbb{S}^\Omega_{L_2},g_2)\big)_k \to F^\Omega_{k+1}(H_1\#H_2,J).$$

The conclusion follows from the following:

4.4. Proposition. *The homomorphism P^K_Υ is a chain homotopy between K^Ω and $\Upsilon^\Omega\circ(\Phi^\Omega_{L_1}\otimes\Phi^\Omega_{L_2})$.*

The proof of the above proposition is contained in section 6.4. It is again a compactness-cobordism argument. The analytical tool is the implicit function theorem together with a suitable family of conformal transformations of the half-strip.

4.3 The Λ ring homomorphism

Let $L_1, L_2 \in C^\infty(\mathbb{T}\times TM)$ be two Lagrangians such that $L_1(0,\cdot) = L_2(0,\cdot)$ with all the time derivatives, and such that L_1 and L_2 satisfy (L0)$^\Lambda$, (L1), (L2), and (25), or equivalently (26). Assume also that the Lagrangian $L_1\#L_2$ defined by (23) satisfies (L0)$^\Lambda$. Let H_1 and H_2 be the Fenchel transforms of L_1 and L_2, so that $H_1\#H_2$ is the Fenchel transform of $L_1\#L_2$, and the three Hamiltonians H_1, H_2, and $H_1\#H_2$ satisfy (H0)$^\Lambda$, (H1), (H2).

The loop product is the compositions of two non-trivial homomorphism: the first one is the exterior homology product followed by the Umkehr map, the second one is the homomorphism induced by concatenation. In section 2.7 we have shown how these two homomorphisms can be expressed in a Morse theoretical way. In other words, we have constructed homomorphisms

$$M_! : M_h(\mathbb{S}^\Lambda_{L_1},g_1)\otimes M_j(\mathbb{S}^\Lambda_{L_2},g_2) \longrightarrow M_{h+j-n}(\mathbb{S}^\Theta_{L_1\oplus L_2},g^\Theta),$$
$$M_\Gamma : M_j(\mathbb{S}^\Theta_{L_1\oplus L_2},g^\Theta) \to M_j(\mathbb{S}^\Lambda_{L_1\#L_2},g^\Lambda)$$

such that the upper squares in the following diagram commute

$$\begin{tikzcd}
H_h(\Lambda(M))\otimes H_j(\Lambda(M)) \arrow[r, "e_!\circ\times"] \arrow[d, "\cong"'] & H_{h+j-n}(\Theta(M)) \arrow[r, "\Gamma_*"] \arrow[d, "\cong"] & H_{h+j-n}(\Lambda(M)) \arrow[d, "\cong"] \\
HM_h(\mathbb{S}^\Lambda_{L_1},g_1)\otimes HM_j(\mathbb{S}^\Lambda_{L_2},g_2) \arrow[r, "HM_!"] \arrow[d, "H\Phi^\Lambda_{L_1}\otimes H\Phi^\Lambda_{L_2}"'] & HM_{h+j-n}(\mathbb{S}^\Theta_{L_1\oplus L_2},g^\Theta) \arrow[r, "HM_\Gamma"] \arrow[d, "H\Phi^\Theta_{L_1\oplus L_2}"] & HM_{h+j-n}(\mathscr{S}^\Lambda_{L_1\#L_2},g^\Lambda) \arrow[d, "H\Phi^\Lambda_{L_1\#L_2}"] \\
HF^\Lambda_h(H_1,J)\otimes HF^\Lambda_j(H_2,J) \arrow[r, "HE"] & HF^\Theta_{h+j-n}(H_1\oplus H_2,J) \arrow[r, "HG"] & HF^\Lambda_{h+j-n}(H_1\#H_2,J)
\end{tikzcd}$$

In the following two sections we will show that also the lower two squares commute. By Theorem 3.11, the composition of the two lower arrows is the pair-of-pants product. We conclude that the pair-of-pants product corresponds to the loop product.

Again, the commutativity of the two lower squares will be seen at the chain level, by proving that the two squares below

$$\begin{tikzcd}
\big(M(\mathbb{S}^\Lambda_{L_1},g_1)\otimes M(\mathbb{S}^\Lambda_{L_2},g_2)\big)_k \arrow[r, "M_!"] \arrow[d, "\Phi^\Lambda_{L_1}\otimes\Phi^\Lambda_{L_2}"'] & M_{k-n}(\mathbb{S}^\Theta_{L_1\oplus L_2},g^\Theta) \arrow[r, "M_\Gamma"] \arrow[d, "\Phi^\Theta_{L_1\oplus L_2}"] & M_{k-n}(\mathscr{S}^\Lambda_{L_1\# L_2},g^\Lambda) \arrow[d, "\Phi^\Lambda_{L_1\# L_2}"] \\
\big(F^\Lambda(H_1,J)\otimes HF^\Lambda(H_2,J)\big)_k \arrow[r, "E"] & F^\Theta_{k-n}(H_1\oplus H_2,J) \arrow[r, "G"] & F^\Lambda_{k-n}(H_1\# H_2,J)
\end{tikzcd} \tag{53}$$

commute up to chain homotopies.

4.4 The left-hand square is homotopy commutative

In this section we show that the chain maps $\Phi^\Theta_{L_1\oplus L_2}\circ M_!$ and $E\circ(\Phi^\Lambda_{L_1}\otimes\Phi^\Lambda_{L_2})$ are homotopic. We start by constructing a one-parameter family of chain maps

$$K^\Lambda_\alpha : \big(M(\mathbb{S}^\Lambda_{L_1},g_1)\otimes M(\mathbb{S}^\Lambda_{L_2},g_2)\big)_* \longrightarrow F^\Theta_{*-n}(H_1\oplus H_2, J_1\oplus J_2),$$

where α is a non-negative number. The definition of K^Λ_α is based on the solution spaces of the Floer equation on the Riemann surface Σ^K_α consisting of a half-cylinder with a slit. More precisely, when α is positive Σ^K_α is the quotient of $[0,+\infty[\times[0,1]$ modulo the identifications

$$(s,0)\sim(s,1)\quad \forall s\in[0,\alpha].$$

with the holomorphic coordinate at $(\alpha,0)\sim(\alpha,1)$ obtained from (43) by a translation by α. When $\alpha=0$, $\Sigma^K_\alpha=\Sigma^K_0$ is just the half-strip $[0,+\infty[\times[0,1]$. Fix $\gamma_1\in\mathscr{P}^\Lambda(L_1)$, $\gamma_2\in\mathscr{P}^\Lambda(L_2)$, and $x\in\mathscr{P}^\Theta(H_1\oplus H_2)$. Let $\mathscr{M}^K_\alpha(\gamma_1,\gamma_2;x)$ be the space of solutions $u\in C^\infty(\Sigma^K_\alpha,T^*M^2)$ of the equation

$$\overline{\partial}_{H_1\oplus H_2,J}(u)=0,$$

satisfying the boundary conditions

$$\pi\circ u(0,\cdot)\in W^u(\gamma_1;-\mathrm{grad}_{g_1}\mathbb{S}^\Lambda_{L_1})\times W^u(\gamma_2;-\mathrm{grad}_{g_2}\mathbb{S}^\Lambda_{L_2}), \tag{54}$$

$$(u(s,0),-u(s,1))\in N^*\Delta^\Theta_M,\quad \forall s\geq\alpha, \tag{55}$$

$$\lim_{s\to+\infty}u(s,\cdot)=x. \tag{56}$$

Let us fix some $\alpha_0\geq 0$. The following result is proved in section 5.10:

4.5. Proposition. *For a generic choice of g_1, g_2, H_1, and H_2, $\mathscr{M}^K_{\alpha_0}(\gamma_1,\gamma_2;x)$ - if non-empty - is a smooth manifold of dimension*

$$\dim\mathscr{M}^K_{\alpha_0}(\gamma_1,\gamma_2;x)=m^\Lambda(\gamma_1;L_1)+m^\Lambda(\gamma_2;L_2)-\mu^\Theta(x)-n.$$

These manifolds carry coherent orientations.

Compactness is again a consequence of the energy estimate

$$\int_{]0,+\infty[\times]0,1[}|\partial_s u(s,t)|^2\,ds dt\leq \mathbb{S}_{L_1}(\gamma_1)+\mathbb{S}_{L_2}(\gamma_2)-\mathbb{A}_{H_1\oplus H_2}(x), \tag{57}$$

implied by (49). When $m^\Lambda(\gamma_1;L_1)+m^\Lambda(\gamma_2;L_2)=k$ and $\mu^\Theta(x;H_1\oplus H_2)=k-n$, the space $\mathscr{M}^K_{\alpha_0}(\gamma_1,\gamma_2;x)$ is a compact zero-dimensional oriented manifold. The usual counting process defines the homomorphism

$$K^\Lambda_{\alpha_0} : \big(M(\mathbb{S}^\Lambda_{L_1},g_1)\otimes M(\mathbb{S}^\Lambda_{L_2},g_2)\big)_k \to F^\Theta_{k-n}(H_1\oplus H_2,J),$$

and a standard gluing argument shows that $K^{\Lambda}_{\alpha_0}$ is a chain map.

Now assume $\alpha_0 > 0$. By standard compactness and gluing arguments, the family of solutions $\mathscr{M}^K_\alpha$ for α varying in the interval $[\alpha_0, +\infty[$ allows to define a chain homotopy between $K^{\Lambda}_{\alpha_0}$ and the composition $E \circ (\Phi^{\Lambda}_{L_1} \otimes \Phi^{\Lambda}_{L_2})$.

Similarly, a compactness and cobordism argument on the Morse side shows that K^{Λ}_0 is chain homotopic to the composition $\Phi^{\Theta}_{L_1 \oplus L_2} \circ M_!$. See Figure 5.

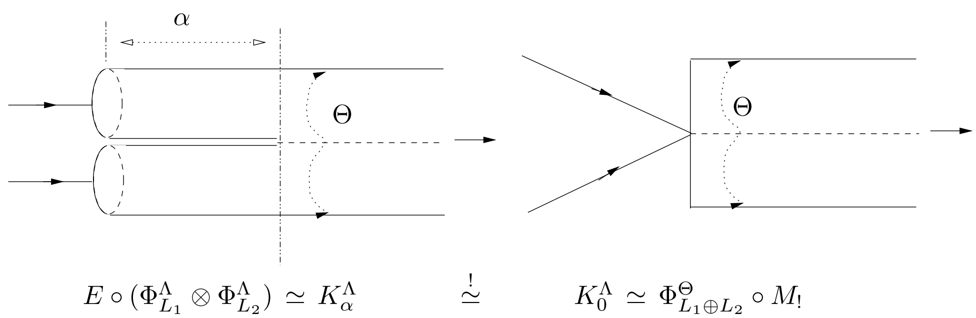

Figure 5: The homotopy through K^{Λ}_α and K^{Λ}_0.

There remains to prove that $K^{\Lambda}_{\alpha_0}$ is homotopic to K^{Λ}_0. Constructing a homotopy between these chain maps by using the spaces of solutions $\mathscr{M}^K_\alpha$ for $\alpha \in [0, \alpha_0]$ presents analytical difficulties: if we are given a solution u of the limiting problem $\mathscr{M}^K_0$, the existence of a (unique) one-parameter family of solutions "converging" to u is problematic, because we do not expect u to be C^0 close to the one-parameter family of solutions, due to the jump in the boundary conditions.

Therefore, we use a detour, starting from the following algebraic observation. If two chain maps $\varphi, \psi : C \to C'$ are homotopic, so are their tensor products $\varphi \otimes \psi$ and $\psi \otimes \varphi$. The converse is obviously not true, as the example of $\varphi = 0$ and ψ non-contractible shows. However, it becomes true under suitable conditions on φ and ψ. Denote by $(\mathbb{Z}, 0)$ the graded group which vanishes at every degree, except for degree zero where it coincides with $\mathbb{Z}$. We see $(\mathbb{Z}, 0)$ as a chain complex with the trivial boundary operator. We have the following:

4.6. LEMMA. *Let (C, ∂) and (C', ∂) be chain complexes, bounded from below. Let $\varphi, \psi : C \to C'$ be chain maps. Assume that there is an element $\epsilon \in C_0$ with $\partial \epsilon = 0$ and a chain map $\delta : C' \to (\mathbb{Z}, 0)$ such that*

$$\delta(\varphi(\epsilon)) = \delta(\psi(\epsilon)) = 1.$$

If $\varphi \otimes \psi$ is homotopic to $\psi \otimes \varphi$ then φ is homotopic to ψ

Proof. Let π be the chain map

$$\pi : C' \otimes C' \to C' \otimes (\mathbb{Z}, 0) \cong C', \quad \pi = \mathrm{id} \otimes \delta.$$

Let $H : C \otimes C \to C' \otimes C'$ be a chain homotopy between $\varphi \otimes \psi$ and $\psi \otimes \varphi$, that is

$$\varphi \otimes \psi - \psi \otimes \varphi = \partial H + H \partial.$$

If we define the homomorphism $h : C \to C'$ by

$$h(a) := \pi \circ H(a \otimes \epsilon), \quad \forall a \in C,$$

we have

$$\begin{aligned} \partial h(a) + h \partial a &= \partial \pi (H(a \otimes \epsilon)) + \pi(H \partial (a \otimes \epsilon)) = \pi(\partial H(a \otimes \epsilon) + H \partial (a \otimes \epsilon)) \\ = \pi(\varphi(a) \otimes \psi(\epsilon) - \psi(a) \otimes \varphi(\epsilon)) &= \varphi(a) \otimes \delta(\psi(\epsilon)) - \psi(a) \otimes \delta(\varphi(\epsilon)) = \varphi(a) - \psi(a). \end{aligned}$$

Hence h is the required chain homotopy. $\square$

We shall apply the above lemma to the complexes

$$C_k = \big(M(\mathbb{S}^\Lambda_{L_1}, g_1) \otimes M(\mathbb{S}^\Lambda_{L_2}, g_2)\big)_{k+n}, \quad C'_k = F^\Theta_k(H_1 \oplus H_2, J_1 \oplus J_2),$$

and to the chain maps K^Λ_0 and $K^\Lambda_{\alpha_0}$. The tensor products $K^\Lambda_0 \otimes K^\Lambda_{\alpha_0}$ and $K^\Lambda_{\alpha_0} \otimes K^\Lambda_0$ are represented by the coupling - in two different orders - of the corresponding elliptic boundary value problems.

4.7. Proposition. *The chain maps $K^\Lambda_0 \otimes K^\Lambda_{\alpha_0}$ and $K^\Lambda_{\alpha_0} \otimes K^\Lambda_0$ are homotopic.*

Constructing a homotopy between the coupled problems is easier than dealing with the original ones: we can keep α_0 fixed and rotate the boundary condition on the initial part of the half-strip. This argument is similar to an alternative way, due to Hofer, to prove the gluing statements in standard Floer homology. Details of the proof of Proposition 4.7 are contained in section 6.5 below.

Here we just construct the cycle ϵ and the chain map δ required in Lemma 4.6. Since changing the Lagrangians L_1 and L_2 (and the corresponding Hamiltonian) changes the chain maps appearing in diagram (53) by a chain homotopy, we are free to choose the Lagrangians so to make the construction easier.

We consider a Lagrangian of the form

$$L_1(t,q,v) := \frac{1}{2}\langle v, v\rangle - V_1(t,q),$$

where the potential $V_1 \in C^\infty(\mathbb{T} \times M)$ satisfies

$$V_1(t,q) < V_1(t,q_0) = 0, \quad \forall t \in \mathbb{T},\ \forall q \in M \setminus \{q_0\}, \tag{58}$$

$$\operatorname{Hess} V_1(t,q_0) < 0, \quad \forall t \in \mathbb{T}. \tag{59}$$

The corresponding Euler-Lagrange equation is

$$\nabla_t \gamma'(t) = -\operatorname{grad} V_1(t,\gamma(t)), \tag{60}$$

where ∇_t denotes the covariant derivative along the curve γ. By (58) and (59), the constant curve q_0 is a non-degenerate minimizer for the action functional $\mathbb{S}^\Lambda_{L_1}$ on the free loop space (actually, it is the unique global minimizer), so

$$m^\Lambda(q_0, L_1) = 0.$$

Notice also that the equilibrium point q_0 is hyperbolic and unstable. We claim that there exists $\omega > 0$ such that

$$\text{every solution } \gamma \text{ of (60) such that } \gamma(0)=\gamma(1), \text{ other than } \gamma(t)\equiv q_0, \text{ satisfies } \mathbb{S}_{L_1}(\gamma) \geq \omega. \tag{61}$$

Assuming the contrary, there exists a sequence (γ_h) of solutions of (60) with $\gamma_h(0) = \gamma_h(1)$ and $0 < \mathbb{S}_{L_1}(\gamma_h) \to 0$. The space of solutions of (60) with action bounded from above is compact - for istance in $C^\infty([0,1], M)$ - so a subsequence of (γ_h) converges to a solution of (60) with zero action. Since q_0 is the only solution with zero action, we find non-constant solutions γ of (60) with $\gamma(0) = \gamma(1)$ in any C^∞-neighborhood of the constant curve q_0. But this is impossible: the fact that the local stable and unstable manifolds of the hyperbolic equilibrium point $(q_0, 0) \in T^*M$ are transverse to the vertical foliation $\big\{T^*_q M \mid q \in M\big\}$ easily implies that if (x_h) is a sequence in the phase space T^*M tending to $(q_0, 0)$ such that the Hamiltonian orbit of x_h at time T_h is on the leaf $T^*_{\pi(x_h)}M$ containing x_h, then the sequence (T_h) must diverge.

A generic choice of the potential V_1 satisfying (58) and (59) produces a Lagrangian L_1 whose associated action functional is Morse on $\Lambda^1(M)$.

Next we consider an autonomous Lagrangian of the form

$$\tilde{L}_2(q,v) := \frac{1}{2}\langle v, v\rangle - V_2(q),$$

where

(i) V_2 is a smooth Morse function on M;

(ii) $0 = V_2(q_0) < V_2(q) < \omega/2$ for every $q \in M \setminus \{q_0\}$;

(iii) V_2 has no local minimizers other than q_0;

(iv) $\|V_2\|_{C^2(M)} < \epsilon$.

Here ϵ is a small positive constant, whose size is to be specified. The critical points of V_2 are equilibrium solutions of the Euler-Lagrange equation associated to $\tilde{L}_2$. The second differential of the action at such an equilibrium solution q is

$$d^2\mathbb{S}^{\Lambda}_{\tilde{L}_2}(q)[\xi,\xi] = \int_0^1 \Big(\langle \xi'(t), \xi'(t) \rangle - \langle \operatorname{Hess} V_2(q)\, \xi(t), \xi(t) \rangle \Big)\, dt.$$

If $0 < \epsilon < 2\pi$, (iv) implies that q is non-degenerate critical point of $\mathbb{S}_{\tilde{L}_2}$ with Morse index

$$m^{\Lambda}(q, \tilde{L}_2) = n - m(q, V_2),$$

a maximal negative subspace being the space of constant vector fields at q taking values into the positive eigenspace of $\operatorname{Hess} V_2(q)$.

The infimum of the energy

$$\frac{1}{2} \int_0^1 |\gamma'(t)|^2\, dt$$

over all non-constant closed geodesics is positive. It follows that if ϵ in (iv) is small enough,

$$\inf \Big\{ \mathbb{S}_{\tilde{L}_2}(\gamma) \mid \gamma \in \mathscr{P}^{\Lambda}(\tilde{L}_2),\ \gamma \text{ non-constant} \Big\} > 0. \tag{62}$$

Since the Lagrangian $\tilde{L}_2$ is autonomous, non-constant periodic orbits cannot be non-degenerate critical points of $\mathbb{S}_{\tilde{L}_2}$. It $W \in C^{\infty}(\mathbb{T} \times M)$ is a generic C^2-small time-dependent potential satisfying:

(v) $0 \leq W(t,q) < \omega/2$ for every $(t,q) \in \mathbb{T} \times M$;

(vi) $W(t,q) = 0$, $\operatorname{grad} W(t,q) = 0$, $\operatorname{Hess} W(t,q) = 0$ for every $t \in \mathbb{T}$ and every critical point q of V_1;

then the action functional associated to the Lagrangian

$$L_2(t,q,v) := \frac{1}{2}\langle v, v \rangle - V_2(q) - W(t,q),$$

is Morse on $\Lambda^1(M)$. By (vi), the critical points of V_1 are still equilibrium solutions of the Euler-Lagrange equation

$$\nabla_t \gamma'(t) = -\operatorname{grad} \big(V_2(t, \gamma(t)) + W(t, \gamma(t)) \big), \tag{63}$$

and

$$m^{\Lambda}(q, L_2) = n - m(q, V_2) \quad \forall q \in \operatorname{crit} V_2. \tag{64}$$

Moreover, (62) implies that if the C^2 norm of W is small enough, then

$$\inf \big\{ \mathbb{S}_{L_2}(\gamma) \mid \gamma \in \mathscr{P}^{\Lambda}(L_2),\ \gamma \text{ non-constant} \big\} > 0. \tag{65}$$

Up to a generic perturbation of the potential W, we may also assume that the equilibrium solution q_0 is the only 1-periodic solution of (63) with $\gamma(0) = \gamma(1) = q_0$ (generically, the set of periodic orbits is discrete, and so is the set of their initial points).

Since the inclusion $c : M \hookrightarrow \Lambda^1(M)$ induces an injective homomorphism between the singular homology groups, the image $c_*([M])$ of the fundamental class of the oriented manifold M does not vanish in $\Lambda^1(M)$. By (ii) and (v), the action $\mathbb{S}_{L_2}$ of every constant curve in M does not exceed 0. So we can regard $c_*([M])$ as a non vanishing element of the homology of the sublevel $\{\mathbb{S}_{L_2} \leq 0\}$. The singular homology of $\{\mathbb{S}_{L_2} \leq 0\}$ is isomorphic to the homology of the subcomplex of the Morse complex $M_*(\mathbb{S}^{\Lambda}_{L_2}, g_2)$ generated by the critical points of $\mathbb{S}^{\Lambda}_{L_2}$ whose action does not exceed 0. By (65), these critical points are the equilibrium solutions q, with $q \in \mathrm{crit} V_2$. By (ii), (iii), and (64), the only critical point of index n in this sublevel is q_0. It follows that the Morse homological counterpart of $c_*([M])$ is $\pm q_0$. In particular, $q_0 \in M_n(\mathbb{S}_{L_2})$ is a cycle. Since $\mathbb{S}^{\Lambda}_{L_2}(q_0) = 0$,

$$\mathbb{S}^{\Lambda}_{L_2}(\gamma) \leq 0, \quad \forall \gamma \in W^u(q_0, -\mathrm{grad}\, \mathbb{S}^{\Lambda}_{L_2}). \tag{66}$$

We now regard the pair (q_0, q_0) as an element of $\mathscr{P}^{\Theta}(L_1 \oplus L_2)$. We claim that if ϵ is small enough, (iv) implies that (q_0, q_0) is a non-degenerate minimizer for $\mathbb{S}_{L_1 \oplus L_2}$ on the space of figure-8 loops $\Theta^1(M)$. The second differential of $\mathbb{S}^{\Theta}_{L_1 \oplus L_2}$ at (q_0, q_0) is the quadratic form

$$d^2\mathbb{S}^{\Theta}_{L_1 \oplus L_2}(q_0, q_0)[(\xi_1, \xi_2)]^2 = \int_0^1 \Big(\langle \xi_1', \xi_1' \rangle - \langle \mathrm{Hess}\, V_1(t, q_0)\, \xi_1, \xi_1 \rangle + \langle \xi_2', \xi_2' \rangle - \langle \mathrm{Hess}\, V_2(q_0)\, \xi_2, \xi_2 \rangle \Big)\, dt,$$

on the space of curves (ξ_1, ξ_2) in the Sobolev space $W^{1,2}(]0,1[, T_{q_0}M \times T_{q_0}M)$ satisfying the boundary conditions

$$\xi_1(0) = \xi_1(1) = \xi_2(0) = \xi_2(1).$$

By (59) we can find $\alpha > 0$ such that that

$$\mathrm{Hess}\, V_1(t, q_0) \leq -\alpha I.$$

By comparison, it is enough to show that the quadratic form

$$Q_\epsilon(u_1, u_2) := \int_0^1 (u_1'(t)^2 + \alpha u_1(t)^2 + u_2'(t)^2 - \epsilon u_2(t)^2)\, dt$$

is coercive on the space

$$\left\{(u_1, u_2) \in W^{1,2}(]0,1[, \mathbb{R}^2) \mid u_1(0) = u_1(1) = u_2(0) = u_2(1)\right\}.$$

When $\epsilon = 0$, the quadratic form Q_0 is non-negative. An isotropic element (u_1, u_2) for Q_0 would solve the boundary value problem

$$\begin{aligned} -u_1''(t) + \alpha u_1(t) &= 0, && (67)\\ -u_2''(t) &= 0, && (68)\\ u_1(0) = u_1(1) &= u_2(0) = u_2(1), && (69)\\ u_1'(1) - u_1'(0) &= u_2'(0) - u_2'(1). && (70) \end{aligned}$$

By (68) u_2 is constant, so by (69) and (70) u_1 is a periodic solution of (67). Since α is positive, u_1 is zero and by (69) so is u_2. Since the bounded self-adjoint operator associated to Q_0 is Fredholm, we deduce that Q_0 is coercive. By continuity, Q_ϵ remains coercive for ϵ small. This proves our claim.

Let H_1 and H_2 be the Hamiltonians which are Fenchel dual to L_1 and L_2. In order to simplify the notation, let us denote by (q_0, q_0) also the constant curve in T^*M^2 identically equal to $((q_0, 0), (q_0, 0))$. Then (q_0, q_0) is a non-degenerate element of $\mathscr{P}^{\Theta}(H_1 \oplus H_2)$, and it has Maslov index

$$\mu^{\Theta}(q_0, q_0) = 0.$$

Let x be an element in $\mathscr{P}^\Theta(H_1 \oplus H_2)$, and let γ be its projection onto $M \times M$. By the definition of the Euler-Lagrange problem for figure-8 loops (see equation (19)), γ_1 is a solution of (60), γ_2 is a solution of (63), and

$$\gamma_1(0) = \gamma_1(1) = \gamma_2(0) = \gamma_2(1), \quad \gamma_2'(1) - \gamma_2'(0) = \gamma_1'(0) - \gamma_1'(1). \tag{71}$$

If γ_1 is the constant orbit q_0, then (71) implies that γ_2 is a 1-periodic solution of (63) such that $\gamma_2(0) = \gamma_2(1) = q_0$, and we have assumed the only curve with these properties is $\gamma_2 \equiv q_0$. If γ_1 is not the constant orbit q_0, (61) implies that

$$\mathbb{S}_{L_1}(\gamma_1) \geq \omega.$$

By (ii) and (v), the infimum of $\mathbb{S}_{L_2}$ is larger than $-\omega$, so we deduce that

$$\mathbb{A}_{H_1 \oplus H_2}(x) = \mathbb{A}_{H_1}(x_1) + \mathbb{A}_{H_2}(x_2) = \mathbb{S}_{L_1}(\gamma_1) + \mathbb{S}_{L_2}(\gamma_2) > 0, \quad \forall x \in \mathscr{P}^\Theta(H_1 \oplus H_2) \setminus \{(q_0, q_0)\}. \tag{72}$$

Now we choose ϵ in the n-th degree component of the chain complex $M(\mathbb{S}^\Lambda_{L_1}, g_1) \otimes M(\mathbb{S}^\Lambda_{L_2}, g_2)$ to be the cycle

$$\epsilon = q_0 \otimes q_0 \in M_0(\mathbb{S}^\Lambda_{L_1}) \otimes M_n(\mathbb{S}^\Lambda_{L_2}).$$

We choose the chain map

$$\delta : F^\Theta(H_1 \oplus H_2, J) \to (\mathbb{Z}, 0)$$

to be the augmentation, that is the homomorphism mapping every 0-degree generator $x \in \mathscr{P}^\Theta_0(H_1 \oplus H_2)$ into 1. We must show that

$$\delta(K^\Lambda_0(q_0 \otimes q_0)) = \delta(K^\Lambda_{\alpha_0}(q_0 \otimes q_0)) = 1. \tag{73}$$

Let $x \in \mathscr{P}^\Theta(H_1 \oplus H_2)$, and let u be an element of either

$$\mathscr{M}^K_0(q_0, q_0; x) \quad \text{or} \quad \mathscr{M}^K_{\alpha_0}(q_0, q_0; x).$$

By the boundary condition (54), the curve $u(0, \cdot)$ projects onto a closed curve in $M \times M$ whose first component is the constant q_0 and whose second component is in the unstable manifold of q_0 with respect to the negative gradient flow of $\mathbb{S}^\Lambda_{L_2}$. By the fundamental inequality (49) between the Hamiltonian and the Lagrangian action and by (66) we have

$$\mathbb{A}_{H_1 \oplus H_2}(x) \leq \mathbb{A}_{H_1 \oplus H_2}(u(0, \cdot)) \leq \mathbb{S}_{L_1 \oplus L_2}(\pi \circ u(0, \cdot)) = \mathbb{S}^\Lambda_{L_1}(q_0) + \mathbb{S}^\Lambda_{L_2}(\pi \circ u_2(0, \cdot)) \leq 0.$$

By (72), x must be the constant curve (q_0, q_0), and all the inequalities in the above estimate are equalities. It follows that u is constant, $u(s, t) \equiv (q_0, q_0)$.

Therefore, the spaces $\mathscr{M}^K_0(q_0, q_0; x)$ and $\mathscr{M}^K_{\alpha_0}(q_0, q_0; x)$ are non-empty if and only if $x = (q_0, q_0)$, and in the latter situation they consist of the unique constant solution $u \equiv (q_0, q_0)$. Automatic transversality holds for such solutions (see [AS06b, Proposition 3.7]), so such a picture survives to the generic perturbations which are necessary to achieve a Morse-Smale situation. Taking also orientations into account, it follows that

$$K^\Lambda_0(q_0 \otimes q_0) = (q_0, q_0), \quad K^\Lambda_\alpha(q_0 \otimes q_0) = (q_0, q_0).$$

In particular, (73) holds.

4.5 The right-hand square is homotopy commutative

In this section we prove that the chain maps $\Phi^{\Lambda}_{L_1 \# L_2} \circ M_{\Gamma}$ and $G \circ \Phi^{\Theta}_{L_1 \oplus L_2}$ are both homotopic to a third chain map, named K^{Θ}. This fact implies that the right-hand square in the diagram (53) commutes up to chain homotopy.

The chain map K^{Θ} is defined by using the following spaces of solutions of the Floer equation on the half-cylinder for the Hamiltonian $H_1 \# H_2$: given $\gamma \in \mathscr{P}^{\Theta}(L_1 \oplus L_2)$ and $x \in \mathscr{P}^{\Lambda}(H_1 \# H_2)$, set

$$\mathscr{M}^{\Theta}_K(\gamma; x) := \Big\{ u \in C^{\infty}([0, +\infty[\times \mathbb{T}, T^*M) \,\Big|\, \overline{\partial}_{J, H_1 \# H_2}(u) = 0,$$
$$\pi \circ u(0, \cdot) \in \Gamma\big(W^u(\gamma; -\mathrm{grad}_{g^{\Theta}} \mathbb{S}^{\Theta}_{L_1 \oplus L_2})\big), \ \lim_{s \to +\infty} u(s, \cdot) = x \text{ uniformly in } t \Big\}.$$

By Theorem 3.2 in [AS06b] (or by the arguments of section 5.10), the space $\mathscr{M}^{\Theta}_K(\gamma; x)$ is a smooth manifold of dimension

$$\dim \mathscr{M}^{\Theta}_K(\gamma; x) = m^{\Theta}(\gamma) - \mu^{\Lambda}(x),$$

for a generic choice of g^{Θ}, H_1, and H_2. These manifolds carry coherent orientations.

Compactness follows from the energy estimate

$$\int_{]0, +\infty[\times \mathbb{T}} |\partial_s u(s,t)|^2 \, ds dt \leq \mathbb{S}_{L_1 \oplus L_2}(\gamma) - \mathbb{A}_{H_1 \# H_2}(x),$$

implied by (49). Counting the elements of the zero-dimensional spaces, we define a homomorphism

$$K^{\Theta} : M_j(\mathbb{S}^{\Theta}_{L_1 \oplus L_2}, g^{\Theta}) \to F^{\Lambda}_j(H_1 \# H_2, J),$$

which is shown to be a chain map.

It is easy to construct a chain homotopy P^{Γ}_K between $\Phi^{\Lambda}_{L_1 \# L_2} \circ M_{\Gamma}$ and K^{Θ} by considering the space

$$\mathscr{M}^{\Gamma}_K(\gamma; x) := \Big\{ (\alpha, u) \in]0, +\infty[\times C^{\infty}([0, +\infty[\times \mathbb{T}, T^*M) \,\Big|\, \overline{\partial}_{J, H_1 \# H_2}(u) = 0,$$
$$\phi^{\Lambda}_{-\alpha}(\pi \circ u(0, \cdot)) \in \Gamma\big(W^u(\gamma; -\mathrm{grad}_{g^{\Theta}} \mathbb{S}^{\Theta}_{L_1 \oplus L_2})\big), \ \lim_{s \to +\infty} u(s, \cdot) = x \text{ uniformly in } t \Big\}.$$

where ϕ^{Λ}_s denotes the flow of $-\mathrm{grad}\, \mathbb{S}^{\Lambda}_{L_1 \# L_2}$ on $\Lambda^1(M)$. As before, we find that generically $\mathscr{M}^{\Gamma}_K(\gamma_1, \gamma_2; x)$ is a manifold of dimension

$$\dim \mathscr{M}^{\Gamma}_K(\gamma; x) = m^{\Lambda}(\gamma) - \mu^{\Theta}(x) + 1.$$

Compactness holds, so an algebraic count of the zero-dimensional spaces produces the homomorphism

$$P^{\Gamma}_K : M_j(\mathbb{S}^{\Theta}_{L_1 \oplus L_2}, g^{\Theta}) \to F^{\Lambda}_{j+1}(H_1 \# H_2, J).$$

A standard gluing argument shows that P^{Γ}_K is the required homotopy.

Finally, the construction of the chain homotopy P^K_G between K^{Θ} and $G \circ \Phi^{\Theta}_{L_1 \oplus L_2}$ is based on the one-parameter family of Riemann surfaces $\Sigma^K_G(\alpha)$, $\alpha > 0$, defined as the quotient of the disjoint union $[0, +\infty[\times [-1, 0] \sqcup [0, +\infty[\times [0, 1]$ under the identifications

$$(s, 0^-) \sim (s, 0^+) \text{ and } (s, -1) \sim (s, 1) \quad \text{for } s \geq \alpha.$$

This object is a Riemann surface with boundary, the holomorphic structure at $(\alpha, 0)$ being given by the map

$$\{\zeta \in \mathbb{C} \mid \mathrm{Re}\, \zeta \geq 0, \ |\zeta| < \epsilon\} \to \Sigma^K_G(\alpha), \quad \zeta \mapsto \alpha + \zeta^2,$$

and the holomorphic structure at $(\alpha,-1)\sim(\alpha,1)$ being given by the map

$$\{\zeta\in\mathbb{C}\mid \operatorname{Re}\zeta\geq 0,\ |\zeta|<\epsilon\}\to\Sigma_G^K(\alpha),\quad \zeta\mapsto\left\{\begin{array}{ll}\alpha-i+\zeta^2 & \text{if } \operatorname{Im}\zeta\geq 0,\\ \alpha+i+\zeta^2 & \text{if } \operatorname{Im}\zeta\leq 0.\end{array}\right.$$

Here ϵ is a positive number smaller than 1 and $\sqrt{\alpha}$.

Given $\gamma\in\mathscr{P}^\Theta(L_1\oplus L_2)$ and $x\in\mathscr{P}^\Lambda(H_1\#H_2)$, we consider the space $\mathscr{M}_G^K(\gamma,x)$ of pairs (α,u) where α is a positive number and $u\in C^\infty(\Sigma_G^K(\alpha),T^*M)$ solves the equation

$$\overline{\partial}_{J,H_1\#H_2}(u)=0,$$

satisfies the boundary conditions

$$\left\{\begin{array}{l}\pi\circ u(s,-1)=\pi\circ u(s,0^-)=\pi\circ u(s,0^+)=\pi\circ u(s,1),\\ u(s,0^-)-u(s,-1)+u(s,1)-u(s,0^+)=0,\end{array}\right.\qquad\forall s\in[0,\alpha],$$
$$(\pi\circ u(0,\cdot-1),\pi\circ u(0,\cdot))\in W^u(\gamma,-\operatorname{grad}\mathbb{S}^\Theta_{L_1\oplus L_2}),$$

and the asymptotic condition

$$\lim_{s\to+\infty}u(s,2t-1)=x(t),$$

uniformly in $t\in[0,1]$. The following result is proved in section 5.10.

4.8. Proposition. *For a generic choice of J_1, J_2, and g^Θ, $\mathscr{M}_G^K(\gamma,x)$ - if non-empty - is a smooth manifold of dimension*

$$\dim\mathscr{M}_G^K(\gamma,x)=m^\Theta(\gamma;L_1\oplus L_2)-\mu^\Lambda(x;H_1\#H_2)+1.$$

The projection $(\alpha,u)\mapsto\alpha$ is smooth on $\mathscr{M}_G^K(\gamma,x)$. These manifolds carry coherent orientations.

The elements (α,u) of $\mathscr{M}_G^K(\gamma,x)$ satisfy the energy estimate

$$\int_{\mathbb{R}\times]-1,1[\setminus\{0\}\times[0,\alpha]}|\partial_s u(s,t)|^2\,ds\,dt\leq\mathbb{S}_{L_1\oplus L_2}(\gamma)-\mathbb{A}_{H_1\#H_2}(x).$$

This provides us with the compactness which is necessary to define the homomorphism

$$P_G^K:M_j(\mathbb{S}^\Theta_{L_1\oplus L_2},g^\Theta)\longrightarrow F^\Lambda_{j+1}(H_1\#H_2,J),$$

by the usual counting procedure applied to the spaces $\mathscr{M}_G^K$. A standard gluing argument shows that P_G^K is a chain homotopy between K^Θ and $G\circ\Phi^\Theta_{L_1\oplus L_2}$.

4.6 Comparison between C, EV, $I_!$ and c, ev, $i_!$

In section 2.5 we have shown that the two upper squares in the diagram

$$\begin{tikzcd}
H_j(M) \arrow[r, "c_*"] \arrow[d, "\cong"'] & H_j(\Lambda(M)) \arrow[r, "\mathrm{ev}_*"] \arrow[d, "\cong"] & H_j(M) \arrow[d, "\cong"] \\
H_jM(f,g_M) \arrow[r, "HMc"] \arrow[dr, "HC"'] & H_jM(\mathbb{S}^\Lambda_L,g^\Lambda) \arrow[r, "HM\mathrm{ev}"] \arrow[d, "H\Phi^\Lambda_L"] & H_jM(f,g_M) \\
& H_jF^\Lambda(H,J) \arrow[ur, "H\mathrm{Ev}"'] &
\end{tikzcd}$$

commute. Our first aim in this section is to show that the lower two triangles commute. We actually work at the chain level, showing that the triangles

$$\begin{tikzcd} M_j(f,g_M) \arrow[r, "Mc"] \arrow[dr, "C"'] & M_j(\mathbb{S}_L^\Lambda, g^\Lambda) \arrow[r, "M\mathrm{ev}"] \arrow[d, "\Phi_L^\Lambda"] & M_j(f,g_M) \\ & F_j^\Lambda(H,J) \arrow[ur, "\mathrm{Ev}"'] & \end{tikzcd}$$

commute up to chain homotopies.

Indeed, a homotopy P^C between C and $\Phi_L^\Lambda \circ Mc$ is defined by the following spaces: given $x \in \mathrm{crit}(f)$ and $y \in \mathscr{P}^\Lambda(H)$, set

$$\mathscr{M}_P^C(x,y) := \Big\{ (\alpha,u) \in]0,+\infty[\times C^\infty([0,+\infty[\times \mathbb{T}, T^*M) \,\Big|\, \overline{\partial}_{J,H}(u) = 0, \\ \phi^\Lambda_{-\alpha}(\pi \circ u(0,\cdot)) \equiv q \in W^u(x) \Big\},$$

where ϕ^Λ is the flow of $-\mathrm{grad}\, \mathbb{S}_L^\Lambda$ on $\Lambda^1(M)$.

Similarly, the definition of the homotopy P^{Ev} between $\mathrm{Ev} \circ \Phi_L^\Lambda$ and $M\mathrm{ev}$ is obtained from the composition of 3 homotopies based on the following spaces: Given $\gamma \in \mathscr{P}^\Lambda(L)$ and $x \in \mathrm{crit}(f)$, set

$$\begin{aligned} \mathscr{M}_{P_1}^{\mathrm{Ev}}(\gamma,x) := \Big\{ (\alpha,u) \,\Big|\, & \alpha \in [1,+\infty[,\ u \in C^\infty([0,\alpha]\times\mathbb{T}, T^*M) \\ & \text{solves } \overline{\partial}_{J,H}(u) = 0,\ u(\alpha,t) \in \mathbb{O}_M\ \forall t \in \mathbb{T}, \\ & u(\alpha,0) \in W^s(x), \\ & \pi \circ u(0,\cdot) \in W^u(\gamma; -\mathrm{grad}\, \mathbb{S}_L^\Lambda) \Big\}, \\ \mathscr{M}_{P_2}^{\mathrm{Ev}}(\gamma,x) := \Big\{ (\alpha,u) \,\Big|\, & \alpha \in [0,1],\ u \in C^\infty([0,1]\times\mathbb{T}, T^*M) \\ & \text{solves } \overline{\partial}_{J,H}(u) = 0,\ u(1,t) \in \mathbb{O}_M\ \forall t \in \mathbb{T}, \\ & u(\alpha,0) \in W^s(x), \\ & \pi \circ u(0,\cdot) \in W^u(\gamma; -\mathrm{grad}\, \mathbb{S}_L^\Lambda) \Big\}, \quad \text{and} \\ \mathscr{M}_{P_3}^{\mathrm{Ev}}(\gamma,x) := \Big\{ (\alpha,u) \,\Big|\, & \alpha \in]0,1],\ u \in C^\infty([0,\alpha]\times\mathbb{T}, T^*M) \\ & \text{solves } \overline{\partial}_{J,H}(u) = 0,\ u(\alpha,t) \in \mathbb{O}_M\ \forall t \in \mathbb{T}, \\ & u(0,0) \in W^s(x), \\ & \pi \circ u(0,\cdot) \in W^u(\gamma; -\mathrm{grad}\, \mathbb{S}_L^\Lambda) \Big\}. \end{aligned}$$

Moreover, recalling the following space on which $M\mathrm{ev}$ is based,

$$\mathscr{M}_{M\mathrm{ev}}(\gamma,x) = W^u(\gamma; -\mathrm{grad}\, \mathbb{S}_L^\Lambda) \cap \mathrm{ev}^{-1}\big(W^s(x; -\mathrm{grad}\, f)\big),$$

we make the following observation.

4.9. Proposition. *Given the finite set $\mathscr{M}_{M\mathrm{ev}}(\gamma,x)$, there exists $\alpha_o > 0$ such that for each $c \in \mathscr{M}_{M\mathrm{ev}}(\gamma,x)$ and $\alpha \in (0,\alpha_o]$ the problem*

$$\begin{aligned} & u \in [0,\alpha]\times\mathbb{T} \to T^*M, \quad \overline{\partial}_{J,H} u = 0, \\ & u(\alpha,t) \in \mathbb{O}_M\ \forall t \in \mathbb{T}, \quad \pi \circ u(0,\cdot) = c, \end{aligned} \tag{74}$$

has a unique solution with the same coherent orientation as c.

Proof. We give a sketch of the proof, details are left to the reader.

First, given a sequence $\alpha_n \to 0$ and associated solutions $u_n \colon [0,\alpha_n]\times\mathbb{T} \to T^*M$ of (74), one can show that $u_n \to (c,0) \in \Lambda T^*M$ uniformly. Here, it is again essential to make a case distinction for the three cases of potential gradient blow-up, $R_n = |\nabla u_n(z_n)| \to \infty$, namely, modulo subsequence, $\alpha_n \cdot R_n \to 0$ or $\to \infty$ or $\to k > 0$.

The most interesting case is $\alpha_n \cdot R_n \to k > 0$ which is dealt with by rescaling $v_n = u_n(\alpha_n\cdot, \alpha_n\cdot)$ as in the proof of Lemma 6.6.

For the converse, we need a Newton type method which is hard to implement for the shrinking domains $[0,\alpha]\times\mathbb{T}$ with $\alpha \to 0$. Instead, we consider the conformally rescaled equivalent problem. Let $v(s,t) = u(\alpha s, \alpha t)$ and consider the corresponding problem for $\alpha \to 0$,

$$\begin{aligned} &v\colon [0,1]\times\mathbb{T}_{\alpha^{-1}} \to T^*M, \quad \overline{\partial}_{J,H_\alpha} v = 0,\\ &\pi(v(0,t)) = c(\alpha t),\ v(1,t) \in \mathbb{O}_M\ \forall t \in \mathbb{T}_{\alpha^{-1}}, \end{aligned} \tag{75}$$

where $H_\alpha(t,\cdot) = \alpha H(\alpha t,\cdot)$ and $\mathbb{T}_{\alpha^{-1}} = \mathbb{R}/\alpha^{-1}\mathbb{Z}$. The proof is now based on the Newton method which requires to show that:

(a) for $v_o(s,t) = 0 \in T^*_{c(\alpha t)}M$ we have $\overline{\partial}_{J,H} v_o \to 0$ as $\alpha \to 0$, which is obvious, and

(b) the linearization D_α of $\overline{\partial}_{J,H_\alpha}$ at v_o is invertible for small $\alpha > 0$ with uniform bound on $\|D_\alpha^{-1}\|_{\mathsf{Op}}$ as $\alpha \to 0$.

We sketch now the proof of this uniform bound.

After suitable trivializations, the linearization D_α of $\overline{\partial}_{J,H_\alpha}$ at v_o with the above Lagrangian boundary conditions can be viewed as an operator D_α on

$$H^{1,p}_{i\mathbb{R}^n,\mathbb{R}^n}(\alpha) \;:=\; \big\{\, v\colon [0,1\times\mathbb{T}_{\alpha^{-1}} \to \mathbb{C}^n \,|\, v(0,\cdot) \in i\mathbb{R}^n,\ v(1,\cdot) \in \mathbb{R}^n \,\big\},$$

with norm $\|\cdot\|_{1,p;\alpha}$. Assuming that D_α^{-1} is not uniformly bounded as $\alpha \to 0$ means that we would have $\alpha_n \to 0$ and $v_n \in H^{1,p}_{i\mathbb{R}^n,\mathbb{R}^n}(\alpha_n)$ with $\|v_n\|_{1,p;\alpha_n} = 1$ such that $\|D_{\alpha_n} v_n\|_{0,p;\alpha_n} \to 0$. The limit operator to compare to is the standard $\overline{\partial}$-operator on maps

$$v\colon [0,1]\times\mathbb{R} \to \mathbb{C}^n, \text{ s.t. } \ v(0,t) \in i\mathbb{R}^n,\ v(1,t) \in \mathbb{R}^n\ \forall t \in \mathbb{R}\,.$$

This comparison operator is clearly an isomorphism (from which one easily shows that D_{α_n} has to be invertible for α_n small).

Let $\beta \in C^\infty(\mathbb{R},[0,1])$ be a cut-off function such that

$$\beta(t) = \begin{cases} 1, & t \le 0,\\ 0, & t \ge 1, \end{cases} \qquad \beta' \le 0,$$

and set $\beta_n(t) = \beta(\alpha_n t - 1)\cdot\beta(-\alpha_n t)$, hence

$$\beta_n{}_{|[0,\alpha_n^{-1}]} \equiv 1 \quad\text{and}\quad \operatorname{supp}\beta_n \subset [-\alpha_n^{-1}, 2\alpha_n^{-1}]\,.$$

We have

$$\|v_n\|_{1,p;\alpha_n} \;\le\; \|\beta_n v_n\|_{1,p;\mathbb{R}} \;\le\; c_1\|v_n\|_{1,p;\alpha_n}\,.$$

Since the linearization D_α is of the form

$$D_\alpha v \;=\; \partial_s v + i\partial_t v + \alpha A(s,t) v$$

with some matrix $A(s,t)$, we observe that

$$\|\overline{\partial}(\beta_n v_n) - D_{\alpha_n}(\beta_n v_n)\|_{0,p;\mathbb{R}} \;=\; \alpha_n\|A\beta_n v_n\|_{0,p;\mathbb{R}} \to 0\,.$$

Moreover,

$$\begin{aligned}\|D_{\alpha_n}(\beta_n v_n)\|_{0,p;\mathbb{R}} &\le \|i\beta_n' v_n\|_{0,p;\mathbb{R}} + \|\beta_n D_{\alpha_n} v_n\|_{0,p;\mathbb{R}} \\ &\le c_2\alpha_n \|v_n\|_{0,p;[-\alpha_n^{-1},0]\cup[\alpha_n^{-1},2\alpha_n^{-1}]} + 3\|D_{\alpha_n} v_n\|_{0,p;\alpha_n} \\ &\le c_2\alpha_n 2\|v_n\|_{1,p;\alpha_n} + 3\|D_{\alpha_n} v_n\|_{0,p;\alpha_n} \to 0\,.\end{aligned}$$

Hence, we find a subsequence such that $\beta_{n_k} v_{n_k} \to v_o \in \ker \overline{\partial} = \{0\}$ which means that $\|v_{n_k}\|_{1,p;\alpha_{n_k}} \to 0$ in contradiction to $\|v_n\| = 1$.

Similarly, we see that the coherent orientation for the determinant of D_{α_n} equals that of $\overline{\partial}$ which is canonically 1. This completes the proof of the proposition. □

From the cobordisms $\mathscr{M}^{\mathrm{Ev}}_{P_i}(\gamma, x)$, $i = 1, 2, 3$, we now obtain the chain homotopy as claimed.

Finally, there remains to prove that the right-hand square in the diagram

$$\begin{tikzcd} H_j(\Lambda(M)) \arrow[r, "\cong"] \arrow[d, "i_!"'] & H_j M(\mathbb{S}_L^\Lambda, g^\Lambda) \arrow[r, "H\Phi_L^\Lambda"] \arrow[d, "HM_{i_!}"] & H_j F(H,J) \arrow[d, "HI_!"] \\ H_{j-n}(\Omega(M,q_0)) \arrow[r, "\cong"] & H_{j-n} M(\mathbb{S}_L^\Omega, g^\Omega) \arrow[r, "H\Phi_L^\Omega"] & H_{j-n} F^\Omega(H,J) \end{tikzcd}$$

commutes, the commutativity of the left-hand square having been established in section 2.5. Again, we work at the chain level, proving that the diagram

$$\begin{tikzcd} M_j(\mathbb{S}_L^\Lambda, g^\Lambda) \arrow[r, "\Phi_L^\Lambda"] \arrow[d, "M_{i_!}"'] & F_j(H,J) \arrow[d, "I_!"] \\ M_{j-n}(\mathbb{S}_L^\Omega, g^\Omega) \arrow[r, "\Phi_L^\Omega"] & F^\Omega_{j-n}(H,J) \end{tikzcd}$$

is homotopy commutative. Indeed, we can show that both $I_! \circ \Phi_L^\Lambda$ and $\Phi_L^\Omega \circ M_{i_!}$ are homotopic to the same chain map $K^!$. The definition of $K^!$ makes use of the following spaces: given $\gamma \in \mathscr{P}^\Lambda(L)$ and $x \in \mathscr{P}^\Omega(H)$, set

$$\begin{aligned}\mathscr{M}_K^!(\gamma, x) := \Big\{ u \in C^\infty([0,+\infty[\times[0,1], T^*M) \,\Big|\, \overline{\partial}_{J,H}(u) = 0, \\ \pi\circ u(s,0) = \pi\circ u(s,1) = q_0 \ \forall s \ge 0,\ \pi\circ u(0,\cdot) \in W^u(\gamma; -\mathrm{grad}\,\mathbb{S}_L^\Lambda), \\ \lim_{s\to+\infty} u(s,\cdot) = x \text{ uniformly in } t \Big\}.\end{aligned}$$

Again, details are left to the reader.

5 Linear theory

5.1 The Maslov index

Let η_0 be the *Liouville one-form* on $T^*\mathbb{R}^n = \mathbb{R}^n \times (\mathbb{R}^n)^*$, that is the tautological one-form $\eta_0 = p\,dq$:

$$\eta_0(q,p)[(u,v)] = p[u], \quad \text{for } q, u \in \mathbb{R}^n,\ p, v \in (\mathbb{R}^n)^*.$$

Its differential $\omega_0 = d\eta_0 = dp \wedge dq$,

$$\omega_0[(q_1,p_1),(q_2,p_1)] = p_1[q_2] - p_2[q_1], \quad \text{for } q_1, q_2 \in \mathbb{R}^n,\ p_1, p_2 \in (\mathbb{R}^n)^*,$$

is the *standard symplectic form* on $T^*\mathbb{R}^n$.

The *symplectic group*, that is the group of linear automorphisms of $T^*\mathbb{R}^n$ preserving ω_0, is denoted by $\mathrm{Sp}(2n)$. Let $\mathscr{L}(n)$ be the Grassmannian of Lagrangian subspaces of $T^*\mathbb{R}^n$, that is the set of n-dimensional linear subspaces of $T^*\mathbb{R}^n$ on which ω_0 vanishes. The *relative Maslov index* assigns to every pair of Lagrangian paths $\lambda_1, \lambda_2 : [a,b] \to \mathscr{L}(n)$ a half integer $\mu(\lambda_1, \lambda_2)$. We refer to [RS93] for the definition and for the properties of the relative Maslov index.

Another useful invariant is the Hörmander index of four Lagrangian subspaces (see [Hör71], [Dui76], or [RS93]):

5.1. DEFINITION. *Let $\lambda_0, \lambda_1, \nu_0, \nu_1$ be four Lagrangian subspaces of $T^*\mathbb{R}^n$. Their* Hörmander index *is the half integer*

$$h(\lambda_0, \lambda_1; \nu_0, \nu_1) := \mu(\nu, \lambda_1) - \mu(\nu, \lambda_0),$$

where $\nu : [0,1] \to \mathscr{L}(n)$ is a Lagrangian path such that $\nu(0) = \nu_0$ and $\nu(1) = \nu_1$.

Indeed, the quantity defined above does not depend on the choice of the Lagrangian path ν joining ν_0 and ν_1.

If V is a linear subspace of $\mathbb{R}^n$, $N^*V \subset T^*\mathbb{R}^n$ denotes its *conormal space*, that is

$$N^*V := \{(q,p) \in \mathbb{R}^n \times (\mathbb{R}^n)^* \mid q \in V, \ V \subset \ker p\} = V \times V^\perp,$$

where $V^\perp$ denotes the set of covectors in $(\mathbb{R}^n)^*$ which vanish on V. Conormal spaces are Lagrangian subspaces of $T^*\mathbb{R}^n$.

Let $C : T^*\mathbb{R}^n \to T^*\mathbb{R}^n$ be the linear involution

$$C(q,p) := (q, -p) \quad \forall (q,p) \in T^*\mathbb{R}^n.$$

The involution C is anti-symplectic, meaning that

$$\omega_0(C\xi, C\eta) = -\omega_0(\xi, \eta) \quad \forall \xi, \eta \in T^*\mathbb{R}^n.$$

In particular, C maps Lagrangian subspaces into Lagrangian subspaces. Since the Maslov index is natural with respect to symplectic transformations and changes sign if we change the sign of the symplectic structure, we have the identity

$$\mu(C\lambda, C\nu) = -\mu(\lambda, \nu), \tag{76}$$

for every pair of Lagrangian paths $\lambda, \nu : [a,b] \to \mathscr{L}(n)$. Since conormal subspaces are C-invariant, we deduce that

$$\mu(N^*V, N^*W) = 0, \tag{77}$$

for every pair of paths V, W into the Grassmannian of $\mathbb{R}^n$. Let V_0, V_1, W_0, W_1 be four linear subspaces of $\mathbb{R}^n$, and let $\nu : [0,1] \to \mathscr{L}(n)$ be a Lagrangian path such that $\nu(0) = N^*W_0$ and $\nu(1) = N^*W_1$. By (76),

$$h(N^*V_0, N^*V_1; N^*W_0, N^*W_1) = \mu(\nu, N^*V_1) - \mu(\nu, N^*V_0) = -\mu(C\nu, N^*V_1) + \mu(C\nu, N^*V_0).$$

But also the Lagrangian path $C\nu$ joins N^*W_0 and N^*W_1, so the latter quantity equals

$$-h(N^*V_0, N^*V_1; N^*W_0, N^*W_1).$$

We deduce the following:

5.2. PROPOSITION. *Let V_0, V_1, W_0, W_1 be four linear subspaces of $\mathbb{R}^n$. Then*

$$h(N^*V_0, N^*V_1; N^*W_0, N^*W_1) = 0.$$

We identify the product $T^*\mathbb{R}^n \times T^*\mathbb{R}^n$ with $T^*\mathbb{R}^{2n}$, and we endow it with its standard symplectic structure. In other words, we consider the product symplectic form, not the twisted one used in [RS93]. Note that the conormal space of the diagonal $\Delta_{\mathbb{R}^n}$ in $\mathbb{R}^n \times \mathbb{R}^n$ is the graph of C,

$$N^*\Delta_{\mathbb{R}^n} = \operatorname{graph} C \subset T^*\mathbb{R}^n \times T^*\mathbb{R}^n = T^*\mathbb{R}^{2n}.$$

The linear endomorphism Ψ of $T^*\mathbb{R}^n$ belongs to the symplectic group $\mathrm{Sp}(2n)$ if and only if the graph of the linear endomorphism ΨC is a Lagrangian subspace of $T^*\mathbb{R}^{2n}$, if and only if the graph of $C\Psi$ is a Lagrangian subspace of $T^*\mathbb{R}^{2n}$. If λ_1, λ_2 are paths of Lagrangian subspaces of $T^*\mathbb{R}^n$ and Ψ is a path in $\mathrm{Sp}(2n)$, Theorem 3.2 of [RS93] leads to the identities

$$\mu(\Psi\lambda_1, \lambda_2) = \mu(\operatorname{graph}(\Psi C), C\lambda_1 \times \lambda_2) = -\mu(\operatorname{graph}(C\Psi), \lambda_1 \times C\lambda_2). \tag{78}$$

The *Conley-Zehnder index* $\mu_{CZ}(\Psi)$ of a symplectic path $\Psi : [0,1] \to \mathrm{Sp}(2n)$ is related to the relative Maslov index by the formula

$$\mu_{CZ}(\Psi) = \mu(N^*\Delta_{\mathbb{R}^n}, \operatorname{graph} C\Psi) = \mu(\operatorname{graph} \Psi C, N^*\Delta_{\mathbb{R}^n}). \tag{79}$$

We conclude this section by fixing some standard identifications, which allow to see $T^*\mathbb{R}^n$ as a complex vector space. By using the Euclidean inner product on $\mathbb{R}^n$, we can identify $T^*\mathbb{R}^n$ with $\mathbb{R}^{2n}$. We also identify the latter space to $\mathbb{C}^n$, by means of the isomorphism $(q,p) \mapsto q + ip$. In other words, we consider the complex structure

$$J_0 := \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

on $\mathbb{R}^{2n}$. With these identifications, the Euclidean inner product $u \cdot v$, respectively the symplectic product $\omega_0(u,v)$, of two vectors $u, v \in T^*\mathbb{R}^n \cong \mathbb{R}^{2n} \cong \mathbb{C}^n$ is the real part, respectively the imaginary part, of their Hermitian product $\langle \cdot, \cdot \rangle$,

$$\langle u, v \rangle := \sum_{j=1}^n u_j \overline{v_j} = u \cdot v + i\,\omega_0(u,v).$$

The involution C is the complex conjugacy. By identifying $V^\perp$ with the Euclidean orthogonal complement, we have

$$N^*V = V \oplus iV^\perp = \left\{ z \in \mathbb{C}^n \mid \operatorname{Re} z \in V,\ \operatorname{Im} z \in V^\perp \right\}.$$

If $\lambda : [0,1] \to \mathscr{L}(n)$ is the path

$$\lambda(t) = e^{i\alpha t}\mathbb{R}, \quad \alpha \in \mathbb{R},$$

the relative Maslov index of λ with respect to $\mathbb{R}$ is the half integer

$$\mu(\lambda, \mathbb{R}) = \begin{cases} -\frac{1}{2} - \lfloor \frac{\alpha}{\pi} \rfloor & \text{if } \alpha \in \mathbb{R} \setminus \pi\mathbb{Z}, \\ -\frac{\alpha}{\pi} & \text{if } \alpha \in \pi\mathbb{Z}. \end{cases} \tag{80}$$

Notice that the sign is different from the one appearing in [RS93] (localization axiom in Theorem 2.3), due to the fact that we are using the opposite symplectic form on $\mathbb{R}^{2n}$. Our sign convention here also differs from the one used in [AS06b], because we are using the opposite complex structure on $\mathbb{R}^{2n}$.

5.2 Elliptic estimates on the quadrant

We recall that a real linear subspace V of $\mathbb{C}^n$ is said *totally real* if $V \cap iV = (0)$. Denote by $\mathbb{H}$ the upper half-plane $\{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$, and by $\mathbb{H}^+$ the upper-right quadrant $\{z \in \mathbb{C} \mid \operatorname{Re} z > 0,\ \operatorname{Im} z > 0\}$. We shall make use of the following Calderon-Zygmund estimates for the Cauchy-Riemann operator $\overline{\partial} = \partial_s + i\partial_t$:

5.3. THEOREM. *Let V be an n-dimensional totally real subspace of $\mathbb{C}^n$. For every $p \in]1,+\infty[$, there exists a constant $c = c(p,n)$ such that*

$$\|Du\|_{L^p} \leq c\|\overline{\partial} u\|_{L^p}$$

for every $u \in C_c^\infty(\mathbb{C}, \mathbb{C}^n)$, and for every $u \in C_c^\infty(\mathrm{Cl}(\mathbb{H}), \mathbb{C}^n)$ such that $u(s) \in V$ for every $s \in \mathbb{R}$.

We shall also need the following regularity result for weak solutions of $\overline{\partial}$. Denoting by $\partial := \partial_s - i\partial_t$ the anti-Cauchy-Riemann operator, we have:

5.4. THEOREM. (Regularity of weak solutions of $\overline{\partial}$) *Let V be an n-dimensional totally real subspace of $\mathbb{C}^n$, and let $1 < p < \infty$, $k \in \mathbb{N}$.*

(i) Let $u \in L^p_{\mathrm{loc}}(\mathbb{C}, \mathbb{C}^n)$, $f \in W^{k,p}_{\mathrm{loc}}(\mathbb{C}, \mathbb{C}^n)$ be such that

$$\mathrm{Re} \int_{\mathbb{C}} \langle u, \partial\varphi \rangle \, dsdt = -\mathrm{Re} \int_{\mathbb{C}} \langle f, \varphi \rangle \, dsdt,$$

for every $\varphi \in C_c^\infty(\mathbb{C}, \mathbb{C}^n)$. Then $u \in W^{k+1,p}_{\mathrm{loc}}(\mathbb{C}, \mathbb{C}^n)$ and $\overline{\partial} u = f$.

(ii) Let $u \in L^p(\mathbb{H}, \mathbb{C}^n)$, $f \in W^{k,p}(\mathbb{H}, \mathbb{C}^n)$ be such that

$$\mathrm{Re} \int_{\mathbb{H}} \langle u, \partial\varphi \rangle \, dsdt = -\mathrm{Re} \int_{\mathbb{H}} \langle f, \varphi \rangle \, dsdt,$$

for every $\varphi \in C_c^\infty(\mathbb{C}, \mathbb{C}^n)$ such that $\varphi(\mathbb{R}) \subset V$. Then $u \in W^{k+1,p}(\mathbb{H}, \mathbb{C}^n)$, $\overline{\partial} u = f$, and the trace of u on $\mathbb{R}$ takes values into the ω_0-orthogonal complement of V,

$$V^{\perp_{\omega_0}} := \{\xi \in \mathbb{C}^n \mid \omega_0(\xi, \eta) = 0 \; \forall \eta \in V\}.$$

5.5. REMARK. *If we replace the upper half-plane $\mathbb{H}$ in (ii) by the right half-plane $\{\mathrm{Re}\, z > 0\}$ and the test mappings $\varphi \in C_c^\infty(\mathbb{C}, \mathbb{C}^n)$ satisfy $\varphi(i\mathbb{R}) \subset V$, then the trace of u on $i\mathbb{R}$ takes value into $V^\perp$, the Euclidean orthogonal complement of V in $\mathbb{R}^{2n}$.*

Two linear subspaces V, W of $\mathbb{R}^n$ are said to be *partially orthogonal* if the linear subspaces $V \cap (V \cap W)^\perp$ and $W \cap (V \cap W)^\perp$ are orthogonal, that is if their projections into the quotient $\mathbb{R}^n / V \cap W$ are orthogonal.

5.6. LEMMA. *Let V and W be partially orthogonal linear subspaces of $\mathbb{R}^n$. For every $p \in]1,+\infty[$, there exists a constant $c = c(p,n)$ such that*

$$\|Du\|_{L^p} \leq c\|\overline{\partial} u\|_{L^p} \tag{81}$$

for every $u \in C_c^\infty(\mathrm{Cl}(\mathbb{H}^+), \mathbb{C}^n)$ such that

$$u(s) \in N^*V \; \forall s \in [0,+\infty[, \quad u(it) \in N^*W \; \forall t \in [0,+\infty[. \tag{82}$$

Proof. Since V and W are partially orthogonal, $\mathbb{R}^n$ has an orthogonal splitting $\mathbb{R}^n = X_1 \oplus X_2 \oplus X_3 \oplus X_4$ such that

$$V = X_1 \oplus X_2, \quad W = X_1 \oplus X_3.$$

Therefore,

$$N^*V = X_1 \oplus X_2 \oplus iX_3 \oplus iX_4, \quad N^*W = X_1 \oplus X_3 \oplus iX_2 \oplus iX_4.$$

Let $U \in \mathrm{U}(n)$ be the identity on $(X_1 \oplus X_2) \otimes \mathbb{C}$, and the multiplication by i on $(X_3 \oplus X_4) \otimes \mathbb{C}$. Then

$$UN^*V = \mathbb{R}^n, \quad UN^*W = X_1 \oplus X_4 \oplus iX_2 \oplus iX_3 = N^*(X_1 \oplus X_4).$$

Up to multiplying u by U, we can replace the boundary conditions (82) by

$$u(s) \in \mathbb{R}^n \ \forall s \in [0,+\infty[, \quad u(it) \in Y \ \forall t \in [0,+\infty[, \tag{83}$$

where Y is a totally real n-dimensional subspace of $\mathbb{C}^n$ such that $\overline{Y} = Y$. Define a $\mathbb{C}^n$-valued map v on the right half-plane $\{\mathrm{Re}\, z \geq 0\}$ by Schwarz reflection,

$$v(z) := \begin{cases} u(z) & \text{if } \mathrm{Im}\, z \geq 0, \\ \overline{u(\overline{z})} & \text{if } \mathrm{Im}\, z \leq 0. \end{cases}$$

By (83) and by the fact that Y is self-conjugate, v belongs to $W^{1,p}(\{\mathrm{Re}\, z > 0\}, \mathbb{C}^n)$, and satisfies

$$v(it) \in Y \quad \forall t \in \mathbb{R}.$$

Moreover,

$$\|\nabla v\|^p_{L^p(\{\mathrm{Re}\, z>0\})} = 2\|\nabla v\|^p_{L^p(\{\mathrm{Re}\, z>0,\ \mathrm{Im}\, z>0\})},$$

and since $\overline{\partial} v(z) = \overline{\overline{\partial} u(\overline{z})}$ for $\mathrm{Im}\, z \leq 0$,

$$\|\overline{\partial} v\|^p_{L^p(\{\mathrm{Re}\, z>0\})} = 2\|\overline{\partial} v\|^p_{L^p(\{\mathrm{Re}\, z>0,\ \mathrm{Im}\, z>0\})}.$$

Then (81) follows from the Calderon-Zygmund estimate on the half plane with totally real boundary conditions (Theorem 5.3). □

Similarly, Theorem 5.4 has the following consequence about regularity of weak solutions of $\overline{\partial}$ on the upper right quadrant $\mathbb{H}^+$:

5.7. Lemma. *Let V and W be partially orthogonal linear subspaces of $\mathbb{R}^n$. Let $u \in L^p(\mathbb{H}^+, \mathbb{C}^n)$, $f \in L^p(\mathbb{H}^+, \mathbb{C}^n)$, $1 < p < \infty$, be such that*

$$\mathrm{Re} \int_{\mathbb{H}^+} \langle u, \partial\varphi \rangle \, ds dt = -\mathrm{Re} \int_{\mathbb{H}^+} \langle f, \varphi \rangle \, ds dt,$$

for every $\varphi \in C^\infty_c(\mathbb{C}, \mathbb{C}^n)$ such that $\varphi(\mathbb{R}) \subset N^*V$, $\varphi(i\mathbb{R}) \subset N^*W$. Then $u \in W^{1,p}(\mathbb{H}^+, \mathbb{C}^n)$, $\overline{\partial} u = f$, the trace of u on $\mathbb{R}$ takes values into N^*V, and the trace of u on $i\mathbb{R}$ takes values into $(N^*W)^\perp = N^*(W^\perp) = iN^*W$.

Proof. By means of a linear unitary transformation, as in the proof of Lemma 5.6, we may assume that $V = N^*V = \mathbb{R}^n$. A Schwarz reflection then allows to extend u to a map v on the right half-plane $\{\mathrm{Re}\, z > 0\}$ which is in L^p and is a weak solution of $\overline{\partial} v = g \in L^p$, with boundary condition in iN^*W on $i\mathbb{R}$. The thesis follows from Theorem 5.4. □

We are now interested in studying the operator $\overline{\partial}$ on the half-plane $\mathbb{H}$, with boundary conditions

$$u(s) \in N^*V, \quad u(-s) \in N^*W \quad \forall s > 0,$$

where V and W are partially orthogonal linear subspaces of $\mathbb{R}^n$. Taking Lemmas 5.6 and 5.7 into account, the natural idea is to obtain the required estimates by applying a conformal change of variable mapping the half-plane $\mathbb{H}$ onto the the upper right quadrant $\mathbb{H}^+$. More precisely, let $\mathscr{R}$ and $\mathscr{T}$ be the transformations

$$\mathscr{R} : \mathrm{Map}(\mathbb{H}, \mathbb{C}^n) \to \mathrm{Map}(\mathbb{H}^+, \mathbb{C}^n), \quad (\mathscr{R}u)(\zeta) = u(\zeta^2), \tag{84}$$

$$\mathscr{T} : \mathrm{Map}(\mathbb{H}, \mathbb{C}^n) \to \mathrm{Map}(\mathbb{H}^+, \mathbb{C}^n), \quad (\mathscr{T}u)(\zeta) = 2\overline{\zeta} u(\zeta^2), \tag{85}$$

where Map denotes some space of maps. Then the diagram

$$\begin{array}{ccc} \mathrm{Map}(\mathbb{H}, \mathbb{C}^n) & \xrightarrow{\overline{\partial}} & \mathrm{Map}(\mathbb{H}, \mathbb{C}^n) \\ \big\downarrow \mathscr{R} & & \mathscr{T} \big\downarrow \\ \mathrm{Map}(\mathbb{H}^+, \mathbb{C}^n) & \xrightarrow{\overline{\partial}} & \mathrm{Map}(\mathbb{H}^+, \mathbb{C}^n) \end{array} \tag{86}$$

commutes. By the elliptic estimates of Lemma 5.6, suitable domain and codomain for the operator on the lower horizontal arrow are the standard $W^{1,p}$ and L^p spaces, for $1 < p < \infty$. Moreover, if $u \in \mathrm{Map}(\mathbb{H}, \mathbb{C}^n)$ then

$$\|\mathscr{R}u\|^p_{L^p(\mathbb{H}^+)} = \frac{1}{4}\int_{\mathbb{H}} \frac{1}{|z|}|u(z)|^p \, dsdt, \tag{87}$$

$$\|D(\mathscr{R}u)\|^p_{L^p(\mathbb{H}^+)} = 2^{p-2}\int_{\mathbb{H}} |Du(z)|^p|z|^{p/2-1} \, dsdt, \tag{88}$$

$$\|\mathscr{T}u\|^p_{L^p(\mathbb{H}^+)} = 2^{p-2}\int_{\mathbb{H}} |u(z)|^p|z|^{p/2-1} \, dsdt. \tag{89}$$

Note also that by the generalized Poincaré inequality, the $W^{1,p}$ norm on $\mathbb{H}^+ \cap \mathbb{D}_r$, where $\mathbb{D}_r$ denotes the open disk of radius r, is equivalent to the norm

$$\|v\|^p_{\tilde{W}^{1,p}(\mathbb{H}^+\cap\mathbb{D}_r)} := \|Dv\|^p_{L^p(\mathbb{H}^+\cap\mathbb{D}_r)} + \int_{\mathbb{H}^+\cap\mathbb{D}_r} |v(\zeta)|^p|\zeta|^p \, d\sigma d\tau, \tag{90}$$

and the $\tilde{W}^{1,p}$ norm of $\mathscr{R}u$ is

$$\|\mathscr{R}u\|^p_{\tilde{W}^{1,p}(\mathbb{H}^+\cap\mathbb{D}_r)} = \frac{1}{4}\int_{\mathbb{H}\cap\mathbb{D}_{r^2}} |u(z)|^p|z|^{p/2-1} \, dsdt + 2^{p-2}\int_{\mathbb{H}\cap\mathbb{D}_{r^2}} |Du(z)|^p|z|^{p/2-1} \, dsdt. \tag{91}$$

So when dealing with bounded domains, both the transformations $\mathscr{R}$ and $\mathscr{T}$ involve the appearance of the weight $|z|^{p/2-1}$ in the L^p norms. Note also that when $p = 2$, this weight is just 1, reflecting the fact that the L^2 norm of the differential is a conformal invariant.

By the commutativity of diagram (86) and by the identities (88), (89), Lemma 5.6 applied to $\mathscr{R}u$ implies the following:

5.8. LEMMA. *Let V and W be partially orthogonal linear subspaces of $\mathbb{R}^n$. For every $p \in]1, +\infty[$, there exists a constant $c = c(p, n)$ such that*

$$\int_{\mathbb{H}} |\nabla u(z)|^p|z|^{p/2-1} \, ds\, dt \leq c^p \int_{\mathbb{H}} |\overline{\partial} u(z)|^p|z|^{p/2-1} \, ds\, dt$$

for every compactly supported map $u : \mathrm{Cl}(\mathbb{H}) \to \mathbb{C}^n$ such that $\zeta \mapsto u(\zeta^2)$ is smooth on $\mathrm{Cl}(\mathbb{H}^+)$, and

$$u(s) \in N^*V \;\forall s \in]-\infty, 0], \quad u(s) \in N^*W \;\forall s \in [0, +\infty[.$$

5.3 Strips with jumping conormal boundary conditions

Let us consider the following data: two integers $k, k' \geq 0$, $k+1$ linear subspaces $V_0, \dots, V_k$ of $\mathbb{R}^n$ such that V_{j-1} and V_j are partially orthogonal, for every $j = 1, \dots, k$, $k'+1$ linear subspaces $V'_0, \dots, V'_k$ of $\mathbb{R}^n$ such that V'_{j-1} and V'_j are partially orthogonal, for every $j = 1, \dots, k'$, and real numbers

$$-\infty = s_0 < s_1 < \dots < s_k < s_{k+1} = +\infty, \quad -\infty = s'_0 < s'_1 < \dots < s'_{k'} < -s'_{k'+1} = +\infty.$$

Denote by $\mathscr{V}$ the $(k+1)$-uple $(V_0, \dots, V_k)$, by $\mathscr{V}'$ the $(k'+1)$-uple $(V'_0, \dots, V'_k)$, and set

$$\mathscr{S} := \{s_1, \dots, s_k, s'_1 + i, \dots, s'_{k'} + i\}.$$

Let Σ be the closed strip

$$\Sigma := \{z \in \mathbb{C} \mid 0 \leq \mathrm{Im}\, z \leq 1\}.$$

The space $C^\infty_{\mathscr{S}}(\Sigma, \mathbb{C}^n)$ is the space of maps $u : \Sigma \to \mathbb{C}^n$ which are smooth on $\Sigma \setminus \mathscr{S}$, and such that the maps $\zeta \mapsto u(s_j + \zeta^2)$ and $\zeta \mapsto u(s'_j + i - \zeta^2)$ are smooth in a neighborhood of 0 in the closed upper-right quadrant

$$\mathrm{Cl}(\mathbb{H}^+) = \{\zeta \in \mathbb{C} \mid \mathrm{Re}\, \zeta \geq 0, \; \mathrm{Im}\, \zeta \geq 0\}.$$

The symbol $C^\infty_{\mathscr{S},c}$ indicates bounded support.

Given $p \in [1,+\infty[$, we define the X^p norm of a map $u \in L^1_{\mathrm{loc}}(\Sigma,\mathbb{C}^n)$ by

$$\|u\|^p_{X^p(\Sigma)} := \|u\|^p_{L^p(\Sigma\setminus B_r(\mathscr{S}))} + \sum_{w\in\mathscr{S}} \int_{\Sigma\cap B_r(w)} |u(z)|^p |z-w|^{p/2-1}\, ds\, dt,$$

where $r<1$ is less than half of the minimal distance between pairs of distinct points in $\mathscr{S}$. This is just a weighted L^p norm, where the weight $|z-w|^{p/2-1}$ comes from the identities (88), (89), and (91) of the last section. Note that when $p>2$ the X^p norm is weaker than the L^p norm, when $p<2$ the X^p norm is stronger than the L^p norm, and when $p=2$ the two norms are equivalent.

The space $X^p_{\mathscr{S}}(\Sigma,\mathbb{C}^n)$ is the space of locally integrable $\mathbb{C}^n$-valued maps on Σ whose X^p norm is finite. The X^p norm makes it a Banach space. We view it as a *real* Banach space.

The space $X^{1,p}_{\mathscr{S}}(\Sigma,\mathbb{C}^n)$ is defined as the completion of the space $C^\infty_{\mathscr{S},c}(\Sigma,\mathbb{C}^n)$ with respect to the norm

$$\|u\|^p_{X^{1,p}(\Sigma)} := \|u\|^p_{X^p(\Sigma)} + \|Du\|^p_{X^p(\Sigma)}.$$

It is a Banach space with the above norm. Equivalently, it is the space of maps in $X^p(\Sigma,\mathbb{C}^n)$ whose distributional derivative is also in X^p. The space $X^{1,p}_{\mathscr{S},\mathscr{V},\mathscr{V}'}(\Sigma,\mathbb{C}^n)$ is defined as the closure in $X^{1,p}_{\mathscr{S}}(\Sigma,\mathbb{C}^n)$ of the space of all $u\in C^\infty_{\mathscr{S},c}(\Sigma,\mathbb{C}^n)$ such that

$$\begin{array}{lll} u(s)\in N^*V_j & \forall s\in[s_j,s_{j+1}], & j=0,\dots,k,\\ u(s+i)\in N^*V'_j & \forall s\in[s'_j,s'_{j+1}], & j=0,\dots,k'. \end{array} \tag{92}$$

Equivalently, it can be defined in terms of the trace of u on the boundary of Σ.

Let $A:\mathbb{R}\times[0,1]\to \mathrm{L}(\mathbb{R}^{2n},\mathbb{R}^{2n})$ be continuous and bounded. For every $\in[1,+\infty[$, the linear operator

$$\overline{\partial}_A : X^{1,p}_{\mathscr{S}}(\Sigma,\mathbb{C}^n)\to X^p_{\mathscr{S}}(\Sigma,\mathbb{C}^n),\quad \overline{\partial}_A u := \overline{\partial} u + Au,$$

is bounded. Indeed, $\overline{\partial}$ is a bounded operator because of the inequality $|\overline{\partial}u|\le|Du|$, while the multiplication operator by A is bounded because

$$\|Au\|_{X^p(\Sigma)} \le \|A\|_\infty \|u\|_{X^p(\Sigma)}.$$

We wish to prove that if $p>1$ and $A(z)$ satisfies suitable asymptotics for $\mathrm{Re}\, z\to\pm\infty$ the operator $\overline{\partial}_A$ restricted to the space $X^{1,p}_{\mathscr{S},\mathscr{V},\mathscr{V}'}(\Sigma,\mathbb{C}^n)$ of maps satisfying the boundary conditions (92) is Fredholm.

Assume that $A\in C^0(\overline{\mathbb{R}}\times[0,1],\mathrm{L}(\mathbb{R}^{2n},\mathbb{R}^{2n}))$ is such that $A(\pm\infty,t)\in\mathrm{Sym}(2n,\mathbb{R})$ for every $t\in[0,1]$. Define $\Phi^+,\Phi^-:[0,1]\to\mathrm{Sp}(2n)$ to be the solutions of the linear Hamiltonian systems

$$\frac{d}{dt}\Phi^\pm(t) = iA(\pm\infty,t)\Phi^\pm(t),\quad \Phi^\pm(0)=I. \tag{93}$$

Then we have the following:

5.9. Theorem. *Assume that $\Phi^-(1)N^*V_0\cap N^*V'_0=(0)$ and $\Phi^+(1)N^*V_k\cap N^*V'_{k'}=(0)$. Then the bounded $\mathbb{R}$-linear operator*

$$\overline{\partial}_A : X^{1,p}_{\mathscr{S},\mathscr{V},\mathscr{V}'}(\Sigma,\mathbb{C}^n)\to X^p_{\mathscr{S}}(\Sigma,\mathbb{C}^n),\quad \overline{\partial}_A u = \overline{\partial} u + Au,$$

is Fredholm of index

$$\begin{aligned} \operatorname{ind}\overline{\partial}_A &= \mu(\Phi^- N^*V_0, N^*V'_0) - \mu(\Phi^+ N^*V_k, N^*V'_{k'}) \\ &-\frac12\sum_{j=1}^{k}(\dim V_{j-1}+\dim V_j - 2\dim V_{j-1}\cap V_j) - \frac12\sum_{j=1}^{k'}(\dim V'_{j-1}+\dim V'_j-2\dim V'_{j-1}\cap V'_j). \end{aligned} \tag{94}$$

The proof of the Fredholm property for Cauchy-Riemann type operators is based on local estimates. By a partition of unity argument, the proof that $\overline{\partial}_A$ is Fredholm reduces to the Calderon-Zygmund estimates of Lemmas 5.6, 5.8, and to the invertibility of $\overline{\partial}_A$ when A does not depend on $\mathrm{Re}\, z$ and there are no jumps in the boundary conditions. Details are contained in the next section. The index computation instead is based on homotopy arguments together with a Liouville type result stating that in a particular case with one jump the operator $\overline{\partial}_A$ is an isomorphism.

5.4 The Fredholm property

The elliptic estimates of section 5.2 have the following consequence:

5.10. Lemma. *For every $p \in]1,+\infty[$, there exist constants $c_0 = c_0(p,n,\mathscr{S})$ and $c_1 = c_1(p,n,k+k')$ such that*

$$\|Du\|_{X^p} \leq c_0 \|u\|_{X^p} + c_1 \|\overline{\partial} u\|_{X^p},$$

for every $u \in C^\infty_{\mathscr{S},c}(\Sigma,\mathbb{C}^n)$ such that

$$u(s) \in N^*V_j \quad \forall s \in [s_{j-1},s_j], \quad u(s+i) \in N^*V'_j \quad \forall s \in [s'_{j-1},s'_j]$$

for every j.

Proof. Let $\{\psi_1,\psi_2\} \cup \{\varphi_j\}_{j=1}^{k+k'}$ be a smooth partition of unity on $\mathbb{C}$ satisfying:

$$\begin{array}{lcll} \operatorname{supp}\psi_1 & \subset & \{z \in \mathbb{C} \mid \mathrm{Im}\, z < 2/3\} \setminus B_{r/2}(\mathscr{S}), & \\ \operatorname{supp}\psi_2 & \subset & \{z \in \mathbb{C} \mid \mathrm{Im}\, z > 1/3\} \setminus B_{r/2}(\mathscr{S}), & \\ \operatorname{supp}\varphi_j & \subset & B_r(s_j) & \forall j = 1,\dots,k, \\ \operatorname{supp}\varphi_{k+j} & \subset & B_r(s'_j + i) & \forall j = 1,\dots,k'. \end{array} \tag{95}$$

By Lemma 5.8,

$$\|D(\varphi_j u)\|_{X^p(\Sigma)} \leq c(p,n)\|\overline{\partial}(\varphi_j u)\|_{X^p(\Sigma)} \leq c(p,n)(\|\overline{\partial}\varphi_j\|_\infty \|u\|_{X^p(\Sigma)} + \|\overline{\partial} u\|_{X^p(\Sigma)}), \quad 1 \leq j \leq k+k'.$$

Since the X^p norm is equivalent to the L^p norm on the subspace of maps whose support does not meet $B_{r/2}(\mathscr{S})$, the standard Calderon-Zygmund estimates on the half-plane (see Theorem 5.3) imply

$$\|D(\psi_j u)\|_{X^p(\Sigma)} \leq c(p,n)\|\overline{\partial}(\psi_j u)\|_{X^p(\Sigma)} \leq c(p,n)(\|\overline{\partial}\psi_j\|_\infty \|u\|_{X^p(\Sigma)} + \|\overline{\partial} u\|_{X^p(\Sigma)}), \quad \forall j = 1,2.$$

We conclude that

$$\|Du\|_{X^p} \leq \|D(\psi_1 u)\|_{X^p(\Sigma)} + \|D(\psi_2 u)\|_{X^p(\Sigma)} + \sum_{j=1}^{k+k'} \|D(\varphi_j u)\|_{X^p(\Sigma)} \leq c_0 \|u\|_{X^p(\Sigma)} + c_1 \|\overline{\partial} u\|_{X^p(\Sigma)},$$

with

$$c_0 := c(p,n)\Big(\|\overline{\partial}\psi_1\|_\infty + \|\overline{\partial}\psi_2\|_\infty + \sum_{j=0}^{k+k'+1} \|\overline{\partial}\varphi_j\|_\infty\Big), \quad c_1 := (k+k'+2)c(p,n),$$

as claimed. □

The next result we need is the following theorem, proved in [RS95, Theorem 7.1]. Consider two continuously differentiable Lagrangian paths $\lambda,\nu : \overline{\mathbb{R}} \to \mathscr{L}(n)$, assumed to be constant on $[-\infty,-s_0]$ and on $[s_0,+\infty]$, for some $s_0 > 0$. Denote by $W^{1,p}_{\lambda,\nu}(\Sigma,\mathbb{C}^n)$ the space of maps $u \in W^{1,p}(\Sigma,\mathbb{C}^n)$ such that $u(s,0) \in \lambda(s)$ and $u(s,1) \in \nu(s)$, for every $s \in \mathbb{R}$ (in the sense of traces). Let $A \in C^0(\overline{\mathbb{R}} \times [0,1], \mathrm{L}(\mathbb{R}^{2n},\mathbb{R}^{2n}))$ be such that $A(\pm\infty,t) \in \mathsf{Sym}(2n,\mathbb{R})$ for any $t \in [0,1]$, and define $\Phi^-,\Phi^+ : [0,1] \to \mathsf{Sp}(2n)$ by (93).

5.11. THEOREM. (Cauchy-Riemann operators on the strip) *Let $p \in]1,+\infty[$, and assume that*

$$\Phi^-(1)\lambda(-\infty) \cap \nu(-\infty) = (0), \quad \Phi^+(1)\lambda(+\infty) \cap \nu(+\infty) = (0).$$

(i) The bounded $\mathbb{R}$-linear operator

$$\overline{\partial}_A : W^{1,p}_{\lambda,\nu}(\Sigma, \mathbb{C}^n) \to L^p(\Sigma, \mathbb{C}^n), \quad \overline{\partial}_A u = \overline{\partial} u + Au,$$

is Fredholm of index

$$\operatorname{ind} \overline{\partial}_A = \mu(\Phi^- \lambda(-\infty), \nu(-\infty)) - \mu(\Phi^+ \lambda(+\infty), \nu(+\infty)) + \mu(\lambda, \nu).$$

(ii) If furthermore $A(s,t) = A(t)$, $\lambda(s) = \lambda$, and $\nu(s) = \nu$ do not depend on s, the operator $\overline{\partial}_A$ is an isomorphism.

Note that under the assumptions of (ii) above, the equation $\overline{\partial} u + Au$ can be rewritten as $\partial_s u = -L_A u$, where L_A is the unbounded $\mathbb{R}$-linear operator on $L^2(]0,1[,\mathbb{C}^n)$ defined by

$$\operatorname{dom} L_A = W^{1,2}_{\lambda,\nu}(]0,1[,\mathbb{C}^n) = \left\{ u \in W^{1,2}(]0,1[,\mathbb{C}^n) \mid u(0) \in \lambda, \ u(1) \in \nu \right\}, \quad L_A = i\frac{d}{dt} + A.$$

The conditions on A imply that L_A is self-adjoint and invertible. These facts lead to the following:

5.12. PROPOSITION. *Assume that A, λ, and ν satisfy the conditions of Theorem 5.11 (ii), and set $\delta := \min \sigma(L_A) \cap [0,+\infty[> 0$. Then for every $k \in \mathbb{N}$ there exists c_k such that*

$$\|u(s,\cdot)\|_{C^k([0,1])} \leq c_k \|u(0,\cdot)\|_{L^2(]0,1[)} e^{-\delta s}, \quad \forall s \geq 0,$$

for every $u \in W^{1,p}(]0,+\infty[\times]0,1[,\mathbb{C}^n)$, $p > 1$, such that $u(s,0) \in \lambda$, $u(s,1) \in \nu$ for every $s \geq 0$, and $\overline{\partial} u + Au = 0$.

Next we need the following easy consequence of the Sobolev embedding theorem:

5.13. PROPOSITION. *Let $s > 0$ and let χ_s be the characteristic function of the set $\{z \in \Sigma \mid |\operatorname{Re} z| \leq s\}$. Then the linear operator*

$$X^{1,p}_{\mathscr{S}}(\Sigma, \mathbb{C}^n) \to X^q_{\mathscr{S}}(\Sigma, \mathbb{C}^n), \quad u \mapsto \chi_s u,$$

is compact for every $q < \infty$ if $p \geq 2$, and for every $q < 2p/(2-p)$ if $1 \leq p < 2$.

Proof. Let (u_h) be a bounded sequence in $X^{1,p}_{\mathscr{S}}(\Sigma, \mathbb{C}^n)$. Let $\{\psi_1, \psi_2\} \cup \{\varphi_j\}_{j=1}^{k+k'}$ be a smooth partition of unity of $\mathbb{C}$ satisfying (95). Then the sequences $(\psi_1 u_h)$, $(\psi_2 u_h)$ and $(\varphi_j u_h)$, for $1 \leq j \leq k+k'$ are bounded in $X^{1,p}(\Sigma, \mathbb{C}^n)$. We must show that each of these sequences is compact in $X^q_{\mathscr{S}}(\Sigma, \mathbb{C}^n)$.

Since the X^q and $X^{1,p}$ norms on the space of maps supported in $\Sigma \setminus B_{r/2}(\mathscr{S})$ are equivalent to the L^q and $W^{1,p}$ norms, the Sobolev embedding theorem implies that the sequences $(\chi_s \psi_1 u_h)$ and $(\chi_s \psi_2 u_h)$ are compact in $X^q_{\mathscr{S}}(\Sigma, \mathbb{C}^n)$.

Let $1 \leq j \leq k$. If u is supported in $B_r(s_j)$, set $v(z) := u(s_j + z)$, so that by (87)

$$\|u\|^q_{X^q(\Sigma)} = \int_{B_r(s_j)\cap\Sigma} |u(z)|^q |z|^{q/2-1} \, dsdt \leq \int_{B_r(s_j)\cap\Sigma} \frac{1}{|z|} |u(z)|^q \, dsdt = 4\|\mathscr{R}u\|^q_{L^q(\mathbb{H}^+\cap\mathbb{D}_{\sqrt{r}})}. \quad (96)$$

Set $v_h(z) := \varphi_j(s_j+z) u_h(s_j+z)$. By (91), the sequence $(\mathscr{R}v_h)$ is bounded in $W^{1,p}(\mathbb{H}^+ \cap \mathbb{D}_{\sqrt{r}})$, hence it is compact in $L^q(\mathbb{H}^+ \cap \mathbb{D}_{\sqrt{r}})$ for every $q < \infty$ if $p \geq 2$, and for every $q < 2p/(2-p)$ if $1 \leq p < 2$. Then (96) implies that $(\varphi_j u_h)$ is compact in $X^q_{\mathscr{S}}(\Sigma, \mathbb{C}^n)$. A fortiori, so is $(\chi_s \varphi_j u_h)$. The same argument applies to $j \geq k+1$, concluding the proof. □

Putting together Lemma 5.10, statement (ii) in Theorem 5.11, and the Proposition above we obtain the following:

5.14. Proposition. *Let $1 < p < \infty$. Assume that the paths of symmetric matrices $A(\pm\infty, \cdot)$ satisfy the assumptions of Theorem 5.9. Then*

$$\overline{\partial}_A : X^{1,p}_{\mathscr{S},\mathscr{V},\mathscr{V}'}(\Sigma, \mathbb{C}^n) \to X^p_{\mathscr{S}}(\Sigma, \mathbb{C}^n)$$

is semi-Fredholm with $\operatorname{ind} \overline{\partial}_A := \dim\ker \overline{\partial}_A - \dim\operatorname{coker} \overline{\partial}_A < +\infty$.

Proof. We claim that there exist $c \geq 0$ and $s \geq 0$ such that, for any $u \in X^{1,p}_{\mathscr{S},\mathscr{V},\mathscr{V}'}(\Sigma, \mathbb{C}^n)$, there holds

$$\|u\|_{X^{1,p}(\Sigma)} \leq c\left(\|(\overline{\partial} + A)u\|_{X^p(\Sigma)} + \|\chi_s u\|_{X^p(\Sigma)}\right), \tag{97}$$

where χ_s is the characteristic function of the set $\{z \in \Sigma \mid |\operatorname{Re} z| \leq s\}$.

By Theorem 5.11 (ii), the asymptotic operators

$$\begin{aligned}\overline{\partial} + A(-\infty, \cdot) &: W^{1,p}_{N^*V_0, N^*V'_0}(\Sigma, \mathbb{C}^n) \to L^p(\Sigma, \mathbb{C}^n),\\ \overline{\partial} + A(+\infty, \cdot) &: W^{1,p}_{N^*V_k, N^*V'_{k'}}(\Sigma, \mathbb{C}^n) \to L^p(\Sigma, \mathbb{C}^n),\end{aligned}$$

are invertible. Since invertibility is an open condition in the operator norm, there exist $s > \max|\operatorname{Re}\mathscr{S}| + 2$ and $c_1 > 0$ such that for any $u \in X^{1,p}_{\mathscr{S},\mathscr{V},\mathscr{V}'}(\Sigma, \mathbb{C}^n)$ with support disjoint from $\{|\operatorname{Re} z| \leq s - 1\}$ there holds

$$\|u\|_{X^{1,p}(\Sigma)} = \|u\|_{W^{1,p}(\Sigma)} \leq c_1\|(\overline{\partial} + A)u\|_{L^p(\Sigma)} = c_1\|(\overline{\partial} + A)u\|_{X^p(\Sigma)}. \tag{98}$$

By Proposition 5.10, there exists $c_2 > 0$ such that for every $u \in X^{1,p}_{\mathscr{S},\mathscr{V},\mathscr{V}'}(\Sigma, \mathbb{C}^n)$ with support in $\{|\operatorname{Re} z| \leq s\}$ there holds

$$\begin{aligned}\|u\|_{X^{1,p}(\Sigma)} &\leq c_2(\|u\|_{X^p(\Sigma)} + \|\overline{\partial} u\|_{X^p(\Sigma)})\\ &\leq (c_2 + \|A\|_\infty)\|u\|_{X^p(\Sigma)} + c_2\|(\overline{\partial} + A)u\|_{X^p(\Sigma)}.\end{aligned} \tag{99}$$

The inequality (97) easily follows from (98) and (99) by writing any $u \in X^{1,p}_{\mathscr{S},\mathscr{V},\mathscr{V}'}(\Sigma, \mathbb{C}^n)$ as $u = (1-\varphi)u + \varphi u$, for φ a smooth real function on Σ having support in $\{|\operatorname{Re} z| < s\}$ and such that $\varphi = 1$ on $\{|\operatorname{Re} z| \leq s - 1\}$.

Finally, by Proposition 5.13 the linear operator

$$X^{1,p}_{\mathscr{S},\mathscr{V},\mathscr{V}'}(\Sigma, \mathbb{C}^n) \to X^p_{\mathscr{S}}(\Sigma, \mathbb{C}^n), \quad u \mapsto \chi_s u,$$

is compact. Therefore the estimate (97) implies that $\overline{\partial}_A$ has finite dimensional kernel and closed range, that is it is semi-Fredholm with index less than $+\infty$. □

It would not be difficult to use the regularity of weak solutions of the Cauchy-Riemann operator to prove that the cokernel of $\overline{\partial}_A$ is finite-dimensional, so that $\overline{\partial}_A$ is Fredholm. However, this will follow directly from the index computation presented in the next section.

5.5 A Liouville type result

Let us consider the following particular case in dimension $n = 1$:

$$k = 1, \quad k' = 0, \quad \mathscr{S} = \{0\}, \quad V_0 = (0), \quad V_1 = \mathbb{R}, \quad V'_0 = \mathbb{R}, \quad A(z) = \alpha,$$

with α a real number. In other words, we are looking at the operator $\overline{\partial} + \alpha$ on a space of $\mathbb{C}$-valued maps u on Σ such that $u(s)$ is purely imaginary for $s \leq 0$, $u(s)$ is real for $s \geq 0$, and $u(s + i)$ is real for every $s \in \mathbb{R}$. Notice that $\Phi^-(t) = \Phi^+(t) = e^{i\alpha t}$, so

$$e^{i\alpha} i\mathbb{R} \cap \mathbb{R} = (0) \quad \forall \alpha \in \mathbb{R} \setminus (\pi/2 + \pi\mathbb{Z}), \quad e^{i\alpha}\mathbb{R} \cap \mathbb{R} = (0) \quad \forall \alpha \in \mathbb{R} \setminus \pi\mathbb{Z},$$

so the assumptions of Theorem 5.9 are satisfied whenever α is not an integer multiple of $\pi/2$. In order to simplify the notation, we set

$$X^p(\Sigma) := X^p_{\{0\}}(\Sigma, \mathbb{C}), \quad X^{1,p}(\Sigma) := X^{1,p}_{\{0\},((0),\mathbb{R}),(\mathbb{R})}(\Sigma, \mathbb{C}).$$

We start by studying the regularity of the elements of the kernel of $\overline{\partial}_\alpha$:

5.15. LEMMA. *Let $p > 1$ and $\alpha \in \mathbb{R} \setminus (\pi/2)\mathbb{Z}$. If u belongs to the kernel of*

$$\overline{\partial}_\alpha : X^{1,p}(\Sigma) \to X^p(\Sigma),$$

then u is smooth on $\Sigma \setminus \{0\}$, it satisfies the boundary conditions pointwise, and the function $(\mathscr{R}u)(\zeta) = u(\zeta^2)$ is smooth on $\mathrm{Cl}(\mathbb{H}^+) \cap \mathbb{D}_1$. In particular, u is continuous at 0, and $Du(z) = O(|z|^{-1/2})$ for $z \to 0$.

Proof. The regularity theory for weak solutions of $\overline{\partial}$ on $\mathbb{C}$ and on the half-plane $\mathbb{H}$ (Theorem 5.4) implies - by a standard bootstrap argument - that $u \in C^\infty(\Sigma \setminus \{0\})$. We just need to check the regularity of u at 0.

Consider the function $f(\zeta) := e^{\alpha\overline{\zeta}^2/2} u(\zeta^2)$ on $\mathbb{H}^+ \cap \mathbb{D}_1$. Since

$$\overline{\partial} f(\zeta) = 2\overline{\zeta} e^{\alpha\overline{\zeta}^2/2} \big(\overline{\partial} u(\zeta^2) + \alpha u(\zeta^2)\big) = 0,$$

f is holomorphic on $\mathbb{H}^+ \cap \mathbb{D}_1$. Moreover, by (91) the function f belongs to $W^{1,p}(\mathbb{H}^+ \cap \mathbb{D}_1)$, and in particular it is square integrable. The function f is real on $\mathbb{R}^+$ and purely imaginary on $i\mathbb{R}^+$, so a double Schwarz reflection produces a holomorphic extension of f to $\mathbb{D}_1 \setminus \{0\}$. Such an extension of f is still square integrable, so the singularity 0 is removable and the function is holomorphic on the whole $\mathbb{D}_1$. It follows that

$$(\mathscr{R}u)(\zeta) = u(\zeta^2) = e^{-\alpha\overline{\zeta}^2/2} f(\zeta)$$

is smooth on $\mathrm{Cl}(\mathbb{H}^+) \cap \mathbb{D}_1$, as claimed. $\square$

The real Banach space $X^p(\Sigma)$ is the space of L^p functions with respect to the measure defined by the density

$$\rho_p(z) := \begin{cases} 1 & \text{if } z \in \Sigma \setminus \mathbb{D}_r, \\ |z|^{p/2-1} & \text{if } z \in \Sigma \cap \mathbb{D}_r. \end{cases}$$

So the dual of $X^p(\Sigma)$ can be identified with the real Banach space

$$\left\{ v \in L^1_{\mathrm{loc}}(\Sigma, \mathbb{C}) \mid \int_\Sigma |v|^q \rho_p(z)\, dsdt < +\infty \right\}, \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1, \tag{100}$$

by using the duality paring

$$\big(X^p(\Sigma)\big)^* \times X^p(\Sigma) \to \mathbb{R}, \quad (v, u) \mapsto \mathrm{Re} \int_\Sigma \langle v, u \rangle \rho_p(z)\, dsdt.$$

We prefer to use the standard duality pairing

$$\big(X^p(\Sigma)\big)^* \times X^p(\Sigma) \to \mathbb{R}, \quad (w, u) \mapsto \mathrm{Re} \int_\Sigma \langle w, u \rangle\, dsdt. \tag{101}$$

With the latter choice, the dual of $X^p(\Sigma)$ is identified with the space of functions $w = \rho_p(z) v$, where v varies in the space (100). From $1/p + 1/q = 1$ we get the identity

$$\begin{aligned} \|w\|^q_{X^q} &= \int_{\Sigma \setminus \mathbb{D}_r} |w|^q\, dsdt + \int_{\Sigma \cap \mathbb{D}_r} |w|^q |z|^{q/2-1}\, dsdt \\ &= \int_{\Sigma \setminus \mathbb{D}_r} |v|^q\, dsdt + \int_{\Sigma \cap \mathbb{D}_r} |v|^q |z|^{(p/2-1)q} |z|^{q/2-1}\, dsdt \\ &= \int_{\Sigma \setminus \mathbb{D}_r} |v|^q\, dsdt + \int_{\Sigma \cap \mathbb{D}_r} |v|^q |z|^{p/2-1}\, dsdt = \int_\Sigma |v|^q \rho_p(z)\, dsdt, \end{aligned}$$

which shows that the standard duality paring (101) produces the identification

$$\big(X^p(\Sigma)\big)^* \cong X^q(\Sigma), \quad \text{for } \frac{1}{p} + \frac{1}{q} = 1.$$

Therefore, we view the cokernel of $\overline{\partial}_\alpha : X^{1,p}(\Sigma) \to X^p(\Sigma)$ as a subspace of $X^q(\Sigma)$. Its elements are a priori less regular at 0 than the elements of the kernel:

5.16. LEMMA. *Let $p>1$ and $\alpha \in \mathbb{R}\setminus(\pi/2)\mathbb{Z}$. If $v \in X^q(\Sigma)$, $1/p+1/q=1$, belongs to the cokernel of*

$$\overline{\partial}_\alpha : X^{1,p}(\Sigma) \to X^p(\Sigma),$$

then v is smooth on $\Sigma \setminus \{0\}$, it solves the equation $\partial v - \alpha v = 0$ with boundary conditions

$$\begin{array}{lll} v(s)\in\mathbb{R}, & v(-s)\in i\mathbb{R} & \forall s>0, \\ v(s+i)\in\mathbb{R} & & \forall s\in\mathbb{R}, \end{array} \tag{102}$$

and the function $(\mathscr{T}v)(\zeta) = 2\overline{\zeta}v(\zeta^2)$ is smooth on $\mathrm{Cl}(\mathbb{H}^+)\cap\mathbb{D}_1$. In particular, $v(z) = O(|z|^{-1/2})$ and $Dv(z) = O(|z|^{-3/2})$ for $z\to 0$.

Proof. Since $v\in X^q(\Sigma)$ annihilates the image of $\overline{\partial}_\alpha$, there holds

$$\mathrm{Re}\int_\Sigma \langle v(z), \overline{\partial}u(z) + \alpha u(z)\rangle\, ds\, dt = 0, \tag{103}$$

for every $u \in X^{1,p}(\Sigma)$. By letting u vary among all smooth functions in $X^{1,p}(\Sigma)$ which are compactly supported in $\Sigma\setminus\{0\}$, the regularity theory for weak solutions of ∂ (the analogue of Theorem 5.4) and a bootstrap argument show that v is smooth on $\Sigma\setminus\{0\}$ and it solves the equation $\partial v - \alpha v = 0$ with boundary conditions (102). There remains to study the regularity of v at 0.

Set $w(\zeta) := (\mathscr{T}v)(\zeta) = 2\overline{\zeta}v(\zeta^2)$. By (89), the function w is in $L^q(\mathbb{H}^+\cap\mathbb{D}_1)$. Let $\varphi \in C^\infty_c(\mathrm{Cl}(\mathbb{H}^+)\cap\mathbb{D}_1)$ be real on $\mathbb{R}^+$ and purely imaginary on $i\mathbb{R}^+$. Then the function u defined by $u(\zeta^2) = \varphi(\zeta)$ belongs to $X^{1,p}(\Sigma)$, and by (103) we have

$$\begin{aligned} 0 = \mathrm{Re}\int_\Sigma \langle v, \overline{\partial}u + \alpha u\rangle\, dsdt &= 4\mathrm{Re}\int_{\mathbb{H}^+\cap\mathbb{D}_1} |\zeta^2| \langle \frac{1}{2\overline{\zeta}} w(\zeta), \frac{1}{2\overline{\zeta}}\overline{\partial}\varphi(\zeta) + \alpha\varphi(\zeta)\rangle\, d\sigma d\tau \\ &= \mathrm{Re}\int_{\mathbb{H}^+\cap\mathbb{D}_1} \langle w(\zeta), \overline{\partial}\varphi(\zeta) + 2\alpha\overline{\zeta}\varphi(\zeta)\rangle\, d\sigma d\tau. \end{aligned}$$

The above identity can be rewritten as

$$\mathrm{Re}\int_{\mathbb{H}^+\cap\mathbb{D}_1} \langle w(\zeta), \overline{\partial}\varphi(\zeta)\rangle\, d\sigma d\tau = -\mathrm{Re}\int_{\mathbb{H}^+\cap\mathbb{D}_1} \langle 2\alpha\zeta w(\zeta), \varphi(\zeta)\rangle\, d\sigma d\tau,$$

so w is a weak solution of $\partial w = 2\alpha\zeta w$ on $\mathrm{Cl}(\mathbb{H}^+)\cap\mathbb{D}_1$ with real boundary conditions. Since w is in $L^q(\mathbb{H}^+\cap\mathbb{D}_1)$, Lemma 5.7 implies that w is in $W^{1,q}(\mathbb{H}^+\cap\mathbb{D}_1)$. In particular, w is square integrable on $\mathbb{H}^+\cap\mathbb{D}_1$, and so is the function

$$f(\zeta) := e^{-\alpha\zeta^2/2} w(\zeta).$$

The function f is anti-holomorphic, it takes real values on $\mathbb{R}^+$ and on $i\mathbb{R}^+$, so by a double Schwarz reflection it can be extended to an anti-holomorhic function on $\mathbb{D}_1\setminus\{0\}$. Since f is square integrable, the singularity 0 is removable and f is anti-holomorphic on $\mathbb{D}_1$. Therefore

$$(\mathscr{T}v)(\zeta) = w(\zeta) = e^{\alpha\zeta^2/2} f(\zeta)$$

is smooth on $\mathrm{Cl}(\mathbb{H}^+)\cap\mathbb{D}_1$, as claimed. □

We can finally prove the following Liouville type result:

5.17. PROPOSITION. *If $0<\alpha<\pi/2$, the operator*

$$\overline{\partial}_\alpha : X^{1,p}(\Sigma) \to X^p(\Sigma)$$

is an isomorphism, for every $1<p<\infty$.

Proof. By Proposition 5.14 the operator $\overline{\partial}_\alpha$ is semi-Fredholm, so it is enough to prove that its kernel and co-kernel are both (0).

Let $u \in X^{1,p}(\Sigma)$ be an element of the kernel of $\overline{\partial}_\alpha$. By Proposition 5.12, $u(z)$ has exponential decay for $|\mathrm{Re}\, z| \to +\infty$ together with all its derivatives. By Lemma 5.15, u is smooth on $\Sigma \setminus \{0\}$, it is continuous at 0, and $Du(z) = O(|z|^{-1/2})$ for $z \to 0$. Then the function $w := u^2$ belongs to $W^{1,q}(\Sigma, \mathbb{C})$ for every $q < 4$. Moreover, w is real on the boundary of Σ, and it satisfies the equation

$$\overline{\partial} w + 2\alpha w = 0.$$

Since $0 < 2\alpha < \pi$, $e^{2\alpha i}\mathbb{R} \cap \mathbb{R} = (0)$, so the assumptions of Theorem 5.11 (ii) are satisfied, and the operator

$$\overline{\partial}_{2\alpha} : W^{1,q}_{\mathbb{R},\mathbb{R}}(\Sigma, \mathbb{C}) \to L^q(\Sigma, \mathbb{C})$$

is an isomorphism. Therefore $w = 0$, hence $u = 0$, proving that the operator $\overline{\partial}_\alpha$ has vanishing kernel.

Let $v \in X^q(\Sigma)$, $1/p + 1/q = 1$, be an element of the cokernel of $\overline{\partial}_\alpha$. By Lemma 5.16, v is smooth on $\Sigma \setminus \{0\}$, $v(s) \in i\mathbb{R}$ for $s < 0$, $v(s) \in \mathbb{R}$ for $s > 0$, v solves $\partial v - \alpha v = 0$, and the function

$$w(\zeta) := 2\overline{\zeta} v(\zeta^2) \tag{104}$$

is smooth in $\mathrm{Cl}(\mathbb{H}^+) \cap \mathbb{D}_1$ and real on the boundary of $\mathbb{H}^+$. In particular, $v(z) = O(|z|^{-1/2})$ and $Dv(z) = O(|z|^{-3/2})$ for $z \to 0$. Furthermore, by Proposition 5.12, v and Dv decay exponentially for $|\mathrm{Re}\, z| \to +\infty$. More precisely, since the spectrum of the operator L_α on $L^2(]0,1[, \mathbb{C})$,

$$\mathrm{dom}\, L_\alpha = W^{1,2}_{\mathbb{R},\mathbb{R}}(]0,1[, \mathbb{C}) = \left\{ u \in W^{1,2}(]0,1[, \mathbb{C}) \mid u(0), u(1) \in \mathbb{R} \right\}, \quad L_\alpha = i\frac{d}{dt} + \alpha,$$

is $\alpha + \pi\mathbb{Z}$, we have $\min \sigma(L_\alpha) \cap [0, +\infty) = \alpha$, hence

$$|v(z)| \leq c e^{-\alpha |\mathrm{Re}\, z|}, \quad \text{for } |\mathrm{Re}\, z| \geq 1. \tag{105}$$

If $w(0) = 0$, the function v vanishes at 0, and $Dv(z) = O(|z|^{-1})$ for $z \to 0$, so v^2 belongs to $W^{1,q}_{\mathbb{R},\mathbb{R}}(\Sigma, \mathbb{C})$ for any $q < 2$, it solves $\partial v^2 - 2\alpha v^2 = 0$, and as before we deduce that $v = 0$. Therefore, we can assume that the real number $w(0)$ is not zero.

Consider the function

$$f : \Sigma \setminus \{0\} \to \mathbb{C}, \quad f(z) := e^{-\alpha \overline{z}/2} \overline{v}(z).$$

Since $\partial v = \alpha v$,

$$\overline{\partial} f(z) = -\alpha e^{-\alpha \overline{z}/2} \overline{v}(z) + e^{-\alpha \overline{z}/2} \overline{\partial v}(z) = e^{-\alpha \overline{z}/2}(-\alpha \overline{v}(z) + \overline{\alpha v}(z)) = 0,$$

so f is holomorphic on the interior of Σ. Moreover, f is smooth on $\Sigma \setminus \{0\}$, and

$$f(s) = e^{-\alpha s/2} \overline{v}(s) \in i\mathbb{R} \quad \text{for } s < 0, \qquad f(s) = e^{-\alpha s/2} \overline{v}(s) \in \mathbb{R} \quad \text{for } s > 0, \tag{106}$$

$$f(s+i) = e^{-\alpha s/2} \overline{v}(s+i) e^{\alpha i/2} \in e^{\alpha i/2}\mathbb{R} \quad \text{for every } s \in \mathbb{R}. \tag{107}$$

Denote by $\sqrt{z}$ the determination of the square root on $\mathbb{C} \setminus \mathbb{R}^-$ such that $\sqrt{z}$ is real and positive for z real and positive, so that $\overline{\sqrt{z}} = \sqrt{\overline{z}}$. By (104),

$$f(z) = -e^{-\alpha \overline{z}/2} \frac{1}{\sqrt{z}} \overline{w}(\sqrt{z}) = \frac{\overline{w}(0)}{\sqrt{z}} + o(|z|^{-1/2}) \quad \text{for } z \to 0. \tag{108}$$

Finally, by (105),

$$\lim_{|\mathrm{Re}\, z| \to +\infty} f(z) = 0. \tag{109}$$

We claim that a holomorphic function with the properties listed above is necessarily zero. By (108), setting $z=\rho e^{\theta i}$ with $\rho>0$ and $0\leq\theta\leq\pi$,

$$f(z)=\frac{\overline{w}(0)}{\sqrt{\rho}}e^{-\theta i/2}+o(|\rho|^{-1/2})\quad\text{for } \rho\to 0.$$

Since $\overline{w}(0)$ is real and not zero, the above expansion at 0 shows that there exists $\rho>0$ such that

$$f(z)\in\bigcup_{\theta\in]-\pi/2-\alpha/4,\alpha/4[} e^{\theta i}\mathbb{R},\quad \forall z\in(B_\rho(0)\cap\Sigma)\setminus\{0\}. \tag{110}$$

If $f=0$ on $\mathbb{R}+i$, then f is identically zero (by reflection and by analytic continuation), so we may assume that $f(\mathbb{R}+i)\neq\{0\}$. By (107) the set $f(\mathbb{R}+i)$ is contained in $\mathbb{R}e^{\alpha i/2}$. Since f is holomorphic on $\mathrm{Int}(\Sigma)$, it is open on such a domain, so we can find $\gamma\in]\alpha/4,\alpha/2[\cup]\alpha/2,3\alpha/4[$ such that $f(\mathrm{Int}(\Sigma))\cap\mathbb{R}e^{\gamma i}\neq\{0\}$. By (109) and (110) there exists $z\in\Sigma\setminus B_\rho(0)$ such that

$$f(z)\in\mathbb{R}e^{\gamma i},\quad |f(z)|=\sup|f(\Sigma\setminus\{0\})\cap\mathbb{R}e^{\gamma i}|>0. \tag{111}$$

By (106) and (107), z belongs to $\mathrm{Int}(\Sigma)$, but since f is open on $\mathrm{Int}(\Sigma)$ this fact contradicts (111). Hence $f=0$. Therefore v vanishes on Σ, concluding the proof of the invertibility of the operator $\overline{\partial}_\alpha$. □

If we change the sign of α and we invert the boundary conditions on $\mathbb{R}$ we still get an isomorphism. Indeed, if we set $v(s,t):=\overline{u}(-s,t)$ we have

$$\overline{\partial}_{-\alpha}v(s,t)=\overline{\partial}v(s,t)-\alpha v(s,t)=-\overline{\overline{\partial}u(-s,t)+\alpha u(-s,t)}=-\overline{\overline{\partial}_\alpha u(-s,t)},$$

so the operators

$$\begin{aligned}\overline{\partial}_\alpha: X^{1,p}_{\{0\},((0),\mathbb{R}),(\mathbb{R})}(\Sigma,\mathbb{C}) &\rightarrow X^p_{\{0\}}(\Sigma,\mathbb{C}),\\ \overline{\partial}_{-\alpha}: X^{1,p}_{\{0\},(\mathbb{R},(0)),(\mathbb{R})}(\Sigma,\mathbb{C}) &\rightarrow X^p_{\{0\}}(\Sigma,\mathbb{C})\end{aligned}$$

are conjugated. Therefore Proposition 5.17 implies:

5.18. Proposition. *If $0<\alpha<\pi/2$, the operator*

$$\overline{\partial}_{-\alpha}: X^{1,p}_{\{0\},(\mathbb{R},(0)),(\mathbb{R})}(\Sigma,\mathbb{C})\rightarrow X^p_{\{0\}}(\Sigma,\mathbb{C})$$

is an isomorphism.

5.6 Computation of the index

The computation of the Fredholm index of $\overline{\partial}_A$ is based on the Liouville type results proved in the previous section, together with the following additivity formula:

5.19. Proposition. *Assume that $A,A_1,A_2\in C^0(\overline{\mathbb{R}}\times[0,1],\mathrm{L}(\mathbb{R}^{2n},\mathbb{R}^{2n}))$ satisfy*

$$A_1(+\infty,t)=A_2(-\infty,t),\quad A(-\infty,t)=A_1(-\infty,t),\quad A(+\infty,t)=A_2(+\infty,t),\quad \forall t\in[0,1].$$

Let $\mathscr{V}_1=(V_0,\dots,V_k)$, $\mathscr{V}_2=(V_k,\dots,V_{k+h})$, $\mathscr{V}_1'=(V_0',\dots,V_{k'}')$, $\mathscr{V}_2'=(V_{k'},\dots,V_{k'+h'}')$ be finite ordered sets of linear subspaces of $\mathbb{R}^n$ such that V_j and V_{j+1}, V_j' and V_{j+1}' are partially orthogonal, for every j. Set $\mathscr{V}=(V_0,\dots,V_k,V_{k+1},\dots,V_{k+h})$ and $\mathscr{V}'=(V_0',\dots,V_{k'}',V_{k'+1}',\dots,V_{k'+h'}')$. Assume that $(A_1,\mathscr{V}_1,\mathscr{V}_1')$ and $(A_2,\mathscr{V}_2,\mathscr{V}_2')$ satisfy the assumptions of Theorem 5.9. Let $\mathscr{S}_1$ be a set consisting of k points in $\mathbb{R}$ and k' points in $i+\mathbb{R}$, let $\mathscr{S}_2$ be a set consisting of h points in $\mathbb{R}$ and h' points in $i+\mathbb{R}$, and let $\mathscr{S}$ be a set consisting of $k+h$ points in $\mathbb{R}$ and $k'+h'$ points in $i+\mathbb{R}$. For $p\in]1,+\infty[$ consider the semi-Fredholm operators

$$\overline{\partial}_{A_1}: X^{1,p}_{\mathscr{S}_1,\mathscr{V}_1,\mathscr{V}_1'}(\Sigma,\mathbb{C}^n)\rightarrow X^p_{\mathscr{S}_1}(\Sigma,\mathbb{C}^n),\quad \overline{\partial}_{A_2}: X^{1,p}_{\mathscr{S}_2,\mathscr{V}_2,\mathscr{V}_2'}(\Sigma,\mathbb{C}^n)\rightarrow X^p_{\mathscr{S}_2}(\Sigma,\mathbb{C}^n)$$
$$\overline{\partial}_A: X^{1,p}_{\mathscr{S},\mathscr{V},\mathscr{V}'}(\Sigma,\mathbb{C}^n)\rightarrow X^p_{\mathscr{S}}(\Sigma,\mathbb{C}^n).$$

Then

$$\operatorname{ind}\overline{\partial}_A = \operatorname{ind}\overline{\partial}_{A_1} + \operatorname{ind}\overline{\partial}_{A_1}.$$

The proof is analogous to the proof of Theorem 3.2.12 in [Sch95], and we omit it. When there are no jumps, that is $\mathscr{S} = \emptyset$ and $\mathscr{V} = (V)$, $\mathscr{V}' = (V')$, Theorem 5.11 shows that the index of the operator

$$\overline{\partial}_A : X^{1,p}_{\emptyset,(V),(V')}(\Sigma, \mathbb{C}^n) = W^{1,p}_{N^*V,N^*V'}(\Sigma, \mathbb{C}^n) \to L^p(\Sigma, \mathbb{C}^n) = X^p_{\emptyset}(\Sigma, \mathbb{C}^n)$$

is

$$\operatorname{ind}\overline{\partial}_A = \mu(\Phi^- N^*V, N^*V') - \mu(\Phi^+ N^*V, N^*V').$$

In the general case, Proposition 5.19 shows that

$$\begin{aligned} &\operatorname{ind}(\overline{\partial}_A : X^{1,p}_{\mathscr{S},\mathscr{V},\mathscr{V}'}(\Sigma, \mathbb{C}^n) \to X^p_{\mathscr{S}}(\Sigma, \mathbb{C}^n)) \\ &= \mu(\Phi^- N^*V_0, N^*V_0') - \mu(\Phi^+ N^*V_k, N^*V'_{k'}) + c(V_0, \dots, V_k; V_0', \dots, V'_{k'}), \end{aligned} \tag{112}$$

where the correction term c satisfies the additivity formula

$$c(V_0, \dots, V_{k+h}; V_0', \dots, V'_{k'+h'}) = c(V_0, \dots, V_k; V_0', \dots, V'_{k'}) + c(V_k, \dots, V_{k+h}; V'_{k'}, \dots, V'_{k'+h'}). \tag{113}$$

Since the Maslov index is in general a half-integer, and since we have not proved that the cokernel of $\overline{\partial}_A$ is finite dimensional, the correction term c takes values in $(1/2)\mathbb{Z} \cup \{-\infty\}$. Actually, the analysis of this section shows that c is always finite, proving that $\overline{\partial}_A$ is Fredholm.

Clearly, we have the following direct sum formula

$$\begin{aligned} &c(V_0 \oplus W_0, \dots, V_k \oplus W_k; V_0' \oplus W_0', \dots, V'_{k'} \oplus W'_{k'}) \\ &= c(V_0, \dots, V_k; V_0', \dots, V'_{k'}) + c(W_0, \dots, W_k; W_0', \dots, W'_{k'}). \end{aligned} \tag{114}$$

Note also that the index formula of Theorem 5.11 produces a correction term of the form

$$c(\lambda; \nu) = \mu(\lambda, \nu), \tag{115}$$

where λ and ν are asymptotically constant paths of Lagrangian subspaces on $\mathbb{C}^n$. The Liouville type results of the previous section imply that

$$c((0), \mathbb{R}^n; \mathbb{R}^n) = -\frac{n}{2} = c(\mathbb{R}^n, (0); \mathbb{R}^n). \tag{116}$$

Indeed, by Proposition 5.17 the operator

$$\overline{\partial}_{\alpha I} : X^{1,p}_{\{0\},((0),\mathbb{R}^n),(\mathbb{R}^n)}(\Sigma, \mathbb{C}^n) \to X^p_{\{0\}}(\Sigma, \mathbb{C}^n)$$

is an isomorphism if $0 < \alpha < \pi/2$. By (80), the Maslov index of the path $e^{i\alpha t}\mathbb{R}^n$, $t \in [0,1]$, with respect to $\mathbb{R}^n$ is $-n/2$. On the other hand, the Maslov index of the path $e^{i\alpha t} i\mathbb{R}^n$, $t \in [0,1]$, with respect to $\mathbb{R}^n$ is 0 because the intersection is (0) for every $t \in [0,1]$. Inserting the information about the Fredholm and the Maslov index in (112), we find

$$0 = \operatorname{ind}\overline{\partial}_{\alpha I} = \frac{n}{2} + c((0), \mathbb{R}^n; (0)),$$

which implies the first identity in (116). The second one is proved in the same way by using Proposition 5.18.

5.20. LEMMA. *Let $(V_0, V_1, \dots, V_k)$ be a $(k+1)$-uple of linear subspaces of $\mathbb{R}^n$, with V_{j-1} and V_j partially orthogonal for every $j = 1, \dots, k$, and let W be a linear subspace of $\mathbb{R}^n$. Then*

$$c(V_0, \dots, V_k; W) = c(W; V_0, \dots, V_k) = -\frac{1}{2} \sum_{j=1}^{k} (\dim V_{j-1} + \dim V_j - 2 \dim V_{j-1} \cap V_j) .$$

Proof. Let us start by considering the case $W = \mathbb{R}^n$. By the additivity formula (113),

$$c(V_0, \dots, V_k; \mathbb{R}^n) = \sum_{j=1}^{k} c(V_{j-1}, V_j; \mathbb{R}^n).$$

Since V_{j-1} and V_j are partially orthogonal, $\mathbb{R}^n$ has an orthogonal splitting $\mathbb{R}^n = X_1^j \oplus X_2^j \oplus X_3^j \oplus X_4^j$ where $V_{j-1} = X_1^j \oplus X_2^j$ and $V_j = X_1^j \oplus X_3^j$. By the direct sum formula (114) and by formula (116),

$$\begin{aligned} c(V_{j-1}, V_j; \mathbb{R}^n) &= c(X_1^j, X_1^j; X_1^j) + c(X_2^j, (0); X_2^j) + c((0), X_3^j; X_3^j) + c((0), (0); X_4^j) \\ &= 0 - \frac{1}{2} \dim X_2^j - \frac{1}{2} \dim X_3^j + 0 = -\frac{1}{2} \dim X_2^j \oplus X_3^j \end{aligned}$$

Since

$$\dim X_2^j \oplus X_3^j = \dim V_{j-1} + \dim V_j - 2 \dim V_{j-1} \cap V_j,$$

the formula for $c(V_0, \dots, V_k; \mathbb{R}^n)$ follows.

Now let $\lambda : \mathbb{R} \to \mathscr{L}(n)$ be a continuous path of Lagrangian subspaces such that $\lambda(s) = \mathbb{R}^n$ for $s \leq -1$ and $\lambda(s) = N^*W$ for $s \geq 1$. By an easy generalization of the additivity formula (113) to the case of non-constant Lagrangian boundary conditions,

$$c(N^*V_0; \lambda) + c(V_0, \dots, V_k; W) = c(V_0, \dots, V_k; \mathbb{R}^n) + c(N^*V_k; \lambda). \tag{117}$$

By (115), $c(N^*V_0; \lambda) = -\mu(\lambda, N^*V_0)$ and $c(N^*V_k; \lambda) = -\mu(\lambda, N^*V_k)$, so (117) leads to

$$\begin{aligned} c(V_0, \dots, V^k; W) &= c(V_0, \dots, V^k; \mathbb{R}^n) - (\mu(\lambda, N^*V_k) - \mu(\lambda, N^*V_0)) \\ &= c(V_0, \dots, V^k; \mathbb{R}^n) - h(N^*V_0, N^*V_k; \mathbb{R}^n, N^*W), \end{aligned}$$

where h is the Hörmander index. By Lemma 5.2, the above Hörmander index vanishes, so we get the desired formula for $c(V_0, \dots, V_k; W)$. The formula for $c(W; V_0, \dots, V_k)$ follows by considering the change of variable $v(s,t) = \overline{u}(s, 1-t)$. □

The additivity formula (113) leads to

$$c(V_0, \dots, V_k; V_0', \dots, V_{k'}') = c(V_0, \dots, V_k; V_0') + c(V_k; V_0', \dots, V_{k'}'),$$

and the index formula in the general case follows from (112) and the above lemma. This concludes the proof of Theorem 5.9.

5.7 Half-strips with jumping conormal boundary conditions

This section is devoted to the analogue of Theorem 5.9 on the half-strips

$$\Sigma^+ := \{z \in \mathbb{C} \mid 0 \leq \operatorname{Im} z \leq 1, \ \operatorname{Re} z \geq 0\} \quad \text{and} \quad \Sigma^- := \{z \in \mathbb{C} \mid 0 \leq \operatorname{Im} z \leq 1, \ \operatorname{Re} z \leq 0\} .$$

In the first case, we fix the following data. Let $k, k' \geq 0$ be integers, let

$$0 = s_0 < s_1 < \cdots < s_k < s_{k+1} = +\infty, \quad 0 = s_0' < s_1' < \cdots < s_{k'}' < s_{k'+1}' = +\infty,$$

be real numbers, and let W, $V_0, \dots, V_k$, $V_0', \dots, V_{k'}'$ be linear subspaces of $\mathbb{R}^n$ such that V_{j-1} and V_j, V_{j-1}' and V_j', W and V_0, W and V_0', are partially orthogonal. We denote by $\mathscr{V}$ the $(k+1)$-uple

$(V_0,\dots,V_k)$, by $\mathscr{V}'$ the $(k'+1)$-uple $(V_0',\dots,V_{k'}')$, and by $\mathscr{S}$ the set $\{s_1,\dots,s_k,s_1'+i,\dots,s_{k'}'+i\}$. The X^p and $X^{1,p}$ norms on Σ^+ are defined as in section 5.3, and so are the spaces $X^p_{\mathscr{S}}(\Sigma^+,\mathbb{C}^n)$ and $X^{1,p}_{\mathscr{S}}(\Sigma^+,\mathbb{C}^n)$. Let $X^{1,p}_{\mathscr{S},W,\mathscr{V},\mathscr{V}'}(\Sigma^+,\mathbb{C}^n)$ be the completion of the space of maps $u\in C^\infty_{\mathscr{S},c}(\Sigma^+,\mathbb{C}^n)$ satisfying the boundary conditions

$$u(it)\in N^*W \quad \forall t\in[0,1],\quad u(s)\in N^*V_j \quad \forall s\in[s_j,s_{j+1}],\quad u(s+i)\in N^*V_j' \quad \forall s\in[s_j',s_{j+1}'],$$

with respect to the norm $\|u\|_{X^{1,p}(\Sigma^+)}$.

Let $A\in C^0([0,+\infty]\times[0,1],\mathrm{L}(\mathbb{R}^{2n},\mathbb{R}^{2n}))$ be such that $A(+\infty,t)$ is symmetric for every $t\in[0,1]$, and denote by $\Phi^+:[0,1]\to \mathrm{Sp}(2n)$ the solutions of the linear Hamiltonian system

$$\frac{d}{dt}\Phi^+(t)=iA(+\infty,t)\Phi^+(t),\quad \Phi^+(0)=I.$$

Then we have:

5.21. Theorem. *Assume that $\Phi^+(1)N^*V_k\cap N^*V_{k'}'=(0)$. Then the $\mathbb{R}$-linear bounded operator*

$$\overline{\partial}_A : X^{1,p}_{\mathscr{S},W,\mathscr{V},\mathscr{V}'}(\Sigma^+,\mathbb{C}^n)\to X^p_{\mathscr{S}}(\Sigma^+,\mathbb{C}^n),\quad \overline{\partial}_A u=\overline{\partial}u+Au,$$

is Fredholm of index

$$\begin{aligned}\operatorname{ind}\overline{\partial}_A=\frac{n}{2}-\mu(\Phi^+N^*V_k,N^*V_{k'}')-\frac{1}{2}(\dim V_0+\dim W-2\dim V_0\cap W)\\ -\frac12(\dim V_0'+\dim W-2\dim V_0'\cap W)-\frac12\sum_{j=1}^{k}(\dim V_{j-1}+\dim V_j-2\dim V_{j-1}\cap V_j)\\ -\frac12\sum_{j=1}^{k'}(\dim V_{j-1}'+\dim V_j'-2\dim V_{j-1}'\cap V_j').\end{aligned}\tag{118}$$

Proof. The proof of the fact that $\overline{\partial}_A$ is semi-Fredholm is analogous to the case of the full strip, treated in section 5.4. There remains compute the index. By an additivity formula analogous to (113), it is enough to prove (118) in the case with no jumps, that is $k=k'=0$, $\mathscr{V}=(V_0)$, $\mathscr{V}'=(V_0')$. In this case, we have a formula of the type

$$\operatorname{ind}\overline{\partial}_A=-\mu(\Phi^+N^*V_0,N^*V_0')+c(W;V_0;V_0'),$$

and we have to determine the correction term c.

Assume $W=(0)$, so that $N^*W=i\mathbb{R}^n$. Let us compute the correction term c when V_0 and V_0' are either (0) or $\mathbb{R}^n$. We can choose the map A to be the constant map $A(s,t)=\alpha I$, for $\alpha\in]0,\pi/2[$, so that $\Phi^+(t)=e^{i\alpha t}$. The Kernel and co-kernel of $\overline{\partial}_{\alpha I}$ are easy to determine explicitly, by separating the variables in the corresponding boundary value PDE's:

(i) If $V_0=V_0'=\mathbb{R}^n$, the kernel and co-kernel of $\overline{\partial}_{\alpha I}$ are both (0). Since $\mu(e^{i\alpha t}\mathbb{R}^n,\mathbb{R}^n)=-n/2$, we have $c((0);\mathbb{R}^n;\mathbb{R}^n)=-n/2$.

(ii) If $V_0=V_0'=(0)$, the kernel of $\overline{\partial}_{\alpha I}$ is $i\mathbb{R}^n e^{-\alpha s}$, while its co-kernel is (0). Since $\mu(e^{i\alpha t}i\mathbb{R}^n,i\mathbb{R}^n)=-n/2$, we have $c((0);(0);(0))=n/2$.

(iii) If either $V_0=\mathbb{R}^n$ and $V_0'=(0)$, or $V_0=(0)$ and $V_0'=\mathbb{R}^n$, the kernel and co-kernel of $\overline{\partial}_{\alpha I}$ are both (0). Since $\mu(e^{i\alpha t}\mathbb{R}^n,(0))=\mu(e^{i\alpha t}(0),\mathbb{R}^n)=0$, we have $c((0);\mathbb{R}^n;(0))=c((0);(0);\mathbb{R}^n)=0$.

Now let W, V_0, and V_0' be arbitrary (with W partially orthogonal to both V_0 and V_0'). Let $U\in\mathrm{U}(n)$ be such that $UN^*W=i\mathbb{R}^n$. Then $UN^*V_0=N^*W_0$ and $UN^*V_0'=N^*W_0'$, where

$$W_0=(V_0\cap W)^\perp\cap(V_0+W),\quad W_0'=(V_0'\cap W)^\perp\cap(V_0'+W).$$

By using the change of variable $v = Uu$, we find

$$c(W; V_0; V_0') = c((0); W_0; W_0'), \tag{119}$$

and we are reduced to compute the latter quantity. By an easy homotopy argument, using the fact that the Fredholm index is locally constant in the operator norm topology, we can assume that W_0 and W_0' are partially orthogonal. Then $\mathbb{R}^n$ has an orthogonal splitting $\mathbb{R}^n = X_1 \oplus X_2 \oplus X_3 \oplus X_4$, where

$$W_0 = X_1 \oplus X_2, \quad W_0' = X_1 \oplus X_3,$$

from which

$$N^* W_0 = X_1 \oplus X_2 \oplus iX_3 \oplus iX_4, \quad N^* W_0' = X_1 \oplus iX_2 \oplus X_3 \oplus iX_4.$$

Then the operator $\overline{\partial}_{\alpha I}$ decomposes as the direct sum of four operators, whose index is computed in cases (i), (ii), and (iii) above. Indeed,

$$\begin{aligned} c((0); W_0; W_0') &= \frac{1}{2} \dim X_4 - \frac{1}{2} \dim X_1 \\ &= \frac{1}{2} \mathrm{codim}(W_0 + W_0') - \frac{1}{2} \dim W_0 \cap W_0' = \frac{1}{2}(n - \dim W_0 - \dim W_0'). \end{aligned}$$

Since

$$\begin{aligned} \dim W_0 &= \dim(V_0 + W) - \dim V_0 \cap W = \dim V_0 + \dim W - 2 \dim W_0 \cap W, \\ \dim W_0' &= \dim(V_0' + W) - \dim V_0' \cap W = \dim V_0' + \dim W - 2 \dim W_0' \cap W, \end{aligned}$$

we find

$$c((0); W_0; W_0') = \frac{n}{2} - \frac{1}{2}(\dim V_0 + \dim W - 2 \dim W_0 \cap W) - \frac{1}{2}(\dim V_0' + \dim W - 2 \dim W_0' \cap W).$$

Together with (119), this proves formula (118). □

We conclude this section by considering the case of the left half-strip Σ^-. Let $k, k' \geq 0$, $\mathscr{V} = (V_0, \dots, V_k)$, and $\mathscr{V}' = (V_0', \dots, V_{k'}')$ be as above. Let

$$-\infty = s_{k+1} < s_k < \cdots < s_1 < s_0 = 0, \quad -\infty = s_{k'+1}' < s_{k'}' < \cdots < s_1' < s_0' = 0,$$

be real numbers, and set $\mathscr{S} = \{s_1, \dots, s_k, s_1' + i, \dots, s_{k'}' + i\}$.

Let $X^{1,p}_{\mathscr{S}, W, \mathscr{V}, \mathscr{V}'}(\Sigma^-, \mathbb{C}^n)$ be the completion of the space of maps $u \in C^\infty_{\mathscr{S},c}(\Sigma^-, \mathbb{C}^n)$ satisfying the boundary conditions

$$u(it) \in N^* W \ \ \forall t \in [0,1], \quad u(s) \in N^* V_j \ \ \forall s \in [s_{j+1}, s_j], \quad u(s+i) \in N^* V_j' \ \ \forall s \in [s_{j+1}', s_j'],$$

with respect to the norm $\|u\|_{X^{1,p}(\Sigma^-)}$.

Let $A \in C^0([-\infty, 0] \times [0,1], \mathrm{L}(\mathbb{R}^{2n}, \mathbb{R}^{2n}))$ be such that $A(-\infty, t)$ is symmetric for every $t \in [0,1]$, and denote by $\Phi^- : [0,1] \to \mathrm{Sp}(2n)$ the solutions of the linear Hamiltonian system

$$\frac{d}{dt} \Phi^-(t) = iA(-\infty, t) \Phi^-(t), \quad \Phi^-(0) = I.$$

Then we have:

5.22. Theorem. *Assume that $\Phi^-(1) N^* V_k \cap N^* V_{k'}' = (0)$. Then the $\mathbb{R}$-linear operator*

$$\overline{\partial}_A : X^{1,p}_{\mathscr{S}, W, \mathscr{V}, \mathscr{V}'}(\Sigma^-, \mathbb{C}^n) \to X^p_{\mathscr{S}}(\Sigma^-, \mathbb{C}^n), \quad \overline{\partial}_A u = \overline{\partial} u + Au, \tag{120}$$

is bounded and Fredholm of index

$$\begin{aligned} \operatorname{ind} \overline{\partial}_A = \frac{n}{2} + \mu(\Phi^- N^* V_k, N^* V'_{k'}) - \frac{1}{2}(\dim V_0 + \dim W - 2 \dim V_0 \cap W) \\ -\frac{1}{2}(\dim V'_0 + \dim W - 2 \dim V'_0 \cap W) - \frac{1}{2} \sum_{j=1}^{k} (\dim V_{j-1} + \dim V_j - 2 \dim V_{j-1} \cap V_j) \\ -\frac{1}{2} \sum_{j=1}^{k'} (\dim V'_{j-1} + \dim V'_j - 2 \dim V'_{j-1} \cap V'_j). \end{aligned} \tag{121}$$

Indeed, notice that if $u(s,t) = \overline{v}(-s,t)$, then

$$-(\overline{\partial} u(s,t) + A(s,t) u(s,t)) = C(\overline{\partial} v(-s,t) - CA(s,t) C v(-s,t)),$$

where C is denotes complex conjugacy. Then the operator (120) is obtained from the operator

$$\overline{\partial}_B : X^{1,p}_{-\mathscr{S},W,\mathscr{V},\mathscr{V}'}(\Sigma^+, \mathbb{C}^n) \to X^p_{-\mathscr{S}}(\Sigma^+, \mathbb{C}^n), \quad \overline{\partial}_B v = \overline{\partial} v + Bv,$$

where $B(s,t) = -CA(-s,t)C$, by left and right multiplication by isomorphisms. In particular, the indices are the same. Then Theorem 5.22 follows from Theorem 5.21, taking into account the fact that the solution Φ^+ of

$$\frac{d}{dt} \Phi^+(t) = iB(+\infty, t) \Phi^+(t), \quad \Phi^+(0) = I,$$

is $\Phi^+(t) = C\Phi^-(t)C$, so that

$$\mu(\Phi^+ N^* V_k, N^* V'_{k'}) = \mu(C \Phi^- C N^* V_k, N^* V'_{k'}) = -\mu(\Phi^- N^* V_k, N^* V'_{k'}).$$

5.8 Non-local boundary conditions

It is useful to dispose of versions of Theorems 5.9, 5.21, and 5.22, involving non-local boundary conditions. In the case of the full strip Σ, let us fix the following data. Let $k \geq 0$ be an integer, let

$$-\infty = s_0 < s_1 < \cdots < s_k < s_{k+1} = +\infty$$

be real numbers, and set $\mathscr{S} := \{s_1, \ldots, s_k, s_1 + i, \ldots, s_k + i\}$. Let $W_0, W_1, \ldots, W_k$ be linear subspaces of $\mathbb{R}^n \times \mathbb{R}^n$ such that W_{j-1} and W_j are partially orthogonal, for $j = 1, \ldots, k$, and set $\mathscr{W} = (W_0, \ldots, W_k)$.

The space $X^{1,p}_{\mathscr{S},\mathscr{W}}(\Sigma, \mathbb{C}^n)$ is defined as the completion of the space of all $u \in C^\infty_{\mathscr{S},c}(\Sigma, \mathbb{C}^n)$ such that

$$(u(s), \overline{u}(s+i)) \in N^* W_j, \quad \forall s \in [s_j, s_{j+1}], \quad j = 0, \ldots, k,$$

with respect to the norm $\|u\|_{X^{1,p}(\Sigma)}$.

Let $A \in C^0(\overline{\mathbb{R}} \times [0,1], \mathrm{L}(\mathbb{R}^{2n}, \mathbb{R}^{2n}))$ be such that $A(\pm\infty, t) \in \mathrm{Sym}(2n, \mathbb{R})$ for every $t \in [0,1]$, and define the symplectic paths $\Phi^+, \Phi^- : [0,1] \to \mathrm{Sp}(2n)$ as the solutions of the linear Hamiltonian systems

$$\frac{d}{dt} \Phi^\pm(t) = iA(\pm\infty, t) \Phi^\pm(t), \quad \Phi^\pm(0) = I.$$

Denote by C the complex conjugacy, and recall from section 5.1 that $\Phi \in \mathrm{L}(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ is symplectic if and only if graph $C\Phi$ is a Lagrangian subspace of $(\mathbb{R}^{2n} \times \mathbb{R}^{2n}, \omega_0 \times \omega_0)$. Then we have the following:

5.23. THEOREM. *Assume that* $\operatorname{graph} C\Phi^-(1) \cap N^*W_0 = (0)$ *and* $\operatorname{graph} C\Phi^+(1) \cap N^*W_k = (0)$. *Then for every* $p \in]1,+\infty[$ *the* $\mathbb{R}$*-linear operator*

$$\overline{\partial}_A : X^{1,p}_{\mathscr{S},\mathscr{W}}(\Sigma,\mathbb{C}^n) \to X^p_{\mathscr{S}}(\Sigma,\mathbb{C}^n), \quad u \mapsto \overline{\partial} u + Au,$$

is bounded and Fredholm of index

$$\begin{aligned}\operatorname{ind} \overline{\partial}_A &= \mu(N^*W_0, \operatorname{graph} C\Phi^-) - \mu(N^*W_k, \operatorname{graph} C\Phi^+) \\ &\quad -\frac{1}{2}\sum_{j=1}^{k}(\dim W_{j-1} + \dim W_j - 2\dim W_{j-1}\cap W_j).\end{aligned}$$

Proof. Given $u : \Sigma \to \mathbb{C}^n$ define $\tilde{u} : \Sigma \to \mathbb{C}^{2n}$ by

$$\tilde{u}(z) := (u(z/2), \overline{u}(\overline{z}/2 + i)).$$

The map $u \mapsto \tilde{u}$ determines a linear isomorphism

$$F : X^{1,p}_{\mathscr{S},\mathscr{W}}(\Sigma,\mathbb{C}^n) \xrightarrow{\cong} X^{1,p}_{\mathscr{S}',\mathscr{W},\mathscr{W}'}(\Sigma,\mathbb{C}^{2n}),$$

where $\mathscr{S}' = \{2s_1,\dots,2s_k, 2s_1+i,\dots,2s_k+i\}$, $\mathscr{W}'$ is the $(k+1)$-uple $(\Delta_{\mathbb{R}^n},\dots,\Delta_{\mathbb{R}^n})$, and we have used the identity

$$N^*\Delta_{\mathbb{R}^n} = \operatorname{graph} C = \{(w,\overline{w}) \mid w \in \mathbb{C}^n\}.$$

The map $v \mapsto \tilde{v}/2$ determines an isomorphism

$$G : X^p_{\mathscr{S}}(\Sigma,\mathbb{C}^n) \xrightarrow{\cong} X^p_{\mathscr{S}'}(\Sigma,\mathbb{C}^{2n}).$$

The composition $G \circ \overline{\partial}_A \circ F^{-1}$ is the operator

$$\overline{\partial}_{\tilde{A}} : X^{1,p}_{\mathscr{S}',\mathscr{W},\mathscr{W}'}(\Sigma,\mathbb{C}^{2n}) \longrightarrow X^p_{\mathscr{S}'}(\Sigma,\mathbb{C}^{2n}), \quad u \mapsto \overline{\partial} u + \tilde{A}u,$$

where

$$\tilde{A}(z) := \frac{1}{2}\left(A(z/2) \oplus CA(\overline{z}/2 + i)C\right).$$

Since

$$\tilde{A}(\pm\infty, t) = \frac{1}{2}\left(A(\pm\infty, t/2) \oplus CA(\pm\infty, 1 - t/2)C\right),$$

we easily see that the solutions $\tilde{\Phi}^\pm$ of

$$\frac{d}{dt}\tilde{\Phi}^\pm(t) = i\tilde{A}(\pm\infty,t)\tilde{\Phi}^\pm(t), \quad \tilde{\Phi}^\pm(0) = I,$$

are given by

$$\tilde{\Phi}^\pm(t) = \Phi^\pm(t/2) \oplus C\Phi^\pm(1-t/2)\Phi(1)^{-1}C.$$

The above formula implies

$$\tilde{\Phi}^\pm(t)^{-1}N^*\Delta_{\mathbb{R}^n} = \operatorname{graph} C\Phi^\pm(1)\Phi^\pm(1-t/2)^{-1}\Phi^\pm(t/2). \tag{122}$$

For $t = 1$ we get

$$\begin{aligned}\tilde{\Phi}^-(1)N^*W_0 \cap N^*\Delta_{\mathbb{R}^n} &= \tilde{\Phi}^-(1)[N^*W_0 \cap \operatorname{graph} C\Phi^-(1)] = (0),\\ \tilde{\Phi}^+(1)N^*W_k \cap N^*\Delta_{\mathbb{R}^n} &= \tilde{\Phi}^+(1)[N^*W_k \cap \operatorname{graph} C\Phi^+(1)] = (0),\end{aligned}$$

so the transversality hypotheses of Theorem 5.9 are fulfilled. By this theorem, the operator $\overline{\partial}_A = G^{-1} \circ \overline{\partial}_{\tilde{A}} \circ F$ is Fredholm of index

$$\begin{aligned} \operatorname{ind} \overline{\partial}_A = \operatorname{ind} \overline{\partial}_{\tilde{A}} = \mu(\tilde{\Phi}^- N^* W_0, N^* \Delta_{\mathbb{R}^n}) - \mu(\tilde{\Phi}^+ N^* W_k, N^* \Delta_{\mathbb{R}^n}) \\ -\frac{1}{2} \sum_{j=1}^{k} (\dim W_{j-1} + \dim W_j - 2 \dim W_{j-1} \cap W_j). \end{aligned} \tag{123}$$

The symplectic paths $t \mapsto \Phi^{\pm}(1)\Phi^{\pm}(1-t/2)^{-1}\Phi^{\pm}(t/2)$ and $t \mapsto \Phi^{\pm}(t)$ are homotopic by means of the symplectic homotopy

$$(\lambda, t) \mapsto \Phi^{\pm}(1)\, \Phi^{\pm} \left(\frac{1+\lambda}{2} - \frac{1-\lambda}{2} t \right)^{-1} \Phi^{\pm} \left(\frac{1+\lambda}{2} t \right),$$

which fixes the end-points I and $\Phi^{\pm}(1)$. By the symplectic invariance and the homotopy invariance of the Maslov index we deduce from (122) that

$$\begin{aligned} \mu(\tilde{\Phi}^- N^* W_0, N^* \Delta_{\mathbb{R}^n}) = \mu(N^* W_0, \tilde{\Phi}^-(\cdot)^{-1} N^* \Delta_{\mathbb{R}^n}) \\ = \mu(N^* W_0, \operatorname{graph} C\Phi^-(1)\Phi^-(1-\cdot/2)^{-1}\Phi^-(\cdot/2)) = \mu(N^* W_0, \operatorname{graph} C\Phi^-). \end{aligned} \tag{124}$$

Similarly,

$$\mu(\tilde{\Phi}^+ N^* W_k, N^* \Delta_{\mathbb{R}^n}) = \mu(N^* W_k, \operatorname{graph} C\Phi^+). \tag{125}$$

The conclusion follows from (123), (124), and (125). □

In the case of the right half-strip Σ^+, we fix an integer $k \geq 0$, real numbers

$$0 = s_0 < s_1 < \cdots < s_k < s_{k+1} = +\infty,$$

a linear subspace $V_0 \subset \mathbb{R}^n$ and a $(k+1)$-uple $\mathscr{W} = (W_0, \dots, W_k)$ of linear subspaces of $\mathbb{R}^n \times \mathbb{R}^n$, such that W_0 and $V_0 \times V_0$ are partially orthogonal, and so are W_{j-1} and W_j, for every $j = 1, \dots, k$. Set $\mathscr{S} = \{s_1, \dots, s_k, s_1 + i, \dots, s_k + i\}$, and let $X^{1,p}_{\mathscr{S}, V_0, \mathscr{W}}(\Sigma^+, \mathbb{C}^n)$ be the completion of the space of maps $u \in C^{\infty}_{\mathscr{S},c}(\Sigma^+, \mathbb{C}^n)$ such that

$$u(it) \in V_0 \quad \forall t \in [0,1], \quad (u(s), \overline{u}(s+i)) \in N^* W_j, \quad \forall s \in [s_j, s_{j+1}], \quad j = 0, \dots, k,$$

with respect to the norm $\|u\|_{X^{1,p}(\Sigma^+)}$.

Let $A \in C^0([0,+\infty] \times [0,1], \mathrm{L}(\mathbb{R}^{2n}, \mathbb{R}^{2n}))$ be such that $A(+\infty, t) \in \mathrm{Sym}(2n, \mathbb{R})$ for every $t \in [0,1]$, and let $\Phi^+ : [0,1] \to \mathrm{Sp}(2n)$ be the solution of the linear Hamiltonian systems

$$\frac{d}{dt} \Phi^+(t) = iA(+\infty, t)\Phi^+(t), \quad \Phi^+(0) = I.$$

Then we have:

5.24. Theorem. *Assume that* $\operatorname{graph} C\Phi^+(1) \cap N^* W_k = (0)$. *Then for every* $p \in]1, +\infty[$ *the* $\mathbb{R}$*-linear operator*

$$\overline{\partial}_A : X^{1,p}_{\mathscr{S}, V_0, \mathscr{W}}(\Sigma^+, \mathbb{C}^n) \to X^p_{\mathscr{S}}(\Sigma^+, \mathbb{C}^n), \quad u \mapsto \overline{\partial} u + Au,$$

is bounded and Fredholm of index

$$\begin{aligned} \operatorname{ind} \overline{\partial}_A = \frac{n}{2} - \mu(N^* W_k, \operatorname{graph} C\Phi^+) - \frac{1}{2}(\dim W_0 + 2\dim V_0 - 2 \dim W_0 \cap (V_0 \times V_0)) \\ -\frac{1}{2} \sum_{j=1}^{k} (\dim W_{j-1} + \dim W_j - 2 \dim W_{j-1} \cap W_j). \end{aligned}$$

Proof. By the same argument used in the proof of Theorem 5.23, the operator $\overline{\partial}_A$ is Fredholm and has the same index of the operator

$$\overline{\partial}_{\tilde{A}} : X^{1,p}_{\mathscr{S}',V_0\times V_0,\mathscr{W},\mathscr{W}'}(\Sigma^+,\mathbb{C}^{2n}) \to X^p_{\mathscr{S}'}(\Sigma^+,\mathbb{C}^{2n}), \quad u\mapsto \overline{\partial}u+\tilde{A}u,$$

where $\mathscr{S}' := \{2s_1,\dots,2s_k,2s_1+i,\dots,2s_k+i\}$, $\mathscr{W}'$ is the $(k+1)$-uple $(\Delta_{\mathbb{R}^n},\dots,\Delta_{\mathbb{R}^n})$, and

$$\tilde{A}(z) := \frac{1}{2}(A(z/2)\oplus CA(\overline{z}/2+i)C).$$

By Theorem 5.21 and by (125), the index of this operator is

$$\begin{aligned}
\operatorname{ind}\overline{\partial}_{\tilde{A}} = n-\mu(N^*W_k,\operatorname{graph}C\Phi^+)-\frac{1}{2}(\dim\Delta_{\mathbb{R}^n}+\dim V_0\times V_0-2\dim\Delta_{\mathbb{R}^n}\cap(V_0\times V_0))\\
-\frac{1}{2}(\dim W_0+\dim V_0\times V_0-2\dim W_0\cap(V_0\times V_0))\\
-\frac{1}{2}\sum_{j=1}^k(\dim W_{j-1}+\dim W_j-2\dim W_{j-1}\cap W_j)\\
= n-\mu(N^*W_k,\operatorname{graph}C\Phi^+)-\frac{n}{2}-\frac{1}{2}(\dim W_0+2\dim V_0-2\dim W_0\cap(V_0\times V_0))\\
-\frac{1}{2}\sum_{j=1}^k(\dim W_{j-1}+\dim W_j-2\dim W_{j-1}\cap W_j).
\end{aligned}$$

The desired formula follows. □

In the case of the left half-strip Σ^-, let k, V_0, $\mathscr{W}$ be as above, and let $\mathscr{S}=\{s_1,\dots,s_k,s_1+i,\dots,s_k+i\}$ with

$$0=s_0>s_1>\dots>s_k>s_{k+1}=-\infty.$$

Let $X^{1,p}_{\mathscr{S},V_0,\mathscr{W}}(\Sigma^-,\mathbb{C}^n)$ the completion of the space of all maps $u\in C^\infty_{\mathscr{S},c}(\Sigma^-,\mathbb{C}^n)$ such that

$$u(it)\in V_0 \ \ \forall t\in[0,1], \quad (u(s),\overline{u}(s+i))\in N^*W_j, \quad \forall s\in[s_{j+1},s_j], \quad j=0,\dots,k,$$

with respect to the norm $\|u\|_{X^{1,p}(\Sigma^-)}$.

Let $A\in C^0([-\infty,0]\times[0,1],\mathrm{L}(\mathbb{R}^{2n},\mathbb{R}^{2n}))$ be such that $A(-\infty,t)$ is symmetric for every $t\in[0,1]$, and let $\Phi^-:[0,1]\to \mathrm{Sp}(2n)$ be the solution of the linear Hamiltonian systems

$$\frac{d}{dt}\Phi^-(t)=iA(-\infty,t)\Phi^+(t), \quad \Phi^-(0)=I.$$

Then we have:

5.25. THEOREM. *Assume that* $\operatorname{graph}C\Phi^-(1)\cap N^*W_k=(0)$. *Then for every* $p\in]1,+\infty[$ *the* $\mathbb{R}$*-linear operator*

$$\overline{\partial}_A : X^{1,p}_{\mathscr{S},V_0,\mathscr{W}}(\Sigma^-,\mathbb{C}^n)\to X^p_{\mathscr{S}}(\Sigma^-,\mathbb{C}^n), \quad u\mapsto\overline{\partial}u+Au,$$

is bounded and Fredholm of index

$$\begin{aligned}
\operatorname{ind}\overline{\partial}_A=\frac{n}{2}+\mu(N^*W_k,\operatorname{graph}C\Phi^-)-\frac{1}{2}(\dim W_0+2\dim V_0-2\dim W_0\cap(V_0\times V_0))\\
-\frac{1}{2}\sum_{j=1}^k(\dim W_{j-1}+\dim W_j-2\dim W_{j-1}\cap W_j).
\end{aligned}$$

5.9 Coherent orientations

As noticed in [AS06b, section 1.4], the problem of giving coherent orientations for the spaces of maps arising in Floer homology on cotangent bundles is somehow simpler than in the case of a general symplectic manifolds, treated in [FH93]. This fact remains true if we deal with Cauchy-Riemann type operators on strips and half-strips with jumping conormal boundary conditions. We briefly discuss this issue in the general case of nonlocal boundary conditions on the strip, the case of the half-strip being similar (see [AS06b, section 3.2]).

We recall that the space $\mathrm{Fred}(E,F)$ of Fredholm linear operators from the real Banach space E to the real Banach space F is the base space of a smooth real non-trivial line-bundle $\det(\mathrm{Fred}(E,F))$, with fibers

$$\det(A) := \Lambda^{\max}(\ker A) \otimes (\Lambda^{\max}(\operatorname{coker} A))^*, \quad \forall A \in \mathrm{Fred}(E,F),$$

where $\Lambda^{\max}(V)$ denotes the component of top degree in the exterior algebra of the finite-dimensional vector space V (see [Qui85]).

Let us recall the setting from section 5.8. We fix the data $k \geq 0$, $\mathscr{S} = \{s_1, \dots, s_k, s_1+i, \dots, s_k+i\}$, with $s_1 < \dots < s_k$, and $\mathscr{W} = (W_0, \dots, W_k)$, where $W_0, \dots, W_k$ are linear subspaces of $\mathbb{R}^n \times \mathbb{R}^n$, such that W_{j-1} is partially orthogonal to W_j, for $j = 1, \dots, k$. Let $A^\pm : [0,1] \to \mathrm{Sym}(\mathbb{R}^n)$ be continuous paths of symmetric matrices such that the linear problems

$$\begin{cases} w'(t) = iA^-(t)w(t), \\ (w(0), Cw(1)) \in N^*W_0, \end{cases} \qquad \begin{cases} w'(t) = iA^+(t)w(t), \\ (w(0), Cw(1)) \in N^*W_k, \end{cases}$$

have only the trivial solution $w = 0$. Such paths are refereed to as *non-degenerate* paths (with respect to W_0 and W_k, respectively). Fix some $p > 1$, and let $\mathscr{D}_{\mathscr{S},\mathscr{W}}(A^-, A^+)$ be the space of operators of the form

$$\overline{\partial}_A : X^{1,p}_{\mathscr{S},\mathscr{W}}(\Sigma, \mathbb{C}^n) \to X^p_{\mathscr{S}}(\Sigma, \mathbb{C}^n), \quad u \mapsto \overline{\partial} u + Au,$$

where $A \in C^0(\overline{\mathbb{R}} \times [0,1], \mathrm{L}(\mathbb{R}^{2n}, \mathbb{R}^{2n}))$ is such that $A(\pm\infty, t) = A^\pm(t)$ for every $t \in [0,1]$. By Theorem 5.23, $\mathscr{D}_{\mathscr{S},\mathscr{W}}(A^-, A^+)$ is a subset of $\mathrm{Fred}(X^{1,p}_{\mathscr{S},\mathscr{W}}(\Sigma, \mathbb{C}^n), X^p_{\mathscr{S}}(\Sigma, \mathbb{C}^n))$. It is actually a convex subset, so the restriction of the determinant bundle to $\mathscr{D}_{\mathscr{S},\mathscr{W}}(A^-, A^+)$ - that we denote by $\det(\mathscr{D}_{\mathscr{S},\mathscr{W}}(A^-, A^+))$ - is trivial.

Let $\mathfrak{S}$ be the family of all subsets of Σ consisting of exactly k pairs of opposite boundary points. It is a k-dimensional manifold, diffeomorphic to an open subsets of $\mathbb{R}^k$. An orientation of $\det(\mathscr{D}_{\mathscr{S},\mathscr{W}}(A^-, A^+))$ for a given $\mathscr{S}$ in $\mathfrak{S}$ uniquely determines an orientation for all choices of $\mathscr{S}' \in \mathfrak{S}$. Indeed, the disjoint unions

$$\bigsqcup_{\mathscr{S} \in \mathfrak{S}} X^{1,p}_{\mathscr{S},\mathscr{W}}(\Sigma, \mathbb{C}^n), \quad \bigsqcup_{\mathscr{S} \in \mathfrak{S}} X^p_{\mathscr{S}}(\Sigma, \mathbb{C}^n),$$

define locally trivial Banach bundles over $\mathfrak{S}$, and the operators $\overline{\partial}_A$ define a Fredholm bundle-morphism between them. Since $\mathfrak{S}$ is connected and simply connected, an orientation of the determinant space of this operator between the fibers of a given point $\mathscr{S}$ induces an orientation of the determinant spaces of the operators over each $\mathscr{S}' \in \mathfrak{S}$.

The space of all Fredholm bundle-morphisms between the above Banach bundles induced by operators of the form $\overline{\partial}_A$ with fixed asymptotic paths A^- and A^+ is denoted by $\mathscr{D}_{\mathscr{W}}(A^-, A^+)$. An orientation of the determinant bundle over this space of Fredholm bundle-morphisms is denoted by $o_{\mathscr{W}}(A^-, A^+)$.

Let $\mathscr{W} = (W_0, \dots, W_k)$, $\mathscr{W}' = (W_k, \dots, W_{k+k'})$ be vectors consisting of consecutively partially orthogonal linear subspaces of $\mathbb{R}^n \times \mathbb{R}^n$, and set

$$\mathscr{W} \# \mathscr{W}' := (W_0, \dots, W_{k+k'}).$$

Let A_0, A_1, A_2 be non-degenerate paths with respect to W_0, W_k, and $W_{k+k'}$, respectively. Then orientations $o_{\mathscr{W}}(A_0, A_1)$ and $o_{\mathscr{W}'}(A_1, A_2)$ of $\det(\mathscr{D}_{\mathscr{W}}(A_0, A_1))$ and $\det(\mathscr{D}_{\mathscr{W}'}(A_1, A_2))$, respectively, determine in a canonical way a *glued orientation*

$$o_{\mathscr{W}}(A_0, A_1) \# o_{\mathscr{W}'}(A_1, A_2)$$

of $\det(\mathscr{D}_{\mathscr{W}\#\mathscr{W}'}(A_0, A_2))$. The construction is analogous to the one described in [FH93, section 3]. This way of gluing orientations is associative. A *coherent orientation* is a set of orientations $o_{\mathscr{W}}(A^-, A^+)$ for each choice of compatible data such that

$$o_{\mathscr{W}\#\mathscr{W}'}(A_0, A_1) = o_{\mathscr{W}}(A_0, A_1) \# o_{\mathscr{W}'}(A_1, A_2),$$

whenever the latter glued orientation is well-defined. The proof of the existence of a coherent orientation is analogous to the proof of Theorem 12 in [FH93].

The choice of such a coherent orientation in this linear setting determines orientations for all the nonlinear objects we are interested in, and such orientations are compatible with gluing. As mentioned above, the fact that we are dealing with the cotangent bundle of an oriented manifold makes the step from the linear setting to the nonlinear one easier. The reason is that we can fix once for all special symplectic trivializations of the bundle $x^*(TT^*M)$, for every solution x of our Hamiltonian problem. In fact, one starts by fixing an orthogonal and orientation preserving trivialization of $(\pi \circ x)^*(TM)$, and then considers the induced unitary trivialization of $x^*(TT^*M)$. Let u be an element in some space $\mathscr{M}(x, y)$, consisting of the solutions of a Floer equation on the strip Σ which are asymptotic to two Hamiltonian orbits x and y and satisfy suitable jumping co-normal boundary conditions. Then we can find a unitary trivialization of $u^*(TT^*M)$ which converges to the given unitary trivializations of $x^*(TT^*M)$ and $y^*(TT^*M)$. We may use such a trivialization to linearize the problem, producing a Fredholm operator in $\mathscr{D}_{\mathscr{S},\mathscr{W}}(A^-, A^+)$. Here A^-, A^+ are determined by the fixed unitray trivializations of $x^*(TT^*M)$ and $y^*(TT^*M)$. The orientation of the determinant bundle over $\mathscr{D}_{\mathscr{S},\mathscr{W}}(A^-, A^+)$ then induced an orientation of the tangent space of $\mathscr{M}(x, y)$ at u, that is an orientation of $\mathscr{M}(x, y)$. See [AS06b, section 1.4] for more details.

When the manifold M is not orientable, one cannot fix once for all trivializations along the Hamiltonian orbits, and the construction of coherent orientations requires understanding the effect of changing the trivialization, as in [FH93, Lemma 15]. The Floer complex and the pair-of-pants product are still well-defined over integer coefficients, whereas the Chas-Sullivan loop product requires $\mathbb{Z}_2$ coefficients.

5.10 Linearization

In this section we recall the nonlinear setting which allows to see the various spaces of solutions of the Floer equation considered in this paper as zeroes of sections of Banach bundles. By showing that the fiberwise derivatives of such sections are conjugated to linear perturbed Cauchy-Riemann operators on a strip with jumping conormal boundary conditions, we prove that these spaces of solutions are generically manifolds, and we compute their dimension. We treat with some details the case of $\mathscr{M}_\Upsilon^\Omega$, the space of solutions of the Floer equation on the holomorphic triangle, defining the triangle product on Floer homology. The functional setting for the other spaces of solutions is similar, so in the other cases we mainly focus on the dimension computation.

The space $\mathscr{M}_\Upsilon^\Omega$. Let us consider the model case of $\mathscr{M}_\Upsilon^\Omega(x_1, x_2; y)$, where $x_1 \in \mathscr{P}^\Omega(H_1)$, $x_2 \in \mathscr{P}^\Omega(H_2)$, and $y \in \mathscr{P}^\Omega(H_1 \# H_2)$ (see section 3.3). This is a space of solutions of the Floer equation on the Riemann surface with boundary Σ_Υ^Ω, described as a strip with a slit in section 3.2.

Let us fix some $p \in]2, +\infty[$, and let us consider the space $\mathscr{W}_\Upsilon^\Omega = \mathscr{W}_\Upsilon^\Omega(x_1, x_2; y)$ of maps $u : \Sigma_\Upsilon^\Omega \to T^*M$ mapping the boundary of Σ_Υ^Ω into $T^*_{q_0}M$, which are of Sobolev class $W^{1,p}$ on compact subsets of Σ_Υ^Ω, for which there exists $s_0 > 0$ such that

$$\begin{aligned} u(s,t-1) &= \exp_{x_1(t)} \zeta_1(s,t), && \forall (s,t) \in]-\infty,-s_0[\times[0,1], \\ u(s,t) &= \exp_{x_2(t)} \zeta_2(s,t), && \forall (s,t) \in]-\infty,-s_0[\times[0,1], \\ u(s,2t-1) &= \exp_{y(t)} \zeta(s,t), && \forall (s,t) \in]s_0,+\infty[\times[0,1], \end{aligned}$$

for suitable $W^{1,p}$ sections ζ_1, ζ_2, ζ of the vector bundles

$$x_1^*(TT^*M) \to]-\infty,-s_0[\times[0,1], \;\; x_2^*(TT^*M) \to]-\infty,-s_0[\times[0,1], \;\; y*(TT^*M) \to]s_0,+\infty[\times[0,1].$$

Here "exp" is the exponential map given by some metric on T^*M, but the space $\mathscr{W}_\Upsilon^\Omega$ does not depend on the choice of this metric. Notice also that when we say "of Sobolev class $W^{1,p}$ on compact subsets of Σ_Υ^Ω", we consider Σ_Υ^Ω endowed with its smooth structure (and not with the structure endowed by the singular coordinate $z = s+it$). Since $p > 2$, the space $\mathscr{W}_\Upsilon^\Omega$ is an infinite dimensional manifold modeled on the real Banach space

$$W^{1,p}_{i\mathbb{R}^n}(\Sigma_\Upsilon^\Omega, \mathbb{C}^n) = W^{1,p}_0(\Sigma_\Upsilon^\Omega, \mathbb{R}^n) \oplus W^{1,p}(\Sigma_\Upsilon^\Omega, i\mathbb{R}^n).$$

Notice that by our definition of the smooth structure of Σ_Υ^Ω, a Banach norm of $W^{1,p}_{i\mathbb{R}^n}(\Sigma_\Upsilon^\Omega, \mathbb{C}^n)$ is

$$\|v\|_1^p := \int_{|\mathrm{Im}\, z|<1,\ |z|>1} (|v(z)|^p + |Dv(z)|^p)\, ds\, dt + \int_{|z|<1} \left(\frac{|v(z)|^p}{|z|} + |Dv(z)|^p |z|^{p/2-1} \right) ds\, dt. \tag{126}$$

Let $\mathscr{E}_\Upsilon^\Omega$ be the Banach bundle over $\mathscr{W}_\Upsilon^\Omega$ whose fiber at u is the space of $u^*(TT^*M)$-valued J-anti-linear one-forms on Σ_Υ^Ω of class L^p. A smooth trivialization (with suitable asymptotics and boundary conditions) of $u^*(TT^*M)$ allows to identify the fiber of $\mathscr{E}_\Upsilon^\Omega$ at u with $\Omega^{0,1}_{L^p}(\Sigma_\Upsilon^\Omega, \mathbb{C}^n)$, the Banach space of $\mathbb{C}^n$-valued complex anti-linear one-forms on Σ_Υ^Ω of class L^p. Again, if an element of $w \in \Omega^{0,1}_{L^p}(\Sigma_\Upsilon^\Omega, \mathbb{C}^n)$ is expressed in terms of the singular global coordinate $z = s+it$ as

$$w = w_0(z)ds - iw_0(z)dt = w_0(z)\, d\overline{z}, \quad w_0 : \Sigma_\Upsilon^\Omega \to \mathbb{C}^n,$$

its L^p-norm on Σ_Υ^Ω is equivalent to

$$\|w\|_0^p := \int_{|\mathrm{Im}\, z|<1,\ |z|>1} |w_0(z)|^p\, ds\, dt + \int_{|z|<1} |w_0(z)|^p |z|^{p/2-1}\, ds\, dt. \tag{127}$$

The perturbed Cauchy-Riemann operator

$$u \mapsto \overline{D}_J u + F_{J,H}(u),$$

defines a smooth section $\overline{D}_{J,H}$ of the Banach bundle $\mathscr{E}_\Upsilon^\Omega$. Elliptic regularity (Theorem 5.3) and exponential estimates at infinity (Proposition 5.12) allow to prove that $\mathscr{M}_\Upsilon^\Omega$ is the set of zeroes of the section $\overline{D}_{J,H} : \mathscr{W}_\Upsilon^\Omega \to \mathscr{E}_\Upsilon^\Omega$.

By choosing a unitary trivialization of $u^*(TT^*M)$, the fiberwise derivative of $\overline{D}_{J,H}$ at $u \in \mathscr{M}_\Upsilon^\Omega$ is easily shown to be conjugated to an operator of the form

$$\overline{D} + G : W^{1,p}_{i\mathbb{R}^n}(\Sigma_\Upsilon^\Omega, \mathbb{C}^n) \to \Omega^{0,1}_{L^p}(\Sigma_\Upsilon^\Omega, \mathbb{C}^n), \quad v \mapsto \frac{1}{2}(Dv + iDv \circ j) + Gv,$$

j denoting the complex structure on Σ_Υ^Ω. Using the singular coordinate $z = s+it$, the Cauchy-Riemann operator $\overline{D}$ and the multiplication operator G take the form

$$\overline{D}v = \frac{1}{2}(\partial_s v + i\partial_t v)\, ds - \frac{i}{2}(\partial_s v + i\partial_t v)\, dt, \quad (Gv)(z) = \frac{1}{2}A(z)v(z)ds - \frac{i}{2}A(z)v(z)dt,$$

where A is a smooth map taking value into $\mathrm{L}(\mathbb{R}^{2n},\mathbb{R}^{2n})$. Since $u(s,t-1)$ converges to $x_1(t)$ for $s\to-\infty$, $u(s,t)$ converges to $x_2(t)$ for $s\to-\infty$, and $u(s,2t-1)$ converges to $y(t)$ for $s\to+\infty$, for any $t\in[0,1]$, the $L(\mathbb{R}^{2n},\mathbb{R}^{2n})$-valued function A has the following asymptotics:

$$A(s+(t-1)i)\to A_1^-(t),\;\; A(s+ti)\to A_2^-(t)\;\text{ for } s\to-\infty,\quad A(s+(2t-1)i)\to A^+(t),\;\text{ for } s\to+\infty,$$

for any $t\in[0,1]$, where $A_1^-(t)$, $A_2^-(t)$, and $A^+(t)$ are symmetric matrices such that the solutions of the linear Hamiltonian systems

$$\frac{d}{dt}\Psi_1^-(t)=iA_1^-(t)\Psi_1^-(t),\quad \frac{d}{dt}\Psi_2^-(t)=iA_2^-(t)\Psi_2^-(t),$$
$$\frac{d}{dt}\Psi^+(t)=2iA^+(t)\Psi^+(t),\quad \Psi_1^-(0)=\Psi_2^-(0)=\Psi^+(0)=I,$$

are conjugated to the differential of the Hamiltonian flows along x_1, x_2, and y:

$$\Psi_1^-(t)\sim D_x\phi^{H_1}(1,x_1(0)),\quad \Psi_2^-(t)\sim D_x\phi^{H_2}(1,x_2(0)),\quad \Psi^+(t)\sim D_x\phi^{H_1\#H_2}(1,y(0)).$$

Then, by the definition of the Maslov index μ^Ω in terms of the relative Maslov index μ, we have

$$\mu^\Omega(x_1)=\mu(\Psi_1^- i\mathbb{R}^n,i\mathbb{R}^n)-\frac{n}{2},\quad \mu^\Omega(x_2)=\mu(\Psi_2^- i\mathbb{R}^n,i\mathbb{R}^n)-\frac{n}{2},\quad \mu^\Omega(y)=\mu(\Psi^+ i\mathbb{R}^n,i\mathbb{R}^n)-\frac{n}{2}. \tag{128}$$

We claim that the linear operator

$$\overline{D}+G: W^{1,p}_{i\mathbb{R}^n}(\Sigma^\Omega_\Upsilon,\mathbb{C}^n)\to\Omega^{0,1}_{L^p}(\Sigma^\Omega_\Upsilon,\mathbb{C}^n).$$

is Fredholm of index

$$\operatorname{ind}(\overline{D}+G)=\mu^\Omega(x_1)+\mu^\Omega(x_2)-\mu^\Omega(y).$$

In order to deduce this claim from Theorem 5.9, we show that the operator $\overline{D}+G$ is conjugated to a linear perturbed Cauchy-Riemann operator on a strip with jumping Lagrangian boundary conditions, in the sense of section 5.3.

Indeed, given $v:\Sigma^\Omega_\Upsilon\to\mathbb{C}^n$ let us consider the $\mathbb{C}^{2n}$-valued map $\tilde v$ on $\Sigma=\{0\le\operatorname{Im}z\le1\}$ defined as

$$\tilde v(z):=(\overline{v}(\overline{z}),v(z)).$$

The map $v\mapsto\tilde v$ gives us an isomorphism

$$W^{1,p}_{i\mathbb{R}^n}(\Sigma^\Omega_\Upsilon,\mathbb{C}^n)\cong X^{1,p}_{\mathscr{S},\mathscr{V},\mathscr{V}'}(\Sigma,\mathbb{C}^{2n}),$$

where

$$\mathscr{S}=\{0\},\quad \mathscr{V}=((0),\Delta_{\mathbb{R}^n}),\quad \mathscr{V}'=(0),$$

$\Delta_{\mathbb{R}^n}$ being the diagonal subspace of $\mathbb{R}^n\times\mathbb{R}^n$, and (0) being the zero subspace of $\mathbb{R}^{2n}$. This follows from comparing the norm (126) to the $X^{1,p}_{\mathscr{S}}$ norm by means of (87), (88), (90) and (91). On the other hand, by comparing the norm (127) to the $X^p_{\mathscr{S}}$ norm by (89), we see that the map $w\mapsto 2\widetilde{w[\partial_s]}$ gives us an isomorphism

$$\Omega^{0,1}_{L^p}(\Sigma^\Omega_\Upsilon,\mathbb{C}^n)\cong X^p_{\mathscr{S}}(\Sigma,\mathbb{C}^{2n}).$$

It is easily seen that composing the operator $\overline{D}+G$ by these two isomorphisms produces the operator

$$\overline{\partial}_{\tilde A}: X^{1,p}_{\mathscr{S},\mathscr{V},\mathscr{V}'}(\Sigma,\mathbb{C}^{2n})\to X^p_{\mathscr{S}}(\Sigma,\mathbb{C}^{2n}),\quad u\mapsto\overline{\partial}u+\tilde A u,$$

where $\tilde{A}(z) = CA(\overline{z})C \oplus A(z)$, C denoting complex conjugacy on $\mathbb{C}^n$.

By Theorem 5.9, the above operator $\overline{\partial}_{\tilde{A}}$ - hence the original operator $\overline{D} + G$ - is Fredholm of index

$$\operatorname{ind}(\overline{\partial}_{\tilde{A}}) = \mu(\Phi^- i\mathbb{R}^{2n}, i\mathbb{R}^{2n}) - \mu(\Phi^+ N^*\Delta_{\mathbb{R}^n}, i\mathbb{R}^{2n}) - \frac{n}{2}, \tag{129}$$

where $\Phi^-, \Phi^+ : [0,1] \to \mathrm{Sp}(4n)$ solve the linear Hamiltonian systems

$$\frac{d}{dt}\Phi^-(t) = i\tilde{A}(-\infty, t)\Phi^-(t), \quad \frac{d}{dt}\Phi^+(t) = i\tilde{A}(+\infty, t)\Phi^+(t), \quad \Phi^-(0) = \Phi^+(0) = I.$$

Since $\tilde{A}(-\infty, t) = CA_1(-t)C \oplus A_2(t)$, we have $\Phi^-(t) = C\Psi_1^-(-t)C \oplus \Psi_2(t)$, and

$$\begin{aligned}\mu(C\Psi_1^-(-\cdot)Ci\mathbb{R}^n, i\mathbb{R}^n) &= \mu(C\Psi_1^-(-\cdot)i\mathbb{R}^n, i\mathbb{R}^n) = -\mu(\Psi_1^-(-\cdot)i\mathbb{R}^n, Ci\mathbb{R}^n) \\ &= -\mu(\Psi_1^-(-\cdot)i\mathbb{R}^n, i\mathbb{R}^n) = \mu(\Psi_1^- i\mathbb{R}^n, i\mathbb{R}^n),\end{aligned}$$

where we have used the fact that C is a symplectic isomorphism from $(\mathbb{R}^{2n}, \omega_0)$ to $(\mathbb{R}^{2n}, -\omega_0)$, and the fact that the Maslov index changes sign when changing the sign of the symplectic form, or when reversing the parameterization of the Lagrangian paths. By the additivity of the Maslov index and by (128) we get

$$\begin{aligned}&\mu(\Phi^- i\mathbb{R}^{2n}, i\mathbb{R}^{2n}) = \mu(C\Psi_1^-(-\cdot)Ci\mathbb{R}^n, i\mathbb{R}^n) + \mu(\Psi_2 i\mathbb{R}^n, i\mathbb{R}^n) \\ &= \mu(\Psi_1^- i\mathbb{R}^n, i\mathbb{R}^n) + \mu(\Psi_2^- i\mathbb{R}^n, i\mathbb{R}^n) = \mu^\Omega(x_1) + \mu^\Omega(x_2) + n.\end{aligned} \tag{130}$$

On the other hand, $\tilde{A}(+\infty, t) = CA^+((1-t)/2)C \oplus A((t+1)/2)$, which implies

$$\Phi^+(t) = C\Psi^+((1-t)/2)\Psi^+(1/2)^{-1}C \oplus \Psi^+((1+t)/2)\Psi^+(1/2)^{-1}.$$

Since $N^*\Delta_{\mathbb{R}^n} = \operatorname{graph} C = \{(z, \overline{z}) \mid z \in \mathbb{C}^n\}$, we easily find

$$\Phi^+(t)N^*\Delta_{\mathbb{R}^n} = \operatorname{graph}\Gamma(t)C, \quad \text{with } \Gamma(t) := \Psi^+\left(\frac{1+t}{2}\right)\Psi^+\left(\frac{1-t}{2}\right)^{-1}. \tag{131}$$

The symplectic paths Γ and Ψ^+ are homotopic by the symplectic homotopy

$$(\lambda, t) \mapsto \Psi^+\left(t + \frac{\lambda}{2}(1-t)\right)\Psi^+\left(\frac{\lambda}{2}(1-t)\right)^{-1},$$

which leaves the end-points $\Gamma(0) = \Psi^+(0) = I$ and $\Gamma(1) = \Psi^+(1)$ fixed. Therefore, by the homotopy invariance of the Maslov index, by (78) and by (128) we have

$$\begin{aligned}\mu(\Phi^+ N^*\Delta_{\mathbb{R}^n}, i\mathbb{R}^{2n}) &= \mu(\operatorname{graph}\Gamma C, i\mathbb{R}^{2n}) = \mu(\operatorname{graph}\Psi^+ C, i\mathbb{R}^{2n}) \\ &= \mu(\Psi^+ i\mathbb{R}^n, i\mathbb{R}^n) = \mu^\Omega(y) + \frac{n}{2}.\end{aligned} \tag{132}$$

Therefore, by (129), (130), and (132), we conclude that

$$\operatorname{ind}(\overline{\partial}_{\tilde{A}}) = \mu^\Omega(x_1) + \mu^\Omega(x_2) + n - \left(\mu^\Omega(y) + \frac{n}{2}\right) - \frac{n}{2} = \mu^\Omega(x_1) + \mu^\Omega(x_2) - \mu^\Omega(y).$$

Hence we have proved that the fiberwise derivative of the section $\overline{D}_{J,H} : \mathscr{W}^\Omega_\Upsilon \to \mathscr{E}^\Omega_\Upsilon$ is a Fredholm operator of index $\mu^\Omega(x_1) + \mu^\Omega(x_2) - \mu^\Omega(y)$.

For a generic choice of the ω-compatible almost complex structure J, the section $\overline{D}_{J,H}$ is transverse to the zero-section (the proof of transversality results of this kind is standard, see [FHS96]). Let us fix such an almost complex structure J. Then, $\mathscr{M}^\Omega_\Upsilon$ - if non-empty - is a smooth submanifold of $\mathscr{W}^\Omega_\Upsilon$ of dimension $\mu^\Omega(x_1) + \mu^\Omega(x_2) - \mu^\Omega(y)$. This proves the Ω part of Proposition 3.4.

The space $\mathscr{M}_\Upsilon^\Lambda$. Let us study the space of solutions $\mathscr{M}_\Upsilon^\Lambda(x_1,x_2;y)$, where $x_1\in\mathscr{P}^\Lambda(H_1)$, $x_2\in\mathscr{P}^\Lambda(H_2)$, and $y\in\mathscr{P}^\Lambda(H_1\#H_2)$ (see section 3.3). It is a space of solutions of the Floer equation on the pair-of-pants Riemann surface Σ_Υ^Λ, described as a quotient of a strip with a slit in section 3.2.

Arguing as in the case of $\mathscr{M}_\Upsilon^\Omega$, it is easily seen that the space $\mathscr{M}_\Upsilon^\Lambda$ is the set of zeroes of a smooth section of a Banach bundle, whose fiberwise derivative at some $u\in\mathscr{M}_\Upsilon^\Lambda$ is conjugated to an operator of the form

$$\overline{D}+G: W^{1,p}(\Sigma_\Upsilon^\Lambda,\mathbb{C}^n)\to\Omega^{0,1}_{L^p}(\Sigma_\Upsilon^\Lambda,\mathbb{C}^n),\quad v\mapsto\frac{1}{2}(Dv+iDv\circ j)+Gv,$$

where

$$(Gv)(z)=\frac{1}{2}A(z)v(z)\,ds-\frac{i}{2}A(z)v(z)\,dt.$$

The smooth map $A:\Sigma_\Upsilon^\Lambda\to L(\mathbb{R}^{2n},\mathbb{R}^{2n})$ has the following asymptotics

$$A(s+(t-1)i)\to A_1^-(t),\ \ A(s+ti)\to A_2^-(t)\ \text{ for } s\to-\infty,\quad A(s+(2t-1)i)\to A^+(t),\ \text{ for } s\to+\infty,$$

for any $t\in[0,1]$, where $A_1^-(t)$, $A_2^-(t)$, and $A^+(t)$ are symmetric matrices such that the solutions of the linear Hamiltonian systems

$$\frac{d}{dt}\Psi_1^-(t)=iA_1^-(t)\Psi_1^-(t),\quad \frac{d}{dt}\Psi_2^-(t)=iA_2^-(t)\Psi_2^-(t),$$
$$\frac{d}{dt}\Psi^+(t)=2iA^+(t)\Psi^+(t),\quad \Psi_1^-(0)=\Psi_2^-(0)=\Psi^+(0)=I,$$

are conjugated to the differential of the Hamiltonian flows along x_1, x_2, and y:

$$\Psi_1^-(t)\sim D_x\phi^{H_1}(1,x_1(0))\quad \Psi_2^-(t)\sim D_x\phi^{H_2}(1,x_2(0))\quad \Psi^+(t)\sim D_x\phi^{H_1\#H_2}(1,y(0)).$$

Then, by the relationship (79) between the Conley-Zehnder index and the relative Maslov index, we have

$$\mu^\Lambda(x_1)=\mu_{CZ}(\Psi_1^-)=\mu(N^*\Delta,\operatorname{graph}C\Psi_1^-),\quad \mu^\Lambda(x_2)=\mu_{CZ}(\Psi_2^-)=\mu(N^*\Delta,\operatorname{graph}C\Psi_2^-),\tag{133}$$
$$\mu^\Lambda(y)=\mu_{CZ}(\Psi^+)=\mu(N^*\Delta,\operatorname{graph}C\Psi^+).\tag{134}$$

Using again the transformation $\tilde v(z):=(\overline{v}(\overline{z}),v(z))$, the operator $\overline{D}+G$ is easily seen to be conjugated to the operator

$$\overline{\partial}_{\tilde A}: X^{1,p}_{\mathscr{S},\mathscr{W}}(\Sigma,\mathbb{C}^{2n})\to X^p_{\mathscr{S}}(\Sigma,\mathbb{C}^{2n}),\quad u\mapsto\overline{\partial}u+\tilde A u,$$

where

$$\mathscr{S}=\{0,i\},\quad \mathscr{W}=(W_0,W_1)=(\Delta_{\mathbb{R}^{2n}},\Delta_{\mathbb{R}^n}\times\Delta_{\mathbb{R}^n}),\quad \tilde A(z)=CA(\overline{z})C\oplus A(z).$$

Notice that the intersection

$$W_0\cap W_1=\{(\xi,\xi,\xi,\xi)\mid \xi\in\mathbb{R}^n\}$$

is an n-dimensional linear subspace of $\mathbb{R}^{4n}$. Then by Theorem 5.23, the operator $\overline{\partial}_{\tilde A}$ is Fredholm of index

$$\operatorname{ind}\overline{\partial}_{\tilde A}=\mu(N^*W_0,\operatorname{graph}C\Phi^-)-\mu(N^*W_1,\operatorname{graph}C\Psi^+)-n,\tag{135}$$

where the symplectic paths Φ^-, $\Phi^+:[0,1]\to\mathrm{Sp}(4n)$ are related to Ψ_1^-, Ψ_2^-, Ψ^+ by the identities

$$\Phi^-(t)=C\Psi_1^-(-t)C\oplus\Psi_2^-(t),\quad \Phi^+(t)=C\Psi^+((1-t)/2)\Psi^+(1/2)^{-1}C\oplus\Psi^+((1+t)/2)\Psi^+(1/2)^{-1}.$$

By the additivity property of the Maslov index with respect to the symplectic splitting of $\mathbb{C}^{4n}$ given by $\mathbb{C}^n \times (0) \times \mathbb{C}^n \times (0)$ times $(0) \times \mathbb{C}^n \times (0) \times \mathbb{C}^n$, by the fact that the action of C changes the sign of the Maslov index and leaves any conormal space invariant, and by (133), we have

$$\begin{aligned}\mu(N^*W_0, \operatorname{graph} C\Phi^-) &= \mu(N^*\Delta_{\mathbb{R}^{2n}}, \operatorname{graph} C\Phi^-) = \mu(N^*\Delta_{\mathbb{R}^n}, \operatorname{graph} \Psi_1^-(-\cdot)C) \\ &+\mu(N^*\Delta_{\mathbb{R}^n}, \operatorname{graph} C\Psi_2^-) = \mu(N^*\Delta_{\mathbb{R}^n}, \operatorname{graph} C\Psi_1^-) + \mu(N^*\Delta_{\mathbb{R}^n}, \operatorname{graph} C\Psi_2^-) \\ &= \mu^\Lambda(x_1) + \mu^\Lambda(x_2).\end{aligned} \tag{136}$$

We recall from (131) that $\Phi^+(t)N^*_{\mathbb{R}^n}$ is the graph of $\Gamma(t)C$, where the symplectic path Γ is homotopic to Ψ^+ by a symplectic homotopy which fixes the end-points. Together with the skew-symmetry of the Maslov index, and identities (78), (79), (134), this implies

$$\begin{aligned}\mu(N^*W_1, \operatorname{graph} C\Phi^+) &= \mu(N^*\Delta_{\mathbb{R}^n} \times N^*\Delta_{\mathbb{R}^n}, \operatorname{graph} C\Phi^+) = -\mu(\operatorname{graph} C\Phi^+, N^*\Delta_{\mathbb{R}^n} \times N^*\Delta_{\mathbb{R}^n}) \\ &= \mu(\Phi^+ N^*\Delta_{\mathbb{R}^n}, N^*\Delta_{\mathbb{R}^n}) = \mu(\operatorname{graph} \Gamma C, N^*\Delta_{\mathbb{R}^n}) = \mu(\operatorname{graph} \Psi^+ C, N^*\Delta_{\mathbb{R}^n}) = \mu^\Lambda(y).\end{aligned} \tag{137}$$

Identities (135), (136), and (137), allow to conclude that

$$\operatorname{ind} \overline{\partial}_{\tilde{A}} = \mu^\Lambda(x_1) + \mu^\Lambda(x_2) - \mu^\Lambda(y) - n.$$

A standard transversality argument then shows that for a generic choice of J the set $\mathscr{M}^\Lambda_\Upsilon(x_1, x_2; y)$ - if non-empty - is a smooth manifold of dimension $\mu^\Lambda(x_1) + \mu^\Lambda(x_2) - \mu^\Lambda(y) - n$, concluding the proof of Proposition 3.4.

The space $\mathscr{M}^\Theta_\partial$. Let x^- and x^+ be elements of $\mathscr{P}^\Theta(H_1 \oplus H_2)$ (see section 3.4). The space $\mathscr{M}^\Theta_\partial(x^-, x^+)$ is easily seen to be the set of zeroes of a section of a Banach bundle whose fiberwise derivative at some $u \in \mathscr{M}^\Theta_\partial$ is conjugated to the operator

$$\overline{\partial}_A : W^{1,p}_{N^*\Delta^\Theta_{\mathbb{R}^n}}(\Sigma, \mathbb{C}^{2n}) = \left\{ v \in W^{1,p}(\Sigma, \mathbb{C}^{2n}) \mid (u(s), \overline{u}(s+i)) \in N^*\Delta^\Theta_{\mathbb{R}^n} \ \forall s \in \mathbb{R} \right\} \to L^p(\Sigma, \mathbb{C}^{2n}),$$

where

$$\Delta^\Theta_{\mathbb{R}^n} = \Delta^\Theta_{\mathbb{R}^n} = \{(\xi, \xi, \xi, \xi) \mid \xi \in \mathbb{R}^n\} \subset \mathbb{R}^{4n}, \tag{138}$$

and the smooth map $A : \Sigma \to L(\mathbb{R}^{4n}, \mathbb{R}^{4n})$ has asymptotics

$$A(s,t) \to A^-(t) \in \mathrm{Sym}(4n) \quad \text{for } s \to -\infty, \quad A(s,t) \to A^+(t) \in \mathrm{Sym}(4n) \quad \text{for } s \to +\infty,$$

such that the symplectic paths Φ^-, Φ^+ solving

$$\frac{d}{dt}\Phi^\pm(t) = iA^\pm(t)\Phi^\pm(t), \quad \Phi^\pm(0) = I,$$

are conjugated to the differential of the flow $\phi^{H_1 \oplus H_2}$ along x^- and x^+,

$$\Phi^-(t) \sim D_x\phi^{H_1 \oplus H_2}(t, x^-(0)), \quad \Phi^-(t) \sim D_x\phi^{H_1 \oplus H_2}(t, x^-(0)).$$

In particular,

$$\mu(N^*\Delta^\Theta_{\mathbb{R}^n}, \operatorname{graph} C\Phi^-) = \mu^\Theta(x^-) + \frac{n}{2}, \quad \mu(N^*\Delta^\Theta_{\mathbb{R}^n}, \operatorname{graph} C\Phi^+) = \mu^\Theta(x^+) + \frac{n}{2}. \tag{139}$$

Theorem 7.1 in [RS95] (or Theorem 5.23 in the special case of no jumps) implies that $\overline{\partial}_A$ is a Fredholm operator of index

$$\operatorname{ind} \overline{\partial}_A = \mu(N^*\Delta^\Theta_{\mathbb{R}^n}, \operatorname{graph} C\Phi^-) - \mu(N^*\Delta^\Theta_{\mathbb{R}^n}, \operatorname{graph} C\Phi^+) = \mu^\Theta(x^-) - \mu^\Theta(x^+).$$

Together with a standard transversality argument, this implies Proposition 3.7.

The space $\mathscr{M}_E$. Let $(x_1,x_2)\in\mathscr{P}^\Lambda(H_1)\times\mathscr{P}^\Lambda(H_2)$ and $y\in\mathscr{P}^\Theta(H_1\oplus H_2)$ (see section 3.5). The study of the space of solutions $\mathscr{M}_E(x_1,x_2;y)$ reduces to the study of an operator of the form

$$\overline{\partial}_A : X^{1,p}_{\mathscr{S},\mathscr{W}}(\Sigma,\mathbb{C}^{2n})\rightarrow X^p_{\mathscr{S}}(\Sigma,\mathbb{C}^{2n}),$$

where

$$\mathscr{S}=\{0,i\},\quad \mathscr{W}=(\Delta_{\mathbb{R}^{2n}},\Delta^\Theta_{\mathbb{R}^n}),$$

the space $\Delta^\Theta_{\mathbb{R}^n}$ being defined in (138). Theorem 5.23 implies that this operator is Fredholm of index

$$\operatorname{ind}\overline{\partial}_A=\mu(N^*\Delta_{\mathbb{R}^{2n}},\operatorname{graph}C\Phi^-)-\mu(N^*\Delta^\Theta_{\mathbb{R}^n},\operatorname{graph}C\Phi^+)-\frac{n}{2}. \tag{140}$$

Arguing as in the study of $\mathscr{M}^\Lambda_\Upsilon$ (identity (136)), we see that

$$\mu(N^*\Delta_{\mathbb{R}^{2n}},\operatorname{graph}C\Phi^-)=\mu^\Lambda(x_1)+\mu^\Lambda(x_1).$$

Arguing as in the study of $\mathscr{M}^\Theta_\partial$ (identities (139), we get

$$\mu(N^*\Delta^\Theta_{\mathbb{R}^n},\operatorname{graph}C\Phi^+)=\mu^\Theta(y)+\frac{n}{2}.$$

We conclude that

$$\operatorname{ind}\overline{\partial}_A=\mu^\Lambda(x_1)+\mu^\Lambda(x_1)-\mu^\Theta(y)-n.$$

The $\mathscr{M}_E$ part of Proposition 3.9 follows.

The space $\mathscr{M}_G$. Let $y\in\mathscr{P}^\Theta(H_1\oplus H_2)$ and $z\in\mathscr{P}^\Lambda(H_1\#H_2)$ (see section 3.5). The study of the space of solutions $\mathscr{M}_G(y,z)$ reduces to the study of an operator of the form

$$\overline{\partial}_A : X^{1,p}_{\mathscr{S},\mathscr{W}}(\Sigma,\mathbb{C}^{2n})\rightarrow X^p_{\mathscr{S}}(\Sigma,\mathbb{C}^{2n}),$$

where

$$\mathscr{S}=\{0,i\},\quad \mathscr{W}=(\Delta^\Theta_{\mathbb{R}^n},\Delta_{\mathbb{R}^n}\times\Delta_{\mathbb{R}^n}).$$

By Theorem 5.23 this operator is Fredholm of index

$$\operatorname{ind}\overline{\partial}_A=\mu(N^*\Delta^\Theta_{\mathbb{R}^n},\operatorname{graph}C\Phi^-)-\mu(N^*(\Delta_{\mathbb{R}^n}\times\Delta_{\mathbb{R}^n}),\operatorname{graph}C\Phi^+)-\frac{n}{2}. \tag{141}$$

As in the study of $\mathscr{M}^\Theta_\partial$, we have

$$\mu(N^*\Delta^\Theta_{\mathbb{R}^n},\operatorname{graph}C\Phi^-)=\mu^\Theta(y)+\frac{n}{2}.$$

As in the study of $\mathscr{M}^\Lambda_\Upsilon$ (identity (137)), we see that

$$\mu(N^*(\Delta_{\mathbb{R}^n}\times\Delta_{\mathbb{R}^n}),\operatorname{graph}C\Phi^+)=\mu^\Lambda(z), \tag{142}$$

and (141) gives us

$$\operatorname{ind}\overline{\partial}_A=\mu^\Theta(y)-\mu^\Lambda(z).$$

This concludes the proof of Proposition 3.9.

The space $\mathscr{M}_{GE}^{\Upsilon}$. Let $x_1 \in \mathscr{P}(H_1)$, $x_2 \in \mathscr{P}(H_2)$, and $z \in \mathscr{P}^{\Lambda}(H_1 \# H_2)$ (see the proof of Theorem 3.11). The space $\mathscr{M}_{GE}^{\Upsilon}(x_1, x_2; z)$ is the set of pairs (α, u) where α is a real positive parameter and u is a solution of the Floer equation on the Riemann surface $\Sigma_{GE}^{\Upsilon}(\alpha)$ with suitable asymptotics and suitable non-local boundary conditions. Linearizing this Floer equation for a fixed $\alpha \in]0, +\infty[$ yields an operator of the form

$$\overline{\partial}_A : X^{1,p}_{\mathscr{S},\mathscr{W}}(\Sigma, \mathbb{C}^{2n}) \to X^p_{\mathscr{S}}(\Sigma, \mathbb{C}^{2n}),$$

where

$$\mathscr{S} = \{-\alpha, \alpha, -\alpha + i, \alpha + i\}, \quad \mathscr{W} = (\Delta_{\mathbb{R}^{2n}}, \Delta^{\Theta}_{\mathbb{R}^n}, \Delta_{\mathbb{R}^n} \times \Delta_{\mathbb{R}^n}).$$

By Theorem 5.23 this operator is Fredholm of index

$$\operatorname{ind} \overline{\partial}_A = \mu(N^* \Delta_{\mathbb{R}^{2n}}, \operatorname{graph} C\Phi^-) - \mu(N^*(\Delta_{\mathbb{R}^n} \times \Delta_{\mathbb{R}^n}), \operatorname{graph} C\Phi^+) - n.$$

As in the discussion of $\mathscr{M}_{\Upsilon}^{\Lambda}$ (identities (136) and (137)),

$$\mu(N^* \Delta_{\mathbb{R}^{2n}}, \operatorname{graph} C\Phi^-) = \mu^{\Lambda}(x_1) + \mu^{\Lambda}(x_2), \quad \mu(N^*(\Delta_{\mathbb{R}^n} \times \Delta_{\mathbb{R}^n}), \operatorname{graph} C\Phi^+) = \mu^{\Lambda}(z).$$

Therefore,

$$\operatorname{ind} \overline{\partial}_A = \mu^{\Lambda}(x_1) + \mu^{\Lambda}(x_2) - \mu^{\Lambda}(z) - n.$$

Considering also the parameter α, we see that for a generic choice of J the space $\mathscr{M}_{GE}^{\Upsilon}(x_1, x_2; z)$ is a smooth manifold of dimension $\mu^{\Lambda}(x_1) + \mu^{\Lambda}(x_2) - \mu^{\Lambda}(z) - n + 1$. This proves Proposition 3.12.

The spaces $\mathscr{M}_C$ and $\mathscr{M}_{\mathrm{Ev}}$. Let f be a Morse function on M, let x be a critical point of f, and let $y \in \mathscr{P}(\Lambda)(H)$. Given $q \in M$, let

$$\widetilde{\mathscr{M}}_C(x, y, q) := \Big\{ u \in C^{\infty}([0, +\infty[\times \mathbb{T}, T^*M) \,\Big|\, u \text{ solves (32)}, \ \pi \circ u(0, t) \equiv q \ \forall t \in \mathbb{T}, \\ \lim_{s \to +\infty} u(s, t) = y(t) \text{ uniformly in } t \in \mathbb{T} \Big\}.$$

The study of such a space involves the study of an operator of the form

$$\overline{\partial}_A : X^{1,p}_{\emptyset,(0),(\Delta_{\mathbb{R}^n})}(\Sigma^+, \mathbb{C}^n) \to X^p_{\emptyset}(\Sigma^+, \mathbb{C}^n).$$

By Theorem 5.24, the Fredholm index of the above operator is

$$\operatorname{ind} \overline{\partial}_A = \frac{n}{2} - \mu(N^* \Delta_{\mathbb{R}^n}, \operatorname{graph} C\Phi^+) - \frac{n}{2} = -\mu^{\Lambda}(y).$$

Therefore, the space

$$\mathscr{M}_C(x, y) = \bigcup_{q \in W^u(x)} \widetilde{\mathscr{M}}_C(x, y, q),$$

has dimension

$$\dim \mathscr{M}_C(x, y) = \dim W^u(x) - \mu^{\Lambda}(y) = m(x) - \mu^{\Lambda}(y),$$

for a generic choice of J and g, proving the first part of Proposition 3.14.

Consider the space of maps

$$\widetilde{\mathscr{M}}_{\mathrm{Ev}}(y) := \Big\{ u \in C^{\infty}(] - \infty, 0] \times \mathbb{T}, T^*M) \,\Big|\, u \text{ solves (32)}, \ u(0, t) \in \mathbb{O}_M \ \forall t \in \mathbb{T}, \\ \lim_{s \to -\infty} u(s, t) = y(t) \text{ uniformly in } t \in \mathbb{T} \Big\}.$$

The study of the this space reduces to the study of an operator of the form

$$\overline{\partial}_A : X^{1,p}_{\emptyset,\mathbb{R}^n,(\Delta_{\mathbb{R}^n})}(\Sigma^-,\mathbb{C}^n) \to X^p_\emptyset(\Sigma^-,\mathbb{C}^n),$$

which has index

$$\operatorname{ind}\overline{\partial}_A = \frac{n}{2} + \mu(N^*\Delta_{\mathbb{R}^n}, \operatorname{graph} C\Phi^-) - \frac{1}{2}(n+2n-2n) = \mu^\Lambda(y),$$

by Theorem 5.25. For a generic choice of J, $\widetilde{\mathscr{M}}_{\mathrm{Ev}}(y)$ is then a manifold of dimension

$$\dim \widetilde{\mathscr{M}}_{\mathrm{Ev}}(y) = \mu^\Lambda(y).$$

For a generic choice of the Riemannian metric g on M, the map

$$\widetilde{\mathscr{M}}_{\mathrm{Ev}}(y) \to M, \quad u \mapsto \pi \circ u(0,0),$$

is transverse to the submanifold $W^s(x)$. For these choices of J and g, the space

$$\mathscr{M}_{\mathrm{Ev}}(y,x) = \left\{ u \in \widetilde{\mathscr{M}}_{\mathrm{Ev}}(y) \mid \pi \circ u(0,0) \in W^s(x) \right\}$$

is a manifold of dimension

$$\dim \mathscr{M}_{\mathrm{Ev}}(y,x) = \dim \widetilde{\mathscr{M}}_{\mathrm{Ev}}(y) - \operatorname{codim} W^s(x) = \mu^\Lambda(y) - m(x),$$

proving the second part of Proposition 3.14.

The space $\mathscr{M}_{I_!}$. Let $x \in \mathscr{P}^\Lambda(H)$ and $y \in \mathscr{P}^\Omega(H)$ (see section 3.6). The study of the space $\mathscr{M}_{I_!}(x,y)$ involves the study of an operator of the form

$$\overline{\partial}_A : X^{1,p}_{\mathscr{S},\mathscr{W}}(\Sigma,\mathbb{C}^n) \to X^p_{\mathscr{S}}(\Sigma,\mathbb{C}^n),$$

where

$$\mathscr{S} = \{0, i\}, \quad \mathscr{W} = (\Delta_{\mathbb{R}^n}, (0)).$$

By Theorem 5.23 this operator is Fredholm of index

$$\operatorname{ind}\overline{\partial}_A = \mu(N^*\Delta_{\mathbb{R}^n}, \operatorname{graph} C\Phi^-) - \mu(i\mathbb{R}^{2n}, \operatorname{graph} C\Phi^+) - \frac{n}{2}. \tag{143}$$

By (79),

$$\mu(N^*\Delta_{\mathbb{R}^n}, \operatorname{graph} C\Phi^-) = \mu_{CZ}(\Phi^-) = \mu^\Lambda(x). \tag{144}$$

By the skew-symmetry of the Maslov index, by the fact that the action of C changes the sign of the Maslov index and leaves $i\mathbb{R}^{2n}$ invariant, and by (78),

$$\begin{aligned}\mu(i\mathbb{R}^{2n}, \operatorname{graph} C\Phi^+) = -\mu(\operatorname{graph} C\Phi^+, i\mathbb{R}^{2n}) &= \mu(\operatorname{graph} \Phi^+ C, i\mathbb{R}^n \times i\mathbb{R}^n) \\ &= \mu(\Phi^+ i\mathbb{R}^n, i\mathbb{R}^n) = \mu^\Omega(y) + \frac{n}{2}.\end{aligned} \tag{145}$$

Therefore by (143), (144), and (145),

$$\operatorname{ind}\overline{\partial}_A = \mu^\Lambda(x) - \mu^\Omega(y) - n.$$

This proves Proposition 3.16.

The space $\mathscr{M}_{\Phi}^{\Theta}$. Let $\gamma \in \mathscr{P}^{\Theta}(L_1 \oplus L_2)$, and let $x \in \mathscr{P}^{\Theta}(H_1 \oplus H_2)$. The study of the space $\mathscr{M}_{\Phi}^{\Theta}(\gamma, x)$ reduces to the study of an operator of the form

$$\overline{\partial}_A : X^{1,p}_{\emptyset,(0),(\Delta^{\Theta}_{\mathbb{R}^n})}(\Sigma^+, \mathbb{C}^{2n}) \to X^p_{\emptyset}(\Sigma^+, \mathbb{C}^{2n}).$$

Indeed, for a generic choice of J, $\mathscr{M}_{\Phi}^{\Theta}(\gamma, x)$ is a manifold of dimension $m^{\Theta}(\gamma)$ plus the Fredholm index of the above operator. By Theorem 5.24, the index of this operator is

$$\operatorname{ind} \overline{\partial}_A = n - \mu(N^* \Delta^{\Theta}_{\mathbb{R}^n}, \operatorname{graph} C\Phi^+) - \frac{1}{2} \dim \Delta^{\Theta}_{\mathbb{R}^n} = \frac{n}{2} - \mu(N^* \Delta^{\Theta}_{\mathbb{R}^n}, \operatorname{graph} C\Phi^+).$$

Since $\mu(N^* \Delta^{\Theta}_{\mathbb{R}^n}, \operatorname{graph} C\Phi^+) = \mu^{\Theta}(x) + n/2$, we conclude that

$$\dim \mathscr{M}_{\Phi}^{\Theta}(\gamma, x) = m^{\Theta}(\gamma) - \mu^{\Theta}(x),$$

proving Proposition 4.2.

The space $\mathscr{M}_{\Upsilon}^{K}$. Let $\gamma_1 \in \mathscr{P}^{\Omega}(L_1)$, $\gamma_2 \in \mathscr{P}^{\Omega}(L_2)$, and $x \in \mathscr{P}^{\Omega}(H_1 \# H_2)$ (see section 4.2). The space $\mathscr{M}_{\Upsilon}^{K}(\gamma_1, \gamma_2; x)$ consists of pairs (α, u) where α is a positive number and $u(s,t)$ is a solution of the Floer equation on the Riemann surface $\Sigma_{\Upsilon}^{K}(\alpha)$, which is asymptotic to x for $s \to +\infty$, lies above some element in the unstable manifold of γ_1 (resp. γ_2) for $s = 0$ and $-1 \leq t \leq 0^-$ (resp. $0^+ \leq t \leq 1$), and lies above q_0 at the other boundary points. Linearizing the Floer equation for a fixed positive α and for fixed elements in the unstable manifolds of γ_1 and γ_2 yields an operator of the form

$$\overline{\partial}_A : X^{1,p}_{\mathscr{S},W,\mathscr{V},\mathscr{V}'}(\Sigma^+, \mathbb{C}^{2n}) \to X^p_{\mathscr{S}}(\Sigma^+, \mathbb{C}^{2n}),$$

where

$$\mathscr{S} = \{\alpha\}, \quad W = (0), \quad \mathscr{V} = ((0), \Delta_{\mathbb{R}^n}), \quad \mathscr{V}' = (0).$$

See the analysis for $\mathscr{M}_{\Upsilon}^{\Omega}$. By Theorem 5.21, the above operator is Fredholm of index

$$\operatorname{ind} \overline{\partial}_A = n - \mu(\Phi^+ N^* \Delta_{\mathbb{R}^n}, i\mathbb{R}^{2n}) - \frac{n}{2} = \frac{n}{2} - \mu(\Phi^+ N^* \Delta_{\mathbb{R}^n}, i\mathbb{R}^{2n}).$$

Hence, by (132), we have

$$\operatorname{ind} \overline{\partial}_A = -\mu^{\Omega}(x; H_1 \# H_2).$$

Letting the elements of the unstable manifolds of γ_1 and γ_2 vary, we increase the index by $m^{\Omega}(\gamma_1; L_1) + m^{\Omega}(\gamma_2; L_2)$. Letting also α vary we further increase the index by 1, and we find the formula

$$\dim \mathscr{M}_{\Upsilon}^{K}(\gamma_1, \gamma_2; x) = m^{\Omega}(\gamma_1; L_1) + m^{\Omega}(\gamma_2; L_2) - \mu^{\Omega}(x; H_1 \# H_2) + 1.$$

See [AS06b], section 3.1, for more details on how to deal with this kind of boundary data. This proves Proposition 4.3.

The space $\mathscr{M}_{\alpha_0}^{K}$. Let $\gamma_1 \in \mathscr{P}^{\Lambda}(L_1)$, $\gamma_2 \in \mathscr{P}^{\Lambda}(L_2)$, and $x \in \mathscr{P}^{\Theta}(H_1 \oplus H_2)$ (see section 4.4). The space $\mathscr{M}_{\alpha_0}^{K}(\gamma_1, \gamma_2; x)$ consists of solutions $u = (u_1, u_2)$ of the Floer equation on the Riemann surface $\Sigma_{\alpha_0}^{K}$, which is asymptotic to x for $s \to +\infty$, u_1 and u_2 lie above some elements in the unstable manifolds of γ_1 and γ_2 for $s = 0$, and u satisfies the figure-8 boundary condition for $s \geq \alpha_0$. Linearizing the Floer for a fixed pair of curves in the unstable manifolds of γ_1 and γ_2 yields an operator of the form

$$\overline{\partial}_A : X^{1,p}_{\mathscr{S},V_0,\mathscr{W}}(\Sigma^+, \mathbb{C}^{2n}) \to X^p_{\mathscr{S}}(\Sigma^+, \mathbb{C}^{2n}),$$

where

$$\mathscr{S} = \{\alpha_0, \alpha_0 + i\}, \quad V_0 = (0), \quad \mathscr{W} = (\Delta_{\mathbb{R}^{2n}}, \Delta^{\Theta}_{\mathbb{R}^n}).$$

See the analysis for $\mathscr{M}_E$. By Theorem 5.24, the above operator is Fredholm of index

$$\operatorname{ind} \overline{\partial}_A = n - \mu(N^*\Delta^{\Theta}_{\mathbb{R}^n}, \operatorname{graph} C\Phi^+) - n - \frac{n}{2} = -\mu(N^*\Delta^{\Theta}_{\mathbb{R}^n}, \operatorname{graph} C\Phi^+) - \frac{n}{2}.$$

Then, by (139), we have

$$\operatorname{ind} \overline{\partial}_A = -\mu^{\Theta}(x) - n.$$

Letting the elements of the unstable manifolds of γ_1 and γ_2 vary, we increase the index by $m^{\Lambda}(\gamma_1; L_1) + m^{\Lambda}(\gamma_2; L_2)$, and we find the formula

$$\dim \mathscr{M}^K_{\alpha_0}(\gamma_1, \gamma_2; x) = m^{\Lambda}(\gamma_1; L_1) + m^{\Lambda}(\gamma_2; L_2) - \mu^{\Theta}(x) - n.$$

This proves Proposition 4.5.

The space $\mathscr{M}^K_G$. Let $\gamma \in \mathscr{P}^{\Theta}(L_1 \oplus L_2)$ and $x \in \mathscr{P}^{\Lambda}(H_1 \# H_2)$ (see section 4.5). The space $\mathscr{M}^K_G(\gamma, x)$ consists of pairs (α, u) where α is a positive number and $u(s,t)$ is a solution of the Floer equation on the Riemann surface $\Sigma^K_G(\alpha)$, which is asymptotic to x for $s \to +\infty$, lies above some element in the unstable manifold of γ for $s = 0$, and satisfies the figure-8 boundary condition for $s \in [0, \alpha]$. Linearizing the Floer equation for a fixed positive α and for a fixed curve in the unstable manifold of γ yields an operator of the form

$$\overline{\partial}_A : X^{1,p}_{\mathscr{S}, V_0, \mathscr{W}}(\Sigma^+, \mathbb{C}^{2n}) \to X^p_{\mathscr{S}}(\Sigma^+, \mathbb{C}^{2n}),$$

where

$$\mathscr{S} = \{\alpha, \alpha + i\}, \quad V_0 = (0), \quad \mathscr{W} = (\Delta^{\Theta}_{\mathbb{R}^n}, \Delta_{\mathbb{R}^n} \times \Delta_{\mathbb{R}^n}).$$

See the analysis for $\mathscr{M}_G$. By Theorem 5.24, the above operator is Fredholm of index

$$\operatorname{ind} \overline{\partial}_A = n - \mu(N^*(\Delta_{\mathbb{R}^n} \times \Delta_{\mathbb{R}^n}), \operatorname{graph} C\Phi^+) - \frac{n}{2} - \frac{n}{2} = -\mu(N^*(\Delta_{\mathbb{R}^n} \times \Delta_{\mathbb{R}^n}), \operatorname{graph} C\Phi^+).$$

Hence, by (142), we have

$$\operatorname{ind} \overline{\partial}_A = -\mu^{\Lambda}(x; H_1 \# H_2).$$

Letting the elements of the unstable manifold of γ vary, we increase the index by $m^{\Theta}(\gamma; L_1 \oplus L_2)$. Letting also α vary we further increase the index by 1, and we find the formula

$$\dim \mathscr{M}^K_G(\gamma, x) = m^{\Theta}(\gamma; L_1 \oplus L_2) - \mu^{\Lambda}(x; H_1 \# H_2) + 1.$$

This proves Proposition 4.8.

6 Compactness and cobordism

6.1 Compactness in the case of jumping conormal boundary conditons

Compactness in the $C^{\infty}_{\mathrm{loc}}$ topology of all the spaces of solutions of the Floer equation considered in this paper can be proved within the following general setting. Let Q be a compact Riemannian manifold, and let $Q_0, Q_1, \dots, Q_k$ be submanifolds of $Q \times Q$. We assume that there is an isometric embedding $Q \hookrightarrow \mathbb{R}^N$ and linear subspaces $V_0, V_1, \dots, V_k$ of $\mathbb{R}^N \times \mathbb{R}^N$, such that V_{j-1} is partially orthogonal to V_j, for every $j = 1, \dots, k$, and

$$Q_j = V_j \cap (Q \times Q).$$

The embedding $Q \hookrightarrow \mathbb{R}^N$ induces an embedding $T^*Q \hookrightarrow T^*\mathbb{R}^N \cong \mathbb{R}^{2N} \cong \mathbb{C}^N$. Since the embedding $Q \hookrightarrow \mathbb{R}^N$ is isometric, the standard complex structure J_0 of $\mathbb{R}^{2N}$ restricts to the metric almost complex structure J on T^*Q. If $z = (q,p)$ is an element of T^*Q, we denote by $\overline{z}$ the element $(q,-p)$. This notation is justified by the fact that in the embedding $T^*Q \subset \mathbb{R}^{2N} \cong \mathbb{C}^N$ the map $z \mapsto \overline{z}$ is the complex conjugacy.

Let $H \in C^\infty([0,1] \times T^*Q)$ be a Hamiltonian satisfying (H1) and (H2). Fix real numbers

$$-\infty = s_0 < s_1 < \cdots < s_k < s_{k+1} = +\infty,$$

and let $u : \mathbb{R} \times [0,1] \to T^*Q$ be a solution of the Floer equation

$$\partial_s u + J(u)(\partial_t u - X_H(t,u)) = 0, \tag{146}$$

satisfying the non-local boundary conditions

$$(u(s,0), \overline{u(s,1)}) \in N^*Q_j \quad \forall s \in [s_{j-1}, s_j], \tag{147}$$

for every $j = 0, \dots, k$.

The map u satisfies the energy identity

$$\begin{aligned}\int_a^b \int_0^1 |\partial_s u(s,t)|^2 \, dt \, ds = \mathbb{A}_H(u(a,\cdot)) - \mathbb{A}_H(u(b,\cdot)) + \int_{[a,b]} (u(\cdot,1)^*\eta - u(\cdot,0)^*\eta) \\ = \mathbb{A}_H(u(a,\cdot)) - \mathbb{A}_H(u(b,\cdot)),\end{aligned} \tag{148}$$

for every $a < b$, where the integral over $[a,b]$ vanishes because $\eta \oplus (-\eta)$ vanishes on N^*Q_j. The following result is proven in [AS06b, Lemma 1.12] (in that lemma different boundary conditions are considered, but the proof makes use only of the energy identity (148) coming from those boundary conditions).

6.1. Lemma. *For every $a > 0$ there exists $c > 0$ such that for every solution $u : \mathbb{R} \times [0,1] \to T^*Q$ of (146), (147), and*

$$\int\int_{\mathbb{R}\times]0,1[} |\partial_s u(s,t)|^2 \, ds \, dt \le a,$$

we have the following estimates:

$$\|u\|_{L^2(I\times]0,1[)} \le c|I|^{1/2}, \quad \|\nabla u\|_{L^2(I\times]0,1[)} \le c(1 + |I|^{1/2}),$$

for every interval I.

The proof of the following result follows the argument of [AS06b, Theorem 1.14], using the above lemma together with the elliptic estimates of Proposition 5.10.

6.2. Proposition. *For every $a > 0$ there is $c > 0$ such that for every solution $u : \mathbb{R}\times[0,1] \to T^*Q$ of (146), (147), with energy bound*

$$\int\int_{\mathbb{R}\times]0,1[} |\partial_s u(s,t)|^2 \, ds \, dt \le a,$$

we have the following uniform estimate:

$$\|u\|_{L^\infty(\mathbb{R}\times]0,1[)} \le c.$$

Proof. By using the above embedding, the Floer equation (146) can be rewritten as

$$\overline{\partial} u = J_0 X_H(t,u). \tag{149}$$

We can pass to local boundary conditions by considering the map

$$v : \mathbb{R} \times [0,1] \to T^*(Q \times Q) \subset \mathbb{C}^{2N}, \quad v(z) := (u(z/2), \overline{u}(\overline{z}/2 + i)).$$

The map v satisfies the boundary conditions

$$v(s,0) \in N^*Q_j \subset N^*V_j, \text{ if } s \in [2s_{j-1}, 2s_j], \quad v(s,1) \in N^*\Delta_Q \subset N^*\Delta_{\mathbb{R}^N}, \ \forall s \in \mathbb{R}. \tag{150}$$

Moreover,

$$\overline{\partial} v(z) = \frac{1}{2} \left(\overline{\partial} u(z/2), \overline{\overline{\partial}(\overline{z}/2 + i)} \right),$$

so by (149) and by the fact that $X_H(t,q,p)$ has quadratic growth in $|p|$ by (29), there is a constant c such that

$$|\overline{\partial} v(z)| \leq c(1 + |v(z)|^2). \tag{151}$$

Let χ be a smooth function such that $\chi(s) = 1$ for $s \in [0,1]$, $\chi(s) = 0$ outside $[-1,2]$, and $0 \leq \chi \leq 1$. Given $h \in \mathbb{Z}$ set

$$w(s,t) := \chi(s-h) v(s,t).$$

Fix some $p > 2$, and consider the norm $\|\cdot\|_{X^p}$ introduced in section 5.3, with $\mathscr{S} = \{2s_1, \dots, 2s_k\}$. The map w has compact support and satisfies the boundary conditions (150), so by Proposition 5.10 we have the elliptic estimate

$$\|\nabla w\|_{X^p} \leq c_0 \|w\|_{X^p} + c_1 \|\overline{\partial} w\|_{X^p}.$$

Since

$$\overline{\partial} w = \chi'(s-h) v + \chi(s-h) \overline{\partial} v = \frac{\chi'}{\chi} w + \chi(s-h) \overline{\partial} v,$$

we obtain, together with (151),

$$\begin{aligned} \|\nabla w\|_{X^p} &\leq (c_0 + c_1 \|\chi'/\chi\|_\infty) \|w\|_{X^p} + c_1 \|\chi(\cdot - h) \overline{\partial} v\|_{X^p} \\ &\leq (c_0 + c_1 \|\chi'/\chi\|_\infty) \|w\|_{X^p} + c_1 c \|\chi(\cdot - h)(1 + |v|^2)\|_{X^p}. \end{aligned}$$

Therefore, we have an estimate of the form

$$\|\nabla w\|_{X^p} \leq a \|w\|_{X^p} + b \|\chi(\cdot - h)(1 + |v|^2)\|_{X^p}. \tag{152}$$

Since w has support in the set $[h-1, h+2] \times [0,1]$, we can estimate its X^p norm in terms of its $X^{1,2}$ norm, by Proposition 5.13. The $X^{1,2}$ norm is equivalent to the $W^{1,2}$ norm, and the latter norm is bounded by Lemma 6.1. We conclude that $\|w\|_{X^p}$ is uniformly bounded. Similarly, the X^p norm of $\chi(\cdot - h)(1 + |v|^2)$ is controlled by its $W^{1,2}$ norm, which is also bounded because of Lemma 6.1. Therefore, (152) implies that w is uniformly bounded in $X^{1,p}$. Since $p > 2$, we deduce that w is uniformly bounded in L^∞. The integer h was arbitrary, hence we conclude that v is uniformly bounded in L^∞, and so is u. □

Let us explain how all the solution spaces $\mathscr{M}$ considered in this paper can be viewed in terms of the above general setting. We describe explicitly the reduction in the case of the spaces $\mathscr{M}_\Upsilon^\Lambda$ associated to the pair-of-pants product as described in sections 3.2 and 3.3, the argument being analogous for all the other solution spaces. The pair-of-pants Riemann surface Σ_Υ^Λ is described as the quotient of the disjoint union of two strips $\mathbb{R} \cup [-1,0]$ and $\mathbb{R} \times [0,1]$ with respect to the identifications

$$(s,-1) \sim (s,0-), \quad (s,0+) \sim (s,1) \quad \forall s \leq 0, \tag{153}$$

$$(s,-1) \sim (s,1), \quad (s,0-) \sim (s,0+) \quad \forall s \geq 0. \tag{154}$$

Given periodic orbits $x_1 \in \mathscr{P}(H_1)$, $x_2 \in \mathscr{P}(H_2)$, and $y \in \mathscr{P}(H_1 \# H_2)$, the space $\mathscr{M}^{\Lambda}_{\Upsilon}(x_1, x_2; y)$ consists of maps

$$u : \Sigma^{\Lambda}_{\Upsilon} \to T^*M,$$

solving the Floer equation $\overline{\partial}_{J,H}(u) = 0$, with asymptotics

$$\lim_{s\to-\infty} u(s, t-1) = x_1(t), \quad \lim_{s\to-\infty} u(s,t) = x_2(t), \quad \lim_{s\to+\infty} u(s, 2t-1) = y(t).$$

We can associate to a map $u : \Sigma^{\Lambda}_{\Upsilon} \to T^*M$ the map $v : \Sigma := \mathbb{R} \times [0,1] \to T^*M^2$ by setting

$$v(s,t) := \big(-u(s,-t), u(s,t)\big).$$

The identifications (153) on the left-hand side of the domain of u are translated into the fact that $v(s,t)$ is 1 periodic in t for $s \leq 0$, or equivalently into the nonlocal boundary condition

$$(v(s,0), -v(s,1)) \in N^*\Delta_{M^2} \quad \forall s \leq 0, \tag{155}$$

where Δ_{M^2} denotes the diagonal in $M^4 = M^2 \times M^2$. The identifications (154) on the right-hand side of the domain of u are translated into the local boundary conditions

$$v(s,0) \in N^*\Delta_M, \quad v(s,1) \in N^*\Delta_M \quad \forall s \geq 0. \tag{156}$$

The map u solves the Floer equation $\overline{\partial}_{J,H}(u) = 0$ if and only if v solves the Floer equation $\overline{\partial}_{J,K}(v) = 0$, where $K : \mathbb{T} \times T^*M^2 \to \mathbb{R}$ is the Hamiltonian

$$K(t, x_1, x_2) := H_1(-t, -x_1) + H_2(t, x_2) \quad \forall (t, x_1, x_2) \in \mathbb{T} \times T^*M^2,$$

which satisfies the growth conditions (H1) and (H2). The asymptotic conditions for u are equivalent to

$$\lim_{s\to-\infty} v(s,t) = (-x_1(-t), x_2(t)), \quad \lim_{s\to+\infty} v(s,t) = \big(-y((1-t)/2), y((1+t)/2)\big). \tag{157}$$

Finally, since

$$|\nabla v(s,t)|^2 = |\nabla u(s,-t)|^2 + |\nabla u(s,t)|^2,$$

the energy of u equals the energy of v,

$$E(u) := \int_{\Sigma^{\Lambda}_{\Upsilon}} |\nabla u|^2 \, dsdt = \int_{\Sigma} |\nabla v|^2 \, dsdt =: E(v).$$

We conclude that $\mathscr{M}^{\Lambda}_{\Upsilon}(x_1, x_2; y)$ can be identified with the space of maps $v : \Sigma \to T^*M^2$ which solve the Floer equation for the Hamiltonian K, satisfy the boundary conditions (155), (156), and the asymptotic conditons (157).

By Nash theorem, we can find an isometric embedding $M \hookrightarrow \mathbb{R}^N$ for N large enough. The induced embedding $M^2 \hookrightarrow \mathbb{R}^{2N}$ is such that

$$\Delta_{M^2} = M^2 \cap \Delta_{\mathbb{R}^{2N}}, \quad \Delta_M \times \Delta_M = M^2 \cap \big(\Delta_{\mathbb{R}^N} \times \Delta_{\mathbb{R}^N}\big).$$

Note that the linear subspaces $\Delta_{\mathbb{R}^{2N}}$ and $\Delta_{\mathbb{R}^N} \times \Delta_{\mathbb{R}^N}$ are partially orthogonal in $\mathbb{R}^{4N}$. Therefore, the problem $\mathscr{M}^{\Lambda}_{\Upsilon}$ reduces to the above setting, with $Q = M^2$, $k = 1$, $s_1 = 0$, $Q_0 = \Delta_{M^2}$, $Q_1 = \Delta_M \times \Delta_M$, $V_0 = \Delta_{\mathbb{R}^{2N}}$, and $V_1 = \Delta_{\mathbb{R}^N} \times \Delta_{\mathbb{R}^N}$.

The energy of solutions in $\mathscr{M}^{\Lambda}_{\Upsilon}(x_1, x_2; y)$ is bounded above from

$$\mathbb{A}_{H_1}(x_1) + \mathbb{A}_{H_2}(x_2) - \mathbb{A}_{H_1 \# H_2}(y),$$

so Proposition 6.2 implies that these solutions have a uniform L^∞ bound.

For the remaining part of the argument leading to $C^\infty_{\rm loc}$ compactness of $\mathscr{M}^{\Lambda}_{\Upsilon}(x_1,x_2;y)$ it is more convenient to use the original definition of this solutions space and the smooth structure of $\Sigma^{\Lambda}_{\Upsilon}$. Then the argument is absolutely standard: If by contradiction there is no uniform C^1 bound, a concentration argument (see e.g. [HZ94, Theorem 6.8]) produces a non-constant J-holomorphic sphere. However, there are no non-constant J-holomorphic spheres on cotangent bundles, because the symplectic form ω is exact. This contradiction proves the C^1 bound. Then the C^k bounds for arbitrary k follow from elliptic bootstrap, as in [HZ94, section 6.4].

Other solutions spaces, such as the space $\mathscr{M}^{\Omega}_{\Upsilon}$ for the traingle products, involve Riemann surfaces with boundary, and the solutions take value on some conormal subbundle of T^*M. In this case the concentration argument for proving the C^1 bound could produce a non-constant J-holomorphic disk with boundary on the given conormal subbundle. However, the Liouville one-form vanishes on conormal subbundles, so such J-holomorphic disks do not exist. Again we find a contradition, leading to C^1 bounds and - by elliptic bootstrap - to C^k bounds for every k.

6.2 Removal of singularities

Removal of singularities results state that isolated singularities of a J-holomorphic map with bounded energy can be removed (see for instance [MS04, section 4.5]). In Proposition 6.4 below, we prove a result of this sort for corner singularities. The fact that we are dealing with cotangent bundles, which can be isometrically embedded into $\mathbb{C}^N$, allows to reduce such a statement to the following easy linear result, where $\mathbb{D}_r$ is the open disk of radius r in $\mathbb{C}$, and $\mathbb{H}^+$ is the quarter plane $\{\mathrm{Re}\, z>0,\ \mathrm{Im}\, z>0\}$.

6.3. Lemma. *Let V_0 and V_1 be partially orthogonal linear subspaces of $\mathbb{R}^n$. Let $u:\mathrm{Cl}(\mathbb{D}_1\cap\mathbb{H}^+)\setminus\{0\}\to\mathbb{C}^n$ be a smooth map such that*

$$u\in L^p(\mathbb{D}_1\cap\mathbb{H}^+,\mathbb{C}^n),\quad \overline{\partial}u\in L^p(\mathbb{D}_1\cap\mathbb{H}^+,\mathbb{C}^n),$$

for some $p>2$, and

$$u(s)\in N^*V_0\quad\forall\, s>0,\quad u(it)\in N^*V_1\quad\forall\, t>0.$$

Then u extends to a continuous map on $\mathrm{Cl}(\mathbb{D}_1\cap\mathbb{H}^+)$.

Proof. Since V_0 and V_1 are partially orthogonal, by applying twice the Schwarz reflection argument of the proof of Lemma 5.6 we can extend u to a continuous map

$$u:\mathbb{D}_1\setminus\{0\}\to\mathbb{C}^n,$$

which is smooth on $\mathbb{D}_1\setminus(\mathbb{R}\cup i\mathbb{R})$, has finite L^p norm on $\mathbb{D}_1$, and satisfies

$$\overline{\partial}u\in L^p(\mathbb{D}_1).$$

Since $p>2$, the L^2 norm of u on $\mathbb{D}_1$ is also finite, and by the conformal change of variables $z=s+it=e^\zeta=e^{\rho+i\theta}$, this norm can be written as

$$\int_{\mathbb{D}_1}|u(z)|^2\,dsdt=\int_{-\infty}^0\int_0^{2\pi}|u(e^{\rho+i\theta})|^2e^{2\rho}\,d\theta d\rho.$$

The fact that this quantity is finite implies that there is a sequence $\rho_h\to-\infty$ such that, setting $\epsilon_h:=e^{\rho_h}$, we have

$$\lim_{h\to\infty}\epsilon_h^2\int_0^{2\pi}|u(\epsilon_h e^{i\theta})|^2d\theta=\lim_{h\to\infty}e^{2\rho_h}\int_0^{2\pi}|u(e^{\rho_h+i\theta})|^2\,d\theta=0. \tag{158}$$

If $\varphi\in C^\infty_c(\mathbb{D}_1,\mathbb{C}^N)$, an integration by parts using the Gauss formula leads to

$$\int_{\mathbb{D}_1}\langle u,\partial\varphi\rangle\,dsdt=\int_{\mathbb{D}_{\epsilon_h}}\langle u,\partial\varphi\rangle\,dsdt+\int_{\mathbb{D}_1\setminus\mathbb{D}_{\epsilon_h}}\langle u,\partial\varphi\rangle\,dsdt$$
$$=\int_{\mathbb{D}_{\epsilon_h}}\langle u,\partial\varphi\rangle\,dsdt-\int_{\mathbb{D}_1\setminus\mathbb{D}_{\epsilon_h}}\langle\overline{\partial}u,\varphi\rangle\,dsdt+i\int_{\partial\mathbb{D}_{\epsilon_h}}\langle u,\varphi\rangle\,dz.$$

Since u and $\overline{\partial} u$ are integrable over $\mathbb{D}_1$, the the first integral in the latter expression tends to zero, while the second one tends to

$$-\int_{\mathbb{D}_1} \langle \overline{\partial} u, \varphi \rangle \, dsdt.$$

As for the last integral, we have

$$\int_{\partial \mathbb{D}_{\epsilon_h}} \langle u, \varphi \rangle \, dz = i\epsilon_h \int_0^{2\pi} \langle u(\epsilon_h e^{i\theta}), \varphi(\epsilon_h e^{i\theta}) \rangle e^{i\theta} \, d\theta,$$

so by the Cauchy-Schwarz inequality,

$$\begin{aligned} \left| \int_{\partial \mathbb{D}_{\epsilon_h}} \langle u, \varphi \rangle \, dz \right| &\leq \epsilon_h \left(\int_0^{2\pi} |u(\epsilon_h e^{i\theta})|^2 \, d\theta \right)^{1/2} \left(\int_0^{2\pi} |\varphi(\epsilon_h e^{i\theta})|^2 \, d\theta \right)^{1/2} \\ &\leq \sqrt{2\pi} \, \epsilon_h \left(\int_0^{2\pi} |u(\epsilon_h e^{i\theta})|^2 \, d\theta \right)^{1/2} \|\varphi\|_\infty. \end{aligned}$$

Then (158) implies that the latter quantity tends to zero for $h \to \infty$. Therefore,

$$\int_{\mathbb{D}_1} \langle u, \partial \varphi \rangle \, dsdt = -\int_{\mathbb{D}_1} \langle \overline{\partial} u, \varphi \rangle \, dsdt,$$

for every test function $\varphi \in C_c^\infty(\mathbb{D}_1, \mathbb{C}^n)$. Since $\overline{\partial} u \in L^p$, by the regularity theory of the weak solutions of $\overline{\partial}$ (see Theorem 5.4 (i)), u belongs to $W^{1,p}(\mathbb{D}_1, \mathbb{C}^n)$. Since $p > 2$, we conclude that u is continuous at 0. □

Let Q_0 and Q_1 be closed submanifolds of Q, and assume that there is an isometric embedding $Q \hookrightarrow \mathbb{R}^N$ such that

$$Q_0 = Q \cap V_0, \quad Q_1 \cap V_1,$$

where V_0 and V_1 are partially orthogonal linear subspaces of $\mathbb{R}^N$.

6.4. PROPOSITION. *Let $X : \mathbb{D}_1 \cap \mathbb{H}^+ \times T^*Q \to TT^*Q$ be a smooth vector field such that $X(z, q, p)$ grows at most polynomially in p, uniformly in (z, q). Let $u : \mathrm{Cl}(\mathbb{D}_1 \cap \mathbb{H}^+) \setminus \{0\} \to T^*Q$ be a smooth solution of the equation*

$$\overline{\partial}_J u(z) = X(z, u(z)) \quad \forall z \in \mathrm{Cl}(\mathbb{D}_1 \cap \mathbb{H}^+) \setminus \{0\}, \tag{159}$$

such that

$$u(s) \in N^*Q_0 \quad \forall\, s > 0, \quad u(it) \in N^*Q_1 \quad \forall\, t > 0.$$

If u has finite energy,

$$\int_{\mathbb{D}_1 \cap \mathbb{H}^+} |\nabla u|^2 \, dsdt < +\infty, \tag{160}$$

then u extends to a continuous map on $\mathrm{Cl}(\mathbb{D}_1 \cap \mathbb{H}^+)$.

Proof. By means of the above isometric embedding, we may regard u as a $\mathbb{C}^N$-valued map, satisfying the equation (159) with $\overline{\partial}_J = \overline{\partial}$, the energy estimate (160), and the boundary condition

$$u(s) \in N^*V_0 \quad \forall\, s > 0, \quad u(it) \in N^*V_1 \quad \forall\, t > 0.$$

By the energy estimate (160), u belongs to $L^p(\mathbb{D}_1 \cap \mathbb{H}^+, \mathbb{C}^N)$ for every $p < +\infty$: for instance, this follows from the Poincaré inequality and the Sobolev embedding theorem on $\mathbb{D}_1$, after applying a Schwarz reflection twice and after multiplying by a cut-off function vanishing on $\partial \mathbb{D}_1$ and equal to 1 on a neighborhood of 0. The polynomial growth of X then implies that

$$X(\cdot, u(\cdot)) \in L^p(\mathbb{D}_1 \cap \mathbb{H}^+, \mathbb{C}^N) \quad \forall p < +\infty. \tag{161}$$

Therefore, Lemma 6.3 implies that u extends to a continuous map on $\mathrm{Cl}(\mathbb{D}_1 \cap \mathbb{H}^+)$. □

The corresponding statement for jumping conormal boundary conditions is the following:

6.5. PROPOSITION. *Let $X : \mathbb{D}_1 \cap \mathbb{H} \times T^*Q \to TT^*Q$ be a smooth vector field such that $X(z,q,p)$ grows at most polynomially in p, uniformly in (z,q). Let $u : \mathrm{Cl}(\mathbb{D}_1 \cap \mathbb{H}) \setminus \{0\} \to T^*Q$ be a smooth solution of the equation*

$$\overline{\partial}_J u(z) = X(z, u(z)) \quad \forall z \in \mathrm{Cl}(\mathbb{D}_1 \cap \mathbb{H}) \setminus \{0\},$$

such that

$$u(s) \in N^*Q_0 \quad \forall\, s > 0, \quad u(s) \in N^*Q_1 \quad \forall\, s < 0.$$

If u has finite energy,

$$\int_{\mathbb{D}_1 \cap \mathbb{H}} |\nabla u|^2 \, dsdt < +\infty,$$

then u extends to a continuous map on the closed half-disk $\mathrm{Cl}(\mathbb{D}_1 \cap \mathbb{H})$.

Proof. The energy is invariant with respect to conformal changes of variable. Therefore, it is enough to apply Proposition 6.4 to the map $v(z) = u(z^2)$, with $z \in \mathbb{D}_1 \cap \mathbb{H}^+$. □

6.3 Proof of Proposition 3.13

Let $x_1 \in \mathscr{P}(H_1)$, $x_2 \in \mathscr{P}(H_2)$, and $z \in \mathscr{P}(H_1 \# H_2)$ be such that

$$\mu^\Lambda(x_1) + \mu^\Lambda(x_2) - \mu^\Lambda(z) = n, \tag{162}$$

so that the manifold $\mathscr{M}^\Upsilon_{GE}(x_1, x_2; z)$ is one-dimensional. By standard arguments, Proposition 3.13 is implied by the following two statements:

(i) for every $y \in \mathscr{P}^\Theta(H_1 \oplus H_2)$ such that

$$\mu^\Theta(y) = \mu^\Lambda(z) = \mu^\Lambda(x_1) + \mu^\Lambda(x_2) - n,$$

and every pair (u_1, u_2) with $u_1 \in \mathscr{M}_E(x_1, x_2; y)$ and $u_2 \in \mathscr{M}_G(y, z)$, there is a unique connected component of $\mathscr{M}^\Upsilon_{GE}(x_1, x_2; z)$ containing a curve $\alpha \mapsto (\alpha, u_\alpha)$ which - modulo translations in the s variable - converges to $(+\infty, u_1)$ and to $(+\infty, u_2)$;

(ii) for every $u \in \mathscr{M}^\Lambda_\Upsilon(x_1, x_2; z)$, there is a unique connected component of $\mathscr{M}^\Upsilon_{GE}(x_1, x_2; z)$ containing a curve $\alpha \mapsto (\alpha, u_\alpha)$ which converges to $(0, u)$.

The first statement follows from standard gluing arguments. Here we prove the second statement, by reducing it to an implicit function type argument. At first the difficulty consists in a parameter dependence of the underlying domain for the elliptic PDE. Using the special form of the occuring conormal type boundary conditions and a suitable localization argument we equivalently translate this parameter dependence into a continuous family of elliptic operators with fixed boundary conditions.

If $(\alpha, v) \in \mathscr{M}^\Upsilon_{GE}(x_1, x_2; z)$, we define $u : \Sigma = \mathbb{R} \times [0,1] \to T^*M^2$ as

$$u(s,t) := (-v(s,-t), v(s,t)),$$

and the Hamiltonian K on $\mathbb{T} \times T^*M^2$ by

$$K(t, x_1, x_2) := H_1(-t, -x_1) + H_2(t, x_2).$$

By this identification, we can view the space $\mathscr{M}^\Upsilon_{GE}(x_1, x_2; z)$ as the space of pairs (α, u), where $\alpha > 0$ and $u : \Sigma \to T^*M^2$ solves the Floer equation

$$\overline{\partial}_{J,K}(u) = 0, \tag{163}$$

with non-local boundary conditions

$$(u(s,0),-u(s,1)) \in \begin{cases} N^*\Delta_{M^2} & \text{if } s \le 0, \\ N^*\Delta_M^{\Theta} & \text{if } 0 \le s \le \alpha, \\ N^*(\Delta_M \times \Delta_M) & \text{if } s \ge \alpha, \end{cases} \tag{164}$$

and asymptotics

$$\lim_{s\to-\infty} u(s,t) = \big(-x_1(-t), x_2(t)\big), \quad \lim_{s\to+\infty} u(s,t) = \big(-z((1-t)/2), z((1+t)/2)\big). \tag{165}$$

Similarly, we can view the space $\mathscr{M}_{\Upsilon}^{\Lambda}(x_1,x_2;z)$ as the space of maps $u:\Sigma \to T^*M^2$ solving the equation (163) with asymptotics (165) and non-local boundary conditions

$$(u(s,0),-u(s,1)) \in \begin{cases} N^*\Delta_{M^2} & \text{if } s \le 0, \\ N^*(\Delta_M \times \Delta_M) & \text{if } s \ge 0. \end{cases} \tag{166}$$

Compactness. We start with the following compactness results, which also clarifies the sense of the convergence in (ii):

6.6. Lemma. *Let (α_h, u_h) be a sequence in $\mathscr{M}_{GE}^{\Upsilon}(x_1,x_2;z)$ with $\alpha_h \to 0$. Then there exists $u_0 \in \mathscr{M}_{\Upsilon}^{\Lambda}(x_1,x_2;z)$ such that up to a subsequence u_h converges to u_0 in $C^\infty_{\mathrm{loc}}(\Sigma \setminus \{0,i\})$, in $C^\infty(\Sigma \cap \{|\mathrm{Re}\, z| > 1\})$, and uniformly on Σ.*

Proof. Since the sequence of maps (u_h) has uniformly bounded energy, Proposition 6.2 implies a uniform L^∞ bound. Then, the usual non-bubbling-off analysis for interior points and boundary points away from the jumps in the boundary condition implies that, modulo subsequence, we have

$$u_h \to u_0 \quad \text{in } C^\infty_{\mathrm{loc}}(\Sigma \setminus \{0,i\}, T^*M^2),$$

where u_0 is a smooth solution of equation (163) on $\Sigma \setminus \{0,i\}$ with bounded energy and satisfying the boundary condions (166), except possibly at 0 and i. By Proposition 6.5, the singularities 0 and i are removable, and u_0 satisfies the boundary condition also at 0 and i. By the index formula (162) and transversality, the sequence u_h cannot split, so u_0 satisfies also the asymptotic conditions (165), and $u_h \to u_0$ in $C^\infty(\Sigma \cap \{|\mathrm{Re}\, z| > 1\})$. Therefore, u_0 belongs to $\mathscr{M}_{\Upsilon}^{\Lambda}(x_1,x_2;z)$, and there remain to prove that $u_h \to u$ uniformly on Σ.

We assume by contraposition that (u_h) does not converge uniformly on Σ. By Ascoli-Arzelà theorem, there must be some blow-up of the gradient. That is, modulo subsequence, we can find $z_h \in \Sigma$ converging either to 0 or to i such that

$$R_h := |\nabla u_h(z_h)| = \|\nabla u_h\|_\infty \to \infty.$$

For sake of simplicity, we only consider the case where $z_h = (s_h, 0) \to 0$, $0 < s_h < \alpha_h \to 0$. The general case follows along analogous arguments using additional standard bubbling-off arguments. For more details, see e.g. [HZ94, section 6.4].

We now have to make a case distinction concerning the behaviour of the quantity $0 < R_h \cdot \alpha_h < \infty$:

(a) The case of a diverging subsequence $R_{h_j} \cdot \alpha_{h_j} \to \infty$ can be handled by conformal rescaling $v_j(s,t) := u_{h_j}(s_{h_j} + s/R_{h_j}, t/R_{h_j})$ which provides us with a finite energy disk with boundary on a single Lagrangian submanifold of conormal type. This has to be constant due to the vanishing of the Liouville 1-form on conormals, contradicting the convergence of $|\nabla v_j(0)| = 1$.

(b) The case of convergence of a subsequence $R_{h_j} \cdot \alpha_{h_j} \to 0$ can be dealt with by rescaling $v_j(s,t) := u_{h_j}(s_{h_j} + \alpha_{h_j}, \alpha_{h_j} t)$. Now v_k has to converge uniformly on compact subsets towards a constant map, since $\|\nabla v_j\|_\infty = |\nabla v_j(0)| = R_{h_j} \cdot \alpha_{h_j} \to 0$. This in particular implies that $u_{h_j}(\cdot, 0)_{|[0,\alpha_{h_j}]}$ converges uniformly to a point contradicting the contraposition assumption.

(c) There remains to study the case $R_h \cdot \alpha_h \to c > 0$. Again we rescale $v_h(s,t) = u_h(\alpha_h s, \alpha_h t)$, which now has to converge to a non-constant J-holomorphic map v on the upper half plane. After applying a suitable conformal coordinate change and transforming the non-local boundary conditions into local ones, we can view v as a map on the half strip $v\colon \Sigma^+ \to T^*M^4$, satisfying the boundary conditions

$$\begin{array}{rcl} v(0,t) \in & N^*\Delta_M^\Theta & \text{for } t \in [0,1], \\ v(s,0) \in & N^*(\Delta_M \times \Delta_M) & \text{for } s \geq 0, \\ v(s,1) \in & N^*\Delta_{M^2} & \text{for } s \geq 0. \end{array}$$

Applying again the removal of singularities for $s \to \infty$ we obtain v as a J-holomorphic triangle with boundary on three conormals. Hence, v would have to be constant, contradicting again the rescaling procedure.

This shows the uniform convergence of a subsequence of (u_h). □

Localization. It is convenient to transform the nonlocal boundary conditions (164) and (166) into local boundary conditions, by the usual method of doubling the space: given $u : \Sigma \to T^*M^2$ we define $\tilde{u} : \Sigma \to T^*M^4$ as

$$\tilde{u}(s,t) := \big(u(s/2, t/2), -u(s/2, 1 - t/2)\big).$$

Then u solves (163) if and only if $\tilde{u}$ solves the equation

$$\overline{\partial}_{J,\tilde{K}}(\tilde{u}) = 0, \tag{167}$$

with upper boundary condition

$$\tilde{u}(s,1) \in N^*\Delta_{M^2} \quad \forall s \in \mathbb{R}, \tag{168}$$

where the Hamiltonian $\tilde{K} : [0,1] \times T^*M^4 \to \mathbb{R}$ is defined by

$$\tilde{K}(t, x_1, x_2, x_3, x_4) := \frac{1}{2} K(t/2, x_1, x_2) + \frac{1}{2} K(1 - t/2, -x_3, -x_4).$$

Moreover, u satisfies (164) if and only if $\tilde{u}$ satisfies

$$\tilde{u}(s,0) \in \left\{ \begin{array}{ll} N^*\Delta_{M^2} & \text{if } s \leq 0, \\ N^*\Delta_M^\Theta & \text{if } 0 \leq s \leq 2\alpha, \\ N^*(\Delta_M \times \Delta_M) & \text{if } s \geq 2\alpha, \end{array} \right. \tag{169}$$

whereas u satisfies (166) if and only if $\tilde{u}$ satisfies

$$\tilde{u}(s,0) \in \left\{ \begin{array}{ll} N^*\Delta_{M^2} & \text{if } s \leq 0, \\ N^*(\Delta_M \times \Delta_M), & \text{if } s \geq 0. \end{array} \right. \tag{170}$$

Finally, the asymptotic condition (165) is translated into

$$\begin{aligned} \lim_{s \to -\infty} \tilde{u}(s,t) &= \big(-x_1(-t/2), x_2(t/2), x_1(t/2 - 1), -x_2(1 - t/2)\big), \\ \lim_{s \to +\infty} \tilde{u}(s,t) &= \big(-z(1/2 - t/4), z(1/2 + t/4), z(t/4), -z(1 - t/4)\big). \end{aligned} \tag{171}$$

Let $u_0 \in \mathscr{M}_\Upsilon^\Lambda(x_1, x_2; z)$. We must prove that there exists a unique connected component of $\mathscr{M}_{GE}^\Upsilon(x_1, x_2; z)$ containing a curve $\alpha \mapsto (\alpha, u_\alpha)$ which converges to $(0, u_0)$, in the sense of Lemma 6.6.

Let $\tilde{u}_0$ be the map from Σ to T^*M^4 associated to u_0: $\tilde{u}_0$ solves (167) with boundary conditions (168), (170), and asymptotic conditions (171). Since we are looking for solutions which converge to $\tilde{u}_0$ uniformly on Σ, we may localize the problem and assume that $M = \mathbb{R}^n$. More precisely, if the projection of $\tilde{u}_0(z)$ onto M^4 is $(q_1, q_2, q_3, q_4)(z)$, we construct open embeddings

$$\Sigma \times \mathbb{R}^n \to \Sigma \times M, \quad (z, q) \mapsto (z, \varphi_j(z, q)), \quad j = 1, \dots, 4,$$

such that $\varphi_j(z, 0) = q_j(z)$ and $D_2\varphi_j(z, 0)$ is an isometry, for every $z \in \Sigma$ (for instance, by composing an isometric trivialization of $q_j^*(TM)$ by the exponential mapping). The induced open embeddings

$$\Sigma \times T^*\mathbb{R}^n \to \Sigma \times T^*M, \quad (z, q, p) \mapsto (z, \psi_j(z, q, p)) := \big(z, \varphi_j(z, q), (D_2\varphi_j(z, q)^*)^{-1}p\big), \quad j = 1, \dots, 4,$$

are the components of the open embedding

$$\Sigma \times T^*\mathbb{R}^{4n} \to \Sigma \times T^*M^4, \quad (z, \xi) \mapsto (z, \psi(z, \xi)) := \big(z, \psi_1(z, \xi_1), \dots, \psi_4(z, \xi_4)\big).$$

Such an embedding allow us to associate to any $\tilde{u} : \Sigma \to T^*M^4$ which is C^0-close to $\tilde{u}_0$ a map $w : \Sigma \to T^*\mathbb{R}^{4n} = \mathbb{C}^{4n}$, by setting

$$\tilde{u}(z) = \psi(z, w(z)).$$

Then $\tilde{u}$ solves (167) if and only if w solves an equation of the form

$$\mathscr{D}(w) := \partial_s w(z) + J(z, w(z))\partial_t w(z) + G(z, w(z)) = 0, \tag{172}$$

where J is an almost complex structure on $\mathbb{C}^{4n}$ parametrized on Σ and such that $J(z, 0) = J_0$ for any $z \in \Sigma$, whereas $G : \Sigma \times \mathbb{C}^{4n} \to \mathbb{C}^{4n}$ is such that $G(z, 0) = 0$ for any $z \in \Sigma$. Moreover, $\tilde{u}$ solves the asymptotic conditions (171) if and only if $w(s, t)$ tends to 0 for $s \to \pm\infty$. The maps $\psi_j(z, \cdot)$ preserve the Liouville form, so they map conormals into conormals. It easily follows that the boundary condition (168) on $\tilde{u}$ is translated into

$$w(s, 1) \in N^*\Delta_{\mathbb{R}^{2n}} \quad \forall s \in \mathbb{R}. \tag{173}$$

Moreover, $\tilde{u}$ satisfies the boundary conditon (169) if and only if w satisfies

$$w(s, 0) \in \begin{cases} N^*\Delta_{\mathbb{R}^{2n}} & \text{if } s \leq 0, \\ N^*\Delta^{\Theta}_{\mathbb{R}^n} & \text{if } 0 \leq s \leq 2\alpha, \\ N^*(\Delta_{\mathbb{R}^n} \times \Delta_{\mathbb{R}^n}) & \text{if } s \geq 2\alpha. \end{cases} \tag{174}$$

Similarly, $\tilde{u}$ satisfies the boundary condition (170) if and only if w satisfies

$$w(s, 0) \in \begin{cases} N^*\Delta_{\mathbb{R}^{2n}} & \text{if } s \leq 0, \\ N^*(\Delta_{\mathbb{R}^n} \times \Delta_{\mathbb{R}^n}) & \text{if } s \geq 0. \end{cases} \tag{175}$$

The element $u_0 \in \mathscr{M}^{\Lambda}_{\Upsilon}(x_1, x_2; z)$ corresponds to the solution $w_0 = 0$ of (172)-(175). By using the functional setting introduced in section 5, we can view the nonlinear operator $\mathscr{D}$ defined in (172) as a continuously differentiable operator

$$\mathscr{D} : X^{1,p}_{\mathscr{S},\mathscr{V},\mathscr{V}'}(\Sigma, \mathbb{C}^{4n}) \to X^p_{\mathscr{S}}(\Sigma, \mathbb{C}^{4n})$$

where $\mathscr{S} := \{0\}$, $\mathscr{V} := (\Delta_{\mathbb{R}^{2n}}, \Delta_{\mathbb{R}^n} \times \Delta_{\mathbb{R}^n})$, $\mathscr{V}' := (\Delta_{\mathbb{R}^{2n}})$, and p is some number larger than 2. Since $J(z, 0) = J_0$, the differential of $\mathscr{D}$ at $w_0 = 0$ is a linear operator of the kind studied in section 5, and by the transversality assumption it is an isomorphism.

Consider the orthogonal decomposition

$$\mathbb{R}^{4n} = W_1 \oplus W_2 \oplus W_3 \oplus W_4,$$

where

$$W_1 := \Delta^{\Theta}_{\mathbb{R}^n} = \Delta_{\mathbb{R}^{2n}} \cap (\Delta_{\mathbb{R}^n} \times \Delta_{\mathbb{R}^n}), \quad \Delta_{\mathbb{R}^{2n}} = W_1 \oplus W_2, \quad \Delta_{\mathbb{R}^n} \times \Delta_{\mathbb{R}^n} = W_1 \oplus W_3,$$

and denote by P_j the the orthogonal projection of $\mathbb{C}^{4n}$ onto $N^*W_j = W_j \oplus iW_j^\perp$. If $\mathscr{T}_\alpha$ is the translation operator mapping some $w : \Sigma \to \mathbb{C}^{4n}$ into

$$(\mathscr{T}_\alpha w)(s,t) := \big(P_1 w(s,t), P_2 w(s,t), P_3 w(s-2\alpha,t), P_4 w(s,t)\big),$$

we easily see that w satisfies the boundary condions (175) if and only if $\mathscr{T}_\alpha w$ satisfies the boundary conditions (174). Therefore, if we define the operator

$$\mathscr{D}_\alpha : X^{1,p}_{\mathscr{S},\mathscr{V},\mathscr{V}'}(\Sigma,\mathbb{C}^{4n}) \to X^p_{\mathscr{S}}(\Sigma,\mathbb{C}^{4n}), \quad \mathscr{D}_\alpha = \mathscr{D} \circ \mathscr{T}_\alpha,$$

we have that $w \in X^{1,p}_{\mathscr{S},\mathscr{V},\mathscr{V}'}(\Sigma,\mathbb{C}^{4n})$ solves $\mathscr{D}_\alpha(w) = 0$ if and only if $\mathscr{T}_\alpha w$ is a solution of (172) satisfying the boundary conditions (173) and (174). The operator

$$[0,+\infty[\times X^{1,p}_{\mathscr{S},\mathscr{V},\mathscr{V}'}(\Sigma,\mathbb{C}^{4n}) \to X^p_{\mathscr{S}}(\Sigma,\mathbb{C}^{4n}), \quad (\alpha,w) \mapsto \mathscr{D}_\alpha(w),$$

is continuous on the product, it is continuously differentiable with respect to the second variable, and this partial differential is continuous on the product. Moreover, $D\mathscr{D}_0(0) = D\mathscr{D}(0)$ is an isomorphism, so the parametric inverse mapping theorem implies that there are a number $\alpha_0 > 0$ and a neighborhood $\mathscr{U}$ of 0 in $X^{1,p}_{\mathscr{S},\mathscr{V},\mathscr{V}'}(\Sigma,\mathbb{C}^{4n})$, such that the set of zeroes in $[0,\alpha_0[\times\mathscr{U}$ of the above operator consists of a continuous curve $[0,\alpha_0[\ni \alpha \to (\alpha,w_\alpha)$ starting at $w_0 = 0$. Then $\alpha \to (\alpha,\mathscr{T}_\alpha w_\alpha)$ provides us with the unique curve in $\mathscr{M}^{\Upsilon}_{GE}(x_1,x_2;z)$ converging to $(0,u_0)$. This concludes the proof of Proposition 3.13.

6.4 Proof of Proposition 4.4

Fix some $\gamma_1 \in \mathscr{P}^\Omega(L_1)$, $\gamma_2 \in \mathscr{P}^\Omega(L_2)$, and $x \in \mathscr{P}^\Omega(H_1\#H_2)$ such that

$$m^\Omega(\gamma_1;L_1) + m^\Omega(\gamma_2;L_2) - \mu^\Omega(x;H_1\#H_2) = 0.$$

By a standard argument in Floer homology, the claim that P^K_Υ is a chain homotopy between K^Ω and $\Upsilon^\Omega \circ (\Phi^\Omega_{L_1} \otimes \Phi^\Omega_{L_2})$ is implied by the following statements:

(i) For every $(u_1,u_2) \in \mathscr{M}^\Omega_\Phi(\gamma_1,y_1) \times \mathscr{M}^\Omega_\Phi(\gamma_2,y_2)$ and every $u \in \mathscr{M}^\Omega_\Upsilon(y_1,y_2;x)$, where $(y_1,y_2) \in \mathscr{P}^\Omega(H_1) \times \mathscr{P}^\Omega(H_2)$ is such that

$$\mu^\Omega(y_1;H_1) + \mu^\Omega(y_2;H_2) = \mu^\Omega(x;H_1\#H_2),$$

there exists a unique connected component of $\mathscr{M}^K_\Upsilon(\gamma_1,\gamma_2;x)$ containing a curve (α,u_α) such that $\alpha \to +\infty$, $u_\alpha(\cdot,\cdot-1)$ and u_α converge to u_1 and u_2 in $C^\infty_{\mathrm{loc}}([0,+\infty[\times[0,1],T^*M)$, while $u_\alpha(\cdot+\sigma(\alpha),2\cdot-1)$ converges to u in $C^\infty_{\mathrm{loc}}(\mathbb{R}\times[0,1],T^*M)$, for a suitable function σ diverging at $+\infty$.

(ii) For every $u \in \mathscr{M}^\Omega_K(\gamma_1,\gamma_2;x)$ there exists a unique connected component of $\mathscr{M}^K_\Upsilon(\gamma_1,\gamma_2;x)$ containing a curve $\alpha \mapsto (\alpha,u_\alpha)$ which converges to $(0,u)$.

Statement (i) can be proved by the standard gluing arguments in Floer theory. Here we prove statement (ii).

Given $u : [0,+\infty[\times[-1,1] \to T^*M$, we define $\tilde{u} : \Sigma^+ \to T^*M^2$ by

$$\tilde{u}(s,t) := (-u(s,-t),u(s,t)).$$

If we define $\tilde{x} : [0,1] \to T^*M^2$ and $\tilde{H} \in C^\infty([0,1]\times T^*M^2)$ by

$$\tilde{x}(t) := \big(-x((1-t)/2),x((1+t)/2)\big), \quad \tilde{H}(t,x_1,x_2) := H_1(1-t,x_1) + H_2(t,x_2),$$

we see that associating $\tilde{u}$ to u produces a one-to-one correspondence between $\mathscr{M}_K^\Omega(\gamma_1,\gamma_2;x)$ and the space $\widetilde{\mathscr{M}}_K^\Omega(\gamma_1,\gamma_2;x)$ consisting of the maps $\tilde{u}:[0,\infty[\times[0,1]\rightarrow T^*M^2$ solving $\overline{\partial}_{J,\tilde{H}}(\tilde{u})=0$ with boundary conditions

$$\tilde{u}(s,0)\in N^*\Delta_M \quad \forall s\geq 0, \tag{176}$$

$$\tilde{u}(s,1)\in T_{q_0}^*M\times T_{q_0}^*M \quad \forall s\geq 0, \tag{177}$$

$$\pi\circ\tilde{u}_1(0,1-\cdot)\in W^u(\gamma_1), \quad \pi\circ\tilde{u}_2(0,\cdot)\in W^u(\gamma_2), \tag{178}$$

$$\lim_{s\rightarrow+\infty}\tilde{u}(s,\cdot)=\tilde{x}. \tag{179}$$

Similarly, we have a one-to-one correspondence between $\mathscr{M}_\Upsilon^K(\gamma_1,\gamma_2;x)$ and the space $\widetilde{\mathscr{M}}_\Upsilon^K(\gamma_1,\gamma_2;x)$ consisting of pairs $(\alpha,\tilde{u})$ where α is a positive number and $\tilde{u}:\Sigma^+\rightarrow T^*M^2$ is a solution of the problem above, with (176) replaced by

$$\tilde{u}(s,0)\in T_{q_0}^*M\times T_{q_0}^*M \quad \forall s\in[0,\alpha], \quad \tilde{u}(s,0)\in N^*\Delta_M \quad \forall s\geq\alpha. \tag{180}$$

Fix some $\tilde{u}^0\in\widetilde{\mathscr{M}}_K^\Omega(\gamma_1,\gamma_2;x)$. Since we are looking for solutions near $\tilde{u}^0$, we can localize the problem as follows. Let $k=m^\Omega(\gamma_1;L_1)+m^\Omega(\gamma_2;L_2)$. Let $q:\mathbb{R}^k\times\Sigma^+\rightarrow M^2$ be a map such that

$$q(0,s,t)=\pi\circ\tilde{u}^0(s,t)\ \ \forall(s,t)\in\Sigma^+, \quad q(\lambda,s,\cdot)\rightarrow\pi\circ\tilde{x}\ \text{ for } s\rightarrow+\infty,\ \forall\lambda\in\mathbb{R}^k,$$

and such that the map

$$\mathbb{R}^k\ni\lambda\mapsto(q_1(\lambda,0,1-\cdot),q_2(\lambda,0,\cdot))\in\Omega^1(M^2)$$

is a diffeomorphism onto a neighborhood of $\pi\circ(\tilde{x}_1(-\cdot),\tilde{x}_2)$ in $W^u(\gamma_1)\times W^u(\gamma_2)$. By means of a suitable trivialization of $q^*(TM^2)$ and using the usual $W^{1,p}$ Sobolev setting with $p>2$, we can transform the problem of finding mas $\tilde{u}$ solving $\overline{\partial}_{J,\tilde{H}}(\tilde{u})=0$ together with (177), (178) and (179) and being close to $\tilde{u}^0$, into the problem of finding pairs $(\lambda,u)\in\mathbb{R}^k\times W^{1,p}(\Sigma^+,T^*\mathbb{R}^{2n})$, solving an equation of the form

$$\overline{\partial}u(z)+f(\lambda,z,u(z))=0 \quad \forall z\in\Sigma^+, \tag{181}$$

with boundary conditions

$$u(0,t)\in N^*(0)\ \ \forall t\in[0,1], \quad u(s,1)\in N^*(0)\ \ \forall s\geq 0. \tag{182}$$

Then the boundary conditon (176) is translated into

$$u(s,0)\in N^*\Delta_{\mathbb{R}^n} \quad \forall s\geq 0, \tag{183}$$

and the solution $\tilde{u}^0$ corresponds to the solution $\lambda=0$ and $u\equiv 0$ of (181), (182), and (183). On the other hand, the problem $\widetilde{\mathscr{M}}_\Upsilon^K(\gamma_1,\gamma_2;x)$ of finding $(\alpha,\tilde{u}^\alpha)$ solving $\overline{\partial}_{J,\tilde{H}}(\tilde{u}^\alpha)=0$ together with (177), (178), (179) and (180) corresponds to the problem of finding $(\lambda,u)\in\mathbb{R}^k\times W^{1,p}(\Sigma^+,T^*\mathbb{R}^{2n})$ solving (181) with boundary conditions (182) and

$$u(s,0)\in N^*(0)\ \ \forall s\in[0,\alpha], \quad u(s,0)\in N^*\Delta_{\mathbb{R}^n}\ \ \forall s\geq\alpha. \tag{184}$$

In order to find a common functional setting, it is convenient to turn the boundary condition (184) into (183) by means of a suitable conformal change of variables on the half-strip Σ^+.

The holomorphic function $z\mapsto\cos z$ maps the half strip $\{0<\operatorname{Re}z<\pi,\ \operatorname{Im}z>0\}$ biholomorphically onto the upper half-plane $\mathbb{H}=\{\operatorname{Im}z>0\}$. It is also a homeomorphism between the closure of these domains. We denote by arccos the determination of the arc-cosine which is the inverse of this function. Then the function $z\mapsto(1+\cos(i\pi z))/2$ is a biholomorphism from the interior of Σ^+ to $\mathbb{H}$, mapping 0 into 1 and i into 0. Let $\epsilon>0$. If we conjugate the linear automorphism $z\mapsto(1+\epsilon)z$ of $\mathbb{H}$ by the latter biholomorphism, we obtain the following map:

$$\varphi_\epsilon(z)=\frac{i}{\pi}\arccos\big((1+\epsilon)\cos(i\pi z)+\epsilon\big).$$

The map φ_ϵ is a homeomorphism of Σ^+ onto itself, it is biholomophic in the interior, it preserves the upper part of the boundary $i + \mathbb{R}^+$, while it slides the left part $i[0,1]$ and the lower part $\mathbb{R}^+$ by moving the corner point 0 into the real positive number

$$\alpha(\epsilon) := -\frac{i}{\pi} \arccos(1+2\epsilon).$$

The function $\epsilon \mapsto \alpha(\epsilon)$ is invertible, and we denote by $\alpha \mapsto \epsilon(\alpha)$ its inverse. Moreover, φ_ϵ converges to the identity uniformly on compact subsets of Σ^+ for $\epsilon \to 0$. An explicit computation shows that

$$\varphi_\epsilon' - 1 \to 0 \quad \text{in } L^p(\Sigma^+), \quad \text{if } 1 < p < 4. \tag{185}$$

If $u : \Sigma^+ \to T^*\mathbb{R}^{2n}$ and $\alpha > 0$, we define

$$v(z) := u(\varphi_{\epsilon(\alpha)}(z)).$$

Since φ_ϵ is holomorphic, $\overline{\partial}(u \circ \varphi_\epsilon) = \varphi_\epsilon' \cdot \overline{\partial} u \circ \varphi_\epsilon$. Therefore, u solves the equation (181) if and only if v solves the equation

$$\overline{\partial} v(z) + \varphi_{\epsilon(\alpha)}'(z) f(\lambda, \varphi_{\epsilon(\alpha)}(z), v(z)) = 0. \tag{186}$$

Given $2 < p < 4$, we set

$$\begin{aligned} W^{1,p}_*(\Sigma^+, T^*\mathbb{R}^{2n}) = \big\{ v \in W^{1,p}(\Sigma^+, T^*\mathbb{R}^{2n}) \,|\, & v(s,0) \in N^*\Delta_{\mathbb{R}^n} \ \forall s \geq 0, \\ & v(s,1) \in N^*(0) \ \forall s \geq 0, \ v(0,t) \in N^*(0) \ \forall t \in [0,1] \big\}, \end{aligned}$$

and we consider the operator

$$F : [0,+\infty[\times \mathbb{R}^k \times W^{1,p}_*(\Sigma^+, T^*\mathbb{R}^{2n}) \to L^p(\Sigma^+, T^*\mathbb{R}^{2n}), \quad F(\alpha, \lambda, v) = \overline{\partial} v + \varphi_{\epsilon(\alpha)}' f(\lambda, \varphi_{\epsilon(\alpha)}(\cdot), v),$$

where $\varphi_0 = \mathrm{id}$. The problem of finding $(\alpha, \tilde{u})$ in $\widetilde{\mathscr{M}}^K_\Upsilon(\gamma_1, \gamma_2; x)$ with $\tilde{u}$ close to $\tilde{u}^0$ is equivalent to finding zeroes of the operator F of the form (α, λ, v) with $\alpha > 0$. By (185), the operator F is continuous, and its differential $D_{(\lambda,v)} F$ with respect to the variables (λ, v) is continuous. The transversality assumption that $\tilde{u}^0$ is a non-degenerate solution of problem $\widetilde{\mathscr{M}}^\Omega_K(\gamma_1, \gamma_2; x)$ is translated into the fact that $D_{(\lambda,v)} F(0,0,0)$ is an isomorphism. Then the parametric inverse maping theorem implies that there is a unique curve $\alpha \mapsto (\lambda(\alpha), v(\alpha))$, $0 < \alpha < \alpha_0$, converging to $(0,0)$ for $\alpha \to 0$, and such that $(\lambda(\alpha), v(\alpha))$ is the unique zero of $F(\alpha, \cdot, \cdot)$ in a neighborhood of $(0,0)$. This concludes the proof of statement (ii).

6.5 Proof of Proposition 4.7

The setting. We recall the setting of section 4.4. Let $\gamma_1 \in \mathscr{P}(L_1)$, $\gamma_2 \in \mathscr{P}(L_2)$, and $x \in \mathscr{P}^\Theta(H_1 \oplus H_2)$. If $\alpha \geq 0$, $\mathscr{M}^K_\alpha(\gamma_1, \gamma_2; x)$ is the space of solutions $u : [0,+\infty[\times [0,1] \to T^*M^2$ of the equation

$$\overline{\partial}_{H_1 \oplus H_2, J}(u) = 0, \tag{187}$$

satisfying the boundary conditions

$$\begin{aligned} \pi \circ u(0,\cdot) \in W^u(\gamma_1; -\mathrm{grad}_{g_1} \mathbb{S}^\Lambda_{L_1}) \times W^u(\gamma_2; -\mathrm{grad}_{g_2} \mathbb{S}^\Lambda_{L_2}), && (188) \\ (u(s,0), -u(s,1)) \in N^*\Delta_{M\times M} \text{ if } 0 \leq s < \alpha, && (189) \\ (u(s,0), -u(s,1)) \in N^*\Delta^\Theta_M \text{ if } s \geq \alpha, && (190) \\ \lim_{s\to+\infty} u(s,\cdot) = x. && (191) \end{aligned}$$

The energy of a solution $u \in \mathscr{M}^K_\alpha(\gamma_1, \gamma_2; x)$ is uniformly bounded:

$$E(u) := \int_{]0,+\infty[\times]0,1[} |\partial_s u|^2 \, ds \, dt \leq \mathbb{S}_{L_1}(\gamma_1) + \mathbb{S}_{L_2}(\gamma_2) - \mathbb{A}_{H_1 \oplus H_2}(x). \tag{192}$$

Let $\alpha_0 > 0$. For a generic choice of g_1, g_2, J_1, and J_2, both $\mathscr{M}^K_0(\gamma_1, \gamma_2; x)$ and $\mathscr{M}^K_{\alpha_0}(\gamma_1, \gamma_2; x)$ are smooth oriented manifolds of dimension

$$m^\Lambda(\gamma_1, L_1) + m^\Lambda(\gamma_2, L_2) - \mu^\Theta(x) - n,$$

for every $\gamma_1 \in \mathscr{P}^\Lambda(L_1)$, $\gamma_2 \in \mathscr{P}^\Lambda(L_2)$, $x \in \mathscr{P}^\Theta(H_1 \oplus H_2)$ (see Proposition 4.5). The usual counting process defines the chain maps

$$K^\Lambda_0, \; K^\Lambda_{\alpha_0} : (M(\mathbb{S}^\Lambda_{L_1}, g_1) \otimes M(\mathbb{S}^\Lambda_{L_2}, g_2))_* \longrightarrow F^\Theta_{*-n}(H_1 \oplus H_2, J_1 \oplus J_2),$$

and we wish to prove that $K^\Lambda_0 \otimes K^\Lambda_{\alpha_0}$ is chain homotopic to $K^\Lambda_{\alpha_0} \otimes K^\Lambda_0$. Since $K^\Lambda_{\alpha_0}$ is homotopic to $K^\Lambda_{\alpha_1}$ for $\alpha_0, \alpha_1 \in]0, +\infty[$, we may as well assume that α_0 is small. Moreover, since the chain maps K^Λ_0 and $K^\Lambda_{\alpha_0}$ preserve the filtrations of the Morse and Floer complexes given by the action sublevels

$$\mathbb{S}^\Lambda_{L_1}(\gamma_1) + \mathbb{S}^\Lambda_{L_2}(\gamma_2) \leq A, \quad \mathbb{A}_{H_1 \oplus H_2}(x) \leq A,$$

we can work with the subcomplexes corresponding to a fixed (but arbitrary) action bound A. We also choose the Lagrangians L_1 and L_2 to be non-negative, so that every orbit has non-negative action.

Convergence. Fix some $\gamma_1 \in \mathscr{P}(L_1)$, $\gamma_2 \in \mathscr{P}(L_2)$, and $x \in \mathscr{P}^\Theta(H_1 \oplus H_2)$, such that

$$m^\Lambda(\gamma_1) + m^\Lambda(\gamma_2) - \mu^\Theta(x) = n. \tag{193}$$

Let (α_h) be an infinitesimal sequence of positive numbers, let u_h be an element of $\mathscr{M}^K_{\alpha_h}(\gamma_1, \gamma_2; x)$, and let c_h be the projection onto $M \times M$ of the closed curve $u(0, \cdot)$. By (188), c_h is an element of $W^u(\gamma_1; -\mathrm{grad}_{g_1} \mathbb{S}^\Lambda_{L_1}) \times W^u(\gamma_2; -\mathrm{grad}_{g_2} \mathbb{S}^\Lambda_{L_2})$. The latter space is pre-compact in $W^{1,2}([0,1], M \times M)$. By the argument of breaking gradient flow lines, up to a subsequence we may assume that (c_h) converges in $W^{1,2}$ to a curve c in $W^u(\tilde{\gamma}_1; -\mathrm{grad}_{g_1} \mathbb{S}^\Lambda_{L_1}) \times W^u(\tilde{\gamma}_2; -\mathrm{grad}_{g_2} \mathbb{S}^\Lambda_{L_2})$, for some $\tilde{\gamma}_1 \in \mathscr{P}(L_1)$ and $\tilde{\gamma}_2 \in \mathscr{P}(L_1)$ such that

$$\text{either } m^\Lambda(\tilde{\gamma}_1) + m^\Lambda(\tilde{\gamma}_2) < m^\Lambda(\gamma_1) + m^\Lambda(\gamma_2) \text{ or } (\tilde{\gamma}_1, \tilde{\gamma}_2) = (\gamma_1, \gamma_2). \tag{194}$$

Similarly, the upper bound (192) on the energy $E(u_h)$ implies that (u_h) converges in C^∞_{loc} on $[0, +\infty[\times[0,1] \setminus \{(0,0), (0,1)\}$, using the standard argument excluding bubbling off of spheres and disks. In particular,

$$(u(s,0), -u(s,1)) \in N^* \Delta^\Theta_M, \quad \forall s > 0, \tag{195}$$

and

$$\pi \circ u(0,t) = c(t), \quad \forall t \in]0,1[. \tag{196}$$

The limit u satisfies equation (187), and

$$\lim_{s \to +\infty} u(s, \cdot) = \tilde{x},$$

with $\tilde{x}$ in $\mathscr{P}^\Theta(H_1 \oplus H_2)$ such that

$$\text{either } \mu^\Theta(\tilde{x}) > \mu^\Theta(x) \text{ or } \tilde{x} = x, \tag{197}$$

by the argument of breaking Floer trajectories. Due to finite energy,

$$E(u) \leq \liminf_{h\to+\infty} E(u_h) \leq \mathbb{S}_{L_1}(\gamma_1) + \mathbb{S}_{L_2}(\gamma_2) - \mathbb{A}_{H_1\oplus H_2}(x),$$

we find by removal singularities (Proposition 6.4) a continuous extension of u to the corner points $(0,0)$ and $(0,1)$. By (195) and (196) we have

$$(u(0,0), -u(0,1)) \in N^*\Delta_M^{\Theta}, \quad \pi\circ u(0,0) = \pi\circ u(0,1) = c(0) = c(1).$$

It follows that the two components of the closed curve c coincide at the starting point, so they describe a figure-eight loop, and u belongs to $\mathscr{M}_0^K(\tilde{\gamma}_1, \tilde{\gamma}_2; \tilde{x})$. Since the latter space is empty whenever

$$m^{\Lambda}(\tilde{\gamma}_1) + m^{\Lambda}(\tilde{\gamma}_2)) - \mu^{\Theta}(\tilde{x}) < n,$$

the index assumption (193) together with (194) and (197) implies that $\tilde{\gamma}_1 = \gamma_1$, $\tilde{\gamma}_2 = \gamma_2$, and $\tilde{x} = x$. We conclude that

$$u \in \mathscr{M}_0^K(\gamma_1, \gamma_2; x).$$

We can also say a bit more about the convergence of (u_h) towards u:

6.7. Lemma. *Let $d_h : [0,1] \to M\times M$ be the curve*

$$d_h(s) := \pi\circ u_h(\alpha_h s, 0) = \pi\circ u_h(\alpha s, 1).$$

Then (d_h) converges uniformly to the constant curve $c(0) = c(1)$.

Proof. It is convenient to replace the non-local boundary conditions (189), (190) by local ones, by setting

$$\tilde{u}_h : [0,+\infty[\times[0,1/2] \to T^*M^4, \quad u_h(s,t) := (u_h(s,t), -u_h(s,1-t)).$$

Then $\tilde{u}_h$ solves an equation of the form

$$\overline{\partial}_{J,\tilde{H}}(\tilde{u}_h) = 0,$$

for a suitable Hamiltonian $\tilde{H}$ on $[0,1/2]\times T^*M^4$, and boundary conditions

$$\pi\circ\tilde{u}_h(0,t) = (c_h(t), c_h(1-t)) \text{ for } 0\leq t\leq 1/2, \tag{198}$$
$$\tilde{u}_h(s,0) \in N^*\Delta_{M\times M} \text{ for } 0\leq s\leq \alpha_h, \tag{199}$$
$$\tilde{u}_h(s,0) \in N^*\Delta_M^{\Theta} \text{ for } s\geq\alpha_h, \tag{200}$$
$$\tilde{u}_h(s,1/2) \in N^*\Delta_{M\times M} \text{ for } s\geq 0, \tag{201}$$
$$\lim_{s\to+\infty} \tilde{u}_h(s,t) = (x(t), -x(1-t)). \tag{202}$$

Then the rescaled map

$$v_h : [0,+\infty[\times[0,1/(2\alpha_h)] \to T^*M^4, \quad v_h(s,t) = \tilde{u}_h(\alpha_h s, \alpha_h t),$$

solves the equation

$$\overline{\partial}_J v_h = \alpha_h J X_{\tilde{H}} = O(\alpha_h) \quad \text{for } h\to 0,$$

with boundary conditions

$$\pi\circ v_h(0,t) = (c_h(\alpha_h t), c_h(1-\alpha_h t)) \text{ for } 0\leq t\leq 1/(2\alpha_h), \tag{203}$$
$$v_h(s,0) \in N^*\Delta_{M\times M} \text{ for } 0\leq s\leq 1, \tag{204}$$
$$v_h(s,0) \in N^*\Delta_M^{\Theta} \text{ for } s\geq 1, \tag{205}$$
$$v_h(s,1/(2\alpha_h)) \in N^*\Delta_{M\times M} \text{ for } s\geq 0, \tag{206}$$
$$\lim_{s\to+\infty} v_h(s,t) = (x(\alpha_h t), -x(1-\alpha_h t)). \tag{207}$$

Since we have applied a conformal rescaling, the energy of v_h is uniformly bounded, so (v_h) converges up to a subsequence to some J-holomorphic map v in the $C^0_{\mathrm{loc}}([0,+\infty[\times[0,+\infty[,T^*M^4)$ topology (more precisely, we have C^∞_{loc} convergence once the domain $[0,+\infty[\times[0,+\infty[$ is transformed by a conformal mapping turning the portion near the boundary point $(1,0)$ into a neighborhood of $(0,0)$ in the upper-right quarter $\mathbb{H}^+ =]0,+\infty[\times]0,+\infty[$). The J-holomorphic map has finite energy, so by removal singularities it has a continuous extension at ∞ (again, by Proposition 6.4 together with a suitable conformal change of variables). By (203), (204), and (205) it satisfies the boundary conditions

$$\pi\circ v(0,t) = (c(0),c(0)) \text{ for } t\geq 0, \tag{208}$$

$$v(s,0)\in N^*\Delta_{M\times M} \text{ for } 0\leq s\leq 1, \tag{209}$$

$$v(s,0)\in N^*\Delta_M^\Theta \text{ for } s\geq 1. \tag{210}$$

Since the boundary conditions are of conormal type and the Liouville one-form η vanishes on conormals, we have

$$\int_{\mathbb{H}^+} |\nabla v|^2\,ds\,dt = \int_{\mathbb{H}^+} v^*(\omega) = \int_{\mathbb{H}^+} v^*(d\eta) = \int_{\mathbb{H}^+} dv^*(\eta) = \int_{\partial\mathbb{H}^+} v^*(\eta) = 0,$$

so v is constant. By (208), $\pi\circ v = (c(0),c(0))$. In particular,

$$\lim_{h\to+\infty} \pi\circ v_h(s,0) = (c(0),c(0)) \quad \text{uniformly in } s\in[0,1].$$

Since

$$\pi\circ v_h(s,0) = \pi\circ\tilde{u}_h(\alpha_h s,0) = (\pi\circ u_h(\alpha_h s,0),\pi\circ u_h(s,1)) = (d_h(s),d_h(s)),$$

the thesis follows. □

Localization. We fix a positive number A, playing the role of the upper bound for the action. Then the union of all spaces of solution $\mathscr{M}_0^K(\gamma_1,\gamma_2;x)$, where $\gamma_1\in\mathscr{P}^\Lambda(L_1)$, $\gamma_2\in\mathscr{P}^\Lambda(L_2)$, and $x\in\mathscr{P}^\Theta(H_1\oplus H_2)$ satisfy the index identity (193) and the action estimates

$$\mathbb{S}_{L_1}(\gamma_1)+\mathbb{S}_{L_2}(\gamma_2)\leq A, \quad \mathbb{A}_{H_1\oplus H_2}(x)\leq A, \tag{211}$$

is a finite set. Let us denote by q_i, $i\in\{1,\dots,m\}$, the points in M such that $\pi\circ u_i(0,0) = \pi\circ u_i(0,1) = (q_i,q_i)$ for some u in the above finite set. We choose the indexing in such a way that the points q_i are pair-wise distinct, and we fix a positive number δ such that

$$B_\delta(q_i)\cap B_\delta(q_j) = \emptyset, \quad \forall\, i\neq j.$$

We may assume that the positive constant δ chosen above is so small that $B_\delta(q_i)\subset M$ is diffeomorphic to $\mathbb{R}^n$. Lemma 6.7 implies the following localization result:

6.8. Lemma. *There exists a positive number $\alpha(A)$ such that for every $\alpha\in]0,\alpha(A)]$, every $\gamma_1\in\mathscr{P}^\Lambda(L_1)$, $\gamma_2\in\mathscr{P}^\Lambda(L_2)$, $x\in\mathscr{P}^\Theta(H_1\oplus H_2)$ satisfying the index identity (193) and the action bounds (195), each solution $u\in\mathscr{M}_\alpha^K(\gamma_1,\gamma_2;x)$ satisfies*

$$\pi\circ u([0,\alpha]\times\{0\}) = \pi\circ u([0,\alpha]\times\{1\})\subset B_{\delta/2}(q_i)\times B_{\delta/2}(q_i),$$

for some $i\in\{1,\dots,m\}$.

The chain homotopy. Since the Grassmannian of subspaces of some given dimension in $\mathbb{R}^n$ is connected and since the δ-ball around each q_i is diffeomorphic to $\mathbb{R}^n$, there exist smooth isotopies

$$\varphi_{ij} : [0,1]\times\mathbb{R}^{3n}\to B_\delta(q_i)^4\times B_\delta(q_j)^4\subset M^8, \quad \forall i,j\in\{1,\dots,m\},$$

such that, setting $V_{ij}^\lambda := \varphi_{ij}(\{\lambda\} \times \mathbb{R}^{3n})$, we have that each V_{ij}^λ is relatively closed in $B_\delta(q_i)^4 \times B_\delta(q_j)^4$ and

$$(\Delta_M^\Theta \times \Delta_M^\Theta) \cap (B_\delta(q_i) \times B_\delta(q_j)) \subset V_{ij}^\lambda, \quad \forall \lambda \in [0,1], \tag{212}$$

$$V_{ij}^\lambda \subset (\Delta_{M\times M} \times \Delta_{M\times M}) \cap (B_\delta(q_i) \times B_\delta(q_j)), \quad \forall \lambda \in [0,1], \tag{213}$$

$$V_{ij}^0 = (\Delta_M^\Theta \times \Delta_{M\times M}) \cap (B_\delta(q_i) \times B_\delta(q_j)), \tag{214}$$

$$V_{ij}^1 = (\Delta_{M\times M} \times \Delta_M^\Theta) \cap (B_\delta(q_i) \times B_\delta(q_j)). \tag{215}$$

Let $\gamma_1, \gamma_3 \in \mathscr{P}^\Lambda(L_1)$, $\gamma_2, \gamma_4 \in \mathscr{P}^\Lambda(L_2)$, and $x_1, x_2 \in \mathscr{P}^\Theta(H_1 \oplus H_2)$ satisfy the index identity

$$m^\Lambda(\gamma_1) + m^\Lambda(\gamma_2) + m^\Lambda(\gamma_3) + m^\Lambda(\gamma_4) - \mu^\Theta(x_1) - \mu^\Theta(x_2) = 2n, \tag{216}$$

and the action bounds

$$\mathbb{S}_{L_1}(\gamma_1) + \mathbb{S}_{L_2}(\gamma_2) + \mathbb{S}_{L_1}(\gamma_3) + \mathbb{S}_{L_2}(\gamma_4) \le A, \quad \mathbb{A}_{H_1\oplus H_2}(x_1) + \mathbb{A}_{H_1\oplus H_2}(x_2) \le A. \tag{217}$$

Given $\alpha > 0$, we define

$$\mathscr{M}_\alpha^P(\gamma_1, \gamma_2, \gamma_3, \gamma_4; x_1, x_2)$$

to be the set of pairs (λ, u) where $\lambda \in [0,1]$ and $u : [0, +\infty[\times[0,1] \to T^*M^4$ is a solution of the equation

$$\overline{\partial}_{H_1\oplus H_2\oplus H_1\oplus H_2, J}(u) = 0, \tag{218}$$

satisfying the boundary conditions

$$\pi \circ u(0, \cdot) \in W^u(\gamma_1) \times W^u(\gamma_2) \times W^u(\gamma_3) \times W^u(\gamma_4), \tag{219}$$

$$(u(s,0), -u(s,1)) \in \bigcup_{i,j=1}^m N^* V_{ij}^\lambda \text{ if } 0 \le s \le \alpha, \tag{220}$$

$$(u(s,0), -u(s,1)) \in N^*(\Delta_M^\Theta \times \Delta_M^\Theta) \text{ if } s \ge \alpha, \tag{221}$$

$$\lim_{s\to+\infty} u(s, \cdot) = (x_1, x_2). \tag{222}$$

Notice that if $(0, u)$ belongs to $\mathscr{M}_\alpha^P(\gamma_1, \gamma_2, \gamma_3, \gamma_4; x_1, x_2)$, then writing $u = (u_1, u_2)$ where u_1 and u_2 take values into T^*M^2, we have

$$u_1 \in \mathscr{M}_0^K(\gamma_1, \gamma_2; x_1), \quad u_2 \in \mathscr{M}_\alpha^K(\gamma_3, \gamma_4; x_2).$$

If transversality holds, we deduce the index estimates

$$m^\Lambda(\gamma_1) + m^\Lambda(\gamma_2) - \mu^\Theta(x_1) \ge n, \quad m^\Lambda(\gamma_3) + m^\Lambda(\gamma_4) - \mu^\Theta(x_2) \ge n.$$

But then (216) implies that the above inequalities are indeed identities. Similarly, if $(1, u)$ belongs to $\mathscr{M}_\alpha^P(\gamma_1, \gamma_2, \gamma_3, \gamma_4; x_1, x_2)$, we deduce that

$$u_1 \in \mathscr{M}_\alpha^K(\gamma_1, \gamma_2; x_1), \quad u_2 \in \mathscr{M}_0^K(\gamma_3, \gamma_4; x_2).$$

and

$$m^\Lambda(\gamma_1) + m^\Lambda(\gamma_2) - \mu^\Theta(x_1) = n, \quad m^\Lambda(\gamma_3) + m^\Lambda(\gamma_4) - \mu^\Theta(x_2) = n.$$

Conversely, we would like to show that pairs of solutions in $\mathscr{M}_0^K \times \mathscr{M}_\alpha^K$ (or $\mathscr{M}_\alpha^K \times \mathscr{M}_0^K$) correspond to elements of $\mathscr{M}_\alpha^P$ of the form $(0, u)$ (or $(1, u)$), at least if α is small. The key step is the following:

6.9. LEMMA. *There exists a positive number $\alpha(A)$ such that for every $\alpha \in]0, \alpha(A)]$, for every $\gamma_1, \gamma_3 \in \mathscr{P}^\Lambda(L_1)$, $\gamma_2, \gamma_4 \in \mathscr{P}^\Lambda(L_2)$, $x_1, x_2 \in \mathscr{P}^\Theta(H_1 \oplus H_2)$ satisfying (216) and (217) and for every $u \in \mathscr{M}^P_\alpha(\gamma_1, \gamma_2, \gamma_3, \gamma_4; x_1, x_2)$ there holds*

$$\pi \circ u([0,\alpha] \times \{0\}) = \pi \circ u([0,\alpha] \times \{1\}) \in B_{\delta/2}(q_i)^2 \times B_{\delta/2}(q_j)^2,$$

for suitable $i, j \in \{1, \dots, m\}$.

Proof. By contradiction, we assume that there are an infinitesimal sequence of positive numbers (α_h) and elements $(\lambda_h, u_h) \in \mathscr{M}^P_{\alpha_h}(\gamma_1, \gamma_2, \gamma_3, \gamma_4; x_1, x_2)$ where $\gamma_1, \gamma_2, \gamma_3, \gamma_4, x_1, x_2$ satisfy (216) and (217), and

$$\pi \circ u_h(s_h \alpha_h, 0) = \pi \circ u_h(s_h \alpha_h, 1) \notin \bigcup_{i,j=1}^m B_{\delta/2}(q_i)^2 \times B_{\delta/2}(q_j)^2, \tag{223}$$

for some $s_h \in [0,1]$. Let $c_h : [0,1] \to M^4$ be the closed curve defined by $c_h(t) = \pi \circ u_h(0,t)$. Arguing as in the *Convergence* paragraph above, we see that up to subsequences

$$\begin{aligned} \lambda_h &\to \lambda \in [0,1], \\ c_h &\to c \in W^u(\gamma_1) \times W^u(\gamma_2) \times W^u(\gamma_3) \times W^u(\gamma_4) \quad \text{in } W^{1,2}([0,1], M^4) \\ u_h &\to u \in \mathscr{M}^K_0(\gamma_1, \gamma_2; x_1) \times \mathscr{M}^K_0(\gamma_3, \gamma_4; x_2) \quad \text{in } C^\infty_{\text{loc}}([0, +\infty[\times [0,1] \setminus \{(0,0),(0,1)\}, T^*M^4). \end{aligned}$$

Since the space $\mathscr{M}^K_0(\gamma_1, \gamma_2; x_1) \times \mathscr{M}^K_0(\gamma_3, \gamma_4; x_2)$ is not empty, we have the index estimates

$$m^\Lambda(\gamma_1) + m^\Lambda(\gamma_2) - \mu^\Theta(x_1) \geq n, \quad m^\Lambda(\gamma_3) + m^\Lambda(\gamma_4) - \mu^\Theta(x_2) \geq n.$$

Together with (216) this implies that

$$m^\Lambda(\gamma_1) + m^\Lambda(\gamma_2) - \mu^\Theta(x_1) = n, \quad m^\Lambda(\gamma_3) + m^\Lambda(\gamma_4) - \mu^\Theta(x_2) = n. \tag{224}$$

Moreover, (217) and the fact that the action of every orbit is non-negative implies that

$$\mathbb{S}_{L_1}(\gamma_1) + \mathbb{S}_{L_2}(\gamma_2) \leq A, \quad \mathbb{A}_{H_1 \oplus H_2}(x_1) \leq A, \tag{225}$$
$$\mathbb{S}_{L_1}(\gamma_3) + \mathbb{S}_{L_2}(\gamma_4) \leq A, \quad \mathbb{A}_{H_1 \oplus H_2}(x_2) \leq A. \tag{226}$$

Furthermore, arguing as in the proof of Lemma 6.7, we find that the curve

$$d_h : [0,1] \times M^4, \quad d_h(s) = \pi \circ u_h(\alpha_h s, 0) = \pi \circ u_h(\alpha_h s, 1),$$

converges uniformly to the constant $c(0) = c(1)$. By (224), (225) and (226),

$$c(0) = c(1) = \pi \circ u(0,0) = \pi \circ u(0,1)$$

is of the form (q_i, q_i, q_j, q_j), for some $i, j \in \{1, \dots, m\}$. But then the uniform convergence of (d_h) to $c(0)$ contradicts (223). □

We fix some $\alpha_0 \in]0, \alpha(A)]$, and we choose the generic data g_1, g_2, H_1, H_2 in such a way that transversality holds for the problems $\mathscr{M}^K_0$, $\mathscr{M}^K_{\alpha_0}$, and $\mathscr{M}^P_{\alpha_0}$. Then each $\mathscr{M}^P_{\alpha_0}(\gamma_1, \gamma_2, \gamma_3, \gamma_4; x_1, x_2)$ is a smooth manifold whose boundary - if non-empty - is precisely the intersection with the regions $\{\lambda = 0\}$ and $\{\lambda = 1\}$. In particular, when (216) and (217) hold, $\mathscr{M}^K_{\alpha_0}(\gamma_1, \gamma_2, \gamma_3, \gamma_4; x_1, x_2)$ is a one-dimensional manifold with possible boundary points at $\lambda = 0$ and $\lambda = 1$, and Lemma 6.9 implies that

$$\mathscr{M}^P_{\alpha_0}(\gamma_1, \gamma_2, \gamma_3, \gamma_4; x_1, x_2) \cap \{\lambda = 0\} = \mathscr{M}^K_0(\gamma_1, \gamma_2; x_1) \times \mathscr{M}^K_{\alpha_0}(\gamma_3, \gamma_4; x_2), \tag{227}$$
$$\mathscr{M}^P_{\alpha_0}(\gamma_1, \gamma_2, \gamma_3, \gamma_4; x_1, x_2) \cap \{\lambda = 1\} = \mathscr{M}^K_{\alpha_0}(\gamma_1, \gamma_2; x_1) \times \mathscr{M}^K_0(\gamma_3, \gamma_4; x_2). \tag{228}$$

We denote by M^A the subcomplex of

$$M(\mathbb{S}_{L_1}^{\Lambda}, g_1) \otimes M(\mathbb{S}_{L_2}^{\Lambda}, g_2) \otimes M(\mathbb{S}_{L_1}^{\Lambda}, g_1) \otimes M(\mathbb{S}_{L_2}^{\Lambda}, g_2)$$

spanned by generators $\gamma_1 \otimes \gamma_2 \otimes \gamma_3 \otimes \gamma_4$ with

$$\mathbb{S}_{L_1}(\gamma_1) + \mathbb{S}_{L_2}(\gamma_2) + \mathbb{S}_{L_1}(\gamma_3) + \mathbb{S}_{L_2}(\gamma_4) \leq A.$$

Similarly, we denote by F^A the subcomplex of

$$F^{\Theta}(H_1 \oplus H_2, J) \otimes F^{\Theta}(H_1 \oplus H_2, J)$$

spanned by generators $x_1 \otimes x_2$ with

$$\mathbb{A}_{H_1 \oplus H_2}(x_1) + \mathbb{A}_{H_1 \oplus H_2}(x_2) \leq A.$$

We define a homomorphism

$$P : M_*^A \to F_{*-2n+1}^A$$

by counting the elements of $\mathscr{M}_{\alpha_0}^{P}(\gamma_1, \gamma_2, \gamma_3, \gamma_4; x_1, x_2)$ in the zero-dimensional case:

$$m^{\Lambda}(\gamma_1) + m^{\Lambda}(\gamma_2) + m^{\Lambda}(\gamma_3) + m^{\Lambda}(\gamma_4) - \mu^{\Theta}(x_1) - \mu^{\Theta}(x_2) = 2n - 1.$$

Using the identities (227) and (228) we see that P is a chain homotopy between the restrictions of $K_0^{\Lambda} \otimes K_{\alpha_0}^{\Lambda}$ and $K_{\alpha_0}^{\Lambda} \otimes K_{\alpha}^{\Lambda}$ to the above subcomplexes. This concludes the proof of Proposition 4.7.

References

[AF07] A. Abbondandolo and A. Figalli, *High action orbits for Tonelli Lagrangians and superlinear Hamiltonians on compact configuration spaces*, J. Differential Equations **234** (2007), 626–653.

[AM06] A. Abbondandolo and P. Majer, *Lectures on the Morse complex for infinite dimensional manifolds*, Morse theoretic methods in nonlinear analysis and in symplectic topology (Montreal) (P. Biran, O. Cornea, and F. Lalonde, eds.), Springer, 2006, pp. 1–74.

[APS08] A. Abbondandolo, A. Portaluri, and M. Schwarz, *The homology of path spaces and Floer homology with conormal boundary conditions*, in preparation, 2008.

[AS06a] A. Abbondandolo and M. Schwarz, *Notes on Floer homology and loop space homology*, Morse theoretic methods in nonlinear analysis and in symplectic topology (Montreal) (P. Biran, O. Cornea, and F. Lalonde, eds.), Springer, 2006, pp. 75–108.

[AS06b] A. Abbondandolo and M. Schwarz, *On the Floer homology of cotangent bundles*, Comm. Pure Appl. Math. **59** (2006), 254–316.

[AS08] A. Abbondandolo and M. Schwarz, *Product structures in Floer homology for cotangent bundles*, in preparation, 2008.

[BCR06] N. A. Baas, R. L. Cohen, and A. Ramírez, *The topology of the category of open and closed strings*, Recent developments in algebraic topology, Contemp. Math., vol. 407, Amer. Math. Soc., Providence, RI, 2006, pp. 11–26.

[BC94] M. Betz and R. L. Cohen, *Moduli spaces of graphs and cohomology operations*, Turkish J. Math. **18** (1994), 23–41.

[ChS99] M. Chas and D. Sullivan, *String topology*, `arXiv:math.GT/9911159`, 1999.

[ChS04] M. Chas and D. Sullivan, *Closed string operators in topology leading to Lie bialgebras and higher string algebra*, The legacy of Niels Henrik Abel, Springer, 2004, pp. 771–788.

[CL07] K. Cieliebak and J. Latschev, *The role of string topology in symplectic field theory*, `arXiv:0706.3284v2 [math.SG]` (2007).

[Coh06] R. L. Cohen, *Morse theory, graphs, and string topology*, Morse theoretical methods in nonlinear analysis and symplectic topology (Montreal) (P. Biran, O. Cornea, and F. Lalonde, eds.), Springer, 2006, pp. 149–184.

[CHV06] R. L. Cohen, K. Hess, and A. A. Voronov, *String topology and cyclic homology*, Birkhäuser, 2006.

[CJ02] R. L. Cohen and J. D. S. Jones, *A homotopy theoretic realization of string topology*, Math. Ann. **324** (2002), 773–798.

[CJY03] R. L. Cohen, J. D. S. Jones, and J. Yan, *The loop homology algebra of spheres and projective spaces*, Categorical decomposition techniques in algebraic topology (Isle of Skye, 2001), Birkhäuser, 2003, pp. 77–92.

[CS08] R. L. Cohen and M. Schwarz, *A Morse theoretic description of string topology*, in preparation, 2008.

[Dol80] A. Dold, *Lectures on algebraic topology*, Springer, Berlin, 1980.

[Dui76] J. J. Duistermaat, *On the Morse index in variational calculus*, Advances in Math. **21** (1976), 173–195.

[Flo88a] A. Floer, *A relative Morse index for the symplectic action*, Comm. Pure Appl. Math. **41** (1988), 393–407.

[Flo88b] A. Floer, *The unregularized gradient flow of the symplectic action*, Comm. Pure Appl. Math. **41** (1988), 775–813.

[Flo89a] A. Floer, *Symplectic fixed points and holomorphic spheres*, Comm. Math. Phys. **120** (1989), 575–611.

[Flo89b] A. Floer, *Witten's complex and infinite-dimensional Morse theory*, J. Differential Geom. **30** (1989), 207–221.

[FH93] A. Floer and H. Hofer, *Coherent orientations for periodic orbit problems in symplectic geometry*, Math. Z. **212** (1993), 13–38.

[FHS96] A. Floer, H. Hofer, and D. Salamon, *Transversality in elliptic Morse theory for the symplectic action*, Duke Math. J. **80** (1996), 251–292.

[Fuk93] K. Fukaya, *Morse homotopy, A^∞-category, and Floer homologies*, Proceedings of GARC Workshop on Geometry and Topology '93 (Seoul, 1993) (Seoul), Lecture Notes Ser., vol. 18, Seoul Nat. Univ., 1993, pp. 1–102.

[Fuk97] K. Fukaya, *Morse homotopy and its quantization*, Geometric topology (Athens, GA, 1993), AMS/IP Stud. Adv. Math., vol. 2, Amer. Math. Soc., Providence, RI, 1997, pp. 409–440.

[FO99] K. Fukaya and K. Ono, *Arnold conjecture and Gromov-Witten invariant*, Topology **38** (1999), 933–1048.

[GH07] M. Goresky and N. Hingston, *Loop products and closed geodesics*, `arXiv:0707.3486v1 [math.AT]` (2007).

[HS95] H. Hofer and D. Salamon, *Floer homology and Novikov rings*, The Floer memorial volume (H. Hofer et al., ed.), Progr. Math., vol. 133, Birkhäuser, Basel, 1995, pp. 483–524.

[HZ94] H. Hofer and E. Zehnder, *Symplectic invariants and Hamiltonian dynamics*, Birkhäuser, Basel, 1994.

[Hör71] L. Hörmander, *Fourier integral operators I*, Acta Math. **127** (1971), 79–183.

[Lan99] S. Lang, *Fundamentals of differential geometry*, Springer, Berlin, 1999.

[LT98] G. Liu and G. Tian, *Floer homology and Arnold conjecture*, J. Differential Geom. **49** (1998), 1–74.

[LT99] G. Liu and G. Tian, *On the equivalence of multiplicative structures in Floer homology and quantum homology*, Acta Mathematica Sinica **15** (1999), 53–80.

[MS04] D. McDuff and D. Salamon, *J-holomorphic curves and symplectic topology*, Colloquium Publications, vol. 52, American Mathematical Society, Providence, R.I., 2004.

[PSS96] S. Piunikhin, D. Salamon, and M. Schwarz, *Symplectic Floer-Donaldson theory and quantum cohomology*, Contact and symplectic geometry (C. B. Thomas, ed.), vol. 8, Cambridge University Press, 1996, pp. 171–200.

[Qui85] D. Quillen, *Determinants of Cauchy-Riemann operators over a Riemann surface*, Functional Anal. Appl. **19** (1985), 31–34.

[Ram06] A. Ramirez, *Open-closed string topology via fat graphs*, `arXiv:math/0606512v1 [math.AT]`, 2006.

[RS93] J. Robbin and D. Salamon, *Maslov index theory for paths*, Topology **32** (1993), 827–844.

[RS95] J. Robbin and D. Salamon, *The spectral flow and the Maslov index*, Bull. London Math. Soc. **27** (1995), 1–33.

[SW06] D. Salamon and J. Weber, *Floer homology and the heat flow*, Geom. Funct. Anal. **16** (2006), 1050–1138.

[Sch93] M. Schwarz, *Morse homology*, Birkhäuser, Basel, 1993.

[Sch95] M. Schwarz, *Cohomology operations from S^1-cobordisms in Floer homology*, Ph.D. thesis, Swiss Federal Inst. of Techn. Zurich, Zurich, Diss. ETH No. 11182, 1995.

[Sma65] S. Smale, *An infinite dimensional version of Sard's theorem*, Amer. J. Math. **87** (1965), 861–866.

[Sul03] D. Sullivan, *Open and closed string field theory interpreted in classical algebraic geometry*, arXiv:math.QA/0302332, 2003.

[Sul07] D. Sullivan, *String topology: Background and present state*, `arXiv: 0710.4141v1 [math.GT]` (2007).

[tDKP70] T. tom Dieck, K. H. Kamps, and D. Puppe, *Homotopietheorie*, Springer, Berlin-Heidlberg-New York, 1970.

[Vit95] C. Viterbo, *The cup-product on the Thom-Smale-Witten complex, and Floer cohomology*, The Floer memorial volume, Progr. Math., vol. 133, Birkhäuser, Basel, 1995, pp. 609–625.

[Vit96] C. Viterbo, *Functors and computations in Floer homology with applications, II*, preprint (revised in 2003), 1996.