

Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

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Euler-Lagrange equations

(revised version: July 2009)

by

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Preprint no.: 46

2008



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Abstract

Let $S \subset \mathbb{R}^2$ be a bounded C^1 domain and let g denote the flat metric in \mathbb{R}^2 . We prove that there exist minimizers of the Willmore functional restricted to a class of isometric immersions of the Riemannian surface (S, g) into \mathbb{R}^3 . We derive the Euler-Lagrange equations satisfied by such constrained minimizers. Our main motivation comes from nonlinear elasticity, where this constrained Willmore functional arises naturally and is called Kirchhoff's plate functional.

1 Introduction

For a surface Σ immersed in \mathbb{R}^3 the Willmore functional is given by

$$\mathcal{W}(\Sigma) = \frac{1}{4} \int_{\Sigma} |H|^2 d\mathcal{H}^2,$$

where H denotes the mean curvature of Σ and \mathcal{H}^2 is the two dimensional Hausdorff measure. One natural question is this: Given a fixed two dimensional Riemannian manifold (S, g) , which is its optimal (isometric) realization as a surface in \mathbb{R}^3 ? Here optimality is understood in the sense that the immersion should minimize the Willmore functional among all isometric immersions of the manifold.

In this paper we prove existence of and derive the Euler-Lagrange equations satisfied by flat minimizers of \mathcal{W} : Let S be a bounded domain in \mathbb{R}^2 and let $g_{ij} = \delta_{ij}$ be the flat metric on \mathbb{R}^2 . We consider minimizers of \mathcal{W} within a subclass of all realizations $\Sigma \subset \mathbb{R}^3$ of the Riemannian surface (S, g) . The minimizers will lie in the set of isometric immersions with finite Willmore energy,

$$W_{\text{iso}}^{2,2}(S; \mathbb{R}^3) = \{u \in W^{2,2}(S; \mathbb{R}^3) : (\nabla u)^T (\nabla u) = Id \text{ almost everywhere.}\}$$

Without boundary conditions the minimizer is the identity. A nontrivial problem arises when one prescribes the values of the immersion and of the surface normal on parts of the boundary.

On $W_{\text{iso}}^{2,2}(S; \mathbb{R}^3)$, the Willmore functional agrees (up to a prefactor) with Kirchhoff's energy functional for thin nonlinearly elastic plates:

$$\mathcal{E}_K(u; S) := \begin{cases} \frac{1}{24} \int_S |\nabla^2 u(x)|^2 dx & \text{if } u \in W_{\text{iso}}^{2,2}(S; \mathbb{R}^3), \\ +\infty & \text{otherwise.} \end{cases} \quad (1)$$

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The Euler-Lagrange equations derived in this article are the key to the regularity analysis of minimizers of the constrained Willmore functional \mathcal{E}_K . They are used in [13] to obtain an optimal regularity result. They can also be used for numerical computations (see [21] for a result in this direction). The main result of this article was announced in [12].

The functional \mathcal{E}_K has been shown to arise as an asymptotic thin film limit from three dimensional nonlinear elasticity [5]. That result has recently lead to an increased interest in $W^{2,2}$ isometric immersions: The papers [18, 6, 3] are directly related to elasticity, whereas more fundamental properties of $W^{2,2}$ isometric immersions are studied in [14] as well as [17, 9, 10, 11] (where it is shown that smooth isometric immersions are dense in $W_{\text{iso}}^{2,2}$) and in [15] (where it is shown that $W^{2,2}$ isometric immersions are C^1 up to the boundary of the domain when the latter is smooth enough).

A problem which is related to the one addressed here is studied in [1]. There the authors consider minimizers of \mathcal{W} within all conformal immersions of a given Riemannian surface. In [20] the author addresses existence and regularity of minimizers within the class of all surfaces with prescribed genus. The Helfrich model for biological membranes deals with the restriction of \mathcal{W} to classes of surfaces with a fixed area. (Incidentally, this constraint is automatically satisfied in the situation considered in this article, but our constraint is far too strong for that model.)

Although related to the results mentioned above, in many respects the problem considered here is quite different, and so are the techniques used to solve it. The source of this difference is that the constancy of the metric g severely restricts the geometry of isometric immersions: A key property of isometric immersions of a flat Riemannian surface is that they are developable surfaces. This was shown in [7, 16] for the case of C^2 immersions. The same remains true when smoothness is replaced by Lipschitz plus finite Willmore energy. So mappings in $W_{\text{iso}}^{2,2}(S; \mathbb{R}^3)$ are (essentially) developable surfaces [14, 17]. Our approach relies heavily on this fact. In many respects it allows to reduce the problem to a problem on curves. An interesting consequence is that the Euler-Lagrange equations derived in this article are ordinary differential equations. This is in sharp contrast to the unconstrained Euler-Lagrange equation, a partial differential equation called the Willmore equation, see e.g. [22] and [19].

The derivation of the Euler-Lagrange equations in the setting considered here is not trivial because the corresponding variations must satisfy two kinds of constraints: The flatness constraint and the boundary conditions. The former is handled by passing to a line of curvature parametrization, which by developability amounts to describing the surface in terms of a single nontrivial line of curvature γ (at least locally). As natural new variables one takes the normal and geodesic curvatures of γ or, equivalently, the normal curvature κ_n of γ and the arclength parametrized preimage Γ of γ . But also the variations of the new variables (Γ, κ_n) must again satisfy two kinds of constraints, some nonlocal ones and a local one. The former arise from the natural “local” boundary conditions for developable surfaces. The latter is the condition that Γ must be “admissible” in a sense made precise later. This condition is a vestige of the fact that the Euler-Lagrange equations describe a surface and not just a curve.

This article is organized as follows. In Section 2 we review some fundamental prop-

erties of flat $W^{2,2}$ isometric immersions. We will take the viewpoint adopted in [10, 11]. At the end of Section 2 we present the main results of this article. In Section 3 we introduce the variations in such a way that they automatically satisfy the local “admissibility” constraint mentioned earlier. At that point they do not yet satisfy the nonlocal constraints coding the boundary conditions. After computing the derivative of the energy functional in Section 4, we introduce the constraint functional in Section 5. By restricting the variations from Section 3 to level sets of the constraint functional one obtains variations which also satisfy the nonlocal constraints. In Section 6 we derive the Euler-Lagrange equations and prove the main results. In the appendix we collect some results related to $W^{2,2}$ isometric immersions (more generally about developable mappings in the sense of [10]). They extend some results from [10].

Notation. Except stated otherwise, $S \subset \mathbb{R}^2$ denotes a bounded C^1 domain. All curves Γ and γ satisfy $|\Gamma'| \equiv 1$ and $|\gamma'| \equiv 1$. By e_i we denote the unit vectors in \mathbb{R}^2 or in \mathbb{R}^3 . The superscript \perp denotes counterclockwise rotation by $\frac{\pi}{2}$.

We write $f(\tau)$ instead of $f \circ \tau$ to denote the composition of mappings. We write $*$ to denote $+$ or $-$, and if $* = +$ then we set $\bar{*} = -$ and viceversa. When referring to pointwise properties of $f \in L^1_{loc}$ we always refer to the precise representative of f . \mathcal{H}^k denotes k -dimensional Hausdorff measure, \mathcal{L}^1 denotes one dimensional Lebesgue measure. If $X \subset \mathbb{R}^2$ then by $\mathcal{C}(X; U)$ we denote the connected component of X that contains the connected set U , and if $U = \{v\}$ then we set $\mathcal{C}(X; v) := \mathcal{C}(X; \{v\})$.

2 Review of $W^{2,2}$ isometric immersions and main results

2.1. Let $S \subset \mathbb{R}^2$ be a bounded Lipschitz domain, let $u_0 \in W^{2,2}_{iso}(S; \mathbb{R}^3)$ and let $\partial_c S \subset \partial S$ be closed. We set

$$\mathcal{A}_{u_0}(S, \partial_c S) = \{u \in W^{2,2}_{iso}(S; \mathbb{R}^3) : u = u_0 \text{ and } \nabla u = \nabla u_0 \text{ on } \partial_c S\}.$$

The equality of the gradients is understood in the trace sense. Clearly $u \in \mathcal{A}_{u_0}(S, \partial_c S)$ implies $\mathcal{A}_u(S, \partial_c S) = \mathcal{A}_{u_0}(S, \partial_c S)$.

For $U \subset S$ Borel we define

$$\mathcal{E}(u; U) := \int_U |\nabla^2 u(x)|^2 dx. \quad (2)$$

The existence of minimizers of \mathcal{E} under the required constraints can be established for different kinds of boundary conditions (e.g. prescribing only the values of u). In our setting it reads as follows:

2.2. Theorem. *Let $S \subset \mathbb{R}^2$ be a bounded Lipschitz domain, let $u_0 \in W^{2,2}_{iso}(S; \mathbb{R}^3)$ and let $\partial_c S \subset \partial S$ be closed. Then there exists $u \in \mathcal{A}_{u_0}(S, \partial_c S)$ satisfying $\mathcal{E}(u; S) = \inf_{\bar{u} \in \mathcal{A}_{u_0}(S, \partial_c S)} \mathcal{E}(\bar{u}; S)$.*

2.3. From now on let $S \subset \mathbb{R}^2$ be a bounded C^1 domain. Let us recall some notions from [10, 11] (see also [9]). We refer to [10, 11] for many more details. For $\mu \in \mathbb{S}^1$

and $x \in S$ we denote by $[x]_\mu$ the connected component of $(x + \text{Span } \mu) \cap S$ which contains x . For $x \in S$ and $\mu \in \mathbb{R}^2 \setminus \{0\}$ we define $\nu(x, \mu) := \inf\{\theta > 0 : x + \theta\mu \notin S\}$.

A pair $(x, \theta) \in S \times (\mathbb{R}^2 \setminus \{0\})$ is said to be transversal if the line segment with endpoints x and $x + \nu(x, \theta)\theta$ intersects ∂S transversally at the point $x + \nu(x, \theta)\theta \in \partial S$. If (x, θ) is transversal then $\nu \in C^1$ near (x, θ) , see Lemma 7.1. We define

$$\nu_1(x, \mu) \cdot e_i := \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (\nu(x + \varepsilon e_i, \mu) - \nu(x, \mu)) \quad (3)$$

$$\nu_2(x, \mu) \cdot e_i := \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (\nu(x, \mu + \varepsilon e_i) - \nu(x, \mu)). \quad (4)$$

By Lemma 7.1 we have $\nu_2(x, \theta) = \nu(x, \theta)\nu_1(x, \theta)$.

For $u \in W_{\text{iso}}^{2,2}(S; \mathbb{R}^3)$ we set

$$C_{\nabla u} := \{x \in S : \nabla u \text{ is constant in a neighbourhood of } x\}.$$

If U is a connected component of $C_{\nabla u}$ then U has finite perimeter, and $S \cap \partial U$ is a disjoint union of straight line segments, see [10].

By [15] we have $\nabla u \in C^0(S; \mathbb{R}^{3 \times 2})$. Following [10], we say that ∇u is S -developable on a set $S_1 \subset S$ if there exists a mapping $q : S_1 \rightarrow \mathbb{S}^1$ such that ∇u is constant on $[x]_{q(x)}$, and $[x]_{q(x)} \cap [y]_{q(y)} \neq \emptyset$ implies $[x]_{q(x)} = [y]_{q(y)}$ whenever $x, y \in S_1$. The mapping q is called a local S -ruling for ∇u near x . The segments $[x]_{q(x)}$ are called rulings as well. In the sequel we will often omit the index q and the prefix S . It is easy to see that q can be chosen (by appropriately choosing antipodal points) to be locally Lipschitz in S , see e.g. [14]. We introduce the set

$$D_{\nabla u} := \{x \in S : \nabla u \text{ is } S\text{-developable in a neighbourhood of } x\}.$$

Thus for all $x_0 \in D_{\nabla u}$ there is a neighbourhood S_q of x_0 and an S -ruling $q : S_q \rightarrow \mathbb{S}^1$ for ∇u .

2.4. Remarks.

- (i) If ∇u is S -developable on $S_1 \subset S$ and on $S_2 \subset S$ then it need not be S -developable on $S_1 \cup S_2$. Therefore, ∇u is not S -developable on $D_{\nabla u}$ in general. See e.g. Figure 1.
- (ii) However, for all $u \in W_{\text{iso}}^{2,2}(S; \mathbb{R}^3)$ the gradient ∇u is S -developable on $S \setminus C_{\nabla u}$, see [14, 17]. (It is in fact S -developable on a larger set, see [10].) In particular, the interior of $S \setminus C_{\nabla u}$ is contained in $D_{\nabla u}$.
- (iii) All S -rulings for ∇u agree on $S \setminus C_{\nabla u}$ if regarded as a mappings into the projective space \mathbb{P}^1 , see Remark 2.2.1 in [10]. If $x \in D_{\nabla u} \cap C_{\nabla u}$, however, then the local S -ruling near x is not unique in general. We denote by $q_{\nabla u} : S \setminus C_{\nabla u} \rightarrow \mathbb{S}^1$ the unique S -ruling for ∇u on $S \setminus C_{\nabla u}$.

If $\Gamma \in W^{2,\infty}([0, T]; S)$ is parametrized by arclength, we set $N := (\Gamma')^\perp$ and $s_\Gamma^*(t) := * \nu(\Gamma(t), N(t))$ for $* = +, -$ and $t \in [0, T]$. We also set $\kappa := \Gamma'' \cdot N$. The curve Γ is said to be locally S -admissible if $1 - s_\Gamma^*(t)\kappa(t) \geq 0$ for almost every $t \in (0, T)$ and

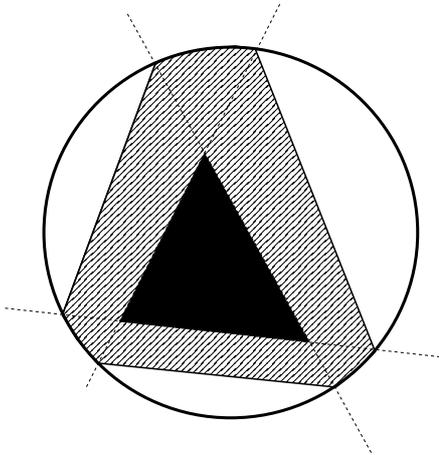


Figure 1: The union of the dashed region and the black triangle is one connected component of $C_{\nabla u}$. The set $D_{\nabla u}$ agrees with the complement of the black triangle. But clearly ∇u is not developable on the complement of the black triangle.

for $* = +, -$. It is called S -admissible if, for all $t_1, t_2 \in [0, T]$, we have $[\Gamma(t_1)]_{N(t_1)} \cap [\Gamma(t_2)]_{N(t_2)} \neq \emptyset$ only if $t_1 = t_2$. It is said to be S -transversal on $J \subset [0, T]$ if $[\overline{\Gamma(t)}]$ intersects ∂S transversally (at both endpoints) for all $t \in J$. If no interval J is specified, then it is understood that J is the whole domain of Γ , i.e. $J = [0, T]$. In what follows we will omit the prefix S .

We define the Frénet frame $R := (\Gamma'|N)^T$. The Frénet equations read

$$R' = \begin{pmatrix} 0 & \kappa \\ -\kappa & 0 \end{pmatrix} R. \quad (5)$$

Now let $\kappa_n \in L^2(0, T)$. Then we define the frame $r \in W^{1,2}((0, T); SO(3))$ as the solution of

$$r' = \begin{pmatrix} 0 & \kappa & \kappa_n \\ -\kappa & 0 & 0 \\ -\kappa_n & 0 & 0 \end{pmatrix} r \quad (6)$$

with some initial value $r(0)$. We define $(\gamma'|v|n) := r^T$ and $\gamma(t) := \gamma(0) + \int_0^t \gamma'$ for some initial value $\gamma(0)$.

2.5. For $s^- < 0 < s^+$ and $x \in (\frac{1}{s^-}, \frac{1}{s^+})$ we introduce $g(s^\pm, x) = \int_{s^-}^{s^+} \frac{1}{1-sx} ds$ and $g_2(s^\pm, x) = -\int_{s^-}^{s^+} \frac{1}{(1-sx)^2} ds$ and $g_3(s^\pm, x) = \int_{s^-}^{s^+} \frac{s}{(1-sx)^2} ds$ and $g_*(s^\pm, x) = * \frac{1}{1-s^*x}$. Let $\Gamma \in W^{2,\infty}([0, T]; S)$ be an arclength parametrized curve which is locally admissible. Define

$$I_0^\Gamma := \{t \in [0, T] : s_\Gamma^+(t)\kappa(t) = 1 \text{ or } s_\Gamma^-(t)\kappa(t) = 1\}. \quad (7)$$

For $\eta > 0$ define

$$I_\eta^\Gamma := \{t \in [0, T] : 1 - s_\Gamma^+(t)\kappa(t) > \eta \text{ and } 1 - s_\Gamma^-(t)\kappa(t) > \eta\}. \quad (8)$$

Let $J \subset [0, T]$ Borel. We define

$$M_{s_{\Gamma}^{\pm}}(J) := \bigcup_{t \in J} (s_{\Gamma}^{-}(t), s_{\Gamma}^{+}(t)) \times \{t\}. \quad (9)$$

For $\kappa_n \in L^2(0, T)$ we define

$$\mathcal{F}(\Gamma, \kappa_n; J) := \int_{M_{s_{\Gamma}^{\pm}}(J)} \frac{\kappa_n^2(t)}{1 - s\kappa(t)} ds dt. \quad (10)$$

Set $[\Gamma(0, T)] := \bigcup \{[\Gamma(t)]_{N(t)} : t \in (0, T)\}$. We define a mapping $(\Gamma, \kappa_n) : [\Gamma(0, T)] \rightarrow \mathbb{R}^3$ by setting

$$(\Gamma, \kappa_n)(\Gamma(t) + sN(t)) := \gamma(t) + sv(t) \text{ for all } t \in [0, T] \text{ and all } s \in (s_{\Gamma}^{-}(t), s_{\Gamma}^{+}(t)). \quad (11)$$

2.6. Remark. If $\Gamma \in W^{2,\infty}([0, T]; S)$ is admissible and $\kappa_n \in L^2(0, T)$ then (Γ, κ_n) is a well defined element of $W_{loc, iso}^{2,2}([\Gamma(0, T)]; \mathbb{R}^3)$, and

$$\int_{[\Gamma(0, T)]} |\nabla^2(\Gamma, \kappa_n)|^2 = \mathcal{F}(\Gamma, \kappa_n; (0, T)). \quad (12)$$

The left-hand side of (12) is finite if and only if its right-hand side is finite, i.e.

$$(s, t) \mapsto \frac{\kappa_n^2(t)}{1 - s\kappa(t)} \in L^1(M_{s_{\Gamma}^{\pm}}(0, T)). \quad (13)$$

If that is the case, then $\kappa_n = 0$ almost everywhere on I_0^{Γ} and

$$\mathcal{F}(\Gamma, \kappa_n; (0, T)) = \int_{(0, T) \setminus I_0^{\Gamma}} \kappa_n^2(t) g(s_{\Gamma}^{\pm}(t), \kappa(t)) dt. \quad (14)$$

Proof. Proposition 2.3 (ii) in [11] implies that (Γ, κ_n) is a well defined mapping in $W_{loc, iso}^{2,2}([\Gamma(0, T)]; \mathbb{R}^3)$. The formula (12) is proven in [11]. It is easy to see that (13) can only hold if $\kappa_n = 0$ almost everywhere on I_0^{Γ} . Away from I_0^{Γ} we can apply Fubini's Theorem to obtain (14). \square

Notice that (10) makes sense for any arclength parametrized curve $\Gamma \in W^{2,\infty}((0, T); S)$ and any $\kappa_n \in L^2(0, T)$, even if the mapping (Γ, κ_n) is not well-defined (e.g. if Γ is not admissible).

The frame r is the Darboux frame of the line of curvature γ on the surface $(\Gamma, \kappa_n)([\Gamma(0, T)])$. One can prove (see [10]) that on $D_{\nabla u}$, the mapping u is locally of the form (Γ, κ_n) , where Γ is the preimage of a nontrivial line of curvature on $u(S)$ and κ_n is its normal curvature. The curve Γ is then called ∇u -integral curve because it satisfies the ODE $\Gamma' = -(q(\Gamma))^{\perp}$ for some local S -ruling q for ∇u that is locally Lipschitz in S . Hence, if $\Gamma([0, T]) \subset S$ then $\Gamma \in W^{2,\infty}([0, T]; S)$.

Denote by χ_* the characteristic function of the set where κ has sign $*$ and set $\sigma := \sum_* \chi_* s_\Gamma^*$. If Γ is transversal, then we define:

$$\bar{h} = \kappa \sum_* \chi_* \nu_1(\Gamma, *N) \text{ and } h = \bar{h} \cdot \Gamma' \quad (15)$$

$$\bar{F}_1 = \sum_* \frac{\nu_1(\Gamma, *N)}{1 - s_\Gamma^* \kappa} + \bar{h} g_2(s_\Gamma^\pm, \kappa) \text{ and } F_1 = \bar{F}_1 \cdot \Gamma' \quad (16)$$

$$\bar{F}_2 = \sum_* \frac{s_\Gamma^* \nu_1(\Gamma, *N)}{1 - s_\Gamma^* \kappa} + \sigma \bar{h} g_2(s_\Gamma^\pm, \kappa) \text{ and } F_2 = \bar{F}_2 \cdot \Gamma'. \quad (17)$$

2.7. Definition. A pair (Γ, κ_n) with $\Gamma \in W^{2,\infty}((0, T); S)$ locally admissible and transversal, and $\kappa_n \in L^2(0, T)$ is said to satisfy the Euler-Lagrange equations if there exist $\lambda_1, \lambda_2 \in \mathbb{R}^3$ and $\lambda_3, \lambda_4 \in \mathbb{R}$ such that the following equations are satisfied for almost every $t \in (0, T)$:

$$2(1 - \chi_{I_0^\Gamma}(t)) \kappa_n(t) g(s_\Gamma^\pm(t), \kappa(t)) = -v(t) \cdot (\lambda_2 - \lambda_1 \wedge \int_t^T \gamma') \quad (18)$$

$$(1 - \chi_{I_0^\Gamma}(t)) \kappa_n^2(t) g_2(s_\Gamma^\pm(t), \kappa(t)) = (1 - \chi_{I_0^\Gamma}(t)) \Omega_2(t) \quad (19)$$

$$(1 - \chi_{I_0^\Gamma}(t)) \kappa_n^2(t) g_3(s_\Gamma^\pm(t), \kappa(t)) = \Omega_3(t) + \chi_{I_0^\Gamma}(t) \frac{\Omega_2(t)}{\kappa(t)} \quad (20)$$

Here, Ω_2 and Ω_3 are the unique Lipschitz continuous solutions to the terminal value problems

$$\Omega_2' = -h\Omega_2 + \kappa_n(\lambda_1 \cdot n) + \kappa_n^2 F_1 \text{ and } \Omega_2(T) = \lambda_3 + \lambda_1 \cdot \gamma'(T) \quad (21)$$

$$\Omega_3' = h\sigma\Omega_2 - \kappa_n \gamma' \cdot (\lambda_2 - \lambda_1 \wedge \int_t^T \gamma') - \kappa_n^2 F_2 \text{ and } \Omega_3(T) = \lambda_4 + \lambda_2 \cdot n(T). \quad (22)$$

2.8. If Γ is admissible and transversal then $S \cap \partial[\Gamma(0, T)] = [\Gamma(0)] \cup [\Gamma(T)]$ and one can define traces on $[\Gamma(0)] \cup [\Gamma(T)]$ for functions in $W^{1,2}([\Gamma(0, T)])$, see Lemma 7.9. Therefore, the following space is well defined for such Γ and for $\kappa_n \in L^2(0, T)$ satisfying (13):

$$\mathcal{A}_{(\Gamma, \kappa_n)} := \left\{ u \in W_{\text{iso}}^{2,2}([\Gamma(0, T)]; \mathbb{R}^3) : (u, \nabla u) = ((\Gamma, \kappa_n), \nabla(\Gamma, \kappa_n)) \right. \\ \left. \text{on } [\Gamma(0)] \cup [\Gamma(T)] \right\}. \quad (23)$$

(The equality of the gradients is understood in the sense of traces.) The main result of this article reads as follows.

2.9. Theorem. Let $S \subset \mathbb{R}^2$ be a bounded C^1 domain, let $T > 0$, let $\Gamma \in W^{2,\infty}([0, T]; S)$ be S -admissible and let $\kappa_n \in L^2(0, T)$ be such that (13) holds. Then $(\Gamma, \kappa_n) \in W_{\text{iso}}^{2,2}([\Gamma(0, T)]; \mathbb{R}^3)$. If, in addition, Γ is transversal and (Γ, κ_n) is a minimizer of $\mathcal{E}(\cdot; [\Gamma(0, T)])$ within the class $\mathcal{A}_{(\Gamma, \kappa_n)}$, then (Γ, κ_n) solves the Euler-Lagrange equations in the sense of Definition 2.7.

The relevance of Theorem 2.9 is that if (Γ, κ_n) is a portion of a minimizer then it is minimizing under its own boundary conditions, i.e. within the class $\mathcal{A}_{(\Gamma, \kappa_n)}$.

To state the main implications of Theorem 2.9 in terms of surfaces, for given $u \in W_{\text{iso}}^{2,2}(S; \mathbb{R}^3)$ we define

$$\begin{aligned}\Sigma_\tau &:= \{x \in D_{\nabla u} \setminus C_{\nabla u} : \overline{[x]}_{q_{\nabla u}(x)} \text{ intersects } \partial S \text{ tangentially}\} \\ \Sigma_c &:= \{x \in D_{\nabla u} \setminus C_{\nabla u} : \overline{[x]}_{q_{\nabla u}(x)} \text{ intersects } \partial_c S\}.\end{aligned}$$

By §2.4 these sets are well defined because all S -rulings for ∇u agree on $S \setminus C_{\nabla u}$ up to identification of antipodal points. And swapping antipodal points does not affect the definition of Σ_τ or Σ_c . Our main result in terms of surfaces reads as follows:

2.10. Theorem. *Let $S \subset \mathbb{R}^2$ be a bounded C^1 domain, let $\partial_c S \subset \partial S$ be closed and let $u \in W_{\text{iso}}^{2,2}(S; \mathbb{R}^3)$. If $x_0 \in D_{\nabla u}$, then there exist $T > 0$ and an admissible curve $\Gamma \in W^{2,\infty}([0, T]; S)$ with $\Gamma(\frac{T}{2}) = x_0$ and a function $\kappa_n \in L^2(0, T)$ such that $u = (\Gamma, \kappa_n)$ on $[\Gamma(0, T)]$. If u minimizes $\mathcal{E}(\cdot; S)$ within $\mathcal{A}_u(S, \partial_c S)$ and if $x_0 \in D_{\nabla u} \setminus (C_{\nabla u} \cup \Sigma_\tau \cup \Sigma_c)$ then one can choose T and (Γ, κ_n) such that, in addition, Γ is transversal on $[0, T]$ and (Γ, κ_n) satisfy the Euler-Lagrange equations in the sense of Definition 2.7.*

Remarks.

- (i) The Euler-Lagrange equations are the basis for the regularity analysis of minimizers of the constrained Willmore functional \mathcal{E}_K defined in the introduction. This analysis is carried out in [13], see also [12].
- (ii) The set $\Sigma_\tau \cup \Sigma_c$ is relatively closed in $D_{\nabla u} \setminus C_{\nabla u}$ (see the proof of Theorem 2.10). If S is convex then clearly $\Sigma_\tau = \emptyset$.
- (iii) On Σ_c the mapping u is fully determined by the condition $u \in \mathcal{A}_u(S, \partial_c S)$, i.e. by its prescribed boundary conditions. If $\mathcal{H}^1(\partial_c S) = 0$ then minimizers are rigid motions and the Euler-Lagrange equations are trivially satisfied.
- (iv) In [13] the geometry of the set $D_{\nabla u}$ is studied when u is a minimizer. The case of general $u \in W_{\text{iso}}^{2,2}(S; \mathbb{R}^3)$ is analyzed in detail in [10].
- (v) The energy of (Γ, κ_n) is given by (12). Since $g(s_\Gamma^\pm, \kappa) = -g_2(s_\Gamma^\pm, \kappa) - \kappa g_3(s_\Gamma^\pm, \kappa)$, the Euler-Lagrange equations show that the energy density agrees with

$$(1 - \chi_{I_0^\Gamma}) \kappa_n^2 g(s_\Gamma^\pm, \kappa) = -\Omega_2 - \kappa \Omega_3. \quad (24)$$

3 The variations

3.1. In the whole article, Γ is a curve that is parametrized by arclength. Unless stated otherwise, in the whole article $S \subset \mathbb{R}^2$ denotes a bounded C^1 domain.

Let $\Gamma \in W^{2,\infty}([0, T]; S)$ and let $\kappa_n \in L^2(0, T)$. Denote by $\kappa := N \cdot \Gamma''$ the curvature of Γ . Let $\varphi \in L^\infty((0, T); \mathbb{R}^3)$. We define $\tilde{\kappa}^\varphi := \kappa + \varphi_1$ and $\tilde{\kappa}_n^\varphi := \kappa_n + \varphi_2$. Further,

we define the frames $\tilde{R}_\varphi : (0, T) \rightarrow SO(2)$, $\tilde{r}_\varphi : (0, T) \rightarrow SO(3)$ to be the solutions to the initial value problems

$$\tilde{R}'_\varphi = (e_1 \otimes e_2 - e_2 \otimes e_1) \tilde{\kappa}^\varphi \tilde{R}_\varphi \text{ with } \tilde{R}_\varphi(0) = R(0) \quad (25)$$

$$\tilde{r}'_\varphi = ((e_1 \otimes e_2 - e_2 \otimes e_1) \tilde{\kappa}^\varphi + (e_1 \otimes e_3 - e_3 \otimes e_1) \tilde{\kappa}_n^\varphi) \tilde{r}_\varphi \text{ with } \tilde{r}_\varphi(0) = r(0). \quad (26)$$

Here the Frénet frame $R(0) \in SO(2)$ is given by the original curve Γ . The choice of $r(0) \in SO(3)$ is irrelevant; for convenience we take $r(0) = R(0)$ (identifying $SO(2)$ with $\{R \in SO(3) : R_{33} = 1\}$).

We define $(\tilde{\gamma}'_\varphi, \tilde{v}_\varphi, \tilde{n}_\varphi) := \tilde{r}'_\varphi{}^T$ and $(\tilde{\Gamma}'_\varphi, \tilde{N}_\varphi) := \tilde{R}'_\varphi{}^T$. We define the curves $\tilde{\Gamma}_\varphi : (0, T) \rightarrow \mathbb{R}^2$, $\tilde{\gamma}_\varphi : (0, T) \rightarrow \mathbb{R}^3$ by setting

$$\tilde{\Gamma}_\varphi(t) = \Gamma(0) + \int_0^t \tilde{\Gamma}'_\varphi(s) ds \text{ and } \tilde{\gamma}_\varphi(t) = \gamma(0) + \int_0^t \tilde{\gamma}'_\varphi(s) ds. \quad (27)$$

Again, $\Gamma(0)$ is given by the original curve, and the choice of $\gamma(0) \in \mathbb{R}^3$ is irrelevant; for simplicity we take $\gamma(0) = \Gamma(0)$ (identifying \mathbb{R}^2 with a subspace of \mathbb{R}^3 in the natural way). Sometimes, $r(0)$ and $\gamma(0)$ will be given in advance by some given curve γ and its Darboux frame r .

If Γ is transversal on $[0, T]$, then by Proposition 3.1.11 in [10] the mapping ν is C^1 in a neighbourhood of $\bigcup_{t \in [0, T]} (\Gamma(t), \pm N(t))$. So if $\|\varphi\|_{L^\infty((0, T); \mathbb{R}^3)}$ is small enough, then the ODE

$$y'_\varphi = [\varphi_3 + \sum_{* \in \{-, +\}} \chi_*(\ast \nu(\tilde{\Gamma} + y_\varphi, \ast \tilde{N}_\varphi) \tilde{\kappa}^\varphi - s_\Gamma^* \kappa)] \tilde{\Gamma}'_\varphi \text{ with } y_\varphi(0) = 0. \quad (28)$$

has a unique Lipschitz continuous solution y_φ . For $t \in [0, T]$ we set

$$\rho_\varphi(t) := y'_\varphi(t) \cdot \tilde{\Gamma}'_\varphi(t) \text{ and } \tau_\varphi(t) := t + \int_0^t \rho_\varphi(s) ds. \quad (29)$$

If $\|\varphi\|_{L^\infty((0, T); \mathbb{R}^3)}$ is small enough, τ_φ is a Bilipschitz homeomorphism of $(0, T)$ onto $\tau_\varphi(0, T) = (0, \tau_\varphi(T))$. Therefore, it makes sense to define R_φ and r_φ by setting (recall our notation that $R_\varphi(\tau_\varphi) = R_\varphi \circ \tau_\varphi$)

$$R_\varphi(\tau_\varphi) := \tilde{R}_\varphi \text{ and } r_\varphi(\tau_\varphi) := \tilde{r}_\varphi \text{ on } [0, T]. \quad (30)$$

With these definitions it is easy to verify that

$$\kappa^\varphi(\tau_\varphi) = \frac{\tilde{\kappa}^\varphi}{\tau'_\varphi} \text{ and } \kappa_n^\varphi(\tau_\varphi) = \frac{\tilde{\kappa}_n^\varphi}{\tau'_\varphi} \text{ almost everywhere on } (0, T). \quad (31)$$

We define $(\gamma'_\varphi, v_\varphi, n_\varphi) := r'_\varphi{}^T$ and $(\Gamma'_\varphi, N_\varphi) := R'_\varphi{}^T$. We define the curves $\Gamma_\varphi : (0, T) \rightarrow \mathbb{R}^2$, $\gamma_\varphi : (0, T) \rightarrow \mathbb{R}^3$ by setting

$$\Gamma_\varphi(t) = \Gamma(0) + \int_0^t \Gamma'_\varphi(s) ds \text{ and } \gamma_\varphi(t) = \gamma(0) + \int_0^t \gamma'_\varphi(s) ds. \quad (32)$$

with the same initial data as above.

With this somewhat implicit dependence of the varied frames R_φ , r_φ on the variation φ , we reduce the uniform admissibility constraints $1 - s_{\Gamma_\varphi}^\pm \kappa^\varphi \geq c$ (needed later)

to $\varphi_3 \geq c'$, where c and c' are positive numbers.

Later, γ will be a line of curvature on the original surface $u(S)$, where $u \in W_{\text{iso}}^{2,2}(S; \mathbb{R}^3)$, and r will be its Darboux frame, see (6). The curve γ_φ will be a line of curvature on the modified surface and r_φ will be the Darboux frame along this curve. Γ_φ and R_φ are the corresponding pulled back quantities. This will yield variations of the original surface $u(S)$.

Notice that $r_\varphi, R_\varphi, \Gamma_\varphi, \gamma_\varphi$ are defined on the interval $\tau_\varphi(0, T)$. Since all of the above curves are parametrized by arc-length, this allows us to change the length of the original curves Γ, γ . One can easily check that

$$\Gamma_\varphi(\tau_\varphi) = \tilde{\Gamma}_\varphi + y_\varphi. \quad (33)$$

Hence $*\nu(\tilde{\Gamma}_\varphi + y_\varphi, *\tilde{N}_\varphi) = s_{\tilde{\Gamma}_\varphi}^*(\tau_\varphi)$, so from (28) we have

$$y'_\varphi = [\varphi_3 + \sum_{* \in \{-, +\}} \chi_*(s_{\tilde{\Gamma}_\varphi}^*(\tau_\varphi) \tilde{\kappa}^\varphi - s_{\tilde{\Gamma}_\varphi}^* \kappa)] \tilde{\Gamma}'_\varphi. \quad (34)$$

For a given one-parameter family $\varphi^{(\varepsilon)} \in L^\infty((0, T); \mathbb{R}^3)$ with $\|\varphi^{(\varepsilon)}\|_{L^\infty((0, T); \mathbb{R}^3)} \leq C\varepsilon$ and $\varepsilon \in [0, 1]$, we define $\dot{\Gamma}(t) := \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} (\tilde{\Gamma}_{\varphi^{(\varepsilon)}}(t) - \Gamma(t))$ and so on. Define $\dot{\tilde{\Gamma}}(t) := \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} (\Gamma_{\varphi^{(\varepsilon)}}(\tau_{\varphi^{(\varepsilon)}}(t)) - \Gamma(t))$.

3.2. Lemma. *Let $\Gamma \in W^{2,\infty}([0, T]; S)$ be transversal and let $\psi, \hat{\psi}^{(\varepsilon)} \in L^\infty((0, T); \mathbb{R}^3)$ be such that $\|\hat{\psi}^{(\varepsilon)}\|_{L^\infty((0, T); \mathbb{R}^3)} \rightarrow 0$ as $\varepsilon \downarrow 0$. Set $\varphi^{(\varepsilon)} := \varepsilon(\psi + \hat{\psi}^{(\varepsilon)})$ and make the definitions from §3.1. Set $\xi := \psi_1 n - \psi_2 v$ and $\Xi(t) := \int_0^t \xi$. Then we have*

$$\dot{\gamma}' = \Xi \wedge \gamma', \quad \dot{v} = \Xi \wedge v, \quad \dot{n} = \Xi \wedge n, \quad \text{and} \quad \dot{\gamma}(t) = \int_0^t \xi(s) \wedge (\gamma(t) - \gamma(s)) ds, \quad (35)$$

$$\dot{N}(t) = -\Gamma'(t) \int_0^t \psi_1 \quad \text{and} \quad \dot{\Gamma}(t) = \int_0^t N(s) \left(\int_0^s \psi_1 \right) ds, \quad (36)$$

$$\dot{y}' = (\Gamma' \otimes \bar{h})(\dot{y} + \dot{\Gamma}) + (\psi_3 + (\bar{h} \cdot \dot{N} + \psi_1)\sigma)\Gamma', \quad (37)$$

$$\dot{\tilde{\Gamma}}' = (\Gamma' \otimes \bar{h})(\dot{y} + \dot{\Gamma}) + (\psi_3 + (\bar{h} \cdot \dot{N} + \psi_1)\sigma)\Gamma' + \dot{\Gamma}. \quad (38)$$

Proof. All except the last two equations follow readily from the variation of constants formula (see e.g. the appendix to [11]). Equation (38) follows from (37) because $\dot{\tilde{\Gamma}} = \dot{y} + \dot{\Gamma}$.

It remains to prove (37). We omit the index (ε) . Since $|y_\varphi|, |\tilde{\Gamma}_\varphi - \Gamma|$ and $|\tilde{N}_\varphi - N|$ are of order ε , we have

$$\begin{aligned} s_{\tilde{\Gamma}_\varphi}^*(\tau_\varphi) - s_\Gamma^* &= *(\nu(\tilde{\Gamma}_\varphi + y, *\tilde{N}_\varphi) - \nu(\tilde{\Gamma}_\varphi + y, *N)) + *(\nu(\tilde{\Gamma}_\varphi + y, *N) - \nu(\Gamma, *N)) \\ &= *s_\Gamma^* \nu_1(\Gamma, *N) \cdot (\tilde{N}_\varphi - N) + *\nu_1(\Gamma, *N) \cdot (y + \tilde{\Gamma}_\varphi - \Gamma) + o(\varepsilon). \end{aligned} \quad (39)$$

(We used Lemma 7.1 (iii) and continuous differentiability of ν .) Hence

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} s_{\tilde{\Gamma}_\varphi}^*(\tau_\varphi) = * \nu_1(\Gamma, *N) \cdot (s_\Gamma^* \dot{N} + \dot{y} + \dot{\Gamma}). \quad (40)$$

So

$$\dot{y}' = [\psi_3 + \kappa \sum * \chi_* \nu_1(\Gamma, *N) \cdot (s_\Gamma^* \dot{N} + \dot{y} + \dot{\Gamma}) + \psi_1 \sigma] \Gamma'.$$

Now (37) follows from the trivial fact that $\chi_* s_\Gamma^* = \chi_* \sigma$ almost everywhere on $\{\kappa \neq 0\}$, which implies that $\kappa \chi_* s_\Gamma^* = \kappa \chi_* \sigma$ almost everywhere. \square

3.3. Denote by $X : (0, T) \rightarrow \mathbb{R}^{2 \times 2}$ the fundamental solution to the homogeneous ordinary differential equation

$$X' = (\Gamma' \otimes \bar{h}) X$$

with initial value $X(0) = Id$. Applying the variation of constants formula to (37) we obtain

$$\dot{y}(t) = X(t) \int_0^t X^{-1}(s) (\psi_3(s) + \sigma(s) \bar{h}(s) \cdot \dot{N}(s) + \sigma(s) \psi_1(s) + \bar{h}(s) \cdot \dot{\Gamma}(s)) \Gamma'(s) ds. \quad (41)$$

Let μ be an \mathbb{R}^2 -valued Radon measure on $[0, T]$ with component measures μ_1 and μ_2 . We introduce the following functions on $[0, T]$:

$$[\mu]_N(t) := - \int_{[t, T]} \Gamma' \cdot d\mu \quad (42)$$

$$[\mu]_\Gamma(t) := \int_t^T N(s) \cdot \mu([s, T]) ds \quad (43)$$

$$\hat{\mu}(t) := \sum_{i=1}^2 X^{-T}(t) \int_{[t, T]} (X^T(s) e_i) d\mu_i(s) \quad (44)$$

$$[\mu]_y := [\mu + (\Gamma' \cdot \hat{\mu}) \bar{h}]_\Gamma + [(\Gamma' \cdot \hat{\mu}) \bar{h} \sigma]_N + (\Gamma' \cdot \hat{\mu}) \sigma \quad (45)$$

Here and in what follows we will frequently identify a Borel function f with the measure $f \mathcal{L}^1$. In (42, 43, 45) the subindices N , Γ and y are only used to label three different operators. They are not directly related to the mappings introduced earlier.

3.4. Lemma. *Suppose that the hypotheses of Lemma 3.2 are satisfied. Let μ be an \mathbb{R}^2 -valued Radon measure on $[0, T]$ of the form $\mu = \mu^a \mathcal{L}^1 + \mu_0 \delta_{\{T\}}$, where $\mu^a \in L^\infty((0, T); \mathbb{R}^2)$ and $\mu_0 \in \mathbb{R}^2$. Then the following hold:*

(i) *We have*

$$\int_{[0, T]} \dot{N} \cdot d\mu = \int_0^T \psi_1(t) [\mu]_N(t) dt \quad (46)$$

$$\int_{[0, T]} \dot{\Gamma} \cdot d\mu = \int_0^T \psi_1(t) [\mu]_\Gamma(t) dt \quad (47)$$

$$\int_{[0, T]} \dot{y} \cdot d\mu = \int_0^T \psi_3(t) \Gamma'(t) \cdot \hat{\mu}(t) dt + \int_0^T \psi_1(t) ([\mu]_y(t) - [\mu]_\Gamma(t)) dt \quad (48)$$

$$\int_{[0, T]} \dot{\Gamma} \cdot d\mu = \int_{[0, T]} (\dot{\Gamma} + \dot{y}) \cdot d\mu = \int_0^T \psi_3(t) \Gamma'(t) \cdot \hat{\mu}(t) dt + \int_0^T \psi_1(t) [\mu]_y(t) dt. \quad (49)$$

(ii) We have $\hat{\mu} \in W^{1,\infty}((0, T); \mathbb{R}^3)$ and

$$\hat{\mu}(t) = X^{-T}(t) \left(X^T(T) \mu(\{T\}) + \int_t^T X^T(s) \mu^a(s) ds \right). \quad (50)$$

Moreover, $\hat{\mu}$ solves the terminal value problem

$$\hat{\mu}' = -(\bar{h} \otimes \Gamma') \hat{\mu} - \mu^a \text{ with } \hat{\mu}(T) = \mu(\{T\}). \quad (51)$$

If $\mu^a(t) \parallel \Gamma'(t)$ for almost every $t \in (0, T)$ then $\hat{\mu} \cdot N$ is constant on $(0, T)$. In particular, if $\mu(\{T\}) \parallel \Gamma'(T)$, then

$$\hat{\mu}(t) \cdot N(t) = 0 \text{ for all } t \in (0, T). \quad (52)$$

(iii) We have

$$[\mu]_y(t) = \int_t^T N(s) \cdot \hat{\mu}(s) ds + [(\Gamma' \cdot \hat{\mu}) \sigma \bar{h}]_N(t) + (\Gamma'(t) \cdot \hat{\mu}(t)) \sigma(t) \quad (53)$$

Proof. Set $w(t) = (\psi_3(t) + (\bar{h}(t) \cdot \dot{N}(t) + \psi_1(t)) \sigma(t) + \bar{h}(t) \cdot \dot{\Gamma}(t)) \Gamma'(t)$. So $\dot{y}' = (\Gamma' \otimes \bar{h}) \dot{y} + w$. Since all involved functions are bounded and since the measure $\mathcal{L}^1 \otimes \mu$ is finite on $[0, T]^2$ we can apply Fubini's Theorem. So from (41) and denoting the characteristic function of the set $\{(t, s) \in [0, T] \times [0, T] : t \leq s\}$ by χ_T , we find

$$\begin{aligned} \int_{[0, T]} \dot{y} \cdot d\mu &= \sum_{i=1}^2 \int \left[\int e_i \cdot X(s) X^{-1}(t) w(t) \chi_T(t, s) dt \right] d\mu_i(s) \\ &= \sum_{i=1}^2 \int \left[\int (X^{-T}(t) X^T(s) e_i) \cdot w(t) \chi_T(t, s) d\mu_i(s) \right] dt \\ &= \int \sum_{i=1}^2 [X^{-T}(t) \int (X^T(s) e_i) \chi_T(t, s) d\mu_i(s)] \cdot w(t) dt \\ &= \int_0^T w(t) \cdot \hat{\mu}(t) dt. \end{aligned} \quad (54)$$

By a similar calculation one proves (46, 47) using (36). To prove (48) we insert the definition of w into (54) to find (we omit the integration measure when it is \mathcal{L}^1):

$$\begin{aligned} \int_{[0, T]} \dot{y} \cdot d\mu &= \int_0^T \psi_3 \Gamma' \cdot \hat{\mu} + \int_0^T (\bar{h} \cdot \dot{\Gamma} + \sigma \bar{h} \cdot \dot{N}) \Gamma' \cdot \hat{\mu} + \int_0^T \psi_1 \sigma \Gamma' \cdot \hat{\mu} \\ &= \int_0^T \psi_3 \Gamma' \cdot \hat{\mu} + \int_0^T (\bar{h} \otimes \Gamma' \hat{\mu}) \cdot \dot{\Gamma} + (\sigma \bar{h} \otimes \Gamma' \hat{\mu}) \cdot \dot{N} + \int_0^T \psi_1 \sigma \Gamma' \cdot \hat{\mu} \\ &= \int_0^T \psi_3 \Gamma' \cdot \hat{\mu} + \int_0^T \psi_1 \left([(\bar{h} \otimes \Gamma') \hat{\mu}]_{\Gamma} + [(\sigma \bar{h} \otimes \Gamma') \hat{\mu}]_N + \sigma \Gamma' \cdot \hat{\mu} \right). \end{aligned}$$

This proves 48. The first equality in (49) follows from (33). The second one follows from (45) by adding (47) and (48).

To prove (ii), notice that (50) follows directly from the hypothesis on μ . By the

definition of X we have $(X^{-T})' = -(\bar{h} \otimes \Gamma')X^{-T}$. So (51) follows from (50) by the product rule. Moreover, if $\mu^a(t) \parallel \Gamma'(t)$ for almost every $t \in (0, T)$ then by $N' = -\kappa\Gamma'$, by $\bar{h} \cdot N = -\kappa$ (see Lemma 7.1), by (51) and by the product rule we have $(\hat{\mu} \cdot N)' = -(\bar{h} \cdot N)(\Gamma' \cdot \hat{\mu}) - \mu^a \cdot N - \kappa\Gamma' \cdot \hat{\mu} = 0$.

To prove (53) notice that by (51) we have $[\mu + (\bar{h} \otimes \Gamma')\hat{\mu}]_\Gamma = [-\hat{\mu}' + \mu_0\delta_{\{T\}}]_\Gamma$. This agrees with $\int_t^T N \cdot \hat{\mu}$ because by $\hat{\mu}(T) = \mu(\{T\})$ both have derivative $-N \cdot \hat{\mu}$, and both are zero at T . \square

4 The energy functional

4.1. Let $\Gamma \in W^{2,\infty}([0, T]; S)$ be locally admissible. Since Γ is continuous, by compactness we have $\tilde{\eta} := \inf \text{dist}_{\partial S}(\Gamma([0, T])) > 0$. Define the open set

$$M_{\tilde{\eta}} = \{(s^-, s^+, \alpha, x) \in \mathbb{R}^4 : \alpha \in (-1, 1), s^- < -\frac{\tilde{\eta}}{2} < \frac{\tilde{\eta}}{2} < s^+ \text{ and } x \in (\frac{\alpha+1}{s^-}, \frac{\alpha+1}{s^+})\}.$$

By local admissibility we have $\kappa \in [\frac{1}{s_\Gamma^-}, \frac{1}{s_\Gamma^+}]$ almost everywhere. Hence from the definition of $\tilde{\eta}$ we have $(s_\Gamma^\pm(t), s_\Gamma^\mp(t), 0, \kappa(t)) \in M_{\tilde{\eta}}$ for almost every $t \in (0, T)$. For $s^- < 0 < s^+$, $\alpha \in (-1, 1)$ and $x \in (\frac{\alpha+1}{s^-}, \frac{\alpha+1}{s^+})$ we define

$$\tilde{g}(s^\pm, \alpha, x) := \int_{s^-}^{s^+} \frac{1}{\alpha + 1 - sx} ds = \begin{cases} -\sum_{* \in \{-, +\}} * \frac{1}{x} \log(\alpha + 1 - s^*x) & \text{if } x \neq 0 \\ \frac{s^+ - s^-}{\alpha + 1} & \text{otherwise.} \end{cases} \quad (55)$$

So $\tilde{g}(s^\pm, 0, x) = g(s^\pm, x)$ with g as defined in §2.5. With the definitions from Section §2.5, for $s^- < 0 < s^+$ and for $x \in (\frac{1}{s^-}, \frac{1}{s^+})$ we have $g_*(s^\pm, x) = * \frac{1}{1 - s^*x}$,

$$xg_2(s^\pm, x) = - \sum_{* \in \{+, -\}} * \frac{1}{1 - s^*x} \text{ and } g_2(s^\pm, x) + xg_3(s^\pm, x) = -g(s^\pm, x). \quad (56)$$

The following facts are also easily verified: We have $\tilde{g} \in C^\infty(M_{\tilde{\eta}})$. In particular, $g(s_\Gamma^\pm, \kappa)$ is uniformly bounded on I_η^Γ for each $\eta > 0$, with I_η^Γ as defined in (8). There is $c > 0$ such that $g(s_\Gamma^\pm, \kappa) \geq c$ and $g_2(s_\Gamma^\pm, \kappa) \leq -c$ almost everywhere on $(0, T)$.

4.2. Let $\varphi \in L^\infty((0, T); \mathbb{R}^3)$ and recall the definitions from §3.1. Then, with \tilde{g} as defined in §4.1,

$$g(s_{\Gamma_\varphi}^\pm(\tau_\varphi), \kappa^\varphi(\tau_\varphi)) = (\rho_\varphi + 1)\tilde{g}(s_{\Gamma_\varphi}^\pm(\tau_\varphi), \rho_\varphi, \tilde{\kappa}^\varphi) \quad (57)$$

because $\tau_\varphi' = \rho_\varphi + 1$ and $\kappa^\varphi(\tau_\varphi) = \frac{\tilde{\kappa}^\varphi}{\tau_\varphi'}$. Since $\kappa_n^\varphi(\tau_\varphi) = \frac{\tilde{\kappa}_n^\varphi}{\tau_\varphi'}$, a change of variables shows that if $\mathcal{L}^1(I_0^{\Gamma_\varphi}) = 0$, then

$$\mathcal{F}(\Gamma_\varphi, \kappa_n^\varphi; \tau_\varphi(0, T)) = \int_0^T (\tilde{\kappa}_n^\varphi)^2(t) \tilde{g}(s_{\Gamma_\varphi}^\pm(\tau_\varphi(t)), \rho_\varphi(t), \tilde{\kappa}^\varphi(t)) dt. \quad (58)$$

4.3. Let Γ be locally admissible and transversal, and recall the definitions of F_1 and F_2 from (16, 17) in Section 2. It is an important fact that $F_1, F_2 \in L^\infty(0, T)$. This is a consequence of the equalities

$$F_1 = \left(\chi_{\{\kappa=0\}} + \sum_* \frac{\chi_*}{1 - s_{\Gamma}^* \kappa} \right) (\nu_1(\Gamma, N) + \nu_1(\Gamma, -N)) \cdot \Gamma', \quad (59)$$

$$F_2 = \left(\chi_{\{\kappa=0\}} + \sum_* \frac{\chi_*}{1 - s_{\Gamma}^* \kappa} \right) (s_{\Gamma}^+ \nu_1(\Gamma, N) + s_{\Gamma}^- \nu_1(\Gamma, -N)) \cdot \Gamma'. \quad (60)$$

To check these equalities, notice that by (15) and by (56) we have $hg_2(s_{\Gamma}^\pm, \kappa) = -(\sum_* (1 - s_{\Gamma}^* \kappa)^{-1}) \cdot (\sum_* \chi_* \nu_1(\Gamma, *N) \cdot \Gamma')$. Using this and (16, 17) it is easy to verify (59, 60).

4.4. Lemma. *Let $\Gamma \in W^{2,\infty}([0, T]; S)$ be locally admissible and transversal and let $\kappa_n \in L^2(0, T)$ be such that (13) holds. Then, for all $\eta > 0$, there is $\varepsilon_\eta > 0$ such that the following holds: If $\varphi \in L^\infty((0, T); \mathbb{R}^3)$ satisfies*

$$\|\varphi\|_{L^\infty((0, T); \mathbb{R}^3)} < \varepsilon_\eta, \quad (61)$$

$$\varphi = 0 \text{ almost everywhere on } \bigcup_* \{0 < 1 - s_{\Gamma}^* \kappa \leq \eta\}, \quad (62)$$

$$\text{there is } c > 0 \text{ such that } \varphi_3 \geq c \text{ a.e. on } I_0^\Gamma, \quad (63)$$

then Γ_φ is locally admissible and transversal on $[0, \tau_\varphi(T)]$, and $\mathcal{F}(\Gamma_\varphi, \kappa_n^\varphi; 0, \tau_\varphi(T)) < \infty$. More precisely, for $* = +, -$ we have the following estimates:

$$1 - s_{\Gamma_\varphi}^* (\tau_\varphi) \kappa^\varphi(\tau_\varphi) \geq \begin{cases} \frac{c}{2} & \text{a.e. on } \{1 - s_{\Gamma}^* \kappa = 0\} \\ \frac{1}{2}(1 - s_{\Gamma}^* \kappa) & \text{a.e. on } \{0 < 1 - s_{\Gamma}^* \kappa \leq \eta\} \\ \frac{\eta}{4} & \text{a.e. on } \{1 - s_{\Gamma}^* \kappa > \eta\}. \end{cases} \quad (64)$$

In particular, $\mathcal{L}^1(I_0^{\Gamma_\varphi}) = 0$.

Proof. By (13) we have $\mathcal{F}(\Gamma, \kappa_n; 0, T) < \infty$. Transversality of Γ_φ on $[0, \tau_\varphi(T)]$ for small enough ε_η is clear from transversality of Γ on $[0, T]$ (see e.g. the proof of Lemma 3.2 in [11]). Let us prove local admissibility of Γ_φ . Recall that $\kappa^\varphi(\tau_\varphi) = \frac{\tilde{\kappa}^\varphi}{\tau_\varphi'}$. Hence by (29) and (34), on $\{\chi_* = 1\}$ we can estimate:

$$1 - s_{\Gamma_\varphi}^* (\tau_\varphi) \kappa^\varphi(\tau_\varphi) = \frac{1}{\tau_\varphi'} (y_\varphi' \cdot \tilde{\Gamma}'_\varphi + 1 - s_{\Gamma_\varphi}^* (\tau_\varphi) \tilde{\kappa}^\varphi) \geq \frac{1}{2} (\varphi_3 + 1 - s_{\Gamma}^* \kappa), \quad (65)$$

because $\tau_\varphi' \in (\frac{1}{2}, 2)$ for small ε_η . For small enough ε_η we have $1 - s_{\Gamma}^* \kappa \geq \frac{\eta}{2}$ on $\{\kappa = 0\}$ and $|\varphi_3| \leq \frac{\eta}{2}$ almost everywhere. Since by hypothesis $\varphi_3 \geq c$ on I_0^Γ and $\varphi_3 = 0$ on $\{0 < 1 - s_{\Gamma}^* \kappa < \eta\}$, we see that (65) implies (64). In particular, Γ_φ is locally admissible since so is Γ . Finally, since $s_{\Gamma_\varphi}^\pm \kappa^\varphi$ is obviously uniformly bounded, the estimates (64) imply that $\mathcal{F}(\Gamma_\varphi, \kappa_n^\varphi; \tau_\varphi(0, T)) < \infty$. \square

4.5. Proposition. *Let $\Gamma \in W^{2,\infty}([0, T]; S)$ be locally admissible and transversal, and let $\kappa_n \in L^2(0, T)$ be such that (13) holds. Let $\hat{\psi}^{(\varepsilon)}, \psi \in L^\infty((0, T); \mathbb{R}^3)$ be such that $\|\hat{\psi}^{(\varepsilon)}\|_{L^\infty((0, T); \mathbb{R}^3)} \rightarrow 0$ as $\varepsilon \downarrow 0$. Set $\varphi^{(\varepsilon)} = \varepsilon(\psi + \hat{\psi}^{(\varepsilon)})$. Let $\eta > 0$ be small and assume that $\psi = \hat{\psi}^{(\varepsilon)} = 0$ almost everywhere on $(0, T) \setminus (I_0^\Gamma \cup I_\eta^\Gamma) = \bigcup_* \{0 < 1 - s_\Gamma^* \kappa \leq \eta\}$ for all $\varepsilon > 0$ and that there is $c > 0$ such that $\psi_3 \geq c$ on I_0^Γ . Then, for all ε small enough, $\Gamma_{\varphi^{(\varepsilon)}}$ is locally admissible and transversal on $[0, \tau_{\varphi^{(\varepsilon)}}(T)]$, and $\mathcal{F}(\Gamma_{\varphi^{(\varepsilon)}}, \kappa_n^{\varphi^{(\varepsilon)}}; 0, \tau_{\varphi^{(\varepsilon)}}(T)) < \infty$ and $\mathcal{L}^1(I_0^{\Gamma_{\varphi^{(\varepsilon)}}}) = 0$. Moreover,*

$$\begin{aligned} & \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \left(\mathcal{F}(\Gamma_{\varphi^{(\varepsilon)}}, \kappa_n^{\varphi^{(\varepsilon)}}; \tau_{\varphi^{(\varepsilon)}}(0, T)) - \mathcal{F}(\Gamma, \kappa_n; (0, T)) \right) \\ &= \int_{I_\eta^\Gamma} 2\psi_2 \kappa_n g(s_\Gamma^\pm, \kappa) \\ & \quad + \int_{I_\eta^\Gamma} \psi_1 \kappa_n^2 \left(g_3(s_\Gamma^\pm, \kappa) + g_2(s_\Gamma^\pm, \kappa) \sigma \right) + \int \psi_1 \left([\kappa_n^2 \bar{F}_1]_y + [\kappa_n^2 \bar{F}_2]_N \right) \\ & \quad + \int_{I_\eta^\Gamma} \psi_3 \kappa_n^2 g_2(s_\Gamma^\pm, \kappa) + \int \psi_3 [\kappa_n^2 \bar{F}_1]^\wedge \cdot \Gamma'. \end{aligned} \quad (66)$$

The functions \bar{F}_1 and \bar{F}_2 were introduced in (16, 17).

Proof. To avoid heavy notation we write φ instead of $\varphi^{(\varepsilon)}$. By the assumptions on ψ and on $\hat{\psi}^{(\varepsilon)}$, we have that, for ε small enough, φ satisfies the hypotheses of Lemma 4.4 with $\varphi_3 \geq \frac{c\varepsilon}{2}$ on I_0^Γ . In particular, $\mathcal{F}(\Gamma_\varphi, \kappa_n^\varphi; \tau_\varphi(0, T)) + \mathcal{F}(\Gamma, \kappa_n; 0, T) < \infty$, and the curve Γ_φ is transversal and locally admissible with

$$1 - s_{\Gamma_\varphi}^* (\tau_\varphi) \kappa^\varphi (\tau_\varphi) \geq \begin{cases} \frac{c\varepsilon}{4} & \text{on } \{1 - s_\Gamma^* \kappa = 0\} \\ \frac{1}{2}(1 - s_\Gamma^* \kappa) & \text{on } \{0 < 1 - s_\Gamma^* \kappa \leq \eta\} \\ \frac{\eta}{4} & \text{on } \{1 - s_\Gamma^* \kappa > \eta\}. \end{cases} \quad (67)$$

Since $s_{\Gamma_\varphi}^\pm \kappa^\varphi$ is obviously uniformly bounded, this implies the following estimates:

$$g(s_{\Gamma_\varphi}^\pm (\tau_\varphi), \kappa^\varphi (\tau_\varphi)) \leq \begin{cases} C |\log \varepsilon| & \text{on } I_0^\Gamma \\ C(g(s_\Gamma^\pm, \kappa) + 1) & \text{on } (0, T) \setminus (I_0^\Gamma \cup I_\eta^\Gamma) \\ C & \text{on } I_\eta^\Gamma. \end{cases} \quad (68)$$

Here and in the rest of this proof C denotes a constant that does not depend on ε or t but might depend on η ; as usual, its value can change from line to line.

Since $|s_{\Gamma_\varphi}^\pm (\tau_\varphi) - s_\Gamma^\pm| \leq C\varepsilon$, we have $\rho_\varphi + 1 - s_{\tilde{\kappa}^\varphi} \geq \frac{\eta}{2}$ on I_η^Γ for small ε and for all $s \in [\min\{s_{\Gamma_\varphi}^-(\tau_\varphi), s_\Gamma^-\}, \max\{s_{\Gamma_\varphi}^+(\tau_\varphi), s_\Gamma^+\}]$. From this one readily deduces that

$$|\tilde{g}(s_{\Gamma_\varphi}^\pm (\tau_\varphi), \rho_\varphi, \tilde{\kappa}^\varphi) - g(s_\Gamma^\pm, \kappa)| \leq C\varepsilon \text{ on } I_\eta^\Gamma. \quad (69)$$

By (13) and by Remark 2.6, we have $\kappa_n = 0$ on I_0^Γ and $\mathcal{F}(\Gamma, \kappa_n; (0, T)) = \mathcal{F}(\Gamma, \kappa_n; (0, T) \setminus I_0^\Gamma)$. Since $\tilde{\kappa}_n^\varphi = \kappa_n$ on $\{0 < 1 - s_\Gamma^* \kappa \leq \eta\}$ and $\tilde{\kappa}_n^\varphi = \varphi_2$ on I_0^Γ ,

we have

$$\begin{aligned}
& \mathcal{F}(\Gamma_\varphi, \kappa_n^\varphi; \tau_\varphi(0, T)) - \mathcal{F}(\Gamma, \kappa_n; (0, T) \setminus I_0^\Gamma) \\
&= \int_{I_0^\Gamma} (\tilde{\kappa}_n^\varphi)^2 \tilde{g} + \int_{(0, T) \setminus I_0^\Gamma} \left(((\tilde{\kappa}_n^\varphi)^2 - \kappa_n^2) \tilde{g} + \kappa_n^2 (\tilde{g} - g) \right) \\
&= \int_{I_0^\Gamma} \varphi_2^2 \tilde{g} + \int_{I_\eta^\Gamma} ((\tilde{\kappa}_n^\varphi)^2 - \kappa_n^2) \tilde{g} + \int_{(0, T) \setminus I_0^\Gamma} \kappa_n^2 (\tilde{g} - g). \quad (70)
\end{aligned}$$

(Here and below we write \tilde{g} instead of $\tilde{g}(s_{\Gamma_\varphi}^\pm(\tau_\varphi), \rho_\varphi, \tilde{\kappa}^\varphi)$ and g instead of $g(s^\pm, \kappa)$.) To estimate the first term in (70) we use $\varphi_2^2 \leq C\varepsilon^2$ and that $\tilde{g} \leq C|\log \varepsilon|$ on I_0^Γ for small ε by (57, 68). Thus $\int_{I_0^\Gamma} \varphi_2^2 \tilde{g} \leq C\varepsilon^2 |\log \varepsilon|$, and this is $o(\varepsilon)$ as $\varepsilon \downarrow 0$.

The second term in (70) equals $\int_{I_\eta^\Gamma} 2\varphi_2 \kappa_n g + O(\varepsilon^2)$ by (68, 69).

Now consider the third term in (70). On $(0, T) \setminus I_0^\Gamma$, the expression $\frac{1}{\varepsilon} \kappa_n^2 (\tilde{g}(s_{\Gamma_\varphi}^\pm(\tau_\varphi), \rho_\varphi, \tilde{\kappa}^\varphi) - g(s_\Gamma^\pm, \kappa))$ converges pointwise to (compare (40)):

$$\kappa_n^2 \left(\sum_* *g_*(s_\Gamma^\pm, \kappa) \nu_1(\Gamma, *N) \cdot (\dot{y} + \dot{\Gamma} + s_\Gamma^* \dot{N}) + g_2(s_\Gamma^\pm, \kappa) \dot{\rho} + g_3(s_\Gamma^\pm, \kappa) \psi_1 \right) \quad (71)$$

as $\varepsilon \downarrow 0$. Here $\dot{\rho}(t) := \lim_{\varepsilon \downarrow 0} \rho_\varphi(t)$ for all t . We claim that

$$|\tilde{g}(s_{\Gamma_\varphi}^\pm(\tau_\varphi), \rho_\varphi, \tilde{\kappa}^\varphi) - g(s_\Gamma^\pm, \kappa)| \leq C\varepsilon \text{ on } (0, T) \setminus (I_0^\Gamma \cup I_\eta^\Gamma). \quad (72)$$

Using (69, 72) we can apply Lebesgue's dominated convergence theorem to conclude that (71) lies in $L^1((0, T) \setminus I_0^\Gamma)$ and that $\frac{1}{\varepsilon} \kappa_n^2 (\tilde{g}(s_{\Gamma_\varphi}^\pm(\tau_\varphi), \rho_\varphi, \tilde{\kappa}^\varphi) - g(s_\Gamma^\pm, \kappa))$ converges to (71) in $L^1((0, T) \setminus I_0^\Gamma)$.

To prove (72) let $* \in \{+, -\}$. On $(0, T) \setminus (I_0^\Gamma \cup I_\eta^\Gamma) = \{0 < 1 - s_\Gamma^* \kappa \leq \eta\}$ we have $\chi_* = 1$ (if $\eta < \frac{1}{2}$) and $\rho_\varphi = s_{\Gamma_\varphi}^*(\tau_\varphi) \tilde{\kappa}^\varphi - s_\Gamma^* \kappa$ because $\varphi_3 = 0$ on this set. Hence $\rho_\varphi + 1 - s_{\Gamma_\varphi}^*(\tau_\varphi) \tilde{\kappa}^\varphi = 1 - s_\Gamma^* \kappa > 0$. Since also $\rho_\varphi + 1 - s_{\Gamma_\varphi}^{\bar{*}}(\tau_\varphi) \tilde{\kappa}^\varphi > 0$ on this set (because $\tilde{\kappa}^\varphi = \kappa$, so $s_{\Gamma_\varphi}^{\bar{*}}(\tau_\varphi) \tilde{\kappa}^\varphi < 0$ provided that ε is small with respect to η) we have:

$$\begin{aligned}
& |\tilde{\kappa}^\varphi \tilde{g}(s_{\Gamma_\varphi}^\pm(\tau_\varphi), \rho_\varphi, \tilde{\kappa}^\varphi) - \kappa g(s_\Gamma^\pm, \kappa)| = \left| \sum_{*'} *' [\log(\rho_\varphi + 1 - s_{\Gamma_\varphi}^{*'}(\tau_\varphi) \tilde{\kappa}^\varphi) - \log(1 - s_\Gamma^{*'} \kappa)] \right| \\
&= \left| * \log \left(\frac{\rho_\varphi + 1 - s_{\Gamma_\varphi}^*(\tau_\varphi) \tilde{\kappa}^\varphi}{1 - s_\Gamma^* \kappa} \right) - * \log \left(\frac{\rho_\varphi + 1 - s_{\Gamma_\varphi}^{\bar{*}}(\tau_\varphi) \tilde{\kappa}^\varphi}{1 - s_\Gamma^{\bar{*}} \kappa} \right) \right| \\
&= \left| \log \left(1 + \frac{(s_\Gamma^{\bar{*}} \kappa - s_{\Gamma_\varphi}^{\bar{*}}(\tau_\varphi) \tilde{\kappa}^\varphi) + (s_{\Gamma_\varphi}^*(\tau_\varphi) \tilde{\kappa}^\varphi - s_\Gamma^* \kappa)}{1 - s_\Gamma^{\bar{*}} \kappa} \right) \right|.
\end{aligned}$$

The second term in the argument of the logarithm is uniformly bounded by $C\varepsilon$ because $1 - s_\Gamma^{\bar{*}} \kappa \geq 1$ on $\{\chi_* = 1\}$. We conclude that $|\tilde{\kappa}^\varphi \tilde{g}(s_{\Gamma_\varphi}^\pm(\tau_\varphi), \rho_\varphi, \tilde{\kappa}^\varphi) - \kappa g(s_\Gamma^\pm, \kappa)| \leq C\varepsilon$, and this implies (72) because $\tilde{\kappa}^\varphi = \kappa$ is uniformly bounded from below by a positive constant on $\{0 < 1 - s_\Gamma^* \kappa \leq \eta\}$.

To conclude the proof of the lemma we use $\dot{\rho} = \dot{y}' \cdot \Gamma' = \psi_3 + \bar{h} \cdot (\dot{\Gamma} + \dot{y}) + (\bar{h} \cdot \dot{N} + \psi_1) \sigma$

by (37). So the integral of (71) over $(0, T) \setminus I_0^\Gamma$ equals

$$\begin{aligned}
& \int_{(0, T) \setminus I_0^\Gamma} \kappa_n^2 \left[\sum_* g_*(s_\Gamma^\pm, \kappa) (*\nu_1(\Gamma, *N) \cdot (\dot{y} + \dot{\Gamma}) + *s_\Gamma^* \nu_1(\Gamma, *N) \cdot \dot{N}) \right. \\
& \quad \left. + g_2(s_\Gamma^\pm, \kappa) (\psi_3 + \bar{h} \cdot (\dot{y} + \dot{\Gamma}) + (\bar{h} \cdot \dot{N} + \psi_1)\sigma) + g_3(s_\Gamma^\pm, \kappa) \psi_1 \right] \\
& = \int_{(0, T) \setminus I_0^\Gamma} \kappa_n^2 \bar{F}_1 \cdot (\dot{y} + \dot{\Gamma}) + \kappa_n^2 \bar{F}_2 \cdot \dot{N} + \kappa_n^2 \psi_3 g_2(s_\Gamma^\pm, \kappa) + \kappa_n^2 \psi_1 (g_3(s_\Gamma^\pm, \kappa) + g_2(s_\Gamma^\pm, \kappa)\sigma) \\
& = \int_0^T \psi_1 \left\{ [\kappa_n^2 \bar{F}_1]_y + [\kappa_n^2 \bar{F}_2]_N + \chi_{(0, T) \setminus I_0^\Gamma} \kappa_n^2 (g_3(s_\Gamma^\pm, \kappa) + g_2(s_\Gamma^\pm, \kappa)\sigma) \right\} \\
& \quad + \int \psi_3 \left\{ [\kappa_n^2 \bar{F}_1]^\wedge \cdot \Gamma' + \chi_{(0, T) \setminus I_0^\Gamma} \kappa_n^2 g_2(s_\Gamma^\pm, \kappa) \right\}.
\end{aligned}$$

We have used that $\kappa_n \psi_1 = \kappa_n \psi_3 = 0$ outside I_η^Γ . \square

5 The constraint functionals

In this section we fix an arclength parametrized curve $\Gamma \in W^{2, \infty}([0, T]; S)$ that is transversal. It defines the Frénet frame R . We also fix $\kappa_n \in L^2(0, T)$ and the solution r to (6) with initial values $\gamma'(0) = \Gamma'(0)$, $v(0) = N(0)$, $n(0) = e_3$ and $\gamma(0) = \Gamma(0)$.

5.1. For given $\lambda_1, \lambda_2 \in \mathbb{R}^3$, and $\lambda_3, \lambda_4 \in \mathbb{R}$ we define the functions $\Lambda_1(t) := \gamma'(t) \cdot (\lambda_2 - \lambda_1 \wedge \int_t^T \gamma'(s) ds)$ and $\Lambda_2(t) = v(t) \cdot (\lambda_2 - \lambda_1 \wedge \int_t^T \gamma'(s) ds)$ and $\Lambda_3(t) = n(t) \cdot (\lambda_2 - \lambda_1 \wedge \int_t^T \gamma'(s) ds)$. The following equalities are obtained directly from the definitions and using the ODEs (6):

$$\Lambda_1' = \kappa \Lambda_2 + \kappa_n \Lambda_3 \quad (73)$$

$$\Lambda_2' = -\kappa \Lambda_1 + \lambda_1 \cdot n \quad (74)$$

$$\Lambda_3' = -\kappa_n \Lambda_1 - \lambda_1 \cdot v \quad (75)$$

5.2. We introduce the \mathbb{R}^2 -valued Radon measure

$$H = (\lambda_3 + \lambda_1 \cdot \gamma'(T)) \Gamma'(T) \delta_{\{T\}} - \kappa_n (\lambda_1 \cdot n) \Gamma'. \quad (76)$$

(As before, we identify a locally \mathcal{L}^1 -integrable function f with the measure $f \mathcal{L}^1$.) For $i = 1, 2, 3$ we introduce the \mathbb{R}^2 -valued Radon measures

$$H_i^{(1)} = \gamma_i'(T) \Gamma'(T) \delta_{\{T\}} - \kappa_n n_i \Gamma' \quad (77)$$

$$H^{(3)} = \Gamma'(T) \delta_{\{T\}}. \quad (78)$$

So $H = \sum_{i=1}^3 (\lambda_1)_i \cdot H_i^{(1)} + \lambda_3 H^{(3)}$. From (51) we deduce that \hat{H} is the unique Lipschitz continuous solution to the terminal value problem

$$\hat{H}' = -(\bar{h} \otimes \Gamma') \hat{H} + \kappa_n (\lambda_1 \cdot n) \Gamma' \text{ and } \hat{H}(T) = (\lambda_3 + \lambda_1 \cdot \gamma'(T)) \Gamma'(T). \quad (79)$$

We claim that

$$\hat{H} \cdot N = 0 \text{ and } [H - \kappa_n^2 \bar{F}_1]^\wedge \cdot N = 0. \quad (80)$$

To prove $\hat{H} \cdot N = 0$ we apply formula (52) to H , using that its absolutely continuous part equals $-\kappa_n(\lambda_1 \cdot n)\Gamma'$. One can easily check that $\bar{F}_1 \cdot N = 0$, so $[H - \kappa_n^2 \bar{F}_1]^\wedge \cdot N = 0$ follows from (52) as well.

5.3. We introduce the following functionals on $L^\infty((0, T); \mathbb{R}^3)$:

$$\begin{aligned} \mathcal{G}_1(\varphi) &= \int_0^T (\tilde{\gamma}'_\varphi(t) \otimes \tilde{\Gamma}'_\varphi(t) + \tilde{v}_\varphi(t) \otimes \tilde{N}_\varphi(t) - v(T) \otimes N(T))(\tilde{\Gamma}'_\varphi(t) + y'_\varphi(t)) dt \\ &\quad - \left(\gamma(T) - (v(T) \otimes N(T))\Gamma(T) \right) \\ \mathcal{G}_2(\varphi) &= \begin{pmatrix} \tilde{\gamma}'_\varphi \cdot v \\ -\tilde{\gamma}'_\varphi \cdot n \\ \tilde{n}_\varphi \cdot v \end{pmatrix} (T) \\ \mathcal{G}_3(\varphi) &= \int_0^T \tilde{\Gamma}'_\varphi(t) dt \cdot \Gamma'(T) + (y_\varphi(T) - \Gamma(T)) \cdot \Gamma'(T) \\ \mathcal{G}_4(\varphi) &= \int_0^T \varphi_1(t) dt. \end{aligned}$$

For $i \in \{1, 2, 3, 4\}$ and $\psi \in L^\infty((0, T); \mathbb{R}^3)$ we define $\dot{\mathcal{G}}_i(\psi) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} (\mathcal{G}_i(\varepsilon\psi) - \mathcal{G}_i(0))$. Comparing the definitions of the \mathcal{G}_i with the definitions of Γ_φ , τ_φ and so on leads to the following observation:

5.4. Remark. Setting $(a_{ij})_{i,j=1,2,3} := r(T)(\tilde{r}_\varphi(T))^T = r(T)(r_\varphi(\tau_\varphi(T)))^T$, we have:

$$\begin{aligned} \mathcal{G}_1(\varphi) &= \gamma_\varphi(\tau_\varphi(T)) - (v(T) \otimes N(T))\Gamma_\varphi(\tau_\varphi(T)) - \left(\gamma(T) - (v(T) \otimes N(T))\Gamma(T) \right) \\ \mathcal{G}_2(\varphi) &= \begin{pmatrix} a_{21} \\ -a_{31} \\ a_{23} \end{pmatrix} \\ \mathcal{G}_3(\varphi) &= \left(\Gamma_\varphi(\tau_\varphi(T)) - \Gamma(T) \right) \cdot \Gamma'(T). \end{aligned}$$

5.5. Lemma. Let $\psi \in L^\infty((0, T); \mathbb{R}^3)$ and set $\xi = \psi_1 n - \psi_2 v$ and $\Xi(t) = \int_0^t \xi$. Then

we have

$$\begin{aligned}\dot{\mathcal{G}}_1(\psi) &= - \sum_{i=1}^3 e_i \int_0^T \psi_2(t) e_i \cdot v(t) \wedge \left(\int_t^T \gamma' \right) dt \\ &\quad + \sum_{i=1}^3 e_i \int_0^T \psi_1(t) \left\{ e_i \cdot n(t) \wedge \left(\int_t^T \gamma' \right) - \left(\int_t^T v_i \right) + [H_i^{(1)}]_y(t) \right\} dt \\ &\quad + \sum_{i=1}^3 e_i \int_0^T \psi_3(t) \Gamma'(t) \cdot [H_i^{(1)}]^\wedge(t) dt\end{aligned}\tag{81}$$

$$\dot{\mathcal{G}}_2(\psi) = \int_0^T \xi(t) dt\tag{82}$$

$$\dot{\mathcal{G}}_3(\psi) = \int_0^T \psi_1(t) [H^{(3)}]_y(t) + \psi_3(t) \Gamma'(t) \cdot [H^{(3)}]^\wedge(t) dt\tag{83}$$

$$\dot{\mathcal{G}}_4(\psi) = \int_0^T \psi_1(t) dt.\tag{84}$$

In particular, for $\lambda_1, \lambda_2 \in \mathbb{R}^3$ and $\lambda_3, \lambda_4 \in \mathbb{R}$, with the definitions from §5.1 we have

$$\begin{aligned}\lambda_1 \cdot \dot{\mathcal{G}}_1(\psi) + \lambda_2 \cdot \dot{\mathcal{G}}_2(\psi) + \lambda_3 \dot{\mathcal{G}}_3(\psi) + \lambda_4 \dot{\mathcal{G}}_4(\psi) \\ = \int_0^T \psi_1(t) \left\{ \lambda_4 + \Lambda_3(t) - \left(\int_t^T \lambda_1 \cdot v \right) + [H]_y(t) \right\} dt \\ - \int_0^T \psi_2(t) \Lambda_2(t) dt + \int_0^T \psi_3(t) \Gamma'(t) \cdot \hat{H}(t) dt.\end{aligned}\tag{85}$$

Proof. Formula (84) is obvious. Formula (82) is an immediate consequence of Lemma 3.2. Recalling (78) we have $\dot{\mathcal{G}}_3(\psi) = \int_{[0,T]} (\dot{\Gamma} + \dot{y}) \cdot dH^{(3)}$, so (83) follows from (49).

To prove (81) we set $\Phi(t) := \int_0^t \psi_1$ and for $\varphi \in L^\infty((0, T); \mathbb{R}^3)$ we introduce

$$\tilde{G}_\varphi := \tilde{\gamma}'_\varphi \otimes \tilde{\Gamma}'_\varphi + \tilde{v}_\varphi \otimes \tilde{N}_\varphi \text{ and } G := \gamma' \otimes \Gamma' + v \otimes N.\tag{86}$$

Set $\dot{G}(t) := \lim_{\varepsilon \downarrow 0} \left(\tilde{G}_{\varepsilon\psi}(t) - G(t) \right)$ for all t . We claim that

$$\dot{G}\Gamma' = \dot{\gamma}' - \Phi v \text{ and } G' = \kappa_n n \otimes \Gamma'.\tag{87}$$

Indeed, the first equation in (87) follows from $\dot{G} = \dot{\gamma}' \otimes \Gamma' + \gamma' \otimes \dot{\Gamma}' + \dot{v} \otimes N + v \otimes \dot{N}$ and from Lemma 3.2. The second one is a straightforward differentiation of the definition in (86), making use of (6).

Now we can write

$$\mathcal{G}_1(\varphi) = \int_0^T (\tilde{G}_\varphi - v(T) \otimes N(T)) (\tilde{\Gamma}'_\varphi + y'_\varphi).\tag{88}$$

Therefore,

$$\begin{aligned}
\dot{\mathcal{G}}_1(\psi) &= \int_0^T \dot{G}(t)\Gamma'(t) dt + \int_0^T (G(t) - v(T) \otimes N(T))(\dot{\Gamma}(t) + \dot{y}(t))' dt \\
&= \int_0^T \dot{G}(t)\Gamma'(t) dt - \int_0^T G'(t)(\dot{\Gamma}(t) + \dot{y}(t)) dt + (\gamma'(T) \otimes \Gamma'(T))(\dot{\Gamma}(T) + \dot{y}(T)) \\
&= \int_0^T \Xi(t) \wedge \gamma'(t) - \Phi(t)v(t) dt + \sum_{i=1}^3 e_i \int_{[0,T]} (\dot{\Gamma} + \dot{y}) \cdot dH_i^{(1)}.
\end{aligned}$$

In the last step we used (77) and the fact that $\dot{G}\Gamma' = \Xi \wedge \gamma' - \Phi v$ by (87) and by (35). Therefore, by the definitions of Ξ and Φ , and using (49) with $\mu = H_i^{(1)}$, we have

$$\begin{aligned}
\dot{\mathcal{G}}_1(\psi) &= \int_0^T (\psi_1(t)n(t) - \psi_2(t)v(t)) \wedge \left(\int_t^T \gamma' \right) dt - \int_0^T \psi_1(t) \left(\int_t^T v \right) dt \\
&\quad + \sum_{i=1}^3 e_i \int_0^T \psi_1(t)[H_i^{(1)}]_y(t) dt + \sum_{i=1}^3 e_i \int_0^T \psi_3(t)\Gamma'(t) \cdot \hat{H}_i^{(1)}(t) dt.
\end{aligned}$$

Rearranging terms we arrive at (81). \square

We define $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4)$, which takes values in \mathbb{R}^8 . The next proposition gives the crucial information that the constraints are not redundant and that we have found a rich enough class of variations. It is based on the following lemma.

5.6. Lemma. *Suppose that there is a Borel set $I \subset \{t \in (0, T) : \kappa_n(t) \neq 0\}$ with $\mathcal{L}^1(I) \neq 0$ and that there are $\lambda_1, \lambda_2 \in \mathbb{R}^3$ and $\lambda_3, \lambda_4 \in \mathbb{R}$ such that*

$$\Lambda_2(t) = 0 \text{ for all } t \in I \quad (89)$$

$$[H]_y(t) + \lambda_4 + \Lambda_3(t) - \int_t^T \lambda_1 \cdot v = 0 \text{ for all } t \in I \quad (90)$$

$$\Gamma'(t) \cdot \hat{H}(t) = 0 \text{ for all } t \in I. \quad (91)$$

(Here, H is as in (76) and Λ_i are as in §5.1 with the λ_i from the hypothesis.)

Then $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = \lambda_4 = 0$.

Proof. From (91) and (80) we deduce that $\hat{H} = 0$ on I . Together with (79) and the fact that $\kappa_n \neq 0$ on I , this implies that

$$\lambda_1 \cdot n = 0 \text{ almost everywhere on } I. \quad (92)$$

By (53) and (80, 91) we have $[H]_y = [(\Gamma' \cdot \hat{H})\sigma\bar{h}]_N$ on I . Thus almost everywhere on I we have $([H]_y)' = 0$. Taking derivatives in (90) and using (75) we thus find

$$\Lambda_1 = 0 \text{ almost everywhere on } I. \quad (93)$$

But by (73), almost everywhere on $\{\Lambda_1 = 0\}$ we have $\kappa_n \Lambda_3 + \kappa \Lambda_2 = \Lambda_1' = 0$. By (89) this implies that

$$\Lambda_3 = 0 \text{ almost everywhere on } I. \quad (94)$$

By (92) we have $0 = -(\lambda_1 \cdot n)' = \kappa_n(\lambda_1 \cdot \gamma')$ almost everywhere on I . Hence $\lambda_1 \cdot \gamma' = 0$ almost everywhere on I . Since also $0 = \Lambda'_3 = -\lambda_1 \cdot v$ almost everywhere on I , we conclude that $\lambda_1 = 0$ because $\mathcal{L}^1(I) > 0$. Since $\Lambda_1 = \Lambda_2 = \Lambda_3 = 0$ on I , this implies that $\lambda_2 = 0$. By (79), the vanishing of λ_1 implies that $\hat{H}' = -(\bar{h} \otimes \Gamma')\hat{H}$ almost everywhere on $(0, T)$. Since $\hat{H} = 0$ on I , we conclude that $\hat{H} = 0$ everywhere on $(0, T)$. Since \hat{H} is Lipschitz by (79), we find $\lambda_3 \Gamma'(T) = \hat{H}(T) = \lim_{t \uparrow T} \hat{H}(t) = 0$. Thus $\lambda_3 = 0$. Since $\hat{H} = 0$ everywhere on $(0, T)$ we conclude from (53) that in fact $[H]_y = 0$ on $(0, T)$. By (90) this implies that $\lambda_4 = 0$. \square

5.7. Proposition. *Assume that $I \subset (0, T)$ is a Borel set with $\mathcal{L}^1(\{t \in I : \kappa_n(t) \neq 0\}) > 0$. Then*

$$\dot{\mathcal{G}}\left(\{\psi \in L^\infty((0, T); \mathbb{R}^3) : \psi = 0 \text{ on } (0, T) \setminus I\}\right) = \mathbb{R}^8.$$

Proof. If the statement were false then there would exist $\lambda_1, \lambda_2 \in \mathbb{R}^3$ and $\lambda_3, \lambda_4 \in \mathbb{R}$ such that $\sum_{i=1}^4 |\lambda_i|^2 = 1$ and such that

$$\sum_{i=1}^4 \lambda_i \cdot \dot{\mathcal{G}}_i(\psi) = 0 \text{ for all } \psi \in L^\infty((0, T); \mathbb{R}^3) \text{ with } \psi = 0 \text{ on } (0, T) \setminus I. \quad (95)$$

Hence, recalling (85) from Lemma 5.5, we would have

$$-\int \psi_2 \Lambda_2 + \int \psi_1 \left\{ \lambda_4 + \Lambda_3 - \int_t^T \lambda_1 \cdot v + [H]_y \right\} + \int \psi_3 \Gamma' \cdot \hat{H} = 0 \quad (96)$$

for all ψ as in (95). Thus (89, 90, 91) would hold on I (and in particular on $\{t \in I : \kappa_n(t) \neq 0\}$). Hence by Lemma 5.6 we would conclude that all λ_i vanish, contradicting $\sum_{i=1}^4 |\lambda_i|^2 = 1$. \square

5.8. Lemma. *Let $\varphi \in L^\infty((0, T); \mathbb{R}^3)$ and assume that Γ and Γ_φ are admissible and transversal, and that the following equations are satisfied:*

$$\gamma_\varphi(\tau_\varphi(T)) - \gamma(T) = (v(T) \otimes N(T))(\Gamma_\varphi(\tau_\varphi(T)) - \Gamma(T)) \quad (97)$$

$$r_\varphi(\tau_\varphi(T)) = r(T) \quad (98)$$

$$\Gamma_\varphi(\tau_\varphi(T)) - \Gamma(T) \parallel N(T) \quad (99)$$

$$R_\varphi(\tau_\varphi(T)) = R(T). \quad (100)$$

Then $[\Gamma_\varphi(0, \tau_\varphi(T))] = [\Gamma(0, T)]$ and $(\Gamma, \kappa_n), (\Gamma_\varphi, \kappa_n^\varphi) \in C^1(S \cap \overline{[\Gamma(0, T)]}; \mathbb{R}^3)$ with $(\Gamma_\varphi, \kappa_n^\varphi) = (\Gamma, \kappa_n)$ and $\nabla(\Gamma_\varphi, \kappa_n^\varphi) = \nabla(\Gamma, \kappa_n)$ on $S \cap \partial[\Gamma(0, T)] = [\Gamma(0)] \cup [\Gamma(T)]$. In particular, if $\mathcal{F}(\Gamma, \kappa_n; 0, T) < \infty$ and $\mathcal{F}(\Gamma_\varphi, \kappa_n^\varphi; 0, \tau_\varphi(T)) < \infty$ then $(\Gamma_\varphi, \kappa_n^\varphi) \in \mathcal{A}_{(\Gamma, \kappa_n)}$

Proof. It is easy to see that (99, 100) imply that $[\Gamma_\varphi(\tau_\varphi(T))] = [\Gamma(T)]$, and by the initial data clearly $[\Gamma_\varphi(0)] = [\Gamma(0)]$.

By transversality $s_{\Gamma_\varphi}^\pm$ and s_{Γ}^\pm are continuous by Proposition 3.1.11 in [10]. Hence Proposition 3.1.4 (iii) in [10] implies that

$$\begin{aligned} [\Gamma(0, T)] &= \mathcal{C}\left(S \setminus ([\Gamma(0)] \cup [\Gamma(T)]); \Gamma(0, T)\right) \\ &= \mathcal{C}\left(S \setminus ([\Gamma_\varphi(0)] \cup [\Gamma_\varphi(\tau_\varphi(T))]); \Gamma_\varphi(0, \tau_\varphi(T))\right) = [\Gamma_\varphi(0, \tau_\varphi(T))]. \end{aligned}$$

We used that $[\Gamma_\varphi(\tau_\varphi(T))] = [\Gamma(T)]$ and $[\Gamma_\varphi(0)] = [\Gamma(0)]$, and that $\Gamma(0, T)$ and $\Gamma_\varphi(0, \tau_\varphi(T))$ intersect (hence by admissibility and connectedness are contained in) the same connected component of $S \setminus ([\Gamma(0)] \cup [\Gamma(T)])$ because $\Gamma_\varphi(0) = \Gamma(0)$ and $\Gamma'_\varphi(0) = \Gamma'(0)$.

Lemma 7.7 implies that $(\Gamma, \kappa_n), (\Gamma_\varphi, \kappa_n^\varphi) \in C^1(S \cap \overline{[\Gamma(0, T)]}; \mathbb{R}^3)$. There it is also shown that $S \cap \partial[\Gamma(0, T)] = [\Gamma(0)] \cup [\Gamma(T)]$ and that the expression (137) holds for the gradient. From this one deduces the boundary conditions on $S \cap \partial[\Gamma(0, T)]$ as in the proof of Proposition 3.2 in [11]. \square

5.9. Lemma. *There is $\varepsilon_0 > 0$ such that the following holds: If $\varphi \in L^\infty((0, T); \mathbb{R}^3)$ is such that $\|\varphi\|_{L^\infty((0, T); \mathbb{R}^3)} < \varepsilon_0$ and such that $\mathcal{G}(\varphi) = 0$, then (97, 98, 99, 100) are satisfied.*

Proof. By Remark 5.4, the condition $\mathcal{G}(\varphi) = 0$ has the following implications: Equation (97) holds because $\mathcal{G}_1(\varphi) = 0$. Equation (99) holds because $\mathcal{G}_3(\varphi) = 0$. Since $\mathcal{G}_2(\varphi) = 0$, the matrix $a := r(T)(r_\varphi(\tau_\varphi(T)))^T$ satisfies $a_{21} = a_{31} = a_{23} = 0$. From this and the fact that $a \in SO(3)$ is close enough to the identity matrix (by choosing ε_0 is small enough), we conclude that $a = I$. Thus (98) holds. By a similar argument, $\mathcal{G}_4(\varphi) = 0$ implies that (100) holds. \square

5.10. Lemma. *Let $I \subset (0, T)$ be a Borel set with $\mathcal{L}^1(\{t \in I : \kappa_n(t) \neq 0\}) > 0$, and let $\psi \in L^\infty((0, T); \mathbb{R}^3)$ satisfy $\dot{\mathcal{G}}(\psi) = 0$. Then there exists a one-parameter family $\hat{\psi}^{(\varepsilon)} \in L^\infty((0, T); \mathbb{R}^3)$ such that the following are satisfied:*

- (i) $\|\hat{\psi}^{(\varepsilon)}\|_{L^\infty((0, T); \mathbb{R}^3)} \rightarrow 0$ as $\varepsilon \downarrow 0$
- (ii) $\hat{\psi}^{(\varepsilon)} = 0$ on $(0, T) \setminus I$ for all ε small enough,
- (iii) $\mathcal{G}\left(\varepsilon(\psi + \hat{\psi}^{(\varepsilon)})\right) = 0$ for all ε small enough.

Proof. One can easily check that the functionals $\mathcal{G}_i : L^\infty((0, T); \mathbb{R}^3) \rightarrow \mathbb{R}^3$ (or $\rightarrow \mathbb{R}$) are continuously Fréchet differentiable in an $L^\infty((0, T); \mathbb{R}^3)$ -neighbourhood of 0. By Proposition 5.7, the restriction of $\dot{\mathcal{G}}$ to the subspace of all $\psi \in L^\infty((0, T); \mathbb{R}^3)$ with $\psi = 0$ on $(0, T) \setminus I$ is surjective. Since the range of $\dot{\mathcal{G}}$ is a linear space and since the target space is finite dimensional, there exist $\hat{\psi}_k^c \in L^\infty((0, T); \mathbb{R}^3)$ with $\hat{\psi}_k^c = 0$ everywhere on $(0, T) \setminus I$, $k = 1, \dots, 8$, such that the matrix $(\dot{\mathcal{G}}_j(\hat{\psi}_i^c))_{i,j=1,\dots,8}$ is invertible. Recall from §5.3 that $\dot{\mathcal{G}}_i = D\mathcal{G}_i(0)$. (Here and below $D\mathcal{G}_i$ denotes the

Fréchet derivative of \mathcal{G}_i .) We define the C^1 function $F : \mathbb{R} \times \mathbb{R}^8 \rightarrow \mathbb{R}^8$ by

$$F(\varepsilon, \delta) = \mathcal{G}(\varepsilon\psi + \sum_{k=1}^8 \delta_k \hat{\psi}_k^c).$$

The partial derivatives $\frac{\partial}{\partial \delta_k} F$, $k = 1, \dots, 8$, exist and are continuous in a neighbourhood of zero. And for $i, j = 1, \dots, 8$ we have

$$\frac{\partial F_j}{\partial \delta_i}(\varepsilon, \delta) = D\mathcal{G}_j(\varepsilon\psi + \sum_{k=1}^8 \delta_k \hat{\psi}_k^c)(\hat{\psi}_i^c).$$

So $\frac{\partial F_j}{\partial \delta_i}(0, 0) = \dot{\mathcal{G}}_j(\hat{\psi}_i^c)$. Hence the matrix-valued function $\left(\frac{\partial F_j}{\partial \delta_i}\right)_{i,j=1,\dots,8}$ is invertible in a neighbourhood of $(0, 0)$. Hence by the implicit function theorem there is $r > 0$ and $\hat{\delta} \in C^1((-r, r); \mathbb{R}^8)$ with $\hat{\delta}(0, 0) = 0$ and such that $F(\varepsilon, \hat{\delta}(\varepsilon)) = 0$ for all $\varepsilon \in (-r, r)$. Taking derivatives with respect to ε in this equation and evaluating at $\varepsilon = 0$ we obtain $\hat{\delta}'(0) = 0$ because by definition we have $\frac{\partial F}{\partial \varepsilon}(0, 0) = \dot{\mathcal{G}}(\psi)$, which is zero by the hypothesis. Since $\hat{\delta} \in C^1$ this implies $\frac{1}{\varepsilon} \hat{\delta}(\varepsilon) \rightarrow 0$ as $\varepsilon \downarrow 0$. Setting

$$\hat{\psi}^{(\varepsilon)} := \frac{1}{\varepsilon} \sum_{k=1}^8 \hat{\delta}_k(\varepsilon) \hat{\psi}_k^c,$$

we obtain the desired functions. \square

The contribution of the next lemma is its part (ii). Without this condition one could simply take $\hat{\psi}^{(\delta)} = \psi - \psi^{(\delta)}$. We omit the proof since it is analogous to the one of Lemma 5.10.

5.11. Lemma. *Let $\psi \in L^\infty((0, T); \mathbb{R}^3)$ be such that $\dot{\mathcal{G}}(\psi) = 0$ and let $\psi^{(\delta)} \in L^\infty((0, T); \mathbb{R}^3)$ satisfy $\|\psi^{(\delta)} - \psi\|_{L^\infty((0, T); \mathbb{R}^3)} \rightarrow 0$ as $\delta \downarrow 0$. Let $I \subset (0, T)$ be a Borel set with $\mathcal{L}^1(\{t \in I : \kappa_n(t) \neq 0\}) > 0$. Then there exist $\hat{\psi}^{(\delta)} \in L^\infty((0, T); \mathbb{R}^3)$ such that the following hold:*

- (i) $\|\hat{\psi}^{(\delta)}\|_{L^\infty((0, T); \mathbb{R}^3)} \rightarrow 0$ as $\delta \downarrow 0$
- (ii) $\hat{\psi}^{(\delta)} = 0$ on $(0, T) \setminus I$ for all δ small enough,
- (iii) $\dot{\mathcal{G}}(\psi^{(\delta)} + \hat{\psi}^{(\delta)}) = 0$ for all δ small enough.

6 The Euler-Lagrange equations

6.1. Proposition. *Let $S \subset \mathbb{R}^2$ be a bounded C^1 -domain. Let $\Gamma \in W^{2,\infty}([0, T]; S)$ be transversal and locally admissible, and let $\kappa_n \in L^2(0, T)$ be such that $\mathcal{F}(\Gamma, \kappa_n; (0, T)) < \infty$. Assume that for all $\eta > 0$ there is $\varepsilon > 0$ such that*

$$\mathcal{F}(\Gamma, \kappa_n; (0, T)) \leq \mathcal{F}(\Gamma_\varphi, \kappa_n^\varphi; \tau_\varphi(0, T)) \quad (101)$$

holds for all $\varphi \in L^\infty((0, T); \mathbb{R}^3)$ satisfying the following conditions:

$$\|\varphi\|_{L^\infty((0, T); \mathbb{R}^3)} < \varepsilon, \quad (102)$$

$$\varphi = 0 \text{ a.e. on } (0, T) \setminus (I_0^\Gamma \cup I_\eta^\Gamma), \quad (103)$$

$$\text{there is } c > 0 \text{ such that } \varphi_3 \geq c \text{ a.e. on } I_0^\Gamma, \quad (104)$$

$$\mathcal{G}(\varphi) = 0. \quad (105)$$

Then $\kappa_n = 0$ almost everywhere on I_0^Γ and (Γ, κ_n) satisfy the Euler-Lagrange equations in the sense of Definition 2.7.

Proposition 6.1 will be a consequence of the following lemma.

6.2. Lemma. *If the hypotheses of Proposition 6.1 are satisfied, then $\kappa_n = 0$ almost everywhere on I_0^Γ and*

$$\begin{aligned} & \int \psi_1 \left\{ [\kappa_n^2 \bar{F}_2]_N + [\kappa_n^2 \bar{F}_1]_y + (1 - \chi_{I_0^\Gamma}) \kappa_n^2 \left(g_3(s_\Gamma^\pm, \kappa) + g_2(s_\Gamma^\pm, \kappa) \sigma \right) \right\} \\ & + \int \psi_3 \left\{ [\kappa_n^2 \bar{F}_1]^\wedge \cdot \Gamma' + (1 - \chi_{I_0^\Gamma}) \kappa_n^2 g_2(s_\Gamma^\pm, \kappa) \right\} \\ & + \int \psi_2 \left\{ 2(1 - \chi_{I_0^\Gamma}) \kappa_n g(s_\Gamma^\pm, \kappa) \right\} \geq 0 \end{aligned} \quad (106)$$

for all $\psi \in L^\infty((0, T); \mathbb{R}^3)$ with the following properties:

$$\text{There is } \eta > 0 \text{ such that } \psi = 0 \text{ a.e. on } (0, T) \setminus (I_0^\Gamma \cup I_\eta^\Gamma) \quad (107)$$

$$\psi_3 \geq 0 \text{ a.e. on } I_0^\Gamma \quad (108)$$

$$\dot{\mathcal{G}}(\psi) = 0. \quad (109)$$

Proof. If $\kappa_n = 0$ almost everywhere on $(0, T)$ then (106) is trivially satisfied. So let us assume that κ_n differs from zero on a set of positive measure. Since $\mathcal{F}(\Gamma, \kappa_n; (0, T)) < \infty$, Remark 2.6 implies that $\kappa_n = 0$ almost everywhere on I_0^Γ . Let $\psi \in L^\infty((0, T); \mathbb{R}^3)$ satisfy (107, 108, 109). Since $\kappa_n = 0$ on I_0^Γ , there is $\eta_0 > 0$ such that $\mathcal{L}^1(\{t \in I_{\eta_0} : \kappa_n(t) \neq 0\}) > 0$. Clearly, we may assume without loss of generality that $\eta < \eta_0$.

Claim #1. If, in addition, $\psi_3 \geq c > 0$ almost everywhere on I_0^Γ , then inequality (106) is satisfied.

In fact, let ψ be as in Claim #1. Applying Lemma 5.10 with $I := \{t \in I_\eta^\Gamma : \kappa_n(t) \neq 0\}$, we obtain $\hat{\psi}^{(\varepsilon)}$ satisfying $\|\hat{\psi}^{(\varepsilon)}\|_{L^\infty((0, T); \mathbb{R}^3)} \rightarrow 0$ as $\varepsilon \downarrow 0$ and $\hat{\psi}^{(\varepsilon)} = 0$ almost everywhere on $(0, T) \setminus I$ and such that, setting $\varphi^{(\varepsilon)} := \varepsilon(\psi + \hat{\psi}^{(\varepsilon)})$, we have $\mathcal{G}(\varphi^{(\varepsilon)}) = 0$ for small ε . Clearly $\varphi^{(\varepsilon)} \geq c\varepsilon$ on I_0^Γ .

Hence by the hypothesis of the lemma we have

$$\mathcal{F}(\Gamma(\varphi^{(\varepsilon)}), \kappa_n^{(\varphi^{(\varepsilon)})}; \tau_{(\varphi^{(\varepsilon)})}(0, T)) \geq \mathcal{F}(\Gamma, \kappa_n; (0, T)) \text{ for all } \varepsilon > 0 \text{ small enough.}$$

Thus $\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} (\mathcal{F}(\Gamma^{(\varphi^{(\varepsilon)})}, \kappa_n^{(\varphi^{(\varepsilon)})}; \tau_{(\varphi^{(\varepsilon)})}(0, T)) - \mathcal{F}(\Gamma, \kappa_n; (0, T))) \geq 0$. The inequality (106) therefore follows from Proposition 4.5. This finishes the proof of Claim #1.

Now let ψ be as in the assumption of Lemma 6.2, i.e. in contrast to Claim #1 we only have $\psi_3 \geq 0$ on I_0^Γ . By Lemma 5.11, for $\delta > 0$ small there exist $\hat{\psi}^{(\delta)}$ which vanish outside I_η^Γ , which satisfy $\|\hat{\psi}^{(\delta)}\|_{L^\infty((0, T); \mathbb{R}^3)} \rightarrow 0$ as $\delta \downarrow 0$ and which are such that

$$\tilde{\psi}_1^{(\delta)} = \psi_1 + \hat{\psi}_1^{(\delta)}, \tilde{\psi}_2^{(\delta)} = \psi_2 + \hat{\psi}_2^{(\delta)} \text{ and } \tilde{\psi}_3^{(\delta)} = \psi_3 + \delta \chi_{I_0^\Gamma} + \hat{\psi}_3^{(\delta)} \quad (110)$$

satisfy $\dot{\mathcal{G}}(\tilde{\psi}^{(\delta)}) = 0$ for all $\delta > 0$. Since $\|\tilde{\psi}^{(\delta)} - \psi\|_{L^2((0, T); \mathbb{R}^3)}$ converges to zero as $\delta \downarrow 0$, we conclude that

$$\begin{aligned} & \int \tilde{\psi}_1^{(\delta)} \left\{ [\kappa_n^2 \bar{F}_2]_N + [\kappa_n^2 \bar{F}_1]_y + \chi_{I_\eta^\Gamma} \kappa_n^2 (g_3(s_\Gamma^\pm, \kappa) + g_2(s_\Gamma^\pm, \kappa) \sigma) \right\} + \\ & + \int \tilde{\psi}_3^{(\delta)} \left\{ [\kappa_n^2 \bar{F}_1]^\wedge \cdot \Gamma' + \chi_{I_\eta^\Gamma} \kappa_n^2 g_2(s_\Gamma^\pm, \kappa) \right\} + \int \tilde{\psi}_2^{(\delta)} 2 \chi_{I_\eta^\Gamma} \kappa_n g(s_\Gamma^\pm, \kappa) \end{aligned} \quad (111)$$

converges to the left-hand side of (106) as $\delta \downarrow 0$. (Notice that $\chi_{I_\eta^\Gamma} g(s_\Gamma^\pm, \kappa)$, $\chi_{I_\eta^\Gamma} g_2(s_\Gamma^\pm, \kappa)$, $\chi_{I_\eta^\Gamma} g_3(s_\Gamma^\pm, \kappa)$ are in $L^\infty(0, T)$ when $\eta > 0$.) On the other hand, $\tilde{\psi}_3^{(\delta)} \geq \delta$ on I_0^Γ by (110) and because $\hat{\psi}_3^{(\delta)} = 0$ on I_0^Γ . Hence by applying Claim #1 with $\psi = \tilde{\psi}^{(\delta)}$, we conclude that (111) is nonnegative for each $\delta > 0$. Thus (106) follows by passing to the limit $\delta \downarrow 0$. \square

6.3. If $\Gamma \in W^{2, \infty}([0, T]; S)$ is transversal, we introduce the following two functions on $[0, T]$:

$$\Omega_2(t) := \Gamma'(t) \cdot [H - \kappa_n^2 \bar{F}_1]^\wedge(t) \quad (112)$$

$$\Omega_3(t) := \Lambda_3(t) + \lambda_4 + [\Omega_2 \bar{h} \sigma - \kappa_n^2 \bar{F}_2]_N(t) - \int_t^T \lambda_1 \cdot v(s) ds. \quad (113)$$

Clearly they are Lipschitz continuous. Recalling (51, 80, 76) as well as (75) and $[\Omega_2 \bar{h} \sigma - \kappa_n^2 \bar{F}_2]_N' = (\Omega_2 \bar{h} \sigma - \kappa_n^2 \bar{F}_2) \cdot \Gamma'$, we see that Ω_2 and Ω_3 satisfy the terminal value problems (21, 22). (Recall that $\Lambda_1(t) = \gamma'(t) \cdot (\lambda_2 - \lambda_1 \wedge \int_t^T \gamma')$.)

Proof of Proposition 6.1. If $\kappa_n = 0$ almost everywhere on $(0, T)$ then (18, 19, 20) hold with $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = \lambda_4 = 0$, since then $H = \Lambda_i = 0$, and so $\Omega_2 = \Omega_3 = 0$, and the left-hand sides vanish identically as well. Let us therefore assume that κ_n differs from zero on a set of positive measure.

Let $\eta > 0$ and $\varphi \in L^\infty((0, T); \mathbb{R}^3)$ such that (102, 103, 104, 105) hold. Clearly, we may assume without loss of generality that $\mathcal{L}^1(\{t \in I_\eta^\Gamma : \kappa_n(t) \neq 0\}) > 0$, since by Lemma 6.2 we have $\kappa_n = 0$ almost everywhere on I_0^Γ . By Lemma 6.2 the inequality (106) holds for all $\psi \in L^\infty((0, T); \mathbb{R}^3)$ which satisfy $\dot{\mathcal{G}}(\psi) = 0$, $\psi_3 \geq 0$ on I_0^Γ and $\psi = 0$ outside $I_0^\Gamma \cup I_\eta^\Gamma$. Set $J := I_0^\Gamma \cup I_\eta^\Gamma$. Denoting the terms in curly brackets on the left-hand side of (106) by h_1 , h_2 and h_3 (i.e. $h_2 := 2(1 - \chi_{I_0^\Gamma}) \kappa_n g(s_\Gamma^\pm, \kappa)$ and so on), inequality (106) has the form

$$\sum_{j=1}^3 \int_J h_j \psi_j \geq 0. \quad (114)$$

Notice that $h_j \in L^\infty(J)$, since $(1 - \chi_{I_0^\Gamma})g(s_\Gamma^\pm, \kappa)$ etc. are essentially bounded on J . Similarly, the system $\dot{\mathcal{G}}(\psi) = 0$ has the form

$$\sum_{j=1}^3 \int_J g_{ij} \psi_j = 0 \text{ for all } i = 1, \dots, 8 \quad (115)$$

for functions $g_{ij} \in L^\infty(0, T)$, $i = 1, \dots, 8$ and $j = 1, 2, 3$. The integration domain is J because $\psi = 0$ on $(0, T) \setminus J$.

By Proposition 5.7, for all $\mu \in \mathbb{R}^8 \setminus \{0\}$ there is $\psi \in L^\infty(J; \mathbb{R}^3)$ such that $\psi = 0$ on I_0^Γ but $0 \neq \sum_{i,j} \int_J \mu_i g_{ij} \psi_j$. Combining this nondegeneracy with (114, 115), we see that the hypotheses of Lemma 7.11 are satisfied (with ψ_3 instead of ψ_1 and with $J_0 = I_0^\Gamma$). Thus there exist $\lambda_1^{(\eta)}, \lambda_2^{(\eta)} \in \mathbb{R}^3$ and $\lambda_3^{(\eta)}, \lambda_4^{(\eta)} \in \mathbb{R}$ such that, writing $\Lambda_2^{(\eta)}$ and $H^{(\eta)}$ instead of Λ_2 and H to make the η -dependence explicit and using (85) from Lemma 5.5, we find that almost everywhere on $I_\eta^\Gamma \cup I_0^\Gamma$ the following equations are satisfied:

$$2(1 - \chi_{I_0^\Gamma})\kappa_n g(s_\Gamma^\pm, \kappa) = -\Lambda_2^{(\eta)} \quad (116)$$

$$[\kappa_n^2 \bar{F}_1]^\wedge \cdot \Gamma' + (1 - \chi_{I_0^\Gamma})\kappa_n^2 g_2(s_\Gamma^\pm, \kappa) \geq \Gamma' \cdot \hat{H}^{(\eta)} \text{ with equality on } I_\eta^\Gamma \quad (117)$$

$$\begin{aligned} & [\kappa_n^2 \bar{F}_2]_N + [\kappa_n^2 \bar{F}_1]_y + (1 - \chi_{I_0^\Gamma})\kappa_n^2 (g_3(s_\Gamma^\pm, \kappa) + g_2(s_\Gamma^\pm, \kappa)\sigma) \\ & = \lambda_4^{(\eta)} + \Lambda_3^{(\eta)} - \left(\int_t^T \lambda_1^{(\eta)} \cdot v \right) + [H^{(\eta)}]_y. \end{aligned} \quad (118)$$

Claim #1. If $\eta' \in (0, \eta]$ then $\lambda_i^{(\eta)} = \lambda_i^{(\eta')}$ for all $i = 1, 2, 3, 4$.

In fact, we have $\mathcal{L}^1(\{t \in I_\eta^\Gamma : \kappa_n(t) \neq 0\}) > 0$. Let $0 < \eta' \leq \eta$ and set $\tilde{\lambda}_i = \lambda_i^{(\eta)} - \lambda_i^{(\eta')}$, $i = 1, 2, 3, 4$. Defining $\tilde{\Lambda}_j$ and \tilde{H} in analogy to Λ_j and H , we find (notice that $I_\eta^\Gamma \subset I_{\eta'}^\Gamma$)

$$\begin{aligned} & \tilde{\Lambda}_2 = 0 \text{ on } I_\eta^\Gamma \\ & [\tilde{H}]_y + \tilde{\lambda}_4 + \tilde{\Lambda}_3 - \int_t^T \tilde{\lambda}_1 \cdot v = 0 \text{ on } I_\eta^\Gamma \\ & \Gamma' \cdot \tilde{H}^\wedge = 0 \text{ on } I_\eta^\Gamma. \end{aligned}$$

But these are just the equations (89, 90, 91). Therefore, Lemma 5.6 implies that $\tilde{\lambda}_i = 0$ for $i = 1, 2, 3, 4$, which proves the claim.

Set $\lambda_i := \lambda_i^{(\eta)}$ and define Λ_j and H with these λ_i , $i = 1, 2, 3, 4$. In view of Claim #1, the equations (116, 117, 118) hold without the index (η) almost everywhere on $(0, T) = I_0^\Gamma \cup \bigcup_{\eta>0} I_\eta^\Gamma$.

By (53) and by (80) we have $[H - \kappa_n^2 \bar{F}_1]_y = [\Omega_2 \sigma \bar{h}]_N + \Omega_2 \sigma$. So equation (118) can be simplified to become (20), since by (117) we know that $\Omega_2 - \kappa_n^2 g_2(s_\Gamma^\pm, \kappa) = 0$ outside I_0^Γ , and since $\sigma = \frac{1}{\kappa}$ on I_0^Γ . \square

Remark. We do not include the inequality (117) on I_0^Γ into the Euler-Lagrange equations because it is trivially satisfied by continuity of Ω_2 and since $\Omega_2' = -h\Omega_2$ on I_0^Γ and $g_2(s_\Gamma^\pm, \kappa) < 0$.

Proof of Theorem 2.9. Let $\Gamma \in W^{2,\infty}([0, T]; S)$ be admissible and let $\kappa_n \in L^2(0, T)$ be such that (13) holds. Then by Remark 2.6 the mapping (Γ, κ_n) is well-defined and belongs to $W_{\text{iso}}^{2,2}([\Gamma(0, T)]; \mathbb{R}^3)$. And $\kappa_n = 0$ almost everywhere on I_0^Γ . Now assume, in addition, that Γ is transversal. Then $\mathcal{A}_{(\Gamma, \kappa_n)}$ is well defined, see §2.8. Assume that $\mathcal{E}((\Gamma, \kappa_n); [\Gamma(0, T)]) \leq \mathcal{E}(u; [\Gamma(0, T)])$ for all $u \in \mathcal{A}_{(\Gamma, \kappa_n)}$. It suffices to show that (Γ, κ_n) satisfies the hypothesis of Proposition 6.1. By (12) in Remark 2.6 we must therefore show that for all $\eta > 0$ there is $\varepsilon > 0$ such that $(\Gamma_\varphi, \kappa_n^\varphi) \in \mathcal{A}_{(\Gamma, \kappa_n)}$ for all $\varphi \in L^\infty((0, T); \mathbb{R}^3)$ satisfying (102, 103, 104, 105).

Let $\varepsilon < \frac{1}{2} \min\{\varepsilon_\eta, \varepsilon_0\}$ (to be specified later), where ε_η is as in Lemma 4.4 and ε_0 is as in Lemma 5.9. By Lemma 4.4 we have that Γ_φ is locally admissible and transversal, and that $\mathcal{F}(\Gamma_\varphi, \kappa_n^\varphi; 0, \tau_\varphi(T)) < \infty$. To prove that

$$(\Gamma_\varphi, \kappa_n^\varphi) \in W_{\text{iso}}^{2,2}([\Gamma_\varphi(0, \tau_\varphi(T)); \mathbb{R}^3), \quad (119)$$

by Remark 2.6 it remains to prove that Γ_φ is admissible.

To prove this, notice that by Lemma 5.9 the formulae (97, 98, 99, 100) are satisfied. By the initial conditions we have $[\Gamma_\varphi(0)] = [\Gamma(0)]$ and by (99, 100) we have $[\Gamma_\varphi(\tau_\varphi(T))] = [\Gamma(T)]$. Since Γ is admissible, this implies that

$$[\Gamma_\varphi(0)] \cap [\Gamma_\varphi(\tau_\varphi(T))] = \emptyset. \quad (120)$$

On the other hand, $\beta_\Gamma^+([0, T]) \cap \beta_\Gamma^-([0, T]) = \emptyset$ by Lemma 7.8. Since Γ, Γ_φ are transversal on $[0, T]$, the mappings $\beta_\Gamma^\pm, \beta_{\Gamma_\varphi}^\pm$ are Lipschitz (see Proposition 3.1.11 in [10]). So $\beta_\Gamma^\pm([0, T])$ and $\beta_{\Gamma_\varphi}^\pm([0, \tau_\varphi(T)])$ are compact. Moreover, using transversality of Γ on $[0, T]$, one readily checks that $\beta_{\Gamma_\varphi}^\pm(\tau_\varphi)$ converge uniformly to β_Γ^\pm as $\|\varphi\|_{L^\infty((0, T); \mathbb{R}^3)} \rightarrow 0$. Hence, choosing ε small enough, we also have

$$\beta_{\Gamma_\varphi}^+([0, \tau_\varphi(T)]) \cap \beta_{\Gamma_\varphi}^-([0, \tau_\varphi(T)]) = \emptyset. \quad (121)$$

Let us assume for the moment that $\beta_\Gamma^\pm(0) \neq \beta_\Gamma^\pm(T)$. Hence for both $* = +, -$, Lemma 3.1.9 in [10] (together with the remark following it) implies that $\beta_\Gamma^*([0, T]) \neq \mathcal{C}(\partial S; \beta_\Gamma^*(0))$. But $\beta_\Gamma^*([0, T])$ is a connected compact subset of the closed Jordan curve $\mathcal{C}(\partial S; \beta_\Gamma^*(0))$. Hence by the uniform convergence $\beta_{\Gamma_\varphi}^*(\tau_\varphi) \rightarrow \beta_\Gamma^*$, we conclude that, choosing ε small enough, we have

$$\beta_{\Gamma_\varphi}^*([0, \tau_\varphi(T)]) \neq \mathcal{C}(\partial S; \beta_{\Gamma_\varphi}^*(0)) \text{ for any } * = +, -. \quad (122)$$

Combining (120, 121, 122) with local admissibility, Lemma 7.5 implies that indeed Γ_φ is admissible. In particular, (119) holds.

Since Γ_φ is admissible and transversal, Lemma 5.8 implies that $[\Gamma_\varphi(0, \tau_\varphi(T))] = [\Gamma(0, T)]$ and that $(\Gamma, \kappa_n), (\Gamma_\varphi, \kappa_n^\varphi) \in C^1(S \cap \overline{[\Gamma(0, T)]}; \mathbb{R}^3)$ with $(\Gamma_\varphi, \kappa_n^\varphi) = (\Gamma, \kappa_n)$ and $\nabla(\Gamma_\varphi, \kappa_n^\varphi) = \nabla(\Gamma, \kappa_n)$ on $S \cap \partial[\Gamma(0, T)]$. Together with (119) this proves that $(\Gamma_\varphi, \kappa_n^\varphi) \in \mathcal{A}_{(\Gamma, \kappa_n)}$. This concludes the proof for the case that $\beta_\Gamma^\pm(0) \neq \beta_\Gamma^\pm(T)$.

Now consider the case when this is not satisfied. Then $\beta_\Gamma^*(0) = \beta_\Gamma^*(T)$ either for $* = +$ or for $* = -$. (If it were true for both $* = +$ and $* = -$ then $[\Gamma(0)] = [\Gamma(T)]$, contradicting admissibility.) Assume without loss of generality that $\beta_\Gamma^+(0) = \beta_\Gamma^+(T)$. Two cases can occur: Either β_Γ^+ is constant on $[0, T]$ or it is not. In the former case, it is easy to see that $\kappa = \frac{1}{s_\Gamma^+}$ almost everywhere on $(0, T)$ (this follows e.g. from

$$(\beta_\Gamma^+)' = (1 - s_\Gamma^+ \kappa) \Gamma' + (s_\Gamma^+)' N, \quad (123)$$

which is formula (82) in [10]). By (13) this implies that $\kappa_n = 0$ almost everywhere on $(0, T)$. Then the Euler-Lagrange equations are trivially satisfied (see e.g. the proof of Proposition 6.1).

Now consider the case that $\beta_\Gamma^+(0) = \beta_\Gamma^+(T)$ but β_Γ^+ is not constant. Then there is $T_1 \in (0, T)$ with $\beta_\Gamma^+(T_1) \neq \beta_\Gamma^+(0) = \beta_\Gamma^+(T)$. Since $\mathcal{L}^1(\{\kappa_n \neq 0\}) > 0$, we may assume that $\mathcal{L}^1(\{t \in (T_1, T) : \kappa_n(t) \neq 0\}) > 0$. (The case where only $\mathcal{L}^1(\{t \in (0, T_1) : \kappa_n(t) \neq 0\}) > 0$ is similar.) Set

$$T' := \min\{t' \in (T_1, T) : \beta_\Gamma^+ \equiv \beta_\Gamma^+(T) \text{ on } [t', T]\}.$$

The minimum is attained by continuity of β_Γ^+ . By (123) we have $\kappa_n = 0$ almost everywhere on $[T', T]$. Hence $\mathcal{L}^1(\{t \in [T_1, T'] : \kappa_n(t) \neq 0\}) > 0$ (in particular $T' > T_1$). Thus by minimality of T' there is $T_2 \in (T_1, T')$ such that

$$\mathcal{L}^1(\{t \in (T_1, T_2) : \kappa_n(t) \neq 0\}) > 0 \text{ and } \beta_\Gamma^+(T_2) \neq \beta_\Gamma^+(T) = \beta_\Gamma^+(0). \quad (124)$$

It is easy to see (e.g. using an extension argument as in the proof of Theorem 2.10) that

$$\mathcal{E}((\Gamma|_J, \kappa_n|_J); [\Gamma(J)]) \leq \mathcal{E}(\tilde{u}; [\Gamma(J)])$$

for all open subintervals $J \subset (0, T)$ and for all $\tilde{u} \in \mathcal{A}_{(\Gamma|_J, \kappa_n|_J)}$. Hence we can apply the first part of this proof to $(\Gamma|_{(0, T_2)}, \kappa_n|_{(0, T_2)})$ and to $(\Gamma|_{(T_1, T)}, \kappa_n|_{(T_1, T)})$. Thus there are multipliers λ_i^0 , $i = 1, \dots, 4$ such that the Euler-Lagrange equations hold on $(0, T_2)$ and multipliers λ_i^1 such that they hold on (T_1, T) . Thus the Euler-Lagrange equations hold on (T_1, T_2) both with λ_i^0 and with λ_i^1 . The functions Ω_i^0 and Ω_i^1 , $i = 2, 3$, are defined by (21, 22) with their respective multipliers and the corresponding terminal points (T_2 and T , respectively). Setting $\tilde{\Omega}_2 := \Omega_2^1 - \Omega_2^0$ and recalling the definitions from §6.3, we have $\tilde{\Omega}_2 = \Gamma' \cdot \tilde{H}^\wedge$. Here, \tilde{H} is defined in analogy to (76) with $\tilde{\lambda}_i := \lambda_i^1 - \lambda_i^0$. Define the functions $\tilde{\Lambda}_i$ as in §5.1 with these $\tilde{\lambda}_i$. We conclude that (89, 90, 91) are satisfied. Hence by (124) Lemma 5.6 implies that $\lambda_i^0 = \lambda_i^1$ for all $i = 1, \dots, 4$. Hence $\tilde{H} = 0$ everywhere on $(0, T)$, and so $\Omega_i^0 = \Omega_i^1$ for $i = 2, 3$ on (T_1, T_2) . Since, moreover, Ω_2^1 and Ω_3^1 satisfy (21, 22), respectively, we conclude that the Euler-Lagrange equations hold on $(0, T)$ with $\lambda_i := \lambda_i^0$ and with

$$\Omega_{2,3} := \begin{cases} \Omega_{2,3}^0 & \text{on } [0, T_1) \\ \Omega_{2,3}^1 & \text{on } [T_1, T]. \end{cases}$$

□

The following remark was not used and so we do not give a proof. We mention it since it answers a natural question: Assume that (Γ, κ_n) has an extension $u \in W_{\text{iso}}^{2,2}(S; \mathbb{R}^3)$ to all of S , i.e. $(\Gamma, \kappa_n) = u|_{[\Gamma(0, T)]}$. (This was not assumed in the proof of Theorem 2.9, but of course it satisfied for (Γ, κ_n) as in Theorem 2.10.) Then $\beta_\Gamma^*(0) = \beta_\Gamma^*(T)$ implies that $\kappa_n = 0$ almost everywhere on $(0, T)$. Without the existence of such an extension u , this conclusion is false in general.

Proof of Theorem 2.10. If $S_1 \subset D_{\nabla u}$ and $q : S_1 \rightarrow \mathbb{S}^1$ is an S -ruling for ∇u then we define

$$\begin{aligned} \Sigma_\tau^q &:= \{x \in S_1 : \overline{[x]}_{q(x)} \text{ intersects } \partial S \text{ tangentially}\} \\ \Sigma_c^q &:= \{x \in S_1 : \overline{[x]}_{q(x)} \text{ intersects } \partial_c S\}. \end{aligned}$$

If $x \in S_1 \setminus \Sigma_\tau^q$, then $[x]_{q(x)}$ intersects ∂S transversally at both ends. By Remark 3.1.6 in [10] this implies that there is $\varepsilon > 0$ such that the segment $[y]_{q(y)}$ intersects ∂S transversally at both ends for all $y \in S_1 \cap B_\varepsilon(x)$. Hence $B_\varepsilon(x) \cap S_1 \subset S_1 \setminus \Sigma_\tau$. So Σ_τ is relatively closed in S_1 . By Lemma 7.1 (or Remark 3.1.6 in [10]), for $x \in S_1 \setminus \Sigma_\tau^q$ we also have that ν is continuous (even C^1) in a neighbourhood of $(x, \pm q(x))$. Hence $\nu(\cdot, \pm q(\cdot))$ are continuous on $B_{\varepsilon_1}(x) \cap S_1$ for some $\varepsilon_1 > 0$. Since $\partial_c S$ is closed, this implies that for all $x \in S_1 \setminus (\Sigma_\tau^q \cup \Sigma_c^q)$ there is $\varepsilon_2 > 0$ such that $B_{\varepsilon_2}(x) \cap S_1 \setminus (\Sigma_\tau^q \cup \Sigma_c^q) = \emptyset$. Thus $\Sigma_\tau^q \cup \Sigma_c^q$ is relatively closed in S_1 . Applying this with $S_1 := D_{\nabla u} \setminus C_{\nabla u}$, we conclude that $\Sigma_\tau \cup \Sigma_c$ is relatively closed in $D_{\nabla u} \setminus C_{\nabla u}$.

If $x_0 \in D_{\nabla u}$ then there exist a neighbourhood S_q of x_0 and a Lipschitz continuous S -ruling $q : S_q \rightarrow \mathbb{S}^1$ for ∇u . By Remark 3.2.1 in [10] there is $T > 0$ and a unique solution $\Gamma \in W^{2,\infty}([0, T]; S_q)$ of the ODE

$$\Gamma' = -(q(\Gamma))^\perp \text{ with } \Gamma\left(\frac{T}{2}\right) = x_0.$$

For small enough T also $\Gamma'(t_0) \cdot \Gamma'(t_1) > 0$ for all $t_0, t_1 \in [0, T]$ (the simple proof can be found e.g. in [10]). So Γ is admissible by Lemma 3.2.3 in [10]. By Proposition 2.2 in [11] there is $\kappa_n \in L^2(0, T)$ such that $u = (\Gamma, \kappa_n)$ on $[\Gamma(0, T)]$.

If the above x_0 is even contained in $D_{\nabla u} \setminus (C_{\nabla u} \cup \Sigma_\tau \cup \Sigma_c)$ then $x_0 \notin \Sigma_\tau^q \cup \Sigma_c^q$, since $q(x_0) = q_{\nabla u}(x_0)$ because $x_0 \notin C_{\nabla u}$ (see §2.4). Hence, choosing $T > 0$ small enough, we have that $\Gamma([0, T]) \subset S_q \setminus (\Sigma_\tau^q \cup \Sigma_c^q)$. In particular, Γ is transversal.

Let $\tilde{u} \in \mathcal{A}_{(\Gamma, \kappa_n)}$. Then by Corollary 7.10 the mapping $\bar{u} : S \rightarrow \mathbb{R}^3$ defined by

$$\bar{u}(x) := \begin{cases} \tilde{u}(x) & \text{if } x \in [\Gamma(0, T)] \\ u(x) & \text{if } x \in S \setminus [\Gamma(0, T)] \end{cases} \quad (125)$$

satisfies $\bar{u} \in W_{\text{iso}}^{2,2}(S; \mathbb{R}^3)$. Since $\Gamma([0, T]) \cap \Sigma_c^q = \emptyset$, we have $\overline{[\Gamma(0, T)]} \cap \partial_c S = \emptyset$. Hence $\bar{u} \in \mathcal{A}_u(S, \partial_c S)$. Thus by the hypothesis $\mathcal{E}(\bar{u}; S) \geq \mathcal{E}(u; S)$. We conclude that $\mathcal{E}(\tilde{u}; [\Gamma(0, T)]) \geq \mathcal{E}((\Gamma, \kappa_n); [\Gamma(0, T)])$. Since $\tilde{u} \in \mathcal{A}_{(\Gamma, \kappa_n)}$ was arbitrary, Theorem 2.9 implies that the Euler-Lagrange equations are satisfied. \square

Proof of Theorem 2.2. We may assume without loss of generality that $\mathcal{H}^1(\partial_c S) > 0$ because otherwise $\mathcal{A}_{u_0}(S, \partial_c S) = W_{\text{iso}}^{2,2}(S; \mathbb{R}^3)$, and so the identity $u(x) = x$ is a minimizer. Clearly $\mathcal{E}(\cdot; S)$ is weakly lower semicontinuous in $W^{2,2}(S; \mathbb{R}^3)$. Let $(u_n) \subset \mathcal{A}_{u_0}(S, \partial_c S)$ satisfy $\mathcal{E}(u_n; S) \rightarrow \inf_{u \in \mathcal{A}_{u_0}(S, \partial_c S)} \mathcal{E}(u; S)$. Then by the isometry constraint clearly $\int_S |\nabla u_n|^2 \leq C$, and by a Poincaré inequality and the fact that we are prescribing u_n on a set of positive boundary measure we conclude that $\|u_n\|_{W^{2,2}(S; \mathbb{R}^3)} \leq C$. Hence there is a subsequence (not relabelled) and a mapping $u \in W^{2,2}(S; \mathbb{R}^3)$ such that $u_n \rightharpoonup u$ weakly in $W^{2,2}(S; \mathbb{R}^3)$. This implies strong convergence in $W^{1,2}$, so $u \in W_{\text{iso}}^{2,2}(S; \mathbb{R}^3)$. Moreover, since $\nabla u_n \rightharpoonup \nabla u$ weakly in $W^{1,2}(S; \mathbb{R}^{3 \times 2})$, by compactness of the trace operator from $W^{1,2}(S)$ to $L^2(\partial S)$ (see e.g. [2] Theorem 6.1-7), the traces $\nabla u_n|_{\partial_c S}$ converge strongly in $L^2(\partial_c S; \mathbb{R}^{3 \times 2})$, whence (for a subsequence) pointwise \mathcal{H}^1 almost everywhere. Since strong L^2 convergence of $u_n|_{\partial_c S}$ already follows from continuity of the trace operator, we have that $u \in \mathcal{A}_{u_0}(S, \partial_c S)$. And from the weak lower semicontinuity of $\mathcal{E}(\cdot; S)$ we conclude that u is a minimizer. \square

7 Appendix

As elsewhere in this article, $S \subset \mathbb{R}^2$ denotes a bounded C^1 domain and $\Gamma \in W^{2,\infty}([0, T]; S)$ is parametrized by arclength.

7.1. Lemma. *For all $x \in S$ and all $\theta \in \mathbb{R}^2 \setminus \{0\}$ we have:*

$$\nu(x, \lambda\theta) = \frac{1}{\lambda}\nu(x, \theta) \text{ for all } \lambda > 0 \quad (126)$$

$$\nu(x + \lambda\theta, \theta) = \nu(x, \theta) - \lambda|\theta| \text{ for all } \lambda \in \mathbb{R} \text{ such that } [x, x + \lambda] \subset S. \quad (127)$$

$$\nu_1(x, \theta) \cdot \theta = -|\theta| \quad (128)$$

$$\nu_2(x, \theta) \cdot \theta = -\nu(x, \theta). \quad (129)$$

Assume, in addition, that $(x, \theta) \in S \times (\mathbb{R}^2 \setminus \{0\})$ is transversal. Then ν is C^1 in a neighbourhood of (x, θ) and there exists $\varepsilon > 0$ such that

$$\nu(x + \nu(x, \theta)(I - R)\theta, R\theta) = \nu(x, \theta) \text{ for all } R \in SO(2) \text{ with } |R - I| < \varepsilon. \quad (130)$$

In particular, if $(x, \theta) \in S \times \mathbb{S}^1$ is transversal then

$$\nu(x, \theta)\nu_1(x, \theta) = \nu_2(x, \theta). \quad (131)$$

Proof. The formulae (126, 127) are obvious consequences of the definition. And (128, 129) follow directly from (126, 127).

That ν is C^1 near (x, θ) under the stated assumptions is proven as in Remark 3.1.6 in [10]. After translating and rotating we may assume that $x + \nu(x, \theta)\theta$ agrees with the origin and that $\theta = -|\theta|e_1$. So $x = \nu(x, \theta)|\theta|e_1$. Under the usual identification of \mathbb{S}^1 with $SO(2)$ and of \mathbb{R}^2 with \mathbb{C} formula (130) reduces to

$$\nu(\nu(x, \theta)|\theta|e^{i\varphi}, -e^{i\varphi}|\theta|) = \nu(\nu(x, \theta)|\theta|, -|\theta|) \text{ for small } |\varphi|. \quad (132)$$

And this formula is a consequence of the definition of ν once we know that the open segment with endpoints 0 and $e^{i\varphi}x$ is contained in S for small $|\varphi|$. But this is satisfied by Remark 3.1.6 in [10]. Finally, taking derivatives with respect to φ in (132) we find $\nu(x, \theta)\nu_1(x, \theta) \cdot \theta^\perp = \nu_2(x, \theta) \cdot \theta^\perp$. Combining this with (128, 129), we obtain (131). \square

7.2. Corollary. *Let $* \in \{+, -\}$ and assume that Γ is transversal. Then ν is C^1 in a neighbourhood of $\bigcup_{t \in [0, T]}(\Gamma(t), *N(t))$ and s_Γ^* is Lipschitz with*

$$(s_\Gamma^*)' = *(1 - s_\Gamma^*\kappa)\nu_1(\Gamma, *N) \cdot \Gamma' \text{ a.e. on } (0, T). \quad (133)$$

Proof. It follows from Proposition 3.1.11 in [10] that $\nu \in C^1$ near $\bigcup_{t \in [0, T]}(\Gamma(t), \pm N(t))$. Thus (133) follows from (131) by differentiating the definition $s_\Gamma^* := *\nu(\Gamma, *N)$. \square

7.3. Lemma. *Assume that Γ is transversal and locally admissible. Then for all $t' \in [0, T]$ there is $\delta_1(t') > 0$ such that*

$$[\Gamma(t)] \cap [\Gamma(t')] = \emptyset \text{ for all } t \in [0, T] \text{ with } |t - t'| \leq \delta_1(t'). \quad (134)$$

Proof. By transversality and Proposition 3.1.11 in [10] the mappings β_Γ^\pm are Lipschitz. By (133) and since $(\beta_\Gamma^*)' = (1 - s_\Gamma^* \kappa) \Gamma' + (s_\Gamma^*)' N$ (see e.g. the proof of Proposition 3.1.11 in [10]), using (133) we conclude

$$(\beta_\Gamma^*)' \cdot \Gamma'(t') = (1 - s_\Gamma^* \kappa) \left[\Gamma' \cdot \Gamma'(t') + (*\nu_1(\Gamma, *N) \cdot \Gamma') N \cdot \Gamma'(t') \right]. \quad (135)$$

Clearly there is $\delta_1(t') > 0$ such that the factor in square brackets is positive for t with $|t - t'| \leq \delta_1(t')$ and for $*$ = +, -. Hence local admissibility and (135) imply that $(\beta_\Gamma^\pm)'(t) \cdot \Gamma'(t') \geq 0$ for all t with $|t - t'| \leq \delta_1(t')$. From this (134) follows immediately. \square

Recalling (9), for brevity we define $M_{s_\Gamma^\pm} := M_{s_\Gamma^\pm}(0, T)$. We define the mapping $\Phi_\Gamma : \mathbb{R} \times [0, T] \rightarrow \mathbb{R}^2$ by $\Phi_\Gamma(s, t) := \Gamma(t) + sN(t)$.

7.4. Lemma. *Let Γ be transversal and locally admissible, and assume that $\Phi_\Gamma(\partial M_{s_\Gamma^\pm})$ is a closed Jordan curve. Then Γ is admissible.*

Proof. The proof is exactly the same as that of Lemma 3.1.5 in [10], except for the following modification: Formula (61) in [10] follows from Lemma 7.3 above. (And not from Lemma 3.1.3 in [10].) \square

7.5. Lemma. *Let Γ be transversal and locally admissible. Assume that $\beta_\Gamma^*([0, T]) \neq \mathcal{C}(\partial S; \beta_\Gamma^*(0))$ for $*$ = +, -, that $\beta_\Gamma^+([0, T]) \cap \beta_\Gamma^-([0, T]) = \emptyset$ and that $[\Gamma(0)] \cap [\Gamma(T)] = \emptyset$. Then Γ is admissible.*

Proof. By transversality, Proposition 3.1.11 in [10] implies that s_Γ^\pm are continuous. As in the proof of Proposition 3.1.8 in [10], the hypotheses therefore imply that $\Phi_\Gamma(\partial M_{s_\Gamma^\pm})$ is a closed Jordan curve. Hence the claim follows from Lemma 7.4. \square

Set $\hat{M}_{s_\Gamma^\pm} := M_{s_\Gamma^\pm}([0, T]) = \bigcup_{t \in [0, T]} (s_\Gamma^-(t), s_\Gamma^+(t)) \times \{t\}$. (Notice that t ranges through the closed interval $[0, T]$.)

7.6. Remark. *If Γ is admissible then Φ_Γ is injective on $\hat{M}_{s_\Gamma^\pm}$ and Φ_Γ^{-1} is continuous on $\Phi_\Gamma(\hat{M}_{s_\Gamma^\pm}) = [\Gamma([0, T])]$.*

Proof. Injectivity of Φ_Γ on $\hat{M}_{s_\Gamma^\pm}$ is just the definition of admissibility. Next notice that

$$\Phi_\Gamma(\bar{M}_{s_\Gamma^\pm} \setminus \hat{M}_{s_\Gamma^\pm}) \cap \Phi_\Gamma(\hat{M}_{s_\Gamma^\pm}) = \emptyset. \quad (136)$$

In fact, $\bar{M}_{s_{\Gamma}^{\pm}} \setminus \hat{M}_{s_{\Gamma}^{\pm}} = \bigcup_{t \in [0, T]} \{s_{\Gamma}^{-}(t), s_{\Gamma}^{+}(t)\} \times \{t\}$. By definition of s_{Γ}^{\pm} this is mapped into ∂S by Φ_{Γ} . Since $\Phi_{\Gamma}(\hat{M}_{s_{\Gamma}^{\pm}}) = [\Gamma([0, T])]$ is contained in S , formula (136) follows from openness of S . Using injectivity of Φ_{Γ} on $\hat{M}_{s_{\Gamma}^{\pm}}$, continuity of Φ_{Γ} on $\bar{M}_{s_{\Gamma}^{\pm}}$ and (136), one easily deduces that Φ_{Γ}^{-1} is continuous on $\Phi_{\Gamma}(\hat{M}_{s_{\Gamma}^{\pm}})$. \square

7.7. Lemma. *If Γ is admissible and transversal and if $\kappa_n \in L^2(0, T)$ then $S \cap \overline{[\Gamma(0, T)]} = [\Gamma([0, T])]$ and $S \cap \partial[\Gamma(0, T)] = [\Gamma(0)] \cup [\Gamma(T)]$. Moreover, $(\Gamma, \kappa_n) \in C^1(S \cap \overline{[\Gamma(0, T)]}; \mathbb{R}^3)$ with*

$$\nabla(\Gamma, \kappa_n)(\Phi_{\Gamma}) = \gamma' \otimes \Gamma' + v \otimes N \text{ on } \hat{M}_{s_{\Gamma}^{\pm}}. \quad (137)$$

Proof. By Remark 2.6 the mapping (Γ, κ_n) is a well-defined element of $W_{\text{loc, iso}}^{2,2}([\Gamma(0, T)]; \mathbb{R}^3)$. It is therefore Lipschitz, so it is continuous up to the boundary of $[\Gamma(0, T)]$. It remains to show that $\nabla(\Gamma, \kappa_n) \in C^0(S \cap \overline{[\Gamma(0, T)]}; \mathbb{R}^{3 \times 2})$. In Lemma 7.9 below we will prove that $S \cap \overline{[\Gamma(0, T)]} = [\Gamma([0, T])]$ and that $S \cap \partial[\Gamma(0, T)] = [\Gamma(0)] \cup [\Gamma(T)]$. Formula (137) is just (13) in [11]. It implies that $\nabla(\Gamma, \kappa_n)(\Phi_{\Gamma})$ is continuous on $\hat{M}_{s_{\Gamma}^{\pm}}$. Hence Remark 7.6 implies that $\nabla(\Gamma, \kappa_n)$ is continuous on $\Phi_{\Gamma}(\hat{M}_{s_{\Gamma}^{\pm}})$. \square

7.8. Lemma. *Assume that Γ is admissible and transversal. Then $\beta_{\Gamma}^{+}([0, T]) \cap \beta_{\Gamma}^{-}([0, T]) = \emptyset$.*

Proof. We omit the index Γ . By transversality, $\beta^{\pm} \in C^0([0, T])$, see e.g. Proposition 3.1.11 in [10]. Thus there are connected components $\partial_+ S$ and $\partial_- S$ of ∂S such that $\beta^{\pm}([0, T]) \subset \partial_{\pm} S$. If $\partial_+ S \neq \partial_- S$ then the lemma follows.

Let us therefore assume that $\beta^{-}([0, T]) \cup \beta^{+}([0, T]) \subset \partial_+ S$. Let $\alpha : \mathbb{S}^1 \rightarrow \partial_+ S$ be a Bilipschitz homeomorphism. By Lemma 3.1.9 in [10] there exist monotone functions $\phi^{\pm} : [0, T] \rightarrow \mathbb{R}$ such that $\beta^{\pm} = \alpha(e^{i\phi^{\pm}})$ and such that $|\phi^{\pm}(T) - \phi^{\pm}(0)| \leq 2\pi$. We claim that ϕ^{\pm} can be chosen to be continuous. In fact, if β^* is constant then we can choose ϕ^* to be constant. Let us therefore assume that β^* is not constant. Since $e^{i\phi^*} = \alpha^{-1}(\beta^*)$ is continuous, if ϕ^* had a discontinuity at $t' \in [0, T]$ (which by monotonicity must be a jump discontinuity), then

$$|\lim_{t \uparrow t'} \phi^*(t) - \lim_{t \downarrow t'} \phi^*(t)| = 2\pi k$$

for some $k \in \mathbb{N}$. Since $|\phi^*(0) - \phi^*(T)| \leq 2\pi$, by monotonicity we must have $|\phi^*(T) - \phi^*(0)| = 2\pi$ and $\phi^*([0, T]) = \{\phi^*(0), \phi^*(T)\}$. Thus β^* must be constant on $[0, T]$, a contradiction proving the claim.

Suppose for contradiction that there were $t_0, t_1 \in [0, T]$ such that $\beta^{+}(t_0) = \beta^{-}(t_1)$. After possibly adding an integer multiple of 2π to ϕ^{-} this implies that $\phi^{+}(t_0) = \phi^{-}(t_1)$. On the other hand, it is easy to verify that if ϕ^{+} is nondecreasing then ϕ^{-} is nonincreasing, and viceversa. (This is shown at the end of this proof.) After

possibly inverting the orientation of α we may assume that ϕ^+ is nondecreasing and that therefore ϕ^- is nonincreasing. Hence by $\phi^+(t_0) = \phi^-(t_1)$, we necessarily have

$$\phi^-(0) \geq \phi^+(0) \text{ and } \phi^-(T) \leq \phi^+(T). \quad (138)$$

By continuity of ϕ^\pm and by (138), the mean value theorem implies that there is $t \in [0, T]$ such that $\phi^-(t) = \phi^+(t)$. So $\beta^-(t) = \beta^+(t)$, a contradiction.

Let us finally prove that ϕ^+ is nonincreasing whenever ϕ^- is nondecreasing and viceversa. Define $\tilde{\alpha}(\varphi) := \alpha(e^{i\varphi})$ and denote by $\eta(x)$ the outer unit normal to S at $x \in \partial S$. After appropriately choosing the orientation of α , we have $(\tilde{\alpha}'(\varphi))^\perp = -\eta(\tilde{\alpha}(\varphi))$ for all $\varphi \in \mathbb{R}$. On the other hand, by transversality, $\eta(\beta^*) \cdot (*N) > 0$ on $[0, T]$. We conclude that

$$*\tilde{\alpha}'(\phi^*(t)) \cdot \Gamma'(t) < 0 \text{ for all } t \in (0, T). \quad (139)$$

Notice that since ϕ^\pm are continuous and since $e^{i\phi^\pm} = \alpha^{-1}(\beta^\pm)$ is Lipschitz, also ϕ^\pm is Lipschitz. Thus $0 \leq 1 - s^*\kappa = (\beta^*)' \cdot \Gamma' = (\tilde{\alpha}(\phi^*))' \cdot \Gamma' = (*\phi^*)' [* \tilde{\alpha}'(\phi^*) \cdot \Gamma']$. The term in square brackets is negative by (139). This proves the claim. \square

7.9. Lemma. *Let Γ be admissible and transversal. Then*

$$[\Gamma(0, T)] = \mathcal{C}\left(S \setminus ([\Gamma(0)] \cup [\Gamma(T)]); \Gamma(0, T)\right). \quad (140)$$

In particular, $S \cap \overline{[\Gamma(0, T)]} = [\Gamma(0, T)]$ and $S \cap \partial[\Gamma(0, T)] = [\Gamma(0)] \cup [\Gamma(T)]$. Moreover, there is a well-defined trace operator $T : W^{1,2}([\Gamma(0, T)]) \rightarrow L^2_{loc}([\Gamma(0)] \cup [\Gamma(T)])$ such that the Gauss-Green formula

$$\int_{[\Gamma(0, T)]} f \operatorname{div} \varphi = - \int_{[\Gamma(0, T)]} \varphi \cdot \nabla f + \int_{[\Gamma(0)] \cup [\Gamma(T)]} (\varphi \cdot \eta) T f \, d\mathcal{H}^1 \quad (141)$$

holds for all $\varphi \in C_0^\infty(S; \mathbb{R}^2)$. Here $\eta(x) := N(T)$ if $x \in [\Gamma(T)]$ and $\eta(x) := -N(0)$ if $x \in [\Gamma(0)]$.

Remarks.

- (i) If $\beta_\Gamma^*(0) = \beta_\Gamma^*(T)$ for some $* \in \{+, -\}$ then $[\Gamma(0, T)]$ is not a Lipschitz domain. (By admissibility it cannot happen that $\beta_\Gamma^*(0) = \beta_\Gamma^*(T)$ holds simultaneously for $* = +$ and for $* = -$.)
- (ii) The second term on the right-hand side of (141) is well defined because $Tf \in L^2_{loc}([\Gamma(0)] \cup [\Gamma(T)])$ and the support of φ lies in S , so $\operatorname{spt} \varphi \cap ([\Gamma(0)] \cup [\Gamma(T)])$ is compact.
- (iii) A trace operator $W^{1,2}(S \setminus \overline{[\Gamma(0, T)]}) \rightarrow L^2_{loc}([\Gamma(0)] \cup [\Gamma(T)])$ is obtained similarly. In what follows we do not display these operators explicitly.

Proof. Set $V := [\Gamma(0, T)]$. Formula (140), $S \cap \overline{[\Gamma(0, T)]} = [\Gamma([0, T])]$ and $S \cap \partial V = [\Gamma(0)] \cup [\Gamma(T)]$ were proven in Proposition 3.1.8 (iii) in [10]. Since by admissibility $[\Gamma(0)] \cap [\Gamma(T)] = \emptyset$, the set V satisfies the hypothesis of Lemma 2.2.2 in [10]. Denote by $B_{r_x}^\pm(x)$ the two connected components (i.e. open half disks) of $B_{r_x}(x) \setminus [x]$, where $[x] := [\Gamma(t)]$ if $x \in [\Gamma(t)]$. By Lemma 2.2.2 in [10], for all $x \in S \cap \partial V$ there is $r > 0$ such that (after choosing the labels \pm appropriately) $B_{r_x}^+(x) \subset V$. Moreover, either $B_{r_x}^-(x) \subset S \setminus \bar{V}$ or $B_{r_x}^-(x) \subset V$.

Assume that $x \in [\Gamma(0)]$. (The case $x \in [\Gamma(T)]$ is analogous.) For all $\varepsilon > 0$ small enough there is $r \in (0, r_x)$ such that $[\Gamma(t)] \cap B_r(x) = \emptyset$ for all $t \geq \varepsilon$. Otherwise there would be $t' \in [\varepsilon, T]$ such that $x \in [\Gamma(t')]$, contradicting admissibility. Thus

$$B_r(x) \cap V = B_r(x) \cap [\Gamma(0, \varepsilon)]. \quad (142)$$

Since $\Gamma \in C^1$ and $\Gamma'(0)$ is perpendicular to $[\Gamma(0)]$, for ε small enough the set $[\Gamma(0, \varepsilon)]$ is contained in one of the two connected components of $\mathbb{R}^2 \setminus (\Gamma(0) + \text{span } N(0))$. In particular, $[\Gamma(0, \varepsilon)]$ does not contain $B_r(x) \setminus [\Gamma(0)]$. By (142) this implies that $B_r(x) \setminus [\Gamma(0)]$ is not contained in V . Thus $B_{r_x}^-(x)$ is not contained in V either.

By the above alternative this implies that $B_{r_x}^-(x) \subset S \setminus \bar{V}$ for all $x \in S \cap \partial V$. Now one can construct the trace operator T from the trace operators on the Lipschitz domains $B_{r_x}^+(x)$ (see Section 4.3 in [4]) by using a partition of unity. Covering $S \cap \partial V$ with countably many such $B_{r_x}(x)$ therefore gives traces in L_{loc}^2 . The formula (141) is easily deduced from the corresponding formula for the local traces. \square

The following corollary is a standard consequence of Lemma 7.9 (its proof uses the fact that the outer unit normals of two adjacent subregions in (141) have opposite signs).

7.10. Corollary. *Let Γ be admissible and transversal, let $u \in W^{2,2}(S; \mathbb{R}^3)$ and $\tilde{u} \in W^{2,2}([\Gamma(0, T)]; \mathbb{R}^3)$ be such that $\nabla u = \nabla \tilde{u}$ on $[\Gamma(0)] \cup [\Gamma(T)]$ in the sense of traces and such that $u = \tilde{u}$ (pointwise) on $[\Gamma(0)] \cup [\Gamma(T)]$. Then the mapping \bar{u} defined by*

$$\bar{u}(x) := \begin{cases} \tilde{u}(x) & \text{if } x \in [\Gamma(0, T)] \\ u(x) & \text{if } x \in S \setminus [\Gamma(0, T)] \end{cases}$$

is in $W^{2,2}(S; \mathbb{R}^3)$ with $\nabla \bar{u} = \chi_{[\Gamma(0, T)]} \nabla \tilde{u} + (1 - \chi_{[\Gamma(0, T)]}) \nabla u$ almost everywhere.

7.11. Lemma. *Let $m, n \in \mathbf{N}$, let $J \subset \mathbb{R}$ be a Borel set with $0 < \mathcal{L}^1(J) < \infty$ and let $J_0 \subset J$ be a Borel set with $\mathcal{L}^1(J_0) < \mathcal{L}^1(J)$. Let $G \in L^\infty(J; \mathbb{R}^{n \times m})$ be such that $\mathcal{L}^1(\{t \in J \setminus J_0 : G^T(t)\mu \neq 0\}) > 0$ for all $\mu \in \mathbb{R}^n \setminus \{0\}$. Assume, moreover, that $h \in L^2(J; \mathbb{R}^m)$ is such that*

$$\int_J h \cdot \psi \geq 0 \text{ for all } \psi \in L^\infty(J; \mathbb{R}^m) \text{ with } \int_J G\psi = 0 \text{ and } \psi_1 \geq 0 \text{ a.e. on } J_0. \quad (143)$$

Then there is $\lambda \in \mathbb{R}^n$ such that the following hold:

$$h_1 \geq \sum_{i=1}^n \lambda_i G_{i1} \text{ a.e. on } J_0 \quad (144)$$

$$h_1 = \sum_{i=1}^n \lambda_i G_{i1} \text{ a.e. on } J \setminus J_0 \quad (145)$$

$$h_k = \sum_{i=1}^n \lambda_i G_{ik} \text{ a.e. on } J \text{ for all } k = 2, \dots, m. \quad (146)$$

Proof. Set $J_1 = J \setminus J_0$ and set $P := \int_{J_1} GG^T \in \mathbb{R}^{n \times n}$. The matrix P is invertible (because $\mu \cdot P\mu = \int_{J_1} |G^T \mu|^2 \neq 0$ for $\mu \neq 0$ by the hypothesis on G). Define the operator $Q : L^2(J; \mathbb{R}^m) \rightarrow L^2(J; \mathbb{R}^m)$ by setting

$$Qf := G^T P^{-1} \int_J Gf.$$

Using symmetry of P one readily checks that Q is self-adjoint. Set

$$T := \{\psi \in L^2(J; \mathbb{R}^m) : \int_J G\psi = 0\},$$

$$K := \{\psi \in L^\infty(J; \mathbb{R}^m) : \psi_1 \geq 0 \text{ a.e. on } J_0\}.$$

Denote by $[\chi_{J_1}]$ the multiplication operator associated to χ_{J_1} . A short calculation shows that $(I - [\chi_{J_1}]Q)\psi \in T \cap K$ whenever $\psi \in K$. But (143) means that $\int_J h \cdot \psi \geq 0$ for all $\psi \in T \cap K$. Hence from self-adjointness of Q and of $[\chi_{J_1}]$ we conclude that

$$\int_J \psi \cdot (I - Q[\chi_{J_1}])h \geq 0 \text{ for all } \psi \in K.$$

This readily implies the claim with $\lambda := P^{-1} \int_{J_1} Gh$. \square

Acknowledgements. Some preliminary results were obtained in my PhD thesis. I would like to thank my advisor Stefan Müller for proposing this problem and for helpful suggestions. Interesting discussions with Sergio Conti and Reza Pakzad are also gratefully acknowledged.

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