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A Non-Self-Adjoint Quadratic Eigenvalue
Problem Describing a Fluid-Solid Interaction.
Part I: Formulation, Analysis, and Computations

by

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A Non-Self-Adjoint Quadratic Eigenvalue Problem Describing a Fluid-Solid Interaction Part I: Formulation, Analysis, and Computations

*This paper is dedicated to Philippe G. Ciarlet
on the occasion of his seventieth birthday*

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Abstract

This two-part paper treats the numerical approximation of a tricky quadratic eigenvalue problem arising from the following generalization of the classical Taylor-Couette problem: A viscous incompressible fluid occupies the region between a rigid inner cylinder and a deformable outer cylinder, which we take to be a nonlinearly viscoelastic membrane. The inner cylinder rotates at a prescribed angular velocity ω , driving the fluid, which in turn drives the deformable outer cylinder. The motion of the outer cylinder is not prescribed, but responds to the forces exerted on it by the moving fluid. A steady solution of this coupled fluid-solid system, analogous to the Couette solution of the classical problem, can be found analytically. Its linearized stability is governed by a non-self-adjoint quadratic eigenvalue problem.

In Part I, we give a careful formulation of the geometrically exact problem. We compute the eigenvalue trajectories in the complex plane as functions of ω by using a Fourier-finite element method. Computational results show that steady solution loses its stability by a process suggestive

of a Takens-Bogdanov bifurcation. In Part II we prove convergence of the numerical method.

1 Introduction

The classical Taylor-Couette problem concerns the motion of a viscous incompressible fluid in the region between two rigid coaxial cylinders, which rotate at prescribed constant angular velocities. This fundamental problem in bifurcation theory and computational fluid dynamics has been the subject of well over 1500 papers. See Chandrasekhar (1981), Chossat & Iooss (1993), Drazin & Reid (2004), Golubitsky, Stewart, & Schaeffer (1988), Joseph (1976), and Lin (1955) for introductions to the Taylor-Couette problem, and see Tagg (1992) for an extensive bibliography.

This paper treats a generalization of this problem in which the outer cylinder is deformable and its motion is not prescribed, but responds to the forces exerted on it by the moving liquid; the inner cylinder rotates at a constant angular velocity ω , driving the liquid, which in turn drives the outer cylinder. See Figure (1.1). We limit our attention here to motions in which the deformable cylinder remains cylindrical, although not necessarily a circular cylinder, and there is no motion of the fluid in the axial direction. Thus we can represent the system by a horizontal cross-section, reducing the problem to two spatial dimensions.

We assume that the deformable outer cylinder is viscoelastic. For simplicity of exposition we model it as a membrane rather than as a shell. Its motion is governed by a quasilinear parabolic-hyperbolic system of partial differential equations. See Section 2. The fluid motion is described by the Navier-Stokes equations. These are coupled to the equations for the deformable membrane through the adherence boundary condition and the traction condition. See Sections 3–5. Our formulation of this problem is an example of a semi-inverse problem of continuum mechanics, which involves, like many such problems, subtleties in specifying subsidiary conditions.

There exists a rigid Couette solution of the coupled fluid-solid system analogous to the Couette solution of the classical Taylor-Couette problem: The fluid streamlines are concentric circles and the deformable membrane rotates rigidly with the same angular velocity ω as the rigid inner cylinder. See Section 6.

We study the stability of the rigid Couette solution with respect to the prescribed angular velocity ω , which is taken to be the bifurcation parameter. Linearizing the equations of motion about the Couette solution and seeking normal mode solutions yields a non-self-adjoint quadratic eigenvalue problem. This can be thought of as a perturbation of the Stokes eigenvalue problem with complicated boundary conditions governing the displacement of the membrane; the eigenvalue parameter λ appears quadratically in the boundary terms. See Section 8. The eigenvalues λ are functions of the angular velocity ω . The behavior of the eigenvalues as ω is increased, in particular the way that they cross the imaginary axis, yields important information about the stability and

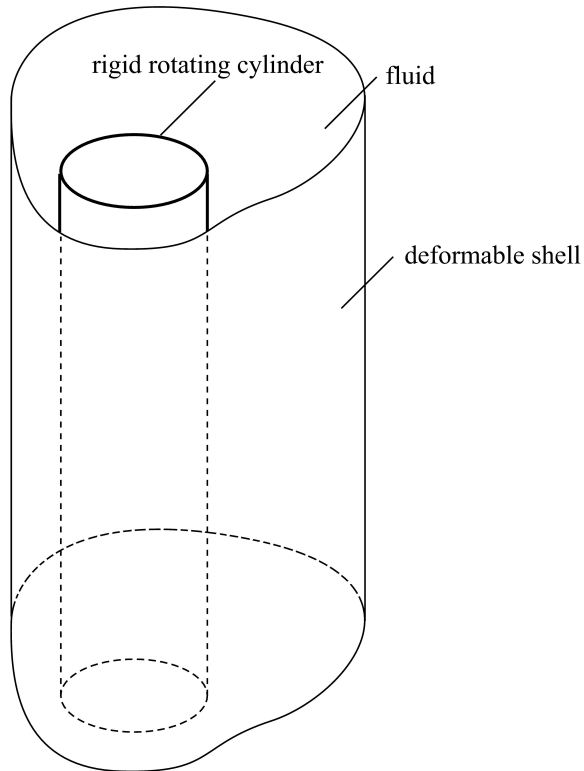


Figure 1.1: The Taylor-Couette problem for flow in a deformable cylinder.

structure of solutions to the fully nonlinear problem. To find out how the eigenvalues cross the imaginary axis requires a numerical study. This paper concludes with the numerical solution of the quadratic eigenvalue problem.

To numerically solve the quadratic eigenvalue problem we first derive a weak formulation of it. A careful choice of test functions must be made in order to eliminate the pressure boundary terms from the equations. From the weak formulation of the quadratic eigenvalue problem we compute the eigenvalues using a Fourier-finite element method: Fourier series are used to reduce the partial differential equations for the fluid on an annulus to ordinary differential equations in the radial variable r , which are discretized using the 1-dimensional finite element method with Taylor-Hood elements. These equations are coupled to algebraic equations for the membrane (obtained from the two-point boundary-value problem for the membrane using Fourier series). Employing the direct QZ eigensolver to solve the resulting matrix eigenvalue problem leads to a fast algorithm. This numerical method is outlined in Section 10 and described in full detail in Part II of the paper, where we also prove convergence. Our numerical results suggest that the Couette solution may lose its stability via a

Takens-Bogdanov bifurcation. See Section 10.

Other works on eigenvalue problems describing fluid-solid interactions include the following and the references cited therein: Bermúdez & Rodríguez (2002), Bermúdez et al. (1995, 2000a, 2000b, 2003a, 2003b, 2006), Conca & Durán (1995), Conca et al. (1992, 1998a, 1998b), Durán et al. (2000), Hamdi et al. (1978), Morand & Ohayon (1979), Schulkes (1992), Voss (2003, 2005), Wang & Bathe (1997). Note that in many of these papers the solid is just a rigid body. The novelty of this paper is that the equations describing the deformable body are nonlinear, we work with a broad class of constitutive functions, and the quadratic eigenvalue problem arises from the linearization of the governing equations about a time dependent state of the deformable body (a relative equilibrium) rather than just a static state.

We use Gibbs notation for vectors and tensors in which the value of a tensor (linear transformation) \mathbf{A} acting on a vector \mathbf{u} is denoted $\mathbf{A} \cdot \mathbf{u}$ and in which in which $\mathbf{ab} \cdot \mathbf{c} := (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$ for vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

2 Formulation of the Equations of Motion for a Cylindrical Membrane

Since the membrane is constrained to remain cylindrical, its motion is determined by that of a typical section. The equations of such a section are those of a string. In this section we summarize the theory of deformable strings from Antman (2005, Chap. 2).

Geometry of deformation. Let $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ be a right-handed orthonormal basis for Euclidean 3-space. For any angle χ we define the vectors

$$(2.1) \quad \mathbf{e}_1(\chi) := \cos \chi \mathbf{i} + \sin \chi \mathbf{j}, \quad \mathbf{e}_2(\chi) := -\sin \chi \mathbf{i} + \cos \chi \mathbf{j} \equiv \mathbf{k} \times \mathbf{e}_1(\chi).$$

The reference configuration of the string is a circle of radius 1, given parametrically by

$$(2.2) \quad \mathbf{r}^\circ(s) = \mathbf{e}_1(s).$$

The arc-length parameter $s \in [0, 2\pi]$ identifies material points of the string, with the points 0 and 2π identified. The position of material point s at time t is $\mathbf{r}(s, t)$. The curve $\mathbf{r}(\cdot, t)$ is assumed to lie in the $\{\mathbf{i}, \mathbf{j}\}$ -plane for each t . The stretch $\nu(s, t)$ is defined by

$$(2.3) \quad \nu(s, t) := |\mathbf{r}_s(s, t)| \equiv \sqrt{\mathbf{r}_s \cdot \mathbf{r}_s}.$$

Note that $\nu = 1$ in the reference configuration. Since ν measures the stretch of the string, i.e., the local ratio of deformed to reference length of the string, we stipulate that $\nu > 0$. We require that \mathbf{r} satisfy the periodicity conditions

$$(2.4) \quad \mathbf{r}(2\pi, t) = \mathbf{r}(0, t), \quad \mathbf{r}_s(2\pi, t) = \mathbf{r}_s(0, t).$$

We represent \mathbf{r} in polar coordinates by

$$(2.5) \quad \mathbf{r}(s, t) =: q(s, t) \mathbf{e}_1(\psi(s, t) + \omega t), \quad q(s, t) := |\mathbf{r}(s, t)|, \quad \psi(s, t) \in [0, 2\pi).$$

Mechanics. Let $\mathbf{n}(\xi, t)$ be the internal contact force exerted at the material point $s = \xi$ at time t . Let $\mathbf{f}(s, t)$ be the force per unit reference length exerted by the fluid on material point s of the string at time t . We give an expression for the force \mathbf{f} in Section 4. Let ρA be the mass density of the string per reference length, which is taken to be constant. The Linear Momentum Law for the string is

$$(2.6) \quad \rho A \mathbf{r}_{tt} = \mathbf{n}_s + \mathbf{f}.$$

(Equation (2.6) is derived by adding up all the forces on a segment $[s_1, s_2]$ of the string, $\mathbf{n}(s_2, t) - \mathbf{n}(s_1, t) + \int_{s_1}^{s_2} \mathbf{f}(s, t) ds$, and setting this equal to the rate of change of linear momentum of the segment, $\partial_t \int_{s_1}^{s_2} \rho A \mathbf{r}_t(s, t) ds$. Differentiating the resulting equation with respect to s_2 gives (2.6).)

We assume that the string can bend and stretch, but that it offers no resistance to bending, only to stretching. Thus the internal contact force $\mathbf{n}(s, t)$ is tangent to the string and has the form

$$(2.7) \quad \mathbf{n} =: N(s, t) \frac{\mathbf{r}_s}{|\mathbf{r}_s|}.$$

$N(s, t)$ is the *tension* at (s, t) .

Constitutive equations. We assume that the membrane is uniform so that ρA is constant and the constitutive function (defined below) is independent of s . The membrane is said to be *viscoelastic of strain-rate type* if there exists a function

$$(2.8) \quad \nu, \dot{\nu} \mapsto \hat{N}(\nu, \dot{\nu})$$

such that

$$(2.9) \quad N(s, t) = \hat{N}(\nu(s, t), \nu_t(s, t)).$$

The superposed dot on ν in (2.8) has no operational significance; it merely identifies the argument of the constitutive function that is to be occupied by the time derivative of ν . The constitutive function \hat{N} is the most general of the form $(\mathbf{r}, \mathbf{r}_s, \mathbf{r}_{st}) \mapsto \tilde{N}(\mathbf{r}, \mathbf{r}_s, \mathbf{r}_{st})$ that is invariant under rigid motion. We assume that \hat{N} has as many continuous derivatives as appear in our analysis. If \hat{N} is independent of $\dot{\nu}$, then the membrane is *elastic*.

To ensure that increases in strain and strain-rate are each accompanied by an increase in tension, we assume that \hat{N} satisfies the monotonicity conditions

$$(2.10) \quad \hat{N}_\nu > 0 \quad \text{and} \quad \hat{N}_{\dot{\nu}} \geq 0.$$

To ensure that an extreme strain be accompanied by an extreme tension, we assume that \hat{N} satisfies the blow-up conditions

$$(2.11) \quad \hat{N}(\nu, \dot{\nu}) \longrightarrow \begin{cases} +\infty \\ -\infty \end{cases} \quad \text{as} \quad \nu \longrightarrow \begin{cases} +\infty \\ 0 \end{cases}$$

for fixed values of $\dot{\nu}$. Finally, we make the nonrestrictive assumption that the tension vanishes when the body is at rest in the reference configuration:

$$(2.12) \quad \hat{N}(1, 0) = 0.$$

Thus the reference configuration of the membrane is a natural configuration.

3 Formulation of the Equations for the Fluid

Let $U(t)$ be the domain occupied by the fluid at time t . This is the region between the rigid cylinder of radius $a < 1$, which rotates at angular velocity ω , and the curve $\mathbf{r}(\cdot, t)$. We assume that the fluid is viscous, incompressible, and homogeneous. We denote by

- ϱ the constant density of the fluid,
- μ the constant dynamic viscosity of the fluid,
- γ the constant kinematic viscosity of the fluid, $\gamma = \mu/\varrho$,
- $\mathbf{v}(\mathbf{x}, t)$ the velocity of the fluid particle occupying position \mathbf{x} at time t ,
- $\varrho p(\mathbf{x}, t)$ the pressure on the fluid particle occupying position \mathbf{x} at time t ,
- $\boldsymbol{\Sigma}(\mathbf{v}, p)$ the Cauchy stress tensor.

(Note that the ϱ for the density of the fluid differs from the ρ appearing in ρA for the mass density of the membrane.) The symmetric part of the velocity gradient is

$$(3.1) \quad \mathbf{D}(\mathbf{v}) := \frac{1}{2} \left[\frac{\partial \mathbf{v}}{\partial \mathbf{x}} + \left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}} \right)^* \right]$$

where the asterisk denotes the transpose. We assume that the fluid is Newtonian, so that the Cauchy stress $\boldsymbol{\Sigma}$ has the Navier-Stokes form

$$(3.2) \quad \boldsymbol{\Sigma}(\mathbf{v}, p) = -\varrho p \mathbf{I} + 2\mu \mathbf{D}(\mathbf{v}).$$

The requirement that the fluid be incompressible is that $\nabla \cdot \mathbf{v} = 0$. In this case,

$$(3.3) \quad \frac{1}{\varrho} \nabla \cdot \boldsymbol{\Sigma} = -\nabla p + \gamma \Delta \mathbf{v},$$

so that the momentum equation for the fluid reduces to the Navier-Stokes equation subject to the incompressibility condition:

$$(3.4) \quad \begin{aligned} \mathbf{v}_t + \frac{\partial \mathbf{v}}{\partial \mathbf{x}} \cdot \mathbf{v} &= \frac{1}{\varrho} \nabla \cdot \boldsymbol{\Sigma} = -\nabla p + \gamma \Delta \mathbf{v} & \text{in } U(t), \\ \nabla \cdot \mathbf{v} &= 0 & \text{in } U(t). \end{aligned}$$

Polar coordinates. We assign polar coordinates (r, ϕ) to a typical fixed point \mathbf{x} in the $\{\mathbf{i}, \mathbf{j}\}$ -plane with respect to the basis $\{\mathbf{e}_1(\omega t), \mathbf{e}_2(\omega t)\}$ rotating with the rigid inner cylinder by

$$(3.5) \quad \mathbf{x} = r\mathbf{e}_1(\phi + \omega t) = r[\cos \phi \mathbf{e}_1(\omega t) + \sin \phi \mathbf{e}_2(\omega t)].$$

Note that ϕ depends on t since \mathbf{x} is fixed, but the basis is rotating. We write the fluid velocity \mathbf{v} in the form

$$(3.6) \quad \mathbf{v}(r\mathbf{e}_1(\phi + \omega t), t) =: u(r, \phi, t)\mathbf{e}_1(\phi + \omega t) + v(r, \phi, t)\mathbf{e}_2(\phi + \omega t).$$

The (transposed) gradient of \mathbf{v} is

$$(3.7) \quad \frac{\partial \mathbf{v}}{\partial \mathbf{x}} = [u_r \mathbf{e}_1 + v_r \mathbf{e}_2] \mathbf{e}_1 + \frac{1}{r} [u_\phi \mathbf{e}_1 + v_\phi \mathbf{e}_2 + u \mathbf{e}_2 - v \mathbf{e}_1] \mathbf{e}_2.$$

where the argument of \mathbf{e}_1 and \mathbf{e}_2 is $\phi + \omega t$. The substitution of (3.6) and (3.7) into (3.4) gives the Navier-Stokes equations in rotating polar coordinates:

$$(3.8) \quad \begin{aligned} u_t - \omega u_\phi + uu_r + \frac{vu_\phi}{r} - \frac{v^2}{r} &= -p_r + \gamma \left[u_{rr} + \frac{u_r}{r} + \frac{u_{\phi\phi}}{r^2} - \frac{2v_\phi}{r^2} - \frac{u}{r^2} \right], \\ v_t - \omega v_\phi + uv_r + \frac{vv_\phi}{r} + \frac{uv}{r} &= -\frac{p_\phi}{r} + \gamma \left[v_{rr} + \frac{v_r}{r} + \frac{v_{\phi\phi}}{r^2} + \frac{2u_\phi}{r^2} - \frac{v}{r^2} \right], \\ (ru)_r + v_\phi &= 0. \end{aligned}$$

4 The Coupling Between the Fluid and the Membrane

The adherence boundary condition for viscous fluids requires that the velocity of the fluid at a solid boundary equal the velocity of the boundary. For our problem, this condition implies that

$$(4.1) \quad u(a, \phi, t) = 0, \quad v(a, \phi, t) = a\omega,$$

$$(4.2) \quad \mathbf{v}(\mathbf{r}(s, t), t) = \mathbf{r}_t(s, t).$$

It follows from (2.5) and (3.6) that (4.2) is equivalent to

$$(4.3) \quad u(q, \psi, t) = q_t, \quad v(q, \psi, t) = q(\psi_t + \omega).$$

The second condition at a fluid-solid interface is that the traction be continuous across it (a version of the law of action and reaction). For our problem, this condition is accounted for by taking \mathbf{f} to be the force per unit reference length exerted by the fluid on the membrane. The outer unit normal to $\mathbf{r}(\cdot, t)$ is $\mathbf{r}_s \times \mathbf{k} / |\mathbf{r}_s|$. The definition of the Cauchy stress tensor says that the force per unit *actual* length exerted by the membrane on the fluid at $\mathbf{r}(s, t)$ is $\boldsymbol{\Sigma} \cdot (\mathbf{r}_s \times \mathbf{k}) / |\mathbf{r}_s|$.

Therefore the force per unit *reference* length exerted by the fluid on the membrane at $\mathbf{r}(s, t)$ is

$$\begin{aligned}
 (4.4) \quad \mathbf{f} &= -\boldsymbol{\Sigma} \cdot (\mathbf{r}_s \times \mathbf{k}) = \boldsymbol{\Sigma} \cdot (\mathbf{k} \times \mathbf{r}_s) = [-\varrho p \mathbf{I} + 2\mu \mathbf{D}(\mathbf{v})] \cdot (\mathbf{k} \times \mathbf{r}_s) \\
 &= -\varrho p (\mathbf{k} \times \mathbf{r}_s) \\
 &\quad + \mu \left[2u_r \mathbf{e}_1 \mathbf{e}_1 + \left(v_r + \frac{1}{q} u_\phi - \frac{1}{q} v \right) (\mathbf{e}_1 \mathbf{e}_2 + \mathbf{e}_2 \mathbf{e}_1) + \frac{2}{q} (v_\phi + u) \mathbf{e}_2 \mathbf{e}_2 \right] \cdot (\mathbf{k} \times \mathbf{r}_s)
 \end{aligned}$$

where \mathbf{e}_1 and \mathbf{e}_2 have argument $\psi(s, t) + \omega t$ and u , v , and p and their derivatives have arguments (q, ψ, t) .

Substituting equations (2.7) and (4.4) into the Linear Momentum Law for the membrane (2.6) yields

$$(4.5) \quad \rho A \mathbf{r}_{tt} = \left[\frac{\hat{N}}{\nu} \mathbf{r}_s \right]_s + \boldsymbol{\Sigma} \cdot (\mathbf{k} \times \mathbf{r}_s).$$

By using (2.5) and (4.4) we find that the $\mathbf{e}_1(\psi + \omega t)$ - and $\mathbf{e}_2(\psi + \omega t)$ -components of equation (4.5) are

$$\begin{aligned}
 (4.6) \quad &\rho A [q_{tt} - q(\psi_t + \omega)^2] \\
 &= \left[\frac{\hat{N}}{\nu} q_s \right]_s - \frac{\hat{N}}{\nu} q \psi_s^2 + \varrho p q \psi_s + \mu \left[-2u_r q \psi_s + \left(v_r + \frac{1}{q} u_\phi - \frac{1}{q} v \right) q_s \right], \\
 &\rho A [q \psi_{tt} + 2q_t(\psi_t + \omega)] \\
 &= \left[\frac{\hat{N}}{\nu} q \psi_s \right]_s + \frac{\hat{N}}{\nu} q_s \psi_s - \varrho p q_s + \mu \left[\frac{2}{q} (v_\phi + u) q_s - \left(v_r + \frac{1}{q} u_\phi - \frac{1}{q} v \right) q \psi_s \right].
 \end{aligned}$$

5 The Area Side Condition

We assume that the fluid completely fills the region between the rigid cylinder and the membrane, so that there are no cavities. Since the fluid is incompressible, the cross-sectional area of the fluid must be a constant, which we denote by A . Fix $R > 1$. We choose $A = \pi(R^2 - a^2)$, the area of the annulus $\{a < |\mathbf{x}| < R\}$. This choice is motivated by the form of the Couette steady solution. See Section 6. Green's Theorem in the Plane implies that the cross-sectional area πR^2 enclosed by the membrane can be written as

$$(5.1) \quad \pi R^2 = \frac{1}{2} \mathbf{k} \cdot \int_0^{2\pi} \mathbf{r}(s, t) \times \mathbf{r}_s(s, t) ds = \frac{1}{2} \int_0^{2\pi} q^2 \psi_s ds.$$

The parameter R is at our disposal. Prescribing R is equivalent to prescribing the area of the fluid. (We could alternatively replace the prescription of R with that of the compressive force at the ends $z = \pm\infty$ of the cylinder.) Our equations determine a definite pressure field. (In contrast, in problems involving an incompressible fluid in a domain with fixed boundary the pressure of the fluid

is determined only up to a constant.) We do not pause to give readily obtained conditions ensuring that this field is everywhere positive in the fluid.

Note that every solution of our governing equations belongs to a family of solutions that differ only by a constant rigid rotation about \mathbf{k} . The rotation can be effected by a relabelling of the coordinates or by a time shift. We identify all such solutions.

6 The Couette Steady Solution

In this section we find a rigid Couette steady solution. The symmetry of our problem suggests that we seek a steady solution in which the membrane is circular and rotates rigidly with constant angular velocity Ω , and the fluid streamlines are concentric circles. Thus in terms of the polar coordinates introduced in Section 3, we seek solutions of the form

$$(6.1) \quad u(r, \phi, t) = 0, \quad v(r, \phi, t) = V(r), \quad p(r, \phi, t) = P(r),$$

$$(6.2) \quad \mathbf{r}(s, t) = R\mathbf{e}_1(s + \Omega t).$$

Note that \mathbf{r} satisfies the side condition (5.1). $R > 1$ is the radius of the circle formed by the string. (Recall that the natural state of the string is a circle of radius 1.) In the notation introduced in equations (2.3) and (2.5),

$$(6.3) \quad \nu = |\mathbf{r}_s| = R, \quad q = |\mathbf{r}| = R, \quad \psi(s, t) = s + (\Omega - \omega)t.$$

The substitution of (6.1) into the Navier-Stokes equations (3.8) yields

$$(6.4) \quad P_r = \frac{V^2}{r}, \quad V_{rr} + \frac{V_r}{r} - \frac{V}{r^2} \equiv \left[V_r + \frac{V}{r} \right]_r \equiv \left[\frac{1}{r}(rV)_r \right]_r = 0.$$

Thus there are constants B, C, D such that

$$(6.5) \quad V(r) = Br + \frac{C}{r}, \quad P(r) = \frac{B^2 r^2}{2} + 2BC \ln r - \frac{C^2}{2r^2} + D.$$

The adherence conditions (4.1), (4.2) imply that

$$(6.6) \quad a\omega = Ba + \frac{C}{a}, \quad R\Omega = BR + \frac{C}{R} \quad \Longleftrightarrow \quad B = \frac{R^2\Omega - a^2\omega}{R^2 - a^2}, \quad C = \frac{R^2a^2(\omega - \Omega)}{R^2 - a^2}.$$

We must obtain equations for the unknown constants Ω and D in terms of the prescribed parameters. By substituting (6.1) and (6.2) into the equation (4.5) for the membrane we find that

$$(6.7) \quad \rho A \Omega^2 R \mathbf{e}_1(s + \Omega t) = \hat{N}(R, 0) \mathbf{e}_1(s + \Omega t) - \varrho R P(R) \mathbf{e}_1(s + \Omega t) - \frac{2C\mu}{R} \mathbf{e}_2(s + \Omega t).$$

Taking the inner product of (6.7) with $\mathbf{e}_2(s + \Omega t)$ yields

$$(6.8) \quad C = 0 \quad \implies \quad \Omega = \omega, \quad B = \omega.$$

Therefore the fluid and the membrane rotate rigidly with the same angular velocity as the rigid cylinder. The system behaves like a rigid body. We call this the rigid Couette solution.

An expression for D can be obtained by taking the inner product of equation (6.7) with $\mathbf{e}_1(s + \Omega t)$. By (6.8), formulas (6.5) for V and P reduce to the simple forms

$$(6.9) \quad V = \omega r, \quad P(r) = \frac{1}{2}\omega^2 r^2 + D,$$

where

$$(6.10) \quad D = D(R, \omega^2) = \frac{\hat{N}(R, 0)}{\varrho R} - \frac{\rho A \omega^2}{\varrho} - \frac{\omega^2 R^2}{2}.$$

Observe that

$$(6.11) \quad P(R) = P(R, \omega^2) = \frac{\hat{N}(R, 0)}{\varrho R} - \frac{\rho A \omega^2}{\varrho},$$

i.e., the pressure at the liquid-solid interface is the balance of the tension in the membrane and the centrifugal force.

In the rest of this paper we analyze how the stability of this (rigid) Couette steady solution depends on ω .

When $\omega = 0$,

$$(6.12) \quad \mathbf{r}(s) = R\mathbf{e}_1(s) + \mathbf{c}, \quad \mathbf{v} = \mathbf{0}, \quad p = D$$

is a steady solution for any $\mathbf{c} \in \mathbb{R}^2$ with $|\mathbf{c}| < R - a$. This is an off-center solution: the rigid cylinder and circular membrane are not concentric. We shall see in Section 9 that this gives rise to an instability.

7 Linearization

To study the linear stability of the Couette steady solution, we first introduce the small parameter ε and perturbation variables, decorated with a superscript 1, by

$$(7.1) \quad \begin{aligned} u(r, \phi, t; \varepsilon) &= 0 + \varepsilon u^1(r, \phi, t) + \mathcal{O}(\varepsilon^2), \\ v(r, \phi, t; \varepsilon) &= \omega r + \varepsilon v^1(r, \phi, t) + \mathcal{O}(\varepsilon^2), \\ p(r, \phi, t; \varepsilon) &= P(r) + \varepsilon p^1(r, \phi, t) + \mathcal{O}(\varepsilon^2), \\ q(s, t; \varepsilon) &= R + \varepsilon q^1(s, t) + \mathcal{O}(\varepsilon^2), \\ \psi(s, t; \varepsilon) &= s + \varepsilon \psi^1(s, t) + \mathcal{O}(\varepsilon^2), \\ \nu(s, t; \varepsilon) &= R + \varepsilon \nu^1(s, t) + \mathcal{O}(\varepsilon^2). \end{aligned}$$

The parameter ε may be regarded as measuring the discrepancy between initial conditions near those for the Couette rigid solution and those for this solution.

We linearize the evolution equations by substituting (7.1) into them, differentiating the resulting equations with respect to ε , and then setting $\varepsilon = 0$.

The linearization of the kinematic relation (2.3) yields

$$(7.2) \quad \nu^1 = q^1 + R\psi_s^1.$$

Define

$$(7.3) \quad N^\circ := \hat{N}(R, 0), \quad N_\nu^\circ := \hat{N}_\nu(R, 0), \quad N_{\dot{\nu}}^\circ := \hat{N}_{\dot{\nu}}(R, 0).$$

Linearizing the components of the membrane equation (4.6) requires care because a fluid-dynamical variable like p has values $p(q(s, t; \varepsilon), \psi(s, t; \varepsilon), t; \varepsilon)$. Thus

$$(7.4) \quad \partial_\varepsilon p(q(s, t; \varepsilon), \psi(s, t; \varepsilon), t; \varepsilon)|_{\varepsilon=0} = \omega^2 R q^1(s, t) + p^1(R, s, t),$$

with similar formulas for u and v and their derivatives. Therefore system (4.6) has linearization

$$(7.5) \quad \begin{aligned} \rho A (q_{tt}^1 - \omega^2 q^1 - 2\omega R \psi_t^1) = & - (N_\nu^\circ - \frac{1}{R} N^\circ) \nu^1 - N_\nu^\circ \nu_t^1 \\ & + \frac{1}{R} N^\circ (q_{ss}^1 - q^1 - 2R \psi_s^1) + \varrho P(R, \omega^2) (R \psi_s^1 + q^1) \\ & + \varrho R^2 \omega^2 q^1 + R \varrho p^1 - 2R \mu u_r^1, \end{aligned}$$

$$(7.6) \quad \begin{aligned} \rho A (R \psi_{tt}^1 + 2\omega q_t^1) = & (N_\nu^\circ - \frac{1}{R} N^\circ) \nu_s^1 + N_\nu^\circ \nu_{st}^1 + \frac{1}{R} N^\circ (R \psi_{ss}^1 + 2q_s^1) \\ & - \varrho P(R, \omega^2) q_s^1 - R \mu (v_r^1 - \frac{1}{R} v^1 + \frac{1}{R} u_\phi^1), \end{aligned}$$

where the fluid variables u^1, v^1, p^1 have arguments (R, s, t) .

Linearizing the Navier-Stokes equations (3.8) about the Couette solution gives

$$(7.7) \quad \begin{aligned} u_t^1 - 2\omega v^1 = & -p_r^1 + \gamma \left(u_{rr}^1 + \frac{u_r^1}{r} + \frac{u_{\phi\phi}^1}{r^2} - \frac{2v_\phi^1}{r^2} - \frac{u^1}{r^2} \right), \\ v_t^1 + 2\omega u^1 = & -\frac{p_\phi^1}{r} + \gamma \left(v_{rr}^1 + \frac{v_r^1}{r} + \frac{v_{\phi\phi}^1}{r^2} + \frac{2u_\phi^1}{r^2} - \frac{v^1}{r^2} \right), \\ (ru^1)_r + v_\phi^1 = & 0. \end{aligned}$$

The domain of the linearized equations (7.7) is fixed (unlike that for the non-linear equations (3.8)) and is $\{(r, \phi) \in [a, R] \times [0, 2\pi)\}$.

The linearization of the adherence boundary conditions (4.1) and (4.3) yields

$$(7.8) \quad u^1(a, \phi, t) = 0, \quad v^1(a, \phi, t) = 0,$$

$$(7.9) \quad q_t^1(s, t) = u^1(R, s, t), \quad R \psi_t^1(s, t) = v^1(R, s, t).$$

Linearizing the area side condition (5.1) gives

$$(7.10) \quad \int_0^{2\pi} q^1(s, t) ds = 0.$$

8 The Quadratic Eigenvalue Problem

We seek solutions of the linearized equations with an exponential time-dependence:

$$(8.1) \quad u^1(r, \phi, t) = u(r, \phi) e^{\lambda t}, \quad q^1(s, t) = q(s) e^{\lambda t}, \quad \text{etc.}$$

Note that the symbols u, v, p, q, ψ, ν have meanings here different from those in the previous sections and that they can assume complex values.

We substitute (8.1) into the equations of Section 7, replacing every ∂_t with λ , to obtain a quadratic eigenvalue problem. We refer to the thus modified equations by the same number. The eigenvalue problem is a perturbation of the Stokes eigenvalue problem (obtained by setting $\omega = 0$ in (7.7)), but with complicated boundary conditions governing the motion of the membrane; the eigenvalue parameter λ appears quadratically in the boundary conditions for the fluid.

Vectorial equations. For the analysis in the subsequent sections it is convenient to write the quadratic eigenvalue problem in a vectorial form. Define $\tilde{\mathbf{v}}(\mathbf{x})$, $\tilde{p}(\mathbf{x})$ and $\tilde{\mathbf{r}}(s)$ by

$$(8.2) \quad \tilde{\mathbf{v}}(r\mathbf{e}_1(\phi)) := u(r, \phi) \mathbf{e}_1(\phi) + v(r, \phi) \mathbf{e}_2(\phi), \quad \tilde{p}(r\mathbf{e}_1(\phi)) := p(r, \phi),$$

$$(8.3) \quad \tilde{\mathbf{r}}(s) := q(s) \mathbf{e}_1(s) + R\psi(s) \mathbf{e}_2(s).$$

We now drop the tildes from these variables. Equations (8.2) and (8.3) can be used to write the quadratic eigenvalue problem (7.5)–(7.10) in the vectorial form

$$(8.4) \quad \begin{aligned} \lambda^2 \rho A \mathbf{r} &= \lambda [N_\nu^\circ (\mathbf{e}_2 \mathbf{e}_2 \cdot \mathbf{r}_s)_s + 2\rho A \omega \mathbf{r} \times \mathbf{k}] + \frac{1}{R} N^\circ \mathbf{r}_{ss} \\ &+ (N_\nu^\circ - \frac{1}{R} N^\circ) (\mathbf{e}_2 \mathbf{e}_2 \cdot \mathbf{r}_s)_s - \varrho P(R, \omega^2) \mathbf{k} \times \mathbf{r}_s + \rho A \omega^2 \mathbf{r} \\ &+ \varrho R^2 \omega^2 (\mathbf{r} \cdot \mathbf{e}_1) \mathbf{e}_1 - R \boldsymbol{\Sigma}(\mathbf{v}, p) \cdot \mathbf{e}_1, \end{aligned}$$

$$(8.5) \quad \lambda \mathbf{v} = \frac{1}{\varrho} \operatorname{div} \boldsymbol{\Sigma}(p, \mathbf{v}) - 2\omega \mathbf{k} \times \mathbf{v}, \quad \nabla \cdot \mathbf{v} = 0,$$

$$(8.6) \quad \mathbf{v} = \mathbf{0} \quad \text{for } |\mathbf{x}| = a,$$

$$(8.7) \quad \mathbf{v}(R\mathbf{e}_1(s)) = \lambda \mathbf{r}(s),$$

$$(8.8) \quad \int_0^{2\pi} \mathbf{r}(s) \cdot \mathbf{e}_1(s) ds = 0.$$

Equations (8.4) and (8.7) have domain $s \in [0, 2\pi)$ and equation (8.5) has domain $a < |\mathbf{x}| < R$. It is easy to check that the substitution of (8.2) and (8.3) into (8.4)–(8.8) yields (7.5)–(7.10) (with every time derivative ∂_t replaced by λ). Note that the stretch variable ν does not appear in equations (8.4)–(8.8); we have written ν in terms of \mathbf{r} using equations (7.2) and (8.3): $\nu = \mathbf{r}_s \cdot \mathbf{e}_2$.

9 Analysis of the Spectrum

If all the eigenvalues λ of (8.4)–(8.8) satisfy $\text{Re}(\lambda) < 0$, then all the perturbations \mathbf{v}^1 , \mathbf{r}^1 , p^1 decay exponentially in time and the rigid Couette solution is linearly stable. On the other hand, if an eigenvalue satisfies $\text{Re}(\lambda) > 0$, then the perturbation corresponding to this eigenvalue grows exponentially in time, in which case the rigid Couette solution is linearly unstable.

Eigenvalue crossings. We now study how the eigenvalues λ cross the imaginary axis as the control parameter ω is varied. Let U denote the annulus $\{\mathbf{x} : a < |\mathbf{x}| < R\}$. We denote the complex conjugate by a superposed bar.

Lemma 9.1 (Energy equation). *A smooth eigenpair $(\lambda, (\mathbf{v}, \mathbf{r}, p))$ of (8.4)–(8.8) satisfies the energy equation*

$$\begin{aligned} \text{Re}(\lambda) & \left(\|\mathbf{v}\|_{L^2(U)}^2 + \frac{N^\circ}{\varrho R} \|\mathbf{r}_s \cdot \mathbf{e}_1\|_{L^2(0,2\pi)}^2 + \frac{N_\nu^\circ}{\varrho} \|\mathbf{r}_s \cdot \mathbf{e}_2\|_{L^2(0,2\pi)}^2 \right. \\ & \quad \left. + \frac{\rho A}{\varrho} |\lambda|^2 \|\mathbf{r}\|_{L^2(0,2\pi)}^2 - \frac{\rho A \omega^2}{\varrho} \|\mathbf{r}\|_{L^2(0,2\pi)}^2 - R^2 \omega^2 \|\mathbf{r} \cdot \mathbf{e}_1\|_{L^2(0,2\pi)}^2 \right) \\ & = -\frac{N_\nu^\circ}{\varrho} |\lambda|^2 \|\mathbf{r}_s \cdot \mathbf{e}_2\|_{L^2(0,2\pi)}^2 - \frac{2\mu}{\varrho} \|\mathbf{D}(\mathbf{v})\|_{L^2(U)}^2 \\ & \quad - P(R, \omega^2) \text{Re}(\lambda) \int_0^{2\pi} (\mathbf{k} \times \mathbf{r}_s) \cdot \bar{\mathbf{r}} \, ds. \end{aligned}$$

Proof. Let $\boldsymbol{\nu}$ be the unit outer normal to ∂U . We take the inner product of (8.5)₁ with $\bar{\mathbf{v}}$ and integrate by parts to obtain

$$\begin{aligned} \lambda \|\mathbf{v}\|_{L^2(U)}^2 & = \frac{1}{\varrho} \int_U \text{div } \boldsymbol{\Sigma} \cdot \bar{\mathbf{v}} \, d\mathbf{x} - 2\omega \int_U (\mathbf{k} \times \mathbf{v}) \cdot \bar{\mathbf{v}} \, d\mathbf{x} \\ & = \frac{1}{\varrho} \int_{\{|\mathbf{x}|=R\}} \boldsymbol{\nu} \cdot \boldsymbol{\Sigma} \cdot \bar{\mathbf{v}} \, dS - \frac{1}{\varrho} \int_U \boldsymbol{\Sigma} : \frac{\partial \bar{\mathbf{v}}}{\partial \mathbf{x}} \, d\mathbf{x} - 2\omega \int_U (\mathbf{k} \times \mathbf{v}) \cdot \bar{\mathbf{v}} \, d\mathbf{x} \\ & = \frac{1}{\varrho} \int_0^{2\pi} \mathbf{e}_1 \cdot \boldsymbol{\Sigma} \cdot \bar{\mathbf{v}} \, R ds - \frac{2\mu}{\varrho} \|\mathbf{D}(\mathbf{v})\|_{L^2(U)}^2 - 2\omega \int_U (\mathbf{k} \times \mathbf{v}) \cdot \bar{\mathbf{v}} \, d\mathbf{x}, \end{aligned}$$

where we have used the adherence boundary condition $\mathbf{v} = \mathbf{0}$ on $\{|\mathbf{x}| = a\}$, the incompressibility condition $\text{div } \bar{\mathbf{v}} = 0$, and the identity $\mathbf{D}(\mathbf{v}) : \partial \bar{\mathbf{v}} / \partial \mathbf{x} = |\mathbf{D}(\mathbf{v})|^2$. We use (8.7) and (8.4) to write the boundary term in (9.3) as

$$\begin{aligned} (9.4) \quad & \frac{1}{\varrho} \int_0^{2\pi} \mathbf{e}_1 \cdot \boldsymbol{\Sigma} \cdot \bar{\mathbf{v}} \, R ds = \frac{1}{\varrho} \int_0^{2\pi} \mathbf{e}_1 \cdot \boldsymbol{\Sigma} \cdot \bar{\lambda} \bar{\mathbf{v}} \, R ds \\ & = -\frac{N_\nu^\circ}{\varrho} |\lambda|^2 \|\mathbf{r}_s \cdot \mathbf{e}_2\|_{L^2(0,2\pi)}^2 + \frac{2\rho A \omega}{\varrho} |\lambda|^2 \int_0^{2\pi} (\mathbf{r} \times \mathbf{k}) \cdot \bar{\mathbf{r}} \, ds - \frac{N^\circ}{\varrho R} \bar{\lambda} \|\mathbf{r}_s\|_{L^2(0,2\pi)}^2 \\ & \quad - \frac{(N_\nu^\circ - R^{-1} N^\circ)}{\varrho} \bar{\lambda} \|\mathbf{r}_s \cdot \mathbf{e}_2\|_{L^2(0,2\pi)}^2 - P(R, \omega^2) \bar{\lambda} \int_0^{2\pi} (\mathbf{k} \times \mathbf{r}_s) \cdot \bar{\mathbf{r}} \, ds \\ & \quad + \frac{\rho A \omega^2}{\varrho} \bar{\lambda} \|\mathbf{r}\|_{L^2(0,2\pi)}^2 + R^2 \omega^2 \bar{\lambda} \|\mathbf{r} \cdot \mathbf{e}_1\|_{L^2(0,2\pi)}^2 - \frac{\rho A}{\varrho} \lambda |\lambda|^2 \|\mathbf{r}\|_{L^2(0,2\pi)}^2. \end{aligned}$$

By substituting (9.4) into (9.3), taking the real part of the resulting equation, and simplifying, we complete the proof. \square

Proposition 9.5 (Eigenvalue crossings). *Eigenvalue problem (8.4)–(8.8) has no nonzero eigenvalues λ on the imaginary axis.*

Proof. By substituting $\text{Re}(\lambda) = 0$ into (9.2) we find that $\|\mathbf{D}(\mathbf{v})\|_{L^2(U)} = 0$, which implies that $\mathbf{v} = \mathbf{0}$ by the Korn and Poincaré inequalities. But $\mathbf{v} = \mathbf{0}$ implies that $\lambda = 0$ by equation (8.7). \square

Proposition 9.6 (Eigenvalue-free regions of \mathbb{C}). *Define*

$$(9.7) \quad M(\omega^2) := \frac{1}{\rho A} (\rho A \omega^2 + \varrho R^2 \omega^2 + \frac{1}{4} \varrho P(R, \omega^2)^2 \max\{R/N^\circ, 1/N_\nu^\circ\}).$$

Eigenvalue problem (8.4)–(8.8) has no eigenvalues λ in the set

$$(9.8) \quad \{\lambda \in \mathbb{C} : |\lambda|^2 \geq M(\omega^2), \text{Re}(\lambda) > 0\} \cup \{\lambda \in \mathbb{C} : \text{Re}(\lambda) = 0, \lambda \neq 0\}.$$

Note that M is an increasing function of ω^2 and that

$$(9.9) \quad M(0) = \frac{\varrho P(R, 0)^2 \max\{R/N^\circ, 1/N_\nu^\circ\}}{4\rho A} = \frac{N^\circ{}^2 \max\{R/N^\circ, 1/N_\nu^\circ\}}{4\rho A \varrho R^2} \rightarrow 0 \text{ as } R \rightarrow 1$$

since $N^\circ = \hat{N}(R, 0) \rightarrow \hat{N}(1, 0) = 0$ and $N_\nu^\circ > 0$.

Proof of Proposition 9.6. By rearranging the energy equality in Lemma 9.1 we obtain

$$(9.10) \quad \begin{aligned} 0 \geq \text{Re}(\lambda) & \left(\|\mathbf{v}\|_{L^2(U)}^2 + \frac{N^\circ}{\varrho R} \|\mathbf{r}_s \cdot \mathbf{e}_1\|_{L^2(0, 2\pi)}^2 + \frac{N_\nu^\circ}{\varrho} \|\mathbf{r}_s \cdot \mathbf{e}_2\|_{L^2(0, 2\pi)}^2 \right. \\ & + \frac{\rho A}{\varrho} |\lambda|^2 \|\mathbf{r}\|_{L^2(0, 2\pi)}^2 - \frac{\rho A \omega^2}{\varrho} \|\mathbf{r}\|_{L^2(0, 2\pi)}^2 \\ & \left. - R^2 \omega^2 \|\mathbf{r} \cdot \mathbf{e}_1\|_{L^2(0, 2\pi)}^2 + P(R, \omega^2) \int_0^{2\pi} (\mathbf{k} \times \mathbf{r}_s) \cdot \bar{\mathbf{r}} \, ds \right). \end{aligned}$$

Let $\epsilon > 0$. Using the Cauchy-Bunyakovskii-Schwarz inequality and the elementary inequality $ab \leq \epsilon a^2 + b^2/4\epsilon$ we obtain

$$(9.11) \quad \int_0^{2\pi} (\mathbf{k} \times \mathbf{r}_s) \cdot \bar{\mathbf{r}} \, ds \geq -\epsilon \|\mathbf{r}_s\|_{L^2(0, 2\pi)}^2 - \frac{1}{4\epsilon} \|\mathbf{r}\|_{L^2(0, 2\pi)}^2.$$

By substituting (9.11) into (9.10) and writing

$$(9.12) \quad \|\mathbf{r}\|_{L^2(0, 2\pi)}^2 = \|\mathbf{r} \cdot \mathbf{e}_1\|_{L^2(0, 2\pi)}^2 + \|\mathbf{r} \cdot \mathbf{e}_2\|_{L^2(0, 2\pi)}^2,$$

we obtain the estimate

$$(9.13) \quad \begin{aligned} 0 \geq \text{Re}(\lambda) & (\|\mathbf{v}\|_{L^2(U)}^2 + c_1 \|\mathbf{r}_s \cdot \mathbf{e}_1\|_{L^2(0, 2\pi)}^2 + c_2 \|\mathbf{r}_s \cdot \mathbf{e}_2\|_{L^2(0, 2\pi)}^2 \\ & + c_3 \|\mathbf{r} \cdot \mathbf{e}_1\|_{L^2(0, 2\pi)}^2 + c_4 \|\mathbf{r} \cdot \mathbf{e}_2\|_{L^2(0, 2\pi)}^2) \end{aligned}$$

where

$$\begin{aligned} c_1 &= \frac{N^\circ}{\varrho R} - \epsilon |P(R, \omega^2)|, & c_2 &= \frac{N_\nu^\circ}{\varrho} - \epsilon |P(R, \omega^2)|, \\ c_3 &= \frac{\rho A |\lambda|^2}{\varrho} - \frac{\rho A \omega^2}{\varrho} - R^2 \omega^2 - \frac{|P(R, \omega^2)|}{4\epsilon}, \\ c_4 &= \frac{\rho A |\lambda|^2}{\varrho} - \frac{\rho A \omega^2}{\varrho} - \frac{|P(R, \omega^2)|}{4\epsilon}. \end{aligned}$$

The choice of $\epsilon = \min\{R^{-1}N^\circ, N_\nu^\circ\}/\varrho |P(R, \omega^2)|$ ensures that c_1 and c_2 are nonnegative and that $c_3 \geq 0$ if and only if $|\lambda|^2 \geq M(\omega^2)$. Note that $c_4 > c_3$.

Let λ be an eigenvalue of (8.4)–(8.8) with $|\lambda|^2 \geq M(\omega^2)$. Then c_1, c_2, c_3 , and c_4 are nonnegative and so $\operatorname{Re}(\lambda) \leq 0$ by inequality (9.13). Therefore there are no eigenvalues λ in the set $\{\lambda \in \mathbb{C} : |\lambda|^2 \geq M(\omega^2), \operatorname{Re}(\lambda) > 0\}$. We had proved that there are no eigenvalues in the set $\{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) = 0, \lambda \neq 0\}$ in Theorem 9.5. \square

Critical values of ω . Although we have shown that there are no nonzero eigenvalues λ on the imaginary axis, we have yet to show that there are values ω_{crit} of ω for which $\lambda = 0$ is indeed an eigenvalue. To show this we substitute $\lambda = 0$ and $\omega = \omega_{\text{crit}}$ into (8.4)–(8.8) to obtain an eigenvalue problem for ω_{crit} :

$$(9.14) \quad \begin{aligned} \mathbf{0} &= \frac{1}{R} N^\circ \mathbf{r}_{ss} + (N_\nu^\circ - \frac{1}{R} N^\circ)(\mathbf{e}_2 \mathbf{e}_2 \cdot \mathbf{r}_s)_s - \varrho P(R, \omega_{\text{crit}}^2) \mathbf{k} \times \mathbf{r}_s \\ &\quad + \rho A \omega_{\text{crit}}^2 \mathbf{r} + \varrho R^2 \omega_{\text{crit}}^2 (\mathbf{r} \cdot \mathbf{e}_1) \mathbf{e}_1 - R \boldsymbol{\Sigma}(\mathbf{v}, p) \cdot \mathbf{e}_1, \end{aligned}$$

$$(9.15) \quad \mathbf{0} = \frac{1}{\varrho} \operatorname{div} \boldsymbol{\Sigma}(p, \mathbf{v}) - 2\omega_{\text{crit}} \mathbf{k} \times \mathbf{v}, \quad \nabla \cdot \mathbf{v} = 0,$$

$$(9.16) \quad \mathbf{v} = \mathbf{0} \quad \text{for } |\mathbf{x}| = a, R,$$

$$(9.17) \quad \int_0^{2\pi} \mathbf{r}(s) \cdot \mathbf{e}_1(s) ds = 0.$$

The fluid equations are now uncoupled from the membrane equation. Clearly $\mathbf{v} = \mathbf{0}$ and $p = \text{constant}$ satisfy (9.15) and (9.16), and a simple energy estimate shows that this is the only solution.

In the membrane equations (7.5) and (7.6) we set $u = v = 0$, $p = \text{constant}$, $\omega = \omega_{\text{crit}}$, $P(R, \omega^2) = (\frac{1}{R} N^\circ - \rho A \omega_{\text{crit}}^2)/\varrho$, set all t -derivatives equal to 0 (corresponding to $\lambda = 0$) and drop all superscripts 1 to obtain

$$(9.18) \quad \begin{aligned} \frac{1}{R} N^\circ q_{ss} + (\frac{1}{R} N^\circ - N_\nu^\circ + \varrho R^2 \omega_{\text{crit}}^2) q - R(N_\nu^\circ + \rho A \omega_{\text{crit}}^2) \psi_s &= -R \varrho p, \\ R N_\nu^\circ \psi_{ss} + (N_\nu^\circ + \rho A \omega_{\text{crit}}^2) q_s &= 0. \end{aligned}$$

We integrate (9.18)₁ over $[0, 2\pi]$ and use (7.10) and the periodicity of q_s and ψ to see that $p = 0$. Let q_k and ψ_k be the Fourier coefficients of q and ψ :

$$(9.19) \quad q(s) = \sum_{k \in \mathbb{Z}} q_k e^{iks}, \quad \psi(s) = \sum_{k \in \mathbb{Z}} \psi_k e^{iks}.$$

The substitution of (9.19) into (9.18) yields a family of matrix equations indexed by $k \in \mathbb{Z}$:

$$(9.20) \quad \begin{bmatrix} \frac{1}{R}N^\circ(1-k^2) - N_\nu^\circ + \varrho R^2\omega_{\text{crit}}^2 & -ikR(N_\nu^\circ + \rho A\omega_{\text{crit}}^2) \\ ik(N_\nu^\circ + \rho A\omega_{\text{crit}}^2) & -k^2RN_\nu^\circ \end{bmatrix} \begin{bmatrix} q_k \\ \psi_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The matrix on the left-hand side of (9.20) has determinant

$$(9.21) \quad \delta_k := -Rk^2[(\rho A)^2\omega_{\text{crit}}^4 + N_\nu^\circ(2\rho A + \varrho R^2)\omega_{\text{crit}}^2 - \frac{1}{R}N^\circ N_\nu^\circ(k^2 - 1)].$$

If $\delta_k = 0$, then (9.20) has nontrivial solutions and so ω_{crit} is an eigenvalue of (9.14)–(9.17), and $\lambda = 0$ is an eigenvalue of (8.4)–(8.8) when $\omega = \omega_{\text{crit}}$. If $k = 0$, then $\delta_0 = 0$ for all $\omega_{\text{crit}} \in \mathbb{R}$. If $|k| = 1$, then $\delta_{\pm 1} = 0$ if and only if $\omega_{\text{crit}} = 0$. For $|k| \geq 2$, $\delta_k = 0$ if and only if

$$(9.22) \quad \begin{aligned} \omega_{\text{crit}}^2 &= \omega_{\text{crit}}^2(k) \\ &= -\frac{1}{2(\rho A)^2}N_\nu^\circ(2\rho A + \varrho R^2) \\ &\quad + \frac{1}{2(\rho A)^2}\{[N_\nu^\circ(2\rho A + \varrho R^2)]^2 + \frac{4}{R}(\rho A)^2N^\circ N_\nu^\circ(k^2 - 1)\}^{1/2}. \end{aligned}$$

We analyze each Fourier mode in turn. Since $\delta_0 = 0$ for all $\omega_{\text{crit}} \in \mathbb{R}$, it follows that $\lambda = 0$ is an eigenvalue of (8.4)–(8.8) for all $\omega \in \mathbb{R}$, with corresponding eigenvector $(\mathbf{v}, p, \mathbf{r}) = (0, 0, R\psi_0\mathbf{e}_2)$. (Note that $q_0 = 0$ by the area side condition (7.10).) This eigenvector, however, corresponds to a rigid rotation about \mathbf{k} of the Couette solution, and so we ignore it.

We saw that $\delta_{\pm 1} = 0$ when $\omega_{\text{crit}} = 0$, so that $\lambda = 0$ is an eigenvalue of (8.4)–(8.8) when $\omega = 0$. This suggests that the rigid Couette solution is linearly unstable for all $\omega > 0$ and so is not observable. (We expect the eigenvalue $\lambda = 0$ to move into the right half-plane when ω is increased from 0. Numerical results in Section 10 confirm this.) To understand how this instability occurs we compute the eigenvector of (9.14)–(9.17) corresponding to eigenvalue $\omega_{\text{crit}} = 0$. Substitute $\omega_{\text{crit}} = 0$ and $k = \pm 1$ into (9.20) to obtain

$$(9.23) \quad \begin{bmatrix} -N_\nu^\circ & \mp iRN_\nu^\circ \\ \pm iN_\nu^\circ & -RN_\nu^\circ \end{bmatrix} \begin{bmatrix} q_{\pm 1} \\ \psi_{\pm 1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which has nontrivial solutions $[q_{\pm 1}, \psi_{\pm 1}] = c[\mp iR, 1]$, c constant. Take $c = 1$ for now. Therefore the unstable perturbations have the form

$$(9.24) \quad \mathbf{r}_{\pm 1}^1(s) = [q_{\pm 1}\mathbf{e}_1(s) + R\psi_{\pm 1}\mathbf{e}_2(s)]e^{\pm is} = [\mp iR\mathbf{e}_1(s) + R\mathbf{e}_2(s)]e^{\pm is},$$

which have real parts

$$(9.25) \quad \text{Re}(\mathbf{r}_{\pm 1}^1) = R\sin(s)\mathbf{e}_1 + R\cos(s)\mathbf{e}_2 = R\mathbf{j}.$$

Thus the unstable perturbation is a translation of the circular cylindrical membrane in the \mathbf{j} direction. (Any other direction can be achieved by choosing the eigenvector scaling c appropriately.) This corresponds to the off-center steady

solution (6.12) and suggests that the rigid Couette solution becomes unstable through the following mechanism: Experimentally it is not possible to exactly align the center of the rigid cylinder with the center of the circular membrane. So when $\omega = 0$ we observe an off-center solution and not the rigid Couette solution. As ω is increased from 0 the misalignment of the centers will cause the membrane to move eccentrically and deform. Even if the membrane were rigid we would still expect the rigid Couette solution to be unstable; in this case the membrane would move eccentrically, but not deform. This type of instability could be avoided by introducing a suitable feedback control to fix the center of mass of the string at the origin. (Something like this is done for certain kinds of ball bearings.) We assume that this is done, and thereby give meaning to our subsequent analysis.

Finally, we consider the case $|k| \geq 2$. Equation (9.22) gives all the values of ω for which $\lambda = 0$ is an eigenvalue of problem (8.4)–(8.8). Observe that each Fourier mode k gives rise to exactly one unstable perturbation (only one eigenvalue crosses the imaginary axis for each k) and that the Fourier modes become unstable in order, i.e.,

$$(9.26) \quad 0 = \omega_{\text{crit}}^2(\pm 1) < \omega_{\text{crit}}^2(\pm 2) < \omega_{\text{crit}}^2(\pm 3) < \dots$$

Also, the critical values of ω do not depend on the viscosities μ and N_ν° . Since $N^\circ \rightarrow 0$ as $R \rightarrow 1$ (by equation (2.12)), we see from formula (9.22) that $\omega_{\text{crit}} \rightarrow 0$ as $R \rightarrow 1$ for all k . The behavior of ω_{crit} for large R depends on the material properties. In summary,

Theorem 9.27 (Critical values of ω). *$\lambda = 0$ is an eigenvalue of problem (8.4)–(8.8) whenever $\omega = \omega_{\text{crit}}(k)$, for $|k| = 1, 2, 3, \dots$, where*

$$\begin{aligned} \omega_{\text{crit}}^2(k) = & -\frac{1}{2(\rho A)^2} N_\nu^\circ (2\rho A + \varrho R^2) \\ & + \frac{1}{2(\rho A)^2} \{ [N_\nu^\circ (2\rho A + \varrho R^2)]^2 + \frac{4}{R} (\rho A)^2 N^\circ N_\nu^\circ (k^2 - 1) \}^{1/2}. \end{aligned}$$

These critical values of ω satisfy

- (i) $0 = \omega_{\text{crit}}^2(\pm 1) < \omega_{\text{crit}}^2(\pm 2) < \omega_{\text{crit}}^2(\pm 3) < \dots$,
- (ii) $\lim_{R \rightarrow 1} \omega_{\text{crit}}(k) = 0$.

In this section we have proved that all the eigenvalues λ of the quadratic eigenvalue problem that cross the imaginary axis cross through the origin, but have not proved anything about the way that they cross. For example, is the first eigenvalue to cross the imaginary axis real in a neighborhood of the origin, which would indicate a steady state bifurcation, or does something more complicated happen? To answer this question we perform a numerical study.

10 Computation of the Eigenvalues

Numerical method. We compute the eigenvalues using a Fourier-finite element method and the direct QZ eigensolver. In Part II of this paper the numerical method is described in detail. Here we just give the basic idea.

The first step is to derive a weak formulation of the quadratic eigenvalue problem (8.4)–(8.8). This is done in the standard way except that the test function \mathbf{w} for the Stokes-type equation (8.5) and the test function \mathbf{q} for the membrane equation (8.4) are related by $\mathbf{w}(R\mathbf{e}_1(s)) = \mathbf{q}(s)$ for all s , i.e., \mathbf{w} takes the value \mathbf{q} on the outer boundary of U . This ensures that the Cauchy stress tensor Σ (and therefore the pressure) does not appear in any of the boundary integrals. The adherence boundary condition (8.7) is enforced in the $H^{1/2}$ -inner product so that the resulting bilinear form satisfies the inf-sup conditions. The test function \mathbf{q} for the membrane equation satisfies (8.8), which ensures that the pressure of the fluid is determined uniquely (not just up to a constant), as it should be; adding a constant to the pressure would cause the deformable boundary to inflate.

As is standard for the finite element solution of the Stokes equations, we do not include the condition $\operatorname{div} \mathbf{v} = 0$ in the function space for the fluid velocity, but instead obtain a mixed weak formulation.

After deriving this weak formulation of the Cartesian equations (8.4)–(8.8) we rewrite it in polar coordinates. Then we introduce Fourier series in ϕ for the fluid variables and in s for the membrane variables and rewrite the weak formulation again to obtain a family of weak problems for the Fourier coefficients (we obtain one weak problem for each Fourier mode). This has the effect of reducing the partial differential equations for the fluid on the annulus U to ordinary differential equations in the radial variable r , and reducing the two-point boundary-value problem for the membrane to algebraic equations.

A 1-dimensional finite element method with Taylor-Hood elements discretizes the fluid equations, producing a matrix quadratic eigenvalue problem, which we solve using the MATLAB function *polyeig*. This function first reduces the quadratic eigenvalue problem to a generalized eigenvalue problem of the form $A\mathbf{x} = \lambda B\mathbf{x}$ by introducing new variables, the same way that a second order ordinary differential equation can be reduced to a pair of first-order ordinary differential equations. Then the generalized eigenvalue problem is solved using the QZ algorithm of Moler & Stewart (1973). Since the Fourier decomposition reduces the problem to one space dimension, the matrices are small and the algorithm is fast. It is also easy to parallelize: A different processor can be used for each Fourier mode.

Constitutive functions and material constants. Up until now we have been working with a broad class of constitutive functions. To compute the spectrum we must choose a constitutive function \hat{N} . We choose the simple

$$(10.1) \quad \hat{N}(\nu) = E\delta(\nu - 1),$$

where E is the modulus of elasticity and δ is the thickness of the membrane. Note that \hat{N} is linear in the strain variable ν , but not in the displacement \mathbf{r} . This constitutive relation does not penalize compression and so does not satisfy hypothesis (2.11). Since we only consider the linearization of \hat{N} about the stretched state $\nu = R$, however, we do not need an accurate model of the tension for materials under compression.

In addition to choosing a constitutive function we must also choose values for all the numerical constants. These are listed in Table (10.1). We chose the fluid to be water and the deformable body to be a soft, rubber-like material. The ratio of the radius of the inner cylinder to the radius of the outer cylinder is close to the value used by G.I. Taylor in his experiments on the classical Taylor-Couette problem in the 1920s.

Recall that the constant ρA is the mass density of the membrane per reference length, i.e., the mass density of the membrane integrated through its thickness in the $\{\mathbf{i}, \mathbf{j}\}$ -plane. If ρ is the mass density of a 3-dimensional rubber membrane then $\rho A = \rho(\delta - \frac{1}{2}\delta^2)$, where δ is the thickness. Since the thickness δ is presumed small we approximate ρA with $\rho\delta$. For the purpose of illustration, this approximation causes no trouble. The value of ρ is given in Table (10.1).

a	0.75 m
R	1.01 m
ϱ	1000 kg/m ³
μ	1.002×10^{-3} kg/ms
δ	$2\pi/1000$ m
ρ	920 kg/m ³
E	0.01 GPa

Table 10.1: The numerical constants used in the computation.

Results. Figures (10.1) and (10.2) show plots of the eigenvalues λ moving around the complex plane as ω is varied. In Section 9 we proved that all the eigenvalues that cross the imaginary axis must cross through the origin, but proved nothing about the way that they cross. From Figure (10.1) we see that, for each Fourier mode, a complex conjugate pair of eigenvalues coalesce at the origin and immediately split thereafter into a complex conjugate pair. Such a trajectory suggests a Takens-Bogdanov bifurcation, which occurs in the problem of fluid flow across an elastic panel, and corresponds to panel flutter. See Guckenheimer & Holmes (1983).

Accuracy check. Theorem 9.27 gives an exact formula for the critical values ω_{crit} of ω , which satisfy $\lambda(\omega_{\text{crit}}) = 0$. We can use this formula to check the accuracy of the code. The computed values of $\lambda(\omega_{\text{crit}})$ are given in Table (10.2) and are of the order of 10^{-11} .

For the Fourier mode $k = 0$ it is possible to use Bessel functions to reduce the quadratic eigenvalue problem to a nonlinear scalar equation for λ . This algebraic equation can then be solved by using the MATLAB function *fsolve*. See Bourne (2007) for details. Table (10.3) displays the eigenvalues computed with the finite element method with $N = 100$ mesh points (left column) against those

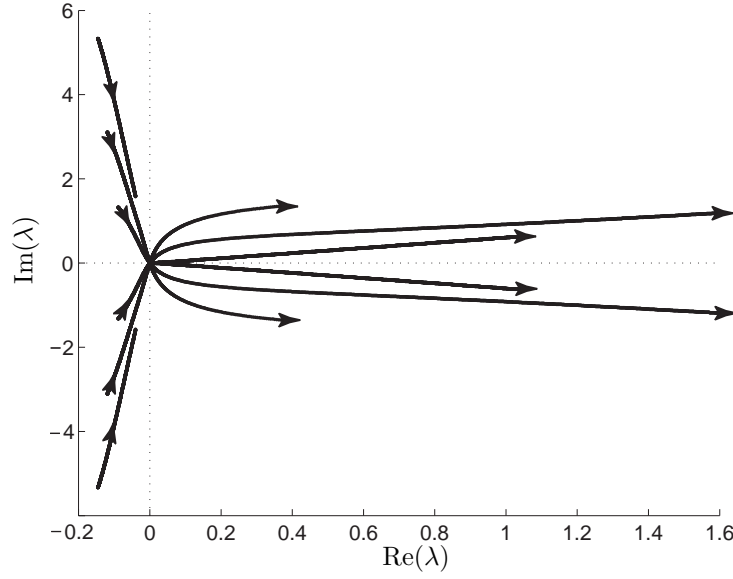


Figure 10.1: Trajectories of the leading eigenvalues λ for Fourier modes $|k| \in \{1, 2, 3, 4\}$. The eigenvalues move from left to right as the angular velocity ω is increased from 0 to 2.5. To obtain this figure the domain $[a, R]$ of the fluid velocity and pressure was partitioned with $N = 25$ equally spaced mesh points.

computed with the Bessel method (right column). We see that the two methods agree to six decimal places, which suggests that the Fourier-finite element algorithm with $N = 100$ mesh points is accurate to six decimal places. This is the best accuracy that we could hope for since the discretization matrices were

k	ω_{crit}	computed value of $\lambda(\omega_{\text{crit}})$
1	0	1.3604×10^{-12}
2	1.3450	$(-0.0126 + 6.6488i) \times 10^{-11}$
3	2.1964	$(3.6084 - 4.3216i) \times 10^{-12}$
4	3.0075	$(-3.9973 - 8.1133i) \times 10^{-12}$

Table 10.2: Accuracy check. Critical values of ω (computed using Theorem 9.27) tabulated against the computed values of $\lambda(\omega_{\text{crit}})$. The exact value of $\lambda(\omega_{\text{crit}})$ is zero. The eigenvalues were computed with $N = 50$ mesh points. In fact, the same order of accuracy can be achieved with only $N = 2$ mesh points since the eigenvector corresponding to eigenvalue $\lambda = 0$ belongs to the finite element approximation spaces for all mesh sizes.

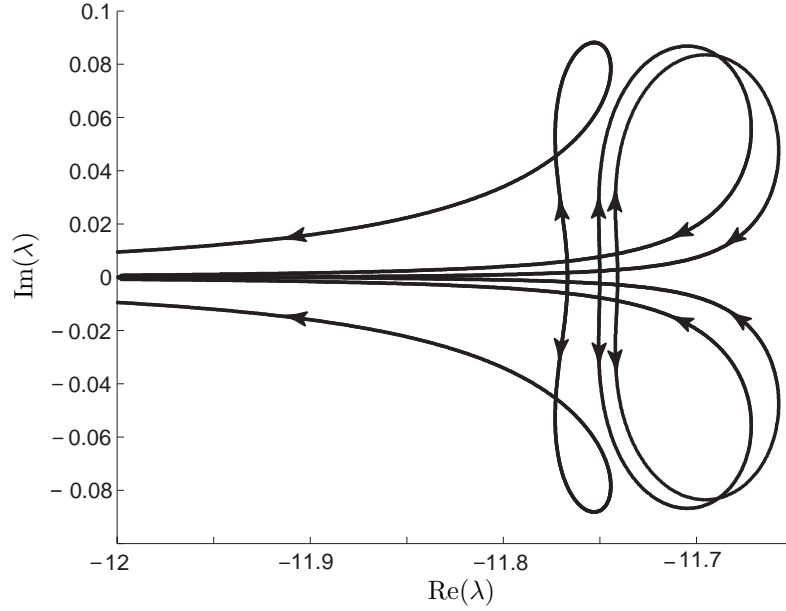


Figure 10.2: Trajectory of the 10th eigenvalue for Fourier modes $|k| \in \{1, 2, 3\}$ (where the eigenvalues are sorted by decreasing real part). The angular velocity ω is varied from 0 to 50. When $\omega = 0$ all the eigenvalues lie on the real axis. As ω is increased the eigenvalues move up (for $k > 0$) or down (for $k < 0$) before looping around and moving to the left.

constructed using a quadrature rule with a tolerance of 10^{-6} .

λ	
FEM	Bessel
-0.02361765	-0.02361795
-0.30262021	-0.30262018
-0.85894235	-0.85894250
-1.69386641	-1.69386600

Table 10.3: Accuracy check. The first few eigenvalues for Fourier mode $k = 0$ (the eigenvalue $\lambda = 0$ is omitted). The eigenvalues were computed using both the finite element method (left column) and by using Bessel functions to obtain a nonlinear equation for λ , which was solved using the MATLAB function *fsolve* (right column). $N = 100$ mesh points were used for the finite element method and we took $\omega = 5$.

11 Comments

In this paper we modelled the deformable body using a membrane theory. The numerical method presented here can easily be applied to the more general case where the deformable body is modelled as a nonlinearly viscoelastic extensible, shearable shell. The convergence proof of Part 2 can be extended without any technical difficulty, only the equations are more complicated.

Bourne and Antman (2009) consider the related Taylor-Couette problem of axisymmetric flow in a deformable cylinder. Here the deformable body is modelled as an axisymmetric shell. The spectrum of the associated quadratic eigenvalue problem, governing the stability of Couette flow, is computed using a numerical method similar to the one developed in this paper, but has a quite different character.

A full-scale perturbation analysis could be based on a generalization of (7.1).

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