Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

Derivation of a Homogenized Von-Kármán Plate Theory from 3d Nonlinear Elasticity

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Preprint no.: 61 2012

DERIVATION OF A HOMOGENIZED VON-KÁRMÁN PLATE THEORY FROM 3D NONLINEAR ELASTICITY

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Abstract. We rigorously derive a homogenized von-Kármán plate theory as a Γ-limit from nonlinear three-dimensional elasticity by combining homogenization and dimension reduction. Our starting point is an energy functional that describes a nonlinear elastic, three-dimensional plate with spatially periodic material properties. The functional features two small length scales: the period ε of the elastic composite material, and the thickness h of the slender plate. We study the behavior as ε and h simultaneously converge to zero in the von-Kármán scaling regime. The obtained limit is a homogenized von-Kármán plate model. Its effective material properties are determined by a relaxation formula that exposes a non-trivial coupling of the behavior of the out-of-plane displacement with the oscillatory behavior in the in-plane directions. In particular, the homogenized coefficients depend on the relative scaling between h and ε, and different values arise for $h \ll \varepsilon$, $\varepsilon \sim h$ and $\varepsilon \ll h$.

Keywords: elasticity, dimension reduction, homogenization, von-Kármán plate theory, two-scale convergence.

2010 Mathematics Subject Classification: 35B27, 49J45, 74E30, 74Q05.

Contents

Date: August 28, 2012.

1. Introduction

We are concerned with the ansatz-free derivation of a homogenized von-Kármán plate theory by simultaneous homogenization and dimension reduction. Our starting point is the energy functional from three-dimensional nonlinear elasticity:

$$\frac{1}{h^4\,|\Omega_h|}\int_{\Omega_h} W_\varepsilon(x,\nabla \boldsymbol{z})\,dx,\qquad \boldsymbol{z}\in H^1(\Omega_h,\mathbb{R}^3). \tag{1}$$

Here $\Omega_h=\omega\times(-\frac{h}{2},\frac{h}{2})\subset\mathbb{R}^3$ is a cylindrical domain with thickness $h\ll 1$, $\boldsymbol{z}:\Omega_h\to\mathbb{R}^3$ a deformation, and W_ε a non-degenerate stored energy function that periodically oscillates in in-plane directions with period $\varepsilon\ll 1$. We are interested in the effective behavior when both the thickness h and the period ε are small. The separate limits $h\to 0$ and $\varepsilon\to 0$ are reasonably well understood: In the seminal work by Friesecke, James and Müller [FJM06] it is shown that (1) Γ-converges for $h\to 0$ (and ε fixed) to a two-dimensional von-Kármán plate theory. Regarding the limit $\varepsilon\to 0$, which is related to homogenization, the first rigorous results relevant in nonlinear elasticity have been obtained by Braides [Bra85] and independently by Müller [Mül87]. They proved that under suitable growth assumptions on W_ε the energy (1) Γ-converges as $\varepsilon\to 0$ (and h fixed) to the functional obtained by replacing W_ε in (1) with the homogenized energy density given by the infinite-cell homogenization formula.

In this paper we study the asymptotic behavior when both the thickness h and the period ε *simultaneously* tend to zero. As a Γ-limit we obtain a two-dimensional von-Kármán plate model with homogenized material properties. It basically takes the form

$$\int_\omega Q_\gamma(\operatorname{sym}\nabla\boldsymbol{u}+\tfrac{1}{2}\nabla v\otimes\nabla v,\nabla^2 v)\,d\hat{x} \tag{2}$$

where the functions $\boldsymbol{u}\in H^1(\omega,\mathbb{R}^3)$ and $v\in H^2(\omega)$ are the scaled in-plane and out-of-plane displacements and monitor the deviation of the deformed plate from a rigid deformation. The expression $\operatorname{sym}\nabla\boldsymbol{u}+\frac{1}{2}\nabla v\otimes\nabla v$ is the membrane strain, while $\nabla^2 v$ corresponds to "infinitesimal" bending. The material properties are encoded in the quadratic energy density Q_γ, which is obtained by a relaxation and homogenization procedure from the quadratic term in the expansion of W_ε at identity. That relaxation exposes a non-trivial coupling of the behavior in the out-of-plane directions with the oscillatory behavior in the in-plane directions. As a consequence, Q_γ depends on the relative scaling of h and ε, and in particular, different expressions arise for $h\ll\varepsilon$, $h\sim\varepsilon$ and $h\gg\varepsilon$. The derived relaxation formulas for Q_γ involve only convex minimization over a single periodicity cell, and thus are easily computable.

The simplicity of the obtained relaxation formulas is surprising at first sight: Since the original three-dimensional, and the obtained two-dimensional models are nonlinear, one would naively expect that an infinite-cell relaxation formula is required. However, since we consider non-degenerate materials, for deformations with low energy (1) is effectively a quadratic function w. r. t. the nonlinear strain $\boldsymbol{E}=\sqrt{(\nabla\boldsymbol{z})^{\mathrm{t}}\nabla\boldsymbol{z}}-\boldsymbol{I}$. As we are going to see, this "hidden convexity" allows to analyze the problem with convex homogenization methods and explains the emergence of a single-cell relaxation formula.

Our analysis follows a scheme developed by the first author in [Neu10, Neu12], and is inspired by [Vel]. Let us briefly describe the basic idea in the following simplified setting: Assume that $W_\varepsilon(x,\boldsymbol{F})=W_0(\frac{\hat{x}}{\varepsilon},\boldsymbol{F})$, $x=(\hat{x},x_3)$, where W_0 denotes a smooth energy density that is $[0,1)^2$-periodic in $\hat{x}$, non-degenerate, i. e. $W_0(x,\boldsymbol{F})\geq\operatorname{dist}^2(\boldsymbol{F},\mathrm{SO}(3))$ for all $\boldsymbol{F}\in\mathbb{M}^3$, frame-indifferent, i. e. $W_0(x,\boldsymbol{RF})=W_0(x,\boldsymbol{F})$ for all $\boldsymbol{F}\in\mathbb{M}^3$ and $\boldsymbol{R}\in\mathrm{SO}(3)$, and minimal for $\boldsymbol{F}=\boldsymbol{I}$. We are interested in the asymptotic behavior of sequences $\boldsymbol{z}^h\in H^1(\Omega_h,\mathbb{R}^3)$ with (1) uniformly bounded in $h\ll 1$. Since W_0 is non-degenerate, the associated nonlinear strain $\boldsymbol{E}^h=\frac{\sqrt{\nabla\boldsymbol{z}^t\nabla\boldsymbol{z}}-\boldsymbol{I}}{h^2}$ is equibounded, in the sense that $\limsup_{h\to 0}\int_{\Omega_h}|\boldsymbol{E}^h|^2\,dx<\infty$. By appealing to the polar factorization for matrices (with positive determinant), frame-indifference and the quadratic expansion $W_0(x_1,x_2,\boldsymbol{I}+\boldsymbol{G}) = Q_0(x_1,x_2,\boldsymbol{G}) + o(|G|^2)$ with $Q_0(x_1,x_2,\boldsymbol{G}) = \frac{\partial^2 W_0(x_1,x_2,\boldsymbol{I})}{\partial F \partial F}(\boldsymbol{G},\boldsymbol{G})$, we get the following formal expansion of (1):

$$\begin{aligned}\frac{1}{h^4\,|\Omega_h|}\int_{\Omega_h} W_\varepsilon(x,\nabla \boldsymbol{y}^h)\,dx &\approx \frac{1}{h^4\,|\Omega_h|}\int_{\Omega_h} W_\varepsilon(x,\sqrt{(\nabla \boldsymbol{y}^h)^T(\nabla \boldsymbol{z}^h)}\,)\,dx\\ &= \frac{1}{h^4\,|\Omega_h|}\int_{\Omega_h} W_0(\tfrac{x_1}{\varepsilon},\tfrac{x_2}{\varepsilon},\boldsymbol{I}+h^2\boldsymbol{E}^h)\,dx\\ &\approx \frac{1}{|\Omega_h|}\int_{\Omega_h} Q(\tfrac{x_1}{\varepsilon},\tfrac{x_2}{\varepsilon},\boldsymbol{E}^h)\,dx + \text{higher order terms.}\end{aligned}$$

We learn that for $h \ll 1$ the essential behavior of the non-convex energy (1) is captured by the functional

$$\boldsymbol{z}^h \mapsto \frac{1}{|\Omega_h|}\int_{\Omega_h} Q(\tfrac{x_1}{\varepsilon},\tfrac{x_2}{\varepsilon},\boldsymbol{E}^h)\,dx. \tag{3}$$

Notice that (3) is non-convex, since $\boldsymbol{z}^h \mapsto \boldsymbol{E}^h$ is nonlinear. However, seen as a function of the the nonlinear strain $\boldsymbol{E}^h$, (3) is convex and quadratic. Our treatment of the homogenization effects crucially relies on the convexity, which, in particular, makes it convenient to apply two-scale convergence (as introduced by [Ngu89, All92]). As a main ingredient, in Proposition 3.3 we prove a two-scale compactness result for the sequence $\{\boldsymbol{E}^h\}_{h>0}$ and precisely identify the structure of its two-scale limits. Since $\boldsymbol{z}^h \mapsto \boldsymbol{E}^h$ is geometrically nonlinear, this task is non-trivial. To overcome this difficulty, we establish in Proposition 3.1, based on the geometric rigidity estimate in [FJM02], a decomposition that shows that any deformation $\boldsymbol{z} \in H^1(\Omega_h,\mathbb{R}^3)$ can be written as the sum of a *von-Kármán ansatz* and a correction that is controlled by the energy. The identification of the two-scale limit of $\{\boldsymbol{E}^h\}_{h>0}$ is then obtained by analyzing the oscillatory behavior of both contributions separately.

Our analysis requires both: techniques from *dimension reduction*, in particular, the quantitative rigidity estimate and approximation schemes developed by Friesecke, James and Müller in their famous work on the derivation of nonlinear plate theories [FJM02, FJM06]; and *homogenization methods*, in particular, two-scale convergence [Ngu89, All92] and periodic unfolding [CDG02, Vis06, MT07].

To our knowledge our result is the first rigorous result combining homogenization and dimension reduction for plates in the von-Kármán regime. Analogue results for the derivation of a homogenized nonlinear bending-torsion rod theory from three-dimensional elasticity and partial results for the more delicate case of nonlinear bending models for plates have been obtained by the first author in [Neu10, Neu12]. A complete analysis regarding nonlinear bending models for plate theory is work in progress. Moreover, wrinkled plates of Föppl-von-Kármán type and nonlinear weakly curved rods have been studied by the second author in [Vel12, Vel]. Results in the membrane regime (where no linearization of the material nonlinearity takes place) are obtained by Braides et al. [BFF00] and Babadijan & Baía [BB06].

1.1. **Notation.**

- $\mathbb{R}^+ := [0,+\infty)$ denotes the set of non-negative real numbers;
- $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$ denotes the standard basis in $\mathbb{R}^3$;
- The components of $x \in \mathbb{R}^3$ and vector fields $\mathbf{c}$ are denoted by $x_\alpha := x\cdot\mathbf{e}_\alpha$ and $\mathbf{c}_\alpha := \mathbf{c}\cdot\mathbf{e}_\alpha$, respectively. We use the shorthand $\hat{x} := (x_1,x_2)$.
- $\mathbb{M}^d$, $\mathbb{M}^d_{\text{sym}}$ and $\mathbb{M}^d_{\text{skw}}$ denote the space of $d\times d$ real matrices, symmetric and skew-symmetric $d\times d$ real matrices, respectively;
- $\operatorname{sym}\boldsymbol{A} = 1/2(\boldsymbol{A}+\boldsymbol{A}^{\mathrm{t}})$, $\operatorname{skw}\boldsymbol{A} = (\boldsymbol{A}-\operatorname{sym}\boldsymbol{A})$ denote the symmetric and skew-symmetric part, respectively:
- $\mathrm{SO}(d) := \{\,\boldsymbol{R}\in\mathbb{M}^d\ :\ \boldsymbol{R}^{\mathrm{t}}\boldsymbol{R} = \boldsymbol{I},\ \det\boldsymbol{R} = 1\,\}$ is the set of rotations of $\mathbb{R}^d$;
- $Y := [0,1)^2$ is the unit cell of periodicity; $\mathcal{Y}$ is the associated torus;

- $\partial_\alpha \boldsymbol{y}$ denotes the partial derivative of $\boldsymbol{y}$ in direction $\mathbf{e}_\alpha$. We set $\hat\nabla \boldsymbol{y} := (\partial_1 \boldsymbol{y}, \partial_2 \boldsymbol{y})$ and define the scaled deformation gradient as $\nabla_h \boldsymbol{y} := (\hat\nabla \boldsymbol{y}, \frac{1}{h}\partial_3 \boldsymbol{y})$.
- ε and h denote generic elements of vanishing sequences of positive numbers $\{\varepsilon\}$ and $\{h\}$, respectively;

$L^p(D,\mathbb{R}^d)$, $H^1(D,\mathbb{R}^d)$, $W^{1,p}(D,\mathbb{R}^d)$, $H^1_0(D,\mathbb{R}^d)$, and $W^{1,p}_0(D,\mathbb{R}^d)$ denote the standard Lebesgue, Hilbert and Sobolev spaces of maps from D to $\mathbb{R}^d$, and the associated subspaces of functions vanishing on the boundary ∂D (in the sense of traces); if no confusion occurs, we tacitly write $L^p(D), H^1(D), \ldots$ or even simply $L^p, H^1, \ldots$.

In this paper we frequently encounter function spaces of periodic functions. We denote by $\mathcal{Y}$ the Euclidean space $\mathbb{R}^2$ equipped with the torus topology, that is for all $z \in \mathbb{Z}^2$ the points $y+z$ and y are identified in $\mathcal{Y}$. We write $C(\mathcal{Y})$ to denote the space of continuous functions $f : \mathbb{R}^2 \to \mathbb{R}$ satisfying $f(y+z) = f(y)$ for all $z \in \mathbb{Z}^2$ and set $C^k(\mathcal{Y}) := C^k(\mathbb{R}^2) \cap C(\mathcal{Y})$. Clearly, $C(\mathcal{Y})$ endowed with the norm $||f||_\infty := \sup_{y\in Y} |f(y)|$ is a Banach space. We denote by $L^2(\mathcal{Y})$, $H^1(\mathcal{Y})$ and $H^1(S\times\mathcal{Y})$ the Banach spaces obtained as the closure of $C^\infty(\mathcal{Y})$ and $C^\infty(\bar S, C^\infty(\mathcal{Y}))$ w. r. t. the norm in $L^2(Y)$, $H^1(Y)$ and $H^1(S\times Y)$, respectively. For $A \subset \mathbb{R}^d$ measurable and X a Banach space, $L^2(A,X)$ is understood in the sense of Bochner. We tacitly identify the spaces $L^2(A, L^2(B))$ and $L^2(A \times B)$ in the sense that whenever $f \in L^2(A \times B)$, then there exists a function $\tilde f \in L^2(A, L^2(B))$ with $f = \tilde f$ almost everywhere in $A \times B$.

2. General framework and main results

The three-dimensional model. Throughout the paper $\Omega_h := \omega \times (hS)$ denotes the reference configuration of a thin plate with mid-surface $\omega \subset \mathbb{R}^2$ and (rescaled) cross-section $S := (-\frac{1}{2}, \frac{1}{2})$. We suppose that ω is a connected Lipschitz domain. For simplicity we assume that ω is centered, that is

$$\int_\omega \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} dx_1\, dx_2 = 0. \tag{4}$$

Deformations of the plate are described by vector fields $\mathbf{z}^h : \Omega_h \to \mathbb{R}^3$. Since we are interested in the behavior $h \to 0$, it is convenient to work on the *canonical reference domain* $\Omega := \omega \times S$. Clearly, if $\mathbf{z}^h \in H^1(\Omega_h, \mathbb{R}^3)$, then $\boldsymbol{y}^h(\hat x, x_3) := \mathbf{z}^h(\hat x, hx_3)$ belongs to $H^1(\Omega, \mathbb{R}^3)$ and we have $\nabla \mathbf{z}^h(\hat x, hx) = \nabla_h \boldsymbol{y}^h(x)$ where $\nabla_h := (\hat\nabla, \frac{1}{h}\partial_3) := (\partial_1, \partial_2, \frac{1}{h}\partial_3)$ denotes the *scaled deformation gradient*.

In finite elasticity the stored energy (per unit volume) of a homogeneous plate with thickness h deformed by $\boldsymbol{z}^h \in H^1(\Omega_h, \mathbb{R}^3)$, resp. by $\boldsymbol{y}^h \in H^1(\Omega, \mathbb{R}^3)$, is given by an integral of the form

$$\fint_{\Omega_h} W(\nabla \boldsymbol{z}^h(x))\, dx = \fint_\Omega W(\nabla_h \boldsymbol{y}^h(x))\, dx. \tag{5}$$

Here and below $\fint_A f\, dx$ stands for $\left(\int_A dx\right)^{-1} \int_A f(x)\, dx$. We consider (composite) materials that are non-degenerate in the sense that $W(F) \geq C \operatorname{dist}^2(F, \mathrm{SO}(3))$. As it was shown in the seminal papers [FJM02] and [FJM06] for non-degenerate materials different higher order plate theories emerge in the zero-thickness-limit from the energy (5) scaled by certain powers of the thickness. In this contribution we are interested in the *von-Kármán regime*, which corresponds to the scaling of (5) by h^{-4}. For future reference let us define for $\boldsymbol{y} \in H^1(\Omega, \mathbb{R}^3)$ and $h > 0$ the quantity

$$e_h(\boldsymbol{y}) := \frac{1}{h^4} \int_\Omega \operatorname{dist}^2(\nabla_h \boldsymbol{y}(x), \mathrm{SO}(3))\, dx. \tag{6}$$

The von-Kármán plate model. In [FJM06] it is shown that a (scaled) deformation $\boldsymbol{y}$ of a plate with small thickness and low energy, i. e. $h \ll 1$ and $e_h(\boldsymbol{y}) \lesssim 1$, approximately behaves as

$$\boldsymbol{y} \approx \fint_\Omega \boldsymbol{y} + \bar{\boldsymbol{R}}\left[\begin{pmatrix} \hat{x} \\ hx_3 \end{pmatrix} + \begin{pmatrix} h^2\boldsymbol{u} \\ hv \end{pmatrix} - h^2 x_3 \begin{pmatrix} \hat{\nabla} v \\ 0 \end{pmatrix}\right]. \tag{7}$$

The deformation on the right-hand side is what is usually called a *von-Kármán ansatz* associated with the triple

$$(\bar{\boldsymbol{R}}, \boldsymbol{u}, v) \in \mathrm{SO}(3) \times \mathcal{A}(\omega), \qquad \mathcal{A}(\omega) := \left\{\, (\boldsymbol{u}, v) \,:\, \boldsymbol{u} \in H^1(\omega, \mathbb{R}^2),\, v \in H^2(\omega) \,\right\}.$$

The expansion (7) can be interpreted as follows: On a large length scale (of the magnitude of the plates diameter) the plate is rigidly deformed, namely translated by $\fint_\Omega \boldsymbol{y}$ and rotated by $\bar{\boldsymbol{R}}$. The deformation on scale h is described by the scaled in-plane displacement $\boldsymbol{u}$ and the scaled out-of-plane displacement v. In [FJM06] it is shown that for $h \ll 1$ the energy (5) scaled by h^{-4} essentially behaves as the von-Kármán plate energy

$$\mathcal{A}(\omega) \ni (\boldsymbol{u}, v) \mapsto \int_\omega Q_2(\operatorname{sym} \hat{\nabla} \boldsymbol{u} + \tfrac{1}{2} \hat{\nabla} v \otimes \hat{\nabla} v)\, d\hat{x} + \frac{1}{12}\int_\omega Q_2(\hat{\nabla}^2 v)\, d\hat{x}, \tag{8}$$

where $\boldsymbol{y}$ and $(\boldsymbol{u}, v)$ are related as in (7), and Q_2 is obtained from W by linearization at identity and a relaxation procedure. The quantity $\hat{\nabla}^2 v$ monitors the curvature of the graph $(\hat{x}, v(\hat{x}))$. In [FJM06] the connection between (5) and (8) is made rigorous in the sense of Γ-convergence.

Precise setup and results. In this contribution we consider plates made of elastic composite materials. Therefore, we define for $\varepsilon, h \in (0,1]$ the energy $I^{\varepsilon,h} : H^1(\Omega, \mathbb{R}^3) \mapsto \mathbb{R}^+ \cup \{+\infty\}$,

$$I^{\varepsilon,h}(\boldsymbol{y}) := \frac{1}{h^4}\int_\Omega W(x, \tfrac{\hat{x}}{\varepsilon}, \nabla_h \boldsymbol{y}(x))\, dx.$$

It models the elastic energy of a plate with mid-plane ω, thickness h and a composite material that oscillates in in-plane directions on scale ε. We assume that ε and the thickness h are coupled with ratio $\gamma \in [0, \infty]$, that is $\varepsilon := \varepsilon(h)$ where

$$\varepsilon(h) \in (0,1] \text{ for all } h \in [0,1], \ \lim_{h\to 0} \varepsilon(h) \to 0 \text{ and } \lim_{h \to 0} \frac{h}{\varepsilon(h)} = \gamma. \tag{9}$$

The elastic properties of the composite are described by the stored energy function $W(x, y, F)$ which is assumed to be $[0,1)^2$-periodic in "$y = \frac{\hat{x}}{\varepsilon}$". The precise assumptions on W are stated in Definition 2.5 below.

In our main result we show that the effective behavior for $I^{\varepsilon(h),h}$ is captured by the limiting functional

$$\begin{aligned} &I^\gamma : \mathcal{A}(\omega) \to \mathbb{R}^+, \\ &I^\gamma(\boldsymbol{u}, v) := \int_\omega Q_\gamma(\hat{x}, \hat{\nabla}\boldsymbol{u} + \tfrac{1}{2}\hat{\nabla} v \otimes \hat{\nabla} v,\, \hat{\nabla}^2 v)\, d\hat{x}. \end{aligned}$$

Here the energy density Q_γ (see Definition 2.7 below) is obtained from the stored energy density W by a linearization at identity and a relaxation procedure that depends on the ratio γ (the relative scaling of h and $\varepsilon(h)$). More precisely, we prove that $I^{\varepsilon(h),h}$ Γ-converges to I^γ w. r. t. the following notion of convergence:

Definition 2.1. We say a sequence $\boldsymbol{y}^h \in H^1(\Omega, \mathbb{R}^3)$ converges to a triple $(\bar{\boldsymbol{R}}, \boldsymbol{u}, v) \in \mathrm{SO}(3) \times H^1(\omega, \mathbb{R}^2) \times H^1(\omega)$, and write $\boldsymbol{y}^h \to (\bar{\boldsymbol{R}}, \boldsymbol{u}, v)$, if there exist rotations $\{\bar{\boldsymbol{R}}^h\}_{h>0}$ and functions $\{\boldsymbol{u}^h\}_{h>0} \subset H^1(\omega, \mathbb{R}^2)$, $\{v^h\}_{h>0} \subset H^1(\omega)$ such that

$$(\bar{\boldsymbol{R}}^h)^T \left(\fint_S \boldsymbol{y}^h(\hat{x}, x_3)\, dx_3 - \fint_\Omega \boldsymbol{y}^h\, dx \right) = \begin{pmatrix} \hat{x} + h^2 \boldsymbol{u}^h(\hat{x}) \\ h v^h(\hat{x}) \end{pmatrix}, \tag{10}$$

$$\boldsymbol{u}^h \rightharpoonup \boldsymbol{u} \text{ weakly in } H^1(\omega, \mathbb{R}^2), \qquad v^h \rightharpoonup v \text{ weakly in } H^1(\omega) \qquad \text{and} \qquad \bar{\boldsymbol{R}}^h \to \bar{\boldsymbol{R}}. \tag{11}$$

A limit in the sense of Definition 2.1 is not unique as it stands. However, uniqueness is obtained modulo the following equivalence relation on $H^1(\omega,\mathbb{R}^2)\times H^1(\omega)$:

$$(\boldsymbol{u}_1,v_1)\sim(\boldsymbol{u}_2,v_2)\qquad :\Leftrightarrow$$
$$\begin{cases}\boldsymbol{u}_2(\hat{x})=\boldsymbol{u}_1(\hat{x})+(\boldsymbol{A}-\frac{1}{2}\boldsymbol{a}\otimes\boldsymbol{a})\hat{x}-v_1(\hat{x})\boldsymbol{a}\\ v_2(\hat{x})=v_1(\hat{x})+\boldsymbol{a}\cdot\hat{x}\end{cases}\qquad\text{for some }\boldsymbol{a}\in\mathbb{R}^2,\,\boldsymbol{A}\in\operatorname{Skew}(2).$$

Lemma 2.2 (uniqueness)**.** *Let $(\bar{\boldsymbol{R}},\boldsymbol{u},v)$, $(\widetilde{\boldsymbol{R}},\tilde{\boldsymbol{u}},\tilde{v})\in\mathrm{SO}(3)\times H^1(\omega,\mathbb{R}^2)\times H^1(\omega)$ and consider a sequence $\boldsymbol{y}^h$ that converges to $(\bar{\boldsymbol{R}},\boldsymbol{u},v)$. Then*

$$\boldsymbol{y}^h\to(\widetilde{\boldsymbol{R}},\tilde{\boldsymbol{u}},\tilde{v})\qquad\Leftrightarrow\qquad\widetilde{\boldsymbol{R}}=\bar{\boldsymbol{R}}\text{ and }(\boldsymbol{u},v)\sim(\tilde{\boldsymbol{u}},\tilde{v}).$$

Now we are ready to state the main result:

Theorem 2.3. *Let Assumption 2.8 (stated below) be satisfied.*

(i) (Compactness). Let $\boldsymbol{y}^h\in H^1(\Omega,\mathbb{R}^3)$ be a sequence with equibounded energy, that is $\limsup_{h\to0}I^{\varepsilon(h),h}(\boldsymbol{y}^h)<\infty$. Then there exists $(\bar{\boldsymbol{R}},\boldsymbol{u},v)\in\mathrm{SO}(3)\times\mathcal{A}(\omega)$ such that $\boldsymbol{y}^h\to(\bar{\boldsymbol{R}},\boldsymbol{u},v)$ up to a subsequence.

(ii) (Invariance). For $(\boldsymbol{u}_1,v_1),(\boldsymbol{u}_2,v_2)\in\mathcal{A}(\omega)$ with $(\boldsymbol{u}_1,v_1)\sim(\boldsymbol{u}_2,v_2)$ we have $I^\gamma(\boldsymbol{u}_1,v_1)=I^\gamma(\boldsymbol{u}_2,v_2)$.

(iii) (Lower bound). Let $\boldsymbol{y}^h\in H^1(\Omega,\mathbb{R}^3)$ be a sequence satisfying $\limsup_{h\to0}I^{\varepsilon(h),h}(\boldsymbol{y}^h)<\infty$. Assume that $\boldsymbol{y}^h\to(\bar{\boldsymbol{R}},\boldsymbol{u},v)$. Then

$$\liminf_{h\to0}I^{\varepsilon(h),h}(\boldsymbol{y}^h)\geq I^\gamma(\boldsymbol{u},v).$$

(iv) (Recovery sequence). For all $(\bar{\boldsymbol{R}},\boldsymbol{u},v)\in\mathrm{SO}(3)\times\mathcal{A}(\omega)$ there exists a sequence $\boldsymbol{y}^h\in H^1(\Omega,\mathbb{R}^3)$ with

$$\boldsymbol{y}^h\to(\bar{\boldsymbol{R}},\boldsymbol{u},v)\qquad\text{and}\qquad\lim_{h\to0}I^{\varepsilon(h),h}(\boldsymbol{y}^h)=I^\gamma(\boldsymbol{u},v).$$

The proof of this and the following results are postponed to Section 4.

Theorem 2.3 is a convergence result in the spirit of Γ-convergence. Based on Theorem 2.3, the convergence of various minimization problems extending $\mathcal{I}^{\varepsilon(h),h}$ (e. g. by additional loading terms and boundary conditions) can be analyzed by appealing to general methods from the theory of Γ-convergence. We refer to [DM93] for further details in that direction.

In the following we introduce the required assumptions and the relaxation formula defining Q_γ properly. We need a couple of definitions.

Definition 2.4 (nonlinear material law)**.** Let $0<\alpha\leq\beta$ and $\rho>0$. The class $\mathcal{W}(\alpha,\beta,\rho)$ consists of all measurable functions $W:\mathbb{M}^3\to[0,+\infty]$ that satisfy the following properties:

(W1) W is frame indifferent, i.e.

$$W(\boldsymbol{R}\boldsymbol{F})=W(\boldsymbol{F})\quad\text{for all }\boldsymbol{F}\in\mathbb{M}^3,\ \boldsymbol{R}\in\mathrm{SO}(3);$$

(W2) W is non degenerate, i.e.

$$W(\boldsymbol{F})\geq\alpha\operatorname{dist}^2(\boldsymbol{F},\mathrm{SO}(3))\quad\text{for all }\boldsymbol{F}\in\mathbb{M}^3;$$
$$W(\boldsymbol{F})\leq\beta\operatorname{dist}^2(\boldsymbol{F},\mathrm{SO}(3))\quad\text{for all }\boldsymbol{F}\in\mathbb{M}^3\text{ with }\operatorname{dist}^2(\boldsymbol{F},\mathrm{SO}(3))\leq\rho;$$

(W3) W is minimal at $\boldsymbol{I}$, i.e.

$$W(\boldsymbol{I})=0;$$

(W4) W admits a quadratic expansion at $\boldsymbol{I}$, i.e.

$$W(\boldsymbol{I}+\boldsymbol{G})=Q(\boldsymbol{G})+o(|\boldsymbol{G}|^2)\qquad\text{for all }\boldsymbol{G}\in\mathbb{M}^3$$

where $Q:\mathbb{M}^3\to\mathbb{R}$ is a quadratic form.

Definition 2.5 (admissible composite material)**.** Let $0 < \alpha \le \beta$ and $\rho > 0$. We say

$$W : \Omega \times \mathbb{R}^2 \times \mathbb{M}^3 \to \mathbb{R}^+ \cup \{+\infty\}$$

describes an admissible composite material of class $\mathcal{W}(\alpha, \beta, \rho)$ if

(i) W is almost surely equal to a Borel function on $\Omega \times \mathbb{R}^2 \times \mathbb{M}^3$,
(ii) $W(\cdot, y, \boldsymbol{F})$ is continuous for almost every $y \in \mathbb{R}^2$ and $\boldsymbol{F} \in \mathbb{M}^3$,
(iii) $W(x, \cdot, \boldsymbol{F})$ is Y-periodic for all $x \in \Omega$ and almost every $\boldsymbol{F} \in \mathbb{M}^3$,
(iv) $W(x, y, \cdot) \in \mathcal{W}(\alpha, \beta, \rho)$ for all $x \in \Omega$ and almost every $y \in \mathbb{R}^2$.

The energy density Q_γ is obtained by relaxing Q w. r. t. certain tensor fields that capture the oscillatory behavior of the nonlinear strain. For the precise definition of Q_γ we need the following:

Definition 2.6. For $\gamma \in [0, \infty]$ we define the following function spaces of *relaxation fields*

$$L_0(S \times \mathcal{Y}, \mathbb{M}^3_{\mathrm{sym}}) := \left\{ \begin{pmatrix} \operatorname{sym} \hat{\nabla}_y \boldsymbol{\zeta} + x_3 \hat{\nabla}^2_y \varphi & \begin{matrix} \boldsymbol{g}_1 \\ \boldsymbol{g}_2 \end{matrix} \\ (\boldsymbol{g}_1,\ \boldsymbol{g}_2) & \boldsymbol{g}_3 \end{pmatrix} : \boldsymbol{\zeta} \in H^1(\mathcal{Y}, \mathbb{R}^2), \right.$$
$$\left. \varphi \in H^2(\mathcal{Y}),\ \boldsymbol{g} \in L^2(S \times \mathcal{Y}, \mathbb{R}^3) \right\}$$

$$L_\infty(S \times \mathcal{Y}, \mathbb{M}^3_{\mathrm{sym}}) := \left\{ \begin{pmatrix} \operatorname{sym} \hat{\nabla}_y \boldsymbol{\zeta} & \begin{matrix} \partial_{y_1} \psi + \boldsymbol{c}_1 \\ \partial_{y_2} \psi + \boldsymbol{c}_2 \end{matrix} \\ \hat{\nabla}_y \psi + (\boldsymbol{c}_1, \boldsymbol{c}_2) & \boldsymbol{c}_3 \end{pmatrix} : \boldsymbol{\zeta} \in L^2(S, H^1(\mathcal{Y}, \mathbb{R}^2)), \right.$$
$$\left. \psi \in L^2(S, H^1(\mathcal{Y})),\ \boldsymbol{c} \in L^2(S, \mathbb{R}^3) \right\}$$

$$L_\gamma(S \times \mathcal{Y}, \mathbb{M}^3_{\mathrm{sym}}) := \left\{ \operatorname{sym}(\hat{\nabla}_y \boldsymbol{\phi}, \tfrac{1}{\gamma} \partial_3 \boldsymbol{\phi}) : \boldsymbol{\phi} \in H^1(S \times \mathcal{Y}, \mathbb{R}^3) \right\} \qquad \text{for } \gamma \in (0, \infty).$$

Now we are in position to present the relaxation formula for Q_γ:

Definition 2.7 (relaxation formula)**.** Let $\gamma \in [0, \infty]$ and let Q be as in Definition 2.5. Define $Q_\gamma : \omega \times \mathbb{M}^2 \times \mathbb{M}^2 \to \mathbb{R}^+$ by

$$Q_\gamma(\hat{x}, \boldsymbol{A}, \boldsymbol{B}) := \inf_{\boldsymbol{U} \in L_\gamma(S \times \mathcal{Y}, \mathbb{M}^3_{\mathrm{sym}})} \iint_{S \times Y} Q\left(\hat{x}, x_3, y,\ \Lambda(\boldsymbol{A}, \boldsymbol{B}) + \boldsymbol{U}\right) dy\, dx_3 \tag{12}$$

where $\Lambda : \mathbb{M}^2 \times \mathbb{M}^2 \to C(S, \mathbb{M}^3)$,

$$\Lambda(\boldsymbol{A}, \boldsymbol{B}) := \left(\sum_{\alpha, \beta = 1}^{2} (\boldsymbol{A}_{\alpha\beta} - x_3 \boldsymbol{B}_{\alpha\beta})(\mathbf{e}_\alpha \otimes \mathbf{e}_\beta) \right). \tag{13}$$

Assumption 2.8. We suppose that

- W is an admissible composite material of class $\mathcal{W}(\alpha, \beta, \rho)$ in the sense of Definition 2.5.
- Q is the quadratic energy density associated to W through expansion (W4) in Definition 2.4.
- the fine-scales h and ε are coupled with ratio $\gamma \in [0, \infty]$ in the sense of (9).
- Q_γ is defined by the relaxation formula in Definition 2.7.

Eventually, we gather some basic properties of admissible W and the associated quadratic forms Q_γ.

Lemma 2.9. *Let W be as in Definition 2.5 and let Q be the quadratic form associated to W through the expansion* (W4)*. Then*

(Q1) $Q(\cdot, y, \cdot)$ is continuous for almost every $y \in \mathbb{R}^2$,

(Q2) $Q(x, \cdot, \boldsymbol{G})$ *is Y-periodic and measurable for all $x \in \Omega$ and all $\boldsymbol{F} \in \mathbb{M}^3$,*
(Q3) for all $x \in \Omega$ and almost every $y \in \mathbb{R}^2$ the map $Q(x, y, \cdot)$ is quadratic and satisfies

$$\alpha|\operatorname{sym}\boldsymbol{G}|^2 \le Q(x,y,\boldsymbol{G}) = Q(x,y,\operatorname{sym}\boldsymbol{G}) \le \beta|\operatorname{sym}\boldsymbol{G}|^2 \qquad \text{for all } \boldsymbol{G} \in \mathbb{M}^3.$$

Furthermore, there exists a monotone function $r : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\}$, only depending on the parameters α, β and ρ, such that $r(\delta) \to 0$ as $\delta \to 0$ and

$$\forall \boldsymbol{G} \in \mathbb{M}^3 \ : \ |W(x,y,\boldsymbol{I}+\boldsymbol{G}) - Q(x,y,\boldsymbol{G})| \le |\boldsymbol{G}|^2 r(|\boldsymbol{G}|) \tag{14}$$

for all $x \in \Omega$ and almost every $y \in \mathbb{R}^2$.

(For a proof see [Neu12, Lemma 2.7].)

In the next lemma we gather some properties of the solution operator associated with the minimization problem in Definition 2.7.

Lemma 2.10. *There exists a bounded linear operator*

$$\boldsymbol{\Pi}_\gamma : L^2(\omega, \mathbb{M}^2) \times L^2(\omega, \mathbb{M}^2) \to L^2(\omega, L_\gamma(S \times \mathcal{Y}, \mathbb{M}^3_{\mathrm{sym}}))$$

such that

(a) for all $\boldsymbol{A}, \boldsymbol{B} \in C(\overline{\omega}, \mathbb{M}^2)$, the field $(x,y) \mapsto \boldsymbol{\Pi}_\gamma[\boldsymbol{A},\boldsymbol{B}](x,y)$ is equivalent (up to a null-set) to a field in $C(\overline{\omega}, L_\gamma(S \times \mathcal{Y}, \mathbb{M}^3_{\mathrm{sym}}))$,
(b) for almost every $\hat{x} \in \omega$ we have

$$Q_\gamma(\hat{x}, \boldsymbol{A}, \boldsymbol{B}) = \iint_{S\times Y} Q\left(\hat{x}, y, x_3,\ \Lambda(\boldsymbol{A},\boldsymbol{B}) + \boldsymbol{\Pi}_\gamma[\boldsymbol{A},\boldsymbol{B}]\right) dx_3\, dy.$$

With the help of the previous lemma we obtain the following properties for Q_γ:

Lemma 2.11. *The mapping $Q_\gamma : \omega \times \mathbb{M}^2 \times \mathbb{M}^2 \to \mathbb{R}^+$ satisfies*

(Q_γ1) Q_γ is continuous
(Q_γ2) $Q_\gamma(\hat{x}, \cdot, \cdot)$ is quadratic and

$$\begin{aligned} \frac{\alpha}{12}(|\operatorname{sym}\boldsymbol{A}|^2 + |\operatorname{sym}\boldsymbol{B}|^2) \quad &\le \quad Q_\gamma(\hat{x},\boldsymbol{A},\boldsymbol{B}) = Q_\gamma(\hat{x}, \operatorname{sym}\boldsymbol{A}, \operatorname{sym}\boldsymbol{B}) \\ &\le \quad \beta(|\operatorname{sym}\boldsymbol{A}|^2 + |\operatorname{sym}\boldsymbol{B}|^2) \end{aligned}$$

for all $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{M}^2$ and $\hat{x} \in \omega$.

3. Two-scale identification of the nonlinear strain

One of the main analytic ingredients in the proof of Theorem 2.3 is a representation of an arbitrary 3d deformation $\boldsymbol{y}^h$ as the sum of a von-Kármán ansatz and a higher order correction term which is estimated by $e_h(\boldsymbol{y}^h)$, see (6). The representation is based on the quantitative analysis developed in [FJM02] and [FJM06], and provides a refined understanding of deformations with equibounded energy in the von-Kármán regime. In the following proposition we establish this representation and provide detailed estimates that build the basis of Proposition 3.3, where the precise structure of the oscillations in the strain are identified.

Proposition 3.1. *Let $\boldsymbol{y} \in H^1(\Omega, \mathbb{R}^3)$ and $h > 0$. There exist $(\bar{\boldsymbol{R}}, \boldsymbol{u}, v) \in \mathrm{SO}(3) \times H^1(\omega, \mathbb{R}^2) \times H^2_{loc}(\omega)$ and correctors $w \in H^1(\omega)$, $\boldsymbol{\phi} \in H^1(\Omega, \mathbb{R}^3)$ with*

$$\int_\omega w = 0, \qquad \int_S \boldsymbol{\phi}(\hat{x}, x_3)\, dx_3 = 0 \quad \text{for almost every } \hat{x} \in \omega$$

such that

$$\bar{\boldsymbol{R}}^t \left(\boldsymbol{y} - \fint_\Omega \boldsymbol{y}\, dx \right) = \begin{pmatrix} \hat{x} \\ hx_3 \end{pmatrix} + \begin{pmatrix} h^2 \boldsymbol{u} \\ h(v + hw) \end{pmatrix} - h^2 x_3 \begin{pmatrix} \hat{\nabla} v \\ 0 \end{pmatrix} + h^2 \boldsymbol{\phi} \tag{15}$$

and

$$||\boldsymbol{u}||^2_{H^1(\omega)} + ||v||^2_{H^1(\omega)} + ||w||^2_{H^1(\omega)} + \frac{1}{h^2}||\boldsymbol{\phi}||^2_{L^2(\Omega)} \lesssim e_h(\boldsymbol{y}) + e_h(\boldsymbol{y})^2. \tag{16}$$

Here $\lesssim$ means $\leq$ up to a multiplicative constant that only depends on ω. In addition, for all $M \subset \omega$ compactly contained in ω we have

$$||\nabla^2 v||^2_{L^2(M)} + ||\nabla_h \boldsymbol{\phi}||^2_{L^2(M\times S)} \lesssim_M e_h(\boldsymbol{y}) + e_h(\boldsymbol{y})^2 \tag{17}$$

where $\lesssim_M$ means $\leq$ up to a multiplicative constant that only depends on M next to ω.

If the boundary of ω is of class $C^{1,1}$, then $(\boldsymbol{u}, v) \in \mathcal{A}(\omega)$ and (17) holds for M replaced by ω.

Remark 1. In the proof of the proposition, the out-of-plane displacement v is defined as the solution to the minimization problem

$$\min_{\substack{v\in H^1(\omega)\\ \int_\omega v=0}} \int_\omega |\hat{\nabla} v - \boldsymbol{p}|^2\, d\hat{x}$$

where $\boldsymbol{p} \in H^1(\omega, \mathbb{R}^2)$ is given by certain entries of a scaled rotation field that approximates $\nabla_h \boldsymbol{y}$. The associated Euler-Lagrange equation reads

$$\begin{cases} -\triangle v = \; -\nabla\cdot \boldsymbol{p} & \text{in } \omega \\ \partial_\nu v = \boldsymbol{p}\cdot \nu & \text{on } \partial\omega, \end{cases}$$

subject to $\int_\omega v\, dx = 0$. Above, ν denotes the normal on $\partial\omega$. Since $\nabla \cdot \boldsymbol{p} \in L^2$, we obtain by standard local regularity estimates that $v \in H^2_{\mathrm{loc}}(\omega)$. (17) holds for $M = \omega$, whenever the regularity of $\partial\omega$ allows for an estimate of the form $||v||_{H^2(\omega)} \lesssim ||\nabla\cdot\boldsymbol{p}||_{L^2(\omega)} + ||\boldsymbol{p}||_{L^2(\omega)}$. In particular, this is the case when $\partial\omega$ is $C^{1,1}$. However, for general Lipschitz domains, we only get $v \in H^{3/2}(\omega)$ up to the boundary.

The effective behavior of composite plates that oscillate on scale ε crucially relies on the oscillatory behavior of the *scaled nonlinear strain* which is defined for $\boldsymbol{y}^h \in H^1(\Omega, \mathbb{R}^3)$ by

$$\boldsymbol{E}^h(\boldsymbol{y}^h) := \frac{\sqrt{(\nabla_h \boldsymbol{y}^h)^{\mathrm{t}}\nabla_h \boldsymbol{y}^h} - \boldsymbol{I}}{h^2}.$$

Consider a sequences of deformations $\boldsymbol{y}^h$ with low energy in the sense that $e^h(\boldsymbol{y}^h) < C$ and suppose that $\boldsymbol{y}^h \to (\bar{\boldsymbol{R}}, \boldsymbol{u}, v)$. In [FJM06] it is shown that (up to a subsequence) the associated nonlinear strain $\boldsymbol{E}^h(\boldsymbol{y}^h)$ weakly converges in L^2 to a *limiting strain* whose "effective" part takes the form

$$\boldsymbol{E}(\boldsymbol{u}, v) := \begin{pmatrix} (\mathrm{sym}\,\hat{\nabla}\boldsymbol{u} - x_3\hat{\nabla}^2 v + \frac{1}{2}\hat{\nabla}v\otimes\hat{\nabla}v) & \begin{matrix}0\\0\end{matrix} \\ \begin{matrix}0 & 0\end{matrix} & 0 \end{pmatrix}. \tag{18}$$

Let us remark that the tensor field $\boldsymbol{E}(\boldsymbol{u}, v)$ defined above is compatible with the equivalence relation on $\mathcal{A}(\omega)$, indeed for $(\boldsymbol{u}, v), (\tilde{\boldsymbol{u}}, \tilde{v}) \in \mathcal{A}(\omega)$ we have

$$(\boldsymbol{u}, v) \sim (\tilde{\boldsymbol{u}}, \tilde{v}) \qquad \Rightarrow \qquad \boldsymbol{E}(\boldsymbol{u}, v) = \boldsymbol{E}(\tilde{\boldsymbol{u}}, \tilde{v}). \tag{19}$$

For homogenization a finer understanding of the limiting strain is required – in particular, a precise understanding of the strain's oscillatory behavior is needed. As we are going to see, the splitting in Proposition 3.1 leads to kinematic constraints for these oscillations. The constraints are non-trivial: they depend on the ratio γ and reflect the coupling between the in-plane and out-of-plane behavior of the plate. Since the limiting energy turns out to be a quadratic function of the strain, it suffices to understand oscillations that emerge precisely on scale ε. For this reason we appeal to two-scale convergence (see [Ngu89, All92] for seminal works on two-scale convergence). More precisely, we are going to identify the two-scale limit of $\boldsymbol{E}^h(\boldsymbol{y}^h)$ along sequences of deformations with equibounded energy. For the statement we need a version of two-scale convergence adapted to thin domains which monitors in-plane oscillations (i. e. w. r. t. $\hat{x} = (x_1, x_2)$) on scale ε.

Definition 3.2 (two-scale convergence)**.** We say a sequence $g^h \in L^2(\Omega)$ weakly two-scale converges in L^2 to the function $g \in L^2(\omega, L^2(S\times\mathcal{Y}))$ as $h \to 0$, if the sequence g^h is bounded in $L^2(\Omega)$ and

$$\lim_{h\to 0} \int_\Omega g^h(x)\,\psi(x, \tfrac{x_1}{\varepsilon(h)}, \tfrac{x_2}{\varepsilon(h)})\,dx = \iint_{\Omega\times Y} g(x,y)\,\psi(x,y)\,dy\,dx$$

for all $\psi \in C_c^\infty(\Omega, C(\mathcal{Y}))$. We say g_h strongly two-scale converges to g if additionally

$$\lim_{h\to 0} ||g^h||_{L^2(\Omega)} = ||g||_{L^2(\Omega\times Y)}.$$

If no confusion occurs, we simply write $g^h \xrightharpoonup{2,\gamma} g$ in L^2 (resp. $g^h \xrightarrow{2,\gamma} g$ in L^2) for weak (resp. strong) two-scale convergence in L^2. For vector fields we define two-scale convergence componentwise. For the reader's convenience we gather basic properties of two-scale convergence in the appendix and refer to [Ngu89, All92, MT07, Vis06] for an introduction to classical two-scale convergence, and to [Neu10], [Neu12] for two-scale convergence adapted to thin domains.

The next proposition states that a two-scale limit of the nonlinear strain $\boldsymbol{E}^h(\boldsymbol{y}^h)$ has a specific form: It can be written as a sum of an effective part in the form of $\boldsymbol{E}(\boldsymbol{u}, v)$, see (18), and a relaxation field of class $L^2(\omega, L_\gamma(S\times\mathcal{Y}, \mathbb{M}^3_{\rm sym}))$, see Definition 2.6.

Proposition 3.3 (Two-scale identification of limiting strain)**.**
(i) (Identification). Let $\boldsymbol{y}^h$ be a sequence in $H^1(\Omega, \mathbb{R}^3)$ that satisfies

$$\limsup_{h\to 0} e^h(\boldsymbol{y}^h) < \infty.$$

Suppose that $\boldsymbol{y}^h \to (\bar{\boldsymbol{R}}, \boldsymbol{u}, v)$ for some triple $(\bar{\boldsymbol{R}}, \boldsymbol{u}, v) \in \mathrm{SO}(3)\times H^1(\omega,\mathbb{R}^2)\times H^1(\omega)$. Then $v \in H^2(\omega)$ and there exists $\boldsymbol{U} \in L^2(\omega, L_\gamma(S\times\mathcal{Y}, \mathbb{M}^3_{\rm sym}))$ such that, up to a subsequence,

$$\boldsymbol{E}^h(\boldsymbol{y}^h) \xrightharpoonup{2,\gamma} \boldsymbol{E}(\boldsymbol{u}, v) + \boldsymbol{U} \quad \text{weakly two-scale in } L^2.$$

(ii) (Approximation). Let $\bar{\boldsymbol{R}} \in \mathrm{SO}(3)$, $(\boldsymbol{u}, v) \in \mathcal{A}(\omega)$ and $\boldsymbol{U} \in L^2(\omega, L_\gamma(S\times\mathcal{Y}, \mathbb{M}^3_{\rm sym}))$. There exists a sequence $\boldsymbol{y}^h$ that converges to $(\bar{\boldsymbol{R}}, \boldsymbol{u}, v)$ such that

$$\boldsymbol{E}^h(\boldsymbol{y}^h) \xrightarrow{2,\gamma} \boldsymbol{E}(\boldsymbol{u}, v) + \boldsymbol{U} \quad \text{strongly two-scale in } L^2(\Omega\times Y, \mathbb{M}^3_{\rm sym}),$$

and

$$\limsup_{h\to 0} h^2 ||\boldsymbol{E}^h(\boldsymbol{y}^h)||_{L^\infty} + ||\operatorname{dist}(\nabla_h \boldsymbol{y}^h, \mathrm{SO}(3))||_{L^\infty} = 0.$$

In the proof of Proposition 3.3 we need the following auxiliary lemma concerning the linearization of the matrix square root.

Lemma 3.4. *Let $\boldsymbol{G}^h, \boldsymbol{K}^h \in L^2(\Omega, \mathbb{M}^3)$ be such that*

$$\boldsymbol{K}^h \text{ is skew-symmetric and} \tag{20}$$

$$\limsup_{h\to 0} ||\boldsymbol{G}^h||_{L^2} + ||\boldsymbol{K}^h||_{L^4} < \infty. \tag{21}$$

Consider

$$\boldsymbol{E}^h := \frac{\sqrt{(\boldsymbol{I} + h\boldsymbol{K}^h + h^2\boldsymbol{G}^h)^t(\boldsymbol{I} + h\boldsymbol{K}^h + h^2\boldsymbol{G}^h)} - \boldsymbol{I}}{h^2}.$$

(i) If $\limsup_{h\to 0}(h||\boldsymbol{K}^h||_{L^\infty} + h^2||\boldsymbol{G}^h||_{L^\infty}) = 0$, then

$$\lim_{h\to 0} ||\boldsymbol{E}^h - (\operatorname{sym}\boldsymbol{G}^h - \frac{1}{2}(\boldsymbol{K}^h)^2)||_{L^2} = 0.$$

(ii) If $\limsup_{h\to 0}||\boldsymbol{E}^h||_{L^2}<\infty$*, then*

$$\lim_{h\to 0}\left|\int_\Omega (\boldsymbol{E}^h-(\operatorname{sym}\boldsymbol{G}^h-\frac{1}{2}(\boldsymbol{K}^h)^2)):\boldsymbol{\Psi}^{\varepsilon(h)}\,dx\right|=0.$$

for all $\boldsymbol{\Psi}^{\varepsilon(h)}(x):=\boldsymbol{\Psi}(x,\frac{\hat{x}}{\varepsilon(h)})$ *with* $\boldsymbol{\Psi}\in C_c^\infty(\Omega,C^\infty(\mathcal{Y},\mathbb{M}^d))$*.*

For part (ii) of the proposition we have to approximate relaxation fields $\boldsymbol{U}\in L^2(\omega,L_\gamma(S\times\mathcal{Y},\mathbb{M}^3_{\text{sym}}))$ by sequences of higher order terms:

Lemma 3.5.

(a) Let $\gamma\in(0,\infty]$ *and* $\boldsymbol{U}\in L^2(\omega,L_\gamma(S\times\mathcal{Y},\mathbb{M}^3_{\text{sym}}))$*. There exists a sequence* $\boldsymbol{\phi}^h\in H^1(\Omega,\mathbb{R}^3)$ *such that*

(a1) $$\limsup_{h\to 0}||\nabla_h\boldsymbol{\phi}^h||_{L^2}<\infty,$$

(a2) $$\limsup_{h\to 0}\left\{||\boldsymbol{\phi}^h||_{L^\infty}+||h^2\nabla_h\boldsymbol{\phi}^h||_{L^\infty}\right\}=0$$

(a3) $$\operatorname{sym}\nabla_h\boldsymbol{\phi}^h\xrightarrow{2,\gamma}\boldsymbol{U}\qquad\text{strongly two-scale in } L^2(\Omega\times Y,\mathbb{M}^3_{\text{sym}}).$$

(b) Let $\gamma=0$ *and* $\boldsymbol{U}\in L^2(\omega,L_0(S\times\mathcal{Y},\mathbb{M}^3_{\text{sym}}))$*. Then there exists a function* $\varphi\in L^2(\omega,H^2(\mathcal{Y}))$ *and a sequence* $\boldsymbol{\phi}^h\in H^1(\Omega,\mathbb{R}^3)$ *such that* (a1)*,* (a2) *hold and*

(a3') $$\operatorname{sym}\nabla_h\boldsymbol{\phi}^h\xrightarrow{2,\gamma}\boldsymbol{U}-\begin{pmatrix} x_3\hat{\nabla}_y^2\varphi & & 0\\ & & 0\\ 0 & 0 & 0\end{pmatrix}\quad\text{strongly two-scale in } L^2(\Omega\times Y,\mathbb{M}^3_{\text{sym}}).$$

4. Proofs

In this section we present the proofs of the previous results in the following ordering: Proposition 3.1, Proposition 3.3, Lemma 3.5, Theorem 2.3 and Lemmas 2.2, 2.10, 2.11 and 3.4.

4.1. Proof of Proposition 3.1.

The proof crucially relies on the following theorem by Friesecke, James and Müller:

Theorem 4.1. *(see* [FJM06, Theorem 6]*) Let* $\omega\subset\mathbb{R}^2$ *be a domain,* $\Omega=\omega\times S$*. Let* $\boldsymbol{y}\in H^1(\Omega;\mathbb{R}^3)$*. Then there exist maps* $\boldsymbol{R}:\omega\to\mathrm{SO}(3)$*,* $\boldsymbol{P}\in H^1(\omega,\mathbb{M}^3)$ *and a matrix* $\bar{\boldsymbol{R}}\in\mathrm{SO}(3)$ *such that*

(22) $$\|\nabla_h\boldsymbol{y}-\boldsymbol{P}\|^2_{L^2(\Omega)}\lesssim h^4e_h(\boldsymbol{y}),$$

(23) $$\|\boldsymbol{P}-\boldsymbol{R}\|^2_{L^2(\omega)}\lesssim h^4e_h(\boldsymbol{y}),$$

(24) $$\|\hat{\nabla}\boldsymbol{P}\|^2_{L^2(\omega)}\lesssim h^2e_h(\boldsymbol{y}),$$

(25) $$\|\boldsymbol{P}-\boldsymbol{R}\|^2_{L^\infty(\omega)}\lesssim h^2e_h(\boldsymbol{y}),$$

(26) $$\|\boldsymbol{P}\|_{L^\infty(\omega)}\lesssim 1,$$

(27) $$\|\nabla_h\boldsymbol{y}-\bar{\boldsymbol{R}}\|^2_{L^2(\Omega)}\lesssim h^2e_h(\boldsymbol{y}),$$

(28) $$\|\boldsymbol{P}-\bar{\boldsymbol{R}}\|^2_{L^p(\omega)}\lesssim_p h^2e_h(\boldsymbol{y}),\quad\forall p<\infty.$$

Here $\lesssim$ *means* $\le$ *up to a multiplicative constant that only depends on* ω*, and* $\lesssim_p$ *means* $\le$ *up to a multiplicative constant that might depend on* p *in addition.*

The theorem is based on the celebrated quantitative geometric rigidity estimate in [FJM02] and yields an approximation of $\nabla_h\boldsymbol{y}^h$ by a rotation field accompanied with quantitative estimates. Based on Theorem 4.1 we prove Proposition 3.1.

Proof of Proposition 3.1. The proof is divided in five steps. In the first four steps we construct the fields $\boldsymbol{u}, v, w$ and $\boldsymbol{\phi}$, establish identity (15) and prove estimate (16) which basically relies on Theorem 4.1. In Step 5 we prove estimate (17) by appealing to elliptic regularity.

Step 1. Construction of the matrix field $\boldsymbol{P}$.

Application of Theorem 4.1 yields a rotation $\bar{\boldsymbol{R}} \in \mathrm{SO}(3)$, a matrix field $\boldsymbol{P} \in H^1(\omega, \mathbb{M}^3)$ and a rotation field $\boldsymbol{R} : \omega \to \mathrm{SO}(3)$ that basically approximate $\nabla_h \boldsymbol{y}$ such that $\nabla_h \boldsymbol{y} \approx \boldsymbol{P} + O(h^4)$ and $\boldsymbol{P} \approx \bar{\boldsymbol{R}} + O(h^2)$ (see (22) – (28) for the precise estimates). By replacing $\boldsymbol{y}$, $\boldsymbol{P}$ and $\boldsymbol{R}$ by $\bar{\boldsymbol{R}}^t(\boldsymbol{y} - \fint_\Omega \boldsymbol{y})$, $\bar{\boldsymbol{R}}^t\boldsymbol{P}$ and $\bar{\boldsymbol{R}}^t\boldsymbol{R}$, we can assume without loss of generality that $\bar{\boldsymbol{R}} = \boldsymbol{I}$ and $\fint_\Omega \boldsymbol{y} = 0$.

We claim that

$$\left\| \frac{\mathrm{sym}\,(\boldsymbol{P} - \boldsymbol{I})}{h^2} \right\|^2_{L^2} \lesssim e_h(\boldsymbol{y}) + e_h(\boldsymbol{y})^2. \tag{29}$$

Indeed, the elementary identity $2\,\mathrm{sym}\,(\boldsymbol{P} - \boldsymbol{I}) = -(\boldsymbol{P}^t - \boldsymbol{I})(\boldsymbol{P} - \boldsymbol{I}) + \boldsymbol{P}^t\boldsymbol{P} - \boldsymbol{I}$ yields

$$\|\,\mathrm{sym}\,(\boldsymbol{P} - \boldsymbol{I})\|^2_{L^2} \le \|(\boldsymbol{P}^t - \boldsymbol{I})(\boldsymbol{P} - \boldsymbol{I})\|^2_{L^2} + \|\boldsymbol{P}^t\boldsymbol{P} - \boldsymbol{I}\|^2_{L^2}.$$

By the Cauchy-Schwarz inequality, the first term on the right-hand side can be estimated as

$$\|(\boldsymbol{P}^t - \boldsymbol{I})(\boldsymbol{P} - \boldsymbol{I})\|^2_{L^2} \le \|\boldsymbol{P} - \boldsymbol{I}\|^4_{L^4} \overset{(28)}{\lesssim} h^4 e_h(\boldsymbol{y})^2,$$

while the second term is treated by appealing to (23) and (26)

$$\begin{aligned} \|(\boldsymbol{P}^t\boldsymbol{P} - \boldsymbol{I})\|^2_{L^2} &\le \|(\boldsymbol{P} - \boldsymbol{R})^t\boldsymbol{P}\|^2_{L^2} + \|\boldsymbol{R}^t(\boldsymbol{P} - \boldsymbol{R})\|^2_{L^2} \\ &\le 2\|\boldsymbol{P}\|^2_{L^\infty}\|\boldsymbol{P} - \boldsymbol{R}\|^2_{L^2} + 2\|\boldsymbol{P} - \boldsymbol{R}\|^2_{L^2} \lesssim h^4 e_h(\boldsymbol{y}). \end{aligned}$$

Step 2. Construction of $\boldsymbol{u}$ and $\tilde{v}$.

We define the scaled in-plane and out-of plane displacement $\boldsymbol{u} \in H^1(\omega, \mathbb{R}^2)$ and $\tilde{v} \in H^1(\omega)$ by the identity

$$\overline{\boldsymbol{y}}(\hat{x}) := \fint_S \boldsymbol{y}(\hat{x}, x_3)\,dx_3 = \begin{pmatrix} \hat{x} + h^2\boldsymbol{u}(\hat{x}) \\ h\tilde{v}(\hat{x}) \end{pmatrix}. \tag{30}$$

We claim that

$$||\boldsymbol{u}||^2_{H^1} + ||\tilde{v}||^2_{H^1} \lesssim e_h(\boldsymbol{y}) + e_h(\boldsymbol{y})^2. \tag{31}$$

The estimate on $\boldsymbol{u}$ can be derived as follows. By appealing to $\int_\Omega \boldsymbol{y}\,dx = 0$ and assumption (4) we find that $\boldsymbol{u}$ has zero integral mean. Hence, Poincaré's and Korn's inequality yield the estimate

$$||\boldsymbol{u}||^2_{H^1} \lesssim \int_\omega |\hat{\nabla}\boldsymbol{u}|^2\,d\hat{x} \lesssim \int_\omega |\hat{\nabla}\boldsymbol{u} - K|^2\,d\hat{x} + |K|^2 \lesssim \int_\omega |\,\mathrm{sym}\,\hat{\nabla}\boldsymbol{u}|^2\,d\hat{x} + |K|^2$$

where $K := \fint_\omega \mathrm{skw}\,\hat{\nabla}\boldsymbol{u}\,d\hat{x}$. Let $\boldsymbol{I}_2$ denote the identity matrix in $\mathbb{M}^2$. We have

$$h^2|K|^2 = \left| \fint_\omega \mathrm{skw}(\hat{\nabla}\overline{\boldsymbol{y}} - \boldsymbol{I}_2)dx \right|^2 \overset{\text{Jensen}}{\lesssim} \fint_\Omega |\nabla_h\boldsymbol{y} - \boldsymbol{I}|^2\,dx \overset{(27)}{\lesssim} h^2 e_h(\boldsymbol{y}). \tag{32}$$

Moreover,

$$\begin{aligned} ||\,\mathrm{sym}\,\hat{\nabla}\boldsymbol{u}||^2_{L^2(\omega)} &= ||\,\mathrm{sym}\left(\frac{\hat{\nabla}\overline{\boldsymbol{y}} - \boldsymbol{I}_2}{h^2}\right)||^2_{L^2(\omega)} \\ &\overset{\triangle\text{-ineq.,Jensen}}{\le} ||\,\mathrm{sym}\left(\frac{\nabla_h\boldsymbol{y} - \boldsymbol{P}}{h^2}\right)||^2_{L^2(\Omega)} + ||\,\mathrm{sym}\left(\frac{\boldsymbol{P} - \boldsymbol{I}}{h^2}\right)||^2_{L^2(\Omega)} \\ &\overset{(22),(29)}{\lesssim} e_h(\boldsymbol{y}) + e_h(\boldsymbol{y})^2, \end{aligned}$$

which completes the estimate on $\hat{\nabla}\boldsymbol{u}$. It remains to estimate $\tilde{v}$. Since $\tilde{v}$ has zero integral mean, Poincaré's inequality and (27) yield

$$||\tilde{v}||^2_{H^1} \lesssim \int_\omega |\hat{\nabla}\tilde{v}|^2\,d\hat{x} \le \frac{1}{h^2}\int_\Omega |\hat{\nabla}\boldsymbol{y}_3|^2\,dx \overset{(27)}{\lesssim} e_h(\boldsymbol{y}).$$

Step 3. Construction of v and w.

In this step we decompose the out-of-plane displacement $\tilde{v}$ into a contribution v (that turns out to be of class H^2_{loc} as we are going to see in Step 5) and a higher order correction w. To this end set $\boldsymbol{p} := \frac{1}{h}(\boldsymbol{P}_{31}, \boldsymbol{P}_{32})$ and let $v \in H^1(\omega)$, $\int_\omega v\,d\hat{x} = 0$, denote the unique minimizer of

$$\int_\omega |\hat{\nabla} v - \boldsymbol{p}|^2\,d\hat{x} \tag{33}$$

amongst all functions in $H^1(\omega)$ with zero integral mean. We define $w \in H^1(\omega)$ via the identity $\tilde{v} = v + hw$ and claim that

$$\int_\omega \left|\frac{\hat{\nabla} v - \boldsymbol{p}}{h}\right|^2 d\hat{x} + ||v||^2_{H^1} + ||w||^2_{H^1} \lesssim e_h(\boldsymbol{y}). \tag{34}$$

Indeed, by the minimality of v we get with the trial function $\hat{x} \mapsto h^{-1}\overline{\boldsymbol{y}}_3$

$$\begin{aligned}\int_\omega \left|\frac{\hat{\nabla} v - \boldsymbol{p}}{h}\right|^2 d\hat{x} \quad &\le \quad \int_\omega \left|\frac{\hat{\nabla}(h^{-1}\overline{\boldsymbol{y}}_3) - \boldsymbol{p}}{h}\right|^2 d\hat{x} \\ &\overset{\text{Jensen}}{\le} \quad \int_\Omega \left|\frac{\hat{\nabla}\boldsymbol{y}_3 - (\boldsymbol{P}_{31}, \boldsymbol{P}_{32})}{h^2}\right|^2 dx \overset{(22)}{\lesssim} e_h(\boldsymbol{y}).\end{aligned} \tag{35}$$

Furthermore, we have

$$\begin{aligned}||v||^2_{H^1} &\overset{\text{Poincaré}}{\lesssim} ||\hat{\nabla} v||^2_{L^2} \lesssim ||\hat{\nabla} v - \boldsymbol{p}||^2_{L^2} + ||\boldsymbol{p}||^2_{L^2} \\ &\overset{(35)}{\lesssim} e_h(\boldsymbol{y}) + \int_\omega \left|\frac{\boldsymbol{P} - \boldsymbol{I}}{h}\right|^2 d\hat{x} \overset{(28)}{\lesssim} e_h(\boldsymbol{y}).\end{aligned}$$

It remains to estimate $w = \frac{\tilde{v} - v}{h}$. We have

$$\begin{aligned}||w||^2_{H^1} &\overset{\text{Poincaré}}{\lesssim} ||\hat{\nabla} w||^2_{L^2} = \frac{1}{h^2}||\hat{\nabla}\tilde{v} - \hat{\nabla} v||^2_{L^2} \\ &\lesssim \frac{1}{h^2}||\hat{\nabla}\tilde{v} - \boldsymbol{p}||^2_{L^2} + \frac{1}{h^2}||\hat{\nabla} v - \boldsymbol{p}||^2_{L^2} \\ &\overset{(30),(35)}{\lesssim} \frac{1}{h^4}\int_\omega \left|\hat{\nabla}\overline{\boldsymbol{y}}_3 - (\boldsymbol{P}_{31}, \boldsymbol{P}_{32})\right|^2 d\hat{x} + e_h(\boldsymbol{y}) \\ &\overset{\text{Jensen},(22)}{\lesssim} e_h(\boldsymbol{y}).\end{aligned}$$

Step 4. Definition of $\boldsymbol{\phi}$ and proof of (15) and (16).

Let us define the corrector field $\boldsymbol{\phi} \in L^2(\Omega, \mathbb{R}^3)$ via the identity

$$\boldsymbol{y} = \begin{pmatrix} \hat{x} \\ hx_3 \end{pmatrix} + \begin{pmatrix} h^2\boldsymbol{u} \\ hv + h^2 w \end{pmatrix} - h^2 x_3 \begin{pmatrix} \hat{\nabla} v \\ 0 \end{pmatrix} + h^2\boldsymbol{\phi}. \tag{36}$$

Note that this is precisely (15), since we have assumed that $\fint_\Omega \boldsymbol{y}\,dx = 0$ and $\bar{\boldsymbol{R}} = \boldsymbol{I}$. In view of (31) and (34), for (16) we only need to prove that $h^{-2}||\boldsymbol{\phi}||^2_{L^2(\Omega)} \lesssim e_h(\boldsymbol{y}) + e_h(\boldsymbol{y})^2$. By construction we have $\fint_S \boldsymbol{\phi}(\hat{x}, x_3)\,dx_3 = 0$ and $\boldsymbol{\phi}(\hat{x}, \cdot) \in H^1(S, \mathbb{R}^3)$ for almost every $\hat{x} \in \omega$. Hence, by Poincaré's inequality, the desired estimate on $\boldsymbol{\phi}$ follows from

$$||\tfrac{1}{h}\partial_3\boldsymbol{\phi}||^2_{L^2(\Omega)} \lesssim e_h(\boldsymbol{y}) + e_h(\boldsymbol{y})^2, \tag{37}$$

which we prove in the following for each component ϕ_α separately. For $\alpha = 1,2$ we have, using (22), (29) and (34):

$$\begin{aligned}
\|\tfrac{1}{h}\partial_3\boldsymbol{\phi}_\alpha\|_{L^2}^2 &\overset{(36)}{=} \left\|\frac{\frac{1}{h}\partial_3\boldsymbol{y}_\alpha + h\partial_\alpha v}{h^2}\right\|_{L^2}^2 \\
&\le 3\left\|\frac{\frac{1}{h}\partial_3\boldsymbol{y}_\alpha - \boldsymbol{P}_{\alpha 3}}{h^2}\right\|_{L^2}^2 + 3\left\|\frac{\boldsymbol{P}_{\alpha 3}+\boldsymbol{P}_{3\alpha}}{h^2}\right\|_{L^2}^2 + 3\left\|\frac{h\partial_\alpha v - \boldsymbol{P}_{3\alpha}}{h^2}\right\|_{L^2}^2 \\
&\lesssim e_h(\boldsymbol{y}) + e_h(\boldsymbol{y})^2.
\end{aligned}$$

For $\alpha = 3$ we have

$$\begin{aligned}
\|\tfrac{1}{h}\partial_3\boldsymbol{\phi}_3\|_{L^2}^2 &= h^{-4}\|\tfrac{1}{h}\partial_3\boldsymbol{y}_3 - 1\|_{L^2}^2 \\
&\le h^{-4}\left\|\frac{1}{h}\partial_3\boldsymbol{y}_3 - \boldsymbol{P}_{33}\right\|_{L^2}^2 + h^{-4}\|\boldsymbol{P}_{33}-1\|_{L^2}^2 \\
&\overset{(22),(23)}{\lesssim} e_h(\boldsymbol{y}).
\end{aligned}$$

Step 5. H^2_{loc}-regularity of v and proof of (17).

As a minimizer of the functional (33) the function $v \in H^1(\omega)$ satisfies the Euler-Lagrange equation

$$\int_\omega \nabla v\cdot\nabla\eta = \int_\omega \boldsymbol{p}\cdot\nabla\eta \qquad \text{for all } \eta \in H^1(\omega).$$

Since $\nabla\cdot\boldsymbol{p} \in L^2(\omega)$ and because ω is a Lipschitz domain, elliptic regularity theory implies that $v \in H^2_{\mathrm{loc}}(\omega)$, i. e. for every set $M \subset \omega$ with $\mathrm{dist}(M,\partial\omega) > 0$ we have $\|\nabla^2 v\|_{L^2(M)} \lesssim_M \big(\|\nabla\cdot\boldsymbol{p}\|_{L^2(\omega)} + \|\boldsymbol{p}\|_{L^2(\omega)}\big)$. By (24) and (28) that estimate turns into

$$\|\nabla^2 v\|_{L^2(M)} \lesssim_M \big(\|\nabla\cdot\boldsymbol{p}\|_{L^2(\omega)} + \|\boldsymbol{p}\|_{L^2(\omega)}\big) \lesssim_M e_h(\boldsymbol{y}). \tag{38}$$

If ω is a domain of class $C^{1,1}$, then H^2-regularity holds up to the boundary, and (38) is fulfilled even for $M = \omega$. After these preliminary remarks we claim that (17) holds for every set $M \subset \omega$ that satisfies (38). In view of (38) and (37) we only have to show

$$\|\hat{\nabla}\boldsymbol{\phi}\|_{L^2(M\times S)}^2 \lesssim_M e_h(\boldsymbol{y}).$$

For the argument notice that by (30) and (36)

$$\partial_\alpha\boldsymbol{y}_\beta = \partial_\alpha\overline{\boldsymbol{y}}_\beta - h^2 x_3\partial^2_{\alpha\beta}v + h^2\partial_\alpha\boldsymbol{\phi}_\beta \qquad \text{for } \alpha,\beta = 1,2.$$

Hence,

$$\begin{aligned}
\|\partial_\alpha\boldsymbol{\phi}_\beta\|_{L^2}^2 &\le 2\left\|\frac{\partial_\alpha\boldsymbol{y}_\beta - \partial_\alpha\overline{\boldsymbol{y}}_\beta}{h^2}\right\|_{L^2}^2 + 2\|x_3\partial^2_{\alpha\beta}v\|_{L^2}^2 \\
&\lesssim \left\|\frac{\partial_\alpha\boldsymbol{y}_\beta - \boldsymbol{P}_{\beta\alpha}}{h^2}\right\|_{L^2}^2 + \left\|\frac{\partial_\alpha\overline{\boldsymbol{y}}_\beta - \boldsymbol{P}_{\beta\alpha}}{h^2}\right\|_{L^2}^2 + \|\hat{\nabla}^2 v\|_{L^2}^2 \\
&\lesssim_M e_h(\boldsymbol{y})
\end{aligned}$$

where we used (22), (31),(28) and (38). Similarly, we have by (22) and (34)

$$\begin{aligned}
\|\hat{\nabla}\boldsymbol{\phi}_3\|_{L^2}^2 &= \left\|\frac{\hat{\nabla}\boldsymbol{y}_3 - (h\hat{\nabla}v + h^2\hat{\nabla}w)}{h^2}\right\|_{L^2}^2 \\
&\lesssim \frac{1}{h^4}\left\|\hat{\nabla}\boldsymbol{y}_3 - (\boldsymbol{P}_{31},\boldsymbol{P}_{32})\right\|_{L^2}^2 + \frac{1}{h^2}\left\|\frac{1}{h}(\boldsymbol{P}_{31},\boldsymbol{P}_{32}) - \hat{\nabla}v\right\|_{L^2}^2 + \|\hat{\nabla}w\|_{L^2}^2 \\
&\lesssim e_h(\boldsymbol{y}).
\end{aligned}$$

□

4.2. **Proof of Proposition 3.3 and Lemma 3.5.** We present the proof of the two-scale identification result for the nonlinear strain. The argument relies on the representation established in Proposition 3.1 and a two-scale identification result for scaled gradients from [Neu10] (see Proposition A.4 in the appendix).

Proof of Proposition 3.3, part (i).
Step 1. Passage to the limit.
The boundedness $\limsup_{h\to 0} I^{\varepsilon(h),h}(\boldsymbol{y}^h) < \infty$ combined with the elementary inequality $|\sqrt{\boldsymbol{F}^t\boldsymbol{F}} - \boldsymbol{I}| \leq \operatorname{dist}^2(\boldsymbol{F},\mathrm{SO}(3))$ shows that the sequence $\boldsymbol{E}^h := \boldsymbol{E}^h(\boldsymbol{y}^h)$ is bounded in L^2. Hence, by Lemma A.3 we can pass to a weakly two-scale convergent subsequence (not relabeled). Let $\boldsymbol{E} \in L^2(\Omega\times\mathcal{Y},\mathbb{M}^3_{\mathrm{sym}})$ denote its limit. In order to identify $\boldsymbol{E}$ we apply Proposition 3.1 and obtain a representation of $\boldsymbol{y}^h$ in terms of functions $\bar{\boldsymbol{R}}^h$, $\boldsymbol{u}^h$, v^h, $\boldsymbol{\phi}^h$ and w^h which, in particular, satisfy the estimate

$$\limsup_{h\to 0}\Big\{ ||\boldsymbol{u}^h||_{H^1(\omega)} + ||v^h||_{H^1(\omega)} + ||w^h||_{H^1(\omega)} + h^{-1}||\boldsymbol{\phi}^h||_{L^2(\Omega)} + ||\hat{\nabla}^2 v^h||_{L^2(M)} + ||\nabla_h\boldsymbol{\phi}^h||_{L^2(M\times S\times Y)}\Big\} < \infty \tag{39}$$

for all M that are compactly contained in ω. As a consequence, there exist a rotation $\widetilde{\boldsymbol{R}} \in \mathrm{SO}(3)$, functions $\tilde{v},\varphi \in H^1(\omega)$ and a vector field $\tilde{\boldsymbol{u}} \in H^1(\omega,\mathbb{R}^2)$ such that, up to a subsequence,

$$\begin{aligned} &\bar{\boldsymbol{R}}^h \to \widetilde{\boldsymbol{R}} \quad \text{in } \mathbb{M}^3, \qquad \boldsymbol{u}^h \rightharpoonup \tilde{\boldsymbol{u}} \quad \text{weakly in } H^1(\omega,\mathbb{R}^2)\\ &v^h \rightharpoonup \tilde{v} \quad \text{and} \quad w^h \rightharpoonup w \quad \text{weakly in } H^1(\omega). \end{aligned}\tag{40}$$

Averaging of (15) in x_3 yields the identity

$$\fint_S \boldsymbol{y}^h\,dx_3 - \fint_\Omega \boldsymbol{y}^h\,dx = \bar{\boldsymbol{R}}^h\begin{pmatrix} \hat{x} + h^2\boldsymbol{u}^h \\ h(v^h + hw^h)\end{pmatrix}.$$

Combined with (40) the identity shows that $\boldsymbol{y}^h \to (\bar{\boldsymbol{R}},\tilde{\boldsymbol{u}},\tilde{v})$. By appealing to Lemma 2.2 we eventually find that $\widetilde{\boldsymbol{R}} = \bar{\boldsymbol{R}}$ and $(\tilde{\boldsymbol{u}},\tilde{v}) \sim (\boldsymbol{u},v)$.

Step 2. Identification of $\boldsymbol{E}$. Localization.

Let us split $\boldsymbol{E}$ into the averaged upper left 2×2 minor and the remainder:

$$\boldsymbol{E}'(x) := \int_Y \sum_{\alpha,\beta=1}^{2} \boldsymbol{E}_{\alpha\beta}(x,y)(\boldsymbol{e}_\alpha\otimes\boldsymbol{e}_\beta)\,dy, \qquad \boldsymbol{U}(x,y) := \boldsymbol{E}(x,y) - \boldsymbol{E}'(x).$$

We claim that statement (i) of the proposition follows, if for all Lipschitz domains M that are compactly contained in ω the following two statements hold:

(a) $\tilde{v}|_M \in H^2(M)$ and $\boldsymbol{E}' = \boldsymbol{E}(\tilde{\boldsymbol{u}},\tilde{v})$ almost everywhere in $M\times S$,
(b) for almost every $\hat{x}\in M$ we have

$$(x_3,y) \mapsto \boldsymbol{U}(\hat{x},x_3,y) \in L_\gamma(S\times\mathcal{Y},\mathbb{M}^3_{\mathrm{sym}}).$$

Indeed, since ω can be exhausted by subsets of the form M, (b) implies that $(x_3,y) \mapsto \boldsymbol{U}(\hat{x},x_3,y) \in L_\gamma(S\times\mathcal{Y},\mathbb{M}^3_{\mathrm{sym}})$ for almost every $\hat{x}\in\omega$. Moreover, by orthogonality, (a) and (b) imply

$$||\boldsymbol{E}'||^2_{L^2(M\times S)} + ||\boldsymbol{U}||^2_{L^2(M\times S\times Y)} \leq ||\boldsymbol{E}||^2_{L^2(\Omega)}.$$

Since the right-hand side does not depend on the set M, we deduce that $\boldsymbol{U} \in L^2(\omega, L_\gamma(S\times\mathcal{Y},\mathbb{M}^3_{\mathrm{sym}}))$ and $\boldsymbol{E}' \in L^2(\Omega)$. Notice that the latter implies $\tilde{v}\in H^2(\omega)$. Since $(\tilde{\boldsymbol{u}},\tilde{v}) \sim (\boldsymbol{u},v)$, we have $v \in H^2(\omega)$ and $\boldsymbol{E}' = \boldsymbol{E}(\boldsymbol{u},v)$ (see (19)). In conclusion statement (i) of the proposition follows.

Step 3. Proof of (a) and (b) in Step 2.
Fix an arbitrary Lipschitz domain M that is compactly contained in ω. In the following all

functions (and domains of integration) are restricted to M (resp. $M\times S$ and $M\times S\times Y$). By virtue of identity (15) it is easy to check that

$$(\bar{\boldsymbol{R}}^h)^t\nabla_h\boldsymbol{y}^h=\boldsymbol{I}+h\boldsymbol{K}^h+h^2(\boldsymbol{G}^h+\nabla_h\boldsymbol{\phi}^h)$$

where

$$\boldsymbol{K}^h=\begin{pmatrix}0&0&-\partial_1\tilde{v}^h\\0&0&-\partial_2\tilde{v}^h\\\partial_1\tilde{v}^h&\partial_2\tilde{v}^h&0\end{pmatrix},\qquad \boldsymbol{G}^h=\begin{pmatrix}\hat{\nabla}\tilde{\boldsymbol{u}}^h-x_3\hat{\nabla}^2\tilde{v}^h&\begin{matrix}0\\0\end{matrix}\\\partial_1 w^h\ \ \partial_2 w^h&0\end{pmatrix}.$$

By estimate (39) we get $v^h\rightharpoonup\tilde{v}$ in $H^2(M)$ and $\boldsymbol{\phi}^h\to0$ in $L^2(M\times S\times\mathcal{Y})$, up to a subsequence. Combined with the embedding $H^1(M)\subset L^4(M)$ we deduce that

$$\limsup_{h\to0}\Big(||\boldsymbol{G}^h||_{L^2}+||\boldsymbol{K}^h||_{L^4}\Big)<\infty.$$

Hence, Lemma A.3 and Proposition A.4 yield, up to a subsequence,

$$\begin{aligned}
&(\boldsymbol{K}^h)^2\to\boldsymbol{K}^2:=\begin{pmatrix}\hat{\nabla}\tilde{v}\otimes\hat{\nabla}\tilde{v}&\begin{matrix}0\\0\end{matrix}\\0\quad 0&|\hat{\nabla}\tilde{v}|^2\end{pmatrix}\qquad\text{strongly in } L^2(M),\\
&\boldsymbol{G}^h\xrightarrow{2,\gamma}\begin{pmatrix}\hat{\nabla}\tilde{\boldsymbol{u}}-x_3\hat{\nabla}^2\tilde{v}&\begin{matrix}0\\0\end{matrix}\\0\quad 0&0\end{pmatrix}+\begin{pmatrix}\hat{\nabla}_y\boldsymbol{\zeta}-x_3\hat{\nabla}^2\varphi&\begin{matrix}0\\0\end{matrix}\\\hat{\nabla}w+\hat{\nabla}_y\psi&0\end{pmatrix}\\
&\nabla_h\boldsymbol{\phi}^h\xrightarrow{2,\gamma}\boldsymbol{H}
\end{aligned}\tag{41}$$

where $\boldsymbol{\zeta}\in L^2(M,H^1(\mathcal{Y},\mathbb{R}^2))$, $\psi\in L^2(M,H^1(\mathcal{Y}))$, $\varphi\in L^2(M,H^2(\mathcal{Y}))$ and $\boldsymbol{H}\in L^2(M\times S\times Y)$ as in Proposition A.4 (98). Application of Lemma 3.4 (ii) yields

$$\operatorname{sym}(\boldsymbol{G}^h+\operatorname{sym}\nabla_h\boldsymbol{\phi}^h)-\frac{1}{2}(\boldsymbol{K}^h)^2\xrightarrow{2,\gamma}\boldsymbol{E}\qquad\text{weakly two-scale in } L^2(M\times S\times\mathcal{Y}).$$

Combined with (41) we get

$$\boldsymbol{E}=\boldsymbol{E}(\tilde{\boldsymbol{u}},\tilde{v})+\boldsymbol{U}',\qquad \boldsymbol{U}':=\operatorname{sym}\begin{pmatrix}\hat{\nabla}_y\boldsymbol{\zeta}-x_3\hat{\nabla}_y^2\varphi&\begin{matrix}0\\0\end{matrix}\\\hat{\nabla}w+\hat{\nabla}_y\psi&-\frac{1}{2}|\hat{\nabla}\tilde{v}|^2\end{pmatrix}+\operatorname{sym}\boldsymbol{H}.$$

Since $\int_Y\boldsymbol{U}'_{\alpha\beta}\,dy=0$ for $\alpha,\beta\in\{1,2\}$, we deduce that $\boldsymbol{E}'=\boldsymbol{E}(\tilde{\boldsymbol{u}},\tilde{v})$ and statement (a) follows. It remains to argue that $\boldsymbol{U}'$ satisfies property (b). We treat the regimes $\gamma=0$, $\gamma\in(0,\infty)$ and $\gamma=\infty$ separately.

Case $\gamma=0$: By (98) we have $\boldsymbol{H}=\Big(\hat{\nabla}_y\boldsymbol{\phi}^{(1)},\partial_3\boldsymbol{\phi}^{(2)}\Big)$ for some $\boldsymbol{\phi}^{(1)}\in L^2(M,H^1(\mathcal{Y}))$ and $\boldsymbol{\phi}^{(2)}\in L^2(M\times Y,H^1(S))$. For $\alpha=1,2$ set

$$\tilde{\boldsymbol{\zeta}}_\alpha:=\boldsymbol{\zeta}_\alpha+\boldsymbol{\phi}^{(1)}_\alpha,\qquad 2\boldsymbol{g}_\alpha:=\partial_\alpha w+\partial_{y_\alpha}\psi+\partial_{y_\alpha}\boldsymbol{\phi}^{(1)}_3+\partial_3\boldsymbol{\phi}^{(2)}_\alpha,\qquad \boldsymbol{g}_3=-\tfrac{1}{2}|\hat{\nabla}\tilde{v}|^2+\partial_3\boldsymbol{\phi}^{(2)}_3.$$

Then $\tilde{\boldsymbol{\zeta}}\in L^2(M,H^1(\mathcal{Y},\mathbb{R}^2))$, $\boldsymbol{g}\in L^2(M\times S\times\mathcal{Y},\mathbb{R}^3)$ and

$$\boldsymbol{U}'=\begin{pmatrix}\operatorname{sym}\hat{\nabla}_y\tilde{\boldsymbol{\zeta}}-x_3\hat{\nabla}_y^2\varphi&\begin{matrix}\boldsymbol{g}_1\\\boldsymbol{g}_2\end{matrix}\\(\boldsymbol{g}_1,\ \boldsymbol{g}_2)&\boldsymbol{g}_3\end{pmatrix}.$$

Hence, $\boldsymbol{U}'\in L^2(M,L_0(S\times\mathcal{Y},\mathbb{M}^3_{\mathrm{sym}}))$.

Case $\gamma=\infty$: By (98) we have $\boldsymbol{H}=\Big(\hat{\nabla}_y\boldsymbol{\phi}^{(1)},\partial_3\boldsymbol{\phi}^{(2)}\Big)$ for some $\boldsymbol{\phi}^{(1)}\in L^2(M\times S,H^1(\mathcal{Y}))$ and $\boldsymbol{\phi}^{(2)}\in L^2(M,H^1(S))$. For $\alpha=1,2$ set

$$\tilde{\boldsymbol{\zeta}}_\alpha:=\boldsymbol{\zeta}_\alpha-x_3\partial_\alpha\varphi+\boldsymbol{\phi}^{(1)}_\alpha,\qquad 2\tilde{\psi}=\psi+\boldsymbol{\phi}^{(1)}_3,\qquad 2\boldsymbol{c}_\alpha=\partial_\alpha w+\partial_3\boldsymbol{\phi}^{(2)}_\alpha,\ \boldsymbol{c}_3=-\tfrac{1}{2}|\hat{\nabla}\tilde{v}|^2+\partial_3\boldsymbol{\phi}^{(2)}_3.$$

Then $\tilde{\boldsymbol{\zeta}} \in L^2(M \times S, H^1(\mathcal{Y},\mathbb{R}^2))$, $\tilde{\psi} \in L^2(M \times S, H^1(\mathcal{Y}))$, $\boldsymbol{c} \in L^2(M \times S, \mathbb{R}^3)$ and

$$\boldsymbol{U}' = \begin{pmatrix} \operatorname{sym} \hat{\nabla}_y \tilde{\boldsymbol{\zeta}} & \begin{matrix} \partial_{y_1}\tilde{\psi} + \boldsymbol{c}_1 \\ \partial_{y_2}\tilde{\psi} + \boldsymbol{c}_2 \end{matrix} \\ \hat{\nabla}_y \tilde{\psi} + (\boldsymbol{c}_1,\ \boldsymbol{c}_2) & \boldsymbol{c}_3 \end{pmatrix}.$$

Hence, $\boldsymbol{U}' \in L^2(M, L_\infty(S \times \mathcal{Y}, \mathbb{M}^3_{\mathrm{sym}}))$.

Case $\gamma \in (0,\infty)$: By (98) we have $\boldsymbol{H} = \left(\hat{\nabla}_y \boldsymbol{\phi}, \frac{1}{\gamma}\partial_3 \boldsymbol{\phi}\right)$ for some $\boldsymbol{\phi} \in L^2(M, H^1(S \times \mathcal{Y}, \mathbb{R}^3))$. Set

$$\widetilde{\boldsymbol{\phi}} := \boldsymbol{\phi} + \begin{pmatrix} \boldsymbol{\zeta} \\ \psi \end{pmatrix} - \begin{pmatrix} x_3 \partial_{y_1}\varphi \\ x_3 \partial_{y_2}\varphi \\ -\frac{1}{\gamma}\varphi \end{pmatrix} + x_3 \gamma \begin{pmatrix} \partial_1 w \\ \partial_2 w \\ -\frac{1}{2}|\hat{\nabla}\tilde{v}|^2 \end{pmatrix}$$

Then $\widetilde{\boldsymbol{\phi}} \in L^2(M, H^1(S \times \mathcal{Y}, \mathbb{R}^3))$ and $\boldsymbol{U}' = \operatorname{sym}(\hat{\nabla}_y \widetilde{\boldsymbol{\phi}}, \frac{1}{\gamma}\partial_3 \widetilde{\boldsymbol{\phi}}) \in L^2(M, L_\gamma(S \times \mathcal{Y}, \mathbb{R}^3))$. □

Next, we prove the approximation result for the two-scale limit of the nonlinear strain.

Proof of Proposition 3.3, part (ii). Without loss of generality we assume that $\bar{\boldsymbol{R}} = \boldsymbol{I}$.

Step 1. Construction for smooth functions and $\gamma > 0$.

Let $\gamma \in (0,\infty]$ and consider smooth functions $\boldsymbol{u}^\delta \in C^1(\bar{\omega}, \mathbb{R}^2)$ and $v^\delta \in C^2(\bar{\omega})$. Since $\gamma > 0$ the relaxation field $\boldsymbol{U}$ can be approximated by a sequence $\boldsymbol{\phi}^h \in H^1(\Omega, \mathbb{R}^3)$ in the sense of Lemma 3.5 (a). Define

$$\boldsymbol{y}^{\delta,h}(x) := \begin{pmatrix} \hat{x} \\ hx_3 \end{pmatrix} + \begin{pmatrix} h^2 \boldsymbol{u}^\delta \\ hv^\delta \end{pmatrix} - h^2 x_3 \begin{pmatrix} \hat{\nabla} v^\delta \\ 0 \end{pmatrix} - h^3 x_3 \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2}|\hat{\nabla} v^\delta(\hat{x})|^2 \end{pmatrix} + h^2 \boldsymbol{\phi}^h. \tag{42}$$

We claim that

(a) $\overline{\boldsymbol{y}}^{\delta,h} := \fint_S \boldsymbol{y}^{\delta,h}\, dx_3$ converges to $(\boldsymbol{I}, \boldsymbol{u}^\delta, v^\delta)$ in the sense of

$$\limsup_{h\to 0} \int_\omega \sum_{\alpha=1,2} \left| \tfrac{\overline{\boldsymbol{y}}^{\delta,h}_\alpha - \hat{x}_\alpha}{h^2} - \boldsymbol{u}^\delta_\alpha \right|^2 + \left| \tfrac{\overline{\boldsymbol{y}}^{\delta,h}_3}{h} - v^\delta \right|^2 d\hat{x} = 0.$$

(b) We have

$$\limsup_{h\to 0} \left(h^2 ||\boldsymbol{E}^{\delta,h}||_{L^\infty} + ||\operatorname{dist}(\nabla_h \boldsymbol{y}^{\delta,h}, \mathrm{SO}(3))||_{L^\infty} \right) = 0.$$

(c) $\boldsymbol{E}^{\delta,h} := \boldsymbol{E}^h(\boldsymbol{y}^{\delta,h})$ converges strongly two-scale in L^2 to

$$\boldsymbol{E}^\delta := \boldsymbol{E}(\boldsymbol{u}^\delta, v^\delta) + \boldsymbol{U}.$$

Argument for (a): We have $\overline{\boldsymbol{y}}^{\delta,h} = \begin{pmatrix} \hat{x} + h^2 \boldsymbol{u}^\delta \\ hv^\delta \end{pmatrix} + h^2 \fint_S \boldsymbol{\phi}^h\, dx_3$. Hence, (a) follows from Lemma 3.5 (a2).

Argument for (b): By construction we have

$$\begin{aligned} &\nabla_h \boldsymbol{y}^h = \boldsymbol{I} + h\boldsymbol{K}^\delta + h^2 \boldsymbol{G}^{\delta,h} \qquad \text{with} \qquad \boldsymbol{K}^\delta := \begin{pmatrix} 0 & 0 & -\partial_1 v^\delta \\ 0 & 0 & -\partial_2 v^\delta \\ \partial_1 v^\delta & \partial_2 v^\delta & 0 \end{pmatrix}, \\ &\boldsymbol{G}^{\delta,h} := \begin{pmatrix} \hat{\nabla}\boldsymbol{u}^\delta - x_3 \hat{\nabla}^2 v^\delta & \begin{matrix} 0 \\ 0 \end{matrix} \\ -\frac{hx_3}{2}\hat{\nabla}(|\hat{\nabla} v^\delta|^2) & -\frac{1}{2}|\hat{\nabla} v^\delta|^2 \end{pmatrix} + \nabla_h \boldsymbol{\phi}^h. \end{aligned} \tag{43}$$

Since $\boldsymbol{u}^\delta$ and v^δ are smooth, and because of Lemma 3.5 (a2) we find that

$$\limsup_{h\to 0} \left\{ h||\boldsymbol{K}^{\delta,h}||_{L^\infty} + h^2 ||\boldsymbol{G}^{\delta,h}||_{L^\infty} \right\} = 0$$

and consequently

$$\lim_{h\to 0} ||\operatorname{dist}(\nabla_h \boldsymbol{y}^h, \mathrm{SO}(3))||_{L^\infty} = 0. \tag{44}$$

Recall that $\operatorname{dist}(\boldsymbol{F}, \mathrm{SO}(3)) = |\sqrt{\boldsymbol{F}^t\boldsymbol{F}} - \boldsymbol{I}|$ holds for matrices $\boldsymbol{F}$ with $\det \boldsymbol{F} > 0$. Hence, (44) yields

$$h^2|\boldsymbol{E}^{\delta,h}| = |\sqrt{(\nabla_h \boldsymbol{y}^{\delta,h})^t \nabla_h \boldsymbol{y}^{\delta,h}} - \boldsymbol{I}| = \operatorname{dist}(\nabla_h \boldsymbol{y}^{\delta,h}, \mathrm{SO}(3)) \qquad \text{for } h \ll 1,$$

and thus (b) follows.

Argument for (c): The combination of identity (43) with Lemma 3.5 (a1), and the smoothness of $\boldsymbol{u}^\delta$ and v^δ implies

$$\limsup_{h\to 0} ||\boldsymbol{G}^{\delta,h}||_{L^2} + ||\boldsymbol{K}^{\delta,h}||_{L^4} < \infty. \tag{45}$$

By appealing to Lemma 3.5 (a3) we get

$$\operatorname{sym} \boldsymbol{G}^{\delta,h} - \frac{1}{2}(\boldsymbol{K}^\delta)^2 \xrightarrow{2,\gamma} \boldsymbol{E}^\delta \qquad \text{strongly two-scale in } L^2 \text{ as } h \downarrow 0. \tag{46}$$

Due to (b) and (45) we can apply Lemma 3.4 (i) and deduce that the difference between $\boldsymbol{E}^{\delta,h}$ and the right-hand side in (46) strongly converges to 0 in L^2. Hence, (c) follows from (46).

Step 2. Conclusion for $\gamma > 0$.

The general approximation scheme is based on Step 1, a density argument and the selection of a diagonal sequence. Let $\boldsymbol{u}^\delta, v^\delta, \boldsymbol{y}^{\delta,h}$ and $\boldsymbol{E}^\delta$ be as in Step 1. By a density argument we may assume that

$$||\boldsymbol{u}^\delta - \boldsymbol{u}||_{H^1(\omega)} + ||v^\delta - v||_{H^2(\omega)} \le \delta. \tag{47}$$

Consider the quantity

$$\begin{aligned} g(\delta, h) := \mathfrak{d}(\varepsilon(h), \boldsymbol{E}^h(\boldsymbol{y}^{\delta,h}), \boldsymbol{E}) + h^2||\boldsymbol{E}^{\delta,h}||_{L^\infty(\Omega)} \\ + \int_\omega \sum_{\alpha=1,2} \left|\frac{\overline{\boldsymbol{y}}^{\delta,h}_\alpha - \hat{x}_\alpha}{h^2} - \boldsymbol{u}_\alpha\right|^2 + \left|\frac{\overline{\boldsymbol{y}}^{\delta,h}_3}{h} - v\right|^2 d\hat{x}, \end{aligned}$$

where $\mathfrak{d}$ is defined in Lemma A.2 and characterizes strong two-scale convergence. By Step 1, the properties of $\mathfrak{d}$ (see Lemma A.2) and (47) we have

$$\begin{aligned} \limsup_{\delta\to 0} \limsup_{h\to 0} g(\delta, h) \quad \le \quad \limsup_{\delta\to 0} \Big\{ ||\boldsymbol{E}^\delta - \boldsymbol{E}||_{L^2(\Omega\times Y)} \\ + \int_\omega \sum_{\alpha=1,2} |\boldsymbol{u}^\delta_\alpha - \boldsymbol{u}_\alpha|^2 + |v^\delta - v|^2 \, d\hat{x} \Big\} = 0. \end{aligned}$$

Hence, by virtue of Lemma A.7 there exists a diagonal sequence, i. e. a monotone function $h \mapsto \delta(h)$ with $\delta(h) \to 0$ as $h \to 0$ such that

$$\lim_{h\to 0} g(\delta(h), h) = 0. \tag{48}$$

Set $\boldsymbol{y}^h := \boldsymbol{y}^{\delta(h),h}$. Then (48) in particular implies that $\mathfrak{d}(\varepsilon(h), \boldsymbol{E}^h(\boldsymbol{y}^h), \boldsymbol{E}) \to 0$ and thus $\boldsymbol{E}^h(\boldsymbol{y}^h) \xrightarrow{2,\gamma} \boldsymbol{E}$. Moreover, (48) implies that $\limsup_{h\to 0} h^2||\boldsymbol{E}^h(\boldsymbol{y}^h)||_{L^\infty} = 0$. It remains to argue that $\boldsymbol{y}^h$ converges to $(\boldsymbol{I}, \boldsymbol{u}, v)$. To this end define $\tilde{\boldsymbol{u}}^h$ and $\tilde{v}^h$ by the identity

$$\fint_S \boldsymbol{y}^h(\hat{x}, x_3)\, dx_3 = \begin{pmatrix} \hat{x} + h^2 \tilde{\boldsymbol{u}}^h(\hat{x}) \\ h\tilde{v}^h(\hat{x}) \end{pmatrix}.$$

By construction we have

$$\limsup_{h\to 0} \{ ||\tilde{\boldsymbol{u}}^h||_{H^1} + ||\tilde{v}^h||_{H^1} \} < \infty,$$

$$\limsup_{h\to 0} \{ ||\boldsymbol{u}^{\delta(h),h} - \tilde{\boldsymbol{u}}^h||_{L^2} + ||v^{\delta(h),h} - \tilde{v}^h||_{L^2} \} = 0.$$

Hence, (48) implies that $\tilde{\boldsymbol{u}}^h \rightharpoonup \boldsymbol{u}$ and $\tilde{v}^h \rightharpoonup v$ weakly in H^1, and therefore $\boldsymbol{y}^h \to (\boldsymbol{I}, \boldsymbol{u}, v)$.

Step 3. Construction for $\gamma = 0$.

The general strategy of the construction is similar to the construction for $\gamma > 0$ presented in Step 1 and Step 2. Therefore, we only indicate the required modifications. In contrast to the regimes $\gamma > 0$, for $\gamma = 0$ application of Lemma 3.5 (b) yields a sequence $\boldsymbol{\phi}^h \in H^1(\Omega, \mathbb{R}^3)$ with

$$\operatorname{sym} \nabla \boldsymbol{\phi}^h \xrightarrow{2,\gamma} \boldsymbol{U} - \begin{pmatrix} x_3 \hat{\nabla}_y^2 \varphi & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for some $\varphi \in L^2(\omega, H^2(\mathcal{Y}))$. In order to capture the oscillations associated with φ we modify the construction of $\boldsymbol{y}^{\delta,h}$ in Step 1. We only consider the smooth case where $\boldsymbol{u}^\delta \in C^1(\bar{\omega}, \mathbb{R}^2)$, $v^\delta \in C^2(\bar{S})$ and $\varphi^\delta \in C_c^\infty(\omega, C^\infty(\mathcal{Y}))$. The approximation in the general case can be obtained as in Step 2 by appealing to a density argument, and the selection of a diagonal sequence as demonstrated in Step 2. Our goal is to construct a sequence of deformations satisfying (a), (b) and (c) of Step 1. To shorten the notation set $\pi_\varepsilon(\hat{x}) := (\hat{x}, \frac{\hat{x}}{\varepsilon(h)})$. Let $\boldsymbol{y}^{\delta,h}$ be given by identity (42) and define

$$\begin{aligned} \widetilde{\boldsymbol{y}}^{\delta,h}(x) \quad := \quad & \boldsymbol{y}^{\delta,h}(x) + x_3 h^2 \varepsilon(h) \begin{pmatrix} \partial_{y_1}\varphi^\delta \circ \pi_{\varepsilon(h)} + \varepsilon(h)\partial_1\varphi^\delta \circ \pi_{\varepsilon(h)} \\ \partial_{y_2}\varphi^\delta \circ \pi_{\varepsilon(h)} + \varepsilon(h)\partial_2\varphi^\delta \circ \pi_{\varepsilon(h)} \\ 0 \end{pmatrix} \\ & - h\varepsilon(h)^2 \begin{pmatrix} 0 \\ 0 \\ \varphi^\delta \circ \pi_{\varepsilon(h)} \end{pmatrix}. \end{aligned}$$

A direct calculation shows that

$$\nabla_h \widetilde{\boldsymbol{y}}^{\delta,h} - \nabla_h \boldsymbol{y}^h = h^2 \begin{pmatrix} x_3 \hat{\nabla}_y^2 \varphi^\delta \circ \pi_{\varepsilon(h)} & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix} + h\widetilde{\boldsymbol{K}}^{\delta,h} + h^2 \boldsymbol{T}^{\delta,h}, \tag{49}$$

where

$$\begin{aligned} \widetilde{\boldsymbol{K}}^{\delta,h} \quad &:= \quad \varepsilon(h) \begin{pmatrix} 0 \quad 0 & (\hat{\nabla}_y \varphi^\delta \circ \pi_{\varepsilon(h)})^t + \varepsilon(h)(\hat{\nabla}\varphi^\delta \circ \pi_{\varepsilon(h)})^t \\ 0 \quad 0 & \\ -\hat{\nabla}_y \varphi^\delta \circ \pi_{\varepsilon(h)} - \varepsilon(h)(\hat{\nabla}\varphi^\delta \circ \pi_{\varepsilon(h)}) & 0 \end{pmatrix}, \\ \boldsymbol{T}^{\delta,h} \quad &:= \quad \varepsilon(h) \begin{pmatrix} x_3 \Big((\partial_\beta \partial_{y_\alpha} \varphi^\delta + \partial_\alpha \partial_{y_\beta} \varphi^\delta) \circ \pi_{\varepsilon(h)} \Big)_{\alpha,\beta=1,2} & 0 \\ & 0 \\ 0 \quad 0 & 0 \end{pmatrix}. \end{aligned}$$

The first term on the right-hand side in (49) strongly two-scale converges to

$$\begin{pmatrix} x_3 \hat{\nabla}_y^2 \varphi^\delta & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so that it remains to argue that the second and third term of the right-hand side in (49) can be viewed as higher order perturbations of $\boldsymbol{K}^{\delta,h}$ and $\boldsymbol{G}^{\delta,h}$ in the expansion (43). Indeed, this the case, since by construction

$$\begin{aligned} & \widetilde{\boldsymbol{K}}^{\delta,h} \text{ is skew-symmetric, } \limsup_{h\to 0} ||\widetilde{\boldsymbol{K}}^{\delta,h}||_{L^\infty} = 0 \text{ and} \\ & \limsup_{h\to 0} ||\boldsymbol{T}^{\delta,h}||_{L^\infty} = 0. \end{aligned}$$

Now, it is easy to check that (a), (b) and (c) of Step 2 hold for $\boldsymbol{y}^{\delta,h}$ replaced by $\widetilde{\boldsymbol{y}}^{\delta,h}$. □

Proof of Lemma 3.5. In order to treat (a) and (b) in parallel, we suppose that for $\gamma = 0$ the relaxation field $\boldsymbol{U} \in L^2(\omega, L_\gamma(S \times \mathcal{Y}, \mathbb{M}^3_{\mathrm{sym}}))$ has the form

$$\boldsymbol{U} = \begin{pmatrix} \operatorname{sym} \hat{\nabla}_y \boldsymbol{\zeta} & \begin{matrix} g_1 \\ g_2 \end{matrix} \\ (g_1,\ g_2) & g_3 \end{pmatrix} \tag{50}$$

for some $\boldsymbol{\zeta} \in L^2(\omega, H^1(\mathcal{Y}, \mathbb{R}^2))$ and $\boldsymbol{g} \in L^2(\Omega \times Y, \mathbb{R}^3)$. Notice that this is not a restriction, since for $\gamma = 0$ we prescribe the limit only up to a term of the form

$$\begin{pmatrix} x_3 \hat{\nabla}^2_y \varphi & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $\varphi \in L^2(\omega, H^2(\mathcal{Y}))$.

We claim that it suffices to prove the following: for all $\delta > 0$ there is $\boldsymbol{U}^\delta \in L^2(\Omega \times Y, \mathbb{M}^3)$ and a sequence $\boldsymbol{\phi}^{\delta,h} \in H^1(\Omega, \mathbb{R}^3)$ such that

$$||\boldsymbol{U}^\delta - \boldsymbol{U}||_{L^2} \leq \delta, \tag{51}$$

$$\operatorname{sym} \nabla_h \boldsymbol{\phi}^{\delta,h} \xrightarrow{2,\gamma} \boldsymbol{U}^\delta \qquad \text{strongly two-scale in } L^2, \tag{52}$$

$$\limsup_{h\to 0} ||\nabla_h \boldsymbol{\phi}^{\delta,h}||_{L^2} \lesssim ||\boldsymbol{U}^\delta||_{L^2}, \tag{53}$$

$$\limsup_{h\to 0} \left\{ ||\boldsymbol{\phi}^{\delta,h}||_{L^\infty} + ||h^2 \nabla_h \boldsymbol{\phi}^{\delta,h}||_{L^\infty} \right\} = 0. \tag{54}$$

Indeed, with the doubly indexed family $\boldsymbol{\phi}^{\delta,h}$ and $\boldsymbol{U}^\delta$ at hand, the conclusion follows by choosing a suitable diagonal sequence. To make this precise, consider the quantity

$$g(\delta, h) := ||\boldsymbol{U}^\delta - \boldsymbol{U}||_{L^2} + \mathfrak{d}(\varepsilon(h), \operatorname{sym} \nabla_h \boldsymbol{\phi}^{\delta,h}, \boldsymbol{U}) + ||\boldsymbol{\phi}^{\delta,h}||_{L^\infty} + ||h^2 \nabla_h \boldsymbol{\phi}^{\delta,h}||_{L^\infty},$$

where $\mathfrak{d}$ is defined in Lemma A.2 and characterizes strong two-scale convergence. By (51), (52) combined with Lemma A.2 and (54), we deduce that

$$\limsup_{\delta\to 0} \limsup_{h\to 0} g(\delta, h) = 0.$$

Hence, by virtue of Attouch's diagonalization lemma (see Lemma A.7), there exists a function $h \mapsto \delta(h)$ such that $g(\delta(h), h) \to 0$. We conclude that the diagonal sequence $\boldsymbol{\phi}^h := \boldsymbol{\phi}^{\delta(h),h}$ satisfies (a2) and (a3) (resp. (a3')). Furthermore, by construction $\nabla_h \boldsymbol{\phi}^{\delta(h),h}$ is bounded in L^2 and thus (a1) follows as well.

It remains to construct $\boldsymbol{U}^\delta$ and $\boldsymbol{\phi}^{\delta,h}$ with the claimed properties. We treat the different regimes for γ separately.

Construction for $\gamma = 0$. Consider the representation (50). Without loss of generality assume that $\int_{S\times Y} \boldsymbol{\zeta}(\hat{x}, \cdot) = 0$ for almost every $\hat{x} \in \omega$. Hence, Korn's inequality yields

$$||\boldsymbol{\zeta}||_{L^2(\omega, H^1(\mathcal{Y}))} \lesssim ||\boldsymbol{U}||_{L^2}$$

Now choose $\boldsymbol{\zeta}^\delta \in C^\infty_c(\omega, C^\infty(\mathcal{Y}, \mathbb{R}^2))$ and $\boldsymbol{g}^\delta \in C^\infty_c(\Omega, C^\infty(\mathcal{Y}, \mathbb{R}^3))$ such that

$$||\boldsymbol{\zeta}^\delta - \boldsymbol{\zeta}||_{L^2(\omega, H^1(\mathcal{Y}))} + ||\boldsymbol{g}^\delta - \boldsymbol{g}||_{L^2} \leq \delta.$$

Define $\boldsymbol{U}^\delta$ by identity (50) with $\boldsymbol{\zeta}$ and $\boldsymbol{g}$ replaced by $\boldsymbol{\zeta}^\delta$ and $\boldsymbol{g}^\delta$, respectively. Notice that (51) is trivially satisfied. Now, define

$$\phi^{\delta,h}_\alpha(x) := \begin{cases} \varepsilon(h) \boldsymbol{\zeta}^\delta_\alpha(\hat{x}, \frac{\hat{x}}{\varepsilon(h)}) + 2h \displaystyle\int_{-\frac{1}{2}}^{x_3} \boldsymbol{g}^\delta_\alpha(\hat{x}, s, \tfrac{\hat{x}}{\varepsilon(h)})\, ds & \text{for } \alpha = 1, 2 \\ h \displaystyle\int_{-\frac{1}{2}}^{x_3} \boldsymbol{g}^\delta_3(\hat{x}, s, \tfrac{\hat{x}}{\varepsilon(h)})\, ds & \text{for } \alpha = 3 \end{cases}$$

We compute

$$\nabla_h \boldsymbol{\phi}^{\delta,h} = \begin{pmatrix} \hat\nabla_y \boldsymbol{\zeta}^\delta(\hat x, \frac{\hat x}{\varepsilon(h)}) & \begin{matrix} 2\boldsymbol{g}_1^\delta(x, \frac{\hat x}{\varepsilon(h)}) \\ 2\boldsymbol{g}_2^\delta(x, \frac{\hat x}{\varepsilon(h)}) \end{matrix} \\ 0 & \boldsymbol{g}_3^\delta(x, \frac{\hat x}{\varepsilon(h)}) \end{pmatrix} + O(\varepsilon(h)) + O(h) + O(\tfrac{h}{\varepsilon(h)}).$$

Here $O(s_{(h)})$ stands for a generic field in $\boldsymbol{T}^h \in L^\infty(\Omega, \mathbb{M}^3)$ with $\limsup_{h\to 0} s_{(h)}^{-1}||\boldsymbol{T}^h||_{L^\infty} \leq C$ where C is independent of h. Since $\gamma = 0$ corresponds to $h \ll \varepsilon(h)$, and because $\boldsymbol{\zeta}^\delta$, $\boldsymbol{g}^\delta$ are smooth, (53) and (54) follow. The strong two-scale convergence statement (52) is a consequence of Lemma A.1.

Construction for $\gamma \in (0,\infty)$. The argument is basically the same as the one for $\gamma = 0$. Therefore, let us only remark that $\boldsymbol{U}$ can be approximated by a relaxation field of the form

$$\boldsymbol{U}^\delta = \operatorname{sym}(\hat\nabla_y \boldsymbol{\phi}^\delta, \tfrac{1}{\gamma}\partial_3\boldsymbol{\phi}^\delta),$$

where $\boldsymbol{\phi}^\delta \in C_c^\infty(\omega, C^1(\bar S, C^\infty(\mathcal{Y}, \mathbb{R}^3)))$ with $\int_{S\times Y} \boldsymbol{\phi}^\delta(\hat x, \cdot) = 0$ for almost every $\hat x \in \omega$. The corresponding sequence of correctors takes the form $\boldsymbol{\phi}^{\delta,h}(x) = \varepsilon(h)\boldsymbol{\phi}^{\delta,h}(x, \frac{\hat x}{\varepsilon(h)})$.

Construction for $\gamma = \infty$. The argument is similar to the previous cases. We only remark that $\boldsymbol{U}$ can be approximated by a relaxation field of the form

$$\boldsymbol{U}^\delta = \begin{pmatrix} \operatorname{sym}\hat\nabla_y \boldsymbol{\zeta}^\delta & \begin{matrix} \partial_{y_1}\psi^\delta + \boldsymbol{c}_1^\delta \\ \partial_{y_2}\psi^\delta + \boldsymbol{c}_2^\delta \end{matrix} \\ \hat\nabla_y\psi^\delta + (\boldsymbol{c}_1^\delta,\ \boldsymbol{c}_2^\delta) & \boldsymbol{c}_3^\delta \end{pmatrix}$$

with $\boldsymbol{\zeta}^\delta \in C_c^\infty(\Omega, C^\infty(\mathcal{Y}, \mathbb{R}^2))$, $\psi^\delta \in C_c^\infty(\Omega, C^\infty(\mathcal{Y}))$, $\boldsymbol{c}^\delta \in C_c^\infty(\Omega, \mathbb{R}^3)$. The corresponding sequence of correctors takes the form

$$\boldsymbol{\phi}^{\delta,h}(x) := \varepsilon(h)\begin{pmatrix} \boldsymbol{\zeta}^\delta(x, \frac{\hat x}{\varepsilon(h)}) \\ 2\psi^\delta(x, \frac{\hat x}{\varepsilon(h)}) \end{pmatrix} + h\int_{-1/2}^{x_3} \boldsymbol{c}(\hat x, s)\, ds.$$

□

4.3. **Proof of Theorem 2.3.**

Proof of Theorem 2.3 part (i) and (ii). Statement (i) follows directly from Proposition 3.3 part (i); indeed, by the non-degeneracy of W (see assumption (W2)), the equi-boundedness of the energy $I^{\varepsilon(h),h}(\boldsymbol{y}^h)$ yields $\limsup_{h\to 0} e^h(\boldsymbol{y}^h) < \infty$. The invariance statement (ii) follows from the elementary identities $\hat\nabla^2 v_1 = \hat\nabla^2 v_2$ and $\Lambda(\boldsymbol{u}_1, v_1) = \Lambda(\boldsymbol{u}_2, v_2)$ where Λ is defined in (13). □

For the lower bound estimate (part (iii)) and the construction of recovery sequences (part (iv)) we need the two following lemmas.

Lemma 4.2 (linearization)**.** *Let* $\{\widetilde{\boldsymbol{E}}^h\}_{h>0} \subset L^2(\Omega, \mathbb{M}^3)$ *satisfy*

$$\limsup_{h\to 0} ||\widetilde{\boldsymbol{E}}^h||_{L^2} < \infty \qquad \textit{and} \qquad \limsup_{h\to 0} h^2||\widetilde{\boldsymbol{E}}^h||_{L^\infty} = 0. \tag{55}$$

Then

$$\limsup_{h\to 0}\left| \frac{1}{h^4}\int_\Omega W(x, \tfrac{\hat x}{\varepsilon(h)}, \boldsymbol{I} + h^2\widetilde{\boldsymbol{E}}(x))\, dx - \int_\Omega Q(x, \tfrac{\hat x}{\varepsilon(h)}, \widetilde{\boldsymbol{E}}(x))\, dx \right| = 0.$$

Proof. We have

$$\begin{aligned}&\left|\frac{1}{h^4}\int_\Omega W(x,\tfrac{\hat{x}}{\varepsilon(h)},\boldsymbol{I}+h^2\widetilde{\boldsymbol{E}}(x))\,dx-\int_\Omega Q(x,\tfrac{\hat{x}}{\varepsilon(h)},\widetilde{\boldsymbol{E}}(x))\,dx\right|\\ &\overset{\triangle\text{-ineq.}}{\leq}\quad \frac{1}{h^4}\int_\Omega\left|W(x,\tfrac{\hat{x}}{\varepsilon(h)},\boldsymbol{I}+h^2\widetilde{\boldsymbol{E}}(x))-h^4Q(x,\tfrac{\hat{x}}{\varepsilon(h)},\widetilde{\boldsymbol{E}}(x))\right|\,dx\\ &\overset{(14)}{\leq}\quad \frac{1}{h^4}\int_\Omega|h^2\widetilde{\boldsymbol{E}}(x)|^2\,r(|h^2\widetilde{\boldsymbol{E}}(x)|)\,dx\\ &\leq\quad r(h^2\|\widetilde{\boldsymbol{E}}\|_{L^\infty})\int_\Omega|\widetilde{\boldsymbol{E}}(x)|^2\,dx,\end{aligned}$$

where in the last line we used that $r(\cdot)$ is monotonically increasing. By appealing to (55) and $\lim_{\delta\to0}r(\delta)=0$, we get $\limsup_{h\to0}r(h^2\|\widetilde{\boldsymbol{E}}\|_{L^\infty})\int_\Omega|\widetilde{\boldsymbol{E}}(x)|^2\,dx=0$ and the proof is complete. □

Lemma 4.3 (convex homogenization)**.** *Let $\{\widetilde{\boldsymbol{E}}^h\}_{h>0}\subset L^2(\Omega,\mathbb{M}^3)$ and $\widetilde{\boldsymbol{E}}\in L^2(\Omega\times Y,\mathbb{M}^3)$.*

(i) If $\widetilde{\boldsymbol{E}}^h\xrightharpoonup{2,\gamma}\widetilde{\boldsymbol{E}}$ weakly two-scale in L^2, then

$$\liminf_{h\to0}\int_\Omega Q(x,\tfrac{\hat{x}}{\varepsilon(h)},\widetilde{\boldsymbol{E}}^h(x))\,dx\geq\iint_{\Omega\times Y}Q(x,y,\widetilde{\boldsymbol{E}}(x,y))\,dy\,dx.$$

(ii) If $\widetilde{\boldsymbol{E}}^h\xrightarrow{2,\gamma}\widetilde{\boldsymbol{E}}$ strongly two-scale in L^2, then

$$\lim_{h\to0}\int_\Omega Q(x,\tfrac{\hat{x}}{\varepsilon(h)},\widetilde{\boldsymbol{E}}^h(x))\,dx=\iint_{\Omega\times Y}Q(x,y,\widetilde{\boldsymbol{E}}(x,y))\,dy\,dx.$$

The proof of Lemma 4.3 is standard. Results of this type go back to [All92]. Since our notion of two-scale convergence is slightly different, we present a short argument (essentially following [Vis06]) in the appendix.

Proof of Theorem 2.3 part (iii) – lower bound. Without loss of generality we assume that

$$\liminf_{h\to0}I^{\varepsilon(h),h}(\boldsymbol{y}^h)=\limsup_{h\to0}I^{\varepsilon(h),h}(\boldsymbol{y}^h)<\infty.\tag{56}$$

Due to the non-degeneracy of W (see (W2)) we have $\limsup_{h\to0}e^h(\boldsymbol{y}^h)<\infty$. Hence, Proposition 3.3 part (i) is applicable, and we deduce that there exists a relaxation field $\boldsymbol{U}\in L^2(\omega,L_\gamma(S\times\mathcal{Y},\mathbb{M}^3_{\rm sym}))$ such that

$$\boldsymbol{E}^h:=\boldsymbol{E}^h(\boldsymbol{y}^h)\xrightharpoonup{2,\gamma}\boldsymbol{E}^0:=\boldsymbol{E}(\boldsymbol{u},v)+\boldsymbol{U}\qquad\text{weakly two-scale in }L^2.\tag{57}$$

In order to apply the linearization Lemma 4.2, we have to truncate the peaks of $\boldsymbol{E}^h$ and the set of points where $\det\nabla_h\boldsymbol{y}^h$ is negative. Therefore, consider the good set $C^h:=\{\,x\in\Omega\,:\,|\boldsymbol{E}^h(x)|\leq h^{-1},\,\det\nabla_h\boldsymbol{y}^h(x)>0\,\}$ and let χ^h denote the indicator function associated with C^h. We claim that restricting $\boldsymbol{E}^h$ to C^h does not affect the two-scale limit, i. e.

$$\chi^h\boldsymbol{E}^h\rightharpoonup\boldsymbol{E}^0\qquad\text{weakly two-scale in }L^2.\tag{58}$$

Indeed, since $\chi^h\boldsymbol{E}^h$ is bounded in L^2, it suffices to argue that $\boldsymbol{E}^h-\chi^h\boldsymbol{E}^h\to0$ in L^1. By appealing to the boundedness of $\boldsymbol{E}^h$ in L^2 and the fact that $\nabla_h\boldsymbol{y}^h$ strongly converges to a rotation, we deduce that $\chi^h\to1$ in $L^p(\Omega)$ for all $1\leq p<\infty$. Hence, Hölder's inequality yields $\|\boldsymbol{E}^h-\chi^h\boldsymbol{E}^h\|_{L^1}=\|1-\chi^h\|_{L^2}\|\boldsymbol{E}^h\|_{L^2}\to0$ and (58) follows.

Now we are ready to prove the lower bound. By appealing to the polar factorization for matrices with non-negative determinant, there exists a matrix field $\boldsymbol{R}^h\,:\,C^h\to\mathrm{SO}(3)$ such that

$$\forall x\in C^h\;:\;\nabla_h\boldsymbol{y}^h(x)=\boldsymbol{R}^h(x)\sqrt{(\nabla_h\boldsymbol{y}^h(x))^t\nabla_h\boldsymbol{y}^h(x)}.$$

Hence, by frame-indifference (see (W1)), non-negativity (see (W2)) and assumption (W3) we have

$$W(x,\tfrac{\hat{x}}{\varepsilon(h)},\nabla_h\boldsymbol{y}^h(x))\geq\chi^h(x)W(x,\tfrac{\hat{x}}{\varepsilon(h)},\nabla_h\boldsymbol{y}^h(x))=W(x,\tfrac{\hat{x}}{\varepsilon(h)},\boldsymbol{I}+h^2\chi^h(x)\boldsymbol{E}^h(x)).$$

Thus,

$$\begin{aligned} I^{\varepsilon(h),h}(\boldsymbol{y}^h) &= \frac{1}{h^4}\int_\Omega W(x, \tfrac{\hat{x}}{\varepsilon(h)}, \nabla_h \boldsymbol{y}^h(x))\,dx \\ &\geq \frac{1}{h^4}\int_\Omega W(x, \tfrac{\hat{x}}{\varepsilon(h)}, \chi^h(x)\boldsymbol{E}^h(x)\,)\,dx. \end{aligned}$$

Due to the truncation we have $\limsup_{h\to 0} h^2||\chi^h \boldsymbol{E}^h||_{L^\infty} = 0$. Hence, with Lemma 4.2 and Lemma 4.3 we get

$$\begin{aligned} \liminf_{h\to 0} I^{\varepsilon(h),h}(\boldsymbol{y}^h) &\geq \liminf_{h\to 0}\int_\Omega Q(x, \tfrac{\hat{x}}{\varepsilon(h)}, \chi^h(x)\boldsymbol{E}^h(x))\,dx \\ &\overset{(58)}{\geq} \iint_{\Omega\times Y} Q(x,y,\boldsymbol{E}^0(x,y))\,dy\,dx \\ &\overset{(57)}{=} \iint_{\Omega\times Y} Q(x,y,\boldsymbol{E}(\boldsymbol{u},v)+\boldsymbol{U}\,)\,dy\,dx \\ &\overset{\text{Def.2.7}}{\geq} \int_\omega Q_\gamma(\hat{x}, \operatorname{sym}\hat{\nabla}\boldsymbol{u} + \tfrac{1}{2}\hat{\nabla}v\otimes\hat{\nabla}v,\, \hat{\nabla}^2 v\,)\,d\hat{x} \\ &= I^\gamma(\boldsymbol{u},v). \end{aligned}$$

□

Proof of Theorem 2.3, part (iv) – recovery sequence. Without loss of generality we assume that $\bar{\boldsymbol{R}} = \boldsymbol{I}$. Application of Lemma 2.10 yields the representation

$$I^\gamma(\boldsymbol{u},v) = \iint_{\Omega\times Y} Q(x,y,\Lambda(\hat{\nabla}\boldsymbol{u} + \tfrac{1}{2}\hat{\nabla}v\otimes\hat{\nabla}v,\, \hat{\nabla}^2 v) + \boldsymbol{U}\,)\,dy\,dx \tag{59}$$

for some relaxation field $\boldsymbol{U} \in L^2(\omega, L_\gamma(S\times\mathcal{Y}, \mathbb{M}^3_{\text{sym}}))$. Set $\boldsymbol{E} := \boldsymbol{E}(\boldsymbol{u},v) + \boldsymbol{U}$ and notice that $\boldsymbol{E} = \Lambda(\hat{\nabla}\boldsymbol{u} + \frac{1}{2}\hat{\nabla}v\otimes\hat{\nabla}v,\, \hat{\nabla}^2 v) + \boldsymbol{U}$.

Now apply Proposition 3.3 (ii) to $(\bar{\boldsymbol{R}} = \boldsymbol{I}, \boldsymbol{u}, v)$ and $\boldsymbol{U}$, and let $\boldsymbol{y}^h$ denote the associated sequence, that is $\boldsymbol{y}^h \to (\boldsymbol{I}, \boldsymbol{u}, v)$ and

$$\boldsymbol{E}^h(\boldsymbol{y}^h) \xrightarrow{2,\gamma} \boldsymbol{E}(\boldsymbol{u},v) + \boldsymbol{U} \quad \text{strongly two-scale in } L^2, \tag{60}$$

and

$$\limsup_{h\to 0} h^2||\boldsymbol{E}^h(\boldsymbol{y}^h)||_{L^\infty} + ||\operatorname{dist}(\nabla_h\boldsymbol{y}^h, \mathrm{SO}(3))||_{L^\infty} = 0. \tag{61}$$

We only need to show that

$$\lim_{h\to 0} I^{\varepsilon(h),h}(\boldsymbol{y}^h) = \iint_{\Omega\times Y} Q(x,y,\Lambda(\hat{\nabla}\boldsymbol{u} + \tfrac{1}{2}\hat{\nabla}v\otimes\hat{\nabla}v,\, \hat{\nabla}^2 v) + \boldsymbol{U}\,)\,dy\,dx. \tag{62}$$

From (61) we learn that $\det\nabla_h\boldsymbol{y}^h > 0$ for h sufficiently small. Hence, by appealing to the polar factorization, we deduce that there exists a rotation field $\boldsymbol{R}^h : \Omega \to \mathrm{SO}(3)$ such that

$$\boldsymbol{R}^h\nabla_h\boldsymbol{y}^h = \sqrt{(\nabla_h\boldsymbol{y}^h)^t\nabla_h\boldsymbol{y}^h} = \boldsymbol{I} + h^2\boldsymbol{E}^h(\boldsymbol{y}^h) \qquad \text{almost everywhere in } \Omega.$$

By appealing to frame-indifference (see (W1)), we get

$$I^{\varepsilon(h),h}(\boldsymbol{y}^h) = \frac{1}{h^4}\int_\Omega W(x, \tfrac{\hat{x}}{\varepsilon(h)}, \boldsymbol{I} + h^2\boldsymbol{E}^h(\boldsymbol{y}^h))\,dx.$$

Now, (62) follows from Lemma 4.2 and 4.3 (ii). □

4.4. **Proofs of Lemmas 2.2, 2.10, 2.11 and 3.4.**

Proof of Lemma 2.2. Without loss of generality assume that $\bar{\boldsymbol{R}} = \boldsymbol{I}$ and $\fint_\Omega \boldsymbol{y}^h\,dx = 0$.

Step 1. Argument for "⇒".

Assume that $\boldsymbol{y}^h \to (\boldsymbol{I}, \boldsymbol{u}, v)$ and $\boldsymbol{y}^h \to (\widetilde{\boldsymbol{R}}, \widetilde{\boldsymbol{u}}, \widetilde{v})$. Then, by definition, there exist two sequences $(\bar{\boldsymbol{R}}^h, \boldsymbol{u}^h, v^h)$ and $(\widetilde{\boldsymbol{R}}^h, \tilde{\boldsymbol{u}}^h, \tilde{v}^h)$ with

$$\begin{gathered}\boldsymbol{u}^h \rightharpoonup \boldsymbol{u},\ \widetilde{\boldsymbol{u}}^h \rightharpoonup \widetilde{\boldsymbol{u}} \text{ weakly in } H^1, \qquad v^h \rightharpoonup v,\ \widetilde{v}^h \rightharpoonup \widetilde{v} \text{ weakly in } H^1,\\ \bar{\boldsymbol{R}}^h \to \boldsymbol{I},\ \widetilde{\boldsymbol{R}}^h \to \widetilde{\boldsymbol{R}},\end{gathered} \tag{63}$$

as $h \to 0$, and

$$\fint_S \boldsymbol{y}^h(\hat{x}, x_3)\,dx_3 = \bar{\boldsymbol{R}}^h \begin{pmatrix} \hat{x} + h^2 \boldsymbol{u}^h \\ h v^h \end{pmatrix} = \widetilde{\boldsymbol{R}}^h \begin{pmatrix} \hat{x} + h^2 \widetilde{\boldsymbol{u}}^h \\ h \widetilde{v}^h \end{pmatrix}.$$

Rearranging terms and introducing $\hat{\boldsymbol{R}}^h := (\widetilde{\boldsymbol{R}}^h)^T \bar{\boldsymbol{R}}^h$ yields

$$\begin{aligned}&(\hat{\boldsymbol{R}}^h - \boldsymbol{I}) \begin{pmatrix} \hat{x} \\ 0 \end{pmatrix} + h \hat{\boldsymbol{R}}^h \begin{pmatrix} \boldsymbol{0} \\ v^h(\hat{x}) \end{pmatrix} + h^2 \hat{\boldsymbol{R}}^h \begin{pmatrix} \boldsymbol{u}^h(\hat{x}) \\ 0 \end{pmatrix}\\ &= h \begin{pmatrix} \boldsymbol{0} \\ \tilde{v}^h(\hat{x}) \end{pmatrix} + h^2 \begin{pmatrix} \tilde{\boldsymbol{u}}^h(\hat{x}) \\ 0 \end{pmatrix}\end{aligned} \tag{64}$$

for almost every $\hat{x} \in \omega$ and all h. In the limit $h \to 0$ we get $(\hat{\boldsymbol{R}} - \boldsymbol{I}) \begin{pmatrix} \hat{x} \\ 0 \end{pmatrix} = 0$. Combined with $\hat{\boldsymbol{R}} \in \mathrm{SO}(3)$, and $\hat{\boldsymbol{R}} = \widetilde{\boldsymbol{R}}^T \bar{\boldsymbol{R}} = \widetilde{\boldsymbol{R}}^T$, this implies $\widetilde{\boldsymbol{R}} = \boldsymbol{I}$.

Set $\hat{\boldsymbol{A}}^h := \frac{\hat{\boldsymbol{R}}^h - \boldsymbol{I}}{h}$. We claim that there exists $\hat{\boldsymbol{A}} \in \mathbb{M}^3_{\mathrm{skw}}$ such that

$$\hat{\boldsymbol{A}}^h \to \hat{\boldsymbol{A}} \qquad \text{with } \operatorname{sym} \hat{\boldsymbol{A}} = 0, \tag{65}$$

$$\frac{\operatorname{sym} \hat{\boldsymbol{A}}^h}{h} \to \frac{1}{2} \hat{\boldsymbol{A}}^2. \tag{66}$$

Here comes the argument. Dividing (64) by h, and rearranging terms, yields

$$\begin{aligned}&\hat{\boldsymbol{A}}^h \begin{pmatrix} \hat{x} \\ 0 \end{pmatrix} + \begin{pmatrix} \boldsymbol{0} \\ v^h(\hat{x}) \end{pmatrix} + h \hat{\boldsymbol{A}}^h \begin{pmatrix} \boldsymbol{0} \\ v^h(\hat{x}) \end{pmatrix} + h \begin{pmatrix} \boldsymbol{u}^h(\hat{x}) \\ 0 \end{pmatrix} + h^2 \hat{\boldsymbol{A}}^h \begin{pmatrix} \boldsymbol{u}^h(\hat{x}) \\ 0 \end{pmatrix}\\ &= \begin{pmatrix} \boldsymbol{0} \\ \tilde{v}^h(\hat{x}) \end{pmatrix} + h \begin{pmatrix} \tilde{\boldsymbol{u}}^h(\hat{x}) \\ 0 \end{pmatrix}.\end{aligned} \tag{67}$$

By applying ∂_α, $\alpha \in \{1,2\}$, we get

$$\begin{aligned}&\hat{\boldsymbol{A}}^h \mathbf{e}_\alpha + \begin{pmatrix} \boldsymbol{0} \\ \partial_\alpha v^h(\hat{x}) \end{pmatrix} + h \hat{\boldsymbol{A}}^h \begin{pmatrix} \boldsymbol{0} \\ \partial_\alpha v^h(\hat{x}) \end{pmatrix} + h \begin{pmatrix} \partial_\alpha \boldsymbol{u}^h(\hat{x}) \\ 0 \end{pmatrix} + h^2 \hat{\boldsymbol{A}}^h \begin{pmatrix} \partial_\alpha \boldsymbol{u}^h(\hat{x}) \\ 0 \end{pmatrix}\\ &= \begin{pmatrix} \boldsymbol{0} \\ \partial_\alpha \tilde{v}^h(\hat{x}) \end{pmatrix} + h \begin{pmatrix} \partial_\alpha \tilde{\boldsymbol{u}}^h(\hat{x}) \\ 0 \end{pmatrix},\end{aligned} \tag{68}$$

and find that $\hat{\boldsymbol{A}}^h \mathbf{e}_\alpha$ converges as $h \to 0$. From the identity $\hat{\boldsymbol{R}}^h \mathbf{e}_3 = \hat{\boldsymbol{R}}^h \mathbf{e}_1 \wedge \hat{\boldsymbol{R}}^h \mathbf{e}_2$, we deduce that

$$\hat{\boldsymbol{A}}^h \mathbf{e}_3 = (\hat{\boldsymbol{A}}^h \mathbf{e}_1 \wedge \hat{\boldsymbol{R}}^h \mathbf{e}_2 + \mathbf{e}_1 \wedge \hat{\boldsymbol{A}}^h \mathbf{e}_2),$$

and thus $\hat{\boldsymbol{A}}^h$ converges to some limit $\hat{\boldsymbol{A}} \in \mathbb{M}^3$. Eventually, the relation $(\hat{\boldsymbol{A}}^h)^T \hat{\boldsymbol{A}}^h = -2 \frac{\operatorname{sym} \hat{\boldsymbol{A}}^h}{h}$ yields (65) and (66).

To complete the argument, it remains to prove that

$$\tilde{v}(\hat{x}) = v(\hat{x}) + \boldsymbol{a} \cdot \hat{x} \qquad \text{where } \boldsymbol{a} := (\hat{\boldsymbol{A}}_{31}, \hat{\boldsymbol{A}}_{32}) \tag{69}$$

$$\tilde{\boldsymbol{u}}(\hat{x}) = \boldsymbol{u}(\hat{x}) + (\boldsymbol{A} - \tfrac{1}{2} \boldsymbol{a} \otimes \boldsymbol{a})\hat{x} - v(\hat{x}) \boldsymbol{a} \tag{70}$$

for some skew symmetric matrix $\boldsymbol{A} \in \mathbb{M}^2_{\rm skw}$. The first identity appears in the limit $h \to 0$ in the third component of identity (67). For the proof of (70) we introduce the skew-symmetric matrix $\boldsymbol{A}^h \in \mathbb{M}^2_{\rm skw}$

$$\boldsymbol{A}^h_{\alpha\beta} = \frac{\hat{\boldsymbol{A}}^h_{\alpha\beta}}{h} - \frac{(\operatorname{sym} \hat{\boldsymbol{A}}^h)_{\alpha\beta}}{h}, \text{ for } \alpha, \beta = 1, 2. \tag{71}$$

Going back to (68) we find that $h^{-1}\hat{\boldsymbol{A}}_{\alpha\beta}$, $\alpha, \beta \in \{1, 2\}$, converges as $h \to 0$. Combined with (66) we deduce that $\boldsymbol{A}^h$ converges to some $\boldsymbol{A} \in \mathbb{M}^2_{\rm skw}$. Now, a calculation yields (70).

Step 2. Argument for "$\Leftarrow$".

Suppose that $\boldsymbol{y}^h \in H^1(\omega; \mathbb{R}^3)$ converges to the triple $(\bar{\boldsymbol{R}}, \boldsymbol{u}, v)$ in the sense of definition (2.1). Let us now take arbitrary $\boldsymbol{A} \in \operatorname{Skew}(2)$ and $\boldsymbol{a} \in \mathbb{R}^2$, and set

$$\tilde{\boldsymbol{R}}^h = \bar{\boldsymbol{R}}^h \exp(-h^2 \boldsymbol{A}_e) \exp(-h \boldsymbol{a}_e), \tag{72}$$

where $\boldsymbol{A}_e, \boldsymbol{a}_e \in \mathbb{M}^3$ are defined by

$$\boldsymbol{A}_e := \begin{pmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & 0 \end{pmatrix}, \qquad \boldsymbol{a}_e := \begin{pmatrix} \boldsymbol{0} & -\boldsymbol{a} \\ \boldsymbol{a}^T & 0 \end{pmatrix}. \tag{73}$$

We define $\tilde{\boldsymbol{u}}^h$, $\tilde{v}^h$ via identity (10). From the expansions

$$\exp(h^2 \boldsymbol{A}_e) = \boldsymbol{I} + h^2 \boldsymbol{A}_e + O(h^4), \qquad \exp(h \boldsymbol{a}_e) = \boldsymbol{I} + h \boldsymbol{a}_e + \frac{h^2}{2} \boldsymbol{a}_e^2 + O(h^3), \tag{74}$$

we conclude that

$$\begin{aligned} \tilde{\boldsymbol{u}}^h(\hat{x}) &= \boldsymbol{u}^h(\hat{x}) + (\boldsymbol{A} - \tfrac{1}{2} \boldsymbol{a} \otimes \boldsymbol{a})\hat{x} - v^h(\hat{x})\boldsymbol{a} + O(h), \\ \tilde{v}^h(\hat{x}) &= v^h(\hat{x}) + \boldsymbol{a} \cdot \hat{x} + O(h), \end{aligned}$$

where $\|O(h)\|_{H^1} \le Ch$, for some $C > 0$. □

Proof of Lemma 2.10. **Step 1.** Completeness of $L_\gamma(S \times \mathcal{Y}, \mathbb{M}^3_{\rm sym})$.

We claim that $L_\gamma := L_\gamma(S \times \mathcal{Y}, \mathbb{M}^3_{\rm sym})$ is a closed subspace of $L^2(S \times \mathcal{Y}, \mathbb{M}^3_{\rm sym})$, and

$$\forall \boldsymbol{A}, \boldsymbol{B} \in \mathbb{M}^2,\, U \in L_\gamma \,:\qquad \iint_{S \times Y} \Lambda(\boldsymbol{A}, \boldsymbol{B}) : U \, dx_3 \, dy = 0. \tag{75}$$

Since the argument is similar for $\gamma = 0$, $\gamma = \infty$ and $\gamma \in (0, \infty)$ we prove the statement only in the most difficult case of $\gamma = 0$. We introduce the space

$$\mathcal{M}_0 := \Big\{ (\boldsymbol{\zeta}, \varphi, \boldsymbol{g}) \in H^1(\mathcal{Y}, \mathbb{R}^2) \times H^1(\mathcal{Y}) \times L^2(S \times \mathcal{Y}, \mathbb{R}^3) \,: \\ \int_Y \boldsymbol{\zeta} \, dy = 0 \quad \text{ and } \quad \int_Y \varphi \, dy = 0 \Big\}.$$

Notice that, due to periodicity, every $(\boldsymbol{\zeta}, \varphi, \boldsymbol{g}) \in \mathcal{M}_0$ also satisfies $\int_Y \hat{\nabla}_y \boldsymbol{\zeta} = \int_Y \hat{\nabla}_y \varphi \, dy = 0$. We also introduce the mapping

$$\mathcal{G}_0 \,:\, \mathcal{M}_0 \to L_0, \qquad (\boldsymbol{\zeta}, \varphi, \boldsymbol{g}) \mapsto \begin{pmatrix} \operatorname{sym} \hat{\nabla}_y \boldsymbol{\zeta} + x_3 \hat{\nabla}^2_y \varphi & \begin{matrix} \boldsymbol{g}_1 \\ \boldsymbol{g}_2 \end{matrix} \\ (\boldsymbol{g}_1,\ \boldsymbol{g}_2) & \boldsymbol{g}_3 \end{pmatrix}. \tag{76}$$

Since $\mathcal{M}_0$ is a closed subspace of $H^1(\mathcal{Y}, \mathbb{R}^2) \times H^2(\mathcal{Y}) \times L^2(S \times \mathcal{Y}, \mathbb{R}^3)$, it suffices to argue that $\mathcal{G}_0$ is an isomorphism. Obviously, $\mathcal{G}_0$ is linear and surjective. We only need to show that

$$\forall (\boldsymbol{\zeta}, \varphi, \boldsymbol{g}) \in \mathcal{M}_0 \,:\qquad \|\boldsymbol{\zeta}\|^2_{H^1} + \|\varphi\|^2_{H^2} + \|\boldsymbol{g}\|^2_{L^2} \sim ||\mathcal{G}_0(\boldsymbol{\zeta}, \varphi, \boldsymbol{g})||^2_{L^2}, \tag{77}$$

where $X \sim Y$ stands for $X \leq c_1 Y$ and $Y \leq c_2 X$ for some universal constants c_1, c_2. To see (77) notice that an application of the Korn– and Poincaré inequality shows that

$$\|\boldsymbol{\zeta}\|_{H^1}^2 + \|\varphi\|_{H^2}^2 + \|\boldsymbol{g}\|_{L^2}^2 \sim \|\operatorname{sym}\hat{\nabla}_y \boldsymbol{\zeta}\|_{L^2}^2 + \|\hat{\nabla}_y^2 \varphi\|_{L^2}^2 + \|\boldsymbol{g}\|_{L^2}^2.$$

Now, using that $\int_{-1/2}^{1/2} x_3 ds_3 = 0$, a direct computation yields

$$\|\operatorname{sym}\hat{\nabla}_y \boldsymbol{\zeta}\|_{L^2}^2 + \|\hat{\nabla}_y^2 \varphi\|_{L^2}^2 + \|\boldsymbol{g}\|_{L^2}^2 \sim \|\mathcal{G}_0(\boldsymbol{\zeta}, \varphi, \boldsymbol{g})\|_{L^2}^2.$$

The validity of (75) follows from the definition of $\Lambda(\boldsymbol{A}, \boldsymbol{B})$ and the fact that gradients of periodic functions have zero integral average.

Step 2. Construction of the solution operator.

Fix $\hat{x} \in \omega$. We claim that for all $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{M}^2$ there exists a unique field $U = U_{\hat{x},\boldsymbol{A},\boldsymbol{B}} \in L_\gamma(S \times \mathcal{Y}, \mathbb{M}^3_{\mathrm{sym}})$ with

$$Q_\gamma(\hat{x}, \boldsymbol{A}, \boldsymbol{B}) = \iint_{S\times Y} Q(\hat{x}, x_3, y, \Lambda(\boldsymbol{A}, \boldsymbol{B}) + U)\, dx_3\, dy. \tag{78}$$

Indeed, since Q is convex, the existence follows by the direct method of the calculus of variations. By property (Q3), cf. Lemma 2.9, and property (75), the integral functional on the right-hand side in (78) is strictly convex on L_γ. Hence, the minimizer is unique, we find that

$$||U||_{L^2(S\times Y)}^2 \leq \frac{\beta}{\alpha}\left(|\boldsymbol{A}|^2 + |\boldsymbol{B}|^2\right). \tag{79}$$

We denote the associated solution operator by

$$P(\hat{x}, \cdot, \cdot) \,:\, \mathbb{M}^2 \times \mathbb{M}^2 \to L_\gamma, \qquad (\boldsymbol{A}, \boldsymbol{B}) \mapsto U_{\hat{x},\boldsymbol{A},\boldsymbol{B}}. \tag{80}$$

Since Q is quadratic and Λ is linear, $P(\hat{x}, \cdot, \cdot)$ is a linear and continuous operator. From property (Q1), cf. Lemma 2.9, we deduce that for all $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{M}^2$ the mapping

$$\omega \ni \hat{x} \mapsto P(\hat{x}, \boldsymbol{A}, \boldsymbol{B}) \in L_\gamma \tag{81}$$

is continuous. Now, let $\boldsymbol{A}(\hat{x})$ and $\boldsymbol{B}(\hat{x})$ be continuous in $\hat{x}$. We have for $\hat{x}, \hat{x}' \in \omega$

$$\begin{aligned} &\|P(\hat{x}, \boldsymbol{A}(\hat{x}), \boldsymbol{B}(\hat{x})) - P(\hat{x}', \boldsymbol{A}(\hat{x}'), \boldsymbol{B}(\hat{x}'))\|_{L^2} \\ &\quad\leq \|P(\hat{x}, \boldsymbol{A}(\hat{x}), \boldsymbol{B}(\hat{x})) - P(\hat{x}, \boldsymbol{A}(\hat{x}'), \boldsymbol{B}(\hat{x}'))\|_{L^2} \\ &\qquad + \|P(\hat{x}', \boldsymbol{A}(\hat{x}'), \boldsymbol{B}(\hat{x}')) - P(\hat{x}, \boldsymbol{A}(\hat{x}'), \boldsymbol{B}(\hat{x}'))\|_{L^2}. \end{aligned}$$

Because of (81), and since $P(\hat{x}, \cdot, \cdot)$ is continuous in its second and third component, the right-hand side vanishes as $\hat{x}' \to \hat{x}$. Hence, $\hat{x} \mapsto P(\hat{x}, \boldsymbol{A}(\hat{x}), \boldsymbol{B}(\hat{x}))$ defines a continuous map from ω to L_γ that we denote by $\Pi_\gamma[\boldsymbol{A}, \boldsymbol{B}]$. Viewed as a function of $(\boldsymbol{A}, \boldsymbol{B})$,

$$\Pi_\gamma \,:\, C(\overline{\omega}, \mathbb{M}^2) \times C(\overline{\omega}, \mathbb{M}^2) \to C(\overline{\omega}, L_\gamma(S \times \mathcal{Y}, \mathbb{M}^3_{\mathrm{sym}})), \qquad (\boldsymbol{A}, \boldsymbol{B}) \mapsto \Pi_\gamma[\boldsymbol{A}, \boldsymbol{B}]$$

defines a linear operator, which, by (79), satisfies

$$\|\Pi_\gamma[\boldsymbol{A}, \boldsymbol{B}]\|_{L^2(\omega\times S\times Y)} \leq \frac{\beta}{\alpha}\int_\omega |\boldsymbol{A}(\hat{x})|^2 + |\boldsymbol{B}(\hat{x})|^2 \, d\hat{x};$$

and thus can be extended to a bounded, linear operator from $L^2(\omega, \mathbb{M}^2) \times L^2(\omega, \mathbb{M}^2)$ to $L^2(\omega, L_\gamma(S \times \mathcal{Y}, \mathbb{M}^3_{\mathrm{sym}}))$. By construction, that operator satisfies all the claimed properties. □

Proof of Lemma 2.11. By the representation of Q_γ via Lemma 2.10 (b) combined with the continuity and linearity properties of $\Pi_\gamma[\boldsymbol{A}, \boldsymbol{B}]$ for $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{M}^2$, we get that Q_γ is continuous, and quadratic in its second and third component. It remains to prove the asserted a priori estimates. By the representation of Q_γ via Lemma 2.10 (b), property (Q3), cf. Lemma 2.9, and the orthogonality (75), we have for all $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{M}^2$ and $\hat{x} \in \omega$

$$\begin{aligned} Q_\gamma(\hat{x}, \boldsymbol{A}, \boldsymbol{B}) &\geq \alpha \iint_{S\times Y} |\operatorname{sym}\Lambda(\boldsymbol{A}, \boldsymbol{B})|^2 \, dx_3\, dy \\ &\geq \frac{\alpha}{12}(|\operatorname{sym}\boldsymbol{A}|^2 + |\operatorname{sym}\boldsymbol{B}|^2). \end{aligned}$$

On the other hand we have

$$\begin{aligned} Q_\gamma(\hat{x}, \boldsymbol{A}, \boldsymbol{B}) &\leq \iint_{S\times Y} Q(\hat{x}, \Lambda(\boldsymbol{A}, \boldsymbol{B}))\, dx_3\, dy \\ &\leq \beta(|\operatorname{sym} \boldsymbol{A}|^2 + |\operatorname{sym} \boldsymbol{B}|^2). \end{aligned}$$

□

Proof of Lemma 3.4. Since $\boldsymbol{K}^h$ is skew-symmetric, we have

(82) $$(\boldsymbol{I} + h\boldsymbol{K}^h + h^2\boldsymbol{G}^h)^t(\boldsymbol{I} + h\boldsymbol{K}^h + h^2\boldsymbol{G}^h) = \boldsymbol{I} + h^2\Big(\boldsymbol{H}^h + h\boldsymbol{B}^h\Big)$$
$$\text{where } \boldsymbol{H}^h = 2\operatorname{sym}\boldsymbol{G}^h - (\boldsymbol{K}^h)^2, \qquad \boldsymbol{B}^h = 2\operatorname{sym}((\boldsymbol{G}^h)^t\boldsymbol{K}^h) + h(\boldsymbol{G}^h)^t\boldsymbol{G}^h.$$

It is easy to check that the assumption in (i) implies

$$\limsup_{h\to 0} ||\boldsymbol{H}^h||_{L^2} < \infty, \qquad \limsup_{h\to 0}\Big(||h^2\boldsymbol{H}^h + h^3\boldsymbol{B}^h||_{L^\infty} + ||h\boldsymbol{B}^h||_{L^2}\Big) = 0.$$

Hence, statement (i) follows from the expansion $\sqrt{\boldsymbol{I}+\boldsymbol{F}} = \boldsymbol{I} + \frac{1}{2}\operatorname{sym}\boldsymbol{F} + o(|\boldsymbol{F}|)$.

Next, we prove statement (ii) by reduction to part (i). To this end we truncate the peaks of $\boldsymbol{G}^h$ and $\boldsymbol{K}^h$: let χ^h be given by

$$\chi^h(x) := \begin{cases} 1 & \text{if } |\boldsymbol{G}^h| \leq h^{-1/2} \text{ and } |\boldsymbol{K}^h| \leq h^{-1/2}, \\ 0 & \text{else.} \end{cases}$$

Because $\boldsymbol{\Psi}^{\varepsilon(h)}$ is uniformly bounded, we get from part (i)

$$\lim_{h\to 0}\left|\int_\Omega \chi^h(\boldsymbol{E}^h - (\operatorname{sym}\boldsymbol{G}^h - \frac{1}{2}(\boldsymbol{K}^h)^2)) : \boldsymbol{\Psi}^{\varepsilon(h)}\, dx\right| = 0.$$

Hence, it remains to show that

(83) $$\limsup_{h\to 0}\left|\int_\Omega (1-\chi^h)(\boldsymbol{E}^h - (\operatorname{sym}\boldsymbol{G}^h - \frac{1}{2}(\boldsymbol{K}^h)^2)) : \boldsymbol{\Psi}^{\varepsilon(h)}\, dx\right| = 0.$$

This can be seen as follows. Because the sequences $\boldsymbol{G}^h$ and $\boldsymbol{K}^h$ are equibounded in L^2 and L^4, respectively, we have $||(1-\chi^h)||^2_{L^2} = \int_{\{\chi^h=0\}} dx \leq \int h|G^h|^2 + h^2|\boldsymbol{K}^h|^4\, dx \to 0$. Hence, $(1-\chi^h)\boldsymbol{\Psi}^{\varepsilon(h)} \to 0$ strongly in L^2 and (83) follows from the equiboundedness of the sequences $\boldsymbol{E}^h$ and $\operatorname{sym}\boldsymbol{G}^h - \frac{1}{2}(\boldsymbol{K}^h)^2$ in L^2. □

5. Continuity of Q_γ in γ

The homogenized quadratic form Q_γ, cf. Definition 2.7, continuously depends on the ratio γ:

Proposition 5.1. *Let Assumption 2.8 be satisfied. Then for any $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{M}^2$ and $\hat{x} \in \omega$ the function*

$$\gamma \mapsto Q_\gamma(\hat{x}, \boldsymbol{A}, \boldsymbol{B})$$

is continuous on $[0,\infty]$.

For the proof we need the following structural continuity result for the space of relaxation fields $L_\gamma(S\times\mathcal{Y}, \mathbb{M}^3_{\text{sym}})$:

Lemma 5.2. *Consider sequences $\{\gamma^n\}_{n\in\mathbb{N}} \subset (0,\infty)$ and $\{\boldsymbol{\phi}^n\}_{n\in\mathbb{N}} \subset H^1(S\times\mathcal{Y}, \mathbb{R}^3)$. Suppose that for $n\to\infty$*

$$\begin{aligned} \gamma^n &\to \gamma && \text{in } [0,\infty], \\ \boldsymbol{U}^n := \operatorname{sym}\Big(\hat{\nabla}_y\boldsymbol{\phi}^n, \tfrac{1}{\gamma_n}\partial_3\boldsymbol{\phi}^n\Big) &\rightharpoonup \boldsymbol{U} && \text{weakly in } L^2(S\times\mathcal{Y}, \mathbb{M}^3_{\text{sym}}). \end{aligned}$$

Then $\boldsymbol{U} \in L^2_\gamma(S\times\mathcal{Y}, \mathbb{M}^3_{\text{sym}})$.

Proof of Lemma 5.2. **Step 1.** Representation of the upper left 2×2 minor of $\boldsymbol{U}$.

Consider the function

$$\boldsymbol{\zeta}^n_\alpha := \boldsymbol{\phi}^n_\alpha - \int_Y \boldsymbol{\phi}^n_\alpha \, dy, \qquad \alpha \in \{1,2\}. \tag{84}$$

We claim that

$$\boldsymbol{\zeta}^n \rightharpoonup \boldsymbol{\zeta} \qquad \text{in } L^2(S, H^1(\mathcal{Y}, \mathbb{R}^2)),$$

where $\boldsymbol{\zeta} \in L^2(S, H^1(\mathcal{Y}, \mathbb{R}^2))$ is characterized by

$$\boldsymbol{U}_{\alpha\beta} = \left(\operatorname{sym} \hat{\nabla}_y \boldsymbol{\zeta}\right)_{\alpha\beta} \qquad \text{for } \alpha, \beta \in \{1,2\}, \tag{85a}$$

$$\int_Y \boldsymbol{\zeta}(x_3, y) \, dy = 0 \qquad \text{for almost every } x_3 \in S. \tag{85b}$$

For the argument first notice that (85a) and (85b) hold for $\boldsymbol{U}^n$ and $\boldsymbol{\zeta}^n$ that take the role of $\boldsymbol{U}^n$ and $\boldsymbol{\zeta}^n$, respectively. Hence, by Korn's inequality for periodic functions on $\mathcal{Y}$, and the Poincaré inequality for functions with zero integral mean on $\mathcal{Y}$, we have

$$||\boldsymbol{\zeta}^n||_{L^2(S\times Y)} \lesssim ||\hat{\nabla}_y \boldsymbol{\zeta}^n||_{L^2(S\times Y)} \lesssim ||\operatorname{sym} \hat{\nabla}_y \boldsymbol{\zeta}^n||_{L^2(S\times Y)} \leq ||\boldsymbol{U}^n||_{L^2(S\times Y)}. \tag{86}$$

Here and below $\lesssim$ stands for $\leq$ up to a universal multiplicative constant. Since the right-hand side is bounded, the sequence $\{\boldsymbol{\zeta}^n\}_{n\in\mathbb{N}}$ is bounded in $L^2(S, H^1(\mathcal{Y}, \mathbb{R}^2))$, and thus $\zeta^n \rightharpoonup \zeta$ for a subsequence, where $\zeta \in L^2(S, H^1(\mathcal{Y}, \mathbb{R}^2))$ satisfies (85a) and (85b). It remains to argue that ζ is uniquely determined by (85a) and (85b), so that we may conclude that the convergence of $\{\boldsymbol{\zeta}^n\}$ holds for the entire sequence, as we claimed. Indeed, (85a) characterizes ζ up to terms of the form $(x_3, y) \mapsto \boldsymbol{S}(x_3)y + c(x_3)$ where $\boldsymbol{S}(x_3)$ is a skew-symmetric 2×2 matrix and $c(x_3) \in \mathbb{R}^2$. However, since ζ is periodic in y, we deduce that $\boldsymbol{S}(x_3) = 0$ for almost every $x_3 \in S$. Likewise (85b) implies $c(x_3) = 0$ for almost every $x_3 \in S$.

Step 2. Conclusion in the case $\gamma = \infty$.

In view of Step 1 we only need to show that

$$\boldsymbol{U}^n_{3\alpha} \rightharpoonup \partial_{y_\alpha}\psi + \boldsymbol{c}_\alpha \qquad \text{weakly in } L^2(S\times Y) \text{ for } \alpha = 1,2, \tag{87a}$$

$$\boldsymbol{U}^n_{33} \rightharpoonup \boldsymbol{c}_3 \qquad \text{weakly in } L^2(S\times Y), \tag{87b}$$

for some $\psi \in L^2(S, H^1(\mathcal{Y}))$ and $\boldsymbol{c} \in L^2(S, \mathbb{R}^3)$.

We start with the argument for (87a). Let $\boldsymbol{\zeta}^n$ be defined as in Step 1, and set $\boldsymbol{c}^n_i := \int_Y \boldsymbol{U}^n_{3i}\, dy$, for $i = 1,2,3$. Since $\boldsymbol{U}^n \rightharpoonup \boldsymbol{U}$ in L^2, we have

$$\boldsymbol{c}^n_i \rightharpoonup \boldsymbol{c}_i := \int_Y \boldsymbol{U}_{3\alpha}\, dy \qquad \text{weakly in } L^2(S, \mathbb{R}^3). \tag{88}$$

By construction we have for $\alpha = 1,2$

$$2\big(\boldsymbol{U}^n_{3\alpha} - \boldsymbol{c}^n_\alpha\big) = \partial_{y_\alpha}\boldsymbol{\phi}^n_3 + \frac{1}{\gamma^n}\partial_3 \boldsymbol{\zeta}^n_\alpha. \tag{89}$$

Note that (86) implies that

$$\lim_{n\to\infty} \iint_{S\times Y} \frac{1}{\gamma^n}\partial_3\boldsymbol{\zeta}^n_\alpha\, \varphi\, dy\, dx_3 = -\lim_{n\to\infty}\iint_{S\times Y} \frac{1}{\gamma^n}\boldsymbol{\zeta}^n_\alpha\, \partial_3\varphi\, dy\, dx_3 = 0,$$

for all $\varphi \in C^\infty_c(S, C^\infty(\mathcal{Y}))$. In combination with $\boldsymbol{U}^n \rightharpoonup \boldsymbol{U}$ in L^2, (88) and (89), we get

$$\iint_{S\times Y} \partial_{y_\alpha}\boldsymbol{\phi}^n_3\, \varphi\, dy\, dx_3 \to 2\iint_{S\times Y} \big(\boldsymbol{U}_{3\alpha} - \boldsymbol{c}_\alpha\big)\, \varphi\, dy\, dx_3 \qquad \text{for all } \varphi \in C^\infty_c(S, C^\infty(\mathcal{Y})). \tag{90}$$

As a consequence, we infer that for all $\varphi_1, \varphi_2 \in C^\infty_c(S, C^\infty(\mathcal{Y}))$ we have

$$0 = 2\iint_{S\times Y} \big(\boldsymbol{U}_{31} - \boldsymbol{c}_1\big)\, \partial_{y_2}\varphi_1 - \big(\boldsymbol{U}_{32} - \boldsymbol{c}_2\big)\, \partial_{y_1}\varphi_2 \; dy\, dx_3.$$

This fact together with the equality (88) implies that $\boldsymbol{U}_{3\alpha}-\boldsymbol{c}_\alpha=\partial_\alpha\psi$ for some $\psi\in L^2(S,H^1(\mathcal{Y}))$, which easily follows by applying Fourier transform. Next, we prove (87b). Since

$$\boldsymbol{U}_{33}^n=\boldsymbol{U}_{33}^n-\int_Y \boldsymbol{U}_{33}^n\,dy+\boldsymbol{c}_3^n=\frac{1}{\gamma^n}\partial_3\left(\boldsymbol{\phi}_3^n-\int_Y \boldsymbol{\phi}_3^n\,dy\right)+\boldsymbol{c}_3^n,$$

we have $\xi^n:=\frac{1}{\gamma^n}\partial_3\left(\boldsymbol{\phi}_3^n-\int_Y\boldsymbol{\phi}_3^n\,dy\right)\rightharpoonup\xi$ weakly in $L^2(S\times Y)$ for some function ξ and we only need to show that ξ is independent of y. To that end, let $\varphi\in C_c^\infty(S,C^\infty(\mathcal{Y}))$ be arbitrary. We have

$$\begin{aligned}\iint_{S\times Y}\xi^n\,\partial_{y_\alpha}\varphi\;dy\,dx_3 &= \iint_{S\times Y}\frac{1}{\gamma^n}\partial_3\boldsymbol{\phi}_3^n\,\partial_{y_\alpha}\varphi\;dy\,dx_3\\ &= \iint_{S\times Y}\frac{1}{\gamma^n}\partial_{y_\alpha}\boldsymbol{\phi}_3^n\,\partial_3\varphi\;dy\,dx_3.\end{aligned}$$

Since the left-hand side converges to $\iint_{S\times Y}\xi\,\partial_{y_\alpha}\varphi\,dy\,dx_3$, and the right-hand side vanishes – as a consequence of (90) combined with $\frac{1}{\gamma_n}\to 0$, we deduce that ξ does not depend on y_α, $\alpha=1,2$.

Step 3. Conclusion for $\gamma=0$.

Let $\boldsymbol{\zeta}$ be defined as in Step 1. By virtue of Lemma A.6 it suffices to prove

$$\iint_{S\times Y}\boldsymbol{\zeta}\cdot(\partial_{y_2}\partial_3\varphi,-\partial_{y_1}\partial_3\varphi)\,dy\,dx_3=0\qquad\text{for all }\varphi\in C_c^\infty(S,C^\infty(\mathcal{Y})),\tag{91}$$

and for $\alpha,\beta=1,2$

$$\iint_{S\times Y}\partial_{y_\beta}\boldsymbol{\zeta}_\alpha\cdot\partial_3\partial_3\varphi\,dy\,dx_3=0\qquad\text{for all }\varphi\in C_c^\infty(S,C^\infty(\mathcal{Y})).\tag{92}$$

For the argument for (91) let $\boldsymbol{\zeta}^n$ be defined as in Step 1. We have

$$\iint_{S\times Y}\boldsymbol{\zeta}\cdot(\partial_{y_2}\partial_3\varphi,-\partial_{y_1}\partial_3\varphi)\,dy\,dx_3=\lim_{n\to\infty}\iint_{S\times Y}\boldsymbol{\zeta}^n\cdot(\partial_{y_2}\partial_3\varphi,-\partial_{y_1}\partial_3\varphi)\,dy\,dx_3.\tag{93}$$

We rewrite the right-hand side and start with an integration by parts:

$$\begin{aligned}&\iint_{S\times Y}\boldsymbol{\zeta}^n\cdot(\partial_{y_2}\partial_3\varphi,-\partial_{y_1}\partial_3\varphi)\,dy\,dx_3=\iint_{S\times Y}\partial_3\boldsymbol{\zeta}_2^n\partial_{y_1}\varphi-\partial_3\boldsymbol{\zeta}_1^n\partial_{y_2}\varphi\,dy\,dx_3\\ &\overset{(84)}{=}\iint_{S\times Y}\partial_3\boldsymbol{\phi}_2^n\partial_{y_1}\varphi-\partial_3\boldsymbol{\phi}_1^n\partial_{y_2}\varphi\,dy\,dx_3\\ &=2\gamma_n\iint_{S\times Y}\boldsymbol{U}_{32}^n\partial_{y_1}\varphi-\boldsymbol{U}_{31}^n\partial_{y_2}\varphi\,dy\,dx_3-\gamma^n\iint_{S\times Y}\partial_{y_2}\boldsymbol{\phi}_3^n\partial_{y_1}\varphi-\partial_{y_1}\boldsymbol{\phi}_3^n\partial_{y_2}\varphi\,dy\,dx_3,\end{aligned}$$

where in the last line we appealed to the identity $2\boldsymbol{U}_{3\alpha}^n=\partial_{y_\alpha}\boldsymbol{\phi}_3^n+\frac{1}{\gamma^n}\partial_3\boldsymbol{\phi}_\alpha^n$ for $\alpha\in\{1,2\}$. It remains to argue that the right-hand side vanishes for $n\to\infty$. Indeed, the first term vanishes since, $\boldsymbol{U}^n$ is bounded and $\gamma^n\to 0$. The second term vanishes, as can be seen by an integration by parts:

$$\iint_{S\times Y}\partial_{y_2}\boldsymbol{\phi}_3^n\partial_{y_1}\varphi-\partial_{y_1}\boldsymbol{\phi}_3^n\partial_{y_2}\varphi\,dy\,dx_3=\iint_{S\times Y}\boldsymbol{\phi}_3^n\Big(-\partial_{y_2}\partial_{y_1}\varphi+\partial_{y_1}\partial_{y_2}\varphi\Big)\,dy\,dx_3=0.$$

To prove (92) let us continue the analysis and conclude that for arbitrary $\varphi\in C_c^\infty(S,C^\infty(\mathcal{Y}))$ and $\alpha,\beta=1,2$ we have:

$$\iint_{S\times Y}\partial_{y_\beta}\boldsymbol{\zeta}_\alpha\partial_3\partial_3\varphi\,dy\,dx_3=\lim_{n\to\infty}\iint_{S\times Y}\partial_{y_\beta}\boldsymbol{\zeta}_\alpha^n\partial_3\partial_3\varphi\,dy\,dx_3.\tag{94}$$

We again rewrite the right-hand side and start with integration by parts

$$\begin{aligned}(95)\qquad \iint_{S\times Y} \partial_{y_\beta}\boldsymbol{\zeta}^n_\alpha \partial_3\partial_3\varphi\,dy\,dx_3 &= \iint_{S\times Y} \partial_3\boldsymbol{\zeta}^n_\alpha \partial_{y_\beta}\partial_3\varphi\,dy\,dx_3\\ &= 2\gamma^n \iint_{S\times Y} \boldsymbol{U}^n_{3\alpha}\partial_{y_\beta}\partial_3\varphi\,dy\,dx_3 - \gamma^n\int_{S\times Y}\partial_{y_\alpha}\boldsymbol{\phi}^n_3\partial_{y_\beta}\partial_3\varphi\,dy\,dx_3\\ &\quad + \iint_{S\times Y}\partial_3\Big(\int_Y \boldsymbol{\phi}^n_\alpha\,dy\Big)\partial_{y_\beta}\partial_3\varphi\,dy\,dx_3.\end{aligned}$$

The first term of the right-hand side vanishes since $\boldsymbol{U}^n$ is bounded in L^2 and the last term is identically equal to 0. In order to analyze the second term, first notice that

$$(96)\qquad \int_{S\times Y}\partial_{y_\alpha}\boldsymbol{\phi}^n_3\partial_{y_\beta}\partial_3\varphi\,dy\,dx_3 = \int_{S\times Y}\partial_3\boldsymbol{\phi}^n_3\partial_{y_\alpha}\partial_{y_\beta}\varphi\,dy\,dx_3.$$

Notice also that $\partial_3\boldsymbol{\phi}^n_3 = \gamma_n\boldsymbol{U}^n_{33}$ converges to 0 in L^2, by the boundedness of $\boldsymbol{U}^n_{33}$. Hence, the second term of the right-hand side in (95) vanishes.

Step 4. Conclusion for $\gamma\in(0,\infty)$.

Without loss of generality we may assume that $\iint_{S\times\mathcal{Y}}\boldsymbol{\phi}^n\,dy\,dx_3 = 0$ and $\{\gamma^n\}_{n\in\mathbb{N}}\subset K$ where K is a compact subset of $(0,\infty)$. By combining the Korn inequality in Proposition A.5 with the scaling argument in the proof of [Vel, Lemma 1], we find that

$$\iint_{S\times Y}|\boldsymbol{\phi}^n|^2 + |(\hat\nabla_y\boldsymbol{\phi}^n,\,\partial_3\boldsymbol{\phi}^n)|^2\,dy\,dx_3 \lesssim \iint_{S\times Y}|\operatorname{sym}(\hat\nabla_y\boldsymbol{\phi}^n,\,\frac{1}{\gamma^n}\partial_3\boldsymbol{\phi}^n)|^2\,dy\,dx_3 = ||\boldsymbol{U}^n||_{L^2},$$

where $\lesssim$ stands for $\leq$ up to a multiplicative constant that only depends on the compact set K. Since $\boldsymbol{U}^n$ is bounded in L^2, we find that $\boldsymbol{\phi}^n\rightharpoonup\boldsymbol{\phi}$ in $H^1(S\times\mathcal{Y},\mathbb{R}^3)$. Since $\gamma^n\to\gamma$, we deduce that

$$\operatorname{sym}\boldsymbol{U}^n = \operatorname{sym}(\hat\nabla_y\boldsymbol{\phi}^n,\,\frac{1}{\gamma^n}\partial_3\boldsymbol{\phi}^n)\rightharpoonup\operatorname{sym}(\hat\nabla_y\boldsymbol{\phi},\,\frac{1}{\gamma}\partial_3\boldsymbol{\phi})\qquad\text{weakly in } L^2.$$

Since the left-hand side converges to $\boldsymbol{U}$, the desired statement follows. □

Proof of Proposition 5.1. Consider the γ-dependent functional

$$\mathcal{Q}_\gamma : L^2(S\times\mathcal{Y},\mathbb{M}^3)\to[0,+\infty],$$
$$\mathcal{Q}_\gamma(\boldsymbol{U}) := \begin{cases}\displaystyle\iint_{S\times Y} Q(\hat x,x_3,y,\Lambda(\boldsymbol{A},\boldsymbol{B})+\boldsymbol{U}) & \text{if } \boldsymbol{U}\in L_\gamma(S\times\mathcal{Y},\mathbb{M}^3_{\rm sym}),\\ +\infty & \text{else.}\end{cases}$$

Note that $Q_\gamma(\hat x,\boldsymbol{A},\boldsymbol{B}) = \inf_{\boldsymbol{U}\in L^2(S\times\mathcal{Y},\mathbb{M}^3)}\mathcal{Q}_\gamma(\boldsymbol{U})$. By definition of $\mathcal{Q}_\gamma$ and property (Q3), the minimum on the right-hand side is attained in $L_\gamma(S\times\mathcal{Y},\mathbb{M}^3_{\rm sym})$. Now, consider $\gamma\in[0,\infty]$ and a sequence $\{\gamma^n\}_{n\in\mathbb{N}}\subset(0,\infty)$ that converges to γ. For $n\in\mathbb{N}$, let $\boldsymbol{U}_n\in L_{\gamma^n}(S\times\mathcal{Y},\mathbb{M}^3_{\rm sym})$ denote a matrix field with $Q_{\gamma^n}(\hat x,\boldsymbol{A},\boldsymbol{B}) = \mathcal{Q}_{\gamma^n}(\boldsymbol{U}^n)$. Due to property (Q3) the sequence $\{\boldsymbol{U}^n\}_{n\in\mathbb{N}}$ is bounded in $L^2(S\times Y,\mathbb{M}^3_{\rm sym})$. Hence, we have, up to a subsequence, $\boldsymbol{U}^n\rightharpoonup\boldsymbol{U}^\infty$ weakly in $L^2(S\times Y,\mathbb{M}^3_{\rm sym})$. From Lemma 5.2 we deduce that $\boldsymbol{U}^\infty\in L_\gamma(S\times Y,\mathbb{M}^3_{\rm sym})$. Hence, by the lower semicontinuity of convex integral functionals w. r. t. weak convergence in L^2, we deduce that

$$\begin{aligned}&\liminf_{n\to\infty} Q_{\gamma^n}(\hat x,\boldsymbol{A},\boldsymbol{B}) = \liminf_{n\to\infty}\mathcal{Q}_{\gamma^n}(\boldsymbol{U}^n)\\ &= \liminf_{n\to\infty}\iint_{S\times Y} Q(\hat x,x_3,y,\Lambda(\boldsymbol{A},\boldsymbol{B})+\boldsymbol{U}^n)\,dy\,dx_3\\ &\geq \iint_{S\times Y} Q(\hat x,x_3,y,\Lambda(\boldsymbol{A},\boldsymbol{B})+\boldsymbol{U}^\infty)\,dy\,dx_3 = \mathcal{Q}_{\gamma^n}(\boldsymbol{U}^\infty)\geq Q_\gamma(\hat x,\boldsymbol{A},\boldsymbol{B}).\end{aligned}$$

It remains to prove the opposite inequality

$$(97)\qquad \limsup_{n\to\infty} Q_{\gamma^n}(\hat x,\boldsymbol{A},\boldsymbol{B})\leq Q_{\gamma^\infty}(\hat x,\boldsymbol{A},\boldsymbol{B}),$$

which we prove by an explicit construction of a recovery sequence. We distinguish the cases $\gamma \in \{0, \infty\}$ and $\gamma \in (0, \infty)$.

The case $\gamma = 0$. Let $\boldsymbol{U} \in L_0(S \times \mathcal{Y}, \mathbb{M}^3)$ satisfy $\mathcal{Q}_0(\boldsymbol{U}) = Q_0(\hat{x}, \boldsymbol{A}, \boldsymbol{B})$. By definition there exist $\boldsymbol{\zeta} \in H^1(\mathcal{Y}, \mathbb{R}^2)$, $\varphi \in H^2(\mathcal{Y})$, $\boldsymbol{g} \in L^2(S \times Y, \mathbb{R}^3)$ such that

$$\boldsymbol{U} = \begin{pmatrix} \operatorname{sym} \hat{\nabla}_y \boldsymbol{\zeta} + x_3 \hat{\nabla}_y^2 \varphi & \begin{matrix} \boldsymbol{g}_1 \\ \boldsymbol{g}_2 \end{matrix} \\ (\boldsymbol{g}_1,\ \boldsymbol{g}_2) & \boldsymbol{g}_3 \end{pmatrix}.$$

Set

$$\begin{aligned} \boldsymbol{\phi}^n_\alpha &:= \boldsymbol{\zeta}_\alpha + x_3 \partial_{y_\alpha} \varphi + \gamma^n \int_{-\frac{1}{2}}^{x_3} \boldsymbol{g}_\alpha(s, y)\, ds \qquad \text{for } \alpha = 1, 2, \\ \boldsymbol{\phi}^n_3 &:= -\frac{1}{\gamma^n} \varphi + \gamma^n \int_{-\frac{1}{2}}^{x_3} \boldsymbol{g}_3(s, y)\, ds. \end{aligned}$$

Then $\boldsymbol{\phi}^n \in H^1(S \times \mathcal{Y}, \mathbb{R}^3)$, $\boldsymbol{U}^n := \operatorname{sym}(\hat{\nabla}_y \boldsymbol{\phi}^n, \frac{1}{\gamma_n} \partial_3 \boldsymbol{\phi}^n) \in L_{\gamma^n}(S \times \mathcal{Y}, \mathbb{M}^3_{\mathrm{sym}})$ and $\boldsymbol{U}^n$ strongly converges to $\boldsymbol{U}$ in L^2, so that

$$\begin{aligned} \mathcal{Q}_{\gamma^n}(\boldsymbol{U}^n) &= \iint_{S \times Y} Q(\hat{x}, x_3, y, \Lambda(\boldsymbol{A}, \boldsymbol{B}) + \boldsymbol{U}^n)\, dy\, dx_3 \\ &\to \iint_{S \times Y} Q(\hat{x}, x_3, y, \Lambda(\boldsymbol{A}, \boldsymbol{B}) + \boldsymbol{U})\, dy\, dx_3 = \mathcal{Q}_0(\boldsymbol{U}), \end{aligned}$$

by the continuity of convex integral functionals w. r. t. strong convergence L^2. By the choice of $\boldsymbol{U}$, (97) follows.

The case $\gamma = \infty$. Let $\boldsymbol{U} \in L_\infty(S \times \mathcal{Y}, \mathbb{M}^3)$ satisfy $\mathcal{Q}_\infty(\boldsymbol{U}) = Q_\infty(\hat{x}, \boldsymbol{A}, \boldsymbol{B})$. By definition there exist $\boldsymbol{\zeta} \in L^2(S, H^1(\mathcal{Y}, \mathbb{R}^2))$, $\psi \in H^1(\mathcal{Y})$, $\boldsymbol{c} \in L^2(S, \mathbb{R}^3)$ such that

$$\boldsymbol{U} = \begin{pmatrix} \operatorname{sym} \hat{\nabla}_y \boldsymbol{\zeta} & \begin{matrix} \partial_{y_1} \psi + \boldsymbol{c}_1 \\ \partial_{y_2} \psi + \boldsymbol{c}_2 \end{matrix} \\ \hat{\nabla}_y \psi + (\boldsymbol{c}_1, \boldsymbol{c}_2) & \boldsymbol{c}_3 \end{pmatrix}.$$

With

$$\begin{aligned} \boldsymbol{\phi}^n_\alpha &:= \boldsymbol{\zeta}_\alpha + \gamma^n \int_{-\frac{1}{2}}^{x_3} \boldsymbol{c}_\alpha(s)\, ds \qquad \text{for } \alpha = 1, 2, \\ \boldsymbol{\phi}^n_3 &:= \psi + \gamma^n \int_{-\frac{1}{2}}^{x_3} \boldsymbol{c}_3(s)\, ds, \end{aligned}$$

the desired convergence (97) follows as in the case $\gamma = 0$.

The case $\gamma \in (0, \infty)$. Choose $\boldsymbol{\phi} \in H^1(S \times \mathcal{Y}, \mathbb{R}^3)$ such that $\mathcal{Q}_\gamma(\boldsymbol{U}) = Q_\gamma(\hat{x}, \boldsymbol{A}, \boldsymbol{B})$ where $\boldsymbol{U} = \operatorname{sym}(\hat{\nabla}_y \boldsymbol{\phi}, \frac{1}{\gamma} \partial_3 \boldsymbol{\phi})$. With $\boldsymbol{U}^n := \operatorname{sym}(\hat{\nabla}_y \boldsymbol{\phi}, \frac{1}{\gamma^n} \partial_3 \boldsymbol{\phi})$ statement (97) follows as in the previous cases. $\square$

Appendix A. Two-scale convergence methods for thin domains

A.1. **Basic properties.** The classical notion of two-scale convergence as introduced in [Ngu89] and systematically studied in [All92] can be characterized by introducing a so called "periodic unfolding" operator. This observation goes back to the work by [ADH90, BLM96]. In [CDG02, Vis06, MT07] periodic unfolding is studied systematically.

As discussed in [Neu10], the version of two-scale convergence introduced in Definition 3.2 can be characterized by "periodic unfolding" in a similar way. For the reader's convenience, we summarize that observation in the following remark.

Recall that $\varepsilon(h)$ and h are coupled with ratio γ in the sense of (9).

Remark 2 (Characterization of strong/weak two-scale convergence by unfolding)**.** Set $\Omega_\infty := \mathbb{R}^2 \times S$. For measurable $g : \Omega_\infty \times S \to \mathbb{R}$ define the tensor field $\mathcal{U}_\varepsilon g : \Omega_\infty \to \mathbb{R}$ by

$$\mathcal{U}_\varepsilon g(x,y) := (\mathcal{U}_\varepsilon g)(\varepsilon\lfloor \tfrac{\hat{x}}{\varepsilon} \rfloor + \varepsilon\{y\}, x_3),$$

where the mappings $\lfloor\cdot\rfloor \in \mathbb{Z}^2$ and $\{\cdot\} \in [0,1)^2 = Y$ yield the integer and fractional part of vectors $\mathbb{R}^2$ and are defined by the identity: $\forall \hat{x} \in \mathbb{R}^2 \; : \; \hat{x} = \lfloor \hat{x} \rfloor + \{\hat{x}\}$. For the following discussion it is convenient to set $r_\varepsilon(\hat{x},y) := \lfloor \hat{x}/\varepsilon(h) \rfloor + \{y\} - \hat{x}/\varepsilon(h)$. Notice that $|r_h(\hat{x},y)| \leq 1$.

(a) Let $1 \leq p,\, q \leq \infty$ be dual integrability exponents. Then a change of variables shows that the identity

$$\int_{\Omega_\infty} g(x)\,\psi(x, \tfrac{\hat{x}}{\varepsilon})\,dx \;=\; \iint_{\Omega_\infty \times Y} (\mathcal{U}_\varepsilon g)(x,y)\,\psi\big(\,\hat{x} + \varepsilon r_\varepsilon(\hat{x},y), x_3, y\,\big)\,dy\,dx$$

holds for all $g \in L^p(\Omega_\infty)$, $\psi \in L^q(\Omega_\infty, C(\mathcal{Y}))$ and $\varepsilon > 0$ (e.g. see Lemma 1.1 in [Vis06]). In particular, we have

$$||g||_{L^2(\Omega_\infty)} = ||\mathcal{U}_\varepsilon g||_{L^2(\Omega_\infty \times Y)}.$$

(b) (Characterization for sequences on Ω_∞). For a sequence g^h in $L^2(\Omega_\infty)$ and $g \in L^2(\Omega_\infty \times Y)$ there holds

$$g^h \xrightharpoonup{2,\gamma} g \qquad \Leftrightarrow \qquad \mathcal{U}_{\varepsilon(h)} g^h \rightharpoonup g \text{ weakly in } L^2(\Omega_\infty \times Y)$$

$$g^h \xrightarrow{2,\gamma} g \qquad \Leftrightarrow \qquad \mathcal{U}_{\varepsilon(h)} g^h \to g \text{ strongly in } L^2(\Omega_\infty \times Y)$$

(see e.g. Proposition 2.5 in [Vis06]).

(c) (Characterization for sequences on the bounded domain Ω). Let $\omega \subset \mathbb{R}^2$ be Lipschitz and set $\Omega := \omega \times Y$. Let $\mathcal{T} : L^2(\Omega) \to L^2(\Omega_\infty)$ denote the operator that associates g with its extension by 0. Then $g^h \in L^2(\Omega)$ strongly/weakly two-scale converges in $L^2(\Omega \times Y)$, if and only if $\mathcal{U}_{\varepsilon(h)}\mathcal{T} g^h$ strongly/weakly converges in $L^2(\Omega_\infty \times Y)$.

The following statement is a direct consequence of Remark 2.

Lemma A.1. *Consider $\varphi \in L^2(\Omega, C(\mathcal{Y}))$. Then $\varphi^h(x) := \varphi(x, \frac{\hat{x}}{\varepsilon(h)})$ strongly two-scale converges in L^2 to φ.*

Lemma A.2 (Characterization of strong two-scale convergence)**.** *There exists a function*

$$\mathfrak{d} \;:\; (0,\infty) \times L^2(\Omega, \mathbb{M}^d) \times L^2(\Omega \times Y, \mathbb{M}^d) \to [0,\infty)$$

such that :

(a) For all $\varepsilon > 0$, $\boldsymbol{E} \in L^2(\Omega, \mathbb{M}^d)$ and $\boldsymbol{F}_1, \boldsymbol{F}_2 \in L^2(\Omega \times Y, \mathbb{M}^d)$ the triangle inequality

$$\mathfrak{d}(\varepsilon, \boldsymbol{E}, \boldsymbol{F}_1 + \boldsymbol{F}_2) \leq \mathfrak{d}(\varepsilon, \boldsymbol{E}, \boldsymbol{F}_1) + \iint_{\Omega \times Y} |\boldsymbol{F}_2|^2\,dydx$$

holds.

(b) For every family $\{\boldsymbol{E}^h\}_{h>0}$ in $L^2(\Omega, \mathbb{M}^d)$ and any function $\boldsymbol{E} \in L^2(\Omega \times Y, \mathbb{M}^d)$ the following two statements are equivalent

(i) $\quad \boldsymbol{E}^h \xrightarrow{2,\gamma} \boldsymbol{E}$ *strongly two-scale converges to $\boldsymbol{E}$ in L^2*

(ii) $\quad \limsup\limits_{h\to 0} \mathfrak{d}(\varepsilon(h), \boldsymbol{E}^h, \boldsymbol{E}) = 0$

Proof. Without loss of generality assume that $d = 1$; the case $d > 1$ can be treated by considering each component of the field separately. As in Remark 2 set $\Omega_\infty := \mathbb{R}^2 \times S$. We identify $\boldsymbol{F}$ and $\boldsymbol{G}$ with their extension by zero to the domain Ω_∞ and $\Omega_\infty \times Y$, respectively. Let $\mathcal{U}_\varepsilon$ denote the operator introduced in Remark 2 and define

$$\mathfrak{d}(\varepsilon, \boldsymbol{F}, \boldsymbol{G}) := \iint_{\Omega_\infty \times S} |(\mathcal{U}_\varepsilon \boldsymbol{F}) - \boldsymbol{G}|^2\,dy\,dx.$$

Then statement (a) is obvious and (b) follows from Remark 2 (c).

□

Lemma A.3 (Compactness)**.**
(a) Let $g^h \in L^2(\Omega)$ be a sequence with equibounded L^2-norm. Then there exists $g \in L^2(\Omega)$ such that $g^h \xrightharpoonup{2,\gamma} g$ in L^2 up to a subsequence.
(b) Let $g \in H^1(\omega)$ be a sequence that weakly converges in $H^1(\omega)$ to a function g. Then there exists $\phi \in L^2(\omega, H^1(\mathcal{Y}))$ such that $\hat{\nabla} g^h \xrightharpoonup{2,\gamma} \hat{\nabla} g + \hat{\nabla}_y \phi$ in L^2 up to a subsequence.
(c) Let $g^h \in H^2(\omega)$ denote a sequence that weakly converges in $H^2(\omega)$ to function g. Then there exists $\phi \in L^2(\omega, H^2(\mathcal{Y}))$ such that, up to a subsequence, $\hat{\nabla}^2 g^h \xrightharpoonup{2,\gamma} \hat{\nabla}^2 g + \hat{\nabla}_y^2 \phi$ weakly two-scale in L^2.

Statement (a) follows from the corresponding compactness result for classical two-scale convergence in $L^2(\Omega \times [0,1)^3)$. Notice that in (b) and (c) the notion of two-scale convergence coincides with classical two-scale convergence, since the involved fields do not depend on the x_3-variable. For a proof of (b) we refer to [All92]. (c) is a special case of Lemma 3 in [Vel].

Next we recall a compactness and identification result for the two-scale limits of scaled gradients:

Proposition A.4 (see Proposition 6.3.5 in [Neu10])**.** *Let $M \subset \mathbb{R}^d$ be a Lipschitz domain.*

– *Let $\boldsymbol{\phi}^h \in H^1(M \times S, \mathbb{R}^3)$ be a sequence satisfying*

$$\limsup_{h \to 0} \left\{ \|\boldsymbol{\phi}^h\|_{L^2} + \|\hat{\nabla}_h \boldsymbol{\phi}^h\|_{L^2} \right\} < \infty.$$

Then there exists a map $\boldsymbol{\phi} \in H^1(M, \mathbb{R}^3)$ and a field

$$\boldsymbol{H} = \begin{cases} \left(\hat{\nabla}_y \boldsymbol{\phi}^{(1)}, \partial_3 \boldsymbol{\phi}^{(2)} \right) & \text{for some } \begin{Bmatrix} \boldsymbol{\phi}^{(1)} \in L^2(M, H^1(\mathcal{Y}, \mathbb{R}^3)) \\ \boldsymbol{\phi}^{(2)} \in L^2(M \times Y, H^1(S, \mathbb{R}^3)) \end{Bmatrix} \\ & \text{if } \gamma = 0, \\ \left(\hat{\nabla}_y \boldsymbol{\phi}, \frac{1}{\gamma} \partial_3 \boldsymbol{\phi} \right) & \text{for some } \boldsymbol{\phi} \in L^2(M, H^1(S \times \mathcal{Y}, \mathbb{R}^3)) \\ & \text{if } \gamma \in (0, \infty), \\ \left(\hat{\nabla}_y \boldsymbol{\phi}^{(1)}, \partial_3 \boldsymbol{\phi}^{(2)} \right) & \text{for some } \begin{Bmatrix} \boldsymbol{\phi}^{(1)} \in L^2(M \times S, H^1(\mathcal{Y}, \mathbb{R}^3)) \\ \boldsymbol{\phi}^{(2)} \in L^2(M, H^1(S, \mathbb{R}^3)) \end{Bmatrix} \\ & \text{if } \gamma = \infty, \end{cases} \tag{98}$$

such that, up to a subsequence, $\boldsymbol{\phi}^h \to \boldsymbol{\phi}$ in L^2 and

$$\hat{\nabla}_h \boldsymbol{\phi}^h \xrightharpoonup{2,\gamma} \left(\hat{\nabla} \boldsymbol{\phi}, 0 \right) + \boldsymbol{H} \qquad \text{weakly two-scale in } L^2.$$

– *Let $\boldsymbol{\phi} \in H^1(M, \mathbb{R}^3)$ and $\boldsymbol{H}$ as in (98). Then there exists a sequence $\boldsymbol{\phi}^h \in H^1(M \times S, \mathbb{R}^3)$ such that $\boldsymbol{\phi}^h \to \boldsymbol{\phi}$ in L^2 and $\hat{\nabla}_h \boldsymbol{\phi}^h \xrightarrow{2,\gamma} \left(\hat{\nabla} \boldsymbol{\phi}, 0 \right) + \boldsymbol{H}$ strongly two-scale in L^2.*

In the space $H^1(S \times \mathcal{Y}, \mathbb{R}^3)$ the following Korn's inequality holds:

Proposition A.5 (cf. Theorem 6.3.7 in [Neu10])**.** *Let $\gamma \in (0, \infty)$. There exists a constant $C = C(\gamma) > 0$ such that for all $\boldsymbol{\phi} \in H^1(S \times \mathcal{Y}, \mathbb{R}^3)$ with $\iint_{S \times Y} \boldsymbol{\phi} = 0$ we have*

$$||\boldsymbol{\phi}||_{L^2} + || \left(\hat{\nabla} \boldsymbol{\phi}, \tfrac{1}{\gamma} \partial_3 \boldsymbol{\phi} \right) ||_{L^2} \le C || \operatorname{sym} \left(\hat{\nabla} \boldsymbol{\phi}, \tfrac{1}{\gamma} \partial_3 \boldsymbol{\phi} \right) ||_{L^2}.$$

Lemma A.6. *Let $\boldsymbol{\zeta} \in L^2(S, H^1(\mathcal{Y}, \mathbb{R}^2))$ be such that*

$$\int_Y \boldsymbol{\zeta}(x_3, y)\, dy = 0 \qquad \text{for almost every } x_3 \in S. \tag{99}$$

Assume that

$$\iint_{S\times\mathcal{Y}} \boldsymbol{\zeta} \cdot (\partial_{y_2}\partial_3\varphi, -\partial_{y_1}\partial_3\varphi)\, dy\, dx_3 = 0 \tag{100}$$

and for $\alpha, \beta = 1, 2$

$$\iint_{S\times Y} \partial_{y_\beta}\boldsymbol{\zeta}_\alpha \cdot \partial_3\partial_3\varphi\, dy\, dx_3 = 0 \tag{101}$$

for all $\varphi \in C_c^\infty(S, C^\infty(\mathcal{Y}))$. Then there exist $\tilde{\boldsymbol{\zeta}} \in H^1(\mathcal{Y}, \mathbb{R}^2)$ and $\tilde{\varphi} \in H^2(\mathcal{Y})$ such that

$$\hat{\nabla}_y \boldsymbol{\zeta} = \hat{\nabla}_y \tilde{\boldsymbol{\zeta}} + x_3 \hat{\nabla}_y^2 \tilde{\varphi}.$$

Proof. Let us define $\tilde{\boldsymbol{\zeta}} := \int_S \boldsymbol{\zeta}\, dx_3$, $\bar{\boldsymbol{\zeta}} := 12 \int_S x_3 \boldsymbol{\zeta}\, dx_3$. By using test functions of the form $\varphi(x_3, y_1, y_2) = \varphi_1(x_3)\varphi_2(y_1, y_2)$, where $\varphi_1 \in C_c(S)$ and $\varphi_2 \in C^\infty(\mathcal{Y})$ we conclude from (101) that

$$\hat{\nabla}_y \boldsymbol{\zeta} = \hat{\nabla}_y \tilde{\boldsymbol{\zeta}} + x_3 \hat{\nabla}_y \bar{\boldsymbol{\zeta}}.$$

Notice that $\tilde{\boldsymbol{\zeta}}, \bar{\boldsymbol{\zeta}} \in H^1(\mathcal{Y}, \mathbb{R}^2)$ and $\int_Y \tilde{\boldsymbol{\zeta}}\, dy = \int_Y \bar{\boldsymbol{\zeta}}\, dy = 0$. Using the property (99) we conclude that $\boldsymbol{\zeta} = \tilde{\boldsymbol{\zeta}} + x_3 \bar{\boldsymbol{\zeta}}$. From the property (100) we conclude that

$$\int_Y \bar{\boldsymbol{\zeta}} \cdot (\partial_{y_2}\varphi, -\partial_{y_1}\varphi) = 0, \quad \forall \varphi \in C^\infty(\mathcal{Y}).$$

This together with the fact that $\int_Y \bar{\boldsymbol{\zeta}}\, dy = 0$ implies that there exists $\tilde{\varphi} \in H^1(\mathcal{Y})$ such that $\bar{\boldsymbol{\zeta}} = \hat{\nabla}_y \varphi$ as can be seen by Fourier transform. Since $\bar{\boldsymbol{\zeta}} \in H^1(\mathcal{Y})$ we have that $\tilde{\varphi} \in H^2(\mathcal{Y})$ which finishes the proof of the lemma.

□

The following diagonalization lemma is due to [Att84, Corollary 1.16]:

Lemma A.7. *Let $g : [0, \infty) \times [0, \infty) \to [0, \infty)$ and suppose that*

$$\limsup_{\delta\to 0} \limsup_{h\to 0} g(\delta, h) = 0.$$

Then there is a monotone function $(0, \infty) \ni h \mapsto \delta(h) \in (0, \infty)$ with $\lim_{h\to 0} \delta(h) = 0$ and $\limsup_{h\to 0} g(\delta(h), h) = 0$.

A.2. **Proof of Lemma 4.3.** We essentially adapt the argument in [Vis06] to our notion of two-scale convergence on thin domains.

Set $\Omega_\infty := \mathbb{R}^2 \times S$. To ease the notation we identify $\widetilde{\boldsymbol{E}}$ with its extension by zero to the domain $\Omega_\infty \times \mathcal{Y}$ and set $Q(x, y, F) := 0$ for $x \in \Omega_\infty \setminus \Omega$, $y \in \mathcal{Y}$ and $F \in \mathbb{M}^3$. Let $\mathcal{U}_\varepsilon$ denote the operator introduced in Remark 2. Since Q is periodic in its y-component (see (Q2) in Lemma 2.9), application of Remark 2 (a) to $x \mapsto Q(x, \frac{\hat{x}}{\varepsilon(h)}, \widetilde{\boldsymbol{E}}^h(x))$ yields

$$\begin{aligned}\int_\Omega Q(x, \tfrac{\hat{x}}{\varepsilon(h)}, \widetilde{\boldsymbol{E}}^h(x))\, dx &= \iint_{\Omega_\infty \times Y} Q\big(\, \hat{x} + \varepsilon(h) r_{\varepsilon(h)}(\hat{x}, y), x_3, y, \mathcal{U}_{\varepsilon(h)} \widetilde{\boldsymbol{E}}^h(x, y)\,\big)\, dy\, dx \\ &= \iint_{\Omega_\infty \times Y} Q(\hat{x}, x_3, y, \mathcal{U}_{\varepsilon(h)} \widetilde{\boldsymbol{E}}^h(x, y))\, dy\, dx + o_h.\end{aligned}$$

for some $o_h \in \mathbb{R}$. By the continuity of Q (see (Q1) in Lemma 2.9), property (Q3) and the boundedness of $\widetilde{\boldsymbol{E}}^h$ in L^2, we have $o_h \to 0$. Thus (i) and (ii) follow from

(i') $$\liminf_{h\to 0} \iint_{\Omega_\infty \times Y} Q(\hat{x}, x_3, y, \mathcal{U}_{\varepsilon(h)} \widetilde{\boldsymbol{E}}^h(x,y))\, dy\, dx \geq \iint_{\Omega_\infty \times Y} Q(\hat{x}, x_3, y, \widetilde{\boldsymbol{E}}(x,y))\, dy\, dx$$

(ii') $$\lim_{h\to 0} \iint_{\Omega_\infty \times Y} Q(\hat{x}, x_3, y, \mathcal{U}_{\varepsilon(h)} \widetilde{\boldsymbol{E}}^h(x,y))\, dy\, dx = \iint_{\Omega_\infty \times Y} Q(\hat{x}, x_3, y, \widetilde{\boldsymbol{E}}(x,y))\, dy\, dx$$

respectively. Since $Q(x, y, F)$ is convex in F, the statement follows from the convergence assumptions on $\widetilde{\boldsymbol{E}}^h$, the characterization (c) in Remark 2 and the well-known continuity (resp. lower semi-continuity) result for convex integral functionals w. r. t. strong (resp. weak) convergence.

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