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Patterns from Bifurcations: A Symmetry Analysis of Networks with Delayed Coupling

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Abstract

We study systems of coupled units in a general network configuration with a coupling delay. We show that the destabilizing bifurcations from an equilibrium are governed by the extreme eigenvalues of the coupling matrix of the network. Based on the equivariant degree method and its computational packages, we perform a symmetry classification of destabilizing bifurcations in bidirectional rings of coupled units, for bifurcating solutions either of steady-states or of oscillating states. We also introduce the concept of secondary dominating orbit types to capture bifurcating solutions of submaximal nature.

Keywords: Symmetry, equivariant degree, bifurcation theory, network, delay, dynamical patterns.

1 Introduction

We consider n identical dynamical systems of form $\dot{x} = f(x)$ coupled together in a general network configuration and possibly with a time delay $\tau \geq 0$:

$$\dot{x_i}(t) = f(x_i(t)) + \kappa g_i(x_1(t-\tau), x_2(t-\tau), \dots, x_n(t-\tau)), \quad i = 1, 2, \dots, n, \tag{1.1}$$

where the scalar $\kappa > 0$ plays the role of coupling strength and g_i describes the interaction among the coupled systems. For simplicity of notations and a clear focus on the structural aspect of the system, we consider only scalar systems, i.e. $x_i \in \mathbb{R}$. The functions $f : \mathbb{R} \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R}$ are assumed to be C^1 and g_i will be assumed equivariant when we consider symmetry. We also assume that f and the g_i vanish at the origin, hence (1.1) admits the zero solution. We will study the stability of the zero solution and its loss of stability through bifurcations, in terms of the time delay and network structure of the system.

The tool we are using for symmetry bifurcation analysis is the equivariant degree and the "Equivariant Degree Maple© Library Package" that performs exact computations of values of equivariant degrees. The idea of using equivariant degree theory for equivariant bifurcation problems has been explored in various texts, see [7, 6] and the references therein. In short, one associates to a given bifurcating equilibrium a *bifurcation invariant* in form of an equivariant degree. Based on the precise value of the bifurcation invariant, one derives a full topological classification of the bifurcating branches respecting their symmetry properties. The calculation task of bifurcation invariants is completely taken over by the "Equivariant Degree Maple© Library Package". This equivariant degree approach together with assistance of the Maple© package has

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been employed for example, in [1, 4, 5, 2]. In the monograph [6], one can find a complete exposition on the subject including the construction of the equivariant degree, its fundamental properties and many applications in equivariant nonlinear problems. The “Equivariant Degree Maple© Library Package” was created by A. Biglands and W. Krawcewicz at the University of Alberta in 2006, supported by an NSERC summer research grant. The package is open source and is free to be downloaded, for example, at `http://www.math.uni-hamburg.de/home/ruan/download`. There is also a newly developed *GAP**-based algorithm as well as a web application† of it available for computation of equivariant degrees for dihedral groups. The results presented in this article are obtained however, using the “Equivariant Degree Maple© Library Package”, which is more straightforward to clarify.

The exact value of bifurcation invariant is computed by calling `showdegree`$[\Gamma]$ for a symmetry group Γ. The command `showdegree`$[\Gamma]$ takes several parameters as input, which are *solely* determined by the critical spectrum of the linearized operator. Corresponding to (1.1), the linearized system around zero is of form

$$\dot{y}(t) = f'(0)y(t) + \kappa C y(t-\tau), \quad y \in \mathbb{R}^n, \tag{1.2}$$

where $C = [c_{ij}] = [\partial g_i(0)/\partial x_j]$ is the *coupling matrix* of the network configuration of the system. In other words, the exact value of the bifurcation invariant associated to the zero solution of (1.1) depends only on the characteristic operator of (1.2).

In fact, all results that one retreat from the bifurcation invariant of (1.1), indeed, remain valid for any Γ-symmetric system whose linearization is of form (1.2). We give several examples of such systems.

The well-known neural network model

$$\dot{x_i}(t) = -x_i(t) + g\left(\sum_{j=1}^{n} a_{ij} x_j(t-\tau)\right), \tag{1.3}$$

where g is typically a sigmoidal function and $a_{ij} \in \mathbb{R}$ are entries of the adjacency matrix A that describes the coupling among the neurons. Linearization about the zero solution has the form (1.2) with $\kappa = g'(0)$ and C can be identified with the adjacency matrix A.

A more general form can be used to model pulse-coupled systems

$$\dot{x_i}(t) = f(x_i(t)) + h(x_i(t)) \cdot g\left(\sum_{j=1}^{n} a_{ij} x_j(t-\tau)\right), \tag{1.4}$$

indicating that the influence of the network on the ith unit may be different depending on the state of the ith unit at that particular time instant. Although (1.4) is not of form (1.1), its linearization is given by (1.2) with $\kappa = h(0)g'(0)$ and $C = A$.

In other models that involve diffusive-type interactions, say of form

$$\dot{x_i}(t) = f(x_i(t)) + \sum_{j=1}^{n} a_{ij} g(x_j(t-\tau) - x_i(t-\tau)), \tag{1.5}$$

**GAP* (“Groups, Algorithms, Programming”) is a system for computational discrete algebra. It provides a programming language and large data libraries of algebraic objects. The system is distributed freely at `http://www.gap-system.org`

†See *Dihedral Calculator* from MuchLearning, `http://dihedral.muchlearning.org`

the linearized equation (1.2) arises with C given by the negative of the Laplacian matrix, i.e. $C = -L = A - D$, where $D = \text{diag}\{k_1, \ldots, k_n\}$ is the diagonal matrix of vertex degrees $k_i = \sum_j a_{ij}$. If the delay originates only from the finite speed of information transmission from j to i, then one has a slightly variant system

$$\dot{x_i}(t) = f(x_i(t)) + \sum_{j=1}^{n} a_{ij} g(x_j(t-\tau) - x_i(t)), \tag{1.6}$$

whose linearization has the form (1.2) (with the identification $f(x_i) \to f(x_i) + g'(0)k_i x_i$) provided that all vertices have the same degree $k_i = k$, i.e., the network is regular.

In this paper we confine ourselves to bi-directional interactions and assume C to be a symmetric matrix; thus C has real eigenvalues. We take two quantities $\alpha := \tau f'(0)$ and $\beta := \tau\kappa\xi$ as bifurcation parameters, where $\xi \in \sigma(C)$. As we shall see, bifurcations, either of steady states or of oscillating states, that destabilize the zero solution, are related only to the extreme eigenvalues of C. Consequently, networks with the same extreme eigenvalues of the coupling matrix will exhibit the same destabilizing bifurcation behavior.

For symmetrically coupled system, we consider systems that are coupled in a bidirectional ring configuration, i.e., they possess dihedral symmetries. We derive the input parameters for computing the bifurcation invariant. To illustrate how to interpret values of bifurcation invariant, we present bifurcation classification results for bidirectional rings of 12 coupled units in Example 6.1. The classification results we present are not restricted to specific ring configurations, and they are derived using extreme eigenvalues of the coupling matrix only. See Table 2 for steady-state bifurcations and Table 3-5 for Hopf bifurcations. For bidirectional rings of larger size, the method can be systematically applied. Computational packages for dihedral symmetry are currently available for D_n up to $n = 200$. See *Dihedral Calculator* from MuchLearning `http://dihedral.muchlearning.org`. Other symmetry groups that are supported by computational packages are the quaternion group Q_8, the alternating groups A_4, A_5 and the symmetric group S_4, using the "Equivariant Degree Maple© Library Package".

It should be mentioned that since the bifurcation invariant is a topological invariant, it remains invariant against all (admissible, equivariant) continuous deformations on the system. As a consequence, the classification result one obtains using the bifurcation invariant remains valid even if the modeling of the system varies within the framework of symmetry. In short, our results are robust against model variations.

2 Preliminaries

2.1 Groups and Group Representations

Throughout we consider groups that are either finite or of form $\Gamma \times S^1$, where Γ is a finite group and S^1 is the group of complex numbers of unit length.

Let G be a group and H be a closed subgroup of G, written as $H \subset G$. Let $N(H) = \{g \in G : gHg^{-1} = H\}$ be the *normalizer* of H and $W(H) = N(H)/H$ the *Weyl group* of H. The set of all closed subgroups of G can be ordered by set inclusion. For subgroups $H, K \subset G$, we write $H \le K$ if $H \subseteq K$; $H < K$ if $H \subsetneq K$. The symbol (H) stands for the conjugacy class of the subgroup H in G; that is $(H) = \{gHg^{-1} : g \in G\}$. The set of all conjugacy classes of closed subgroups of G affords a partial order given by: $(H) \le (K)$ if $H \subseteq gKg^{-1}$ for some $g \in G$; similarly, $(H) < (K)$ if $H \subsetneq gKg^{-1}$ for some $g \in G$.

Example 2.1 (cf. [6]) Let $\Gamma = D_{12}$ be the dihedral group of order 24, which is represented as the group of 12 rotations: 1, η, η^2, $\dots$, η^{11} and 12 reflections: κ, $\kappa\eta$, $\kappa\eta^2$, $\dots$, $\kappa\eta^{11}$ of the complex plane $\mathbb{C}$, where η stands for the complex multiplication by $e^{\frac{i\pi}{6}}$ and κ denotes the complex conjugation. There are two kinds of subgroups in D_{12}: cyclic and dihedral. The cyclic subgroups are $\mathbb{Z}_1, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6, \mathbb{Z}_{12}$, where $\mathbb{Z}_k$ denotes the cyclic subgroup generated by η^l with $l = \frac{12}{k}$. The dihedral subgroups are

$$D_{k,j} = \{1, \eta^l, \eta^{2l}, \dots, \eta^{(k-1)l}, \kappa\eta^j, \kappa\eta^{j+l}, \kappa\eta^{j+2l}, \dots, \kappa\eta^{j+(k-1)l}\}, \quad \text{for } 0 \le j < l := \frac{12}{k},$$

where $k \in \{1, 2, 3, 4, 6, 12\}$. If l is odd, then all subgroups $D_{k,j}$ for $0 \le j < l$ are conjugate to $D_{k,0} := D_k$. If l is even, then all subgroups $D_{k,j}$ with j being even are conjugate to $D_{k,0} = D_k$; all subgroups $D_{k,j}$ with j being odd are conjugate to $D_{k,1} := \tilde{D}_k$. Thus, up to conjugacy relation, we have the dihedral subgroups: D_1, $\tilde{D}_1$, D_2, $\tilde{D}_2$, D_3, $\tilde{D}_3$, D_4, D_6 $\tilde{D}_6$, D_{12}. $\diamond$

A *real* (resp. *complex*) *representation* of G is a finite-dimensional real (resp. complex) vector space X with a continuous map, or *action*, $\psi : G \times X \to X$ such that the map $\psi(g, \cdot) : X \to X$ is linear, for every $g \in G$. Banach representations are similarly defined for Banach spaces with an action for which $\psi(g, \cdot)$ is bounded linear. We abbreviate $\psi(g, x)$ with gx.

A subset $\Omega \subset X$ is called *invariant*, if $gx \in \Omega$ whenever $x \in \Omega$ for all $g \in G$. A representation X of G is called *irreducible*, if $\{0\}$ and X are the only invariant subspaces in X. An action is called *free*, if $gx = x$ for some $x \in X$ implies $g = e$ is the neutral element.

Example 2.2 (cf. [6]) The dihedral group D_n, for $n \in \mathbb{N}$ even, has the following real irreducible representations:

(i) The trivial representation $\mathcal{V}_0 \simeq \mathbb{R}$, where every element acts as the identity map.

(ii) For $1 \le i \le \frac{n}{2} - 1$, there is the representation $\mathcal{V}_i \simeq \mathbb{R}^2 \simeq \mathbb{C}$ given by the following actions:

$$\eta z = \eta^i \cdot z, \quad \kappa z = \bar{z},$$

where "$\cdot$" is the complex multiplication and "$\bar{\ }$" is the complex conjugation.

(iii) The representation $\mathcal{V}_{\frac{n}{2}} \simeq \mathbb{R}$ given by: $\eta x = x$ and $\kappa x = -x$.

(iv) The representation $\mathcal{V}_{\frac{n}{2}+1} \simeq \mathbb{R}$ given by: $\eta x = -x$ and $\kappa x = x$.

(v) The representation $\mathcal{V}_{\frac{n}{2}+2} \simeq \mathbb{R}$ given by: $\eta x = -x$ and $\kappa x = -x$.

It has the following complex irreducible representations:

(i) The trivial representation $\mathcal{U}_0 \simeq \mathbb{C}$, where every element acts as the identity map.

(ii) For $1 \le j \le \frac{n}{2} - 1$, there is the representation $\mathcal{U}_j \simeq \mathbb{C} \times \mathbb{C}$ given by the following actions:

$$\eta(z_1, z_2) = (\eta^j \cdot z_1, \eta^{-j} \cdot z_2), \quad \kappa(z_1, z_2) = (z_2, z_1),$$

where "$\cdot$" is the complex multiplication.

(iii) The representation $\mathcal{U}_{\frac{n}{2}} \simeq \mathbb{C}$ given by: $\eta z = z$ and $\kappa z = -z$.

(iv) The representation $\mathcal{U}_{\frac{n}{2}+1} \simeq \mathbb{C}$ given by: $\eta z = -z$ and $\kappa z = z$.

(v) The representation $\mathcal{U}_{\frac{n}{2}+2} \simeq \mathbb{C}$ given by: $\eta z = -z$ and $\kappa z = -z$.

For $n \in \mathbb{N}$ odd, the dihedral group D_n has the above listed irreducible representations (i)-(iii), where n is replaced with $(n+1)$. $\diamond$

Let $x \in X$. By the *symmetry* of x, we mean the *isotropy* subgroup of x given by $\mathrm{Iso}\,(x) := \{g \in G \,:\, gx = x\}$ with respect to the group action on X. The set $\mathrm{Orb}\,(x) := \{gx \,:\, g \in G\}$ is called the *orbit* of x and the *symmetry* of the orbit is defined by the *orbit type* of x which is the conjugacy class $(\mathrm{Iso}\,(x))$of $\mathrm{Iso}\,(x)$. Note that $\mathrm{Iso}\,(gx) = g\,\mathrm{Iso}\,(x)g^{-1}$ for $g \in G$, thus the symmetry of the orbit is independent of the choice of x from the orbit.

Let $\Omega \subset X$ be a subset and $H \subset G$ be a closed subgroup. Define $\Omega_H = \{x \in X : \mathrm{Iso}\,(x) = H\}$. It can be verified that the Weyl group $W(H)$ acts freely on Ω_H. Denote the H*-fixed point subspace* in Ω by $\Omega^H = \{x \in X : gx = x, \forall\, g \in H\}$. Note that $\Omega_H \subset \Omega^H$. Moreover, Ω^H is the disjoint union of $\Omega_{\tilde{H}}$ for all $\tilde{H} \supseteq H$.

Example 2.3 Let $\Gamma = D_{12}$ and $X = \mathcal{V}_1$ be the real irreducible representation of D_{12} given in Example 2.2. Then, orbit types that occur in X are: (D_{12}), (D_1), $(\tilde{D}_1)$ and $(\mathbb{Z}_1)$ (cf. Example 2.1 for notations), with the corresponding fixed point subspaces:

$$X^{D_{12}} = \{(0,0)\}, \quad X^{D_1} = \{(x,0) : x \in \mathbb{R}\}, \quad X^{\tilde{D}_1} = \{re^{-\frac{i\pi}{12}} : r \in \mathbb{R}\}, \quad X^{\mathbb{Z}_1} = X.$$

Note that X^{D_1} is the disjoint union of subsets $X_{D_1} = \{(x,0) : x \in \mathbb{R},\, x \neq 0\}$ and $X_{D_{12}} = \{(0,0)\}$. On the subset X_{D_1}, the Weyl group $W(D_1) = D_2/D_1 \simeq \mathbb{Z}_2$ acts freely by the reflection. On the subset X_{D_12}, the Weyl group $W(D_{12}) = D_{12}/D_{12} \simeq \mathbb{Z}_1$ acts freely by the neutral element. $\diamond$

Finally, we remark that there is a natural way of "converting" a complex Γ-representation into a real $\Gamma \times S^1$-representation. Let U be a complex Γ-representation. Define a $\Gamma \times S^1$-action on U by

$$(\gamma, z)u = z \cdot (\gamma u), \quad \text{for } (\gamma, z) \in \Gamma \times S^1,\, u \in U, \tag{2.7}$$

where $\cdot$ stands for the complex multiplication. The obtained representation is denoted by $\bar{U}$ and called the $\Gamma \times S^1$*-representation induced* from U. Note that $\bar{U}$ is irreducible as a real $\Gamma \times S^1$-representation if U is irreducible as a complex Γ-representation.

2.2 Equivariant Maps and Equivariant Degree

Let X, Y be two Banach representations of G. A continuous map $f : X \to Y$ is called *equivariant*, if $f(g_\circ x) = g_* f(x)$, for all $x \in X$ and $g \in G$, where $\circ$ and $*$ stand for the G-actions on X and Y, respectively. A subset $\Omega \subset X$ is called *invariant*, if $g_\circ x \in \Omega$ whenever $x \in \Omega$ for all $g \in G$. In equivariant nonlinear analysis, one is interested in finding zeros of an equivariant map f in an invariant domain Ω. Note that by equivariance, the set of all zeros of f in Ω is composed of disjoint group orbits, thus one speaks of *zero orbits*, instead of zeros, of f.

A map f is called *admissible* on Ω, if $f(x) \neq 0$ for all $x \in \partial\Omega$. A homotopy $h : [0,1] \times X \to Y$ is called *admissible*, if $h(t, \cdot)$ is admissible for all $t \in [0,1]$. An equivariant degree, intuitively speaking, is an algebraic count of zero orbits of an admissible f in Ω with respect to orbit types, which remains unchanged against all admissible (equivariant) homotopies from f.

In the next two subsections, we review from [6] two types of equivariant degrees that will be used in Section 5 for bifurcation analysis. In both cases, the equivariant degree is first defined in finite-dimensional representations for continuous maps, and then extended to infinite-dimensional Banach representations for compact vector fields.

2.2.1 Equivariant Degree without Parameters

Let $G = \Gamma$ be a finite group acting on a finite-dimensional Γ-representation X. Let Φ be the set of all orbit types that appear in X. That is, every element of Φ is a conjugacy class of a finite subgroup of Γ. Consider a continuous equivariant map $f : X \to X$ on an open bounded invariant domain $\Omega \subset X$ such that f is admissible on Ω. Define an *equivariant degree (without parameter)* of f in Ω by a finite sum of integer-indexed orbit types:

$$\Gamma\text{-Deg}\,(f, \Omega) = \sum_{(K)\in\Phi} n_K \cdot (K), \tag{2.8}$$

where $n_K \in \mathbb{Z}$ is an integer counting zero orbits of orbit type (K). The precise definition of n_K can be given by the following *recurrence formula*:

$$n_K = \frac{\deg\,(f|_{\Omega^K}, \Omega^K) - \sum_{(\tilde{K})>(K)} n_{\tilde{K}} \cdot |W(\tilde{K})| \cdot n(K, \tilde{K})}{|W(K)|}. \tag{2.9}$$

We explain the notations used in (2.9) and their geometric meanings. Recall that Ω^K denotes the fixed point subspace of K in Ω. By restricting f on Ω^K, one obtains an (admissible) map $f|_{\Omega^K} : \Omega^K \to \Omega^K$. Using the classical Brouwer degree "deg", the integer "$\deg\,(f|_{\Omega^K}, \Omega^K)$" counts the zeros of f in Ω^K. Since not every element in Ω^K has the precise isotropy K, one needs to subtract those zeros of larger isotropies. This is done by subtracting the summands in (2.9). Within each summand, $n_{\tilde{K}}$ is the integer counting zero orbits of orbit type $(\tilde{K})$. Since the Weyl group $W(\tilde{K})$ acts freely on $\Omega_{\tilde{K}}$, the integer $n_{\tilde{K}} \cdot |W(\tilde{K})|$ then counts the zeros of isotropy $\tilde{K}$. The number $n(K, \tilde{K})$ is defined as the number of distinct conjugate copies of $\tilde{K}$ that contain K, formally by

$$n(K, \tilde{K}) = \left|\frac{\{g \in \Gamma : K \subset g\tilde{K}g^{-1}\}}{N(\tilde{K})}\right|.$$

Thus, the number $n_{\tilde{K}} \cdot |W(\tilde{K})| \cdot n(K, \tilde{K})$ counts the zeros of isotropy K' for all K' with $(K') = (\tilde{K})$. It follows that the expression of the numerator in (2.9) gives the count of zeros of f having precise isotropy K. Again, since $W(K)$ acts freely on Ω_K, we have then the total expression on the right hand side of (2.9) gives the count of zero orbits of f having orbit type (K).

Example 2.4 Let $\Gamma = D_{12}$ and $X = \mathcal{V}_1$ be the real irreducible representation of D_{12} given in Example 2.2. Consider the antipodal map $f = -\text{Id} : X \to X$ on the unit disc $B \subset X$, which is D_{12}-equivariant and B-admissible. As mentioned in Example 2.3, orbit types that occur in $\mathcal{V}_1$ are: (D_{12}), (D_1), $(\tilde{D}_1)$ and $(\mathbb{Z}_1)$. Thus,

$$\Gamma\text{-Deg}\,(-\text{Id}, B) = n_{D_{12}} \cdot (D_{12}) + n_{D_1} \cdot (D_1) + n_{\tilde{D}_1} \cdot (\tilde{D}_1) + n_{\mathbb{Z}_1} \cdot (\mathbb{Z}_1).$$

We compute n_{D_1} using (2.9). To do so, we first need to compute $n_{D_{12}}$:

$$n_{D_{12}} = \frac{\deg\,(-\text{Id}, B^{D_{12}})}{|W(D_{12})|} = \frac{1}{1} = 1,$$

where we used the fact $B^{D_{12}} = X^{D_{12}} \cap B = \{(0,0)\}$, $W(D_{12}) = \mathbb{Z}_1$ from Example 2.3 and $\deg\,(-\text{Id}, \mathbb{R}^m) = (-1)^m$ for $m \in \{0\} \cup \mathbb{N}$. Thus, we have

$$n_{D_1} = \frac{\deg\,(-\text{Id}, B^{D_1}) - 1 \cdot 1 \cdot 1}{|W(D_1)|} = \frac{-1-1}{2} = -1,$$

where we used the fact $n(D_1, D_{12}) = \left|\frac{D_{12}}{D_{12}}\right| = 1$ and $W(D_1) = \mathbb{Z}_2$. Following (2.9) further, one shows that

$$\Gamma\text{-Deg}\,(-\mathrm{Id}, B) = (D_{12}) - (D_1) - (\tilde{D}_1) + (\mathbb{Z}_1).$$

$\diamond$

The definition of equivariant degree can be extended, in a standard way, to infinite-dimensional Banach representations for *compact equivariant fields*, namely, equivariant maps of form $f = \mathrm{Id} - F : D \subset X \to X$ that are admissible on a bounded domain D such that $\overline{F(D)}$ is compact. It was shown in [3] that the equivariant degree defined by (2.8)-(2.9), as well as its infinite-dimensional extension, satisfies usual properties of a degree theory such as the *existence property*, which states that

$$n_K \neq 0 \text{ in } \ (2.8) \quad \Rightarrow \quad f^{-1}(0) \cap \Omega^K \neq \emptyset,$$

which can be useful for predicting zero orbits of orbit type at least (K).

2.2.2 Equivariant Degree with One Parameter

Let $G = \Gamma \times S^1$ be the product of a finite group Γ and the circle group S^1. There are two types of closed subgroups in G: those subgroups that are of form $K \times S^1$ for some subgroups $K \subset \Gamma$, or otherwise, they are the *twisted subgroups* of G, defined as follows.

Definition 2.5 A subgroup $H \subset \Gamma \times S^1$ is called a *twisted l-folded subgroup*, if there exists a subgroup $K \subset \Gamma$, an integer $l \geq 0$ and a group homomorphism $\varphi : K \to S^1$ such that

$$H = K^{\varphi,l} := \{(\gamma, z) \,:\, \varphi(\gamma) = z^l\}.$$

Conjugacy classes of twisted subgroups are called *twisted orbit types.* $\diamond$

Example 2.6 Let $G = D_{12} \times S^1$ be the product group of the dihedral D_{12} and the unit circle $S^1 \subset \mathbb{C}$. We describe its twisted subgroups $H = K^\phi$. Clearly, all subgroups of D_{12} are twisted subgroups with $\phi \equiv 1 \in S^1$. Besides that, there are twisted subgroups that are not contained in D_{12}. These can be classified into two categories: those for which $K = \mathbb{Z}_k$ and those for which $K = D_{k,j}$ (cf. Example 2.1 for notations).

Let $K = \mathbb{Z}_k$ for some $k \in \{1, 2, 3, 4, 6, 12\}$ and $\phi : K \to S^1$ be given by $\phi(\eta^l) = \eta^{jl}$ for some j with $1 \leq j < k$. Then,

$$K^\phi = \{(1,1), (\eta^l, \eta^{jl}), (\eta^{2l}, \eta^{2jl}), \dots, (\eta^{(k-1)l}, \eta^{j(k-1)l})\} := \mathbb{Z}_k^{t_j}, \quad \text{for } 1 \leq j < k.$$

Among these subgroups, $\mathbb{Z}_k^{t_j}$ and $\mathbb{Z}_k^{t_{k-j}}$ are conjugate to each other, for $1 \leq j < k$. Thus, for k even, up to conjugacy relation, we have the twisted subgroups $\mathbb{Z}_k^{t_1}$, $\mathbb{Z}_k^{t_2}, \dots,$ $\mathbb{Z}_k^{t_{\frac{k}{2}}} := \mathbb{Z}_k^d$; for k odd, $\mathbb{Z}_k^{t_1}$, $\mathbb{Z}_k^{t_2}, \dots,$ $\mathbb{Z}_k^{t_{\frac{k-1}{2}}}$. That is, we have $\mathbb{Z}_2^d$, $\mathbb{Z}_3^{t_1}$, $\mathbb{Z}_4^{t_1}$, $\mathbb{Z}_4^d$, $\mathbb{Z}_6^{t_1}$, $\mathbb{Z}_6^{t_2}$, $\mathbb{Z}_6^d$, $\mathbb{Z}_{12}^{t_1}$, $\mathbb{Z}_{12}^{t_2}$, $\mathbb{Z}_{12}^{t_3}$, $\mathbb{Z}_{12}^{t_4}$, $\mathbb{Z}_{12}^{t_5}$, $\mathbb{Z}_{12}^d$.

Let $K = D_{k,j}$ for some $k \in \{1, 2, 3, 4, 6, 12\}$ and $0 \leq j < l = \frac{12}{k}$. Up to conjugacy, it is sufficient to consider $K = D_k$ in case l is odd; and $K = D_k$, $K = \tilde{D}_k$ in case l is even (cf. Example 2.1). Let $\phi : K \to S^1$ be the group homomorphism such that $\ker \phi = \mathbb{Z}_k$. Then,

$$D_k^\phi = \{(1,1), (\eta^l, 1), \dots, (\eta^{(k-1)l}, 1), (\kappa, -1), (\kappa\eta^l, -1), \dots, (\kappa\eta^{(k-1)l}, -1)\} := D_k^z,$$

and

$$\tilde{D}_k^{\phi} = \{(1,1),(\eta^l,1),\dots,(\eta^{(k-1)l},1),(\kappa\eta,-1),(\kappa\eta^{1+l},-1),\dots,(\kappa\eta^{1+(k-1)l},-1)\} := \tilde{D}_k^z, \quad \text{if } l \text{ is even.}$$

Thus, we have D_1^z, $\tilde{D}_1^z$, D_2^z, $\tilde{D}_2^z$, D_3^z, $\tilde{D}_3^z$, D_4^z, D_6^z, $\tilde{D}_6^z$, D_{12}^z.

In the case k is even, there is a group homomorphism $\phi : K \to S^1$ for which $\ker\phi = D_{\frac{k}{2}}$. Then,

$$D_k^{\phi} = \{(1,1),(\eta^l,-1),(\eta^{2l},1),\dots,(\eta^{(k-1)l},-1),(\kappa,1),(\kappa\eta^l,-1),\dots,(\kappa\eta^{(k-1)l},-1)\} := D_k^d,$$

and

$$\tilde{D}_k^{\phi} = \{(1,1),(\eta^l,-1),(\eta^{2l},1),\dots,(\eta^{(k-1)l},-1),(\kappa\eta,1),(\kappa\eta^{1+l},-1),\dots,(\kappa\eta^{1+(k-1)l},-1)\} := \tilde{D}_k^d, \quad \text{if } l \text{ is even.}$$

Also, there is a group homomorphism $\phi : K \to S^1$ for which $\ker\phi = \tilde{D}_{\frac{k}{2}}$. Then,

$$D_k^{\phi} = \{(1,1),(\eta^l,-1),(\eta^{2l},1),\dots,(\eta^{(k-1)l},-1),(\kappa,-1),(\kappa\eta^l,1),\dots,(\kappa\eta^{(k-1)l},1)\} := D_k^{\hat{d}},$$

and

$$\tilde{D}_k^{\phi} = \{(1,1),(\eta^l,-1),(\eta^{2l},1),\dots,(\eta^{(k-1)l},-1),(\kappa\eta,-1),(\kappa\eta^{1+l},1),\dots,(\kappa\eta^{1+(k-1)l},1)\} := \tilde{D}_k^{\hat{d}}, \quad \text{if } l \text{ is even.}$$

One shows that for l even, D_k^d and $D_k^{\hat{d}}$ are conjugate; $\tilde{D}_k^d$ and $\tilde{D}_k^{\hat{d}}$ are conjugate. Thus, in the case k is even, up to conjugacy relation, we have the twisted subgroups D_k^d and $D_k^{\hat{d}}$ if l is odd; D_k^d and $\tilde{D}_k^d$ if l is even. That is, for D_{12}, we have D_1^d, $\tilde{D}_1^d$, D_2^d, $\tilde{D}_2^d$, D_3^d, $\tilde{D}_3^d$, D_4^d, $D_4^{\hat{d}}$, D_6^d, $\tilde{D}_6^d$, D_{12}^d, $D_{12}^{\hat{d}}$. $\diamond$

Let X be a finite-dimensional representation of G and $\mathbb{R}$ be the one-dimensional *parameter space* on which G acts trivially. Let Φ_1 be the set of all twisted orbit types that appear in $\mathbb{R}\times X$. Consider a continuous equivariant map $f : \mathbb{R}\times X \to X$ on an open bounded invariant domain $\Omega \subset \mathbb{R}\times X$ such that f is admissible on Ω. Define an *equivariant degree (with one parameter)* of f in Ω by a finite sum of integer-indexed twisted orbit types:

$$\Gamma \times S^1\text{-Deg}\,(f,\Omega) = \sum_{(H)\in\Phi_1} n_H \cdot (H), \tag{2.10}$$

where $n_H \in \mathbb{Z}$ is an integer counting zero orbits of the twisted orbit type (H). There is another recurrence formula, in resemblance of (2.9), that can be used to define the coefficients n_H's. We omit the precise formula here and refer to [6]. It is sufficient to mention that this degree can be extended to infinite-dimensional Banach representations for compact equivariant fields. The resulting degree satisfies all classical properties of an equivariant degree theory, among which the *existence property* plays an important role for our purpose:

$$n_H \neq 0 \text{ in } (2.10) \quad \Rightarrow \quad f^{-1}(0)\cap\Omega^H \neq \emptyset.$$

3 Coupled Systems of Identical Cells

We consider the main equation (1.1), which describes n identical dynamical systems of form $\dot{x} = f(x)$ with $f(0)=0$ coupled together in a general network configuration including a possible time delay $\tau \geq 0$. For convenience, we repeat (1.1) here:

$$\dot{x}_i(t) = f(x_i(t)) + \kappa g_i(x_1(t-\tau), x_2(t-\tau),\dots,x_n(t-\tau)), \quad i = 1,2,\dots,n,$$

where $x_i \in \mathbb{R}$, $\kappa > 0$ is the coupling strength, $f : \mathbb{R} \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R}$ are assumed to be C^1. Furthermore, the g_i are assumed to vanish at the origin. Thus, (1.1) admits the zero solution, for which we study the stability and bifurcations in terms of the time delay and symmetry of the system.

By linearizing (1.1) at zero, we obtain (1.2) which is

$$\dot{y}(t) = f'(0)y(t) + \kappa C y(t-\tau)$$

where $y = (y_1, \ldots, y_n)$ and $C = [c_{ij}] = [\partial g_i(0)/\partial x_j]$. The network structure is encoded in the coupling matrix C. The component $c_{ij} \in \mathbb{R}$ describes how strongly the j-th cell influences the i-th cell. The influence is enhancing if $c_{ij} > 0$ or inhibiting if $c_{ij} < 0$.

As mentioned earlier in the Introduction, (1.1) and (1.2) represent a broad class of coupled systems, for which our bifurcation analysis applies. Based on the linearized system (1.2), we obtain bifurcation existence results that are valid for systems whose linearization has the form (1.2), such as (1.3), (1.4), (1.5) and (1.6). For simplicity, we assume the coupling matrix C to be symmetric so that it has only real eigenvalues. All the time delay are assumed to be identical. These assumptions allow us to carry out a stability and bifurcation analysis of manageable size.

4 Stability Analysis and the Bifurcation Diagram

For $\tau > 0$, the time in the linearized equation (1.2) can be scaled $t \to t/\tau$ to yield

$$\dot{y}(t) = \tau f'(0)y(t) + \tau\kappa C y(t-1) \tag{4.11}$$

Thus, the characteristic operator for (4.11) is

$$\Delta(\lambda) = (\lambda - \tau f'(0))I_n - \tau\kappa e^{-\lambda} C : \mathbb{C}^n \to \mathbb{C}^n \tag{4.12}$$

and the corresponding characteristic equation is

$$\det \Delta(\lambda) = \prod_{\xi \in \sigma(C)} \left(\lambda - \tau f'(0) - \tau\kappa e^{-\lambda}\xi\right) = 0. \tag{4.13}$$

Since C is assumed to be a symmetric matrix, we have $\sigma(C) \subset \mathbb{R}$. Let $\xi \in \sigma(C)$ and consider the corresponding factor in (4.13). If $\lambda = u + iv$ is a characteristic root, then separating real and imaginary parts leads to

$$\begin{cases} u - \alpha - \beta e^{-u}\cos v = 0 \\ v + \beta e^{-u}\sin v = 0, \end{cases} \tag{4.14}$$

where $\alpha = \tau f'(0)$ and $\beta = \tau\kappa\xi$. For purely imaginary roots, we have $u = 0$, giving

$$\begin{cases} -\alpha - \beta\cos v = 0 \\ v + \beta\sin v = 0. \end{cases} \tag{4.15}$$

For $v = 0$ the solution is the line L1 defined by $\beta = -\alpha$, which corresponds to parameter values for which $\lambda = 0$ is a characteristic root. Over the intervals $v \in (k\pi, (k+1)\pi)$, $k \in \mathbb{Z}$, the solution can be expressed in the parametric form $(\alpha(v), \beta(v)) = (v/\tan(v), -v/\sin(v))$, which gives parametric curves for which there exists a pair of purely imaginary characteristic roots of the form $\lambda = \pm iv$. These bifurcation curves are depicted in Figure 1. Knowing that the zero

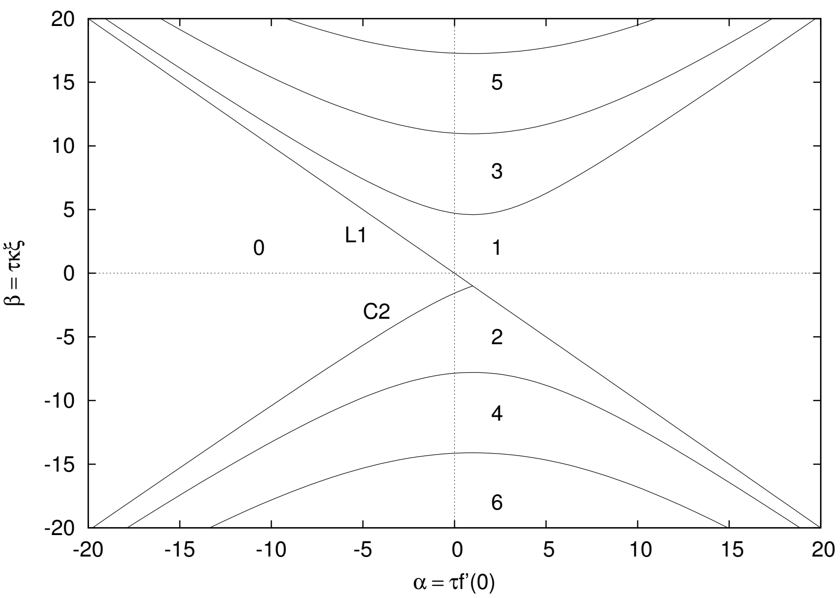


Figure 1: Bifurcation diagram of the characteristic equation. The curves indicate the parameter values for which the characteristic equation has a root on the imaginary axis. The curves separate the α-β parameter plane into regions in which the number of characteristic roots with positive real parts is a constant, the value of which is indicated in the figure. Hence "0" indicates the region where the origin is stable, which is bounded from above by the straight line L1 and from below by the curve C2.

solution is stable for $\beta = 0$ and $\alpha < 0$, and because characteristic roots can cross the imaginary axis only for parameter values belonging to the bifurcation curves, one can then move vertically in the parameter plane, increasing the number of roots with positive real parts appropriately each time a bifurcation curve is crossed. Implicit differentiation on bifurcation curves shows that the characteristic roots on the imaginary axis move to the right as $|\beta|$ increases, yielding the picture shown in Figure 1.

The region of stability is indicated in Figure 1 by the label "0". It is bounded from above by the straight line L1 and from below by the curve C2. The latter is given by the parametric branch $(\alpha, \beta) = (v/\tan(v), -v/\sin(v))$, $v \in (0, \pi)$, and meets the line L1 at the point $(1, -1)$. This is of course for one particular spatial mode corresponding to the eigenvalue ξ. One can then repeat the same argument for all eigenmodes $\xi \in \sigma(C)$. If a parameter pair (α, β) escapes the stable region by crossing the line L1, a bifurcation of steady states may occur. If it crosses the curve C2, then a bifurcation of oscillating states can take place. The codimension of these bifurcations is related to the multiplicity of the eigenvalue ξ given by the critical value of $\beta = \tau\kappa\xi$.

4.1 Effect of Network Structure

Suppose we start with stable systems ($f'(0) < 0$) without coupling, so we are initially on the negative α-axis. As we increase the coupling κ stability may be lost via a stationary or an oscillatory bifurcation through the first eigenmode ξ to hit L1 or C2. The important observation is that this first bifurcation depends only on the extremal eigenvalues ξ of the coupling matrix C. Hence, the number of relevant parameters is greatly reduced and one needs to check only the two extremal eigenvalues of the coupling matrix regardless of the network size. Thus one can classify networks by defining equivalence classes according to the extreme eigenvalues: networks having the same smallest and largest eigenvalues will have identical stability properties with regard to the class .

It is possible to give more precise statements. For diffusively coupled systems such as (1.5) or (1.6), the coupling matrix C is given by the negative of the Laplacian matrix; therefore, all its eigenvalues are non-positive, the largest one always being zero. In fact, for connected networks, all eigenvalues of C are strictly negative, except for a single zero eigenvalue. In this case, it is the smallest eigenvalue of C (i.e., the largest Laplacian eigenvalue) that determines the first bifurcation. As far as the network structure is concerned, this is the only relevant quantity.

For systems of the form (1.3) or (1.4), C is given by the adjacency matrix A, which can have both negative and positive eigenvalues. Thus both $\xi_{\min}$ and $\xi_{\max}$ should be considered for the first bifurcation.

4.2 Effect of Delays

For $\tau = 0$, the characteristic equation for (1.2) is

$$\prod_{\xi \in \sigma(C)} (\lambda - f'(0) - \kappa\xi) = 0. \tag{4.16}$$

from which the characteristic roots can be directly read off as $\lambda = f'(0)+\kappa\xi$, $\xi \in \sigma(C)$. The roots are real for real network eigenvalues ξ; hence the only critical root is $\lambda = 0$, which occurs when $f'(0) = -\kappa\xi$. The corresponding critical curve is a straight line on the parameter plane of $f'(0)$ versus $\kappa\xi$, which can be identified with the line L1 of Figure 1. Thus, one has stability below this line and one real positive characteristic root above, for a given spatial mode corresponding to ξ. In particular, Hopf bifurcations are not possible.

Hence, stationary bifurcations given by L1 of Figure 1 are independent of the delay, whereas the remaining set of curves of oscillatory bifurcation are a result of delay. In the following sections we will consider both stationary and oscillatory bifurcations in our symmetry analysis. The former type will be relevant for both delayed and undelayed systems, whereas the latter will be a feature of delayed systems only.

5 Symmetry Aspect and Equivariant Bifurcations

By a *symmetry* of a dynamical system, we mean a group of elements acting on the phase space that keep the system invariant.

Let S_n be the group of all permutations of n symbols. For $\Sigma \in S_n$ and consider its natural action on $\mathbb{R}^n$ by $(x_1, \ldots, x_n) \mapsto (x_{\Sigma(1),\ldots,x_{\Sigma(n)}})$. Consider a subgroup $\Gamma \subset S_n$.

Lemma 5.1 Let $\kappa \neq 0$. Then Γ is a symmetry of systems of form (1.1) if and only if

$$g_{\Sigma(i)}(x_1, x_2, \ldots, x_n) = g_i(x_{\Sigma(1)}, x_{\Sigma(2)}, \ldots, x_{\Sigma(n)}), \tag{5.17}$$

for all $\Sigma \in \Gamma$ and $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

Proof Let $\Sigma \in \Gamma$ and apply its action on (1.1). We obtain

$$\dot{x}_{\Sigma(i)}(t) = f(x_{\Sigma(i)}(t)) + \kappa g_i(x_{\Sigma(1)}(t-\tau), x_{\Sigma(2)}(t-\tau), \ldots, x_{\Sigma(n)}(t-\tau)). \tag{5.18}$$

Comparing with (1.1), we see that (5.18) is the same system as (1.1) if and only if

$$\kappa g_i(x_{\Sigma(1)}(t-\tau), x_{\Sigma(2)}(t-\tau), \ldots, x_{\Sigma(n)}(t-\tau)) = \kappa g_{\Sigma(i)}(x_1(t-\tau), x_2(t-\tau), \ldots, x_n(t-\tau)).$$

This leads to (5.17), since $\kappa \neq 0$. □

Remark 5.2 Note that a necessary condition for (5.17) to hold is

$$c_{ij} = c_{\Sigma(i)\Sigma(j)}, \quad \forall\, \Sigma \in \Gamma, \tag{5.19}$$

is satisfied for the coupling matrix C in the linearization (1.2). For systems (1.3), (1.4), (1.5) and (1.6), however, it is sufficient, since (5.17) reduces to $a_{ij} = a_{\Sigma(i)\Sigma(j)}$ for all $\Sigma \in \Gamma$. ◇

5.1 Bifurcation Analysis Using showdegree[Γ]

In what follows, $\Gamma \subset S_n$ stands for the group of symmetries of the system (**??**), which is determined by (5.17). In case the system takes special form of (1.3), (1.4), (1.5) or (1.6), the symmetry is determined directly by the coupling matrix C in (**??**) (cf. Remark 5.2). We are interested in studying the bifurcations that destabilize the equilibrium $x = 0$.

The tool we are using for bifurcation analysis is the equivariant degree and the "Equivariant Degree Maple© Library Package" that performs exact computations of values of equivariant degrees. This package is free to be downloaded at `http://www.math.uni-hamburg.de/home/ruan/download`.

We provide details of using the Maple© package for our computations. In fact, all computations of exact values of the associated bifurcation invariants are done by calling

$$\texttt{showdegree}[\Gamma](n_0, n_1, \ldots, n_r, m_0, m_1, \ldots, m_s), \quad \text{for } n_i, m_j \in \mathbb{Z}, \tag{5.20}$$

where Γ is a finite group describing the permutational symmetry of the coupled system, n_i's and m_j's are integers to be determined by the critical spectrum of the linearized system at the equilibrium.

The number r and s in (5.20) are predetermined by Γ. They are the number of all distinct (nontrivial) irreducible representations of Γ over reals and over complex numbers, respectively. In what follows, we use $\mathcal{V}_0, \mathcal{V}_1, \ldots, \mathcal{V}_r$ for the distinct real irreducible representations and $\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_s$ for the complex ones, where $\mathcal{V}_0$ and $\mathcal{U}_0$ are reserved for the trivial representations.

5.2 Steady-State Bifurcations

Assume that (α, β) crosses L1 through (α_o, β_o) from the shaded region in Figure 1. Then,

$$\alpha_o = -\beta_o = \tau\kappa\xi_o, \tag{5.21}$$

for an eigenvalue $\xi_o \in \sigma(C)$. For $\tau, \kappa > 0$, ξ_o is the maximal eigenvalue of C. Let $E(\xi_o)$ be the generalized eigenspace of ξ_o. Given the Γ-action on $\mathbb{R}^n$, we decompose $\mathbb{R}^n$ into pieces of $\mathcal{V}_i$'s:

$$\mathbb{R}^n = V_0 \times V_1 \times \cdots \times V_r,$$

where every V_i

$$V_i = \underbrace{\mathcal{V}_i \times \cdots \times \mathcal{V}_i}_{n_i \text{ times}} \tag{5.22}$$

is a product of n_i copies of $\mathcal{V}_i$ for some integer $n_i \in \mathbb{N} \cup \{0\}$. Also, since $E(\xi_o)$ is a Γ-invariant subspace of $\mathbb{R}^n$, we can decompose $E(\xi_o)$ as:

$$E(\xi_o) = E_0 \times E_1 \times \cdots \times E_r,$$

where every E_i is given by

$$E_i = \underbrace{\mathcal{V}_i \times \cdots \times \mathcal{V}_i}_{e_i \text{ times}} \tag{5.23}$$

is a product of e_i copies of $\mathcal{V}_i$ for some integer $e_i \in \mathbb{N} \cup \{0\}$. Using (5.22)-(5.23), define

$$u_i := n_i - e_i, \tag{5.24}$$

for $i = 0, 1, \ldots, r$. Then, the bifurcation invariant around (α_o, β_o) is given by

$$\omega_0 := \texttt{showdegree}[\Gamma](n_0, \ldots, n_r, 1, 0, \ldots, 0) - \texttt{showdegree}[\Gamma](u_0, \ldots, u_r, 1, 0, \ldots, 0). \tag{5.25}$$

Running the Maple© package, we obtain the value of ω_0 which is of form

$$c_1(K_1) + c_2(K_2) + \cdots + c_p(K_p),$$

for integers $c_i \in \mathbb{Z}$ and conjugacy classes (K_i) of subgroups K_i in Γ.

Theorem 5.3 *Let (α_o, β_o) be such that $\alpha_o = -\beta_o$ and $\xi_o \in \sigma(C)$ be given by (5.21). If ω_0 given by (5.25) is of form*

$$\omega_0 = c_1(K_1) + c_2(K_2) + \cdots + c_p(K_p),$$

for some $c_i \neq 0$, then there exists a bifurcating branch of steady states of symmetry at least (K_i).

Proof The formula (5.25) of the bifurcation invariant was established in [8] (cf. Theorem 8.5.2). We provide a different and more straightforward proof in the appendix (cf. Appendix A). The rest of the statement follows from the existence property of equivariant degree. □

Corollary 5.4 *Under the hypotheses of Theorem 5.3, if moreover, the subgroup K_i satisfies*

$$\xi_o \notin \sigma(C|_{Fix(H)}), \quad \forall H \supsetneq K_i, \tag{5.26}$$

then there exists a bifurcating branch of steady states of symmetry precisely (K_i).

Proof By Theorem 5.3, there exists a bifurcating branch of steady states of symmetry *at least* (K_i). Let (H) be the symmetry of this branch of solutions. Then, $(H) \geq (K_i)$. Up to the group conjugacy, we have $H \supseteq K_i$. We need to show $H = K_i$. Assume to the contrary that $H \supsetneq K_i$. Then, by (5.26), we have that when restricted to $\mathrm{Fix}\,(H)$, the characteristic operator $\Delta(0)|_{\mathrm{Fix}\,(H)} : \mathrm{Fix}\,(H) \to \mathrm{Fix}\,(H)$ is an isomorphism, for (α, β) in a neighborhood of (α_o, β_o). By the theorem of implicit functions, there can be no additional solution in neighborhood of the trivial solution $x = 0 \in \mathrm{Fix}\,(H)$, which is a contradiction. □

5.3 Hopf Bifurcations

Assume that (α, β) crosses C2 through (α_o, β_o) from the shaded region in Figure 1. Since C2 bounds the region from below and $\tau, \kappa > 0$, the first parameter pair that crosses C2 must be related to the minimal eigenvalue $\xi_{\min}$ of C.

Let $\xi_o \in \sigma(C)$ be the corresponding eigenvalue, i.e.

$$\beta_o = \tau\kappa\xi_o. \tag{5.27}$$

That is, $\xi_o = \xi_{\min}$ becomes critical. Consider the complexification $\mathbb{C}^n = \mathbb{C} \otimes \mathbb{R}^n$ of the phase space $\mathbb{R}^n$ and extend the Γ-action on $\mathbb{C}^n$ by defining

$$\gamma(z \otimes x) = z \otimes (\gamma x), \quad \text{for} \quad \gamma \in \Gamma,\, x \in \mathbb{R}^n. \tag{5.28}$$

The (generalized) eigenspace $E(\xi_o)$ remains Γ-invariant as a complex subspace of $\mathbb{C}^n$. Thus, we decompose $E(\xi_o)$ into irreducible representations $\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_s$ as:

$$E(\xi_o) = F_0 \times F_1 \times \cdots \times F_s,$$

where every F_j is given by

$$F_j = \underbrace{\mathcal{U}_j \times \cdots \times \mathcal{U}_j}_{m_j \text{ times}} \tag{5.29}$$

is a product of m_j copies of $\mathcal{U}_j$ for some integer $m_j \in \mathbb{N} \cup \{0\}$. Then, the bifurcation invariant around (α_o, β_o) for Hopf bifurcation is given by

$$\omega_1 := \texttt{showdegree}[\Gamma](0, 0, \ldots, 0, -m_0, -m_1, \ldots, -m_s). \tag{5.30}$$

Running the Maple© package, we obtain the value of ω_1 being of form

$$c_1(H_1) + c_2(H_2) + \cdots + c_q(H_q),$$

for integer coefficients $c_j \in \mathbb{Z}$ and conjugacy classes (H_j) of subgroups $H_j \subset \Gamma \times S^1$.

Theorem 5.5 *Let (α_o, β_o) be such that $(\alpha_o, \beta_o) \in C2$ in Figure 1 and ω_1 be given by (5.30). If*

$$\omega_1 = c_1(H_1) + c_2(H_2) + \cdots + c_q(H_q),$$

contains a non-zero coefficient $c_j \neq 0$ for some (H_j), then there exists a bifurcating branch of oscillating states of symmetry at least (H_j).

Proof Using equivariant degree theory, the bifurcation invariant is computed by (cf. [6])

$$\omega_1 = \texttt{showdegree}[\Gamma](k_0, k_1, \ldots, k_r, t_0, t_1, \ldots, t_s),$$

where k_i's are related to the *positive spectrum* of the right hand side of (4.11) in the constant function space, and the t_j's are the *crossing numbers* which are equal to either m_j or $-m_j$, depending on the direction of the crossing of the critical characteristic roots.

Consider (4.11) in the constant function space. Then,

$$\big(\tau f'(0)\mathrm{Id} + \tau\kappa C\big)x = 0, \quad x \in \mathbb{R}^n.$$

The positive spectrum σ_+ of the linear operator $(\tau f'(0)\mathrm{Id} + \tau\kappa C)$ is

$$\sigma_+ = \{\tau f'(0) + \tau\kappa\xi \,:\, \tau f'(0) + \tau\kappa\xi > 0, \quad \xi \in \sigma(C)\} = \{\alpha + \beta(\xi) \,:\, \alpha + \beta(\xi) > 0, \quad \xi \in \sigma(C)\},$$

which is an empty set, since the curve C2 lies in the area $\alpha + \beta < 0$. Since the integer k_i is the total number of copies of $\mathcal{V}_i$ in $E(\mu)$ for $\mu \in \sigma_+$, we have $k_i = 0$ for all $i = 0, 1, \ldots, r$.

The crossing numbers are positive if the critical characteristic roots cross from the right to the left of the complex plane; and negative otherwise. As (a, β) crosses C2 at (α_o, β_o) from the shaded region in Figure 1, the count of characteristic roots with positive real part increases by 2, thus all nonzero t_j's are negative and equal to $-m_j$. □

Theorem 5.5 gives an existing result of bifurcating branches together with their *least* symmetry. To sharpen to the *precise* symmetry, one can work with orbit types that satisfy certain maximal condition. Here, we recall the concept of *dominating orbit types* from [6] and introduce a new complementing definition of *secondary dominating orbit types.*

Definition 5.6 Let $\{\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_m\}$ be the set of irreducible Γ-representations that occur in $\mathbb{C}^n$, where $\mathbb{C}^n$ is the complexification of the phase space $\mathbb{R}^n$ of the system (1.1). Let $\bar{\mathcal{U}}_j$ be the $\Gamma \times S^1$-representation induced from $\mathcal{U}_j$, for $j = 1, 2, \ldots, m$ (cf. (2.7)). Collect maximal orbit types from $\bar{\mathcal{U}}_j$, for $j = 1, 2, \ldots, m$. Denote this collection by $\mathcal{M}$. An orbit type $(H) \in \mathcal{M}$ is called *dominating*, if (H) is maximal in $\mathcal{M}$. A non-dominating orbit type $(L) \in \mathcal{M}$ is called *secondary dominating*, if all orbit types $(H) \in \mathcal{M}$ satisfying $(L) < (H)$ are dominating. ◇

Proposition 5.7 *Let (α_o, β_o) be such that $(\alpha_o, \beta_o) \in C2$ in Figure 1 and ξ_o be the corresponding eigenvalue of C given by (5.27). Assume that ω_1 defined by (5.30) contains (H) with a nonzero coefficient. Then, the following holds:*

(i) If (H) is a dominating orbit type, then there exists a bifurcating branch of oscillating states of symmetry precisely equal to (H).

(ii) If (H) is a secondary dominating orbit type and for every dominating orbit type $(\tilde{H})$ with $(H) < (\tilde{H})$, there exists a C-invariant subspace $S \subset \mathbb{R}^n$ such that

(a) S contains every state of symmetry $\tilde{H}$; and

(b) $\xi_o \notin \sigma(C|_S)$,

then there exists a bifurcating branch of oscillating states of symmetry precisely being (H).

Proof The statement (i) follows from [6]. (ii) follows from the theorem of implicit functions, in the same spirit as Corollary 5.4. More precisely, let (H) be a secondary dominating orbit type with a no-zero coefficient in ω_1. By Theorem 5.5, there exists a a bifurcating branch of oscillating states of symmetry at least (H). Let $(\tilde{H})$ be the precise symmetry of this branch and suppose that $(H) < (\tilde{H})$. By definition of secondary dominating orbit types, the only orbit types that are strictly larger than (H) are dominating orbit types. Thus, $(\tilde{H})$ is dominating and so there exists a C-invariant subspace S in $\mathbb{R}^n$ satisfying (a)-(b). Note that this subspace S is also flow invariant for the system (1.1), thus one can consider the restricted flow on S. The bifurcating branch of oscillating states, by condition (a), is contained in S. However, by condition (b) and the Implicit Function Theorem, there can be no bifurcation taking place in S. This leads to a contradiction. □

6 Bidirectional Ring Configuration

In this section, we study the bifurcations of the system (1.1) with a bidirectional ring configuration. That is, we assume g_i's satisfy (5.17) for $\Gamma = D_n$. If the system takes form of (1.3), (1.4), (1.5) or (1.6), this assumption can be weakened to (5.19). In either case, the coupling matrix C in (1.2) satisfies (5.19), which in case of dihedral configuration implies that C is a *circulant matrix*‡ with $c_{1j} = c_{1,(n+2-j)}$ for $1 \leq j \leq n$. In particular, C is a symmetric matrix.

It is known that a circulant matrix with first row entries $c_0, c_1, \dots, c_{n-1}$ has the following eigenvalues

$$\xi_j = c_0 + c_1\omega_j + c_2\omega_j^2 + \cdots + c_{n-1}\omega_j^{n-1}, \quad j = 0, 1, 2, \dots, n-1,$$

with their eigenvectors $v_j = (1, \omega_j, \omega_j^2, \dots, \omega_j^{n-1})^T$, where $\omega_j = e^{\frac{2\pi i j}{n}}$'s are the n-th roots of unity. Moreover, if the circulant matrix is D_n-symmetric, then $\xi_j = \xi_{n-j}$ for $0 < j, k < n$, which is essentially induced by the D_n-symmetry. In fact, we have

$$\begin{cases} E(\xi_0) = \mathcal{V}_0, \\ E(\xi_j) = E(\xi_{n-j}) = \mathcal{V}_j \quad \text{for } 0 < j < \frac{n}{2} \\ E(\xi_{\frac{n}{2}}) = \mathcal{V}_{(\frac{n}{2}+2)}, \quad \text{if } n \text{ is even} \end{cases} \tag{6.31}$$

(cf. Example 2.2 for notations $\mathcal{V}_j$). An eigenvalue $\xi \in \sigma(C)$ is called *simple*, if $E(\xi)$ is irreducible. To a critical eigenvalue ξ_o, we associate an index set

$$I = \{i \,:\, \xi_i = \xi_o\} \tag{6.32}$$

(in case n is even and $\xi_{\frac{n}{2}} = \xi_o$, we put $\frac{n}{2} + 2$ into I instead of $\frac{n}{2}$), which collects all indices of irreducible representations that have to do with the critical eigenvalue ξ_o.

6.1 Steady-State Bifurcations for Bidirectional Rings

Recall that D_n acts on the phase space $\mathbb{R}^n$ by

$$\eta(x_1, x_2, \dots, x_n) = (x_n, x_1, x_2, \dots, x_{n-1}) \tag{6.33}$$

$$\kappa(x_1, x_2, \dots, x_n) = (x_n, x_{n-1}, \dots, x_1), \tag{6.34}$$

for $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. Using characters of representations, $\mathbb{R}^n$ can be decomposed into irreducible representations of D_n. In case of even n, we have

$$\mathbb{R}^n = \mathcal{V}_0 \times \mathcal{V}_1 \times \mathcal{V}_2 \times \cdots \times \mathcal{V}_{\frac{n}{2}-1} \times \mathcal{V}_{\frac{n}{2}+2} \tag{6.35}$$

and in case of odd n, we have

$$\mathbb{R}^n = \mathcal{V}_0 \times \mathcal{V}_1 \times \mathcal{V}_2 \times \cdots \times \mathcal{V}_{\frac{n-1}{2}}, \tag{6.36}$$

(cf. Example 2.2 for notations $\mathcal{V}_j$). It follows that the non-zero n_i's in (5.25) are (cf. (5.22))

$$\begin{cases} n_0 = n_1 = n_2 = \cdots = n_{\frac{n}{2}-1} = n_{\frac{n}{2}+2} = 1, \quad \text{if } n \text{ is even}, \\ n_0 = n_1 = n_2 = \cdots = n_{\frac{n-1}{2}} = 1, \quad \text{if } n \text{ is odd}. \end{cases} \tag{6.37}$$

‡Recall that an $n \times n$-matrix is called *circulant*, if every row is the right shift of the previous row (mod n). A circulant matrix $C = (c_{ij})$ is also denoted by circ $[c_1 1, c_1 2, \dots, c_{1n}]$ using the entries of its first row.

The integers u_i's in (5.25) are determined by the critical eigenvalue ξ_o and the corresponding I (cf. (6.32)). Based on (6.31) and the definition (5.24) of u_i, we have the non-zero u_i's are

$$\begin{cases} u_i = 1, & \text{for } i \in \{0, 1, 2, \dots, \frac{n}{2} - 1, \frac{n}{2} + 2\} \setminus I, \quad \text{if } n \text{ is even,} \\ u_i = 1, & \text{for } i \in \{0, 1, 2, \dots, \frac{n-1}{2}\} \setminus I, \quad \text{if } n \text{ is odd.} \end{cases} \tag{6.38}$$

Thus, the bifurcation invariant ω_0 can be computed using (5.25), accompanied by (6.37)-(6.38).

Example 6.1 (Simple critical eigenvalues for bidirectional rings) Let C be a coupling matrix satisfying (5.19) for $\Gamma = D_n$. Then, C is determined by $(\frac{n}{2} + 1)$ resp. $(\frac{n+1}{2})$ different entries if n is even resp. odd. These entries decide which eigenvalue is maximal. Let $\xi_o \in \sigma(C)$ be the maximal eigenvalue. Assume that ξ_o is simple, i.e. $E(\xi_o)$ is irreducible. Then, the index set I is a singleton and there are only possibly $\frac{n}{2}$ or $\frac{n-1}{2}$ different values of ω_0, depending on if n is even or odd. As an example, for $n = 12$, we have

$$\omega_0 = \begin{cases} -2(D_{12}) + 2(\tilde{D}_6) + 4(D_4) - 2(\tilde{D}_3) + 2(D_3) - 2(\tilde{D}_2) - 2(D_2) - 2(\mathbb{Z}_4) + 2(\mathbb{Z}_2), & \text{if } \xi_o = \xi_0 \\ (D_1) - (\tilde{D}_1), & \text{if } \xi_o = \xi_1 \\ -(D_2) + (\tilde{D}_2) + 2(D_1) - 2(\tilde{D}_1), & \text{if } \xi_o = \xi_2 \\ -(\tilde{D}_3) + (D_3), & \text{if } \xi_o = \xi_3 \\ 2(D_4) - 2(D_2) - (\mathbb{Z}_4) + (\mathbb{Z}_2) - 2(\tilde{D}_1) + 2(D_1), & \text{if } \xi_o = \xi_4 \\ -(\tilde{D}_1) + (D_1), & \text{if } \xi_o = \xi_5 \\ (\tilde{D}_6) - 2(\tilde{D}_3) + (\mathbb{Z}_3), & \text{if } \xi_o = \xi_6 \end{cases} \tag{6.39}$$

These values combined with fixed point subspaces of subgroups of D_{12} (cf. Table 1) lead to the classification result summarized in Table 2.To illustrate, in case $\xi_o = \xi_1$, we have two orbit types

K	$\mathrm{Fix}\,(K)$	$\sigma(C\vert_{\mathrm{Fix}\,(K)})$
D_{12}	$\{x_1 = x_2 = \cdots = x_{12}\}$	ξ_0
D_6	$\{x_1 = x_2 = \cdots = x_{12}\}$	ξ_0
$\tilde{D}_6$	$\{x_1 = x_3 = \cdots = x_{11}, x_2 = x_4 = \cdots = x_{12}\}$	ξ_0, ξ_6
$\mathbb{Z}_6$	$\{x_1 = x_3 = \cdots = x_{11}, x_2 = x_4 = \cdots = x_{12}\}$	ξ_0, ξ_6
D_4	$\{x_1 = x_3 = x_4 = x_6 = x_7 = x_9 = x_{10} = x_{12}, x_2 = x_5 = x_8 = x_{11}\}$	ξ_0, ξ_4
$\mathbb{Z}_4$	$\{x_1 = x_4 = x_7 = x_{10}, x_2 = x_5 = x_8 = x_{11}, x_3 = x_6 = x_9 = x_{12}\}$	ξ_0, ξ_4, ξ_4
D_3	$\{x_1 = x_4 = x_5 = x_8 = x_9 = x_{12}, x_2 = x_3 = x_6 = x_7 = x_{10} = x_{11}\}$	ξ_0, ξ_3
$\tilde{D}_3$	$\{x_1 = x_3 = x_5 = x_7 = x_9 = x_{11}, x_2 = x_6 = x_{10}, x_4 = x_8 = x_{12}\}$	ξ_0, ξ_3, ξ_6
$\mathbb{Z}_3$	$\{x_1 = x_5 = x_9, x_2 = x_6 = x_{10}, x_3 = x_7 = x_{11}, x_4 = x_8 = x_{12}\}$	$\xi_0, \xi_3, \xi_3, \xi_6$
D_2	$\{x_1 = x_6 = x_7 = x_{12}, x_2 = x_5 = x_8 = x_{11}, x_3 = x_4 = x_9 = x_{10}\}$	ξ_0, ξ_2, ξ_4
$\tilde{D}_2$	$\{x_1 = x_5 = x_7 = x_{11}, x_2 = x_4 = x_8 = x_{10}, x_3 = x_9, x_6 = x_{12}\}$	$\xi_0, \xi_2, \xi_4, \xi_6$
$\mathbb{Z}_2$	$\{x_1 = x_7, x_2 = x_8, x_3 = x_9, x_4 = x_{10}, x_5 = x_{11}, x_6 = x_{12}\}$	$\xi_0, \xi_2, \xi_2, \xi_4, \xi_4, \xi_6$
D_1	$\{x_1 = x_{12}, x_2 = x_{11}, x_3 = x_{10}, x_4 = x_9, x_5 = x_8, x_6 = x_7\}$	$\xi_0, \xi_1, \xi_2, \xi_3, \xi_4, \xi_5$
$\tilde{D}_1$	$\{x_1 = x_{11}, x_2 = x_{10}, x_3 = x_9, x_4 = x_8, x_5 = x_7\}$	$\xi_0, \xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6$
$\mathbb{Z}_1$	$\mathbb{R}^{12}$	$\xi_0, \xi_1, \xi_1, \xi_2, \xi_2, \xi_3, \xi_3, \xi_4, \xi_4, \xi_5, \xi_5, \xi_6$

Table 1: Fixed point subspaces of $K \subset D_{12}$ and eigenvalues of the coupling matrix $C|_{\mathrm{Fix}\,(K)}$: $\mathrm{Fix}\,(K) \to \mathrm{Fix}\,(K)$ (up to conjugacy classes of subgroups).

(D_1) and $(\tilde{D}_1)$ with non-zero coefficients in ω_0. Using Table 1, we have that $\xi_1 \notin \sigma(C|_{\mathrm{Fix}\,(H)})$ for all $H > D_1$, thus by Corollary 5.4, there exists at least one bifurcating branch of steady states of symmetry precisely (D_1). Since (D_1) consists of 6 isotropy subgroups: $\eta^k D_1 \eta^{-k}$ for

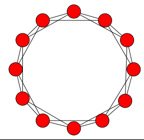
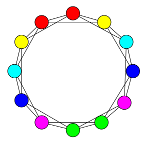
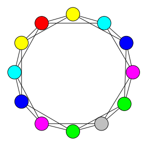
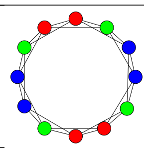
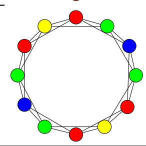
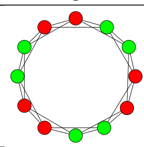
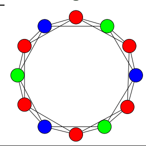
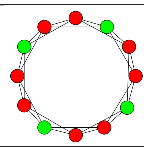
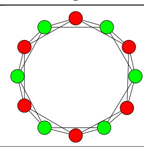
Critical Eigenvalue	**Symmetry**	**Form of Bifurcating Steady-States** (for distinct $a, b, c, d, e, f, g \in \mathbb{R}$)	**Figure**
ξ_0	D_{12}	$(a, a, a, a, a, a, a, a, a, a, a, a)$	
ξ_1 or ξ_5	D_1	$(a, b, c, d, e, f, f, e, d, c, b, a)$	
	$\eta D_1 \eta^{-1}$	$(a, a, b, c, d, e, f, f, e, d, c, b)$	
	$\eta^2 D_1 \eta^{-2}$	$(b, a, a, b, c, d, e, f, f, e, d, c)$	
	$\eta^3 D_1 \eta^{-3}$	$(c, b, a, a, b, c, d, e, f, f, e, d)$	
	$\eta^4 D_1 \eta^{-4}$	$(d, c, b, a, a, b, c, d, e, f, f, e)$	
	$\eta^5 D_1 \eta^{-5}$	$(e, d, c, b, a, a, b, c, d, e, f, f)$	
	$\tilde{D}_1$	$(a, b, c, d, e, f, e, d, c, b, a, g)$	
	$\eta \tilde{D}_1 \eta^{-1}$	$(g, a, b, c, d, e, f, e, d, c, b, a)$	
	$\eta^2 \tilde{D}_1 \eta^{-2}$	$(a, g, a, b, c, d, e, f, e, d, c, b)$	
	$\eta^3 \tilde{D}_1 \eta^{-3}$	$(b, a, g, a, b, c, d, e, f, e, d, c)$	
	$\eta^4 \tilde{D}_1 \eta^{-4}$	$(c, b, a, g, a, b, c, d, e, f, e, d)$	
	$\eta^5 \tilde{D}_1 \eta^{-5}$	$(d, c, b, a, g, a, b, c, d, e, f, e)$	
ξ_2	D_2	$(a, b, c, c, b, a, a, b, c, c, b, a)$	
	$\eta D_2 \eta^{-1}$	$(a, a, b, c, c, b, a, a, b, c, c, b)$	
	$\eta^2 D_2 \eta^{-2}$	$(b, a, a, b, c, c, b, a, a, b, c, c)$	
	$\tilde{D}_2$	$(a, b, c, b, a, d, a, b, c, b, a, d)$	
	$\eta \tilde{D}_2 \eta^{-1}$	$(d, a, b, c, b, a, d, a, b, c, b, a)$	
	$\eta^2 \tilde{D}_2 \eta^{-2}$	$(a, d, a, b, c, b, a, d, a, b, c, b)$	
ξ_3	D_3	$(a, b, b, a, a, b, b, a, a, b, b, a)$	
	$\eta D_3 \eta^{-1}$	$(a, a, b, b, a, a, b, b, a, a, b, b)$	
	$\tilde{D}_3$	$(a, b, a, c, a, b, a, c, a, b, a, c)$	
	$\eta \tilde{D}_3 \eta^{-1}$	$(c, a, b, a, c, a, b, a, c, a, b, a)$	
ξ_4	D_4	$(a, b, a, a, b, a, a, b, a, a, b, a)$	
	$\eta D_4 \eta^{-1}$	$(a, a, b, a, a, b, a, a, b, a, a, b)$	
	$\eta^2 D_4 \eta^{-2}$	$(b, a, a, b, a, a, b, a, a, b, a, a)$	
ξ_6	$\tilde{D}_6$	$(a, b, a, b, a, b, a, b, a, b, a, b)$	

Table 2: Summary of distinct forms of steady states bifurcating from the equilibrium $x = 0$ of the system (1.1) for $n = 12$.

$k = 0, 1, \ldots, 5$, we derive the form of the solution for each of these isotropies. The same can be applied to $(\tilde{D}_1)$.

Note that the all possible values of ω_0 do not depend on the entries of C directly, but rather their choice of the maximal eigenvalue. For example, if every cell is connected only with its 2 nearest neighbors, then $\xi_o = \xi_0$ if the coupling is enhancing; and $\xi_o = \xi_6$ if it is inhibiting. That is, this configuration does not allow ξ_o to be ξ_i for $i \in \{1, 2, 3, 4, 5\}$. However, if every cell is connected with its 4 nearest neighbors, then every eigenvalue can be maximal for some choices of the two coupling strength d_1, d_2. See Figure 2 for their precise relation.

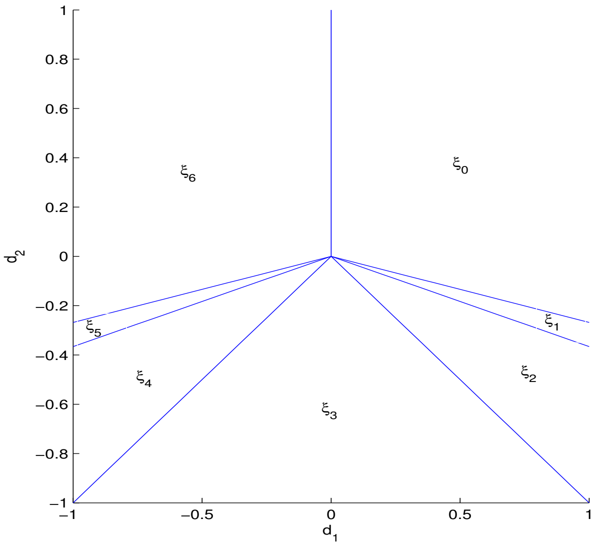


Figure 2: The maximal eigenvalue of the coupling matrix C, if every cell is connected with its 4 nearest neighbors, in relation to d_1, d_2.

$\diamond$

Besides those values listed in (6.39), ω_0 can take other values if ξ_o is non-simple. For example, the coupling configuration with 4 nearest neighbors allows double critical eigenvalues as shown in Figure 2, when the relation between d_1, d_2 follows one of the lines there. In this case, one can work out the index set I and compute ω_0 individually. The same result using Theorem 5.3 and Corollary 5.4 applies.

6.2 Hopf Bifurcations for Bidirectional Rings

The complexification of $E(\xi_j)$ for $\xi_j \in \sigma(C)$ satisfies

$$\begin{cases} E^c(\xi_0) = \mathcal{U}_0, \\ E^c(\xi_j) = E^c(\xi_{n-j}) = \mathcal{U}_j \quad \text{for } 0 < j < \frac{n}{2} \\ E^c(\xi_{\frac{n}{2}}) = \mathcal{U}_{(\frac{n}{2}+2)}, \quad \text{if } n \text{ is even} \end{cases} \tag{6.40}$$

(cf. Example 2.2 for notations $\mathcal{U}_j$). It follows that the non-zero integers m_j's in (5.30) are

$$m_j = 1, \quad \text{for } j \in I. \tag{6.41}$$

where I is given by (6.32). The bifurcation invariant ω_1 can then be computed using (5.30) together with (6.41).

Example 6.2 (Simple critical eigenvalues for bidirectional rings) Following Example 6.1, we take C that satisfies (5.19) with $\Gamma = D_n$. The $(\frac{n}{2}+1)$ resp. $(\frac{n+1}{2})$ different entries of C decide which eigenvalue is minimal. Let $\xi_o \in \sigma(C)$ be the minimal eigenvalue. Assume that ξ_o is simple. Then, the index set I is a singleton and there are only possibly $\frac{n}{2}$ or $\frac{n-1}{2}$ different values of ω_0, depending on if n is even or odd. Again for $n = 12$, we have

$$\omega_1 = \begin{cases} -(D_{12}), & \text{if } \xi_o = \xi_0 \\ -(\mathbb{Z}_{12}^{t_1}) - (D_2^d) - (\tilde{D}_2^d) + (\mathbb{Z}_2^d), & \text{if } \xi_o = \xi_1 \\ -(\mathbb{Z}_{12}^{t_2}) - (D_4^d) - (D_4^{\hat{d}}) + (\mathbb{Z}_4^d), & \text{if } \xi_o = \xi_2 \\ -(\mathbb{Z}_{12}^{t_3}) - (D_6^d) - (\tilde{D}_6^d) + (\mathbb{Z}_6^d), & \text{if } \xi_o = \xi_3 \\ -(\mathbb{Z}_{12}^{t_4}) - (D_4^z) - (D_4) + (\mathbb{Z}_4), & \text{if } \xi_o = \xi_4 \\ -(\mathbb{Z}_{12}^{t_5}) - (D_2^d) - (\tilde{D}_2^d) + (\mathbb{Z}_2^d), & \text{if } \xi_o = \xi_5 \\ -(D_{12}^{\hat{d}}), & \text{if } \xi_o = \xi_6 \end{cases}$$

To find dominating and secondary dominating orbit types, consider the maximal orbit types in $\mathcal{U}_i$'s. They are (D_{12}) in $\mathcal{U}_0$; $(\mathbb{Z}_{12}^{t_1})$, (D_2^d) $(\tilde{D}_2^d)$ in $\mathcal{U}_1$; $(\mathbb{Z}_{12}^{t_2})$, (D_4^d) $(D_4^{\hat{d}})$ in $\mathcal{U}_2$; $(\mathbb{Z}_{12}^{t_3})$, (D_6^d), $(\tilde{D}_6^d)$ in $\mathcal{U}_3$; $(\mathbb{Z}_{12}^{t_4})$, (D_4^z) (D_4) in $\mathcal{U}_4$; $(\mathbb{Z}_{12}^{t_5})$, (D_2^d) $(\tilde{D}_2^d)$ in $\mathcal{U}_5$; $(D_{12}^{\hat{d}})$ in $\mathcal{U}_6$. Among these orbit types, we find the dominating orbit types: (D_{12}), $(D_{12}^{\hat{d}})$, $(\mathbb{Z}_{12}^{t_1})$, $(\mathbb{Z}_{12}^{t_2})$, $(\mathbb{Z}_{12}^{t_3})$, $(\mathbb{Z}_{12}^{t_4})$, $(\mathbb{Z}_{12}^{t_5})$, (D_6^d), $(\tilde{D}_6^d)$, (D_4^d), (D_4^z) and the secondary dominating orbit types: (D_2^d), $(\tilde{D}_2^d)$, (D_4), $(D_4^{\hat{d}})$. The values of ω_1 together with the dominating and secondary dominating orbit types lead to the classification result summarized in Table 3-5 using Proposition 5.7.

$\diamond$

A The Proof of Theorem 5.3

Theorem 5.3 *Let (α_o, β_o) be such that $\alpha_o = -\beta_o$ and $\xi_o \in \sigma(C)$ be given by (5.21). If ω_0 given by (5.25) is of form*

$$\omega_0 = c_1(K_1) + c_2(K_2) + \cdots + c_p(K_p),$$

for some $c_i \neq 0$, then there exists a bifurcating branch of steady states of symmetry at least (K_i).

Proof The parameter pair (α, β) escapes the shaded region in Figure 1 by crossing over L1 through (α_o, β_o) (cf. Figure 3).

Let $c : [\lambda_-, \lambda_+] \subset \mathbb{R} \to \mathbb{R}^2$ be a parametrization of the crossing curve such that $c(\lambda_-) = (\alpha_-, \beta_-)$, $c(\lambda_o) = (\alpha_o, \beta_o)$ and $c(\lambda_+) = (\alpha_+, \beta_+)$. Then, the initial bifurcation problem becomes a bifurcation problem around λ_o. More precisely, we have a Γ-equivariant map $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ such that $F(\lambda, 0) = 0$ for all $\lambda \in [\lambda_-, \lambda_+]$. The spectrum of $D_xF(\lambda, 0)$ belongs to $\mathbb{C}_-$ (the left half of the complex plane) for all $\lambda \in [\lambda_-, \lambda_o]$ and as λ crosses λ_o, the spectrum of $D_xF(\lambda_o, 0)$ intersects with $i\mathbb{R}$ at 0.

Without loss of generality, let $\lambda_\pm = \pm 4$ and $\lambda_o = 0$. Define a box around the bifurcation point $(0, 0) \in \mathbb{R} \times \mathbb{R}^n$ by (cf. Figure 4)

$$\Omega_1 := \{(\lambda, x) : |\lambda| < 4, \quad \|x\| < \rho\},$$

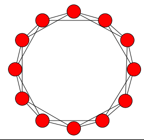
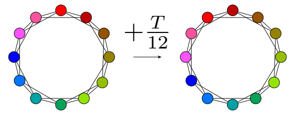
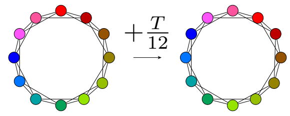
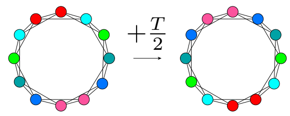
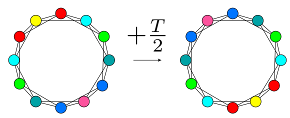
Critical Eigenvalue	**Symmetry**	**Form of Oscillating-States** (for some period T)	**Figure**
ξ_0	D_{12}	$(x(t), x(t), x(t), \ldots, x(t))$	
ξ_1	$\mathbb{Z}_{12}^{t_1}$	$(x_1(t), x_1(t+\frac{T}{12}), x_1(t+\frac{2T}{12}) \ldots, x_1(t+\frac{11T}{12}))$	
	$\kappa\mathbb{Z}_{12}^{t_1}\kappa^{-1}$	$(x_1(t), x_1(t+\frac{11T}{12}), x_1(t+\frac{10T}{12}) \ldots, x_1(t+\frac{T}{12}))$	
ξ_1 or ξ_5	D_2^d	$(x_1(t), x_2(t), x_3(t), x_3(t+\frac{T}{2}), x_2(t+\frac{T}{2}), x_1(t+\frac{T}{2}),$ $x_1(t+\frac{T}{2}), x_2(t+\frac{T}{2}), x_3(t+\frac{T}{2}), x_3(t), x_2(t), x_1(t))$	
	$\eta D_2^d\eta^{-1}$	$(x_1(t), x_1(t), x_2(t), x_3(t), x_3(t+\frac{T}{2}), x_2(t+\frac{T}{2}),$ $x_1(t+\frac{T}{2}), x_1(t+\frac{T}{2}), x_2(t+\frac{T}{2}), x_3(t+\frac{T}{2}), x_3(t), x_2(t))$	
	$\eta^2 D_2^d\eta^{-2}$	$(x_2(t), x_1(t), x_1(t), x_2(t), x_3(t), x_3(t+\frac{T}{2}), x_2(t+\frac{T}{2}),$ $x_1(t+\frac{T}{2}), x_1(t+\frac{T}{2}), x_2(t+\frac{T}{2}), x_3(t+\frac{T}{2}), x_3(t))$	
	$\eta^3 D_2^d\eta^{-3}$	$(x_3(t), x_2(t), x_1(t), x_1(t), x_2(t), x_3(t), x_3(t+\frac{T}{2}),$ $x_2(t+\frac{T}{2}), x_1(t+\frac{T}{2}), x_1(t+\frac{T}{2}), x_2(t+\frac{T}{2}), x_3(t+\frac{T}{2}))$	
	$\eta^4 D_2^d\eta^{-4}$	$(x_3(t+\frac{T}{2}), x_3(t), x_2(t), x_1(t), x_1(t), x_2(t), x_3(t),$ $x_3(t+\frac{T}{2}), x_2(t+\frac{T}{2}), x_1(t+\frac{T}{2}), x_1(t+\frac{T}{2}), x_2(t+\frac{T}{2}))$	
	$\eta^5 D_2^d\eta^{-5}$	$(x_2(t+\frac{T}{2}), x_3(t+\frac{T}{2}), x_3(t), x_2(t), x_1(t), x_1(t),$ $x_2(t), x_3(t), x_3(t+\frac{T}{2}), x_2(t+\frac{T}{2}), x_1(t+\frac{T}{2}), x_1(t+\frac{T}{2}))$	
	$\tilde{D}_2^d$	$(x_1(t), x_2(t), x_3(t), x_2(t+\frac{T}{2}), x_1(t+\frac{T}{2}), x_4(t),$ $x_1(t+\frac{T}{2}), x_2(t+\frac{T}{2}), x_3(t), x_2(t), x_1(t), x_4(t+\frac{T}{2}))$	
	$\eta\tilde{D}_2^d\eta^{-1}$	$(x_4(t+\frac{T}{2}), x_1(t), x_2(t), x_3(t), x_2(t+\frac{T}{2}), x_1(t+\frac{T}{2}),$ $x_4(t), x_1(t+\frac{T}{2}), x_2(t+\frac{T}{2}), x_3(t), x_2(t), x_1(t))$	
	$\eta^2\tilde{D}_2^d\eta^{-2}$	$(x_1(t), x_4(t+\frac{T}{2}), x_1(t), x_2(t), x_3(t), x_2(t+\frac{T}{2}),$ $x_1(t+\frac{T}{2}), x_4(t), x_1(t+\frac{T}{2}), x_2(t+\frac{T}{2}), x_3(t), x_2(t))$	
	$\eta^3\tilde{D}_2^d\eta^{-3}$	$(x_2(t), x_1(t), x_4(t+\frac{T}{2}), x_1(t), x_2(t), x_3(t), x_2(t+\frac{T}{2}),$ $x_1(t+\frac{T}{2}), x_4(t), x_1(t+\frac{T}{2}), x_2(t+\frac{T}{2}), x_3(t))$	
	$\eta^4\tilde{D}_2^d\eta^{-4}$	$(x_3(t), x_2(t), x_1(t), x_4(t+\frac{T}{2}), x_1(t), x_2(t), x_3(t),$ $x_2(t+\frac{T}{2}), x_1(t+\frac{T}{2}), x_4(t), x_1(t+\frac{T}{2}), x_2(t+\frac{T}{2}))$	
	$\eta^5\tilde{D}_2^d\eta^{-5}$	$(x_2(t+\frac{T}{2}), x_3(t), x_2(t), x_1(t), x_4(t+\frac{T}{2}), x_1(t),$ $x_2(t), x_3(t), x_2(t+\frac{T}{2}), x_1(t+\frac{T}{2}), x_4(t), x_1(t+\frac{T}{2}))$	

Table 3: Summary of distinct forms of oscillating states bifurcating from the equilibrium $x = 0$ of the system (1.1), where cells are coupled to their nearest and next nearest neighbors (Part I).

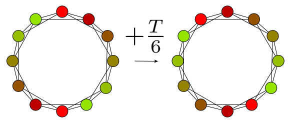
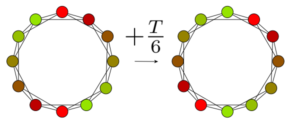
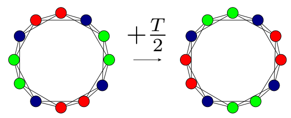
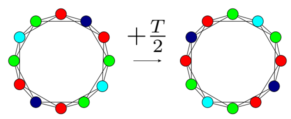
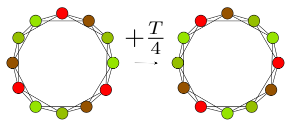
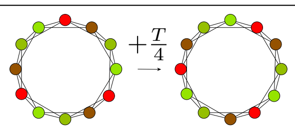
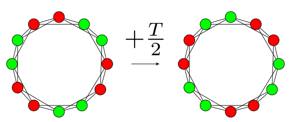
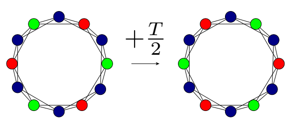
Critical Eigenvalue	**Symmetry**	**Form of Oscillating-States** (for some period T)	**Figure**
ξ_2	$\mathbb{Z}_{12}^{t_2}$	$(x_1(t), x_1(t+\frac{T}{6}), x_1(t+\frac{2T}{6}), \dots, x_1(t+\frac{5T}{6}),$ $x_1(t), x_1(t+\frac{T}{6}), x_1(t+\frac{2T}{6}), \dots, x_1(t+\frac{5T}{6}))$	
	$\kappa\mathbb{Z}_{12}^{t_2}\kappa^{-1}$	$(x_1(t), x_1(t+\frac{5T}{6}), x_1(t+\frac{4T}{6}) \dots, x_1(t+\frac{T}{6}),$ $x_1(t), x_1(t+\frac{5T}{6}), x_1(t+\frac{4T}{6}) \dots, x_1(t+\frac{T}{6}))$	
	D_4^d	$(x_1(t), x_2(t), x_1(t+\frac{T}{2}), x_1(t+\frac{T}{2}), x_2(t), x_1(t),$ $x_1(t), x_2(t), x_1(t+\frac{T}{2}), x_1(t+\frac{T}{2}), x_2(t), x_1(t))$	
	$\eta D_4^d\eta^{-1}$	$(x_1(t), x_1(t), x_2(t), x_1(t+\frac{T}{2}), x_1(t+\frac{T}{2}), x_2(t),$ $x_1(t), x_1(t), x_2(t), x_1(t+\frac{T}{2}), x_1(t+\frac{T}{2}), x_2(t))$	
	$\eta^2 D_4^d\eta^{-2}$	$(x_2(t), x_1(t), x_1(t), x_2(t), x_1(t+\frac{T}{2}), x_1(t+\frac{T}{2}),$ $x_2(t), x_1(t), x_1(t), x_2(t), x_1(t+\frac{T}{2}), x_1(t+\frac{T}{2}))$	
	$D_4^{\hat{d}}$	$(x_1(t), x_2(t), x_1(t), x_1(t+\frac{T}{2}), x_2(t+\frac{T}{2}), x_1(t+\frac{T}{2}),$ $x_1(t), x_2(t), x_1(t), x_1(t+\frac{T}{2}), x_2(t+\frac{T}{2}), x_1(t+\frac{T}{2}))$	
	$\eta D_4^{\hat{d}}\eta^{-1}$	$(x_1(t+\frac{T}{2}), x_1(t), x_2(t), x_1(t), x_1(t+\frac{T}{2}), x_2(t+\frac{T}{2}),$ $x_1(t+\frac{T}{2}), x_1(t), x_2(t), x_1(t), x_1(t+\frac{T}{2}), x_2(t+\frac{T}{2}))$	
	$\eta^2 D_4^{\hat{d}}\eta^{-2}$	$(x_2(t+\frac{T}{2}), x_1(t+\frac{T}{2}), x_1(t), x_2(t), x_1(t), x_1(t+\frac{T}{2}),$ $x_2(t+\frac{T}{2}), x_1(t+\frac{T}{2}), x_1(t), x_2(t), x_1(t), x_1(t+\frac{T}{2}))$	
ξ_3	$\mathbb{Z}_{12}^{t_3}$	$(x_1(t), x_1(t+\frac{T}{4}), x_1(t+\frac{T}{2}), x_1(t+\frac{3T}{4}),$ $x_1(t), x_1(t+\frac{T}{4}), \dots, x_1(t+\frac{3T}{4}))$	
	$\kappa\mathbb{Z}_{12}^{t_3}\kappa^{-1}$	$(x_1(t), x_1(t+\frac{3T}{4}), x_1(t+\frac{T}{2}), x_1(t+\frac{T}{4}),$ $x_1(t), x_1(t+\frac{3T}{4}), \dots, x_1(t+\frac{T}{4}))$	
	D_6^d	$(x_1(t), x_1(t+\frac{T}{2}), x_1(t+\frac{T}{2}), x_1(t), x_1(t), x_1(t+\frac{T}{2}),$ $x_1(t+\frac{T}{2}), x_1(t), x_1(t), x_1(t+\frac{T}{2}), x_1(t+\frac{T}{2}), x_1(t))$	
	$\eta D_6^d\eta^{-1}$	$(x_1(t), x_1(t), x_1(t+\frac{T}{2}), x_1(t+\frac{T}{2}), x_1(t), x_1(t),$ $x_1(t+\frac{T}{2}), x_1(t+\frac{T}{2}), x_1(t), x_1(t), x_1(t+\frac{T}{2}), x_1(t+\frac{T}{2}))$	
	$\tilde{D}_6^d$	$(x_1(t), x_2(t), x_1(t), x_2(t+\frac{T}{2}), x_1(t), x_2(t),$ $x_1(t), x_2(t+\frac{T}{2}), x_1(t), x_2(t), x_1(t), x_2(t+\frac{T}{2}))$	
	$\eta\tilde{D}_6^d\eta^{-1}$	$(x_2(t+\frac{T}{2}), x_1(t), x_2(t), x_1(t), x_2(t+\frac{T}{2}),$ $x_1(t), x_2(t), x_1(t), x_2(t+\frac{T}{2}), x_1(t), x_2(t), x_1(t))$	

Table 4: Summary of distinct forms of oscillating states bifurcating from the equilibrium $x = 0$ of the system (1.1), where cells are coupled to their nearest and next nearest neighbors (Part II).

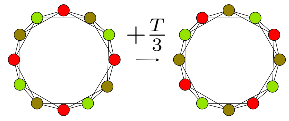
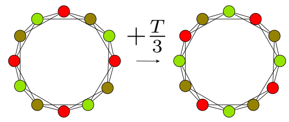
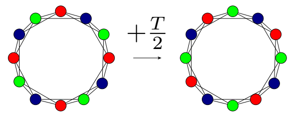
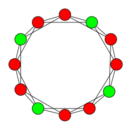
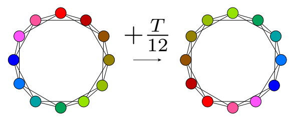
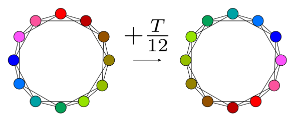
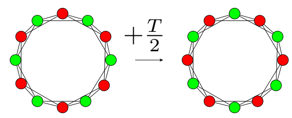
Critical Eigenvalue	Symmetry	Form of Oscillating-States (for some period T)	Figure
ξ_4	$\mathbb{Z}_{12}^{t_4}$	$(x_1(t), x_1(t+\frac{T}{3}), x_1(t+\frac{2T}{3}),$ $x_1(t), x_1(t+\frac{T}{3}), \ldots, x_1(t+\frac{2T}{3}))$	
	$\kappa\mathbb{Z}_{12}^{t_4}\kappa^{-1}$	$(x_1(t), x_1(t+\frac{2T}{3}), x_1(t+\frac{T}{3}),$ $x_1(t), x_1(t+\frac{2T}{3}), \ldots, x_1(t+\frac{T}{3}))$	
	D_4^z	$(x_1(t), x_2(t), x_1(t+\frac{T}{2}), x_1(t), x_2(t), x_1(t+\frac{T}{2}),$ $x_1(t), x_2(t), x_1(t+\frac{T}{2}), x_1(t), x_2(t), x_1(t+\frac{T}{2}))$	
	$\eta D_4^z\eta^{-1}$	$(x_1(t+\frac{T}{2}), x_1(t), x_2(t), x_1(t+\frac{T}{2}), x_1(t), x_2(t),$ $x_1(t+\frac{T}{2}), x_1(t), x_2(t), x_1(t+\frac{T}{2}), x_1(t), x_2(t))$	
	$\eta^2 D_4^z\eta^{-2}$	$(x_2(t), x_1(t+\frac{T}{2}), x_1(t), x_2(t), x_1(t+\frac{T}{2}), x_1(t),$ $x_2(t), x_1(t+\frac{T}{2}), x_1(t), x_2(t), x_1(t+\frac{T}{2}), x_1(t))$	
	D_4	$(x_1(t), x_2(t), x_1(t), x_1(t), x_2(t), x_1(t),$ $x_1(t), x_2(t), x_1(t), x_1(t), x_2(t), x_1(t))$	
	$\eta D_4\eta^{-1}$	$(x_1(t), x_1(t), x_2(t), x_1(t), x_1(t), x_2(t),$ $x_1(t), x_1(t), x_2(t), x_1(t), x_1(t), x_2(t))$	
	$\eta^2 D_4\eta^{-2}$	$(x_2(t), x_1(t), x_1(t), x_2(t), x_1(t), x_1(t),$ $x_2(t), x_1(t), x_1(t), x_2(t), x_1(t), x_1(t))$	
ξ_5	$\mathbb{Z}_{12}^{t_5}$	$(x_1(t), x_1(t+\frac{5T}{12}), x_1(t+\frac{10T}{12}), x_1(t+\frac{3T}{12}),$ $x_1(t+\frac{8T}{12}), x_1(t+\frac{T}{12}), x_1(t+\frac{6T}{12}), x_1(t+\frac{11T}{12}),$ $x_1(t+\frac{4T}{12}), x_1(t+\frac{9T}{12}), x_1(t+\frac{2T}{12}), x_1(t+\frac{7T}{12})$	
	$\kappa\mathbb{Z}_{12}^{t_5}\kappa^{-1}$	$(x_1(t), x_1(t+\frac{7T}{12}), x_1(t+\frac{2T}{12}), x_1(t+\frac{9T}{12}),$ $x_1(t+\frac{4T}{12}), x_1(t+\frac{11T}{12}), x_1(t+\frac{6T}{12}), x_1(t+\frac{T}{12}),$ $x_1(t+\frac{8T}{12}), x_1(t+\frac{3T}{12}), x_1(t+\frac{10T}{12}), x_1(t+\frac{5T}{12})$	
ξ_6	$D_{12}^{\hat{d}}$	$(x_1(t), x_1(t+\frac{T}{2}), x_1(t), x_1(t+\frac{T}{2}), \ldots, x_1(t+\frac{T}{2}))$	

Table 5: Summary of distinct forms of oscillating states bifurcating from the equilibrium $x = 0$ of the system (1.1), where cells are coupled to their nearest and next nearest neighbors (Part III).

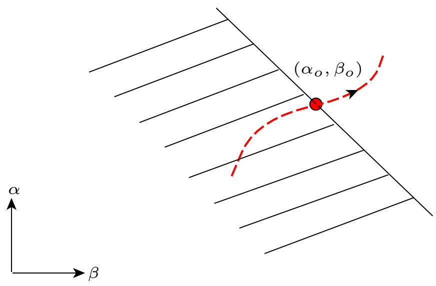


Figure 3: The crossing of (α, β) through $(\alpha_o, \beta_o) \in$ L1.

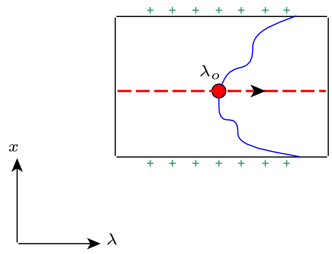


Figure 4: An isolating box Ω_1 around the bifurcating point $\lambda = \lambda_o$, where the red line is the equilibrium, the blue curves are potential bifurcating solutions and the plus signs "+" are the signs of auxiliary function ζ_1.

where $\rho > 0$ is such that $F(\pm 4, \cdot)$ is a homeomorphism on $\{x \in \mathbb{R}^n \,:\, \|x\| < \rho\}$. Without loss of generality, let $\rho = 2$. Define $\mathcal{F}_1 : \overline{\Omega}_1 \to \mathbb{R} \times \mathbb{R}^n$ by

$$\mathcal{F}_1(\lambda, x) := \big(|\lambda|(\|x\| - 2) + \|x\| - 1, F(\lambda, x)\big) := (\zeta_1(\lambda, x), F(\lambda, x)).$$

Note that $\zeta_1 > 0$ for $\|x\| = 2$ and $\zeta_1 < 0$ for $\|x\| = 0$. Functions with such property are called *auxiliary functions* on Ω_1. Thus, by construction, zeros of $\mathcal{F}_1$ in Ω_1 are contained properly in Ω_1, i.e. $\mathcal{F}_1^{-1}(0) \cap \overline{\Omega}_1 \subset \Omega_1$, and if $\mathcal{F}_1(\lambda, x) = 0$, then $x \neq 0$. In other words, zeros of $\mathcal{F}_1$ correspond precisely those non-trivial zeros of F in Ω_1. The bifurcation invariant ω_0 is defined by

$$\omega_0 = \Gamma\text{-Deg}\,(\mathcal{F}_1, \Omega_1).$$

To compute ω_0, we perform several homotopies on $\mathcal{F}_1$. Define $\mathcal{F}_2 : \overline{\Omega}_1 \to \mathbb{R} \times \mathbb{R}^n$ by

$$\mathcal{F}_2(\lambda, x) := \big(|\lambda|(\|x\| - 1) + \|x\| + 1, F(\lambda, x)\big) := (\zeta_2(\lambda, x), F(\lambda, x)).$$

Since $\zeta_2 > 0$ for $\|x\| = 2$, we have $\mathcal{F}_1$ and $\mathcal{F}_2$ are homotopic on Ω_1 by a linear homotopy. Also, $\zeta_2 > 0$ for $|\lambda| \leq \frac{1}{2}$. Thus, zeros of $\mathcal{F}_2$ in Ω_1 are contained in the following subset of Ω_1

$$\Omega_2 := \{(\lambda, x) \,:\, \frac{1}{2} < \lambda < 4,\ \|x\| < 2\}.$$

By excision property, we have $\Gamma\text{-Deg}\,(\mathcal{F}_1, \Omega_1) = \Gamma\text{-Deg}\,(\mathcal{F}_2, \Omega_2)$.

Moreover, $\mathcal{F}_2$ is homotopic to $\mathcal{F}_3 : \overline{\Omega}_2 \to \mathbb{R} \times \mathbb{R}^n$ defined by

$$\mathcal{F}_3(\lambda, x) := (\zeta_2(\lambda, x), D_x F(\lambda, 0)).$$

Decompose $\mathbb{R}^n$ into the sum of the critical eigenspace and the eigenspaces of the rest (all negative) eigenvalues of $D_x F(\lambda_o, 0)$, say $\mathbb{R}^n = R_0 \times R_1$. Then, for $x = (x_1, x_2) \in R_0 \times R_1$, the linear map $D_x F(\lambda, 0)(x_1, x_2)$ is homotopic to $(\lambda x_1, -x_2)$. Thus, $\mathcal{F}_3$ is homotopic to $\mathcal{F}_4 : \overline{\Omega}_2 \to \mathbb{R} \times \mathbb{R}^n$ defined by

$$\mathcal{F}_4(\lambda, x) := (\zeta_2(\lambda, x), (\lambda x_1, -x_2)), \quad \text{for } x = (x_1, x_2) \in R_0 \times R_1 .$$

Note that $\mathcal{F}_4(\lambda, x) = 0$ only if $x = 0$. Substituting $x = 0$ into $\zeta_2(\lambda, x)$, we have $\zeta_2(\lambda, 0) = 0$ if and only if $\lambda = \pm 1$. That is,

$$\mathcal{F}_4^{-1}(0) \cap \Omega_2 = \{(-1, 0), (1, 0)\}.$$

It follows that

$$\Gamma\text{-Deg}\,(\mathcal{F}_2, \Omega_2) = \Gamma\text{-Deg}\,(\mathcal{F}_4, \Omega_2) = \Gamma\text{-Deg}\,(\mathcal{F}_4, \mathcal{N}_{-1}) + \Gamma\text{-Deg}\,(\mathcal{F}_4, \mathcal{N}_1),$$

where $\mathcal{N}_{-1}$ resp. $\mathcal{N}_1$ is a small neighborhood of $(-1, 0)$ resp. $(1, 0)$. On $\mathcal{N}_{-1}$, we have $\mathcal{F}_4$ is homotopic to $(1 + \lambda, -x_1, -x_2)$. By suspension, we obtain

$$\Gamma\text{-Deg}\,(\mathcal{F}_4, \mathcal{N}_{-1}) = \Gamma\text{-Deg}\,(-\text{Id}, B_1(\mathbb{R}^n)),$$

where $B_1(\cdot)$ denotes the unit ball. On the other hand, $\mathcal{F}_4$ is homotopic to $(1 - \lambda, x_1, -x_2)$ on $\mathcal{N}_1$, so by multiplication, we have

$$\Gamma\text{-Deg}\,(\mathcal{F}_4, \mathcal{N}_1) = -\Gamma\text{-Deg}\,(-\text{Id}, B_1(R_1)).$$

Therefore,

$$\omega_0 = \Gamma\text{-Deg}\,(-\text{Id}, B_1(\mathbb{R}^n)) - \Gamma\text{-Deg}\,(-\text{Id}, B_1(R_1)).$$

Using `showdegree`$[\Gamma]$, it is expressed as

$$\omega_0 = \texttt{showdegree}[\Gamma](n_0, n_1, \ldots, n_r, 1, 0, \ldots, 0) - \texttt{showdegree}[\Gamma](u_0, u_1, \ldots, u_r, 1, 0, \ldots, 0),$$

where n_i's and u_i's are defined by (5.22)-(5.24).

The statement then follows from the existence property of degree.

□

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