DONALDSON AND SEIBERG-WITTEN INVARIANTS

THE WITTEN CONJECTURE

GREG NABER

DONALDSON INVARIANTS $\hookrightarrow$ SEIBERG-WITTEN INVARIANTS

$SU(2)$ $\qquad\qquad$ $U(1)$

$$*F_\omega = -F_\omega \qquad\qquad \begin{cases} \not{D}_A \psi = 0 \\ F_A^+ = \sigma^+((\psi \otimes \psi^*)_0) \end{cases}$$

$$\mathcal{D}_M(x) = \exp\left(Q_M(x,x)/2\right) \sum_{\alpha \in \Lambda} 2^{m(M)} SW_0(M,\alpha) \exp\left(c_1(L^0(\alpha))(x)\right)$$

- DONALDSON THEORY
- EQUIVARIANT COHOMOLOGY AND THE WITTEN LAGRANGIAN
- SEIBERG-WITTEN INVARIANTS AND THE CONJECTURE
- ADDENDA

1. A SIMPLE EXAMPLE

QUATERNIONIC HOPF BUNDLE :

$$\begin{array}{ccc} Sp(1) & \hookrightarrow & S^7 \\ & & \downarrow \\ & & S^4 \end{array}$$

$$Sp(1) = \{ g \in \mathbb{H} : |g| = 1 \} \hookrightarrow S^7 = \{ p = (q^1, q^2) \in \mathbb{H}^2 : |q^1|^2 + |q^2|^2 = 1 \}$$

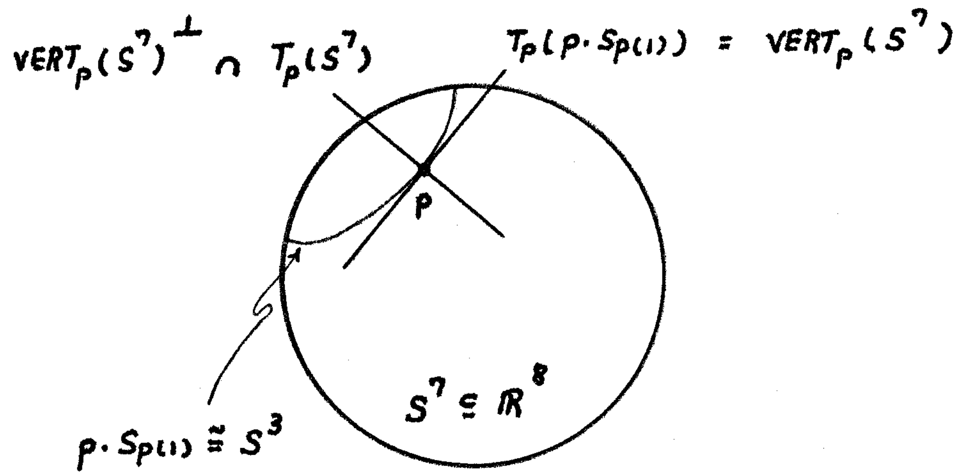
$$\cong SU(2)$$

$$\cong S^3$$

$$p \cdot g = (q^1, q^2) \cdot g = (q^1 g, q^2 g)$$

$$\downarrow$$

$$S^7 / Sp(1) := \mathbb{HP}^1 \cong S^4$$



NATURAL CONNECTION ON $Sp(1) \hookrightarrow S^7 \to S^4$: $\omega \in \Omega^1(S^7, sp(1))$

$$KER\ \omega_p = VERT_p(S^7)^\perp \cap T_p(S^7)$$

$$= HOR_p^\omega(S^7)$$

$$\omega = IM(\bar{q}^1 dq^1 + \bar{q}^2 dq^2) \quad \text{(RESTRICTED TO } S^7\text{)}$$

CORRESPONDING <u>GAUGE</u> <u>POTENTIAL</u> ON $\mathbb{H} \cong \mathbb{R}^4$:

$$\mathcal{A} = s^* \omega = \operatorname{Im}\left(\frac{\bar{q}}{1+|q|^2}\, dq \right)$$

MORE GENERALLY, $(\lambda, n) \in (0,\infty) \times \mathbb{H}$ GIVES

$$\omega_{\lambda,n} \in \Omega^1(S^7, sp(1))$$

UNIQUELY DETERMINED BY

$$\mathcal{A}_{\lambda,n} = s^* \omega_{\lambda,n} = \operatorname{Im}\left(\frac{\bar{q} - \bar{n}}{\lambda^2 + |q-n|^2}\, dq \right)$$

<u>BPST</u> <u>INSTANTON</u> WITH <u>CENTER</u> n AND <u>SCALE</u> λ $\quad (\omega = \omega_{1,0})$

<u>CURVATURES</u>

$$\Omega_{\lambda,n} = d\omega_{\lambda,n} + \tfrac{1}{2}[\omega_{\lambda,n}, \omega_{\lambda,n}]$$

UNIQUELY DETERMINED BY <u>GAUGE</u> <u>FIELD</u> <u>STRENGTHS</u>

$$\mathcal{F}_{\lambda,n} = s^* \Omega_{\lambda,n} = d\mathcal{A}_{\lambda,n} + \tfrac{1}{2}[\mathcal{A}_{\lambda,n}, \mathcal{A}_{\lambda,n}] = \cdots = \frac{\lambda^2}{(\lambda^2 + |q-n|^2)^2}\, d\bar{q} \wedge dq$$

EACH IS <u>ANTI</u>-<u>SELF</u>-<u>DUAL</u> (<u>ASD</u>)

$$* \mathcal{F}_{\lambda,n} = - \mathcal{F}_{\lambda,n}$$

($*$ = HODGE STAR ON $\mathbb{H} \cong \mathbb{R}^4$ DETERMINED BY THE STANDARD ORIENTATION AND RIEMANNIAN METRIC)

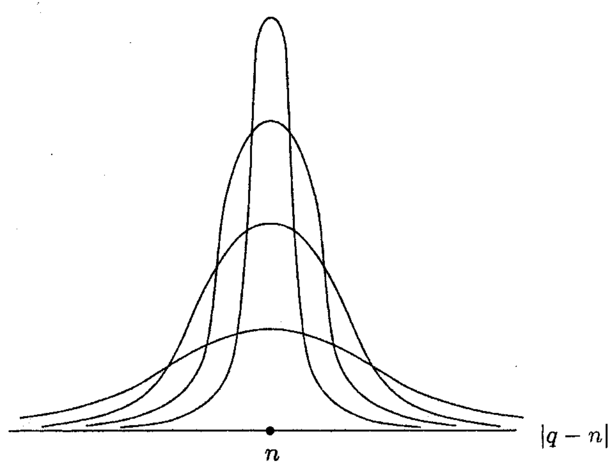
COMPUTE

$$\frac{1}{8\pi^2}\int_{\mathbb{R}^4} -\,Tr\left(\mathcal{F}_{\lambda,n}\wedge{}^*\mathcal{F}_{\lambda,n}\right) = \frac{1}{8\pi^2}\int_{\mathbb{R}^4} Tr\left(\mathcal{F}_{\lambda,n}\wedge\mathcal{F}_{\lambda,n}\right) = \cdots$$

$$= \frac{1}{8\pi^2}\int_{\mathbb{R}^4} \frac{48\lambda^4}{(\lambda^2+|q-n|^2)^4}\, dVol_{\mathbb{R}^4}$$

$$= \;1\; = \text{CHERN NUMBER OF } Sp(1) \hookrightarrow S^7 \to S^4$$

$\boxed{\text{ADDENDUM 1 : CHERN-WEIL CHARACTERISTIC CLASSES}}$



$-\,Tr\left(\mathcal{F}_{\lambda,n}\wedge{}^*\mathcal{F}_{\lambda,n}\right)$ FOR FIXED n

AND VARIOUS λ

INCREASINGLY CONCENTRATED AT

CENTER n AS SCALE $\lambda \to 0$

<u>GAUGE</u> <u>GROUP</u> : $\mathcal{G}$ = ALL DIFFEOMORPHISMS SATISFYING

$$f(p\cdot g) = f(p)\cdot g$$

AND COVERING id_{S^4}

$$\begin{tikzcd} S^7 \arrow[r, "f"] \arrow[d] & S^7 \arrow[d] \\ S^4 \arrow[r, dotted, "id_{S^4}"'] & S^4 \end{tikzcd}$$

TWO CONNECTIONS ω_1 AND ω_2 ON $Sp(1) \hookrightarrow S^7 \to S^4$ ARE <u>GAUGE</u> <u>EQUIVALENT</u> IF, FOR SOME $f \in \mathcal{G}$,

$$\omega_2 = f^*\omega_1$$

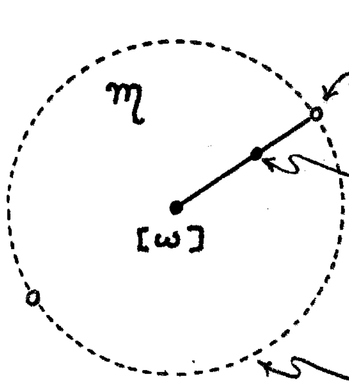
ω_{λ_1, n_1} GAUGE EQUIVALENT TO ω_{λ_2, n_2} $\iff$ $(\lambda_1, n_1) = (\lambda_2, n_2)$

$\mathcal{M}$ = <u>MODULI</u> <u>SPACE</u> OF GAUGE EQUIVALENCE CLASSES $[\omega]$ OF ASD CONNECTIONS ω ON $Sp(1) \hookrightarrow S^7 \longrightarrow S^4$

<u>ATIYAH</u>-<u>HITCHIN</u>-<u>SINGER</u> :

$$\begin{aligned} \mathcal{M} &= \{ [\omega_{\lambda,n}] : (\lambda, n) \in (0,\infty) \times \mathbb{R}^4 \} \\ &\cong (0,\infty) \times \mathbb{R}^4 \\ &\cong \{ x \in \mathbb{R}^5 : \|x\| < 1 \} \end{aligned}$$

FIXES THE "CENTER" n



"SCALE" $\lambda \to 0$ AS $r \to 1$

$\partial\mathcal{M}$ AS A COPY OF S^4, THOUGHT OF AS "INSTANTONS CONCENTRATED AT A POINT"

2. DONALDSON'S GENERALIZATION

M = COMPACT, SIMPLY CONNECTED, ORIENTED, SMOOTH (C^∞) 4-MANIFOLD WITH

$$" \; b_2^+(M) = 0 \; "$$

REMARKS ON $b_2^+(M)$:

M CONNECTED $\Rightarrow H_0(M;\mathbb{Z}) \cong \mathbb{Z}$ $\qquad$ $\pi_1(M) \cong 0 \Rightarrow H_1(M;\mathbb{Z}) \cong 0$

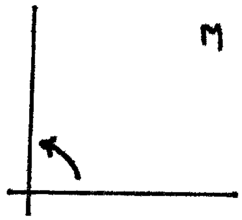
POINCARÉ DUALITY $\Rightarrow H_4(M;\mathbb{Z}) \cong \mathbb{Z}$ AND $H_3(M;\mathbb{Z}) \cong 0$

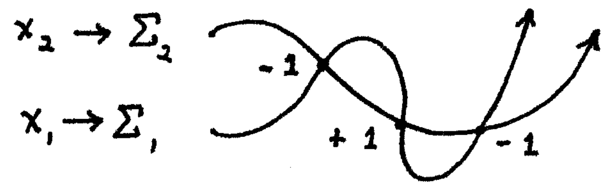
$H_2(M;\mathbb{Z})$ IS A FINITELY GENERATED, FREE ABELIAN GROUP (ISOMORPHIC TO $\mathbb{Z} \oplus \overset{b_2(M)}{\cdots} \oplus \mathbb{Z}$) EACH GENERATOR OF WHICH CAN BE REPRESENTED BY A SMOOTHLY EMBEDDED, ORIENTED, CLOSED SURFACE Σ IN X

INTERSECTION FORM (ASSUMING $H_2(M;\mathbb{Z}) \not\cong 0$) :

$$Q_M : H_2(M;\mathbb{Z}) \oplus H_2(M;\mathbb{Z}) \to \mathbb{Z}$$

A UNIMODULAR, SYMMETRIC, $\mathbb{Z}$-VALUED BILINEAR FORM ON $H_2(M;\mathbb{Z})$ DEFINED AS FOLLOWS :





$Q_M(x_1, x_2)$ = SUM OF SIGNED INTERSECTION POINTS

<u>ALTERNATIVE</u> : $\quad Q_M : H^2(M;\mathbb{Z}) \oplus H^2(M;\mathbb{Z}) \to \mathbb{Z}$

$$Q_M(\alpha_1, \alpha_2) = \int_M \alpha_1 \wedge \alpha_2$$

$b_2^+(M)$ = MAXIMAL DIMENSION OF A SUBSPACE OF $H_2(M;\mathbb{Z})$ ON WHICH Q_M IS POSITIVE DEFINITE

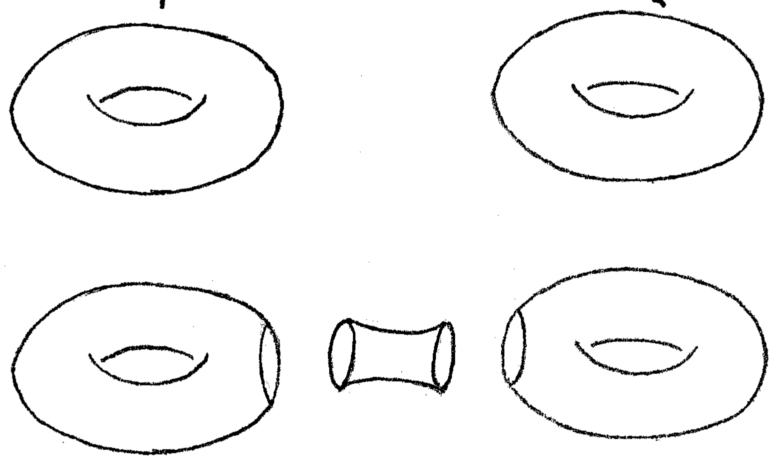
= DIMENSION OF THE SPACE OF SELF-DUAL HARMONIC 2-FORMS ON M (FOR ANY CHOICE OF RIEMANNIAN METRIC ON M)

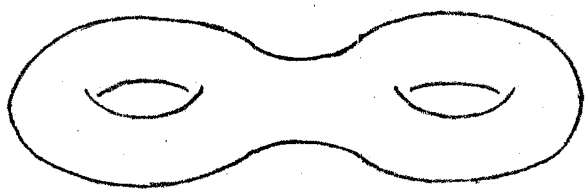
<u>EXAMPLES</u> :

M	$H_2(M;\mathbb{Z})$	Q_M	$b_2^+(M)$
S^4	0	ϕ	-
$\mathbb{CP}^2$	$\mathbb{Z}$	(1)	1
$\overline{\mathbb{CP}}^2$	$\mathbb{Z}$	(-1)	0
$S^2 \times S^2$	$\mathbb{Z}\oplus\mathbb{Z} = 2\mathbb{Z}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	1
$K3$	$22\mathbb{Z}$	$3\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 2(-E_8)$	3

MORE EXAMPLES FROM CONNECTED SUMS :

M_1 M_2





$M_1 \# M_2$

$$H_2(M_1 \# M_2 ; \mathbb{Z}) \cong H_2(M_1 ; \mathbb{Z}) \oplus H_2(M_2 ; \mathbb{Z})$$

$$Q_{M_1 \# M_2} = Q_{M_1} \oplus Q_{M_2}$$

E.G.,

$$Q_{m\,\mathbb{CP}^2 \# n\,\overline{\mathbb{CP}}^2} = \mathrm{DIAG}\,(1 \overset{m}{\cdots} 1 \;\; -1 \overset{n}{\cdots} -1)$$

$$b_2^+ (m\,\mathbb{CP}^2 \# n\,\overline{\mathbb{CP}}^2) = m$$

<u>DONALDSON'S</u> <u>THEOREM</u> (1983): M A COMPACT, SIMPLY CONNECTED, ORIENTED, SMOOTH (C^∞) 4-MANIFOLD. THEN

$$b_2^+(M) = 0 \implies Q_M = -id.$$

<u>REMARK</u>: Q_M CAN ALSO BE DEFINED FOR TOPOLOGICAL 4-MANIFOLDS. FREEDMAN (1982) PROVED THAT EVERY UNIMODULAR, SYMMETRIC, $\mathbb{Z}$-VALUED BILINEAR FORM ON A FINITELY GENERATED, FREE ABELIAN GROUP IS THE INTERSECTION FORM OF AT LEAST ONE (AND AT MOST TWO) SUCH MANIFOLDS. COROLLARY OF DONALDSON + FREEDMAN:

> THERE ARE OVER 10 MILLION TOPOLOGICALLY DISTINCT COMPACT, ORIENTED, SIMPLY CONNECTED, TOPOLOGICAL 4-MANIFOLDS M WITH $b_2(M) = 32$, NONE OF WHICH ADMIT A SMOOTH STRUCTURE.

"PROOF" $\qquad SU(2) \hookrightarrow P_1 \xrightarrow{\pi_1} M \qquad$ (CHERN CLASS 1)

CHOOSE RIEMANNIAN METRIC g ON M. GIVES HODGE STAR $*$.

A CONNECTION ω ON P_1 IS g-ANTI-SELF-DUAL (g-ASD) IF

$$*F_\omega = -F_\omega$$

(SOLUTIONS ALSO SATISFY THE YANG-MILLS EQUATIONS)

NOTE : TAUBES HAS PROVED THAT SUCH CONNECTIONS EXIST. HE DOES THIS BY LOCALLY "GRAFTING" A BPST INSTANTON ONTO M. THE GRAFTING PROCEDURE INTRODUCES A SMALL SELF-DUAL PART WHICH, PROVIDED $b_2^+(M) = 0$, CAN BE KILLED BY A SMALL PERTURBATION.

GAUGE GROUP :

$\mathcal{G}(P_1)$ = ALL DIFFEOMORPHISMS SATISFYING

$$f(p \cdot g) = f(p) \cdot g$$

AND COVERING id_M.

$$\begin{tikzcd} P_1 \arrow[r, "f"] \arrow[d] & P_1 \arrow[d] \\ M \arrow[r, dotted, "id_M"'] & M \end{tikzcd}$$

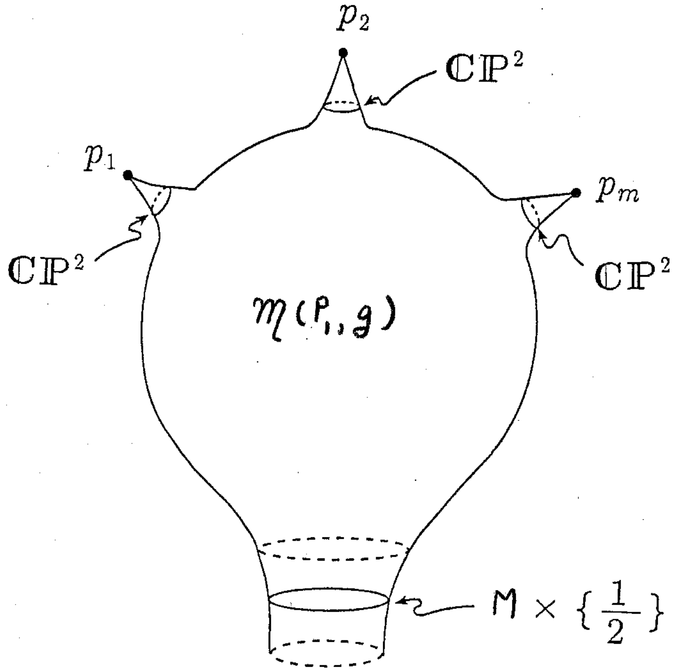
TWO CONNECTIONS ω_1 AND ω_2 ARE GAUGE EQUIVALENT IF, FOR SOME $f \in \mathcal{G}(P_1)$,

$$\omega_2 = f^* \omega_1 .$$

$\mathcal{M}(P_1, g)$ = MODULI SPACE OF GAUGE EQUIVALENCE CLASSES OF g-ASD CONNECTIONS ON P_1

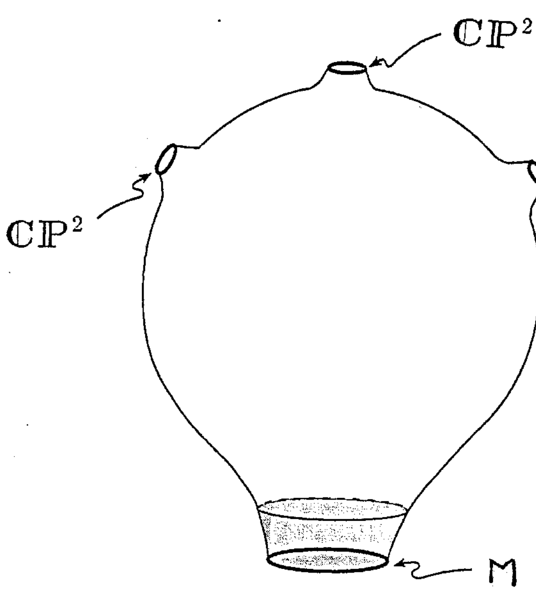
ADDENDUM 2 : THE MODULI SPACES

DONALDSON HAS SHOWN THAT, FOR A "GENERIC" RIEMANNIAN METRIC g, $\mathcal{M}(P_1, g)$ LOOKS LIKE THIS :



- $\exists\ p_1, \dots, p_m \in \mathcal{M}(P_1, g)$ SUCH THAT $\mathcal{M}(P_1, g) - \{p_1, \dots, p_m\}$ IS A SMOOTH, ORIENTED 5 - MANIFOLD
- EACH p_i HAS A NEIGHBORHOOD HOMEOMORPHIC TO A CONE OVER $\mathbb{CP}^2$.
- $\exists\ K \overset{\text{COMPACT}}{\subseteq} \mathcal{M}(P_1, g)$ SUCH THAT $\mathcal{M}(P_1, g) - K$ IS DIFFEOMORPHIC TO THE CYLINDER $M \times (0,1)$.

NOW CUT OFF THE TOP HALF OF EACH CONE AND THE BOTTOM HALF OF THE CYLINDER.



- M IS COBORDANT TO A DISJOINT UNION OF $\mathbb{CP}^2$'s.

- SIGNATURE OF THE INTERSECTION FORM IS A COBORDISM INVARIANT.

ETC. $\square$

ADDENDUM 3 : "ETC."

3. <u>DONALDSON POLYNOMIAL INVARIANTS</u> $\gamma_d(M)$, $d = 0, 1, 2, \ldots$

M = COMPACT, SIMPLY CONNECTED, ORIENTED, SMOOTH (C^∞) 4-MANIFOLD WITH

" VARIOUS ASSUMPTIONS ON $b_2^+(M)$ AS THE NEED ARISES."

$$\gamma_0(M) \in \mathbb{Z}$$

$$\gamma_d(M) : H_2(M;\mathbb{Z}) \to \mathbb{Z}[\tfrac{1}{2}], \quad d = 1, 2, \ldots$$

<u>REMARKS ON THE CONSTRUCTION</u> :

$$SU(2) \hookrightarrow P_k \xrightarrow{\pi_k} M \qquad (\text{CHERN CLASS } k \geq 0)$$

CHOOSE A RIEMANNIAN METRIC g ON M. GIVES HODGE STAR $*$.

$\mathcal{M}(P_k, g)$ = MODULI SPACE OF GAUGE EQUIVALENCE CLASSES OF g-ASD CONNECTIONS ON P_k

$$\boxed{\text{ADDENDUM 2 : THE MODULI SPACES}}$$

<u>EXAMPLES</u> :

1. $M = S^4$, g = STANDARD RIEMANNIAN METRIC, $k = 1$

$\mathcal{M}(P_1, g)$ = MODULI SPACE OF BPST INSTANTONS

$\cong$ OPEN 5-BALL

2. M ARBITRARY , g ARBITRARY , $k \leq 0$

$$k = \frac{1}{8\pi^2} \int_M Tr(F_\omega \wedge F_\omega) = \frac{1}{8\pi^2} \int_M (|F_\omega^-|^2 - |F_\omega^+|^2) dVol_g$$

WHERE $F_\omega^\pm = \frac{1}{2}(F_\omega \pm {}^*F_\omega)$.

$$k < 0 \Rightarrow F_\omega^+ \neq 0 \Rightarrow \omega \text{ CANNOT BE } g\text{-ASD}$$

$$\mathcal{M}(P_{-1}, g) = \mathcal{M}(P_{-2}, g) = \cdots = \phi$$

$k = 0 \Rightarrow$ ANY g-ASD ω ($F_\omega^+ = 0$) IS FLAT ($F_\omega^- = 0$ AS WELL)

CONVERSELY, FLAT $\Rightarrow$ g-ASD

THE $k = 0$ BUNDLE IS TRIVIAL AND FLAT CONNECTIONS EXIST ON ANY TRIVIAL BUNDLE SO $\mathcal{M}(P_0, g) \neq \phi$.

M SIMPLY CONNECTED $\Rightarrow$ ANY TWO FLAT CONNECTIONS ON THE TRIVIAL BUNDLE ARE GAUGE EQUIVALENT SO

$$\mathcal{M}(P_0, g) = \{ \text{POINT} \}$$

HENCEFORTH CONSIDER ONLY

$$k \geq 1$$

3. $M = S^2 \times S^2$, g = STANDARD METRIC , $k = 1$

$$\mathcal{M}(P_1, g) = \phi$$

4. $M = \mathbb{CP}^2$, g = STANDARD (FUBINI-STUDY) METRIC , $k = 1$

$$\mathcal{M}(P_1, g) = \emptyset$$

5. $M = \overline{\mathbb{CP}}^2$, g = STANDARD (FUBINI-STUDY) METRIC , $k = 1$

$$\mathcal{M}(P_1, g) \cong \text{OPEN CONE OVER } \overline{\mathbb{CP}}^2$$

<u>REMARK</u>: COMPARE # 4 AND 5. DONALDSON THEORY IS HIGHLY SENSITIVE TO ORIENTATIONS.

ASSUMING

$$b_2^+(M) > 0$$

DONALDSON SHOWS THAT

FOR A "GENERIC" RIEMANNIAN METRIC g,
$\mathcal{M}(P_k, g)$ IS EITHER EMPTY OR A SMOOTH (C^∞), ORIENTABLE MANIFOLD OF DIMENSION

$$8k - 3(1 + b_2^+(M))$$

(AN ORIENTATION IS CANONICALLY DETERMINED BY ORIENTING THE VECTOR SPACE $H_+^2(M; \mathbb{R})$).

REMARKS ON $b_2^+(M) > 0$:

THE SINGULARITIES (CONES OVER $\mathbb{CP}^2$) IN THE MODULI SPACE WHEN $b_2^+(M) = 0$ ARISE FROM "REDUCIBLE" CONNECTIONS (LARGE ISOTROPY GROUP UNDER $\mathcal{G}$-ACTION). IN THE SPACE $\mathcal{R}$ OF RIEMANNIAN METRICS ON M THE SET OF THOSE g FOR WHICH THERE EXIST REDUCIBLE g-ASD ω IS A COUNTABLE UNION OF SUBMANIFOLDS OF CODIMENSION $b_2^+(M)$.
$b_2^+(M) > 0 \Rightarrow$ CAN FIND A "GOOD" g.

TO DEFINE INVARIANTS FROM THE MODULI SPACES REQUIRES THE STRONGER ASSUMPTION

$$b_2^+(M) > 1.$$

TO ENSURE INDEPENDENCE OF THE CHOICE OF g, NEED THE SET OF METRICS WHICH INTRODUCE REDUCIBLES TO BE SUFFICIENTLY "THIN" THAT A GENERIC VARIATION OF g (PATH IN $\mathcal{R}$) CAN AVOID REDUCIBLES.

DEFINING THE INVARIANTS :

CASE 1 : $8k - 3(1 + b_2^+(M)) = 0$

GENERICALLY, $\mathcal{M}(P_k, g)$ IS EITHER EMPTY OR 0-DIMENSIONAL, ORIENTED, AND COMPACT (THIS IS THE ONLY CASE IN WHICH THE MODULI SPACE IS COMPACT).

$$\gamma_0(M) = \begin{cases} \displaystyle\sum_{[\omega] \in \mathcal{M}(P_k, g)} (-1)^{[\omega]} & , \ \mathcal{M}(P_k, g) \neq \emptyset \\ 0 & , \ \mathcal{M}(P_k, g) = \emptyset \end{cases}$$

CASE 2 : $8k - 3(1 + b_2^+(M)) > 0$

ONE MORE ASSUMPTION REQUIRED. HENCEFORTH,

$$b_2^+(M) > 1 \text{ AND } \underline{\text{ODD}}.$$

WRITE

$$8k - 3(1 + b_2^+(M)) = 2d_k$$

FOR SOME POSITIVE INTEGER d_k.

GENERICALLY, $\mathcal{M}(P_k, g)$ IS EITHER EMPTY OR $2d_k$-DIMENSIONAL, ORIENTED, AND NEVER COMPACT.

<u>DONALDSON</u> μ - <u>MAP</u> :

$$\mu : H_2(M;\mathbb{Z}) \longrightarrow H^2(\mathcal{M}(P_k, g);\mathbb{Z})$$

$\boxed{\text{ADDENDUM 4 : THE } \mu\text{-MAP}}$

NAIVE DEFINITION OF

$$\gamma_{d_k}(M) : H_2(M;\mathbb{Z}) \longrightarrow \mathbb{Z}[\tfrac{1}{2}]$$

$$x \in H_2(M;\mathbb{Z}) \implies \mu(x) \wedge \overset{d_k}{\cdots} \wedge \mu(x) \in H^{2d_k}(\mathcal{M}(P_k, g);\mathbb{Z})$$

$$\gamma_{d_k}(M)(x) = \int_{\mathcal{M}(P_k,g)} \mu(x) \wedge \overset{d_k}{\cdots} \wedge \mu(x)$$

HERE ARE A FEW OF THE THINGS WRONG WITH THIS "DEFINITION" :

- $\mathcal{M}(P_k, g)$ IS NOT COMPACT SO YOU CAN'T INTEGRATE OVER IT

- DEEP ANALYTICAL RESULTS OF UHLENBECK AND TAUBES LED DONALDSON TO A CANONICAL COMPACTIFICATION $\overline{\mathcal{M}}(P_k, g)$ OF $\mathcal{M}(P_k, g)$ AND AN EXTENSION $\bar{\mu} : H_2(M;\mathbb{Z}) \to H^2(\overline{\mathcal{M}}(P_k, g);\mathbb{Z})$ OF THE μ-MAP. HOWEVER,

- $\bar{\mathcal{M}}(P_k, g)$ IS NOT A MANIFOLD SO INTEGRATION OVER IT MUST BE REPLACED BY PAIRING WITH THE FUNDAMENTAL CLASS.

- REGRETTABLY, $\bar{\mathcal{M}}(P_k, g)$ ADMITS A FUNDAMENTAL CLASS ONLY FOR SUFFICIENTLY LARGE k, I.E., IN THE <u>STABLE</u> <u>RANGE</u>

$$k > \frac{3}{4}(1 + b_2^+(M))$$

OR EQUIVALENTLY

$$d_k > \frac{3}{2}(1 + b_2^+(M)).$$

IF d IS AN INTEGER SATISFYING

$$d \equiv -\frac{3}{2}(1 + b_2^+(M)) \bmod 4$$

AND

$$d > \frac{3}{2}(1 + b_2^+(M))$$

WE HAVE

$$\gamma_d(M) : H_2(M; \mathbb{Z}) \to \mathbb{Z}$$

- REMOVING THE STABLE RANGE CONDITION AND THE MOD 4 CONGRUENCE REQUIRES
  - ANOTHER μ-MAP $\mu : H_0(M; \mathbb{Z}) \to H^4(\mathcal{M}(P_k, g); \mathbb{Z})$
  - A DETOUR AROUND THE FACT THIS μ-MAP DOES NOT FULLY EXTEND TO $\bar{\mathcal{M}}(P_k, g)$
  - A BLOW-UP FORMULA RELATING $\gamma_d(M)$ AND $\gamma_d(M \# n\overline{\mathbb{CP}}^2)$

ADDENDUM 5 : DONALDSON POLYNOMIALS

THE END RESULT IS A SEQUENCE

$$\gamma_d(M) \ , \ d = 0, 1, 2, \ldots$$

OF ORIENTATION PRESERVING DIFFEOMORPHISM INVARIANTS OF M.

<u>DONALDSON</u> (FORMAL POWER) <u>SERIES</u>

$$\mathcal{D}_M(x) = \sum_{d=0}^{\infty} \frac{\gamma_d(M)(x)}{d!}$$

E.G., FOR $M = K3$

$$\mathcal{D}_{K3}(x) = \sum_{d=0}^{\infty} \frac{(Q_{K3}(x,x)/2)^d}{d!} = \exp\left(Q_{K3}(x,x)/2\right).$$

SPRING, 1994 : A BREAKTHROUGH

<u>KRONHEIMER</u> - <u>MROWKA</u> <u>STRUCTURE</u> <u>THEOREM</u> : IF M IS OF "D-SIMPLE TYPE", THEN THERE EXIST COHOMOLOGY CLASSES

$K_1, \ldots, K_s \in H^2(M;\mathbb{Z})$ (<u>D</u>-<u>BASIC</u> <u>CLASSES</u>)

AND RATIONAL NUMBERS

$a_1, \ldots, a_s \in \mathbb{Q}$ (<u>COEFFICIENTS</u>)

SUCH THAT

$$\mathcal{D}_M(x) = \exp(Q_M(x,x)/2) \sum_{r=1}^{s} a_r \exp(K_r(x))$$

MOREOVER, EACH K_r IS AN INTEGRAL LIFT OF THE SECOND STIEFEL-WHITNEY CLASS $w_2(M) \in H^2(M;\mathbb{Z}_2)$.

FALL, 1994 : RENDERED MOOT (DISCOVERY OF SEIBERG-WITTEN INVARIANTS)