

EQUIVARIANT COHOMOLOGY AND THE WITTEN LAGRANGIAN

WITTEN'S MAGICAL FORMULA

$$\mathcal{D}_M(x) = \exp(\mathcal{Q}_M(x,x)/2) \sum_{\mathcal{L} \in \Lambda} 2^{m(M)} SW_0(M, \mathcal{L}) \exp(c_1(L^0(\mathcal{L}))(x))$$

↑
DONALDSON
INVARIANTS

↑
SEIBERG-WITTEN
INVARIANTS

AND THE PHYSICS IT CAME FROM

$$\Phi = (\omega, \phi, \lambda, \eta, \psi, \zeta)$$

$$S_{DW}[\Phi] = \int_M \text{Tr} \left\{ \frac{1}{4} F_\omega \wedge *F_\omega + \frac{1}{4} F_\omega \wedge F_\omega - \frac{1}{2} \psi \wedge [\phi, \psi] \right. \\ \left. - i d^\omega \zeta \wedge \psi - 2i [\zeta, * \zeta] \lambda \right. \\ \left. + i \phi d^\omega * d^\omega \lambda - \zeta \wedge * d^\omega \eta \right\}$$

$$Z_{DW} = \int \exp(-S_{DW}[\Phi]/e^2) \mathcal{D}\Phi$$

- MATHAI-QUILLEN (1986)

" SUPERSYMMETRIC " FORMULAS FOR THE EULER CHARACTERISTIC OF FINITE-DIMENSIONAL VECTOR BUNDLES FROM EQUIVARIANT COHOMOLOGY.

- ATIYAH-JEFFREY (1990)

FORMAL EXTENSION OF MATHAI-QUILLEN TO CERTAIN INFINITE-DIMENSIONAL VECTOR BUNDLE REPRODUCES $S_{DW}[\Phi]$ AND EXHIBITS Z_{DW} AS AN " EULER CHARACTERISTIC ".

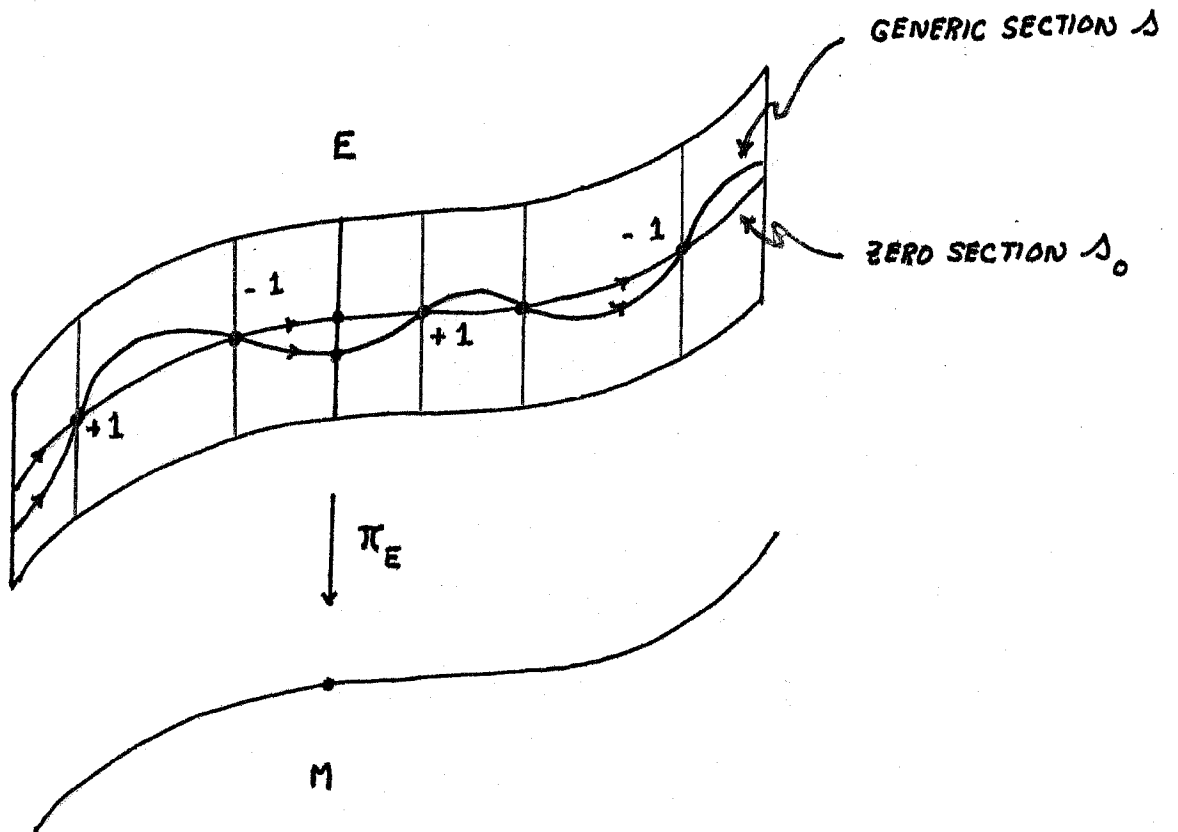
- MATHAI-QUILLEN FORMALISM FOR TQFT OF " COHOMOLOGICAL TYPE " .

EULER CHARACTERISTIC OF AN ORIENTED, REAL VECTOR BUNDLE

$$E \xrightarrow{\pi_E} M$$

OF FIBER DIMENSION $2k$ OVER A COMPACT, ORIENTED MANIFOLD M

OF DIMENSION $2k$ IS DEFINED AS FOLLOWS :



$\chi(E) :=$ INTERSECTION NUMBER OF ANY GENERIC SECTION

INTEGRAL REPRESENTATION:
$$e(E) = \int_M e(E)$$

$e(E)$ = EULER CLASS OF $E \xrightarrow{\pi_E} M$

TWO APPROACHES TO $e(E)$:

1. (CHERN-WEIL) FIBER METRIC ON $E \xrightarrow{\pi_E} M$ GIVES

$$SO(2k) \hookrightarrow F_{SO}(E) \xrightarrow{\pi_{SO}} M.$$

CHOOSE CONNECTION ω WITH CURVATURE Ω . CHOOSE THE ad -INVARIANT POLYNOMIAL

PFAF : $so(2k) \rightarrow \mathbb{R}$

$$(2\pi)^{-k} \text{PFAF}(\Omega) = \frac{1}{2^{2k} \pi^k k!} \sum_{\sigma \in S_{2k}} (-1)^\sigma \Omega_{\sigma(1)\sigma(2)} \wedge \dots \wedge \Omega_{\sigma(2k-1)\sigma(2k)}$$

IS A CLOSED FORM ON $F_{SO}(E)$ AND IS BASIC ($SO(2k)$ -INVARIANT AND HORIZONTAL) SO IT DESCENDS TO A CLOSED $2k$ -FORM ON M WHOSE COHOMOLOGY CLASS

$$e(E)$$

DOES NOT DEPEND ON THE CHOICE OF ω .

NOTE: $e(TS^2)$ IS COMPUTED IN APPENDUM 6.

2. (THOM CLASS) THERE EXISTS A UNIQUE $U(E) \in H_{CV}^{2k}(E; \mathbb{R})$
 WHOSE INTEGRAL OVER EACH FIBER $\pi_E^{-1}(p)$ IS 1. THEN

$$c(E) = \Delta^*(U(E))$$

FOR ANY SECTION Δ OF $E \xrightarrow{\pi_E} M$.

MATHAI-QUILLEN WILL YIELD VARIOUS EXPLICIT REPRESENTATIVES OF THE THOM CLASS $U(E)$.

ANOTHER VIEW OF THE PFAFFIAN $PF_{AF} : \Delta\sigma(2k) \rightarrow \mathbb{R} :$

$V =$ TYPICAL FIBER OF $E \xrightarrow{\pi_E} M$

$\{\psi^1, \dots, \psi^{2k}\} =$ ORIENTED, ORTHONORMAL BASIS FOR V

$=$ ODD GENERATORS OF THE SUPERCOMMUTATIVE SUPERALGEBRA

$$\Lambda V = \bigoplus_{i=0}^{2k} \Lambda^i V = (\Lambda V)_0 \oplus (\Lambda V)_1,$$

$$VOL = \psi^1 \dots \psi^{2k} \in \Lambda^{2k} V$$

$$A = (A_{ij}) \in \Delta\sigma(2k) \rightarrow \alpha_A = \frac{1}{2} A_{ij} \psi^i \psi^j \in \Lambda^2 V$$

$$= \frac{1}{2} \psi^T A \psi$$

$$= -\frac{1}{2} \sum_{l=1}^{2k} \psi^l A \psi^l$$

$$\frac{1}{h!} \alpha_A^h = \text{PFAF}(A) \text{ VOL}$$

AS A BEREZIN (OR FERMIONIC) INTEGRAL :

ANY $f \in \Lambda V$ IS A POLYNOMIAL WITH REAL COEFFICIENTS IN THE ODD "VARIABLES" ψ^1, \dots, ψ^{2k} , DEFINE

$$\int f \partial\psi = f_{\text{VOL}} \in \mathbb{R}$$

= COEFFICIENT OF VOL = $\psi^1 \dots \psi^{2k}$

THUS,

$$\text{PFAF}(A) = \int e^{\frac{1}{2} \psi^T A \psi} \partial\psi = \int e^{-\frac{1}{2} \sum \psi^l A \psi^l} \partial\psi$$

SUBSTITUTING FOR A THE CURVATURE Ω OF A CONNECTION ON $\text{SO}(2k) \hookrightarrow F_{\text{SO}}(E) \rightarrow M$ GIVES A BEREZIN INTEGRAL REPRESENTATION OF A FORM WHICH DESCENDS TO A REPRESENTATIVE OF THE EULER CLASS ON M .

MATHAI-QUILLEN REPRESENTATIVES OF THE THOM CLASS ARE LIKEWISE GIVEN AS BEREZIN INTEGRALS.

EXTENDED BEREZIN INTEGRATION :

\mathcal{A} = ANOTHER SUPERCOMMUTATIVE SUPERALGEBRA

E.G., $\Omega^*(V)$

$\mathcal{A} \otimes \wedge V$ = SUPER TENSOR PRODUCT

$$(a_1 \otimes f_1)(a_2 \otimes f_2) = (-1)^{\deg f_1 \deg a_2} (a_1 a_2) \otimes (f_1 f_2)$$

= POLYNOMIALS F WITH COEFFICIENTS IN \mathcal{A} IN THE ODD "VARIABLES" ψ^1, \dots, ψ^{2k}

$$\int F \otimes \psi = F_{\text{VOL}} \in \mathcal{A}$$

EXAMPLES :

1. $\mathcal{A} = \Omega^*(V) =$ COMPLEX-VALUED DIFFERENTIAL FORMS ON V

$$\Omega^*(V) \otimes \wedge V$$

$\{\mu_1, \dots, \mu_{2k}\} =$ BASIS FOR $V^* \subseteq \Omega^0(V)$ DUAL TO $\{\psi^1, \dots, \psi^{2k}\}$

OPINING THE " \otimes " FOR ELEMENTS OF $\Omega^*(V) \otimes \wedge V$ WE HAVE

$$-i d\mu_j \psi^j = i \psi^j d\mu_j = i \psi^T d\mu \in \Omega^*(V) \otimes \wedge V$$

$$-\frac{1}{2} \|\mu\|^2 \mathbf{1} = -\frac{1}{2} \|\mu\|^2 = -\frac{1}{2} (\mu_1^2 + \dots + \mu_{2k}^2) \in \Omega^*(V) \otimes \wedge V$$

$$(2\pi)^{-k} e^{-\frac{1}{2} \|\mu\|^2 + i \psi^T d\mu} \in \Omega^*(V) \otimes \wedge V$$

$$\int (2\pi)^{-k} e^{-\frac{1}{2}\|u\|^2 + i\psi^T u} \Theta\psi =$$

$$(2\pi)^{-k} \int e^{-\frac{1}{2}\|u\|^2} e^{i\psi^T u} \Theta\psi =$$

$$(2\pi)^{-k} e^{-\frac{1}{2}\|u\|^2} \int e^{i\psi^1 u_1 + \dots + i\psi^{2k} u_{2k}} \Theta\psi =$$

$$(2\pi)^{-k} e^{-\frac{1}{2}\|u\|^2} \int e^{i\psi^1 u_1} \dots e^{i\psi^{2k} u_{2k}} \Theta\psi =$$

$$(2\pi)^{-k} e^{-\frac{1}{2}\|u\|^2} \int (1 + i\psi^1 u_1) \dots (1 + i\psi^{2k} u_{2k}) \Theta\psi =$$

$$(2\pi)^{-k} e^{-\frac{1}{2}\|u\|^2} \int (i\psi^1 u_1) \dots (i\psi^{2k} u_{2k}) \Theta\psi =$$

$$(2\pi)^{-k} e^{-\frac{1}{2}\|u\|^2} \cdot 2k \int (-1)^{\frac{1}{2}(2k)(2k+1)} du_1 \dots du_{2k} \psi^1 \dots \psi^{2k} \Theta\psi =$$

$$(2\pi)^{-k} e^{-\frac{1}{2}\|u\|^2} du_1 \dots du_{2k} \in \Omega^*(V)$$

INTEGRATES TO 1 OVER V . THINK OF IT AS A "GAUSSIAN" REPRESENTATIVE OF THE THOM CLASS OF V , REGARDED AS A VECTOR BUNDLE OVER A POINT.

$$2. \quad \mathcal{A} = \mathbb{C}[\mathcal{L}\sigma(V)] \otimes \Omega^*(V)$$

$$\{\xi_1, \dots, \xi_n\} = \text{BASIS FOR } \mathcal{L}\sigma(V) \quad (n = k(2k-1))$$

$$\{x^1, \dots, x^n\} = \text{DUAL BASIS FOR } \mathcal{L}\sigma(V)^*, \text{ REGARDED AS LINEAR FUNCTIONS ON } \mathcal{L}\sigma(V)$$

$$\mathbb{C}[\mathcal{L}\sigma(V)] = \mathbb{R}[x^1, \dots, x^n] \otimes \mathbb{C}$$

$\mathbb{C}[\mathcal{A}\sigma(V)] \otimes \Omega^*(V) = \text{SUMS OF TERMS}$

$$\alpha = \rho \otimes \varphi$$

WHICH ARE TO BE REGARDED AS

$\Omega^*(V)$ -VALUED POLYNOMIALS ON $\mathcal{A}\sigma(V)$

$$\alpha(\xi) = (\rho \otimes \varphi)(\xi) = \rho(\xi)\varphi$$

GRADING : $\deg(\rho \otimes \varphi) = 2 \deg \rho + \deg \varphi$

WE DESCRIBE AN ELEMENT ν OF $\mathbb{C}[\mathcal{A}\sigma(V)] \otimes \Omega^*(V)$, CALLED THE UNIVERSAL THOM FORM OF V , AS THE BEREZIN INTEGRAL OF AN ELEMENT OF

$$\mathbb{C}[\mathcal{A}\sigma(V)] \otimes \Omega^*(V) \otimes \wedge V.$$

NOTATION : $\xi \in \mathcal{A}\sigma(V)$ GIVES A LINEAR TRANSFORMATION

$$M_\xi : V \rightarrow V$$

$$M_\xi(\psi) = \left. \frac{d}{dt} (\exp(t\xi)(\psi)) \right|_{t=0}$$

IF $M_a = M_{\xi_a}$, THEN

$$M_\xi = x^a(\xi) M_a$$

$$\text{PFAF}(M_\xi) = \int e^{-\frac{1}{2} \sum_i \psi^i x^a(\xi) M_a \psi^i} \mathcal{D}\psi$$

NOW NOTICE THAT

$$-\frac{1}{2} \sum_{\alpha} \psi^{\alpha} x^{\alpha} \pi_{\alpha} \psi^{\alpha} \in \mathbb{C}[\mathfrak{so}(V)] \otimes \Omega^*(V) \otimes \wedge V$$

$$e^{-\frac{1}{2} \sum_{\alpha} \psi^{\alpha} x^{\alpha} \pi_{\alpha} \psi^{\alpha}} \in \mathbb{C}[\mathfrak{so}(V)] \otimes \Omega^*(V) \otimes \wedge V$$

$$(2\pi)^{-k} e^{-\frac{1}{2} \|\mu\|^2 + i \psi^T d\mu - \frac{1}{2} \sum_{\alpha} \psi^{\alpha} x^{\alpha} \pi_{\alpha} \psi^{\alpha}} \in \mathbb{C}[\mathfrak{so}(V)] \otimes \Omega^*(V) \otimes \wedge V$$

SO WE CAN DEFINE

$$\nu = (2\pi)^{-k} \int e^{-\frac{1}{2} \|\mu\|^2 + i \psi^T d\mu - \frac{1}{2} \sum_{\alpha} \psi^{\alpha} x^{\alpha} \pi_{\alpha} \psi^{\alpha}} \Theta \psi \in \mathbb{C}[\mathfrak{so}(V)] \otimes \Omega^*(V).$$

EXAMPLE : $V = \mathbb{R}^2$ (USUAL ORIENTATION AND INNER PRODUCT)

$\{\psi^1, \psi^2\} =$ STANDARD BASIS

$\{\mu_1, \mu_2\} =$ DUAL BASIS (COORDINATE FUNCTIONS)

$\mathfrak{so}(V) \cong \mathfrak{so}(2)$

$\{\xi_1\} = \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$

$\{x^i\} =$ DUAL BASIS

$$\nu = (2\pi)^{-1} e^{-\frac{1}{2}(\mu_1^2 + \mu_2^2)} d\mu_1 d\mu_2 + (2\pi)^{-1} x^i e^{-\frac{1}{2}(\mu_1^2 + \mu_2^2)}$$

↑
 $\mathbb{C}^0[\mathfrak{so}(2)] \otimes \Omega^2(\mathbb{R}^2)$

↑
 $\mathbb{C}^1[\mathfrak{so}(2)] \otimes \Omega^0(\mathbb{R}^2)$

INTEGRATES TO 1 OVER \mathbb{R}^2

NOTE : THE PROOF IS IN ADDENDUM 7.

WE WILL SEE SHORTLY HOW THE UNIVERSAL THOM FORM OF V CAN BE USED TO PRODUCE (GAUSSIAN) REPRESENTATIVES OF THE THOM CLASS FOR ANY VECTOR BUNDLE WITH TYPICAL FIBER V .

FIRST, HOWEVER,

WHAT KIND OF "THING" IS V ?

BRIEF ASIDE ON THE CARTAN MODEL OF EQUIVARIANT COHOMOLOGY:

M = SMOOTH MANIFOLD

G = COMPACT, CONNECTED LIE GROUP

LEFT ACTION : $\sigma : G \times M \rightarrow M$

$$\sigma(g, m) = g \cdot m = \sigma_g(m)$$

OBJECTIVE : COHOMOLOGY OF M/G

$\Omega^*(M)$ = GRADED ALGEBRA OF COMPLEX-VALUED DIFFERENTIAL FORMS ON M

$\mathbb{C}[\mathfrak{g}]$ = GRADED ALGEBRA OF COMPLEX-VALUED POLYNOMIALS ON THE LIE ALGEBRA \mathfrak{g} OF G

$\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M)$ = SUMS OF $\alpha = \rho \otimes \psi = \Omega^*(M)$ -VALUED POLYNOMIALS ON \mathfrak{g}

$$\alpha(\xi) = (\rho \otimes \psi)(\xi) = \rho(\xi)\psi$$

GRADING : $\deg(\alpha) = \deg(\rho \otimes \psi) = 2 \deg(\rho) + \deg(\psi)$

INDUCED ACTION OF G ON $\mathbb{C}[g] \otimes \Omega^*(M)$:

$$(g \cdot \alpha)(\xi) = (g \cdot (\rho \circ \alpha))(\xi) = \rho(g^{-1} \xi g) \sigma_{g^{-1}}^*(\alpha)$$

$$\begin{aligned} \Omega_G^*(M) &= [\mathbb{C}[g] \otimes \Omega^*(M)]^G \\ &= G\text{-INVARIANT ELEMENTS OF } \mathbb{C}[g] \otimes \Omega^*(M) \\ &= G\text{-EQUIVARIANT DIFFERENTIAL FORMS ON } M \end{aligned}$$

G-EQUIVARIANT EXTERIOR DERIVATIVE:

$$d_G : \Omega_G^*(M) \rightarrow \Omega_G^*(M)$$

$$(d_G \alpha)(\xi) = d(\alpha(\xi)) - \xi_{\#}(\alpha(\xi))$$

WHERE

$$\xi_{\#}(m) = \left. \frac{d}{dt} (\exp(-t\xi) \cdot m) \right|_{t=0}$$

$(\Omega_G^*(M), d_G)$ IS A COCHAIN COMPLEX AND ITS COHOMOLOGY

$$H_G^*(M)$$

IS THE CARTAN MODEL OF THE G-EQUIVARIANT COHOMOLOGY OF M .

THEOREM (CARTAN): IF THE ACTION OF G ON M

IS FREE, THEN

$$H_G^*(M) \cong H_{\text{deRham}}^*(M/G; \mathbb{C}).$$

NOTE: MORE DETAILS AND $H_{S^1}^*(S^3)$ CAN BE FOUND IN
ADDENDUM 8.

INTEGRATION OF EQUIVARIANT DIFFERENTIAL FORMS :

$$\int_M : \Omega_G^*(M) \rightarrow \mathbb{C}[g]$$

$$\left(\int_M \alpha \right) (\xi) = \int_M \alpha(\xi) := \int_M \alpha(\xi)_{[2k]}$$

UNIVERSAL THOM FORM

$$\nu = (2\pi)^{-k} \int e^{-\frac{1}{2} \|u\|^2 + i \psi^T du - \frac{1}{2} \sum_{\alpha} \psi^{\alpha} x^{\alpha} \eta_{\alpha} \psi^{\alpha}} \partial \psi \in \Omega_{SO(V)}^*(V)$$

SATISFIES

$$d_{SO(V)} \nu = 0$$

AND

$$\int_V \nu = 1$$

NOTE : PROOFS FOR $V = \mathbb{R}^2$ IN ADDENDUM 9.

THOM FORMS FROM THE UNIVERSAL THOM FORM :

ANY ORIENTED REAL VECTOR BUNDLE WITH TYPICAL FIBER V CAN BE VIEWED AS THE VECTOR BUNDLE

$$P \times_{\rho} V$$

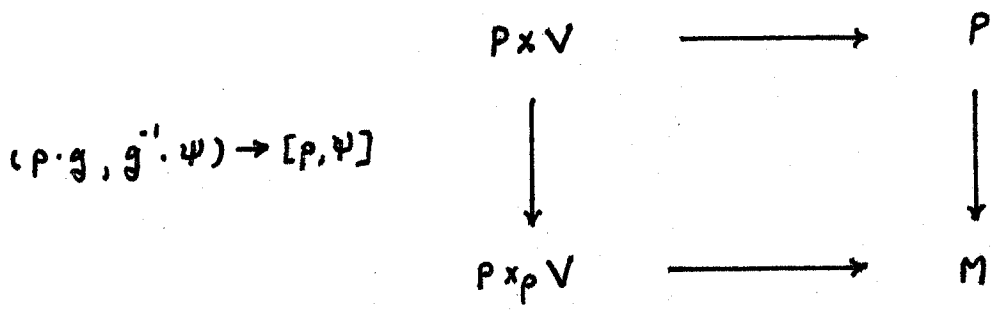
ASSOCIATED TO SOME PRINCIPAL G -BUNDLE

$$G \hookrightarrow P \longrightarrow M$$

BY SOME REPRESENTATION

$$\rho : G \rightarrow SO(V)$$

$$(g \cdot \psi = \rho(g)(\psi))$$



$$\rho_* : \mathfrak{g} \rightarrow \mathfrak{so}(V)$$

$$\begin{aligned}
 \rho \otimes \psi &\in [C[\mathfrak{so}(V)] \otimes \Omega^*(V)]^{SO(V)} \\
 &\rightarrow (\rho \circ \rho_*) \otimes \psi \in [C[\mathfrak{g}] \otimes \Omega^*(V)]^G
 \end{aligned}$$

E.G., $\nu \in \Omega_{SO(V)}^*(V) \rightarrow \nu_G \in \Omega_G^*(V)$

CHERN-WEIL MAP :

CHOOSE ω ON P

$$C[\mathfrak{g}]^G \rightarrow \Omega^*(P) \stackrel{\text{BASIC}}{\cong} \Omega^*(M)$$

$$\rho \rightarrow \rho(\Omega)$$

NATHAI-QUILLEN MAP :

$$[C[\mathfrak{g}] \otimes \Omega^*(V)]^G \rightarrow \Omega^*(P \times V) \stackrel{\text{BASIC}}{\cong} \Omega^*(P \times_{\rho} V)$$

$$\rho \otimes \psi \rightarrow \text{HOR}_{\omega}(\rho(\Omega) \wedge \psi)$$

$$d_G\text{-CLOSED} \rightarrow d\text{-CLOSED}$$

NOTE : WE USE THE SAME SYMBOLS FOR FORMS ON P AND V AND THEIR PULLBACKS TO $P \times V$ BY A PROJECTION.

UNDER THE PATHAI-QUILLEN MAP

$$\mathcal{V}_G = (2\pi)^{-k} \int e^{-\frac{1}{2} \|\mu\|^2 + i \psi^T d\mu - \frac{1}{2} \sum \psi^l (x^a \circ \rho_*) \eta_a \psi^l} \mathcal{D}\psi$$

GOES TO

$U =$ THE HORIZONTAL PROJECTION OF

$$\begin{aligned} & (2\pi)^{-k} \int e^{-\frac{1}{2} \|\mu\|^2 + i \psi^T d\mu - \frac{1}{2} \sum \psi^l (x^a(\rho_* \Omega)) \eta_a \psi^l} \mathcal{D}\psi \\ &= (2\pi)^{-k} \int e^{-\frac{1}{2} \|\mu\|^2 + i \psi^T d\mu + \frac{1}{2} \psi^T (\rho_* \Omega) \psi} \mathcal{D}\psi \end{aligned}$$

WHERE $(\rho_* \Omega)$ IS THE SKEW-SYMMETRIC MATRIX IMAGE OF THE G -CURVATURE UNDER ρ_* .

EXAMPLE : $V = \mathbb{R}^2$ AS IN THE EXAMPLE ON PAGE 10.

$$\mathcal{V} = (2\pi)^{-1} e^{-\frac{1}{2} (\mu_1^2 + \mu_2^2)} (x' + d\mu, d\mu_2)$$

VECTOR BUNDLE = TS^2 (USUAL STRUCTURE ON S^2)

$$G = SO(2)$$

$$SO(2) \hookrightarrow F_{SO}(TS^2) \rightarrow S^2$$

$$\rho = \text{id}_{SO(2)} \quad (\text{so } \rho_* = \text{id}_{SO(2)})$$

$$\omega = \omega' \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \Omega = \Omega' \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$U = (2\pi)^{-1} e^{-\frac{1}{2} (\mu_1^2 + \mu_2^2)} (\Omega' + d\mu_1, d\mu_2 + \omega' \wedge (\mu_1, d\mu_1 + \mu_2, d\mu_2))$$

NOTE : THE PROOF IS IN ADDENDUM 10.

IN GENERAL,

$$\omega = \text{HOR}_\omega (2\pi)^k \int e^{-\frac{1}{2} \|u\|^2 + i \psi^T du + \frac{1}{2} \psi^T (\rho_* \Omega) \psi} \Theta \psi$$

IS A BASIC FORM ON $P \times V$ WHICH DESCENDS TO A GAUSSIAN REPRESENTATIVE OF THE THON CLASS ON $P \times_\rho V$.

PULLING BACK TO M BY A SECTION OF $P \times_\rho V \rightarrow M$ GIVES A REPRESENTATIVE OF THE EULER CLASS.

BUT EVERY SECTION OF $P \times_\rho V \rightarrow M$ IS OF THE FORM

$$m \rightarrow (\Delta(m), S(\Delta(m))) \rightarrow [\Delta(m), S(\Delta(m))]$$

WHERE Δ IS A SECTION OF $G \hookrightarrow P \rightarrow M$ AND $S : P \rightarrow V$ IS EQUIVARIANT ($S(p \cdot g) = \rho(g^{-1}) (S(p))$).

THUS, ONE CAN PULL ω BACK DIRECTLY BY

$$(\Delta, S \circ \Delta)^* = ((1, S) \circ \Delta)^* = \Delta^* \circ (1, S)^*$$

I.E., PULL V -FACTORS BACK BY S TO GET A FORM ON P AND THEN PULL THIS BACK TO M BY Δ .

EXAMPLE : $TS^2 = F_{SO}(TS^2) \times_{\text{id}_{SO(2)}} \mathbb{R}^2$ AS IN THE EXAMPLE ON PAGE 16.

ω = LEVI-CIVITA CONNECTION ON $F_{SO}(TS^2)$

Δ = SECTION OF $F_{SO}(TS^2)$

= ORIENTED, ORTHONORMAL FRAME FIELD ON S^2

$$\Delta(\phi, \theta) = \left(\phi, \theta, \frac{\partial}{\partial \phi}, \frac{1}{\sin \phi} \frac{\partial}{\partial \theta} \right)$$

$S : F_{S^0}(TS^2) \rightarrow \mathbb{R}^2$ AN EQUIVARIANT MAP

(I.E., A VECTOR FIELD ON S^2)

$$S(\Delta(\phi, \theta)) = (\gamma \sin \theta, \gamma \cos \theta \cos \phi)$$

WHERE $\gamma \in \mathbb{R}$ IS ARBITRARY

THE RESULTING REPRESENTATIVE OF THE EULER CLASS IS

$$(2\pi)^{-1} \int e^{-\frac{1}{2} \gamma^2 (\sin^2 \theta + \cos^2 \theta \cos^2 \phi)} \sin \phi (1 + \gamma^2 \cos^2 \theta \sin^2 \phi) d\phi d\theta$$

NOTE : THE PROOF IS IN ADDENDUM II.

IN PARTICULAR, WE OBTAIN THAT, FOR ANY $\gamma \in \mathbb{R}$,

$$\frac{1}{2\pi} \int_0^{2\pi} \int_0^{\pi} e^{-\frac{1}{2} \gamma^2 (\sin^2 \theta + \cos^2 \theta \cos^2 \phi)} \sin \phi (1 + \gamma^2 \cos^2 \theta \sin^2 \phi) d\phi d\theta$$

$$= \int(S^2)$$

$$= 2$$

ATIYAH - JEFFREY TRANSFORMATION OF THE UNIVERSAL THOM FORM (SKETCH) :

NOTE : DETAILED CALCULATIONS CAN BE
FOUND IN ADDENDUM 12.

$$(1) \quad U = (2\pi)^{-k} e^{-\frac{1}{2} \|\mu\|^2} \int \exp(i \psi^T du + \frac{1}{2} \psi^T (\rho_* \Omega) \psi) \Theta \psi$$

(EVALUATED ON HORIZONTAL PARTS)

ADDITIONAL ASSUMPTIONS AND CHOICES :

1. $G \hookrightarrow P \xrightarrow{\pi_P} M$ IS ORIENTABLE (I.E., \exists n -FORM Ψ ON P ($n = \dim G$) SUCH THAT, IF $m \in M$ AND $\iota_m : \pi_P^{-1}(m) \hookrightarrow P$, THEN $\iota_m^* \Psi$ IS AN ORIENTATION FOR $\pi_P^{-1}(m) \cong G$). FOLLOWS THAT P IS ORIENTED BY THE "LOCAL PRODUCT ORIENTATION" $\pi_P^* (\text{VOL}_M) \wedge \Psi$.
2. THE ACTION OF G ON P IS ORIENTATION PRESERVING ($\sigma_g : P \rightarrow P$, $\sigma_g(p) = p \cdot g$, IS AN ORIENTATION PRESERVING DIFFEOMORPHISM $\forall g \in G$).
3. A G -INVARIANT RIEMANNIAN METRIC $\langle \cdot, \cdot \rangle$ HAS BEEN CHOSEN FOR P .

4. THE CONNECTION ω FOR P IS THE ONE WHOSE DISTRIBUTION OF HORIZONTAL SPACES IS THE FAMILY OF \langle, \rangle -ORTHOGONAL COMPLEMENTS TO THE $G \hookrightarrow P \xrightarrow{\pi_P} M$ VERTICAL SPACES
5. AN ad -INVARIANT INNER PRODUCT $(,)$ ON \mathfrak{g} HAS BEEN CHOSEN AND NORMALIZED SO THAT THE VOLUME OF G (ARISING FROM THE CORRESPONDING BI-INVARIANT RIEMANNIAN METRIC ON G) IS 1.

NOW WE BEGIN TO MANIPULATE \mathcal{U} , GIVEN BY (1) :

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega] \text{ AND}$$

$[\omega, \omega]$ VANISHES ON HORIZONTAL PARTS SO

$$(2) \quad \mathcal{U} = (2\pi)^{-k} e^{-\frac{1}{2}\|\mu\|^2} \int \exp(i\psi^T d\mu + \frac{1}{2}\psi^T(\rho_*(d\omega))\psi) \mathcal{D}\psi$$

(EVALUATED ON HORIZONTAL PARTS)

TOOLS :

$$C_p : \mathfrak{g} \rightarrow \text{VERT}_p(P) \subseteq T_p(P)$$

$$C_p(\xi) = \xi^{\#}(p) = \left. \frac{d}{dt} (p \cdot \exp(t\xi)) \right|_{t=0}$$

$$C_p^* : T_p(P) \rightarrow \mathfrak{g}$$

$$\langle \nu_p, C_p(\eta) \rangle_p = (C_p^*(\nu_p), \eta) \quad \forall \nu_p \in T_p(P) \quad \forall \eta \in \mathfrak{g}$$

$$R_p = C_p^* \circ C_p : \mathfrak{g} \rightarrow \mathfrak{g} \quad (\text{INVERTIBLE AND SELF-ADJOINT})$$

$$R_p^{-1} : \mathfrak{g} \rightarrow \mathfrak{g}$$

THEN

$$\omega_p = R_p^{-1} \circ C_p^*$$

AS LIE ALGEBRA-VALUED 1-FORMS,

$$\omega = R^{-1} \circ C^*$$

SO

$$d\omega = dR^{-1} \wedge C^* + R^{-1} dC^*$$

C^* VANISHES ON HORIZONTAL VECTORS AND THEREFORE SO DOES $dR^{-1} \wedge C^*$ SO

$$(3) \quad U = (2\pi)^{-\frac{n}{2}} e^{-\frac{1}{2}\|\mu\|^2} \int \exp(i\psi^T d\mu + \frac{1}{2}\psi^T (\rho_*(R^{-1} dC^*)) \psi) \Theta \psi$$

(EVALUATED ON HORIZONTAL PARTS)

TO ELIMINATE THE EXPLICIT APPEARANCE OF THE INVERSE, USE THE FOURIER INVERSION FORMULA (ON \mathfrak{g}) AND A CHANGE OF VARIABLE.

$\phi = (\phi_1, \dots, \phi_n)$ AND $\lambda = (\lambda_1, \dots, \lambda_n)$ COORDINATES ON \mathfrak{g}

AND f A REAL-VALUED FUNCTION ON \mathfrak{g}

$$(\mathcal{F}f)(\phi) = (2\pi)^{-n/2} \int_{\mathfrak{g}} e^{-i(\phi, \lambda)} f(\lambda) d\lambda$$

$$(\tilde{\mathcal{F}}f)(\lambda) = (2\pi)^{-n/2} \int_{\mathfrak{g}} e^{i(\lambda, \phi)} f(\phi) d\phi$$

NOW, $F = \mathcal{F}(\mathcal{F}F)$ IMPLIES

$$f(\mu) = (2\pi)^{-n} \int_{\mathfrak{g}} \int_{\mathfrak{g}} e^{-i(\mu, \lambda)} e^{i(\lambda, \phi)} f(\phi) d\phi d\lambda$$

USE THIS TO EVALUATE $f(R^{-1}\mu)$ AND MAKE THE CHANGE OF VARIABLE $\lambda \rightarrow R\lambda$ TO OBTAIN

$$f(R^{-1}\mu) = (2\pi)^{-n} \int_{\mathfrak{g}} \int_{\mathfrak{g}} e^{-i(\mu, \lambda)} e^{i(\phi, R\lambda)} f(\phi) \det R d\lambda d\phi$$

APPLY THIS TO THE PATHA) - QUILLEN FORM WITH $\mu = dc^*$ TO OBTAIN

$$(4) \quad \mathcal{U} = (2\pi)^{-n} (2\pi)^{-k} e^{-\frac{1}{2} \| \mu \|^2} \iiint \exp \left(\frac{1}{2} \psi^T (\rho_* \phi) \psi + i \psi^T d\mu - i (dc^*, \lambda) + i (\phi, R\lambda) \right) \det R \otimes \psi d\phi d\lambda$$

(EVALUATED ON HORIZONTAL PARTS)

TO INCORPORATE "EVALUATED ON HORIZONTAL PARTS" DIRECTLY INTO THE INTEGRAL WE NEED:

A NORMALIZED VERTICAL VOLUME FORM FOR A PRINCIPAL

G-BUNDLE $G \hookrightarrow Q \xrightarrow{\pi} X$ (WE HAVE IN MIND

$G \hookrightarrow P \times V \rightarrow P \times_{\rho} V$) IS AN n -FORM ($n = \dim G$)

W ON Q SUCH THAT, IF $\iota_x : \pi^{-1}(x) \hookrightarrow Q$, THEN

$$\int_{\pi^{-1}(x)} \iota_x^* W = 1$$

$\forall x \in X.$

ONE CAN ALWAYS CONSTRUCT SUCH A THING AND, IN OUR CASE, IT CAN BE WRITTEN AS A BEREZIN INTEGRAL

$$W = (\det R)^{-1} \int e^{(C^*, \eta)} \partial \eta$$

WHERE η IS A VARIABLE IN \mathfrak{g} AND (C^*, η) IS THE 1-FORM ON $P \times V$ DEFINED BY

$$(C^*, \eta)(\xi) = (C^*(\xi), \eta) = \langle \xi, C(\eta) \rangle = \langle \xi, \eta^{\#} \rangle$$

FOR ANY VECTOR FIELD ξ .

WEDGING WITH W KILLS ANY VERTICAL PARTS AND LEAVES ONLY TERMS WITH HORIZONTAL PARTS AND A FACTOR OF W . THEN FIBER INTEGRATION (I.E., IGNORING W) LEAVES ONLY HORIZONTAL PARTS.

DOING THIS FOR U IN (4) ADDS (C^*, η) TO THE EXPONENT, CANCELS THE $\det R$ AND INTRODUCES ONE MORE BEREZIN INTEGRAL:

$$(5) \quad U = (2\pi)^{-n} (2\pi)^{-k} e^{-\frac{1}{2} \|\mu\|^2} \iiint \exp \left(\frac{1}{2} \psi^T(\rho, \phi) \psi + i \psi^T d\mu - i (dC^*, \lambda) + i (\phi, R\lambda) + (C^*, \eta) \right) \partial \eta \partial \psi d\phi d\lambda$$

PULLING BACK THE V -PARTS OF THIS FORM ON $P \times V$ BY AN EQUIVARIANT MAP

$$s : P \rightarrow V$$

GIVES A FORM

$$(6) \quad (2\pi)^{-n} (2\pi)^{-k} e^{-\frac{1}{2}\|s\|^2} \iiint \exp\left(\frac{1}{2}\psi^T(\rho_*\phi)\psi + i\psi^T ds - i(dc^*, \lambda) + i(\phi, R\lambda) + (c^*, \eta)\right) \partial\eta \partial\psi d\phi d\lambda$$

WHOSE INTEGRAL OVER P IS THE EULER NUMBER OF $P \times_p V$.

FINALLY, WE ADOPT THE PRACTICE IN (SUPERSYMMETRIC) PHYSICS OF WRITING THE INTEGRAL OF A FORM AS A BEREZIN INTEGRAL FOLLOWED BY A VOLUME INTEGRAL:

$$\int_P \alpha = \int_P \int \alpha(x^i, \xi_i) \partial\xi d\omega$$

APPLY THIS TO (6) AND (FINALLY!) SUPPRESS ALL BUT ONE OF THE INTEGRAL SIGNS TO GET THE ATIYAH-JEFFREY FORMULA FOR THE EULER NUMBER OF $P \times_p V$:

$$(7) \quad (2\pi)^{-n} (2\pi)^{-k} e^{-\frac{1}{2}\|s\|^2} \int \exp\left(\frac{1}{2}\psi^T(\rho_*\phi)\psi + i\psi^T ds - i(dc^*, \lambda) + i(\phi, R\lambda) + (c^*, \eta)\right) \partial\xi \partial\eta \partial\psi d\lambda d\phi d\omega$$

NOW WE WANT TO APPLY THIS (FORMALLY) TO AN INFINITE-DIMENSIONAL VECTOR BUNDLE THAT ARISES NATURALLY IN DONALDSON THEORY.

$M =$ COMPACT, SIMPLY CONNECTED, ORIENTED,
SMOOTH 4-MANIFOLD WITH $b_2^+(M) > 1$
AND ODD

$g =$ GENERIC RIEMANNIAN METRIC ON M

$SU(2) \hookrightarrow P \xrightarrow{\pi} M$ WITH

$$8c_2(P) - 3(1 + b_2^+(M)) = 0$$

PRINCIPAL BUNDLE :

$$\hat{\mathcal{H}}(P) \hookrightarrow \hat{\mathcal{A}}(P) \rightarrow \hat{\mathcal{B}}(P)$$

(NOTATION FROM APPENDIX 2)

$\hat{\mathcal{H}}(P)$ - ACTION ON $\Omega_+^2(M, ad P)$ GIVES AN ASSOCIATED VECTOR BUNDLE

$$\hat{\mathcal{A}}(P) \times_{\hat{\mathcal{H}}(P)} \Omega_+^2(M, ad P) \rightarrow \hat{\mathcal{B}}(P)$$

$\Delta_0 =$ 0-SECTION

$\Delta =$ SELF-DUAL CURVATURE SECTION $= F^+$

$$\Delta([\omega]) = [\omega, F_\omega^+]$$

$$\eta(P, g) = \Delta(\hat{\mathcal{B}}(P)) \cap \Delta_0(\hat{\mathcal{B}}(P))$$

$$\chi_0(M) = \text{"EULER NUMBER"}$$

ATIYAH-JEFFREY (SKETCH) :

NOTE : DETAILS ARE AVAILABLE IN ADDENDUM 13.

REQUIRES A $\hat{\mathcal{G}}$ -INVARIANT RIEMANNIAN METRIC FOR $\hat{\mathcal{A}}$ (I WILL DROP THE "(P)" EVERYWHERE FROM THIS POINT ON).

$\hat{\mathcal{A}}$ OPEN IN \mathcal{A} , WHICH IS AN AFFINE SPACE

$$T_\omega(\hat{\mathcal{A}}) \cong \Omega^1(M, \text{ad } P)$$

RECALL : ON $\Omega^k(M, \text{ad } P)$,

$$\langle \mu, \nu \rangle_{\mathbb{R}} = \int_M -2 \text{tr}(\mu \wedge * \nu)$$

IS INVARIANT UNDER ADJOINT ACTION.

THUS, $\langle \cdot, \cdot \rangle_{\mathbb{R}}$ ON $T_\omega(\hat{\mathcal{A}})$ GIVES $\hat{\mathcal{G}}$ -INVARIANT RIEMANNIAN METRIC ON $\hat{\mathcal{A}}$.

THIS GIVES A CORRESPONDING CONNECTION ON $\hat{\mathcal{G}} \hookrightarrow \hat{\mathcal{A}} \rightarrow \hat{\mathcal{B}}$:

$$T_\omega(\hat{A}) \cong T_\omega(\omega, \hat{A}) \oplus \text{KER}(S^\omega) \cong \text{IM}(d^\omega) \oplus \text{KER}(S^\omega)$$

$$d^\omega : \Omega^0(M, \text{ad } P) \rightarrow \Omega^1(M, \text{ad } P)$$

$$S^\omega : \Omega^1(M, \text{ad } P) \rightarrow \Omega^0(M, \text{ad } P)$$

(SEE APPENDIX 2)

$$\text{HOR}_\omega(\hat{A}) \cong \text{KER}(S^\omega)$$

NOW EXAMINE EACH TERM IN THE ATIYAH-JEFFREY EXPONENT

$$\begin{aligned} & -\frac{1}{2} \|s(\omega)\|^2 + \frac{1}{2} \psi^T(\rho_* \phi) \psi + i \psi^T ds_\omega(\zeta) \\ & - i (dc_\omega^+(\zeta), \lambda) + i(\phi, R_\omega \lambda) + (c_\omega^+(\zeta), \eta) \end{aligned}$$

SEPARATELY.

$$\begin{aligned} 1. \quad -\frac{1}{2} \|s(\omega)\|^2 &= -\frac{1}{2} \|F_\omega^+\|^2 = \int_M \text{tr}(F_\omega^+ \wedge * F_\omega^+) \\ &= \int_M \text{tr}(F_\omega^+ \wedge F_\omega^+) \end{aligned}$$

$-\frac{1}{2} \ s(\omega)\ ^2 = \frac{1}{2} \int_M \text{tr}(F_\omega^+ \wedge * F_\omega^+) + \frac{1}{2} \int_M \text{tr}(F_\omega \wedge F_\omega)$
$\begin{array}{ccc} \uparrow & & \uparrow \\ \text{YANG-MILLS TERM} & & \text{TOPOLOGICAL TERM} \end{array}$

DONALDSON THEORY ANALOGUES OF C , C^* AND R :

$$\text{LIE ALGEBRA OF } \hat{\mathcal{H}} \cong \Omega^0(M, \text{ad } P)$$

$$T_\omega(\hat{\mathcal{A}}) \cong \Omega^1(M, \text{ad } P)$$

SO $\forall \omega \in \hat{\mathcal{A}}$

$$C_\omega : \Omega^0(M, \text{ad } P) \rightarrow \Omega^1(M, \text{ad } P)$$

$$C_\omega(\xi) = \left. \frac{d}{dt} (\omega \cdot \exp(t\xi)) \right|_{t=0} = d^\omega \xi$$

$$C_\omega = d^\omega$$

THUS, RELATIVE TO \langle , \rangle_0 AND \langle , \rangle_1 ,

$$C_\omega^* : \Omega^1(M, \text{ad } P) \rightarrow \Omega^0(M, \text{ad } P)$$

$$C_\omega^* = S^\omega = - * d^\omega *$$

AND

$$R_\omega : \Omega^0(M, \text{ad } P) \rightarrow \Omega^0(M, \text{ad } P)$$

$$R_\omega = C_\omega^* \circ C_\omega = S^\omega \circ d^\omega = \Delta_0^\omega = - * d^\omega + d^\omega$$

= SCALAR LAPLACIAN OF ω

$$2. \quad i(\phi, R_\omega \lambda)$$

BOTH ϕ AND λ ARE IN THE LIE ALGEBRA OF $\hat{\mathcal{H}}$ SO INTRODUCE TWO "BOSONIC" FIELDS

$$\phi, \lambda \in \Omega^0(M, \text{ad} P)$$

AND INTERPRET $(,)$ AS \langle , \rangle_0 . FOR EACH $\omega \in \hat{\mathcal{A}}$,

$$i(\phi, R_\omega \lambda) = 2i \int_M \text{tr}(\phi d^\omega * d^\omega \lambda)$$

3. $(C_\omega^*(\xi), \eta)$

$C_\omega^* : \Omega^1(M, \text{ad} P) \rightarrow \Omega^0(M, \text{ad} P)$ SO INTRODUCE TWO "FERMIONIC" FIELDS

$$\xi \in \Omega^1(M, \text{ad} P)$$

$$\eta \in \Omega^0(M, \text{ad} P)$$

AND IDENTIFY $(,) = \langle , \rangle_0$.

$$\begin{aligned} (C_\omega^*(\xi), \eta) &= \langle C_\omega^*(\xi), \eta \rangle_0 = \langle \xi, C_\omega(\eta) \rangle, \\ &= \langle \xi, d^\omega \eta \rangle, \end{aligned}$$

$$(C_\omega^*(\xi), \eta) = -2 \int_M \text{tr}(\xi \wedge d^\omega \eta)$$

$$4. \quad i \psi^T ds_\omega(\zeta)$$

$$s = F^+ : \hat{A} \rightarrow \Omega_+^2(n, adP)$$

$$ds_\omega = d_+^\omega : \Omega^1(n, adP) \rightarrow \Omega_+^2(n, adP)$$

$$ds_\omega(\zeta) = d_+^\omega \zeta$$

INTRODUCE "FERMIONIC" FIELD

$$\psi \in \Omega_+^2(n, adP)$$

AND INTERPRET FINITE-DIMENSIONAL EXPRESSIONS

$$A^T B = (A^1 \dots A^r) \begin{pmatrix} B^1 \\ \vdots \\ B^r \end{pmatrix}$$

AS THE APPROPRIATE INNER PRODUCT ON FIELDS.

$$i \psi^T ds_\omega(\zeta) = i \langle \psi, d_+^\omega(\zeta) \rangle_2$$

$$i \psi^T ds_\omega(\zeta) = -2i \int_M d^m \zeta \wedge \psi$$

$$5. \quad \frac{1}{2} \psi^T (\rho + \phi) \psi$$

ρ IS THE REPRESENTATION OF $\hat{\mathcal{G}}$ ON $\Omega^2(M, \text{ad } P)$ GIVING RISE TO THE ASSOCIATED VECTOR BUNDLE. REGARDING $\hat{\mathcal{G}}$ AS SECTIONS OF $\text{Ad } P$ THIS IS, POINTWISE, THE ADJOINT ACTION OF $\text{SU}(2)$ ON $\mathfrak{su}(2)$. INFINITESIMAL ACTION ρ_* IS BRACKET.

$$(\rho_* \phi) \psi = [\phi, \psi]$$

SO

$$\frac{1}{2} \psi^T (\rho_* \phi) \psi = \frac{1}{2} \langle [\phi, \psi], \psi \rangle_2$$

$$\frac{1}{2} \psi^T (\rho_* \phi) \psi = - \int_M \text{tr} (\psi \wedge [\phi, \psi])$$

$$6. \quad -i (dC_\omega^*(\zeta), \lambda)$$

dC^* IS A 2-FORM ON \hat{A} WITH VALUES IN $\Omega^0(M, \text{ad } P)$

COMPUTING dC_ω^* REQUIRES A LITTLE WORK (APPENDIX 13),

BUT THE RESULT IS

$$dC_\omega^*(\zeta_1, \zeta_2) = -2^* [\zeta_1, \zeta_2]$$

$$-i (dC_\omega^*(\zeta), \lambda) = 2i \langle {}^* [\zeta, \zeta], \lambda \rangle_0$$

$$-i (dC_{\omega}^*(z), \lambda) = -4i \int_M \text{tr} ([z, *z] \lambda)$$

ADDING TOGETHER ALL OF THE TERMS IN THE BOXES AND (IN DEFERENCE TO THE PHYSICS LITERATURE) SWITCHING TO

$$\text{Tr} = -2 \text{tr}$$

GIVES

$$\int_M \text{Tr} \left\{ -\frac{1}{4} F_{\omega} \wedge *F_{\omega} - \frac{1}{4} F_{\omega} \wedge F_{\omega} + \frac{1}{2} \psi \wedge [\phi, \psi] + i d^{\omega} z \wedge \psi \right. \\ \left. + 2i [z, *z] \lambda - i \phi d^{\omega} *d^{\omega} \lambda + z \wedge *d^{\omega} \eta \right\}$$

WRITING THE INTEGRAL IN (7) IN THE FORM

$$\int e^{-\int_M \text{Tr} \mathcal{L}_{\text{DW}}} \mathcal{D}\Phi$$

WE IDENTIFY THE DONALDSON-WITTEN LAGRANGIAN:

$$\mathcal{L}_{\text{DW}}[\Phi] = \frac{1}{4} F_{\omega} \wedge *F_{\omega} + \frac{1}{4} F_{\omega} \wedge F_{\omega} - \frac{1}{2} \psi \wedge [\phi, \psi] - i d^{\omega} z \wedge \psi \\ - 2i [z, *z] \lambda + i \phi d^{\omega} *d^{\omega} \lambda - z \wedge *d^{\omega} \eta$$

ACTION : $S_{\text{DW}}[\Phi] = \int_M \text{Tr} \mathcal{L}_{\text{DW}}$

PARTITION FUNCTION : $Z_{\text{DW}} = \int e^{-S_{\text{DW}}[\Phi]/e^2} \mathcal{D}\Phi$