

SEIBERG-WITTEN THEORY : PROLOGUE

WITTEN'S 1988 TOPOLOGICAL QUANTUM FIELD THEORY (TQFT) :

M AS BEFORE WITH RIEMANNIAN METRIC g . FIX SOME

$$SU(2) \hookrightarrow P \xrightarrow{\pi} M$$

FIELD CONTENT :

GAUGE FIELD (CONNECTION) ω WITH CURVATURE $F_\omega \in \Omega^2(M, \text{ad } P)$
+ "MATTER FIELDS"

BOSONIC

$$\phi \in \Omega^0(M, \text{ad } P)$$

$$\lambda \in \Omega^0(M, \text{ad } P)$$

FERMIONIC

$$\eta \in \Omega^0(M, \text{ad } P)$$

$$\psi \in \Omega^1(M, \text{ad } P)$$

$$\zeta \in \Omega_+^2(M, \text{ad } P)$$

$$\Phi = (\omega, \phi, \lambda, \eta, \psi, \zeta)$$

+ "GHOST NUMBERS" + "SUPERSYMMETRY OPERATOR"

+ DONALDSON-WITTEN ACTION $S_{\text{DW}}[\Phi]$

$$S_{DW}[\Phi] = \int_M \text{Tr} \left\{ \frac{1}{4} F_\omega \wedge {}^* F_\omega + \frac{1}{4} F_\omega \wedge F_\omega - \frac{1}{2} \psi \wedge [\phi, \psi] \right. \\ \left. - i d^\omega \zeta \wedge \psi - 2i [\zeta, {}^* \zeta] \lambda + i \phi d^\omega {}^* d^\omega \lambda \right. \\ \left. - \zeta \wedge {}^* d^\omega \eta \right\}$$

PARTITION FUNCTION : $Z_{DW} = \int e^{-S_{DW}[\Phi]/e^2} \mathcal{D}\Phi$

EXPECTATION VALUES OF OBSERVABLES \mathcal{O} :

$$\langle \mathcal{O} \rangle = \int e^{-S_{DW}[\Phi]/e^2} \mathcal{O}[\Phi] \mathcal{D}\Phi$$

SYMMETRIES BUILT INTO S_{DW} + FORMAL PATH INTEGRAL MANIPULATIONS
 " PROVE " THAT THESE ARE INDEPENDENT OF THE CHOICES OF g AND e .

FORMALLY COMPUTING Z_{DW} IN THE "WEAK COUPLING LIMIT" $e \rightarrow 0$,
 THE STATIONARY PHASE APPROXIMATION HAPPENS TO BE EXACT AND THE
 INTEGRAL LOCALIZES TO THE ANTI-SELF-DUAL MODULI SPACE GIVING,
 WHEN $8k - 3(1 + b_2^+(M)) = 0$, THE 0-DIMENSIONAL DONALDSON INVARIANT.

$$Z_{DW} = \delta_0(M)$$

WITTEN SIMILARLY OBTAINS THE REMAINING DONALDSON INVARIANTS
(FORMALLY) AS EXPECTATION VALUES FOR A FAMILY OF OBSERVABLES
PARAMETRIZED BY $\chi \in H_2(M; \mathbb{Z})$.

DUALITY IN WITTEN'S TQFT : 1988 - SPRING, 1994

$e \rightarrow 0$

WEAK COUPLING

ULTRAVIOLET

NON-ABELIAN

PERTURBATIVE

COMPUTABLE

DONALDSON INVARIANTS

$e \rightarrow \infty$

STRONG COUPLING

INFRARED

ABELIAN

NONPERTURBATIVE

INTRACTIBLE

?

SEIBERG-WITTEN (FALL, 1994) : EXACT SOLUTIONS IN THE INFRARED

SEIBERG-WITTEN INVARIANTS

THESE SEIBERG-WITTEN INVARIANTS ARISE IN MUCH THE SAME WAY AS
THE DONALDSON INVARIANTS FROM MODULI SPACES OF SOLUTIONS
TO THE "SEIBERG-WITTEN EQUATIONS".

MATRIX MODEL OF THE (REAL) ALGEBRA \mathbb{H} OF QUATERNIONS :

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbb{I} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathbb{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbb{K} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$q = q^1 \mathbb{1} + q^2 \mathbb{I} + q^3 \mathbb{J} + q^4 \mathbb{K}$$

$$\bar{q} = q^1 \mathbb{1} - q^2 \mathbb{I} - q^3 \mathbb{J} - q^4 \mathbb{K}$$

= CONJUGATE TRANSPOSE OF THE MATRIX q

MATRIX MODEL OF \mathbb{R}^4 : ALL 4×4 COMPLEX MATRICES OF THE FORM

$$\chi = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}$$

WITH

$$\|x\|^2 = \det \chi$$

$$\langle x, y \rangle = \frac{1}{4} (\|x+y\|^2 - \|x-y\|^2).$$

ORTHONORMAL BASIS FOR \mathbb{R}^4 :

$$E_1 = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & \mathbb{J} \\ \mathbb{J} & 0 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 0 & \mathbb{K} \\ \mathbb{K} & 0 \end{pmatrix}$$

SATISFIES

$$E_i E_j + E_j E_i = -2 \langle E_i, E_j \rangle \mathbb{1}, \quad i, j = 1, 2, 3, 4$$

REAL SUBALGEBRA OF $\mathbb{C}^{4 \times 4}$ GENERATED BY $\{E_1, E_2, E_3, E_4\}$ IS THE REAL CLIFFORD ALGEBRA OF \mathbb{R}^4 AND IS DENOTED $Cl(\mathbb{R}^4)$.

COMPLEX SUBALGEBRA OF $\mathbb{C}^{4 \times 4}$ GENERATED BY $\{E_1, E_2, E_3, E_4\}$ IS THE COMPLEX CLIFFORD ALGEBRA OF \mathbb{R}^4 AND IS DENOTED $Cl(\mathbb{R}^4) \otimes \mathbb{C}$.

BASIS : $E_0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} = \mathbb{1}$

$$E_1 = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & \mathbb{J} \\ \mathbb{J} & 0 \end{pmatrix}, E_4 = \begin{pmatrix} 0 & \mathbb{K} \\ \mathbb{K} & 0 \end{pmatrix}$$

$$E_1 E_2 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}, E_1 E_3 = \begin{pmatrix} \mathbb{J} & 0 \\ 0 & -\mathbb{J} \end{pmatrix}, E_1 E_4 = \begin{pmatrix} \mathbb{K} & 0 \\ 0 & -\mathbb{K} \end{pmatrix}$$

$$E_2 E_3 = \begin{pmatrix} \mathbb{K} & 0 \\ 0 & \mathbb{K} \end{pmatrix}, E_2 E_4 = \begin{pmatrix} -\mathbb{J} & 0 \\ 0 & -\mathbb{J} \end{pmatrix}, E_3 E_4 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix}$$

$$E_1 E_2 E_3 = \begin{pmatrix} 0 & \mathbb{K} \\ -\mathbb{K} & 0 \end{pmatrix}, E_1 E_2 E_4 = \begin{pmatrix} 0 & -\mathbb{J} \\ \mathbb{J} & 0 \end{pmatrix}$$

$$E_1 E_3 E_4 = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}, E_2 E_3 E_4 = \begin{pmatrix} 0 & -\mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$$

$$E_1 E_2 E_3 E_4 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

\mathbb{Z}_2 -GRADINGS :

$$\mathcal{C}l(\mathbb{R}^4) \cong \mathcal{C}l_0(\mathbb{R}^4) \oplus \mathcal{C}l_1(\mathbb{R}^4)$$

DIAGONAL \oplus ANTI-DIAGONAL

EVEN \oplus ODD

AND SIMILARLY FOR $\mathcal{C}l(\mathbb{R}^4) \otimes \mathbb{C}$

CENTERS : $\mathcal{Z}(\mathcal{C}l(\mathbb{R}^4)) = \text{SPAN}_{\mathbb{R}}\{E_0\} \cong \mathbb{R}$

$$\mathcal{Z}(\mathcal{C}l(\mathbb{R}^4) \otimes \mathbb{C}) = \text{SPAN}_{\mathbb{C}}\{E_0\} \cong \mathbb{C}$$

IDENTIFY $\mathcal{U}(1)$ WITH THE SUBSET

$$\mathcal{U}(1) = \{e^{\theta i} E_0 : \theta \in \mathbb{R}\}$$

OF $\mathcal{C}l(\mathbb{R}^4) \otimes \mathbb{C}$ (AND GENERALLY WRITE $e^{\theta i}$ FOR $e^{\theta i} E_0 = e^{\theta i} \mathbb{1}$).

IDENTIFY $\mathbb{R}^4 = \text{SPAN}_{\mathbb{R}} \{E_1, E_2, E_3, E_4\} \subseteq \text{Cl}(\mathbb{R}^4) \subseteq \text{Cl}(\mathbb{R}^4) \otimes \mathbb{C}$.

$$xy + yx = -2\langle x, y \rangle \mathbb{1} \quad \forall x, y \in \mathbb{R}^4$$

$\text{Cl}^{\times}(\mathbb{R}^4) = \text{MULTIPLICATIVE GROUP OF UNITS IN } \text{Cl}(\mathbb{R}^4)$

$x \in \mathbb{R}^4$ WITH $\|x\| = 1 \Rightarrow x \in \text{Cl}^{\times}(\mathbb{R}^4)$ AND $x^{-1} = -x$

$\text{PIN}(\mathbb{R}^4) = \text{SUBGROUP OF } \text{Cl}^{\times}(\mathbb{R}^4) \text{ GENERATED BY } \{x \in \mathbb{R}^4 : \|x\| = 1\} = S^3$

$$\text{SPIN}(\mathbb{R}^4) = \text{PIN}(\mathbb{R}^4) \cap \text{Cl}_0(\mathbb{R}^4)$$

(USUALLY WRITTEN $\text{SPIN}(4)$)

THEOREM : $\text{SPIN}(4) = \left\{ \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} : u_1, u_2 \in \text{SU}(2) \right\}$ IS THE UNIVERSAL DOUBLE COVER OF $\text{SO}(4)$.

$$\text{SPIN} : \text{SPIN}(4) \rightarrow \text{SO}(\mathbb{R}^4) \cong \text{SO}(4)$$

$$(\text{SPIN}(\mu))(x) = \mu x \mu^{-1}$$

COMPLEX ANALOGUE OF $\text{SPIN}(4)$:

$\text{SPIN}^{\mathbb{C}}(4) = \text{SUBGROUP OF } \text{Cl}^{\times}(\mathbb{R}^4) \otimes \mathbb{C}$ GENERATED BY $\text{SPIN}(4)$ AND $\text{U}(1)$.

$$= \left\{ e^{\theta i} \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} : \theta \in \mathbb{R}, u_1, u_2 \in \text{SU}(2) \right\}$$

$$= \left\{ \begin{pmatrix} u_+ & 0 \\ 0 & u_- \end{pmatrix} : u_{\pm} \in \text{U}(2), \det u_+ = \det u_- \right\}$$

$SPIN(4)$ AND $SPIN^c(4)$ ARE COMPACT LIE GROUPS.

$$SPIN(4) \times U(1) \longrightarrow SPIN^c(4)$$

$$(\mu, e^{\theta i}) \longrightarrow e^{\theta i} \mu$$

SURJECTIVE HOMOMORPHISM WITH KERNEL $\mathbb{Z}_2 = \pm(1, 1)$

$$SPIN^c(4) \cong SPIN(4) \times U(1) / \mathbb{Z}_2$$

SOME MAPPINGS:

1. $S : SPIN^c(4) \rightarrow U(1)$

$$\begin{aligned} S(\xi) &= S \begin{pmatrix} \nu_+ & 0 \\ 0 & \nu_- \end{pmatrix} = S \begin{pmatrix} e^{\theta i} \nu_+ & 0 \\ 0 & e^{\theta i} \nu_- \end{pmatrix} \\ &= \det \nu_+ = \det \nu_- = e^{2\theta i} \end{aligned}$$

SURJECTIVE HOMOMORPHISM WITH KERNEL $SPIN(4)$

2. $\pi : SPIN^c(4) \rightarrow SO(4)$

$$\pi(\xi) = \pi(e^{\theta i} \mu) = \text{ad}_\xi = \text{ad}_\mu \in \text{SO}(\mathbb{R}^4) \cong SO(4)$$

3. $SPIN^c : SPIN^c(4) \rightarrow SO(4) \times U(1)$

$$\begin{aligned} SPIN^c(\xi) &= SPIN^c(e^{\theta i} \mu) = (\pi(\xi), S(\xi)) \\ &= (SPIN(\mu), e^{2\theta i}) \end{aligned}$$

SURJECTIVE HOMOMORPHISM WITH KERNEL $\mathbb{Z}_2 = \pm 1$

LIE ALGEBRAS OF $SPIN(4)$ AND $SPIN^C(4)$:

8.

$$\mathfrak{spin}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2) \cong \left\{ \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} : A_1, A_2 \in \mathfrak{su}(2) \right\}$$

$$\mathfrak{spin}^C(4) \cong \mathfrak{spin}(4) \oplus \mathfrak{u}(1) \cong \left\{ \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} + t i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : A_1, A_2 \in \mathfrak{su}(2), t \in \mathbb{R} \right\}$$

$$Cl(\mathbb{R}^4) \otimes \mathbb{C} \cong \mathbb{C}^{4 \times 4} \quad \text{AND} \quad \dim_{\mathbb{C}}(Cl(\mathbb{R}^4) \otimes \mathbb{C}) = 16 \Rightarrow$$

$$Cl(\mathbb{R}^4) \otimes \mathbb{C} = \mathbb{C}^{4 \times 4} \cong \text{END}_{\mathbb{C}}(S_{\mathbb{C}})$$

$$S_{\mathbb{C}} = \mathbb{C}^4 \quad \text{WITH} \quad \langle z, w \rangle = \bar{z}^1 w^1 + \bar{z}^2 w^2 + \bar{z}^3 w^3 + \bar{z}^4 w^4$$

THE ELEMENTS OF $Cl(\mathbb{R}^4) \otimes \mathbb{C}$ (IN PARTICULAR, THOSE IN \mathbb{R}^4 , $Cl(\mathbb{R}^4)$, $SPIN(4)$ AND $SPIN^C(4)$) ALL ACT ON $S_{\mathbb{C}}$.

THIS ACTION IS CALLED CLIFFORD MULTIPLICATION

AND IS DENOTED WITH A DOT .

RESULTING REPRESENTATIONS :

1. $Cl(\mathbb{R}^4) \longrightarrow \text{END}_{\mathbb{C}}(S_{\mathbb{C}}) \quad (\text{IRREDUCIBLE})$
2. $\Delta_{\mathbb{C}} : SPIN(4) \longrightarrow \text{AUT}_{\mathbb{C}}(S_{\mathbb{C}})$

THIS IS NOT IRREDUCIBLE AS ONE SEES BY WRITING

$$S_{\mathbb{C}} \cong S_{\mathbb{C}}^{+} \oplus S_{\mathbb{C}}^{-}$$

$$\begin{pmatrix} z^1 \\ z^2 \\ z^3 \\ z^4 \end{pmatrix} = \begin{pmatrix} z^1 \\ z^2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ z^3 \\ z^4 \end{pmatrix}$$

CLIFFORD MULTIPLICATION BY ELEMENTS OF $Cl_0(\mathbb{R}^4)$ PRESERVES $S_{\mathbb{C}}^{+}$ AND $S_{\mathbb{C}}^{-}$, WHEREAS CLIFFORD MULTIPLICATION BY ELEMENTS OF $Cl_1(\mathbb{R}^4)$ REVERSES $S_{\mathbb{C}}^{+}$ AND $S_{\mathbb{C}}^{-}$. $SPIN(4) \subseteq Cl_0(\mathbb{R}^4) \Rightarrow$

$$\Delta_{\mathbb{C}} = \Delta_{\mathbb{C}}^{+} \oplus \Delta_{\mathbb{C}}^{-}$$

$$\Delta_{\mathbb{C}}^{\pm} : SPIN(4) \rightarrow SU(S_{\mathbb{C}}^{\pm})$$

$\Delta_{\mathbb{C}}^{+}$ AND $\Delta_{\mathbb{C}}^{-}$ ARE INEQUIVALENT, IRREDUCIBLE REPRESENTATIONS OF $SPIN(4)$ ON \mathbb{C}^2 .

3.

$$\hat{\Delta}_{\mathbb{C}} : SPIN^{\mathbb{C}}(4) \rightarrow AUT_{\mathbb{C}}(S_{\mathbb{C}})$$

$$\hat{\Delta}_{\mathbb{C}} = \hat{\Delta}_{\mathbb{C}}^{+} \oplus \hat{\Delta}_{\mathbb{C}}^{-}$$

$$\hat{\Delta}_{\mathbb{C}}^{\pm} : SPIN^{\mathbb{C}}(4) \rightarrow U(S_{\mathbb{C}}^{\pm})$$

A FEW MORE IMPORTANT ACTIONS :

4. $\mathbb{R}^4 \cong Cl_0(\mathbb{R}^4)$ ACTS BY CLIFFORD MULTIPLICATION ON $S_{\mathbb{C}}$, REVERSING $S_{\mathbb{C}}^+$ AND $S_{\mathbb{C}}^-$.

$$\mathbb{R}^4 \cdot () : S_{\mathbb{C}}^{\pm} \rightarrow S_{\mathbb{C}}^{\mp}$$

5. $SPIN^{\mathbb{C}}(4)$ ACTS ON \mathbb{C} VIA $S : SPIN^{\mathbb{C}}(4) \rightarrow U(1)$ AND COMPLEX MULTIPLICATION.

$$\xi \cdot z = (e^{\theta i} \mu) \cdot z = S(\xi)z = e^{2\theta i} z$$

6. $SOL(4)$ ACTS ON $Cl(\mathbb{R}^4)$:

$SPIN(4)$ ACTS ON $Cl(\mathbb{R}^4)$ BY CONJUGATION

$$\mu \cdot p = \mu p \mu^{-1}$$

BUT $(-\mu) p (-\mu)^{-1} = \mu p \mu^{-1}$ SO THIS DESCENDS TO AN ACTION OF $SOL(4) \cong SPIN(4)/\mathbb{Z}_2$ ON $Cl(\mathbb{R}^4)$.

7. COMPLEX-VALUED 2-FORMS $\Omega^2(\mathbb{R}^4)$ ON \mathbb{R}^4 ACT ON $S_{\mathbb{C}}$ VIA THE LINEAR ISOMORPHISM

$$\rho : \Omega^2(\mathbb{R}^4) \hookrightarrow Cl_0(\mathbb{R}^4) \otimes \mathbb{C}$$

$$\rho(\eta) = \rho\left(\sum_{i < j} \eta_{ij} E^i \wedge E^j\right) = \sum_{i < j} \eta_{ij} E_i E_j =$$

$$\left(\begin{array}{cc} (\eta_{12} + \eta_{34})I + (\eta_{13} - \eta_{24})J + (\eta_{14} + \eta_{23})K & 0 \\ 0 & (-\eta_{12} + \eta_{34})I + (-\eta_{13} - \eta_{24})J + (-\eta_{14} + \eta_{23})K \end{array} \right)$$

η REAL-VALUED $\Rightarrow \rho(\eta)$ SKEW-HERMITIAN

η $\text{Im } \mathbb{C}$ -VALUED $\Rightarrow \rho(\eta)$ HERMITIAN

$S_{\mathbb{C}}^{+}$ AND $S_{\mathbb{C}}^{-}$ INVARIANT UNDER ANY $\rho(\eta)$

$$\rho^{\pm}(\eta) = \rho(\eta) | S_{\mathbb{C}}^{\pm}$$

E.G., SUPPRESSING THE TWO ZERO ENTRIES

$$\rho^{+}(\eta) = (\eta_{12} + \eta_{34})I + (\eta_{13} + \eta_{42})J + (\eta_{14} + \eta_{23})K$$

$$\rho^{\pm} : \Omega^2(\mathbb{R}^4) \rightarrow \text{END}_{\mathbb{C}}(S_{\mathbb{C}}^{\pm})$$

$$\Omega^2(\mathbb{R}^4) \cong \Omega_{+}^2(\mathbb{R}^4) \oplus \Omega_{-}^2(\mathbb{R}^4)$$

$$\rho^{\pm} | \Omega_{\pm}^2(\mathbb{R}^4) : \Omega_{\pm}^2(\mathbb{R}^4) \longrightarrow \text{END}_{\mathbb{C}}(S_{\mathbb{C}}^{\pm})$$

(COMPLEX LINEAR ISOMORPHISM)

INVERSES :

$$\sigma^{\pm} : \text{END}_{\mathbb{C}}(S_{\mathbb{C}}^{\pm}) \rightarrow \Omega_{\pm}^2(\mathbb{R}^4)$$

EXAMPLE : LET $\psi = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} \in S_{\mathbb{C}}^{+}$. GIVES AN ENDDORPHISM OF $S_{\mathbb{C}}^{+}$:

$$\psi \otimes \psi^{\dagger} = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} (\bar{\psi}^1 \quad \bar{\psi}^2) = \begin{pmatrix} |\psi^1|^2 & \psi^1 \bar{\psi}^2 \\ \bar{\psi}^1 \psi^2 & |\psi^2|^2 \end{pmatrix}$$

TRACE FREE PART IS

$$\begin{aligned}
 (\psi \otimes \psi^*)_{\circ} &= \psi \otimes \psi^* - \frac{1}{2} \text{tr}(\psi \otimes \psi^*) \mathbb{1} \\
 &= \begin{pmatrix} \frac{1}{2}(|\psi^1|^2 - |\psi^2|^2) & \psi^1 \bar{\psi}^2 \\ \bar{\psi}^1 \psi^2 & \frac{1}{2}(|\psi^2|^2 - |\psi^1|^2) \end{pmatrix}
 \end{aligned}$$

ONE VERIFIES THAT

$$\begin{aligned}
 \sigma^+(\psi \otimes \psi^*)_{\circ} &= -\frac{1}{4} \left\{ (\psi^* \mathbb{I} \psi)(E^1 \wedge E^2 + E^3 \wedge E^4) + \right. \\
 &\quad (\psi^* \mathbb{J} \psi)(E^1 \wedge E^3 + E^4 \wedge E^2) + \\
 &\quad \left. (\psi^* \mathbb{K} \psi)(E^1 \wedge E^4 + E^2 \wedge E^3) \right\}
 \end{aligned}$$

NOW GLOBALIZE THESE ALGEBRAIC CONSTRUCTIONS TO BUNDLES OVER

M = COMPACT, SIMPLY CONNECTED, ORIENTED
SMOOTH 4-MANIFOLD + RESTRICTIONS ON $b_2^+(M)$

ANY CHOICE OF RIEMANNIAN METRIC g FOR M GIVES

$$\text{SO}(4) \hookrightarrow F_{\text{SO}}(M) \xrightarrow{\pi_{\text{SO}}} M$$

WE FIX THE LEVI-CIVITA CONNECTION ω_{LC} ON THIS FRAME BUNDLE.

NOTE : FOR MORE DETAILS ON THE CONSTRUCTIONS WHICH FOLLOW, SEE APPENDIX 15.

A SPIN^c STRUCTURE \mathcal{L} FOR M CONSISTS OF A $\text{SPIN}^c(4)$ -BUNDLE

$$\text{SPIN}^c(4) \hookrightarrow S^c(M) \xrightarrow{\pi_{S^c}} M$$

OVER M AND A SMOOTH MAP $\Lambda : S^c(M) \rightarrow F_{SO}(M)$ SATISFYING

$$\begin{array}{ccc} S^c(M) & \xrightarrow{\Lambda} & F_{SO}(M) \\ \pi_{S^c} \searrow & & \swarrow \pi_{SO} \\ & M & \end{array} \quad \pi_{SO} \circ \Lambda = \pi_{S^c}$$

AND

$$\Lambda(p \cdot \xi) = \Lambda(p) \cdot \pi(\xi)$$

WHERE $\pi : \text{SPIN}^c(4) \rightarrow \text{SO}(4)$ IS MAP #2 ON P. 7.

NOTE: ANY COMPACT, ORIENTED 4-MANIFOLD ADMITS A SPIN^c STRUCTURE FOR ANY CHOICE OF g (HIRZEBRUCH-HOPF) AND, IN THE SIMPLY CONNECTED CASE, EQUIVALENCE CLASSES OF SPIN^c STRUCTURES ON M ARE IN ONE-TO-ONE CORRESPONDENCE WITH THE ELEMENTS OF $H^2(M; \mathbb{Z})$ WHOSE MOD 2 REDUCTION IS THE SECOND STIEFEL-WHITNEY CLASS $w_2(M)$ OF M .

GIVEN A SPIN^c STRUCTURE WE HAVE THE FOLLOWING ASSOCIATED BUNDLES:

$$S(\mathcal{L}) = S^c(M) \times_{\hat{A}_c} S_c \quad (\text{SPINOR BUNDLE})$$

$$S^\pm(\mathcal{L}) = S^c(M) \times_{\hat{A}_c^\pm} S_c^\pm \quad (\text{POSITIVE AND NEGATIVE SPINOR BUNDLES})$$

$$L(\mathcal{L}) = S^c(M) \times_S \mathbb{C} \quad (\text{DETERMINANT LINE BUNDLE})$$

SOME OTHER RELEVANT BUNDLES :

$$U(1) \hookrightarrow L^0(\mathcal{L}) \xrightarrow{\pi_{L^0}} M \quad (\text{ORIENTED, ORTHONORMAL FRAME BUNDLE FOR SOME FIBER METRIC ON } L(\mathcal{L}))$$

$$Cl(M) = F_{SO}(M) \times_{SO(4)} Cl(4) \quad (\text{CLIFFORD BUNDLE})$$

$$Cl(M) \otimes \mathbb{C} = F_{SO}(M) \times_{SO(4)} (Cl(4) \otimes \mathbb{C}) \quad (\text{COMPLEXIFIED CLIFFORD BUNDLE})$$

NOTE : CAN SHOW THAT $w_2(M) = c_1(L^0(\mathcal{L})) \pmod 2$ AND, CONVERSELY, GIVEN A PRINCIPAL $U(1)$ -BUNDLE L^0 OVER M WITH $w_2(M) = c_1(L^0) \pmod 2$, THERE IS A $Spin^c$ STRUCTURE \mathcal{L} FOR M WITH $L^0(\mathcal{L}) = L^0$.

GLOBAL VERSION OF $\sigma^+ : \text{END}_0(S_c^+) \rightarrow \Omega_+^2(\mathbb{R}^4) :$

$$\sigma^+ : T^*(\text{END}_0(S^+(\mathcal{L}))) \rightarrow \Omega_+^2(M)$$

IDENTIFIES SECTIONS OF THE TRACE FREE ENDOMORPHISM BUNDLE OF THE POSITIVE SPINOR BUNDLE WITH SELF-DUAL 2-FORMS ON M .

FIELD CONTENT OF SEIBERG - WITTEN THEORY :

GAUGE FIELD : CONNECTION A ON $U(1) \hookrightarrow L^0(\mathcal{L}) \rightarrow M$

POSITIVE SPINOR FIELD : SECTION $\psi \in T(S^+(\mathcal{L}))$

CURVATURE OF A IS THE $U(1)$ -VALUED 2-FORM dA ON $L^0(\mathcal{L})$, BUT $U(1)$ IS ABELIAN SO THIS IS UNIQUELY DETERMINED BY A $U(1)$ -VALUED 2-FORM F_A ON M (GAUGE POTENTIAL)

$$\psi \rightarrow (\psi \otimes \psi^*) \in T(\text{END}_0(S^+(\mathcal{L}))) \rightarrow \sigma^+((\psi \otimes \psi^*) \circ) \in \Omega_+^2(M)$$

$$1^{\text{ST}} \text{ SEIBERG-WITTEN EQUATION : } F_A^+ = \sigma^+((\psi \otimes \psi^*) \circ)$$

FOR THE 2ND SEIBERG-WITTEN EQUATION WE INTRODUCE A DIRAC OPERATOR ON SPINOR FIELDS :

$\text{SPIN}^c(4)$ DOUBLE COVERS $\text{SO}(4) \times U(1)$ BY THE MAP $\text{SPIN}^c = (\pi, \delta)$ SO $S^c(M)$ DOUBLE COVERS THE FIBER PRODUCT $F_{\text{SO}}(M) \times L^0(\mathcal{L})$. DENOTE THE MAP BY SPIN^c ALSO.

$$\begin{array}{ccccc} \text{SPIN}^c(4) & \hookrightarrow & S^c(M) & \longrightarrow & M \\ & & \downarrow \text{SPIN}^c & & \\ \text{SO}(4) \times U(1) & \hookrightarrow & F_{\text{SO}}(M) \times L^0(\mathcal{L}) & \longrightarrow & M \end{array}$$

THIS GIVES

$$\begin{array}{ccc}
 \text{SPIN}^c(4) \hookrightarrow S^c(M) \rightarrow M & & \text{SPIN}^c(4) \hookrightarrow S^c(M) \rightarrow M \\
 \downarrow \text{Pr}_F \circ \text{SPIN}^c & & \downarrow \text{Pr}_{L^0} \circ \text{SPIN}^c \\
 \text{SO}(4) \hookrightarrow F_{\text{SO}}(M) \rightarrow M & & \mathcal{U}(1) \hookrightarrow L^0(\mathcal{L}) \rightarrow M
 \end{array}$$

FIXED CONNECTION : ω_{LC}

SW CONNECTION : A

SPIN^c CONNECTION

$$\omega_A = (\text{SPIN}^c)^* (\text{Pr}_F^* \omega_{LC} + \text{Pr}_{L^0}^* A)$$

ω_A GIVES COVARIANT DERIVATIVES ON SECTIONS OF ASSOCIATED BUNDLES, E.G.,

$$\nabla_A : T(S(\mathcal{L})) \rightarrow \Omega^1(M) \otimes T(S(\mathcal{L}))$$

$$\psi \in T(S(\mathcal{L})) \rightarrow \nabla_A \psi \in \Omega^1(M) \otimes T(S(\mathcal{L}))$$

$$\nabla_A \psi(V) \in T(S(\mathcal{L}))$$

FOR EVERY VECTOR FIELD V ON M

LET $\{E_1, E_2, E_3, E_4\}$ BE A LOCAL ORIENTED, ORTHONORMAL FRAME FIELD ON M , I.E., A SECTION OF $F_{\text{SO}}(M)$.

EACH E_i CAN BE REGARDED EITHER AS A VECTOR FIELD ON M (SO $\nabla_A \psi(E_i)$ MAKES SENSE) OR AS A SECTION OF THE CLIFFORD BUNDLE $Cl(M)$ WHICH THEREFORE ACTS BY CLIFFORD MULTIPLICATION ON SECTIONS OF $S(\mathcal{L})$.

$$\tilde{\nabla}_A \psi = \sum_{i=1}^4 E_i \cdot \nabla_A \psi(E_i)$$

$$S(L) = S^+(L) \oplus S^-(L)$$

CLIFFORD MULTIPLICATION BY E_i INTERCHANGES $S^\pm(L)$. RESTRICTING \tilde{D}_A TO $S^\pm(L)$ THEREFORE GIVES

$$D_A : T(S^+(L)) \rightarrow T(S^-(L))$$

$$D_A^* : T(S^-(L)) \rightarrow T(S^+(L))$$

D_A IS OUR DIRAC OPERATOR (AND D_A^* IS ITS ADJOINT RELATIVE TO THE NATURAL INNER PRODUCTS ON SPACES OF SECTIONS).

2ND SEIBERG-WITTEN EQUATION : $D_A \psi = 0$

SEIBERG-WITTEN : GIVEN M , SELECT RIEMANNIAN METRIC g AND $SU(2)$ STRUCTURE L FOR THE CORRESPONDING $F_{SU(2)}(M)$. SW CONFIGURATION SPACE IS

$$\mathcal{A}(L) = \{ (A, \psi) : A \text{ IS A CONNECTION ON } U(1) \hookrightarrow L^0(L) \rightarrow M \text{ AND } \psi \in T(S^+(L)) \text{ IS A POSITIVE SPINOR FIELD ON } M \}$$

$(A, \psi) \in \mathcal{A}(L)$ IS A SW MONOPOLE IF IT SATISFIES

(SW1) $F_A^+ = \sigma^+(\psi \otimes \psi^*)$ (CURVATURE EQUATION)

(SW2) $D_A \psi = 0$ (DIRAC EQUATION)

NOTE : TO SEE WHAT THESE EQUATIONS LOOK LIKE ON \mathbb{R}^4 , SOME EXPLICIT SOLUTIONS, AND SOME BOUNDEDNESS RESULTS, CONSULT APPENDIX 16.

SEIBERG-WITTEN GAUGE GROUP $\mathcal{G}(\mathcal{L})$:

1. ALL AUTOMORPHISMS σ OF $S^c(M)$
THAT COVER THE IDENTITY ON $F_{SO}(M)$

$$Pr_F \circ SPIN^c \circ \sigma = Pr_F \circ SPIN^c$$

$$SPIN^c(4) \hookrightarrow S^c(M) \rightarrow M$$

$$\downarrow$$

$$SO(4) \hookrightarrow F_{SO}(M) \rightarrow M$$

2. $C^\infty(M, \mathcal{U}(1))$

$$\gamma \in C^\infty(M, \mathcal{U}(1)) \iff \sigma_\gamma : S^c(M) \rightarrow S^c(M)$$

$$\sigma_\gamma(p) = p \cdot \gamma(\pi_{S^c}^{-1}(p))$$

ACTION OF $\mathcal{G}(\mathcal{L})$ ON $(A, \psi) \in \mathcal{A}(\mathcal{L})$:

CONNECTION A : σ_γ INDUCES AN AUTOMORPHISM OF $L^0(\mathcal{L})$

$$\begin{array}{ccc} S^c(M) & \xrightarrow{\sigma_\gamma} & S^c(M) \\ \downarrow & & \downarrow \\ L^0(\mathcal{L}) & \xrightarrow{\sigma'_\gamma} & L^0(\mathcal{L}) \end{array}$$

$$A \cdot \gamma = (\sigma'_\gamma)^* A$$

SPINOR FIELD ψ : IDENTIFY THE SECTION ψ OF $S^+(\mathcal{L})$ WITH
AN EQUIVARIANT MAP $\psi : S^c(M) \rightarrow S^c_+$

$$\psi \cdot \gamma = \sigma_\gamma^* \psi = \psi \circ \sigma_\gamma$$

$$(A, \psi) \cdot \gamma = (A \cdot \gamma, \psi \cdot \gamma) = ((\sigma'_\gamma)^* A, \sigma_\gamma^* \psi)$$

THEOREM: THE ACTION OF $\mathcal{G}(\mathcal{L})$ ON THE SW CONFIGURATION SPACE $\mathcal{A}(\mathcal{L})$ CARRIES SOLUTIONS TO (SW) ONTO OTHER SOLUTIONS TO (SW), I.E., IF $(A, \psi) \in \mathcal{A}(\mathcal{L})$ SATISFIES

$$\begin{cases} F_A^+ = \sigma^+((\psi \otimes \psi^*)_0) \\ \not{D}_A \psi = 0 \end{cases}$$

AND $\gamma \in C^\infty(M, U(1))$, THEN $(A, \psi) \cdot \gamma = (A \cdot \gamma, \psi \cdot \gamma)$ SATISFIES

$$\begin{cases} F_{A \cdot \gamma}^+ = \sigma^+(((\psi \cdot \gamma) \otimes (\psi \cdot \gamma)^*)_0) \\ \not{D}_{A \cdot \gamma}(\psi \cdot \gamma) = 0 \end{cases}$$

TWO MODULI SPACES :

$$\mathcal{B}(\mathcal{L}) = \mathcal{A}(\mathcal{L}) / \mathcal{G}(\mathcal{L})$$

$$\mathcal{M}(\mathcal{L}) = \{ (A, \psi) \in \mathcal{A}(\mathcal{L}) : F_A^+ = \sigma^+((\psi \otimes \psi^*)_0), \not{D}_A \psi = 0 \} / \mathcal{G}(\mathcal{L})$$

ASSUMING THAT EACH IS REPLACED BY AN "APPROPRIATE SOBOLEV COMPLETION" THE ANALYSIS IS VERY SIMILAR TO THAT OF THE DONALDSON MODULI SPACES.

WE WILL MENTION ONLY A FEW POINTS AT WHICH THERE ARE DIFFERENCES.

1. THE REDUCIBLE ELEMENTS OF $\mathcal{X}(\mathcal{L})$ ARE EASILY IDENTIFIED.

LEMMA: AN ELEMENT (A, ψ) OF $\mathcal{X}(\mathcal{L})$ IS LEFT FIXED BY SOME NON-IDENTITY ELEMENT γ OF $\mathcal{Y}(\mathcal{L})$ IF AND ONLY IF $\psi \equiv 0$ AND, IN THIS CASE, $\gamma: M \rightarrow U(1)$ MUST BE A CONSTANT MAP.

2. ASSOCIATED WITH ANY SOLUTION (A, ψ) TO (SW) IS A FUNDAMENTAL ELLIPTIC COMPLEX $\mathcal{E}(A, \psi)$ WITH FINITE DIMENSIONAL COHOMOLOGY GROUPS ADMITTING INTERPRETATIONS ANALOGOUS TO THOSE IN DONALDSON THEORY.

$$H^0(A, \psi) = 0 \iff (A, \psi) \text{ IRREDUCIBLE} \iff \psi \neq 0$$

$$H^1(A, \psi) = \text{FORMAL TANGENT SPACE TO } \mathcal{M}(\mathcal{L}) \text{ AT } [A, \psi]$$

$$H^2(A, \psi) = \text{OBSTRUCTION SPACE} \quad (H^2(A, \psi) = 0 \iff \text{IMPLICIT}$$

FUNCTION THEOREM GIVES A LOCAL MANIFOLD STRUCTURE FOR THE SET OF SOLUTIONS TO (SW) NEAR (A, ψ) OF DIMENSION

$$\dim H^1(A, \psi))$$

IF $H^0(A, \psi) = 0$ AND $H^2(A, \psi) = 0$ THEN THE ATIYAH-SINGER THEOREM GIVES

$$\frac{1}{4} (c_1(L^0(\mathcal{L}))^2 - 2\tau(B) - 3\sigma(B))$$

AS THE DIMENSION OF $\mathcal{M}(\mathcal{L})$ NEAR $[A, \psi]$.

$H^0(A, \psi) = 0$ AND $H^2(A, \psi) = 0$ ARE THE GENERIC SITUATION, BUT NOW "GENERIC" MEANS SOMETHING DIFFERENT THAN IT DID IN DONALDSON THEORY.

(A) AS IN DONALDSON THEORY, IF $b_2^+(M) > 0$ THERE IS A DENSE G_δ SET $\text{Gen}(\mathcal{Q})$ OF RIEMANNIAN METRICS ON M SUCH THAT

FOR ANY $g \in \text{Gen}(\mathcal{Q})$ AND ANY CORRESPONDING Spin^c STRUCTURE \mathcal{L} ANY SOLUTION (A, ψ) TO (SW) HAS

$$H^0(A, \psi) = 0.$$

IF $b_2^+(M) > 1$ THIS IS TRUE FOR A GENERIC PATH OF RIEMANNIAN METRICS.

(B) THERE IS NO ANALOGOUS "GENERIC METRICS THEOREM" FOR $H^2(A, \psi)$. IN THIS CASE ONE MUST PERTURB NOT THE METRIC, BUT THE EQUATIONS THEMSELVES.

FIX g AND \mathcal{L} . FOR ANY $\eta \in \Omega_+^2(M, \mathbb{C})$, THE η -PERTURBED SW EQUATIONS ARE

$$F_A^+ = \sigma^+(\psi \otimes \psi^*) + \eta$$

$$D_A \psi = 0$$

EVERYTHING WE HAVE SAID ABOUT (SW) IS TRUE OF (PSW). ALSO

THERE IS A DENSE G_δ SET $\text{Gen}(\Omega_+^2)$ IN $\Omega_+^2(M, \mathbb{C})$ SUCH THAT FOR ANY $\eta \in \text{Gen}(\Omega_+^2)$ EVERY SOLUTION (A, ψ) TO (PSW) HAS

$$H^2(A, \psi) = 0.$$

THEOREM: IF $b_2^+(M) > 0$, $g \in \text{Gen}(\mathcal{R})$, \mathcal{L} IS ANY CORRESPONDING Spin^c STRUCTURE AND $\eta \in \text{Gen}(\Omega_+^2)$, THEN THE MODULI SPACE $\mathcal{M}(\mathcal{L}, \eta)$ OF SOLUTIONS TO THE η -PERTURBED SEIBERG-WITTEN EQUATIONS IS EITHER EMPTY OR A SMOOTH MANIFOLD OF DIMENSION

$$\frac{1}{4} (c_1(L^0(\mathcal{L})))^2 - 2\tau(M) - 3\sigma(M).$$

A CHOICE OF ORIENTATION FOR $H_+^2(M; \mathbb{R})$ CANONICALLY ORIENTS $\mathcal{M}(\mathcal{L}, \eta)$.

NOTE: IF $b_2^+(M) > 1$ THERE IS ALSO A "COBORDISM" RESULT FOR GENERIC PATHS OF METRICS AND PERTURBATIONS ANALOGOUS TO THAT IN DONALDSON THEORY (ONE CAN EFFECTIVELY "FIX" THE Spin^c STRUCTURE ALONG A PATH OF METRICS BECAUSE THE FRAME BUNDLES ARE ALL NATURALLY ISOMORPHIC).

3. THE SEIBERG-WITTEN MODULI SPACES ARE ALWAYS COMPACT!

THIS IS THE MOST SIGNIFICANT DIFFERENCE BETWEEN DONALDSON AND SEIBERG-WITTEN THEORY.

ASD MODULI SPACES ARE NOT COMPACT BECAUSE THE ASD-EQUATIONS ARE CONFORMALLY INVARIANT IN DIMENSION 4.

BY CONTRAST, SEIBERG-WITTEN SOLUTIONS SATISFY A PRIORI UNIFORM BOUNDS, E.G., IF (A, ψ) SATISFIES (SW), THEN, FOR EVERY $x \in M$,

$$\|\psi(x)\|^2 \leq K(B) = \max \left\{ -\frac{1}{2} \chi(x_0) : x_0 \in M \right\}.$$

NOW FIX A GENERIC METRIC g , GENERIC PERTURBATION η AND ORIENTATION FOR $H_+^2(M; \mathbb{R})$. SUPPOSE THERE IS A CORRESPONDING Spin^c STRUCTURE \mathcal{L} FOR WHICH

$$c_1(L^0(\mathcal{L}))^2 = 2\tau(M) + 3\sigma(M).$$

THE MODULI SPACE IS EITHER EMPTY OR A FINITE SET OF ISOLATED POINTS, EACH EQUIPPED WITH A SIGN ± 1 .

DEFINE THE 0-DIMENSIONAL SEIBERG-WITTEN INVARIANT

$$SW_0(M, \mathcal{L})$$

OF M ASSOCIATED WITH \mathcal{L} TO BE ZERO IN THE FIRST CASE AND THE SUM OF THE SIGNS IN THE SECOND.

NOTE : WHEN $b_2^+(M) > 1$ A COBORDISM ARGUMENT SHOWS THAT $SW_0(M, \mathcal{L})$ IS INDEPENDENT OF g AND η AND IS, IN FACT, A DIFFERENTIAL TOPOLOGICAL INVARIANT OF M .

ANOTHER CONSEQUENCE OF THE A PRIORI BOUNDS :

IF g AND η ARE FIXED, THEN THERE ARE ONLY FINITELY MANY (EQUIVALENCE CLASSES OF) Spin^c STRUCTURES \mathcal{L} ON B THAT SATISFY

1. $c_1(L^0(\mathcal{L}))^2 - 2\tau(M) - 3\sigma(M) \geq 0$, AND

2. $\eta(\mathcal{L}, \eta) \neq \emptyset$.

WHEN THE DIMENSION OF THE MODULI SPACE IS NOT ZERO, SEIBERG-WITTEN INVARIANTS ARE DEFINED BY INTEGRATING A CERTAIN COHOMOLOGY CLASS OVER IT:

FIX $p_0 \in M$. LET $\mathcal{G}_0(\mathcal{L}) =$ SUBGROUP OF $\mathcal{G}(\mathcal{L})$
OF THOSE ELEMENTS THAT
ACT TRIVIAALLY ON FIBER
OF $S^c(M)$ ABOVE p_0 .

$\mathcal{M}_0(\mathcal{L}, \eta) =$ SW-SOLUTIONS MODULO $\mathcal{G}_0(\mathcal{L})$

NATURAL PROJECTION : $\mathcal{M}_0(\mathcal{L}, \eta) \rightarrow \mathcal{M}(\mathcal{L}, \eta)$

U(1)-ACTION ON $\mathcal{M}_0(\mathcal{L}, \eta)$: $[A, \psi]_0 \cdot e^{\theta i} = [A, e^{i\theta} \psi]_0$

PRINCIPAL U(1)-BUNDLE :

$$U(1) \hookrightarrow \mathcal{M}_0(\mathcal{L}, \eta) \rightarrow \mathcal{M}(\mathcal{L}, \eta)$$

1ST CHERN CLASS :

$$c_1(\mathcal{M}_0(\mathcal{L}, \eta)) \in H^2(\mathcal{M}(\mathcal{L}, \eta); \mathbb{Z})$$

$$SW(M, \mathcal{L}) = \int_{\mathcal{M}(\mathcal{L}, \eta)} c_1(\mathcal{M}_0(\mathcal{L}, \eta))^{d_{\mathcal{L}}}$$

$$\text{WHERE } 2d_{\mathcal{L}} = \frac{1}{4} (c_1(L^0(\mathcal{L}))^2 - 2\chi(M) - 3\sigma(M)).$$

EMPIRICAL EVIDENCE SUGGESTS THAT $SW(M, \mathcal{L}) \neq 0$ ONLY FOR THOSE \mathcal{L} SATISFYING $c_1(L^0(\mathcal{L}))^2 - 2\chi(M) - 3\sigma(M) = 0$.

THE ELEMENTS $c, (L^0(\mathcal{L})) \in H^2(M; \mathbb{Z})$ CORRESPONDING TO THOSE \mathcal{L} FOR WHICH $c, (L^0(\mathcal{L}))^2 - 2\tau(M) - 3\sigma(M) = 0$ ARE CALLED SW-BASIC CLASSES.

M IS SAID TO BE OF SW SIMPLE TYPE IF NONZERO SW-INVARIANTS ARISE ONLY FROM 0-DIMENSIONAL MODULI SPACES, I. E., IF

$$SW(M, \mathcal{L}) \neq 0 \Rightarrow c, (L^0(\mathcal{L}))^2 - 2\tau(M) - 3\sigma(M) = 0.$$

THE WITTEN CONJECTURE

1. M IS OF D-SIMPLE TYPE $\Leftrightarrow M$ IS OF SW-SIMPLE TYPE

2. D-BASIC CLASSES COINCIDE WITH SW-BASIC CLASSES

$$\begin{aligned} 3. \mathcal{D}_M(x) &= \exp(Q_M(x, x)/2) \sum_{r=1}^3 a_r \exp(K_r(x)) \\ &= \exp(Q_M(x, x)/2) \sum_{\mathcal{L} \in \Lambda} 2^{m(M)} SW_0(M, \mathcal{L}) \exp(c, (L^0(\mathcal{L}))(x)) \end{aligned}$$

WHERE Λ IS THE SET OF ALL (EQUIVALENCE CLASSES OF)

SPIN^c-STRUCTURES FOR WHICH $c, (L^0(\mathcal{L}))^2 - 2\tau(M) - 3\sigma(M) = 0$ AND

$$m(M) = 2 + \frac{1}{4} (7\tau(M) + 11\sigma(M)).$$

NOTE : FOR A BRIEF TOUR OF HOW WITTEN ARRIVED AT THE CONJECTURE, SEE APPENDIX 17 (BY MATILDE PARCOLLI).

ATTITUDES ONE MIGHT ADOPT TOWARD THIS CONJECTURE :

1. IT SHOULD BE RIGOROUSLY PROVED

- PIDSTRIGATCH AND TYURIN
- FEEHAN AND LENESE (PARTIAL RESULTS HAVE ALREADY PAID DIVIDENDS)
- VAJAC

2. RIGOROUSLY TRUE OR NOT, THE SW INVARIANTS PROVIDE A MUCH MORE TRACTABLE TOOL FOR THE STUDY OF 4-MANIFOLDS, SO IT MAKES GOOD, PRACTICAL SENSE TO ABANDON THE ASD EQUATIONS IN FAVOR OF THE SW EQUATIONS.

- ESSENTIALLY EVERYONE ELSE

3. IF PHYSICS IS TRULY CAPABLE OF CASTING SUCH A PENETRATING LIGHT UPON MATHEMATICS AT THE DEEPEST LEVELS, MATHEMATICIANS WILL WANT TO TAKE HEED AND TURN THEIR ATTENTION ONCE AGAIN TO THEIR HISTORICAL ROOTS IN PHYSICS.

- ATIYAH