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Rigidity and Geometry of Microstructures

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CHAPTER 1

Introduction

The concept of a differential and the corresponding calculus certainly present one, if not the central idea of mathematical analysis. Since their first formulation these methods how to decompose and approximate nonlinear objects by linear and affine pieces have been refined and generalized considerably. Consequently, this "affine analysis", in particular in its applications to mappings between higherdimensional spaces, is of fundamental importance not just for mathematics itself but also for physics and engeneering. This is convincingly demonstrated by the fact that everybody who studies one of these subjects will be quite soon confrontated with the differential calculus.

In this light, the inverse problem of an "affine synthesis" seems surprisingly neglected. Whereas the fundamental theorem of calculus answers this problem for the case of one variable completely, in higher dimenions much less is known. In particular, the question which given linear or affine pieces can be combined into one single mapping as well as the more precise question concerning possible distributions for the gradients of maps between Euclidean spaces remains essentially open.

For a systematic approach to this kind of problems the following consideration, which is at the heart of the work presented here, seems important. Which finite sets of matrices can be the image of a gradient and their individual members be combined together? In other words, what geometric patterns are produced by the mappings corresponding to these gradients.

These questions are in a very interesting and stimulating way linked to recent investigations of modern materials which posses a very rich internal microstructure. The patterns observed experimentally in these alloys have a large similarity with the already mentioned mathematical ones. In fact, also they describe a global deformation which results out of a phase transition and which uses only certain deformation gradients that are specific for the material given.

As shown in the pioneering work by J.M. Ball and R.D. James ([**6**], [**7**]) and in the paper ([**21**]) by M. Chipot and D. Kinderlehrer, these gradients minimize a strongly nonconvex energy density. Therefore, the underlying variational problem cannot provide the good control about the minimizing map that one usually gets from the elliptic partial differential equations studied in classical mechanics. In many cases, not even the existence of a minimizer is ensured. An understanding of the arising microstructures, therefore, has to be derived from the above stated more general mathematical question.

The description of the in both cases arising very complex and highly nonsmooth structures requires tools that are closely related to L. Tartar's ideas (see [**95**]) how to capture oscillation phenomena in nonlinear partial differential equations. This links our work to the broader framework of the nonconvex calculus of variations, where one studies not only the exact distributions of the individual gradients but introduces also a suitable compactification of them (which we call approximate distributions) in the space of all probabilities on the matrices $\mathbb{M}^{n\times m}$.

In control theory, these concepts were invented by L.C. Young already in the thirties, see [**101**] and [**102**]. For the vectorial case the notion of quasiconvexity, which was introduced in the fundamental contributions by C.B. Morrey ([**60**] and [**61**]), is the decisive one. It allows to obtain compactness and separation results that characterize this non-scalar situation.

The impact of this concept also in other parts of mathematics, as e.g. harmonic or quasiconformal analysis, can be seen from the papers [**2**],[**10**] und [**91**].

Unfortunately, the usefulness of quasiconvexity is limited by the fact, that we are still lacking a fundamental understanding of this notion as well as efficiently managable characterizations of quasiconvex functions. Therefore, other convexity notions that can serve as good approximations of quasiconvexity are studied intensively. Here, rank-one convexity has to be mentioned in particular. It shows up, if we restrict the considerations to gradients of so-called laminations. These are mappings which are constructed in quite specific way which turns out to cause that rank-one convexity has on the space of all matrices a much more local and even geometrical nature than this is the case for quasiconvexity. Before rank-one convexity was understood more from the analytical point of view in terms of the Legendre-Hadamard condition (which is obtained from quasiconvexity by linearization), we will emphasize this new geometric aspect in Chapter 4.

In the presented work we therefore start some more systematic investigation into the geometric properties of rank-one convexity. Led by this understanding, we found new stability results, which of course also apply to some extent to quasiconvexity. This, in turn enabled us to answer a question of J.M. Ball and R.D. James about some characteristic feature of finite sets of gradients which was formulated in [**5**]. In order to understand properly how complex examples of this kind have to be, we derive on the one hand new existence results for partial differential inclusions and establish on the other hand regularity results for the hyperbolic Monge-Ampère equation. In this way we are finally able to show a small but definite difference between sets on which exact and approximative gradient distributions can live.

1. Rank-one connections and Rigidity

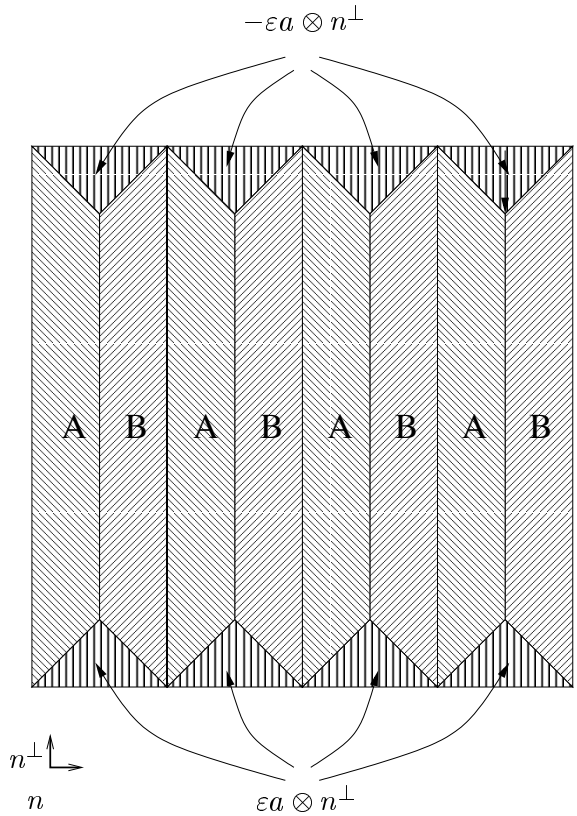
Here we briefly present the known results and constraints concerning the domain of exact and approximate gradient distributions, in order to clarify our goal and the way in which we are going to approach it. A very nice and broad introduction to this subject can be found in [**64**], therefore we limit ourself to the topics essential for our specific situation.

The main question is "which macroscopic behaviour" can be realized by a map respecting given restrictions on its "microscopical structure". Here, "microstructure" is understood in terms of the gradients used, whereas "macroscopical behaviour" refers to the boundary values of the map. At the begin, let us consider the simple but in the same moment very fundamental example of two compatible gradients. In this case, two rank-one connected matrices $A, B \in \mathbb{M}^{n\times m}$ are given. In other words, $\operatorname{rank}(A-B)=1$ and hence we find nonzero vectors $a \in \mathbb{R}^n$ and $n \in \mathbb{R}^m$ such that

$$A - B = a \otimes n \text{ or, equivalently, } (A-B)x = a\langle n, x\rangle \text{ for all } x \in \mathbb{R}^m.$$

The meaning of this condition is that the linear mappings corresponding to A and B agree along a whole hyperplane of $\mathbb{R}^m$. Therefore, domains on which an affine map has gradient A can touch with domains on which the gradient equals B along this hyperplane without violating the tangential continuity. Then it is of course possible to construct a Lipschitz map on a connected domain that uses precisely the gradients A and B. Since the number of such flat interfaces as well as the thickness of the layers can be chosen arbitrarily we see from Figure 1 the following. Each linear map corresponding to a matrix C in the rank-one segment $[A, B] \subset \mathbb{M}^{n\times m}$ can be, up to a quickly oscillating but arbitrarily small error, obtained as the macroscopic behaviour of a map with microstructure $\{A, B\}$. Because a detailed mathematical description of this and similar construction is given in Section 3.1 of this work we mention here only one feature. Using an arbitarily small interpolation layer but making the construction scale sufficiently fine, we can realize the boundary datum C precisely and still keep the Lipschitz constants uniformly bounded. Viewing this from the outside, it means that we can without changing the boundary datum modify the gradient distribution which originally was a Dirac sitting in C into a distribution which up to an arbitarily small error lives at the end points of a freely chosen rank-one segment passing through C. The

preservation of the boundary data implies, due to Fubini's Theorem, that during this modification the barycentre of the distribution does not change.



The picture on the left illustrates the value of the gradient in the different parts of the domain. After adding a suitable affine function we can suppose $C = 0$ for the approximated matrix. The size of the error in the interpolation layer as well as the area of the latter becomes arbitrarily small if the finess of the construction increases.
The gradient distributions before and after the distribution look as pictured below.

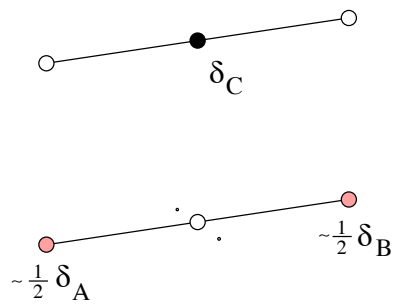


FIGURE 1. Lamination and gradient distributions

Of course, the question arises whether this simple, so called lamination construction is very special or if it already covers or at least in some sense generates the whole spectrum of possible constructions. To ask more precisely, we introduce the following notions.

DEFINITION 1.1. *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain. The gradient distribution of a Lipschitz map $f : \Omega \to \mathbb{R}^n$ is the probability measure on the space of all $(n \times m)$-matrices given by*

$$\mathcal{M} \subset \mathbb{M}^{n\times m} \to \mathscr{H}^m(\{x \in \Omega \,;\, \nabla f(x) \in \mathcal{M}\})/\mathscr{H}^m(\Omega) \text{ for all } \mathcal{M} \subset \mathbb{M}^{n\times m}.$$

A compactly supported probability μ on $\mathbb{M}^{n\times m}$ is an approximate gradient distribution if μ is the weak* limit of the gradient distributions of a sequence $\{f_k\}$ of uniformly Lipschitz mappings that are defined on a fixed domain and all have the same affine boundary data.

We say that the compact set $\mathcal{K} \subset \mathbb{M}^{n\times m}$ is

- rigid for exact solutions *if each on a domain defined Lipschitz map f with $\nabla f \in \mathcal{K}$ a.e. is necessarily affine. Equivalently, each exact gradient distribution with support in $\mathcal{K}$ is a Dirac measure,*
- rigid for approximate solutions *if each approximate gradient distribution living in $\mathcal{K}$ is a Dirac measure.*

The just defined approximate gradient distributions agree with the in the literature more frequently studied Gradient Young measure $\mathscr{M}_{\mathrm{qc}}$. We also consider their localization $\mathscr{M}_{\mathrm{qc}}(\mathcal{K}) = \{\mu \in \mathscr{M}_{\mathrm{qc}} \,;\, \mathrm{spt}(\mu) \subset \mathcal{K}\}$ and emphasize that for a given $\mu \in \mathscr{M}_{\mathrm{qc}}(\mathcal{K})$ the approximating exact distributions do not need to have their supports in $\mathcal{K}$ but only close to it. This is precisely what

characterizes them as a minimizing sequence of a corresponding variational problem. Another important aspect is, that mappings whose gradient distributions generate measure in $\mathscr{M}_{\mathrm{qc}}$ respect the same affine boundary datum. This enables us to extend such maps in a periodic way and, after suitable rescalings, to obtain highly oscillating and hence weakly converging gradients and puts approximate gradient distributions in perfect duality with suitable convexity notions on the space of matrices.

In fact, our remark about the barycentre of the gradient distribution already implies that all possible macroscopical boundary values are contained in the convex hull of the microscopical gradients. In the scalar case this statement cannot be strengthened. In the situation of higher dimensions, however, nonlinear Null-Lagrangians do exists. These are mappings $F : \mathbb{M}^{n\times m} \to \mathbb{R}$ such that $\int_\Omega F(\nabla u(x))\,dx$ is determined by $u_{|\partial\Omega}$ already. We denote this class by $\mathcal{N}\,\mathcal{L}(n,m)$ and it consists precisely of all affine combinations of the minors of $(n \times m)$-matrices. Consequently, each approximate gradient distribution also satisfies

$$F(\bar{\mu}) = \int F(X)\,d\mu(X) \text{ for all } F \in \mathcal{N}\mathcal{L}(n,m).$$

We write $\mathscr{M}_{\mathrm{pc}}$ for the family of all measures which satisfy these so-called minor conditions because they also satisfy Jensen's inequality not only for convex, but also for all polyconvex functions. Such functions are the pointwise supremum over some family of Null-Lagrangians or, equivalently, can be written as a convex function of all minors. Therefore, $\mathscr{M}_{\mathrm{pc}}$ gives an upper bound for $\mathscr{M}_{\mathrm{qc}}$ and it is clear that microstructures in $\mathcal{K}$ have their boundary values in $\mathcal{K}^{pc} = \{\bar{\mu}\ ;\ \mu \in \mathscr{M}_{\mathrm{pc}}(\mathcal{K})\}$. More interesting, a Hahn-Banach argument gives the dual representation

$$\mathcal{K}^{pc} = \{X\ ;\ F(X) \le \sup F(\mathcal{K}) \text{ for all } F : \mathbb{M}^{n\times m} \to \mathbb{R} \ \text{ polyconvex }\}.$$

The originally derived duality concerned approximate gradient distributions and quasiconvex functions $F : \mathbb{M}^{n\times m} \to \mathbb{R}$, which are characterized by fulfilling

$$\int_\Omega F(A + \nabla\varphi(y))\,dy \ge \int_\Omega F(A)\,dy \text{ for all } A \in \mathbb{M}^{n\times m} \text{ and } \varphi \in \mathrm{Lip}_0(\Omega, \mathbb{R}^n).$$

It is quite easy to see that the condition is indepedent of the domain considered. D. Kinderlehrer and P. Pedregal showed (see [**43**], also for the inhomogeneous case), that approximate gradient distributions are among probabilities distinguished by satisfying Jensen's inequality for all quasiconvex functions. Expressed formally, we have

$$\mathscr{M}_{\mathrm{qc}} = \{\mu\ ;\ \mathrm{spt}(\mu) \text{ compact and } \int F(Y)\,d\mu(Y) \ge F(\bar{\mu}) \text{ for all } F : \mathbb{M}^{n\times m} \to \mathbb{R} \text{ quasiconvex }\},$$

and again the quasiconvex hull $\mathcal{K}^{qc}$ which now consists only of the really possible boundary values among all those given by $\mathcal{K}^{pc}$ can be described in terms of separation properties of quasiconvex functions. Since the definition of quasiconvexity is much more difficult to verify, a good inner estimate for $\mathcal{K}^{qc}$ is needed.

Here the already introduced lamination construction comes into the game. It shows that $\mathcal{K}^{qc}$ at least contains $\mathcal{K}^{lc}$. The latter is the smallest superset of $\mathcal{K}$ which together with two rank-one connected matrices A and B contains also the rank-one segment $[A, B]$. But what happens in case that there exists no rank-one connection at all in $\mathcal{K}$? More precisely, we ask:

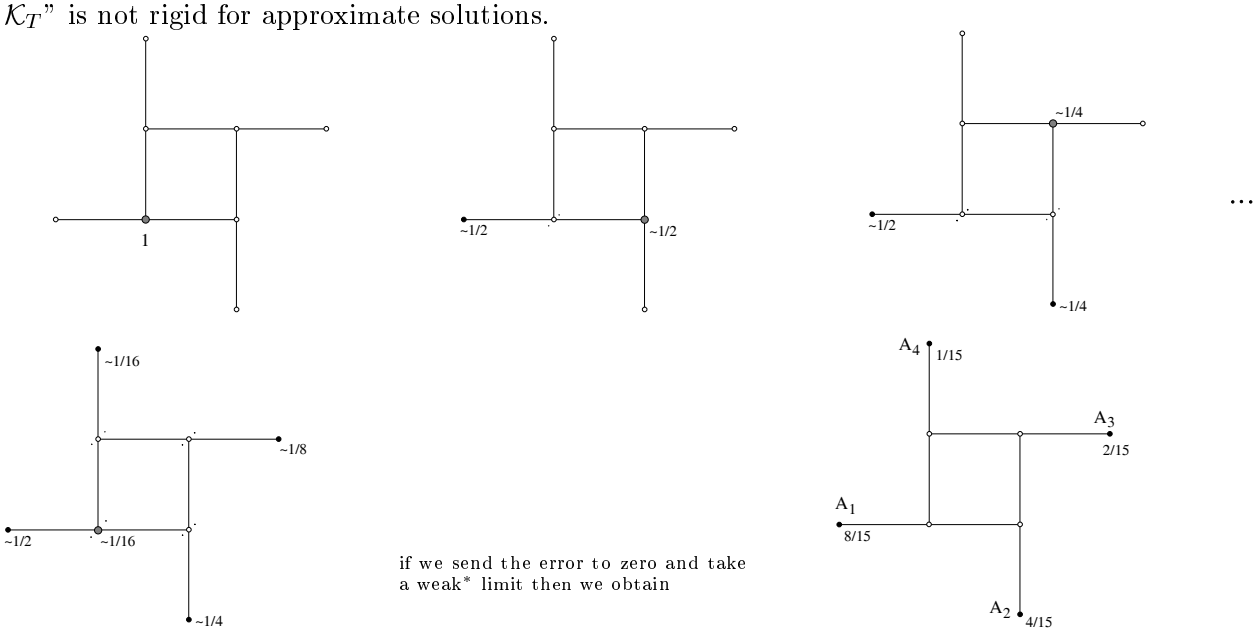
> Let $\mathcal{K} \subset \mathbb{M}^{n\times m}$ be compact and suppose $\mathrm{rank}(A - B) \ne 1$ for all $A, B \in \mathcal{K}$. Does this imply that $\mathcal{K}$ is necessarily rigid for approximate or for exact solutions?

We want to stress that these two rigidity notions are independent of each other. The seemingly plausible implication that rigidity for approximate solution implies also rigidity for exact ones is refuted by the example of holomorphic functions whose gradient run through an infinite compact subset K of the space of all conformal matrices $\{F \in \mathbb{M}^{2\times 2}\ ;\ F_{11} = F_{22}, F_{12} = -F_{21}\}$. On the other hand, this space does not contain rank-one connections and is rigid for approximate solutions.

The first, and moreover finite counterexample concerning rigidity of approximate solutions was given by L. Tartar in 1983, see [**96**]. A related idea appeared already in [**81**] and similar examples were independently found in [**4**] and [**73**]. The whole construction takes place in the diagonal (2×2)-matrices. In order to illustrate it, we identify the point $(x, y) \in \mathbb{R}^2$ with $\mathrm{diag}(x, y) \in \mathbb{M}^{2\times 2}$. Therefore, rank-one connections precisely correspond to vertical or horizontal segments in the plane. Using this, it is easily checked that the set consisting of the four matrices

$$A_1 = \mathrm{diag}(-3, -1), A_2 = \mathrm{diag}(1, -3), A_3 = \mathrm{diag}(3, 1) \text{ und } A_4 = \mathrm{diag}(-1, 3)$$

does not have any rank-one connection. If we now start from a Dirac measure in $\mathrm{diag}(-1, 1)$, i.e. the affine map $(x_1, x_2) \to (-x_1, x_2)$, and perform the lamination construction, then we gradually obtain the exact gradient distributions shown on the picture. All of them have again the same barycentre, and in the weak* limit they give an approximate gradient distribution supported in $\mathcal{K}_T = \{A_1, \ldots, A_4\}$ which is, therefore, not a Dirac measure. This shows that the "Tartar square $\mathcal{K}_T$" is not rigid for approximate solutions.



This example also makes clear, that the class of prelaminates (also called laminates of finite order) which is the class of probabilities that can be obtained from Dirac measures via the just shown iterative splitting procedure in rank-one directions plays, together with its weak* closure $\mathscr{M}_{\mathrm{rc}}$, a very important role. Because this construction cares only about rank-one directions, it is natural to expect that $\mathscr{M}_{\mathrm{rc}}$ is in duality with the cone of rank-one convex functions. These are the $F : \mathbb{M}^{n\times m} \to \mathbb{R}$ which are convex along all rank-one lines. In the paper [**76**] this and the two dual representations of the rank-one convex hull $\mathcal{K}^{rc}$ were established. Since all quasiconvex functions are rank-one convex, it is clear that

$$\mathcal{K}^{rc} = \{X \; ; \; F(X) \leq \sup F(\mathcal{K}) \text{ for all } F : \mathbb{M}^{n\times m} \to \mathbb{R} \;\; \text{rank-one convex } \}$$

provides an inner and moreover often numerically managable approximation for $\mathcal{K}^{qc}$. This shows that Morrey's question "qc=rc?", which is up to Šverák's beautiful counterexample [**86**] completely open, addresses the relation between general mappings and those which are constructed or at least approximated by an iterated use of the lamination construction. The same fundamental question occurs also in other mathematical branches, for instance it is central in the studies of D. Preiss about fine differentiablity results for Lipschitz maps, see [**77**]. A first positive result that established equality of the two classes, but restricted to distributions in $\mathrm{diag}(2 \times 2)$, was obtained in [**67**].

In any case, we can now reformulate the question stated above in a more precise manner and ask for examples similar to Tartar's square $\mathcal{K}_T$ but concerning exact solutions.

> THE N-GRADIENT PROBLEM: Given a set $\mathcal{K} \subset \mathbb{M}^{n\times m}$ consisting of N matrices without rank-one connections, is then $\mathcal{K}$ necessarily rigid for exact solutions? In particular, what happens for $N = 4$?

It was known already for a long time that $\mathcal{K}_T$ itself is rigid (which in the sequel without further comments means "rigid for exact solutions"), see Section 2.5 in [**64**]. Indeed, the condition $\nabla f \in \mathrm{diag}(2\times 2)$ enforces that $f = \nabla F$ where $F : \Omega \to \mathbb{R}$ is a solution of the wave equation $\partial^2 F/\partial x_1 \partial x_2 = 0$. The occuring characteristics now show the existence of rank-one connections. For the further understanding it is also helpful to mention that for a fixed N generic projection and restrictions show that the N-gradient problem is decided in $\mathbb{M}^{2\times 2}$ already. Moreover, previous work by J.M. Ball and R.D. James ([**6**]) for $N = 2$ and V. Šverák ([**87**]) and K. Zhang ([**104**]) for $N = 3$ demonstrated that in these cases the nonexistence of rank-one connections implies rigidity for both exact and approximate solutions. In particular it was shown that for $N = 2$ all exact solutions necessarily have the structure coming from the lamination construction.

In the first case open in the literature ($N = 4$), the matrices $A_1, \dots, A_4$ can always, except one easy degenerate case, be transformed so that they satisfy

$$A_1, \dots, A_4 \in \mathcal{H}_D = \{X \in \mathbb{M}^{2\times 2}_{sym} \ ; \ \det(X) = D\}, \quad D \text{ a real.}$$

However, the situation varies significantly with the sign of D. In case $D > 0$ we have a complete absence of rank-one connections, and this problem was already answered by V. Šverák. He showed rigidity for the approximate and the exact solutions, for this purpose he established in [**88**] regularity results for the elliptic Monge-Ampère equation without a priori convexity assumptions. The last previously open case $D = -1$ is again more similar to the already mentioned situation in $\mathbb{R}^2$, which reflects $\mathrm{diag}(2\times 2)$. Indeed, also each point $P \in \mathcal{H}_{-1}$ is on two rank-one lines which entirely stay in the one-sheeted hyperboloid* $\mathcal{H}_{-1}$. Because these two rank-one fibrations are twisted against each other, we observe new global properties and problems. One important question is the following. If the gradient of a Lipschitz vectorfield $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ stays in one half of the hyperboloid then an exchange of coordinates that was already used in the work of J. Moser ([**62**]), of R. Schoen and J. Wolfson ([**82**]), and of L.C. Evans und R.F. Gariepy ([**33**]) transforms f again in the solution of the wave equation. Therefore, both for answering the 4-Gradient problem as well as for further attempts to extend the "rc=qc" given in [**67**] from $\mathbb{R}^2$ to $\mathcal{H}_{-1}$ it seems important to understand whether exact solutions of $\nabla f \in \mathcal{H}_{-1}$ can, at least locally, be transfered into solutions of the wave equation.

Our regularity results for the hyperbolic Monge-Ampère equation in Chapter 2 answer these questions to a large extent. They show that this transformation can be impossible in some parts of the domain. The set of such singular points, however, is always discrete in Ω. Our results give a good description of solutions to $\nabla f \in \mathcal{H}_{-1}$ and complete a research project that started by a joint work [**22**] with M. Chlebík which was only concerned with the rigidity question in the last previously open case of the 4-gradient problem. On the other hand, jointly with D. Preiss we found in the course of similar investigations a 4-matrices configuration in $\mathcal{H}_{-1}$ which is in some sense even less rigid than the Tartar square $\mathcal{K}_T$. It will be presented in Section 4.2 and relies on a further difference between the geometries of $\mathbb{R}^2$ and $\mathcal{H}_{-1}$ that allow us to obtain out of only four matrices in the same moment four different Tartar squares. Because of the already mentioned rigidity result for four matrices, also this set does not allow nontrivial exact solutions. The first nonrigid finite sets with rank-one connections were found using a geometric approach to stability questions for rank-one and quasiconvex hulls, compare with [**44**]. In order to reduce the number of gradients used, we have to be able to solve partial differential inclusions $\nabla f \in \mathcal{K}$ for new kinds of sets $\mathcal{K}$ that

*This geometric description will be justified in Section 4.2 using conformal coordinates.

distinguish themselfes by a complicated geometry of their rank-one convex hull. For this purpose, in Chapter 3 we develop a method that unifies previous methods and reduces the problem to a very natural stability question for gradients. Finally, in Chapter 4 we present a geometric approach to rank-one convexity and, armed with the previous results, also a nonrigid set consisting of five matrices without rank-one connections that was found together with D. Preiss as a modification of our four matrices configuration improving the Tartar square. This answers the N-gradient problem definitely.

2. Some preliminary considerations

In this section we present the basic and to a large extent well-known facts and concepts that will be utilized throughout the whole work.

We start with conventions from linear algebra, and give also other notation that might deserve some explanation here since it comes from general topology and measure theory.

2.1. Generalities. The symbol $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with its unit sphere $\mathbb{S}^{n-1}$. The $(n \times m)$-matrices form the space $\mathbb{M}^{n\times m}$ and by $\mathbb{M}^{n\times n}_{sym}$ we denote all symmetric $(n\times n)$-matrices. As usually, $\mathbb{M}^{n\times m}$ is equiped with the scalar product $A : B = \mathrm{tr}(A^T B)$, where A^T is the transpose of A and tr the trace operator, the scalar product in turn induces the Euclidean norm on $\mathbb{M}^{n\times m}$. The determinant of $M \in \mathbb{M}^{n\times m}$ is denoted by $\det(M)$ and the cofactor $\mathrm{cof}(F)$ is the matrix consisting of all $(n-1)\times(n-1)$-minors. It satifies in general Cramer's rule $\mathrm{cof}(F)^T \cdot F = \det(F) \cdot \mathrm{Id}$, but for us actually only the case $n = 2$ is interesting. Then we can use that

$$\det(A + B) = \det(A) + \mathrm{cof}(A) : B + \det(B) \quad \text{for } A, B \in \mathbb{M}^{2\times 2}, \tag{1.1}$$

and hence $\det(A + B) = \det(A) + \det(B)$ if $A \in \mathbb{M}^{2\times 2}_{sym}$ and $B = -B^T$. (But we can derive this fact also using the conformal coordinates to be introduced in Notation 1.10). Identifying matrices with the linear mappings they present, we write $\mathrm{rank}(A)$ for the linear dimension of the image of A. In particular, matrices in $\mathbb{M}^{n\times m}$ *of rank* at most *one* are precisely those of the form $a \otimes b$, with $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. We think of them both as

$$a \otimes b \in \mathbb{M}^{n\times m} \text{ with } (a \otimes b)_{ij} = a_i b_j \text{ and as maps } a \otimes b : x \in \mathbb{R}^m \to a\langle b, x\rangle.$$

Because the only problems invariant under the action of a group which we consider are placed in $\mathbb{M}^{2\times 2}$, we only introduce the group of all rotations of the plane $\mathrm{SO}(2)$ which we usually identify via complex multiplication with the unit circle $\mathbb{S}^1$ in the plane.

Now we turn to the metric and topological concepts. Given a space X with a metric ϱ, we denote as usual $\mathrm{dist}(x, M) = \mathrm{dist}_\varrho(x, M) = \inf(\{\varrho(x, y)\ ;\ y \in M\})$ for $x \in X$ and $M \subset X$. Depending on the degree of ambiguity possible in the given context, we write $B(M, r) = B_\varrho(M, r) = B_\varrho(M, r; X)$ for the open metric r-neighbourhood $\{y \in X\ ;\ \mathrm{dist}(M, y) < r\}$, $r \geq 0$, of a given set M. Similarly, $\bar{B}(M, r) = \{y \in X\ ;\ \mathrm{dist}(M, y) \leq r\}$ with the same meaning for the other variants and if M is a singleton then we replace $B(\{x\}, r)$ by $B(x, r)$ and similarly. The Hausdorff distance of two sets $M, N \subset X$ is given by $\mathrm{dist}_{\mathcal{H}}(M, N) = \inf\{r > 0\ ;\ M \subset B(N, r) \text{ and } N \subset B(M, r)\}$. As usual, a map $f : (X, \varrho) \to (Y, \sigma)$ between two metric spaces is *Lipschitz*, if there is a finite constant c such that $\sigma(f(x), f(y)) \leq c \cdot \varrho(x, y)$ for all $x, y \in X$ and its *Lipschitz constant* $\mathrm{lip}(f)$ is the smallest such (nonnegative) constant. If, in addition, the inverse map f^{-1} is Lipschitz on the image of f, we call f *bilipschitz* . For the space of all Lipschitz functions we use the symbol $\mathrm{Lip}(X, Y)$ and if Y is a (complete) linear space and M is understood as a subset and a subspace of (X, ϱ), then $\mathrm{Lip}_0(M, Y)$ is the set of all $f \in \mathrm{Lip}(M, Y)$ whose unique extension to the closure of M vanishes on the boundary of M (in X).

The closure of a set M in a general topological space (X, τ) will be denoted by $\bar{M}$ or $\mathrm{clos}(M)$, sometimes we will use a subscript to emphasize which topology we use. The relative interior (where the ambient space will be clear from the context) is written as $\mathrm{int}_{\mathrm{rel}}(M)$ and ∂M stays as usual

for the boundary. We say that M is *nowhere dense* if the interior of $\operatorname{clos} M$ is empty and that M is *of the first Baire category in* X if M can be covered by countably many nowhere dense sets. Otherwise, we say that M is *of the second Baire category* , note that it is certainly nonempty in this case. Finally, a subset is *residual* in X if its complement is of the first category. We will also use the phrase, see [**15**] for a nice presentation of the whole concept, that *the typical point x in X has property $\mathcal{P}$* if $\{x \in X \; ; \; \mathcal{P}(x) \text{ holds}\}$ is residual in X. Of course, such a statement is interesting only if the residual sets in the space X are nonempty. This is true for instance if X is a complete metric space, or more general a Baire space. We will not consider this very general class of spaces, but several times we deal with G_δ-sets. These are by definition those sets which are the intersection of a countable family of open sets and are interesting for us, since Alexandroff's theorem tells that (X, τ) is (homeomorphic to) a G_δ-subset of a complete metric space if and only if it can be metrized with a complete metric. We denote by $f_k \rightrightarrows$ the uniform convergence of a sequence of functions.

Concerning measure theory, we use the standard notation $\mathscr{H}^m$ for the m-dimensional Hausdorff measure as defined e.g. in Section 2.10 of [**35**], the case $m = 1$ and therefore the length measure will be particularly useful for us. The m-dimensional Lebesgue measure $\mathscr{L}^m$ is then just the measure $\mathscr{H}^m$ operating on the m-dimensional Euclidean space and if no ambiguity is possible we use the abbreviation $|\cdot|$. Beside this, we will use the restriction operator $\llcorner$ which defines the *restriction of a measure μ to a set M* to be the new measure $\mu \llcorner M$ satisfying $\mu \llcorner M(A) = \mu(M \cap A)$. The only other perhaps less commonly known notion is the *pushforward of a measure μ under the measurable map f* which is by definition the measure $f_\# \mu$ on the target space of f given by $f_\# \mu(M) = \mu(f^{-1}(M))$. This notion is particularly interesting for us, since the formula

$$\int F(f(x))\, d\mu(x) = \int F(y)\, d f_\# \mu(y)$$

holds. In other words, to evaluate integrals of the type $\int_\Omega F(\nabla u(x))\, dx$, we only need to know the already mentioned gradient distribution of u which is simply the (normalized) pushforward of the Lebesgue measure restricted to Ω under the map $x \to \nabla u(x)$. Finally, we say that a measure μ *lives on a set M* if μ restricted to the complement of M vanishes. The smallest closed set on which μ lives is as usually said to be the support $\operatorname{spt}(\mu)$ of the measure μ. Similarly, we say that *the essential image* of a function f defined on a space X with measure μ *is contained in the set M* if μ lives on $f^{-1}(M)$. For a probability measure living on a finite dimensional linear space V , we define its barycentre (also called centre of mass or first momentum) to be the point $\bar{\mu} = \int_V x\, d\mu(x)$ provided any of the norms on V is μ-integrable, so in particular if μ has a compact support.

Being more specific about our spaces, we say that a Lipschitz map f is a (countably) piecewise affine map if it maps a closed subset of an $\mathbb{R}^m$ into some $\mathbb{R}^n$ and, $\mathscr{L}^m$-almost all of its domain is covered by the union of (countably many) open sets with the property that f is an affine function on each of them. (In particular, it is hidden in this definition that the boundary of the domain of f has Lebesgue measure zero). I believe the alternative definition calling such maps almost everywhere locally affine would have several advantages, for instance that ambiguity about the finite or countable number of affine pieces would disappear. We stick, however, in this work to the conventional notation.

2.2. Convexity notions for vectorial variational problems. Based on the introductory discussion in Section 1 we give a rather brief, but more formal, survey of the different generalized convexities used in the calculus of variations with several variables. Nice general references are [**23**] and [**64**]. We will not give any proofs, but try to indicate the underlying duality for all of these notions, even if we focus in this work on the already mentioned rank-one convexity, and refer to the relevant literature. We consider real valued functions only, for the other case a nice summary can be found in Section 2.2 of [**32**].

Definition 1.2. *For a given function $f : \mathbb{M}^{n\times m} \to \mathbb{R}$ we say, that*

a) *f is* rank-one convex *, or briefly rc, if for each $A \in \mathbb{M}^{n\times m}$ and all $a \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$ the function $t \to f(A + t \cdot a \otimes b)$ is convex on the real line.*
b) *f is* quasiconvex*, or briefly qc, if for each open bounded subset Ω of $\mathbb{R}^m$, each $A \in \mathbb{M}^{n\times m}$ and any $\varphi \in \mathrm{Lip}_0(\Omega, \mathbb{R}^n)$ the Jensen type inequality*
$$f(A) \leq \frac{1}{\mathscr{L}^m(\Omega)} \int_\Omega f(A + \nabla u(x))\, dx$$
holds.
c) *f is polyconvex, or briefly pc, if there is a convex function $g : \mathbb{R}^{\tau(n,m)} \to \mathbb{R}$ such that $f(X) = g(\hat{X})$ for all $X \in \mathbb{M}^{n\times m}$. Here $\hat{X}$ is the* minor vector *consisting of all subdeterminants (of order one up to $\min(n, m)$) of X and has the easily computable length $\tau(n, m)$.*
d) *f is convex if for all $A, B \in \mathbb{M}^{n\times m}$ the function $t \to f(A+t\cdot B)$ is convex on the real line.*

Then it is well-known, and using the arguments from our foregoing discussion also easy to verify that
$$f \text{ convex } \Rightarrow f \text{ polyconvex } \Rightarrow f \text{ quasiconvex } \Rightarrow f \text{ rank-one convex.}$$
All these notions agree if $\min(m, n) = 1$ but in general none of the converse implications holds. However, the relation between rank-one and quasiconvexity is very difficult to understand and only fourty years after C.B. Morrey posed the question, V. Šverák in [**86**] proved that for $n \geq 3$ and $m \geq 2$ these two do not agree.

To give a better picture of this kind of functions we just notice that they are at least locally Lipschitz (see e.g. [**23**] or Lemma 2.2 in [**8**]).

We also introduce the following classes of measures that will be in duality with the just defined cones of generalized convex functions via the Jensen inequality. To emphasize the similar nature of all these dualities, we introduce the metasymbol $\square$ which stays for any of *rc*, *qc* or *pc* and the case $\square = co$ refers to ordinary convexity.

Definition 1.3. *For $\square \in \{rc, qc, pc, co\}$ we define $\mathscr{M}_\square$ to be the class of all compactly supported probability measures on $\mathbb{M}^{n\times m}$ that satisfy the inequality $f(\bar{\mu}) \leq \int f(X)\, d\mu(X)$ for all $f : \mathbb{M}^{n\times m} \to \mathbb{R}$ which are $\square$. Measures in $\mathscr{M}_{\mathrm{rc}}$ are called* laminates*, the members of $\mathscr{M}_{\mathrm{qc}}$ form the* Gradient Young measures *and $\mathscr{M}_{\mathrm{pc}}$ consists of the polyconvex measures. Given a compact set $\mathcal{K} \subset \mathbb{M}^{n\times m}$ we introduce the localization*
$$\mathscr{M}_\square(\mathcal{K}) = \{\mu \in \mathscr{M}_\square \ ; \ \mathrm{spt}(\mu) \in \mathcal{K}\}.$$

Due to Jensen's inequality, in the convex case there is no other restriction on μ, so we write then rather $\mathscr{M}$ for the class of all compactly supported probability measures. Now, of course the question arises if the member $\mathscr{M}_\square$ can be also described in a more constructive way. We get, depending on the class considered, more or less explicite answers but need a bit more notation.

Definition 1.4. *By $\mathscr{PL}$ we denote the class of* prelaminates*, sometimes also called laminates of finite order. This is the smallest family of probability measures on $\mathbb{M}^{n\times m}$ such that*
(a) *it contains all Dirac measures in $\mathbb{M}^{n\times m}$, and*
(b) *if $\mu = \sum_{i=1}^k \lambda_i \delta_{X_i}$ (X_i's not necessarily distinct) is in the class and $X_1 = \lambda\tilde{X}_0 + (1-\lambda)\tilde{X}_1$ where $\lambda \in [0, 1]$ and $[\tilde{X}_0, \tilde{X}_1] \subset \mathcal{U}$ is a rank-one segment, then also $\tilde{\mu} = \lambda_1(\lambda\delta_{\tilde{X}_0} + (1-\lambda)\delta_{\tilde{X}_1}) + \sum_{i=2}^k \lambda_i \delta_{X_i}$ belongs to that class.*

In other words, the class $\mathscr{PL}$ is closed under splitting the atoms of its members into rank-one directions. Therefore, each member μ in $\mathscr{PL}$ is obtained from a Dirac mass $\mu_0 = \delta_{X_1}$ following*

*which just means closed under the modifaction we can achieve with the lamination construction

a generating splitting sequence $\{\mu_l\}_{l=0}^k$ *which is constructed using (b) from above. For the further understanding it is helpful to note that all the barycentres along a generating splitting sequence do agree. We use the expression* order of the prelaminate $\mu \in \mathscr{PL}$ *for the length* k *of the shortest splitting sequence generating* μ.

Having in mind our applications to approximate gradient distributions, we have to be more specific than in Definition 1.1 about the exact gradient distributions we will deal with.

DEFINITION 1.5. *The pre-Gradient Young measures are all probabilities on* $\mathbb{M}^{n\times m}$ *of the form*

$$\mu(\mathcal{A}) = |\{x \in \Omega \ ; \ A + \nabla\varphi(x) \in \mathcal{A}\}|/|\Omega| \text{ for } \mathcal{A} \subset \mathbb{M}^{n\times m},$$

where $A \in \mathbb{M}^{n\times m}$, $\varphi \in \mathrm{Lip}_0(\Omega, \mathbb{R}^n)$ *and* $\Omega \subset \mathbb{R}^m$ *is a bounded domain.*

Now we get the important

THEOREM 1.6. *The measures defined in Definition 1.3 can be obtained in the following more specific ways.*

(i) *Each* $\mu \in \mathscr{M}_{\mathrm{rc}}(\mathcal{K})$ *is in the weak* closure of*

$$\mathscr{PL} \cap \{\lambda \ ; \ \mathrm{spt}(\lambda) \subset \mathcal{U} \text{ and } \bar{\lambda} = \bar{\mu}\},$$

where $\mathcal{U}$ *is any open set containing the ordinary convex hull* $\mathcal{K}^{co}$ *of* $\mathcal{K}$ *in* $\mathbb{M}^{n\times m}$. *In Subsection 4.1.1 we will see that actually any neighbourhood* $\mathcal{U}$ *of the smaller rank-one convex hull* $\mathcal{K}^{rc}$ *to be defined in a moment would work as well.*

(ii) *Each* $\mu \in \mathscr{M}_{\mathrm{qc}}(\mathcal{K})$ *is in the weak* closure of the pre-Gradient Young measures living in any open neighbourhood of* $\mathcal{K}^{co}$ *and having barycentre* $\bar{\mu}$.

(iii) *Each* $\mu \in \mathscr{M}_{\mathrm{pc}}(\mathcal{K})$ *can be weakly* approximated by convex combinations of "elementary" polyconvex measures with barycentre* $\bar{\mu}$ *whose support consist of at most* $\tau(n,m)+1$ *points contained in* $\mathcal{K}$.

A very important tool to establish these results is, together with the convexity of the subclasses we want to approximate with and the Hahn-Banach theorem, the notion of a generalized convex envelope a function.

DEFINITION 1.7. *For* $\square \in \{rc, qc, pc, co\}$ *and* $f : \mathbb{M}^{n\times m} \to \mathbb{R}$ *we write* $f^\square$ *for the* $\square$*-envelope of* f, *this is the pointwise supremum of all* $g : \mathbb{M}^{n\times m} \to [-\infty, \infty)$ *which are* $\square$ *and satisfy* $g(X) \le f(X)$ *for all* $X \in \mathbb{M}^{n\times m}$.

The key idea how to prove Theorem 1.6 is to show that we do not only have the almost obvious representation

$$f^\square(X) = \inf\{\int f\, d\mu \ ; \ \mu \in \mathscr{M}_\square \text{ and } \bar{\mu} = X\}$$

but that we can also obtain $f^\square$ utilizing as test measures only those from the smaller subclasses appearing in Theorem 1.6. For polyconvexity this essentially uses only the Caratheodory number of $\mathbb{R}^{\tau(n,m)}$ and was mentioned already in [**16**]. The rank-one convex case is even more simple, the problems coming from the constraints on the support were handled in [**68**] or see subsection 4.1.1 of this work. Certainly, the quasiconvex situation is the technically most demanding - it was firstly treated in [**43**], more recent presentations can be found in [**52**] or [**92**]. The fact that $f^{qc}(X)$ can be obtained as the infimum over $\int_\Omega f(X + \nabla\varphi(x))\, dx/|\Omega|$ where φ runs through $\mathrm{Lip}_0(\Omega, \mathbb{R}^n)$ is due to B. Dacorogna, see e.g. [**23**], and becomes quite transparent if we further reduce it to the case of piecewise affine φ. To handle the localization of the supports, a more sophisticated tool which is usually called "Zhang's Lemma", see [**103**], is needed - but in its sharp version as given in Theorem 4 of [**63**]. It roughly says if a sequence $\{\mu_k\}_k$ of pre-gradient Young measures with a common barycentre satisfies $\lim_k \int \mathrm{dist}(X, \mathcal{K})\, d\mu_k(X) \to 0$, then given any open set $\mathcal{U}$ containing

the convex hull of $\mathcal{K}$ we can find a new sequence of pre-Gradient Young measures with the same barycentre and the same weak*-limits but living all the time in $\mathcal{U}$. This is established by a carefully modification of the functions giving the pre-Gradient Young measures.

Finally, we introduce the generalized convex hulls of sets, a notion we are going to use throughout this work.

DEFINITION 1.8. *For $\square \in \{rc, qc, pc, co\}$ and $\mathcal{K} \subset \mathbb{M}^{n\times m}$ compact we set*

$$K^{\square} = \{\bar{\mu}\ ;\ \mu \in \mathscr{M}_{\square}(\mathcal{K})\}.$$

A set $\mathcal{K}$ is said to be $\square$ if $\mathcal{K} = \mathcal{K}^{\square}$ - so we get the notion of rank-one, quasi- and polyconvex sets and hulls.

After the foregoing preparations we now easily get.

THEOREM 1.9. *For $\square \in \{rc, qc, pc, co\}$ and $\mathcal{K} \subset \mathbb{M}^{n\times m}$ compact the hull satsifies*

$$\mathcal{K}^{\square} = \{X\ ;\ f(X) \leq \sup f(\mathcal{K}) \text{ for all } f : \mathbb{M}^{n\times m} \to \mathbb{R} \text{ which are } \square\}.$$

Indeed, $\mathcal{K}^{\square}$ is contained in the right hand side just by the definition of $\mathscr{M}_{\square}$. For the converse inclusion it is now enough to consider the zero set of the envelope $f = (\mathrm{dist}(\mathcal{K}, \cdot)^p)^{\square}$, where $p \in (1, \infty)$, and prove that $f(X) = 0$ implies $0 = \min\{\int f\, d\mu\ ;\ \mu \in \mathscr{M}_{\square}(K)\}$. In case of the quasiconvex envelope we again have to use Zhang's Lemma mentioned before. Also here, the polyconvex situation is formally much simpler since we have the description, see e.g. [**89**],

$$\mathcal{K}^{pc} = \{X\ ;\ \hat{X} \in \{\hat{Y}\ ;\ Y \in \mathcal{K}\}^{co}\},$$

which hides two problems typical in this situation. Firstly, the convexification in $\mathbb{R}^{\tau(n,m)}$ is due to the very high dimensions ($\tau(3,3) = 19$ already) quite complex and, secondly, then to understand which matrices X have their minor vector $\hat{X}$ in this convex set is because of the nonlinearities in the subdeterminants often even more complicated.

2.3. Conformal coordinates and the two well problem.

NOTATION 1.10. (Conformal coordinates) One of the themes underlying this work are the consequences of the theory of quasiconformal mappings on the existence and regularity results for mappings with prescribed values of the gradient. Therefore, the usefulness of conformal coordinates comes at no surprise and will be exploited several times. To be more precise, the idea simply is to express each linear selfmap of the plane $\mathbb{C}$ in terms of its two orthogonal components, the holomorphic and the antiholomorphic part.

So, with a complex number $w = u + \mathrm{i}v$ we associate the following two maps

- the holomorphic map $w_{\mathcal{H}} : z \in \mathbb{C} \to w \cdot z$ or written in matrix language: $w_{\mathcal{H}} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$,
- the antiholomorphic map $w_{\bar{\mathcal{H}}} : z \in \mathbb{C} \to w \cdot \bar{z}$ or as matrix $w_{\bar{\mathcal{H}}} = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$.

Now it is easy to check that each matrix $A \in \mathbb{M}^{2\times 2}$ can be represented as the sum of 2 such maps

$$A = (A_{\mathcal{H}})_{\mathcal{H}} + (A_{\bar{\mathcal{H}}})_{\bar{\mathcal{H}}}, \text{ where } A_{\mathcal{H}} = \frac{1}{2}\begin{pmatrix} A_{11} + A_{22} \\ A_{21} - A_{12} \end{pmatrix}, A_{\bar{\mathcal{H}}} = \frac{1}{2}\begin{pmatrix} A_{11} - A_{22} \\ A_{21} + A_{12} \end{pmatrix} \in \mathbb{C}, \tag{1.2}$$

so $A(z) = (A_{\mathcal{H}})z + (A_{\bar{\mathcal{H}}})\bar{z}$ for all $z \in \mathbb{C}$. Moreover, we have the useful formulae

$$\det(A) = |A_{\mathcal{H}}|^2 - |A_{\bar{\mathcal{H}}}|^2 \text{ and } \|A\|^2 = 2(|A_{\mathcal{H}}|^2 + |A_{\bar{\mathcal{H}}}|^2) \tag{1.3}$$

and can control how much A distorts length since the geometric interpretation of complex multiplication yields that

$$\max_{|z|=1}|A(z)| = |A_{\mathcal{H}}| + |A_{\bar{\mathcal{H}}}| \text{ and } \min_{|z|=1}|A(z)| = |(|A_{\mathcal{H}}| - |A_{\bar{\mathcal{H}}}|)|.$$

Finally, we will use the following, easily verified chain rule

$$(A\circ B)_{\mathcal{H}} = A_{\mathcal{H}}\cdot B_{\mathcal{H}} + A_{\bar{\mathcal{H}}}\cdot\overline{(B_{\bar{\mathcal{H}}})},\ \ (A\circ B)_{\bar{\mathcal{H}}} = A_{\mathcal{H}}\cdot B_{\bar{\mathcal{H}}} + A_{\bar{\mathcal{H}}}\cdot\overline{(B_{\mathcal{H}})}. \tag{1.4}$$

NOTATION 1.11. ("Generalized conformal coordinates" and two SO(2) wells) The most simple (and in some sense the only rigorously understood) nontrivial example of a set of matrices invariant under the natural physical frame indifference is the union of two copies of the group of rotations in the plane

$$\mathcal{K} = \mathcal{K}_{A,B} = \mathrm{SO}(2)A + \mathrm{SO}(2)B.$$

Therefore, $\mathcal{K}$ serves as a prototype for the questions about existence of exact and approximate solutions. For this purpose it is very helpful that we know all its semi-convex hulls occuring in the calculus of variations. Their description becomes more easy if we introduce on $\mathbb{M}^{2\times 2}$ coordinates especially adapted to the set $\mathcal{K}$. Presumably inspired by the foregoing example of classical conformal coordinates, V. Šverák in [**89**] indentified

$$(y,z)\in\mathbb{C}\times\mathbb{C} \text{ with } GCC_{A,B}(y,z) = y_{\mathcal{H}}\circ A + z_{\mathcal{H}}\circ B.$$

In fact, in the foregoing Notation 1.10 we just considered the case $A = \mathrm{Id}$, $B = \mathrm{diag}(1,-1)$. However, due to our motivation coming in most of the cases from problems in continuum mechanics we will usually assume that $\det(A), \det(B) > 0$. It is clear that $GCC_{A,B}$ is a linear isomorphism of $\mathbb{C}^2$ onto $\mathbb{M}^{2\times 2}$ iff $(A\circ B^{-1})_{\bar{\mathcal{H}}} \neq 0$. Using the chain rule, in this case the inverse isomorphism could be given by

$$(GCC_{A,B})^{-1}(M) = ((M\circ B^{-1})_{\bar{\mathcal{H}}}/(A\circ B^{-1})_{\bar{\mathcal{H}}}), (M\circ A^{-1})_{\bar{\mathcal{H}}}/(B\circ A^{-1})_{\bar{\mathcal{H}}}) \text{ for any } M\in\mathbb{M}^{2\times 2}.$$

Moreover, the following results from [**89**] and [**65**] now provide a very satisfactory understanding of the hulls of $\mathcal{K}$. We denote by $\lambda_1(BA^{-1}) \le \lambda_2(BA^{-1})$ the singular values of the matrix BA^{-1}, which are just the eigenvalues of $((BA^{-1})^T(BA^{-1}))^{1/2}$. Since we assume from the very outset that $(BA^{-1})_{\bar{\mathcal{H}}} \neq 0$ and that $\det A, \det B > 0$, we have $0 < \lambda_1 < \lambda_2$. In case that $\lambda_1 > 1$, we conclude $\inf_{|z|=1}|Bz| > \sup_{|z|=1}|Az|$ and hence there is no rank-one connection between different matrices in $\mathcal{K}$. Similarly for $\lambda_2 < 1$, and in both cases we easily check that even $\det(X-Y) > 0$ for different X, Y in K. Now it follows from [**90**] that all $\mu\in\mathscr{M}_{pc}(\mathcal{K})$ are Dirac measures and hence $\mathcal{K}^{pc} = \mathcal{K}^{qc} = \mathcal{K}^{rc} = \mathcal{K}^{lc} = \mathcal{K}$.

In the other stable case, which will be the one of main interest for us, we have $\lambda_1 < 1 < \lambda_2$. Using polar decomposition together with the diagonalization of symmetric matrices and the SO(2)-invariance of the problem, we can restrict our attention to the situation that $A = \mathrm{Id}$ and $B = \mathrm{diag}(\lambda_1,\lambda_2)$. It is then easy to check that $\max_{Q\in\mathrm{SO}(2)}\det(Q-B) > 0 > \min_{Q\in\mathrm{SO}(2)}\det(Q-B)$. From this one directly obtains, e.g. since (1.3) gives $\det(z_{\mathcal{H}} - B) = |B_{\mathcal{H}} - z|^2 - |B_{\bar{\mathcal{H}}}|^2$, that each $M\in\mathcal{K}$ is rank-one connected to two different matrices on the other well in $\mathcal{K}$. Therefore, we speak then of two compatible wells. Working in the generalized conformal coordinates it is not difficult to see that for $\det A < \det B$ we have

$$(\mathcal{K}_{A,B})^{pc} = \big\{F = GCC_{A,B}(y,z)\ ;\ |y| \le \frac{\det B - \det F}{\det B - \det A} \text{ and } |z| \le \frac{\det F - \det A}{\det B - \det A}\big\}, \tag{1.5}$$

which collapses to

$$(\mathcal{K}_{A,B})^{pc} = \{F = GCC_{A,B}(y,z)\ ;\ |y| + |z| \le 1 \text{ and } \det F = \det A\} \text{ if } \det B = \det A. \tag{1.6}$$

A very nice geometrical argument in [**89**] shows the more surprising fact that

$$(\mathcal{K}_{A,B})^{pc} = L_3(\mathcal{K}_{A,B}) \text{ if } \det A < \det B \text{ and } (\mathcal{K}_{A,B})^{pc} = L_2(\mathcal{K}_{A,B}) \text{ if } \det A = \det B. \tag{1.7}$$

For further applications that we have in mind, it will be important to understand which matrices are in the (relative) interior of the hull $\mathcal{K}^{pc}$ and, in particular, that this interior is dense in all of the hull. Results of this kind are by now folklore already, but since their proofs available in the literature are quite brief, we prefer to give the full argument here.

LEMMA 1.12. *Let us consider the set*

$$\mathcal{W} = \{(A,B) \in (\mathbb{M}^{2\times 2})^2 \ ; \ \det(A), \det(B) > 0 \text{ and } \lambda_1(BA^{-1}) < 1 < \lambda_2(BA^{-1})\}.$$

Then,

i) *$\mathcal{W}$ is open in $(\mathbb{M}^{2\times 2})^2$ and if $(A,B) \in \mathcal{W}$ then each $X \in \mathrm{SO}(2)A$ is rank-one connected to two different matrices in $\mathrm{SO}(2)B$,*
ii) *if $\det(A) < \det(B)$, then $\mathrm{int}_{\mathbb{M}^{2\times 2}}(\mathcal{K}^{pc}_{A,B})$ is the set of those $F = GCC_{A,B}(y,z)$ which make both inequalities in (1.5) strict,*
iii) *if $\det(A) = \det(B)$, then $\mathrm{int}_{\{\det(\cdot)=\det(A)\}}(\mathcal{K}^{pc}_{A,B})$ is the set of those $F = GCC_{A,B}(y,z) \in \mathcal{K}^{pc}_{A,B}$ which satisfy $|y| + |z| < 1$,*
iv) *in both cases, $\mathcal{K}_{A,B}$ (and hence all of $\mathcal{K}^{pc}_{A,B}$) is contained in the closure of this interior. Moreover, this interior completely contains each open rank-one segment $(X,Y) \subset \mathcal{K}_{A,B}$ which intersects it.*
v) *The mapping which takes $(A,B) \in \mathcal{W}$ to $\mathcal{K}_{A,B}$ is continuous when the target is equipped with the Hausdorff metric.*

Proof. Of course, openess of $\mathcal{W}$ is obvious and the rest of statement i) was already discussed. Concerning ii), it is clear that our condition on F is sufficient for being in the interior. For the converse implication, we pick any $F_0 = GCC(y_0, z_0) \in \mathrm{int}(\mathcal{K}^{pc})$. For $\varepsilon > 0$ but small we have $(y_\varepsilon, z_\varepsilon) = (1+\varepsilon)(y_0, z_0) \in \mathcal{K}^{pc}$ and $|y_\varepsilon| \geq |y_0|$ but $\det(F_\varepsilon) > \det(F_0)(\geq \det(B) > 0)$. Hence, we infer that $|y_0| < (\det(B) - \det(F_0))/(\det(B) - \det(A))$. Moreover, $p(\lambda) = [(\det(GCC(\lambda y_0, \lambda z_0) - \det(A))/(\det(B) - \det(A))] - \lambda|z_0|$ is a strictly convex quadratic polynomial with $p(0) < 0$. Hence, $p(1) = 0$ would imply that $P(1-\varepsilon) < 0$ for $\varepsilon > 0$ but small. Since then $(1-\varepsilon)(y_0, z_0) \in (\mathcal{K}_{A,B})^{pc}$, this is possible and hence $p(1) > 0$ which had to be shown.

Next, we want to prove iii) - of course, again the necessary part only. The nice geometric description of $\mathcal{K}^{pc}$ given in [**89**] makes this statement very intuitiv. However, the formal argument is a little bit longer. For this purpose, we can of course assume $A = \mathrm{Id}$ and rewrite the generalized conformal coordinates into classical ones. So,

$$GCC_{A,B}(y,z) = y_{\mathcal{H}}A + z_{\mathcal{H}}B = (y_{\mathcal{H}} + z_{\mathcal{H}}(B_{\mathcal{H}})_{\mathcal{H}}) + z_H(B_{\bar{\mathcal{H}}})_{\bar{\mathcal{H}}} = (y + (B_{\mathcal{H}})z)_{\mathcal{H}} + (zB_{\bar{\mathcal{H}}})_{\bar{\mathcal{H}}}, \text{ and hence}$$

$$\det(GCC_{A,B}(y,z)) = |y + (B_{\mathcal{H}})z|^2 - |B_{\bar{\mathcal{H}}}|^2|z|^2.$$

It is clear that $F_0 = GCC(y_0, z_0) \in \mathrm{int}_{\mathrm{rel}}(\mathcal{K}^{pc})$ excludes the existence of a sequence $(y_k, z_k) \to (y_0, z_0)$ such that $|y_k| + |z_k| = 1$ and $\det(F_k) < 1$. Hence $\langle y_0, (B_{\mathcal{H}})z_0\rangle = -|y_0||B_{\mathcal{H}}||z_0|$ since $|y_0 + B_{\mathcal{H}}z_0|$ has to be minimal keeping the length of y and z constant. Consequently, $(y_0, B_{\mathcal{H}}z_0)$ is of the form $((1-t)e, -t|B_{\mathcal{H}}|e)$ for some $t \in [0,1]$ and $e \in \mathbb{S}^1$. We set

$$g(t) = \det(GCC(y_t, z_t)) = \det(GCC((1-t)e, -|B_{\mathcal{H}}|t(B_{\mathcal{H}})^{-1}e) = (1 - (|B_{\mathcal{H}}| + 1)t)^2 - |B_{\bar{\mathcal{H}}}|^2t^2.$$

Since $\det(B) > 0$ we have $|B_{\mathcal{H}}| > |B_{\bar{\mathcal{H}}}|$. Therefore, g is a strictly convex quadratic function and a short reflection on the meaning of g shows that $g(0) = g(1) = \det(A)$. Moreover, by what we already explained, the parametre t_0 corresponding to (y_0, z_0) must be a (weak) local mininum of g on $[0,1]$ but must also satisfy $g(t_0) = 1$. As this is impossible, statment iii) is verified.

To show that $\mathcal{K} \subset \mathrm{clos}(\mathrm{int}_{\mathrm{rel}}(\mathcal{K}^{pc}))$, we could use some elementary but not very enlightening calculations giving close points or whole rays in the interior. Instead, we prefer to argue in a purely geometric manner, just using the following two basic facts.

a) If one of the inequalities (1.5) or (1.6) is strict in the point $(y,z) \in GCC^{-1}_{A,B}(\mathcal{K}^{pc})$, and if $[(y,z),(y',z')]$ corresponds to a rank-one segment in $\mathcal{K}^{pc}$, then this inequality remains sharp in all of the open $((y,z),(y',z'))$. Indeed, since the determinant is affine along rank-one segments and the norm is convex, this is obvious.
b) The function $t \in \mathbb{R} \to |x+tx'|$ is strictly convex provided the two points x and x' are not colinear. In fact, this is nothing but the strict convexity of the euclidean plane.

Note that statement a) together with iii) and iv) already proves the second part of v). Now, we start from $X_0 \cong (y_0,0) \in \mathrm{SO}(2)A$ rank-one connected to $X_1 \cong (0,z_1) \in \mathrm{SO}(2)B$ and choose $Y_1 = (\lambda y_0, (1-\lambda)z_1) \in (X_0, X_1)$ close to X_1. Hence, by the statements i) and ii) we know that Y_1 is rank-one connected to $X_2 \cong (y_2,0) \in \mathrm{SO}(2)A \setminus \{X_0\}$. Since λy_0 and y_2 are not colinear, by b) the considered inequality containing $|y|$ becomes strict on all of the interval (Y_1, X_2). In case of equal determinants we are done, and otherwise we just have to choose $Y_2 \in (Y_1, X_2)$ close to X_2 and hence again rank-one connected to a suitable $X_3 \in \mathrm{SO}(2)B$. Then on (Y_2, X_3) the inequality containing $|y|$ remains strict by a) and the second one becomes strict by b). In both cases, these segments reach arbitrarily close to both wells in $\mathcal{K}$, so iv) follows.

Finally, we consider v). Due to ii), continuity is clear in those $(A,B) \in \mathcal{W}$ with $\det(A) \neq \det(B)$. The case $\det(A) = \det(B)$ is more subtle, the description in ii) and iii) are not really helpful if we consider nearby (A',B') with different determinant. Nevertheless, using the description given in [**89**] if follows that $\mathcal{K}^{pc}_{A,B}$ consists of all rank-one segments from $L_1(\mathcal{K}_{A,B})$ to $\mathcal{K}_{A,B}$. Moreover, $\mathcal{K}_{A,B}$ intersects the cone $\{\det(\cdot) = 0\} + Y$ for $Y \in L_1(\mathcal{K}_{A,B})$ transversally. This shows that neither $L_1(\mathcal{K}_{A',B'})$ nor $L_2(\mathcal{K}_{A',B'})$ can suddenly become much smaller, so the latter will still be close to all of $\mathcal{K}^{pc}_{A,B}$. Because even more is shown in Theorem 4.2 of [**108**], we skip the details. The same paper provides also a formal proof of the rather obvious fact that $(A,B) \to \mathcal{K}^{pc}_{A,B}$ is upper semicontinuous, hence we are done.

□

CHAPTER 2

The hyperbolic Monge-Ampère Equation and singularities of mappings with two constraints

1. Main results

In this chapter we study the equation

$$\nabla f = (\nabla f)^T \tag{2.1}$$

$$\det(\nabla f) = -1 \tag{2.2}$$

for a Lipschitz map $f : \Omega \to \mathbb{R}^2$, where Ω is the unit disc in $\mathbb{R}^2$. Since (2.1) is equivalent to the existence of a potential $u : \Omega \to \mathbb{R}$ with $\nabla u = f$, the system (2.1), (2.2) is the hyperbolic Monge-Ampère equation

$$\det(\nabla^2 u) = -1.$$

Surprisingly, (2.1), (2.2) can, at least formally, be transformed into a linear wave equation by a substitution which mixes the dependent and the independent variables. In this way the the dependent variables in the determinant get seperated from each other and the nonlinearity in the equation disappears. This substitution is known eg. already from the theory of Lagrangian submanifolds ([**82**]), but found its applications also in the theory of Hamiltonian systems ([**62**]) and investigations on partial regularity of minimizers in the class of area preserving mappings ([**33**]). For this purpose we have to assume that

$$\partial_1 f_1 \geq c > 0 \tag{2.3}$$

and write $y = f(x)$. Then, formally (2.1), (2.2) can be formulated as (see Section 7 below for the details)

$$dy_1 \wedge dx_1 + dy_2 \wedge dx_2 = 0 \tag{2.4}$$

$$dy_1 \wedge dy_2 + dx_1 \wedge dx_2 = 0. \tag{2.5}$$

In view of (2.3) we can write $x_1 = g_1(y_1, x_2)$, $y_2 = g_2(y_1, x_2)$ and in these variables (2.5), (2.4) becomes

$$\partial_2 g_1 + \partial_1 g_2 = 0, \quad \partial_2 g_2 + \partial_1 g_1 = 0$$

which implies that there is a potential G satisfying

$$\nabla G = (g_1, -g_2) \text{ and so } \Box G = 0, \text{ where } \Box = \partial_1^2 - \partial_2^2. \tag{2.6}$$

Using the explicit solution of the wave equation we deduce that

$$y_2 + x_1 = h_1(y_1 - x_2) \text{ and } y_2 - x_1 = h_2(y_1 + x_2). \tag{2.7}$$

If we assume in addition that $\partial_2 f_2 \geq c > 0$ we similarly obtain

$$y_1 - x_2 = \tilde{h}_1(y_2 + x_1), y_1 + x_2 = \tilde{h}_2(y_2 - x_1). \tag{2.8}$$

This raises the following questions

(i) Are all Lipschitz solutions of (2.1),(2.2) locally or even globally of the form (2.7),(2.8), at least after a suitable rotation of the coordinate axes?

(ii) Are Lipschitz solutions of (2.1),(2.2) which satisfy affine boundary conditions necessarily trivial?

(iii) Are the solutions of (2.1),(2.2) trivial if in addition $\nabla f \in \mathcal{A}$ a.e. and $\mathcal{A}$ does not contain any rank-one connection? (Note, the representation (2.7) or (2.8) provides strong evidence for such a result.)

In this chapter, we will show that the representation (2.7), (2.8) holds locally away from a discrete set of "branch" points - see Theorem 2.20 and Theorem 2.34 below. Moreover, such singular branch points can indeed exist as Example 2.26 shows, and hence (2.7),(2.8) does in general hold only up to the first singularity. This fact is described in Corollary 2.23 and is reminiscent of the behaviour of meromorphic functions. Affirmative answers to the questions (ii) and (iii) are given in Corollary 2.7 and Theorem 2.28. In Proposition 2.30 we establish related results for the degenerate equation

$$(\nabla f) = (\nabla f)^T, \det(\nabla f) = 0 \tag{2.9}$$

which we study without the usual convexity assumptions made e.g. in [**17**] and [**98**].

Apart from this independent interest, in Chapter 4 we will see that these results play a crucial role in the characterization of maps having only four gradients which possess no rank-one connections.

2. Overview of the proof

To treat (2.1), (2.2) and the degenerate case (2.9) simultaneously, we consider the system

$$\nabla f = (\nabla f)^T, \det(\nabla f) = -D \text{ where } D \geq 0 \text{ constant}$$

and partially motivated by (2.7), (2.8) we introduce the modified maps

$$f_+(z) = f(z) + \mathrm{i}\sqrt{D}z, f_-(z) = f(z) - \mathrm{i}\sqrt{D}z.$$

Since the gradient of the added terms is antisymmetric, the Jacobian of this sum is just the sum of the Jacobians and hence vanishes identically. We can even perturb f_+ or f_- a little bit more and then conclude that both f_+ and f_- are singular limits of mappings of bounded distortion. Because such maps are known to be open, our first major observation (contained already in the joint paper [**22**] with Miroslav Chlebík) is Proposition 2.6 stating that $f_\pm$ satisfy the degenerate monotonicity condition

$$f_*(\bar{U}) = f_*(\partial U) \text{ for } U \subset \Omega \text{ and } * \in \{-, +\}.$$

Hence, $\mathrm{im}(f_*)$ is not just a Lebesgue zero set as follows from the area formula, but even a Lipschitz curve, perhaps selfintersecting. In Proposition 2.8 we establish further basic properties. In particular, if we exclude by a generic choice of the coordinate system some singularities, then

- connected components of level sets of one coordinate $(f_*)_i$ are also level sets of the other coordinate function $(f_*)_{i'}$, where $\{i, i'\} = \{1, 2\}$
- for $D > 0$ is $(f_+ - f_-)$ a nontrivial similitude and hence, all level sets of f_+ essentially fit into the fixed lipschitz curve $\mathrm{im}(f_-)$ and the same holds if we interchange f_+ and f_-. Moreover, the level sets of f_+ and f_- do intersect transversally.

Another conclusion of the degenerate monotonicity is that connected components of level sets reach up to the boundary of any open set intersecting them, see Lemma 2.10. In fact, a crucial step in our proof is to strengthen this result, and to show that the level set can never terminate at an interior point - in other words, we need to prove that each of its components intersects the boundary twice. In Proposition 2.12 we establish that this fact holds generically. The proof of this Proposition is a little bit involved and utilizes deeper results about the nice topological structure of Jordan domains.

At this point the proof splits into two cases. If $D > 0$ then ∇f is contained in the set $\{A \in \mathbb{M}^{2\times 2}_{sym}\ ;\ \det(M) = -D\}$ which is in conformal coordinates just the one-sheeted hyperboloid $\{x \in$

$\mathbb{R}^3$; $x_3^2 - x_1^2 - x_2^2 = -D\}$. In the second, degenerate case the gradient ∇f stays in the rank-one cone intersected with $\mathbb{M}^{2\times 2}_{sym}$.

In the first case the seemingly simple Lemma 2.13 contains the key observation that the class of injective subpaths of level sets of f_* is closed under pointwise convergence. This allows us to show

a) no level set of f_* can terminate in an inner point of Ω, see Theorem 2.16
b) given a point $z \in \Omega$ and a "sector" determined by two paths in $f_+^{-1}(f_+(z))$ from z to $\partial\Omega$, then there exists a path in the f_--level set which starts at z and goes into that sector. The same holds if we interchange $+$ and $-$, see Lemma 2.17.

Now we introduce the notion of a branch point.These are the points were level sets split, so at least 3 disjoint curves inside the same level set start at such points. By what we told before, z is a branch point for f_+ if and only if it is a branch point for f_-. Because each f_* level set is contained in the fixed Lipschitz curve $\mathrm{im}(f_{\bar{*}})$ of controlled length, a first and simple combinatorial argument shows that the branch points contained in one fixed level set can not cluster. Next, a refined version of statement b) shows that the branch points contained in any of the sectors can not cluster. Indeed, this would force two of the three branches to collapse and contradict our injectivity result Lemma 2.13. Figure 1 summarizes these ideas which lead to Theorem 2.20 ensuring that branch points can not cluster inside Ω.

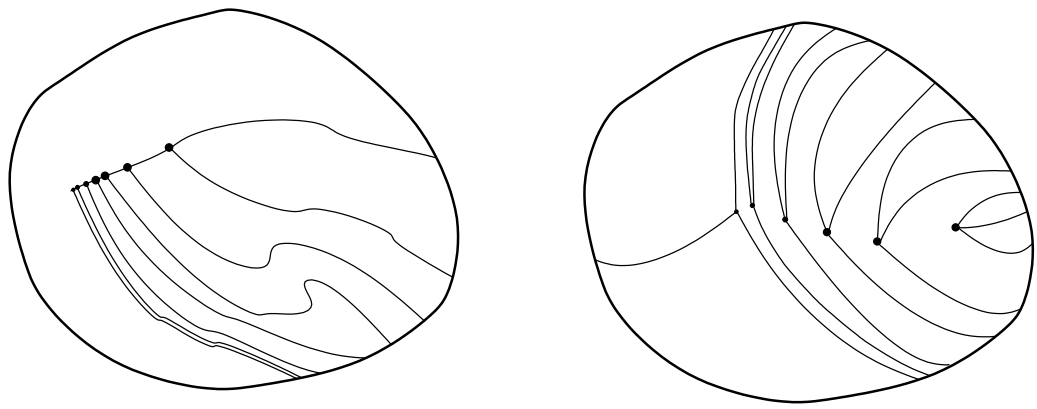


FIGURE 1. Clustering of branch points

Finally, in Theorem 2.22 we give a precise geometrical description of f near points where level sets do not branch. Proposition 2.24 shows that this description indicates also a way to construct such mappings. In this way we establish a correspondence between regular parts of f and pairs of transversal Lipschitz curves. Gluing together examples constructed in this way suitably, we establish in Example 2.26 that branch points can indeed occur. We conclude the discussion of the nondegenerate case with Theorem 2.28 which completely answers questions about the existence of rank-one connections in the essential image of the gradient of our solution f.

Next, we turn to the degenerate case $D = 0$. In this situation $f = f_+ = f_-$, so it might appear that we loose too much information since we can not use the interplay between f_+, f_- and the transversality of their level sets. We gain, however, the information that gradients are now singular and symmetric, hence their kernel is determined by their image which is in turn given by the tangent to $\mathrm{im}(f)$ at $f(z)$. This leads, similar to the situation in Theorem 2.28, to the conclusion that "almost all" level sets are lines - the remaining singular level sets can be controlled by more sophisticated arguments. In this manner, we establish also for $D = 0$ a description of solutions f. Again, the construction given in Proposition 2.32 shows that these necessary conditions characterize the solutions. Moreover, we see that in this degenerate case an uncountable range of the gradients does not need to contain rank-one connections.

At the very end of this chapter we discuss in Section 7 in full detail the relations between our results and the formal calculations done above. It turns out that the substitutions made there are possible if and only if we are away from the branch points. We give an example demonstrating the interesting fact that in the degenerate case $D = 0$ such a substitution might be everywhere impossible.

3. The topological toolkit

Since we need to gain good control about all level sets of our functions we can not use a priori knowledge about their regularity. Therefore, we recall some useful facts - mainly about general continuous curves.

The first result states, roughly speaking, that inside any curve an injective path connecting the endpoints can be found by cutting out loops in a sufficiently sophisticated way. The statement is well known in the theory of arcwise connected spaces, see e.g. [**55**], §45, II Theorem 1 and Theorem 2. For the convenience of the reader we include a simple proof based on an idea presumably first time used by M.J.L.Kelley. A similar presentation can be found in Theorem 4.1 of Chapter II in [**100**]

LEMMA 2.1. *Let $\varphi : [0,1] \to (X,\tau)$ be continuous and suppose $\varphi(1) \neq \varphi(0)$. Then there exists $\psi : [0,1] \to \varphi([0,1])$ continuous and injective such that $\psi(0) = \varphi(0)$ and $\psi(1) = \varphi(1)$*

Proof. We consider the family $\mathscr{U}$ of all sets U open in $[0,1]$ such that each connected component V of U satisfies $\varphi(\sup(V)) = \varphi(\inf(V))$. Ordering $\mathscr{U}$ by inclusion, it is fairly easy to check that for any chain $\mathscr{C} \subset \mathscr{U}$ the relatively open set $\bigcup \mathscr{C}$ belongs to $\mathscr{U}$ again. So the Hausdorff maximal principle ensures the existence of a maximal member U_0 of $\mathscr{U}$. It is obvious that maximality of U_0 implies the following two properties of the set $C = [0,1] \setminus U_0$

- $(0,1) \cap C$ is a nonempty compact set without isolated points.
- If $x, y \in C$ and $\psi(x) = \psi(y)$ then $C \cap (x,y) = \emptyset$.

Note that $\mathrm{clos}(C \cap (0,1))$ is the support of a continuous probability μ on $[0,1]$, in fact this is a simple consequence of the well-known result that any perfect subset of a Polish space contains a homeomorphic image of the standard Cantor set, see e.g. Theorem 6.2 in [**42**]. Now we can define our function ψ by

$$\psi(t) = \varphi(\max(\{s \in [0,1] \ ; \ \mu([0,s]) \leq t\})).$$

It is rather routine to verify that ψ has all required properties, just observe that we have also $\psi(t) = \varphi(\min(\{s \in [0,1] \ ; \ \mu([0,s]) \geq t\}))$.

DEFINITION 2.2. We will use the following notations to distinguish different classes of arcs. *A curve is a continuous mapping φ of a compact interval, for which we can of course always choose $[0,1]$. If there is no danger of ambiguity, we will call also the image $\mathrm{im}(\varphi)$ of φ, which is in our situation always a subset of the plane, a curve. A path is a homeomorphic image of $[0,1]$, and a regular path φ is a path $\varphi : [c_1, c_2] \to \mathbb{R}^2$ such that φ is 1-Lipschitz and $\mathscr{H}^1(\varphi(A)) = |A|$ for any Borel set $A \subset [c_1, c_2]$.* (Note that due to Lemma 2.1 any curve contains a path connecting its endpoints, and that any path of finite $\mathscr{H}^1$-measure can be turned into a regular path using a reparametrization by arclength).

For a set M (understood as a topological subspace of $\mathbb{R}^2$) and $x \in M$ we denote by $\mathrm{CC}(M,x)$ the connected component of M containing the point x. As usual, this is the largest connected subset of M containing x and, therefore, it is a set relatively closed in M.

Finally, we will have to construct many closed curves and for this purpose we will form products of curves in the way used e.g. to define fundamental groups. *Let φ_1 be a curve from x to y (i.e.*

$\varphi_1(0) = x$ and $\varphi_1(1) = y$) and φ_2 be a curve from y to z then $\mathrm{conc}(\varphi_1, \varphi_2)$ *is the curve defined by*

$$\mathrm{conc}(\varphi_1, \varphi_2)(t) = \begin{cases} \varphi_1(2t) & \text{if } t \leq \frac{1}{2} \\ \varphi_2(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

The curve inverse to φ is defined by $\mathrm{inv}(\varphi)(t) = \varphi(1-t)$.

As already mentioned, in the sequel we will have to deal with level sets which a priori do not have much regularity. The following lemma guarantees that these sets anyhow contain sufficiently large regular paths.

LEMMA 2.3. *Let $M \subset \mathbb{R}^2$ be a connected G_δ-set of finite $\mathscr{H}^1$-measure. Then M is pathwise and locally connected, in fact for all $x, y \in M$ there is a regular path in M from x to y and for all $x \in M$ and ε positive we find an open U with $x \in U \subset B(x, \varepsilon)$ such that $M \cap U$ is connected.*

Proof. Due to 2B Corollary of [**37**] the space M is locally connected and as connected components of open subsets of such spaces are open again, also the required set U is found easily. By 3A Lemma in [**37**] for any $x, y \in M$ there is a curve from x to y in M. Lemma 2.1 now ensures the existence of a subpath α from x to y and as $\mathscr{H}^1(\mathrm{im}(\alpha)) < \infty$, it can be turned into a regular path.

We will also have to work with open sets of a complicated geometry. Therefore, Jordan domains will be helpful to control at least the topological properties of such sets.

NOTATION 2.4. Following the terminology in Chapter 14 of [**80**], we say that $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ is a *Jordan curve* if γ is a homeomorphism and that an open set $G \subset \mathbb{R}^2$ is a *Jordan domain* if it is bounded and ∂G is the image of a Jordan curve. A point x in the boundary of a bounded open set $G \subset \mathbb{R}^2$ is said to be a simple boundary point if for every sequence $x_k \to x$, $x_k \in G$ there is a continuous map $\varphi : (0, 1] \to G$ such that $\varphi(1/k) = x_k$ and that $\lim_{t \to 0_+} \varphi(t) = x$. We recall the following results.

(a) If $G \subset \mathbb{R}^2$ is open and bounded, both G and $\mathbb{R}^2 \setminus G$ are connected and each point $x \in \partial G$ is a simple boundary point, then there is a homeomorphism h mapping $\{z \in \mathbb{C}\ ;\ |z| \leq 1\}$ onto $\bar{G}$ (and hence $h(\{z \in \mathbb{C}\ ;\ |z| < 1\}) = G$) , see [**80**], Theorem 14.19.
(b) If γ is a Jordan curve, then $\mathbb{R}^2 \setminus \mathrm{im}(\gamma)$ has one bounded and one unbounded connected component, usually denoted by $U_b(\gamma)$ and $U_\infty(\gamma)$, and $\mathrm{im}(\gamma) = \partial U_b(\gamma) = \partial U_\infty(\gamma)$, see Theorem 3.29 in [**36**].
(c) The situation described in (a) is equivalent to G being a Jordan domain, see Remark 14.20 in [**80**].

4. The basic setting and tricks

NOTATIONS AND BASIC SETTING 2.5. In the sequel we consider a Lipschitz map $f : \bar{\Omega} \to \mathbb{R}^2$ where $\Omega = B(0, 1)$ for simplicity - any bounded Jordan domain with boundary of finite length would work as well. We assume that, for some $D \geq 0$ fixed,

$$\nabla f(x) \in \mathcal{H}_{-D} = \{M \in \mathbb{M}^{2\times 2}_{sym}\ ;\ \det(M) = -D\} \text{ for a.e. } x \in \Omega.$$

With the usual identification $z = x_1 + \mathrm{i}x_2$, we define for $* \in \{+, -\}$, $i \in \{1, 2\}$ the auxiliary maps

$$f_*(z) = f(z) * \mathrm{i}\sqrt{D}z \text{ and } f_{*,i}(z) = \langle f_*(z), e_i \rangle,$$

e_1, e_2 being the canonical unit vectors in the plane.

Moreover, for $x \in \bar{\Omega}$ we denote

$$l_*(x) = \{y \in \bar{\Omega}\ ;\ f_*(y) = f_*(x)\} \text{ and } l_{*,i}(x) = \{y \in \bar{\Omega}\ ;\ f_{*,i}(y) = f_{*,i}(x)\}.$$

Finally, it will be convenient to agree on the notation

$$\bar{*} = \begin{cases} - & \text{if } * = + \\ + & \text{if } * = - \end{cases}$$

PROPOSITION 2.6. *(see the proof of Theorem 6 in* [**22**]*) For* $* \in \{+,-\}$ *and* $M \subset \Omega$ *we have* $f_*(\bar{M}) = f_*(\partial M)$.

Proof. We define for $\delta > 0$

$$g_\delta(x) = f(x) * \mathrm{i}\sqrt{D+\delta}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x,$$

so we have the uniform convergence $g_\delta \rightrightarrows f_*$ as $\delta \to 0_+$. Because ∇f is symmetric and hence orthogonal to the cofactor of the linear part added, it follows from 1.1 that $\det(\nabla g_\delta)(x) = \det(\nabla f)(x) + (D+\delta) = \delta$ a.e. in Ω and hence g_δ is a mapping of bounded distortion. A fundamental result about such maps, see e.g. Theorem 6.4 in Chapter II of Reshetnyak's general treatment of such maps [**79**], ensures that each g_δ is an open mapping. In particular, $\partial g_\delta(N) \subset g_\delta(\partial N)$ for all $N \subset \Omega$.

Now suppose there is $y \in f_*(\bar{M}) \setminus f_*(\partial M)$. Because $g_\delta \rightrightarrows f_*$, there is an $\varepsilon > 0$ such that $B(y,\varepsilon) \cap \partial(g_\delta(\operatorname{int}(M))) \subset B(y,\varepsilon) \cap g_\delta(\partial M) = \emptyset$ if $\delta \in (0,\varepsilon)$. Obviously, $y \in f_*(\operatorname{int} M)$ and hence $B(y,\varepsilon) \cap g_\delta(\operatorname{int} M) \neq \emptyset$ which shows $B(y,\varepsilon) \subset g_\delta(\operatorname{int} M)$ and in particular $|B(y,\varepsilon)| \leq |g_\delta(M)| \leq \delta|\bar{\Omega}|$ for all $\delta \in (0,\varepsilon)$. The obvious contradiction for δ very small finishes the proof.

COROLLARY 2.7. *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded domain,* $D \geq 0$ *and let* $f : \Omega \to \mathbb{R}^2$ *be Lipschitz such* $\nabla f(x) \in \mathcal{H}_D$ *a.e. in* Ω. *If there is an affine map* A *which agrees with* f *on* $\partial\Omega$ *then* $f \equiv A$ *on the whole* Ω.

Proof. Adding a constant to f if necessary, we might assume that A is even a linear map. Since the determinant is a Null-Lagrangian and since $A = \frac{1}{|\Omega|}\int_\Omega \nabla f(x)\,dx$ we conclude that also $A \in \mathcal{H}_{-D}$. Therefore, we see that f_+ agrees on $\partial\Omega$ with the singular matrix $A_+ = A + \sqrt{D}\mathrm{i}$ which can be written as $a \otimes b$. The foregoing Lemma 2.6 ensures that $f_+(\bar{\Omega}) = f_+(\partial\Omega)$ and therefore $\nabla f_+(x) = a \otimes b_x$ for almost every $x \in \Omega$. But we also know that $a \otimes (b_x - b) = \nabla f_+(x) - a \otimes b = \nabla f(x) - A$ is a symmetric matrix. Hence $b_x - b \| a$ almost everywhere. Consider first the case $a = 0$. For $D = 0$ this implies that f_+ is a constant function, which proves the desired assertion $f = f_+ \equiv 0 = A$. If $D > 0$ we obtain the contradiction that $A = A_+ - \mathrm{i}\sqrt{D} = -\mathrm{i}\sqrt{D}$ is non symmetric. Therefore, we can assume $a \neq 0$ and hence $b_x - b = \lambda_x a$. So, we see that $f(x) - Ax$ is constant in the direction $\mathrm{i}a$. As this function vanishes on the boundary of Ω it vanishes also on the whole domain and the proof is finished.

PROPOSITION 2.8. *After a generic orthogonal change of coordinates the maps* f_+ *and* f_- *have the following properties:*

(a) *The sets* $\operatorname{im}(f_+)$ *and* $\operatorname{im}(f_-)$ *do not contain any vertical nor horizontal segments*
(b) *If* $\nabla f(x) \neq 0$ *almost everywhere, so in particular if* $D \neq 0$, *then for a.e.* $x \in \Omega$ *we have* $\nabla f_{*,i}(x) \neq 0$ *whenever* $* \in \{+,-\}$ *and* $i = 1,2$.

Consequently, we obtain that

(c) *If* $y \in M \subset l_{*,i}(x)$ *and* M *is connected, then* $M \subset l_*(y)$. *Moreover, as* $f_*(y) \bar{*} 2\mathrm{i}\sqrt{D} \cdot l_*(y) \subset f_{\bar{*}}(\partial\Omega)$, *we infer that for* $D > 0$ *any* $l_*(y)$ *is of finite measure and any such* M *is pathwise and locally connected.*

Finally, we have the following "transversality" between the f_+*- and the* f_-*-level sets:*

(d) *If* $y \in l_{\bar{*}}(x)$, $x, y \in \bar{\Omega}$ *then* $\operatorname{dist}(y, l_*(x)) \geq 2\sqrt{D}|y-x|/\operatorname{lip}(f_*)$.

Proof. First, pick any $\theta \in \mathbb{R}$ and consider $F_\theta(z) = e^{-\mathrm{i}\theta} f(e^{\mathrm{i}\theta} z)$. Obviously, we have $\nabla F_\theta(z) = e^{-\mathrm{i}\theta}\nabla f(e^{\mathrm{i}\theta} z)e^{\mathrm{i}\theta}$ and hence, since ∇f was symmetric ∇F_θ is symmetric as well. Moreover, also

$\det(\nabla F_\theta(z)) = \det(\nabla f(e^{\mathrm{i}\theta}z)) = -D$ almost everywhere in Ω. Notice that $F_{\theta,*} = F_\theta(z) * \mathrm{i}\sqrt{D}z = e^{-\mathrm{i}\theta}(f(e^{\mathrm{i}\theta}z) * \mathrm{i}\sqrt{D}e^{\mathrm{i}\theta}z) = e^{-\mathrm{i}\theta}f_*(e^{\mathrm{i}\theta}z)$, so $\mathrm{im}(F_{\theta,*}) = e^{-\mathrm{i}\theta}\cdot\mathrm{im}(f_*)$.

Let S_* be the set of all $d \in \mathbb{S}^1$ for which there exist x, y with $0 \neq (x-y)\|d$ and $[x,y] \subset \mathrm{im}(f_*)$. Since Proposition 2.6 implies that $\mathscr{H}^1(\mathrm{im}(f_*)) = \mathscr{H}^1(f_*(\partial\Omega)) \leq \mathrm{lip}(f_*)\mathscr{H}^1(\partial\Omega) < \infty$, it is easy to see that S_* is at most countable. Therefore, for all but countably many $\theta \in [0,\pi)$ the rotated map F_θ fulfills (a). Similarly, if $\nabla f(x) \neq 0$ almost everywhere, then the set S'_* of all $d \in \mathbb{S}^1$ for which $|\{x \in \Omega\ ;\ \mathrm{im}(\nabla f_*)(x) \perp d\}| > 0$ is at most countable, so selecting θ from a co-countable subset of $[0,\pi)$ and replacing f by F_θ we can ensure that both (a) and (b) are satisfied.

Now, if $M \subset l_{*,i}(x)$ is connected, then $f_*(M)$ is a connected subset of $\mathrm{im}(f_*) \cap (f_*(x) + e_i^\perp)$. The latter set is totally disconnected by (i) and hence $f_*(M)$ is a singleton. This shows $M \subset l_*(y)$. Next, we choose any $z \in l_*(y)$ and see that $f_{\bar{*}}(z) = f_*(z)\bar{*}2\mathrm{i}\sqrt{D}z = f_*(y)\bar{*}2\mathrm{i}\sqrt{D}z$ and consequently, $f_*(y) \bar{*} 2\mathrm{i}\sqrt{D}\cdot l_*(y) \subset f_{\bar{*}}(\Omega) \subset f_{\bar{*}}(\partial\Omega)$ by Proposition 2.6. This together with Lemma 2.3 proves part (c).

Finally, to verify (d) we pick an arbitrary $z \in l_*(x)$ and estimate $\mathrm{lip}(f_*)|y-z| \geq |f_*(y)-f_*(z)| = |f_*(y) - f_*(x)| = |f_{\bar{*}}(y) * 2\mathrm{i}\sqrt{D}y - f_{\bar{*}}(x) \bar{*} 2\mathrm{i}\sqrt{D}x| = 2\sqrt{D}|y-x|$. □

Next, we exclude that the level curves of f_* can create loops.

LEMMA 2.9. *Let $N \subset \bar{\Omega} \cap l_*(x)$ be a connected G_δ-set and let $\varphi : [0,1] \to \bar{\Omega} \cap l_*(x)$ be a path. Suppose $\varphi(0), \varphi(1) \in N$. If $\mathscr{H}^1(l_*(x)) < \infty$, so in particular if $D > 0$, then $\mathrm{im}(\varphi) \subset N$.*

Proof. Due to our assumption $\mathscr{H}^1(N) < \infty$, so Lemma 2.3 ensures the existence of a regular path ψ in N from $\varphi(1)$ to $\varphi(0)$. If there is $z = \varphi(t) \notin N$, then $t \in (0,1)$. We set $t_+ = \min\{s > t\ ;\ \varphi(s) \in \mathrm{im}(\psi)\}$ and $t_- = \max\{s < t\ ;\ \varphi(s) \in \mathrm{im}(\psi)\}$ and define $\bar{t}_+ = \psi^{-1}(\varphi(t_+))$, $\bar{t}_- = \psi^{-1}(\varphi(t_-))$. It is easy to check that $\gamma = \mathrm{conc}(\psi_{|[\bar{t}_+ \to \bar{t}_-]}, \varphi_{|[t_- \to t_+]})$ is a Jordan curve, moreover $\mathrm{im}(\gamma) \subset l_*(x) \cap \bar{\Omega}$. Let U_b be the bounded component of $\mathbb{R}^2 \setminus \mathrm{im}(\gamma)$, we claim that $U_b \subset \Omega$. In fact, if $y \in \mathbb{R}^2 \setminus (\mathrm{im}(\gamma) \cup \Omega)$ then there is a path $\alpha : [0,1] \to \mathbb{R}^2 \setminus \mathrm{im}(\gamma)$ connecting y to a point very far from $\bar{\Omega}$ and hence both are in the unbounded component of $\mathbb{R}^2 \setminus \mathrm{im}(\gamma)$ - for this it is enough to make sure that $\mathrm{im}(\alpha) \cap \bar{\Omega} \subset \{y\}$. Now Proposition 2.6 tells us that $f_*(U_b) \subset f_*(\mathrm{im}(\gamma)) = \{x\}$, so $U_b \subset l_*(x)$ which contradicts $\mathscr{H}^1(l_*(x))$ finite.

LEMMA 2.10. *Let $U \subset \Omega$ be open in the plane and $x \in U$. Then for any $* \in \{+,-\}$ the set $\mathrm{clos}(\mathrm{CC}(l_*(x) \cap U, x))$ intersects ∂U.*

Proof. Assume the conclusion to fail. We set $M = \mathrm{CC}(l_*(x) \cap U, x)$ and claim $M = \bar{M}$. Indeed, if $y \in \bar{M} \setminus M$, then $M \cup \{y\} \subset l_*(x)$ and is also a connected set strictly larger than M. Hence $y \notin U$, so $\bar{M} \cap \partial U \neq \emptyset$ contrary to our assumption.

Therefore, we find an $\varepsilon > 0$ such that $B(M, 2\varepsilon) \subset U$ and define $V = \{y\ ;\ \mathrm{dist}(y, \mathbb{R}^2 \setminus U) \geq \varepsilon\}$. Then $M \subset \mathrm{int}(V)$, $V \cap l_*(x)$ is compact and M is a connected component of the space $V \cap l_*(x) \subset U \cap l_*(x)$. In fact, this inclusion was the reason why V was introduced as it seems more complicated to prove that M is a component of the larger $\bar{U} \cap l_*(x)$. Anyhow, living now in a compact space we know from §42, II.2 in [**55**] that M is also a quasicomponent of this space, which is by definition the intersection of some family of sets which are both closed and open in $V \cap l_*(x)$. So for each $y \in \partial V \cap l_*(x)$ we find a set $M_y \supset M$ closed and open in $V \cap l_*(x)$ such that $y \notin M_y$. Due to the compactness of $\partial V \cap l_*(x)$ we obtain a set $\tilde{M} \supset M$ which is a closed and open subspace of $V \cap l_*(x)$ and fulfills $\tilde{M} \cap \partial V \cap l_*(x) = \emptyset$. As $\tilde{M} \subset l_*(x)$ we conclude $\tilde{M} \cap \partial V = \emptyset$. Because $\tilde{M}$ and $(V \cap l_*(x)) \setminus \tilde{M}$ are compact, there is a $\delta > 0$ such that $2\delta < \mathrm{dist}(\tilde{M}, \partial V \cup ((V \cap l_*(x)) \setminus \tilde{M}))$, in particular $B(\tilde{M}, 2\delta) \subset \mathrm{int}(V)$.

We set $G = \{y\ ;\ \mathrm{dist}(y, \tilde{M}) < \delta\}$, then $M \subset G \subset \mathrm{int}(V)$. Moreover, $y \in \partial G$ implies that $y \notin \tilde{M}$ as well as $y \notin (V \cap l_*(x)) \setminus \tilde{M}$ and consequently $y \notin V \cap l_*(x)$. This shows that $f_*(y) \neq f_*(x)$ whenever $y \in \partial G$. But as $x \in G$, we see that $f_*(\bar{G}) \neq f_*(\partial G)$. This contradiction to Proposition 2.6 finishes our proof.

DEFINITION 2.11. *We say that $t \in \mathbb{R}$ is a* regular *value for $f_{*,i}$ if for any $U \subset \Omega$ open in the plane and any $x \in U$ with $f_{*,i}(x) = t$ the cardinality of* $\operatorname{clos}(\mathrm{CC}(l_*(x) \cap U, x)) \cap \partial U$ *is at least two.*

PROPOSITION 2.12. *Suppose that for a given $(*, i)$ and $t \in \mathbb{R}$ the set $C = f_{*,i}^{-1}(t)$ is of finite $\mathscr{H}^1$-measure and that $x \notin C$ whenever $f_{*,i}$ has a local extremum at $x \in \Omega$. Then t is a regular value for $f_{*,i}$.*

Proof. We argue by contradiction again, which means that due to Lemma 2.10 we suppose $\partial U \cap \bar{M} = \{p\}$ where $M = \mathrm{CC}(l_*(x_0) \cap U, x_0)$, $f_{*,i}(x_0) = t$ and $\bar{M} = M \cup \{p\}$ since we already mentioned that connected components are always relatively closed. The first observation we make is that

$$\text{for all } x \in M \text{ there exists } r_x > 0 \text{ such that } B(x, 2r_x) \cap l_{*,i}(x_0) \subset M \text{ and } B(x, 2r_x) \subset U. \tag{2.10}$$

Of course, the second inclusion is easily arranged. To obtain the first one, notice that Lemma 2.10 gives the estimate $\mathscr{H}^1(\mathrm{CC}(l_{*,i}(y) \cap U, y)) \geq \operatorname{dist}(y, \partial U)$ for any $y \in U$. As all different connected components are disjoint and closed in U and since $\mathscr{H}^1(l_{*,i}(x_0)) < \infty$ by assumption, we conclude that only finitely many connected components of $l_{*,i}(x_0) \cap U$ can intersect $B(x, {}^1\!/\!_2 \operatorname{dist}(x, \partial U))$. But now (2.10) is obvious.

In the rest of the proof we will modify the set U into a set G satisfying

$$G \subset \Omega \text{ is a Jordan domain, } \bar{M} \cap \partial G = \{p\} \text{ and } M = G \cap l_{*,i}(x_0). \tag{2.11}$$

Suppose for a moment that such a domain was found. We show that the open set $G \setminus M = G \setminus l_{*,i}(x_0)$ is connected, therefore $f_{*,i}(G \setminus l_{*,i}(x_0)) \subset (t, \infty)$ or $f_{*,i}(G \setminus l_{*,i}(x_0)) \subset (-\infty, t)$. In any case, we would conclude that $f_{*,i}$ has a local extremum at x_0, which contradicts our assumptions. To verify the connectedness, we choose any two points $x_1, x_2 \in G \setminus M$. Using Lemma 2.10, we infer that for both sets $N_j = l_{*,i}(x_j) \cap G \setminus M$ and their components $C_j = \mathrm{CC}(N_j, x_j)$ there are $p_j \in \bar{C}_j \cap \partial(G \setminus M)$. As $x_j \notin l_{*,i}(x)$, we see that $\bar{C}_j \cap \bar{M} = \emptyset$ and hence $p_j \in \bar{C}_j \cap \partial G$. Using the fact that G is a Jordan domain, we find a homeomorphism h mapping $\bar{G}$ on the closed unit disc $\bar{D}$, which necessarily also fulfills $h(G) = D$. We fix $\varepsilon > 0$ with $h(p_1), h(p_2) \notin B(h(p), 3\varepsilon)$ and as $\partial D \cap \overline{h(M)} = h(\partial G \cap \bar{M}) = \{h(p)\}$ we also find a positive δ such that $h(M) \cap \{z \in D \ ; \ |z| > 1 - \delta\} \subset B(h(p), \varepsilon)$. Since $h(p_j) \in \overline{h(C_j)}$, we find $\tilde{p}_j \in h(C_j) \cap D \cap B(h(p_j), \delta)$ and an arc α in $\{z \in D \ ; \ |z| > 1 - \delta\} \setminus h(M)$ from $\tilde{p}_1$ to $\tilde{p}_2$. Consequently, $h^{-1} \circ \alpha$ connects C_1 to C_2 inside $G \setminus M$, which was to be shown.

It remains to give the construction of a set G satisfying (2.11). Based on (2.10) and using that M is locally compact, we find a sequence of open balls $B(x_j, r_j)$, $j = 1, \dots$ such that

a) $x_j \in M$, $x_j \to p$ as $j \to \infty$ and $|x_i - x_j| \neq r_i + r_j$ for all i, j,
b) $r_j < {}^1\!/\!_3 \min\{|x_j - p|, r_{x_j}\}$ with r_{x_j} from (2.10),
c) $M \subset G_0$ where $G_0 = \bigcup_j B(x_j, r_j)$.

We now obtain the desired domain in the usual way by "filling" the holes in G_0 - since G_0 as well as all the holes are connected, the new G is connected without holes and hence simply connected. To be more formal, let G be the complement of the unbounded connected component of $\mathbb{R}^2 \setminus G_0$. It is clear that G is open and that $\mathbb{R}^2 \setminus G$ is connected. Because $G_0 \subset U \subset \Omega$, we see that $\mathbb{R}^2 \setminus G$ contains the connected set $\mathbb{R}^2 \setminus \Omega$. Moreover, since M and each individual disk $B(x_j, r_j)$ are connected, the same is true for G_0 and the set $(\bar{G}_0 \setminus \partial G)$ which is contained between G_0 and $\bar{G}_0$. Because

$$G = (\bar{G}_0 \setminus \partial G) \cup \bigcup \{K \ ; \ K \text{ a bounded connected component of } \mathbb{R}^2 \setminus G_0\},$$

we see that G is also connected, and therefore simply connected.

Before verifying that it is indeed a Jordan domain, we check the two other statements in (2.11). For this purpose we use that $\partial G \subset \partial G_0$ and the later consists only of part of the boundaries of the disks that together G_0 plus the limit point p. Therefore it intersects $l_{*,i}(x_0)$ only in this one point p - meaning that $l_{*,i}(x_0)$ has nothing in common with the boundaries, and hence neither

with the interior, of the holes of G_0. More precisely, we see that $\bar{G}_0 \subset \bar{U}$, so it is obvious that $\partial G \subset \partial G_0 \subset \bigcup_j \partial B(x_j, r_j) \cup \{p\}$ and $\bar{M} \cap \partial G \subset \bar{M} \setminus G = \{p\}$. We notice that due to the definition of G_0 and the conditions a), b) we have even $\bar{G}_0 = \bigcup_j \bar{B}(x_j, r_j) \cup \{p\}$ and hence (2.10) ensures that $C \cap \bar{G}_0 = M \cup \{p\}$. To check that p is in fact a boundary point, we use again Lemma 2.3 to find a regular path $\varphi : [0, s_1] \to C$ with $\varphi(0) = x_0$, $\varphi(s_1) \in \partial\Omega$ and $\varphi([0, s_1)) \subset \Omega$. Set $s_0 = \inf\{s \ ; \ \varphi(s) \notin U\}$ then $\varphi(s_0) \in \partial U \cap \bar{M} = \{p\}$. Next, we notice that Lemma 2.9 implies $\varphi([s_0, s_1]) \cap M = \emptyset$. Since $G_0 \cap C = M$, this yields $\varphi([s_0, s_1]) \cap G_0 = \emptyset$. Hence, $p = \varphi(s_0)$ belongs to the unbounded component of $\mathbb{R}^2 \setminus G_0$ and this ensures $p \in \bar{G}_0 \setminus G \subset \partial G$. It is also clear that $M = C \cap G_0 \subset l_{*,i}(x_0) \cap G$. If there would be a $y \in l_{*,i}(x_0) \cap G \setminus M$ then $y \notin G_0$ and $K = \mathrm{CC}(\mathbb{R}^2 \setminus G_0, y)$ is bounded. So, $p \notin K$ and hence $\partial K \cap C \subset K \cap \partial G_0 \cap C \subset K \cap (M \cup \{p\}) = \emptyset$. Because this obviously contradicts Lemma 2.10 we conclude that indeed $l_{*,i}(x_0) \cap G = M$.

Finally, we show that each point $y \in \partial G$ is a simple boundary point. Then G is a Jordan domain due to Theorem 2.4 and we are done. It is helpful to make the following simple geometric observation. Given any point $z \in G$ there is always $z' \in [z, y] \cap G_0$ such that $[z, z'] \subset G$. This is obvious if $z \in G_0$. Else, $K = \mathrm{CC}(\mathbb{R}^2 \setminus G_0, z)$ is nonvoid and compact with $y \notin K$. Let $\tilde{z}$ be the point in $\mathrm{CC}([z, y] \cap K, z)$ with maximal distance to z. Then $[z, \tilde{z}] \subset K \subset G$ but in the same time $\tilde{z} \in \mathrm{clos}([z, y] \cap G_0)$. Hence a point $z' \in [\tilde{z}, y]$ sufficiently close to $\tilde{z}$ but in G_0 does the job. Now we distinguish two cases.

First, suppose $y \neq p$ and let us be given a sequence $y_l \in G$, $y_l \to y$. We need to find a continuous curve $\varphi : (0, 1] \to G$ with $\varphi(1/l) = y_l$ and $\lim_{t \to 0_+} \varphi(t) = y$. Using that $\partial G \setminus \{p\}$ is contained in the boundaries of the disks $B(x_j, r_j)$ we find a positive ε such that $B(y, 2\varepsilon) \cap B(x_j, r_j) \neq \emptyset$ implies $y \in \partial B(x_j, r_j)$. As G is pathwise connected, we can of course suppose that $y_l \in B(y, \varepsilon)$ for all l. Due to the observation just made, we can assume in addition that $y_l \in G_0$. For each such y_l there is a j with $y_l \in B(x_j, r_j)$ and so $y \in \partial B(x_j, r_j)$ and hence $[y_l, y) \subset G_0$. Moreover, for each $\delta > 0$ the set $S(j, \delta) = B(y, \delta) \cap B(x_j, r_j)$ is pathwise connected. Using condition a), it is easy to check that $y \in \partial B(x_j, r_j) \cap \partial B(x_{j'}, r_{j'})$ implies $S(j, \delta) \cap S(j', \delta) \neq \emptyset$ and consequently there are no problems to construct the requested φ.

In the very last step, we consider the case when $y = p$. Because we can again assume $y_l \in B(x_{j_l}, r_{j_l}) \subset G_0$ with $r_{j_l} \to 0$, it is clearly enough to find given any sequence $j_l \to \infty$ a suitable path φ for the $y_l = x_{j_l}$. As $M \cup \{p\}$ is a connected compact set of finite $\mathscr{H}^1$-measure, it is a locally connected space due to Lemma 2.3. So we find open sets $U_k \subset B(p, 1/k)$ with $p \in U_k$ and $(\{p\} \cup M) \cap U_k$ connected. Now we claim that the G_δ-set $U_k \cap M$ is connected as well, which by Lemma 2.3 suffices to obtain the paths needed to built the required curve φ. But indeed, otherwise we would find $z, z' \in M \cap U_k$ such that $z' \notin \mathrm{CC}(M \cap U_k, z)$. Since M itself is pathwise connected and $\mathscr{H}^1(M) < \infty$, there is a regular path $\psi_1 : [0, c_1] \to M$ with $\psi_1(0) = z$ and $\psi_1(c_1) = z'$. On the other hand, because $U_k \cap (M \cup \{p\})$ is (pathwise) connected, there is a regular path $\psi_2 : [0, c_2] \to U_k \cap (M \cup \{p\})$ connecting z to z'. Obviously, we have $p \in \mathrm{im}(\psi_2) \setminus \mathrm{im}(\psi_1)$. This final contradiction to Lemma 2.9 completes the proof of the proposition.

5. On the hyperboloid

It is clear that the nondegenerate case $D > 0$ in our basic assumption $\nabla f \in \mathcal{H}_{-D}$ can be reduced by a simple rescaling to the situation $D = 1$ in which we will work in the sequel. The following Lemma captures a crucial feature of the level sets of the functions under investigation. Even though it looks quite simple, it will prove to be a very useful tool.

LEMMA 2.13. *Let $\varphi_j : [a_j, b_j] \to \bar{\Omega}$ be 1-Lipschitz, injective and suppose $f_* \circ \varphi_j \equiv z_j$ is constant and $\mathscr{H}^1(\varphi_j(A)) = |A|$ for all Borel sets $A \subset [0, 1]$. If $a_j \to a$, $b_j \to b$ and $\varphi_j(t) \to \varphi(t)$ for all $t \in (a, b)$ then the 1-Lipschitz extension of φ to $[a, b]$ is injective.*

Proof. If $a = b$ then the statement is trivial and hence true. Otherwise φ has a unique Lipschitz extension to $[a, b]$ which we denote with the same symbol. Obviously, $z_0 = \lim_j z_j$ is the constant value of f_* along the image of φ. If the statement fails, we find $\alpha < \beta$ in $[a, b]$ with $\varphi(\alpha) = \varphi(\beta)$.

First we consider the situation when $\varphi_{|[\alpha,\beta]} \equiv c_0$ is constant. Due to Proposition 2.8(c)

$$z_j \mathbin{\bar{*}} 2\mathrm{i}\varphi_j(t) = f_{\bar{*}}(\varphi_j(t)) \subset f_{\bar{*}}(\partial\Omega) \text{ for all } j. \tag{2.12}$$

Thus, by assumption

$$2|\beta - \alpha| = \mathscr{H}^1(2\mathrm{i}\varphi_j([\alpha, \beta])) \leq \mu(f_{\bar{*}}(\varphi_j([\alpha, \beta]))), \text{ where } \mu = \mathscr{H}^1 \llcorner f_{\bar{*}}(\partial\Omega).$$

Hence, sending j to infinity we conclude that this finite measure μ charges sets of arbitrarily small diameter with measure at least $2(\beta - \alpha)$. This is a clear contradiction to the continuity of μ.

Therefore, we can assume the existence of $\gamma \in (\alpha, \beta)$ with $\varphi(\gamma) \neq \varphi(\alpha)$. Based on Lemma 2.1 we select continuous and injective maps

$$\psi_1 : [0, 1] \to \varphi([\alpha, \gamma]), \psi_2 : [0, 1] \to \varphi([\gamma, \beta]) \text{ with } \psi_1(0) = \psi_2(1) = \varphi(\alpha) \text{ and } \psi_1(1) = \psi_2(0) = \varphi(\gamma).$$

Observe that Lemma 2.9 implies $\psi_1([0, 1]) = \psi_2([0, 1])$.

We denote $K = \varphi^{-1}(\psi_1([0, 1])) \cap ((\alpha, \beta) \setminus \{\gamma\})$, from what we just showed it is obvious that for each $x \in K$ there is an $x' \in K \setminus \{x\}$ which depends on x in a Borel measurable way and fulfills $\varphi(x) = \varphi(x')$. Moreover, the Lipschitz continuity of φ ensures that $|K| > 0$. The non injectivity of φ on K together with the assumptions made on the φ_j show that

$$\text{for all } x \in K, r \in (0, r_x) : \liminf_{j\to\infty} \mathscr{H}^1(B(\varphi_j(x), r) \cap \mathrm{im}(\varphi_j)) \geq 4r,$$

here $r_x = \min\{{}^1\!/_2|x - x'|, \mathrm{dist}(\{x, x'\}, \{a, b\})\}$. From (2.12) and the measure preserving property of the φ_j we infer therefore the existence of $r_0 > 0$ such that

$$\mu(\{z \; ; \; \mu(B(z, 2r)) \geq 7r\}) > |K| \text{ whenever } r \in (0, r_0).$$

Consequently, we obtain $\mu(\{z \; ; \; \bar{D}^1(\mu, z) \geq {}^7\!/_4\}) \geq |K| > 0$. This contradicts the well known upper bound on the density of (finite) Hausdorff measures (see e.g. 2.10.19(5) in [**35**]). Hence, we are done.

LEMMA 2.14. *Let $t \in \mathbb{R}$ be a regular value for $f_{*,i}$. Suppose $U \subset \Omega$ is open in the plane, $\varphi : [0, c] \to U$ is a regular path and $f_{*,i} \circ \varphi \equiv t$. Then there exist a a regular path $\tilde{\varphi} : [0, \tilde{c}] \to \bar{U}$, $c < \tilde{c} < \infty$ which extends φ and fulfills $\tilde{\varphi}([0, \tilde{c})) \subset U$ and $\tilde{\varphi}(\tilde{c}) \in \partial U$.*

Proof. Set $y = \varphi(0)$ and $x = \varphi(c)$. For $k \geq 1$ we consider the open set $U_k = U \setminus \varphi([0, (k-1)c/k])$ and the component $M_k = \mathrm{CC}(l_{*,i}(x) \cap U_k, x)$. As t was supposed to be a regular value, we infer the existence of $p_k \in \bar{M}_k \cap \partial U_k \setminus \{q_k\}$ where $q_k = \varphi((k-1)c/k) \in \bar{M}_k \cap \partial U_k$. We claim that $p_k \notin \varphi([0, (k-1)c/k])$, and hence $p_k \in \partial U$. Indeed, by Proposition 2.8 the set $M_k \subset l_*(y)$ is of finite $\mathscr{H}^1$-measure and we conclude from Lemma 2.3 that there is a regular path $\psi_k : [0, d_k] \to M_k \cup \{q_k, p_k\}$ with $d_k \leq \mathscr{H}^1(l_*(y))$, $\psi_k(0) = p_k$ and $\psi_k(1) = q_k$. Because $\varphi([0, c]) \subset l_*(x)$ as well, we see that $p_k \in \varphi([0, (k-1)c/k]$ implies by Lemma 2.9 $\mathrm{im}(\psi_k) \subset \varphi([0, (k-1)c/k])$ so $\mathrm{im}(\psi_k) \subset \{p_k, q_k\}$ which is of course impossible. The paths

$$\tilde{\varphi_k}(t) = \begin{cases} \varphi(t) & \text{if } t \leq \frac{k-1}{k}c \\ \psi_k(t - \frac{k-1}{k}c) & \text{if } \frac{k-1}{k}c \leq t \leq \frac{k-1}{k}c + d_k \end{cases}$$

are obviously regular. As their total length stays bounded, we can select a converging subsequence φ_{k_l} and using Lemma 2.13 it is easy to check that the limit curve φ_∞ is injective, 1-Lipschitz, extends φ and ends in ∂U. Due to the general lower semicontinuity result for $\mathscr{H}^1$-measure on curves, φ_∞ has finite length. Therefore, its reparametrization by arclength is what we just looked for.

COROLLARY 2.15. *Let $t \in \mathbb{R}$ be a regular value for $f_{*,i}$. Assume $U \subset \Omega$ is open, $x \in U$ and $f_{*,i}(x) = t$. Then there exist an injective, 1-Lipschitz and measure preserving $\varphi : [0, C] \to l_*(x) \cap \bar{U}$ such that $\varphi(0), \varphi(C) \in \partial U$, $\varphi((0, C)) \subset U \cap l_*(x)$ and $x \in \mathrm{im}(\varphi)$.*

Proof. It is clear from Lemma 2.10, Proposition 2.8 and Lemma 2.3 that there is regular path $\psi : [0, c] \to l_*(x)$ with $\psi([0, c)) \subset U$ and $\varphi(c) \in \partial U$. We consider the regular path $\tilde{\psi}(t) = \psi(c/2 - t)$ for $t \in [0, c/2]$. By the foregoing Lemma 2.14 we can find an extension, still denoted by $\tilde{\psi}$ which connects $\tilde{\psi}(0)$ to ∂U. Due to Lemma 2.9 we have $\mathrm{im}(\psi) \cap \mathrm{im}(\tilde{\psi}) = \psi([0, c/2])$ which allows us to join ψ and $\tilde{\psi}$ into the required path φ.

THEOREM 2.16. *Given any $(*, i) \in \{+, -\} \times \{1, 2\}$, then each real number t is regular for $f_{*,i}$.*

Proof. We choose an arbitrary $x_0 \in \Omega$, and an open $U \subset \Omega$ containing x_0 and have to show that $\mathrm{clos}(\mathrm{CC}(l_*(x_0) \cap U, x_0)) \cap \partial U$ consists of more than one point.

First we note that almost each $t \in \mathbb{R}$ is a regular value for $f_{*,i}$. Indeed, already the generalized Fubini formula, see Theorem 2.10.25 in [**35**], implies that $\mathscr{H}^1(f_{*,i}^{-1}(t)) < \infty$ for almost each t. Let E be the set of all local minima of $f_{*,i}$ in $\bar{\Omega}$. Then $E = \bigcup_{k \leq 1} E_k$, where E_k consists of all x such that $f_{*,i}(x) = \min f_{*,i}(B(x, 1/k))$. If $f_{*,i}(E)$ is s uncountable, then for some $N \in \mathbb{N}$ the set $f_{*,i}(E_N)$ is uncountable. Hence, there exists an uncountable set $M \subset E_N$ such that $f_{*,i}(x) \neq f_{*,i}(y)$ whenever $x, y \in M$ are different. As M is infinite, we find $x, y \in M$ with $0 < |x - y| < 1/2N$, contradiction. Because the same argument applies to local maxima, Proposition 2.12 shows that almost each t is a regular value.

If $f_{*,i}(x_0)$ itself is regular, we are done. Else consider $B(x_0, 1/k) \subset U$. Due to Proposition 2.8(c) the set $f_{*,i}(B(x_0, 1/k))$ contains an interval and hence a regular value t_k. We find $x_k \in B(x_0, 1/k) \cap f_{*,i}^{-1}(t_k)$ and, using Corollary 2.15, a regular path φ_k in $\bar{U} \cap l_*(x_k)$ from ∂U to ∂U trough x_k. Since $\mathscr{H}^1(\mathrm{im}(\varphi_k)) \leq 1/2 \mathscr{H}^1(f_*(\partial\Omega)) < \infty$ by Proposition 2.8(c), we can select in the space of curves a cluster point φ_∞ of the sequence $\{\varphi_k\}_{k=1}^\infty$. Lemma 2.13 ensures that φ_∞ is injective, passes in $\bar{U} \cap l_*(x)$ from ∂U trough x to ∂U. As this in particular means $\mathrm{card}(\mathrm{im}(\varphi_\infty) \cap \partial U) \geq 2$, we are done.

LEMMA 2.17. *Let $x \in \Omega$ and let the regular paths φ_1, φ_2 start from x, run in $l_*(x)$ and satisfy $\mathrm{im}(\varphi_1) \cap \mathrm{im}(\varphi_2) = \{x\}$. Let α be any path in $\bar{\Omega}$ which intersects both φ_1 and φ_2 but does not contain x. Denote $t_i = \min\{t \,;\, \varphi_i(t) \in \mathrm{im}(\alpha)\}$ and $s_i = \alpha^{-1}(\varphi_i(t_i))$. We consider the Jordan curve*

$$\gamma = \mathrm{conc}(\varphi_1|[0 \to t_1], \alpha_{|[s_1 \to s_2]}, \varphi_2|[t_2 \to 0])$$

and the bounded component U_b of $\mathbb{R}^2 \setminus \mathrm{im}(\gamma)$. Then there is a regular path $\tilde{\varphi} : [0, \tilde{c}] \to l_{\bar{*}}(x)$ with $\tilde{\varphi}(0) = x$, $\tilde{\varphi}(\tilde{c}) \in \alpha(s_1, s_2)$ and $\tilde{\varphi}((0, \tilde{c})) \subset U_b$.

Proof. It is easy to check that γ is indeed a Jordan curve, and hence $x \in \overline{U_b}$. As in the proof of Lemma 2.9 we also see that $U_b \subset \Omega$. We take a sequence $x_k \in U_b$ converging to x and use Corollary 2.15 to find for each k a regular path $\psi_k : [0, c_k] \to \overline{U_b} \cap l_{\bar{*}}(x_k)$ with $\psi_k(0), \psi_k(c_k) \in \mathrm{im}(\gamma)$ and $x_k \in \psi_k((0, c_k)) \subset U_b$. Because of Proposition 2.8(d), the sets $l_*(x)$ and $l_{\bar{*}}(x_k)$ can intersect at most once. Hence, there is a suitable regular subpath $\tilde{\psi}_k$ of ψ_k connecting x_k to $\alpha((s_1, s_2))$. Since the length of these $\tilde{\psi}_k$ is again uniformly bounded by $1/2 \mathscr{H}^1(f_*(\partial\Omega))$, we see that a cluster point $\tilde{\psi}_\infty$ starts in x, runs in $\overline{U_b} \cap l_{\bar{*}}(x)$ and reaches $\alpha([s_1, s_2])$. Moreover, as it can intersect $l_*(x)$ only once, it has to stay all the time in U_b before it hits $\alpha(s_1, s_2)$. □

DEFINITION 2.18. *Given $* \in \{+, -\}$ we say that $x \in \Omega$ is a branch point for f_* if there exist three, not necessarily regular, paths $\varphi_i : [0, 1] \to l_*(x)$, $i = 1, 2, 3$ such that $\varphi_i(0) = x$ and $\mathrm{im}(\varphi_i) \cap \mathrm{im}(\varphi_j) = \{x\}$ if $i \neq j$.*

REMARK 2.19. *The point $x \in \Omega$ is a branch point for f_* if and only if it is a branch point for $f_{\bar{*}}$. Moreover, in this case each of the "different branches" can be extended until $\partial\Omega$ and also the extended paths will intersect only in x.*

The extensions can of course be found using Lemma 2.14 and Lemma 2.9 ensures that they stay essentially disjoint. The statement comparing branch points for f_+ and f_- is an immediate consequence of Lemma 2.17. Note that a more refined version of the simple argument used in the proof of this Lemma will play an important role in the proofs of Theorem 2.20 and of Theorem 2.22.

THEOREM 2.20. *The set of branch points is discrete.*

Proof. Let T be the set of all branch points in Ω. Suppose $\{x_k\}_1^\infty \subset T$ with $x_k \to x \in \Omega$. Then a simple application of Lemma 2.13 shows that x is a branch point as well. Fix $* \in \{+,-\}$. Using an argument similar to the one proving statement (2.10) we select

$$R \in (0, {}^1\!/_8 \operatorname{dist}(x, \partial\Omega)) \text{ such that } B(x, 4R) \cap l_*(x) \subset \mathrm{CC}(l_*(x) \cap \Omega, x).$$

First we consider the case when the set $T_x = B(x, 4R) \cap T \cap l_*(x)$ is infinite. Because of the connectedness property just ensured and since connected components are relatively closed, Lemma 2.3 guarantees for each $y \in T_x$ the existence of a regular path $\varphi_y : [0, c_y] \to l_*(x)$ with $\varphi_y(0) = x$, $\varphi_y(t_y) = y$, $\varphi_y(c_y) \in \partial\Omega$ and $\varphi_y([0, c_y)) \subset \Omega$. In order to avoid the natural but more demanding arguments how to order the branching points and build a tree of infinite length in the level set, we boil the reasoning down to the two cases of interest. Assume first, that for some $N \subset T_x$ finite the inclusion $T_x \subset \bigcup\{\operatorname{im}(\varphi_y)\ ;\ y \in N\}$ holds. Then we find $y_0 \in N$ such that the set $M = T_x \cap \operatorname{im}(\varphi_{y_0})$ is infinite. For each $y \in M$ we find a regular path $\psi_y : [0, d_y] \to l_*(x)$ such that $\psi_y(0) = y$, $\psi_y(d_y) \in \partial\Omega$ and, since y is a branch point, $\psi_y((0, d_y)) \cap \varphi_{y_0}([0, c_{y_0}]) = \emptyset$. Notice that for $y \neq y'$ we conclude from Lemma 2.9 that $\operatorname{im}(\psi_y) \cap \operatorname{im}(\psi_{y'}) = \emptyset$. As $\mathscr{H}^1(\operatorname{im}(\psi_y)) \geq \operatorname{dist}(y, \partial\Omega) \geq 4R$ for all $y \in M$ we would conclude $\mathscr{H}^1(l_*(x)) = \infty$. On the other hand, if $z \in T_x \setminus \bigcup\{\operatorname{im}(\varphi_y)\ ;\ y \in N\}$ [*] then, again due to Lemma 2.9 we have $\varphi_z((t_z, c_z]) \cap \bigcup\{\operatorname{im}(\varphi_y)\ ;\ y \in N\} = \emptyset$. In particular, as $\operatorname{dist}(z, \partial\Omega) \geq 4R$, we get

$$\mathscr{H}^1\big(\bigcup\{\operatorname{im}(\varphi_y)\ ;\ y \in N \cup \{z\}\}\big) \geq 4R + \mathscr{H}^1\big(\bigcup\{\operatorname{im}(\varphi_y)\ ;\ y \in N\}\big).$$

Combining these two considerations, we see that T_x being an infinite set always implies $\mathscr{H}^1(l_*(x)) = \infty$, which clearly contradicts Proposition 2.8.

Therefore, we will in addition suppose that $B(x, 4R) \cap T_x = \{x\}$. In the sequel, we denote $M = \mathrm{CC}(l_*(x) \cap \Omega, x)$. We know that for each $y \in M$ there is a regular path $\varphi_y : [0, c_y) \to l_*(x) \cap \Omega$ with $\varphi_y(0) = x$, $\varphi_y(t_y) = y$ and $\varphi_y(c_y) \in \partial\Omega$. Lemma 2.9 ensures that $\varphi_{y|[0,2R]} \neq \varphi_{\tilde{y}|[0,2R]}$ enforces $\varphi_y((2R, c_y]) \cap \varphi_{\tilde{y}}((2R, c_{\tilde{y}}]) = \emptyset$. As each of these arcs is of length at least $6R$, we conclude that the family

$$\mathcal{I} = \{\varphi_{y|[0,2R]}\ ;\ y \in M\}$$

is finite. To each $y \in M$ we associate an $\tilde{\varphi}_y \in \mathcal{I}$, as there are no branch points in $l_*(x) \cap B(x, 2R) \setminus \{x\}$ the $\tilde{\varphi}_y$ is unique. Fix any $\gamma \in \mathcal{I}$ and consider the set $M_\gamma = \{y \in M\ ;\ \tilde{\varphi}_y = \gamma\}$. Obviously, M_γ is closed in M and hence in Ω. So $M_\gamma \setminus \gamma([0, R])$ is a connected G_δ-subset of $l_*(x)$ and now Lemma 2.9 implies that $x \notin \operatorname{clos}(M_\gamma \setminus \gamma([0, R]))$. In particular, there is a $\delta_\gamma \in (0, R)$ such that $B(x, 2\delta_\gamma) \cap M_\gamma \subset \gamma([0, R])$. As $\gamma(R) \neq x$, for δ_γ small enough this inclusion remains true if we decrease R a little bit. Finally, we choose $\delta = \min_{\gamma \in \mathcal{I}} \delta_\gamma > 0$.

Now we return to our original sequence $\{x_k\}_1^\infty$ of branch points and can of course assume that it is entirely contained in $B(x, \delta)$. Using Lemma 2.15, we start from x_k a regular path ψ_k in $l_*(x_k)$ connecting to $\partial\Omega$. To avoid more complicated topological considerations, we choose the radius R

[*] this concerns in particular the other, easier, case when we have infinitely many branches starting from one point

more carefully. Observe that due to Theorem 2.10.25 in [**35**] for any K with $\mathscr{H}^1(K) < \infty$ there is only a Lebesgue zero set of $r > 0$ for which $K \cap \partial B(x,r))$ is infinite. Moreover, for any of the curves ψ_k and a.e $r > 0$ the following condition holds:

$$\text{whenever } |\psi_k(t) - x| = r \text{ then } \psi_k'(t) \text{ exists and } \langle \psi_k'(t), \psi_k(t) - x\rangle \neq 0.$$

Consequently, we can in addition to all the properties of R already ensured also assume that

- $\partial B(x,R) \cap M$ is finite
- for each $k = 1, \dots$ is the set $S_k = \{t\ ;\ |\psi_k(t) - x| = R\}$ finite and $|\psi_k(t) - x| - R$ changes sign at each point in S_k. Moreover, as ψ_k starts in x and ends in $\partial\Omega$, we see that $\mathrm{card}(S_k)$ is odd.

Now we fix an arbitrary k. By what was just told, there is a connected component C_k of $\partial B(x,R)\backslash M$ with endpoints p_k and q_k such that $\mathrm{card}(S_k \cap C_k)$ is odd. We claim that then $\tilde{\varphi}_{p_k} \neq \tilde{\varphi}_{q_k}$.

Otherwise there exists a regular path ψ in $M_{\tilde{\varphi}_{p_k}} \setminus \tilde{\varphi}_{p_k}([0,R])$ connecting p_k to q_k. Because $\mathrm{im}(\psi) \subset M$, we see that "concatenating" ψ with the path given by the arc C_k gives a Jordan curve γ. As usual, we denote the components of $\mathbb{R}^2 \setminus \mathrm{im}(\gamma)$ by U_b and U_∞. If $x \in U_b$, we consider a regular path $\hat{\varphi}$ in $M \setminus \mathrm{im}(\psi)$ connecting x to $\partial\Omega$, for this purpose we could choose any extension due to Lemma 2.14 of an arc in $\mathcal{I} \setminus \{\tilde{\varphi}_{p_k}\}$. Since $\partial\Omega \subset U_\infty$, we find $\hat{p} \in \mathrm{im}(\hat{\varphi}) \cap \mathrm{im}(\gamma)$. So $\hat{p} \in (M \setminus \mathrm{im}(\psi)) \cap \mathrm{im}(\gamma) = \emptyset$. Therefore, x must belong to U_∞ and since $B(x,\delta) \cap \mathrm{im}(\gamma) = \emptyset$, we have also $x_k \in U_\infty$. As $\mathrm{im}(\psi) \cap C_k = \emptyset$, we see that there is a $\sigma \in \{+1, -1\}$ such that for each $y \in C_k$ and some η_y positive $y' \in B(y, \eta_y)$ belongs to U_b if and only if $\sigma(|y' - x| - R) > 0$ and $y' \in U_\infty$ precisely if $\sigma(|y' - x| - R) < 0$. Because $\mathrm{card}(S_k)$ is odd, we see that for some $\beta > 0$ small $\psi_k(\min(S_k) - \beta)$ and $\psi_k(\max(S_k) + \beta)$ belong to different components of $\mathbb{R}^2 \setminus \mathrm{im}(\gamma)$. Since $\mathrm{im}(\psi_k) \cap \mathrm{im}(\gamma) \subset C_k$ we have that $\psi_k(\min(S_k) - \beta)$ belongs like x_k itself to U_∞. On the other hand, $\psi_k(\max(S_k) + \beta)$ must be in the same component like the end of ψ_k and hence in U_∞ again, contradiction.

So $\tilde{\varphi}_{p_k} \neq \tilde{\varphi}_{q_k}$ indeed, and replacing $\{x_k\}_k$ by a subsequence (without changing notations) if necessary we can also suppose $p = p_k$ and $q = q_k$ do in fact not depend on k. As the $\tilde{\psi}$'s are different, there exists a regular path $\varphi_0 : [0,c] \to M$ with $\varphi_0(0) = p$, $\varphi_0(c) = q$ and $\varphi_0(t_0) = x$ for some $t_0 \in (0,c)$. Again we "concatenate" φ_0 with the path given by the arc $C = C_k$ (independent of k) to a Jordan curve $\hat{\gamma}$. Using the same arguments as for γ, we see that x_k is always in the bounded component $\hat{U}_b$ of $\mathbb{R}^2 \setminus \mathrm{im}(\hat{\gamma})$. We know that each x_k is a branch point, and hence there are three "disjoint" regular paths $\psi_k^1, \psi_k^2, \psi_k^3$ in $l_*(x_k)$ starting from x_k and reaching $C = C_k$. Because of the general uniform length estimate we can, perhaps once more switching to a subsequence, assume that $\psi_k^j \rightrightarrows \psi^j$. It is easy to see that each ψ^j connects x in $l_*(x)$ to $\bar{C}_k \cap l_*(x) = \{p,q\}$. On the other hand, applying Lemma 2.13 to the regular paths $\mathrm{conc}(\mathrm{inv}(\psi_k^i), \psi_k^j)$ we see that

$$\mathrm{im}(\psi^i) \cap \mathrm{im}(\psi^j) = \{x\} \text{ for } 1 \leq i < j \leq 3.$$

This obvious contradiction finishes the proof. □

LEMMA 2.21. *Suppose φ_*^j, $(*, j) \in \{+,-\} \times \{1,2\}$ are regular paths in $\bar{\Omega}$ such that*

- $f_* \circ \varphi_*^j \equiv z_j^*$
- φ_-^j *connects* x_{j1} *to* x_{j2}
- φ_+^j *connects* x_{1j} *to* x_{2j}.

Then we can conclude that

(a) $x_{22} - x_{21} = x_{12} - x_{11}$,
(b) $\mathrm{im}(\varphi_+^2) = \mathrm{im}(\varphi_+^1) + (x_{12} - x_{11})$ *and* $\mathrm{im}(\varphi_-^2) = \mathrm{im}(\varphi_-^1) + (x_{21} - x_{11})$.

Proof. As $2\mathfrak{i}(x_{22} - x_{21}) = f_+(x_{22}) - f_+(x_{21}) = f_+(x_{12}) - f_+(x_{11}) = 2\mathfrak{i}(x_{12} - x_{11})$, part (a) is obvious.

If $z_+^1 = z_+^2$ then moreover $f_+(x_{22}) = f_+(x_{21})$ and hence $x_{22} = x_{21}$ and $x_{12} = x_{11}$. We see that $\mathrm{im}(\varphi_-^2) = \{x_{21}\} = \{x_{11}\} + (x_{21} - x_{11}) = \mathrm{im}(\varphi_-^1) + (x_{21} - x_{11})$. The statement concerning $\mathrm{im}(\varphi_+^j)$ is an immediate consequence of Lemma 2.9. As the same reasoning applies in the case when $z_-^1 = z_-^2$, we can assume in the sequel that $(z_+^1 - z_+^2)(z_-^1 - z_-^2) \neq 0$.

It is easy to check that under the assumptions just made the curve

$$\gamma = \mathrm{conc}(\varphi_+^1, \varphi_-^2, \mathrm{inv}(\varphi_+^2), \mathrm{inv}(\varphi_-^1))$$

is in fact a Jordan curve. Again we denote by U_b the bounded component of $\mathbb{R}^2 \setminus \mathrm{im}(\gamma)$ which is contained in Ω. Now we fix any $x \in \mathrm{im}(\varphi_+^1) \setminus \{x_{11}, x_{21}\}$. As $\mathrm{im}(\varphi_+^1) \subset l_+(x)$, we see that $f_-(x) \notin \{f_-(x_{11}), f_-(x_{21})\}$. Because $x \in \bar{U}_b$, we find $x_k \in U_b$ converging to x and Lemma 2.15 ensures the existence of regular paths $\tilde{\psi}_k$ in $l_-(x_k)$ passing from ∂U_b to ∂U_b inside U_b and through x_k. By what was just observed, $\mathrm{im}(\tilde{\psi}_k) \cap (\mathrm{im}(\varphi_-^1) \cup \mathrm{im}(\varphi_-^2)) = \emptyset$ for k sufficiently large. Again due to Proposition 2.8(d) we have $\mathrm{card}(\mathrm{im}(\tilde{\psi}_k) \cap \mathrm{im}(\psi_+^1)) \leq 1$. So, a suitable regular subpath ψ_k of $\tilde{\psi}_k$ connects x_k inside $U_b \cap l_-(x_k)$ to $\mathrm{im}(\varphi_+^2)$. Passing to a cluster point of $\{\psi_k\}_k$ we find a regular path ψ in $l_-(x) \cap U_b$ from x to $x' \in \mathrm{im}(\varphi_+^2)$. Because the assumptions of this Lemma are fulfilled for x_{11}, x, x', x_{12} we conclude from the already established part (a) that $x' - x = x_{12} - x_{11}$. Therefore $\mathrm{im}(\varphi_+^2) \supset \mathrm{im}(\varphi_+^1) + (x_{12} - x_{11})$. Because the role of $\mathrm{im}(\varphi_+^2)$ and $\mathrm{im}(\varphi_+^1)$ can be interchanged and since the same argument applies to $\mathrm{im}(\varphi_-^j)$, the Lemma is proven. □

Note that it follows from the Example 2.27 at the end of this section that the part of the statement (b) which concerns $\mathrm{im}(\varphi_-^j)$ is in general not true if we only have the existence of the paths φ_-^j, $j = 1, 2$ and know that $f_+(x_{11}) = f_+(x_{21})$, $f_+(x_{12}) = f_+(x_{22})$.

THEOREM 2.22. *Suppose $x_0 \in \Omega$ is not a branch point for f_+, f_-. Then there exist $\varepsilon > 0$ and two regular paths $\varphi_+, \varphi_- : (-\varepsilon, \varepsilon) \to \mathbb{R}^2$ with $\varphi_+(0) = \varphi_-(0) = 0$ and such that*

1) *The map $\Phi : t \to \varphi_+(t_1) + \varphi_-(t_2)$ is a bilipschitz map of $(-\varepsilon, \varepsilon)^2$ onto a neighbourhood of the origin, and $\mathrm{lip}(\Phi^{-1}) \leq const(\mathrm{lip}(f) + 1)^2$.*
2) *For all $t \in (-\varepsilon, \varepsilon)^2$ the formula*

$$f(x_0 + \Phi(t)) = f(x_0) + \mathrm{i}(\varphi_-(t_2) - \varphi_+(t_1))$$

holds.

Proof. Obviously, we can suppose that $x_0 = 0$ and that for some fixed $R > 0$ there are no branch points in $B(0, 2R) \subset \Omega$. For φ_-, φ_+ we choose the now unique (up to taking the inverse) regular paths in $l_-(0)$ and $l_+(0)$ passing from $\partial B(0, 2R)$ inside $B(x, 2R)$ through 0 to $\partial B(0, 2R)$, see Lemma 2.15 and we shift the parametrisation so that zero is mapped onto the origin. As the paths are 1-Lipschitz, both are defined on intervals containing at least $(-2R, 2R)$. We denote by p_*^u the intersection of $\varphi_{*|[0,\infty)}$ with $\partial B(0, 2R)$ and similar for p_*^l using $\varphi_{*|(-\infty,0]}$. We set $U = B(0, 2R)$ and define U_-^u to be the bounded component belonging to the Jordan curve obtained by joining φ_+ with $\mathrm{arc}_{\partial U}(p_+^u, p_-^u, p_+^l)$, which is the subarc of ∂U from p_+^u through p_-^u to p_+^l. Analogous we define U_-^l using $\mathrm{arc}_{\partial U}(p_+^u, p_-^l, p_+^l)$. It follows from Lemma 2.17 that $U_-^l \cap U_-^u = \emptyset$ and it is easy to check that $U_-^l \cup U_-^u \cup \mathrm{im}(\varphi_+) \supset U$. We now define a whole family of U_+^l- and U_+^u-sets. For this purpose, if $r \in (-2R, 2R)$ let ψ_r be the unique regular path such that

- $\psi_r(0) = \varphi_+(r)$,
- $\psi_r((0, c_r^+)) \subset l_-(\psi_r(0)) \cap U_-^u$, $\psi_r(c_r^+) \in \mathrm{arc}_{\partial U}(p_+^u, p_-^u, p_+^l)$
- $\psi_r((c_r^-, 0)) \subset l_-(\psi_r(0)) \cap U_-^l$, $\psi_r(c_r^-) \in \mathrm{arc}_{\partial U}(p_+^u, p_-^l, p_+^l)$
- $c_r^- \leq r - 2R$ and $c_r^+ \geq 2R - r$.

We choose $U_+^l(r)$ to be the bounded component belonging to the Jordan curve consisting of ψ_r and $\mathrm{arc}(\psi_r(c_r^+), p_+^l, \psi_r(c_r^-))$ and to obtain U_+^u simply replace this arc by $\mathrm{arc}(\psi_r(c_r^+), p_+^u, \psi_r(c_r^-))$. As before, Lemma 2.17 implies $U_+^l(r) \cap U_+^u(r) = \emptyset$ and that $U_+^l(r) \cup U_+^u(r) \cup \mathrm{im}(\psi_r) \supset U$. We

also note that $\varphi_{+|(r,\infty)} \subset U_+^u(r)$ and $\varphi_{+|(-\infty,r)} \subset U_+^l(r)$ which in turn implies $U_+^l(r) \subset U_+^l(r')$ and $U_+^u(r) \supset U_+^u(r')$ if $-2R < r < r' < 2R$.

Next, for $r \in (0,R)$ the points $\psi_r(r)$ and $\psi_r(-r)$ are well defined and in U. We denote by $\alpha_r^u : [a_r^{u,-}, a_r^{u,+}] \to l_+(\varphi_r(r))$ the unique regular path with $\alpha_r^u(0) = \psi_r(r)$, $\alpha_r^u(t) \in \partial U$ if $t \in \{a_r^{u,-}, a_r^{u,+}\}$, $\alpha_r^u(t) \in U_+^u(r)$ if $t \in (0, a_r^{u,+})$ and $\alpha_r^u(t) \in U_+^l(r)$ if $t \in (a_r^{u,-}, 0)$. Completely analogously, just replacing $\psi_r(r)$ by $\psi_r(-r)$ we define α_r^l. Note that α_r^l lives in U_-^l and α_r^u stays in U_-^u. If $\mathrm{im}(\alpha_r^u) \cap \mathrm{im}(\psi_{-r}) \cap U \neq \emptyset$ then it consists by Proposition 2.8(d) of precisely one point which we will denote by z_r^u. Analogously we define $z_r^l \in \mathrm{im}(\alpha_r^l) \cap \mathrm{im}(\psi_{-r}) \cap U$ whenever it exists.

In fact, next we will show that for $r > 0$ but sufficiently small both z_r^u and z_r^l exist. First we conclude from Lemma 2.15, Lemma 2.17, the absence of branch points and the usual compactness arguments the following. There is a δ positive such that for any $y \in B(0,\delta)$ we can find a regular path ω_y in $B(0,R) \cap l_-(y)$ which connects y to $\mathrm{im}(\varphi_+)$. Now, consider any $y \in B(0,\delta) \cap U_+^l(0)$ which is by ω_y connected to $\varphi_+(r_y)$. As $\mathrm{im}(\omega_y) \subset l_-(y) \cap U$, we see that $\mathrm{im}(\omega_y) \cap \mathrm{im}(\varphi_-) = \emptyset$ and hence $\varphi_+(r_z) \in U_+^l(0)$. Hence $r_y < 0$ and $y \in U_+^l(r)$ for $r \in (r_y, 0)$. This shows $\bigcap_{r>0} \bigcup_{r' \in (0,r)} \overline{U_+^u(-r')} \cap B(0,\delta) \subset \overline{U_+^u(0)}$.

If there is a sequence $r_k \to 0_+$ such that $z_{r_k}^u$ does not exists, then we see that $\mathrm{im}(\alpha_{r_k}^u) \subset U_+^u(-r_k)$. Therefore, any cluster point α_0^u of $\{\alpha_{r_k}^u\}$ fulfills $B(0,\delta) \cap \mathrm{im}(\alpha_0^u) \subset \overline{U_+^u(0)}$. On the other hand, α_0^u is a path (of finite) length in $l_+(0) \cap U$ with $\alpha_0^u(0) = 0 = \lim_k \psi_{r_k}(r_k)$ which connects on both end to ∂U. As $\varphi_{+|(-\infty,0)} \subset U_+^l(0)$, this clearly shows that 0 has to be a branch point, contradiction. Since the argument works a well for $z_{r_k}^l$, we infer that for some $r_0 > 0$ both $z_{r_0}^u$ and $z_{r_0}^l$ exist and will now get into a situation where we can apply Lemma 2.21.

We define $s^u = (\alpha_{r_0}^u)^{-1}(z_{r_0}^u)$, $s^l = (\alpha_{r_0}^l)^{-1}(z_{r_0}^l)$ and $t^u = (\psi_{-r_0})^{-1}(z_{r_0}^u)$, $t^l = (\psi_{-r_0})^{-1}(z_{r_0}^l)$ and finally the Jordan curve

$$\gamma = \mathrm{conc}(\psi_{r_0|[-r_0 \to r_0]}, \alpha_{r_0|[0 \to s^u]}^u, \psi_{-r_0|[t^u \to t^l]}, \alpha_{r_0|[s^l \to 0]}^l).$$

Next we need to verify that $0 \in U_b(\gamma)$. Else $\varphi_+([0,r_0)) \subset U_\infty(\gamma)$ and we obviously find a $\delta > 0$ with $B(x_1,\delta) \cap (\mathrm{im}(\alpha_{r_0}^u) \cup \mathrm{im}(\alpha_{r_0}^l) \cup \mathrm{im}(\psi_{-r_0})) = \emptyset$, here $x_1 = \varphi_+(r_0)$. Because $U_+^l(r_0)$ is a Jordan domain, we find a neighbourhood V of x_1 which is contained in $B(x_1,\delta)$ such that $V \cap U_+^l(r_0)$ is connected. Since $(V \cap U_+^l(r_0)) \cap \varphi_+([0,r_0)) \neq \emptyset$ but $V \cap U_+^l(r_0) \cap \mathrm{im}(\gamma) = V \cap U_+^l(r_0) \cap \mathrm{im}(\psi_{r_0}) = \emptyset$ we see that $V \cap U_+^l(r_0) \subset U_\infty(\gamma)$. On the other hand, it is clear that $\mathrm{im}(\psi_{-r_0}) \subset \mathrm{clos}(U_+^l(r_0))$ and $\alpha_{r_0|(0 \to s^u]}^u$, $\alpha_{r_0|[s^l \to 0)}^l$ are connected to this arc. Therefore, $\mathrm{im}(\gamma) \subset \mathrm{clos}(U_+^l(r_0))$ and hence $V \cap U_+^u(r_0) \subset U_\infty(\gamma)$. But since $V \subset \mathrm{im}(\gamma) \cup U_+^u(r_0) \cup U_+^l(r_0)$ we infer $x_1 \notin \mathrm{clos}(U_b(\gamma))$, contradiction.

It will be helpful to name a few more points. We set $x_1^u = \psi_{r_0}(r_0)$, $x_1^l = \psi_{r_0}(-r_0)$, $x_{-1} = \varphi_+(-r_0)$, $x_{-1}^u = z_{r_0}^u$ and $x_{-1}^l = z_{r_0}^l$. Further we pick the uniquely determined points x_0^u, x_0^l such that

$$\{x_0^u\} = \mathrm{im}(\alpha_{r_0}^u) \cap \mathrm{im}(\varphi_-), \{x_0^l\} = \mathrm{im}(\alpha_{r_0}^l) \cap \mathrm{im}(\varphi_-).$$

Both points exist since the α-paths change from $U_+^u(0)$ to $U_+^l(0)$.

Applying Lemma 2.21, we gradually obtain $x_1^u - x_1 = x_0^u - x_0 = x_{-1}^u - x_{-1}$, $x_1^l - x_1 = x_0^l - x_0 = x_{-1}^l - x_{-1}$, and $x_1 - x_0 = x_1^u - x_0^u = x_1^l - x_0^l$, $x_{-1} - x_0 = x_{-1}^u - x_0^u = x_{-1}^l - x_0^l$. Moreover, part (b) of the same Lemma implies that $\varphi_-([0, \varphi_-^{-1}(x_0^u)]) = \psi_{r_0}([0,r_0]) - x_1$ and as both φ_-, ψ_{r_0} are parametrized by arclength and since $x_0^u \in U_-^u$, we see $x_0^u = \varphi_-(r_0)$. Similar $x_0^u = \alpha_{r_0}^u(r_0)$ and $x_0^l = \varphi_-(-r_0) = \alpha_{r_0}^l(r_0)$, $x_{-1}^u = \psi_{-r_0}(r_0) = \alpha_{r_0}^u(2r_0)$, $x_{-1}^l = \psi_{-r_0}(-r_0) = \alpha_{r_0}^l(2r_0)$.

Due to Lemma 2.15, for any point $y \in U_b(\gamma)$ we can start a twosided (unique) regular path $\tilde{\varphi}_+^y : (a_+(y), b_+(y)) \to U_b \cap l_+(y)$ such that $a_+(y) < 0 < b_+(y)$, $\tilde{\varphi}_+^y(0) = y$, $\tilde{\varphi}_+^y(a_+(y)) \in \mathrm{im}(\psi_{-r_0})$ and $\tilde{\varphi}_+^y(b_+(y)) \in \mathrm{im}(\psi_{r_0})$. In the same way we find $\tilde{\varphi}_-^y : (a_-(y), b_-(y)) \to U_b \cap l_-(y)$ such that $\tilde{\varphi}_-^y(0) = y$, $\tilde{\varphi}_-^y(a_-(y)) \in \mathrm{im}(\alpha_{r_0}^l)$ and $\tilde{\varphi}_-^y(b_-(y)) \in \mathrm{im}(\alpha_{r_0}^u)$. Because $\tilde{\varphi}_+^y$ passes from $U_+^l(0)$ to $U_+^u(0)$, it must intersect $\mathrm{im}(\varphi_-)$, say at the point $\varphi_-(t_-^y)$. Applying the arclength argument

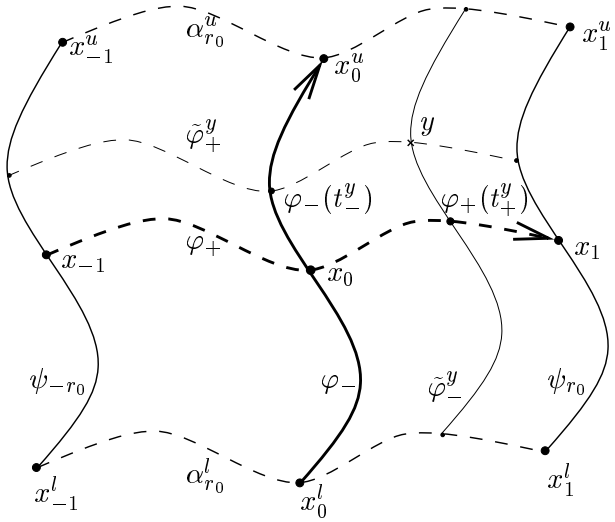


FIGURE 2. The local coordinate system

used already, now to the points $x_1, \tilde\varphi^y_+(b_+(y)), \varphi_-(t^y_-), x_0$ we see $t^y_- \in (-r_0, r_0)$. In the same way $t^y_+ \in (-r_0, r_0)$ is obtained.

The following simple transformation rule between φ_-, φ_+ and the individual level sets $\tilde\varphi^y_+$, $\tilde\varphi^y_-$ hold:

$$\tilde\varphi^y_+(t) = y + \varphi_+(t^y_+ + t) - \varphi_+(t^y_+), t \in (-r_0 - t^y_+, r_0 - t^y_+),$$
$$\tilde\varphi^y_-(t) = y + \varphi_-(t^y_- + t) - \varphi_-(t^y_-), t \in (-r_0 - t^y_-, r_0 - t^y_-).$$

As concerns the endpoints of the domains, we have $a_+(y) = -r_0 - t^y_+$, $b_+(y) = r_0 - t^y_+$ and similar for $a_-(y), b_-(y)$. Indeed, the same based on Lemma 2.21 and used to find the "coordinates" of $x^u_0, x^u_{-1}, \dots$ works here as well. Obviously, part (a) of this Lemma also says that $y = \varphi_+(t^y_+) + \varphi_-(t^y_-)$ which together with formulae just obtained imply that $\Phi((-r_0, r_0)^2) = U_b(\gamma)$. It will be useful to notice that $t^{\Phi(t)}_+ = t_1$ and $t^{\Phi(t)}_- = t_2$. For instance the first equation is consequence of $t^{\varphi_+(t_1)}_- = 0$ which leads to $\tilde\varphi^{\varphi_+(t_1)}_-(t_2) = \varphi_+(t_1) + \varphi_-(t_2 + 0) - \varphi_-(0) = \Phi(t)$ and shows that $\varphi_+(t_1) \in \mathrm{im}(\tilde\varphi^{\Phi(t)}_-)$. Of course, this also gives the existence of Φ^{-1}. Now we finish the proof of part (a) by estimating $\mathrm{lip}(\Phi^{-1})$.

Assume for a moment we would know that φ_-, φ_+ are bilipschitz with constant c. Given $t, \bar t \in (-r_0, r_0)^2$ we use Proposition 2.8(d) and $\Phi(t) \in l_+(\Phi(\bar t_1, t_2))$, $\Phi(\bar t) \in l_-(\Phi(\bar t_1, t_2))$ to estimate

$$\begin{aligned} |\Phi(t) - \Phi(\bar t)| &\geq \mathrm{dist}(\Phi(t), l_-(\Phi(\bar t_1, t_2)) \geq 2|\Phi(t) - \Phi(\bar t_1, t_2)|/\,\mathrm{lip}(f_-) \\ &\geq 2|\varphi_+(t_1) - \varphi_+(\bar t_1)|/\,\mathrm{lip}(f_-) \geq \frac{2c}{\mathrm{lip}(f_-)}|t_1 - \bar t_1|. \end{aligned}$$

As the same estimate can be done for $|t_2 - \bar t_2|$, we get $|\Phi(t) - \Phi(\bar t)| \geq c|t - \bar t|/(\mathrm{lip}(f) + 1)$.

To see that φ_+ is bilipschitz, we will actually show that it is the graph of a Lipschitz function in a suitable coordinate system, of course the same holds for φ_-. For this purpose, we fix $t_0 \in (-r_0, r_0)$ such that $\varphi'_-(t_0)$ exist and has length 1. Let d_0 be a unit vector orthogonal to $\varphi'_-(t_0)$. Note that there exists an $\varepsilon > 0$ such that $t_1, t_2 \in (-r_0, r_0)$ with $|t_1 - t_2| < \varepsilon$ implies $|\langle d_0, \varphi_+(t_1) - \varphi_+(t_2)\rangle| \geq |\varphi_+(t_1) - \varphi_+(t_2)|/\,\mathrm{lip}(f_-)$. Indeed, for $y = \Phi(t_1, t_0)$ we see that if ε is small enough then $\tilde\varphi^y_-(-2\varepsilon, 2\varepsilon) \subset l_-(y)$ very well approximates a segment I in direction $\varphi'_-(t_0)$, radius 2ε and centre y. Because $\varphi_+(t_2) + \varphi_-(t_0) \in l_+(y)$, we infer from Proposition 2.8(d) that $|\langle d_0, \varphi_+(t_1) - \varphi_+(t_2)\rangle| = \mathrm{dist}(\varphi_+(t_2) + \varphi_-(t_0), I) \geq |\varphi_+(t_2) + \varphi_-(t_0) - y|/\,\mathrm{lip}(f_-) = |\varphi_+(t_1) - \varphi_+(t_2)|/\,\mathrm{lip}(f_-)$. It is quite easy to see that the estimate just obtained excludes the existence of local extrema of the function $t \to \langle d_0, \varphi_+(t)\rangle$, so this function must be strictly monotone. Therefore, $\varphi_+(t)$ can be expressed as

a continuous function g of $\langle d_0, \varphi_+(t)\rangle$. Using this estimate once more, we see that g is locally and hence also globally $\mathrm{lip}(f_-)$-Lipschitz. Therefore the length of the graph of g over an interval (a,b) is at most $\sqrt{1+\mathrm{lip}(f_-)^2}|b-a|$ which in turn implies $|\varphi_+(t_1)-\varphi_+(t_2)| \geq |\langle d_0, \varphi_+(t_1)-\varphi_+(t_2)\rangle| \geq |t_1-t_2|/\sqrt{1+\mathrm{lip}(f_-)^2}$.

Finally, we prove the formula given for f in part (b). Since $\varphi_-(t_2) \in l_+(\Phi(t))$ and $\varphi_+(t_1) \in l_-(\Phi(t))$, we compute

$$f_-(\Phi(t)) = f_-(\varphi_+(t_1)) = f_+(\varphi_+(t_1)) - 2\mathrm{i}\varphi_+(t_1) = f_+(0) - 2\mathrm{i}\varphi_+(t_1)$$

and similar

$$f_+(\Phi(t)) = f_+(\varphi_-(t_2)) = f_-(\varphi_-(t_2)) + 2\mathrm{i}\varphi_-(t_2) = f_-(0) + 2\mathrm{i}\varphi_-(t_2).$$

Because $f = {}^1\!/\!_2(f_+ + f_-)$ we get as required

$$f(\Phi(t)) = f(0) + \mathrm{i}(\varphi_-(t_2) - \varphi_+(t_1)).$$

□

COROLLARY 2.23. *For $* \in \{+,-\}$ let $\varphi_* : (a_*, b_*) \to \mathbb{R}^2$ be a regular path such that $a_* < 0 < b_*$, $\varphi_*(0) = 0$ and $t \to f_*(\varphi_*(t)+x)$ is constant. If $\{\varphi_+(t_+)+\varphi_-(t_-)+x\ ;\ t_* \in (a_*, b_*)\}$ is a subset of Ω not containing any branch points, then $\Phi : [a_+, b_+]\times[a_-, b_-] \to \bar{\Omega}$ defined by $\Phi(t) = \varphi_+(t_1)+\varphi_-(t_2)$ is a bilipschitz map and $f(\Phi(t)+x) = f(x)+\mathrm{i}(\varphi_-(t_2)-\varphi_+(t_1))$ holds for all t in the domain of Φ.*

Proof. For $\delta > 0$ let M_δ be the set of all $t \in R_\delta = [a_+ + \delta, b_+ - \delta] \times [a_- + \delta, b_- - \delta]$ for which the representation claimed above for f fails. Assume that our conclusion is not true. It is obvious from Theorem 2.22.1) that we can find a δ positive with $M_\delta \neq \emptyset$. Hence we can choose a sequence $t^k \in M_\delta$ converging to t^0 satisfying $\min(|t_1^0|, |t_2^0|) = \inf\{\min(|t_1|, |t_2|)\ ;\ t \in M_\delta\}$. Without loss of generality, we can assume $|t_1^0| \leq |t_2^0|$ and, as $M_\delta \cap \{t\ ;\ t_1 t_2 = 0\} = \emptyset$, we see that $M_\delta \cap \{t\ ;\ t_1 = t_1^0\} = \emptyset$. Consequently, we have $f(\varphi_+(t_1^0)+x+\varphi_-(t_2)) = f(x)+\mathrm{i}(\varphi_-(t_2)-\varphi_+(t_1^0))$ if $t_2 \in I = [a_- + \delta, b_- - \delta]$. A standard compactness argument shows that there is a common positive ε_0 such that for all $x_t = \varphi_+(t_1^0)+x+\varphi_-(t)$, $t \in I$ the conclusion of Theorem 2.22 holds for this ε_0.

Based on this, it is quite easy to check that

$$\sup(\{p\ ;\ f(x+\varphi_+(t_1)+\varphi_-(q\cdot t_2^0)) = f(x)+\mathrm{i}(\varphi_-(q\cdot t_2^0)-\varphi_+(t_1)) \text{ if } 0 \leq q \leq p, |t_1 - t_1^0| < \varepsilon_0\}) \geq 1.$$

This shows that the "local coordinates" at $\Phi(t^0)+x$ are obtained from φ_+, φ_- via the canonical transformation. Therefore, the local representation of f near $\Phi(t^0)+x$ agrees with the formula stated above. In particular, $t^0 \notin \overline{M_\delta}$ - contradiction. □

PROPOSITION 2.24. *Let $\varphi_+ : (a_+, b_+) \to \mathbb{R}^2$, $\varphi_- : (a_-, b_-) \to \mathbb{R}^2$ be Lipschitz paths with $a_* < 0 < b_*$, $\varphi_*(0) = 0$ and such that $\Phi : (a_+, b_+)\times(a_-, b_-) \to \mathbb{R}^2$ defined by $\Phi(t) = \varphi_+(t_1)+\varphi_-(t_2)$ is a bilipschitz mapping.*

Denote by Ω the open domain $\mathrm{im}(\Phi)$, then the mapping f defined by $f(\varphi_+(t_1)+\varphi_-(t_2)) = a+\mathrm{i}(\varphi_-(t_2)-\varphi_+(t_1))$, $a \in \mathbb{R}^2$ arbitrary is a Lipschitz map such that $\nabla f(x) \in \mathcal{H}_D$ a.e. in Ω.

REMARK 2.25. *It is easy to check that Φ is bilipschitz provided there are two direction $d_+, d_- \in \mathbb{S}^1$ and two closed cones $C_* = \{y\ ;\ \langle y, d_*\rangle \geq \varepsilon_*|y|\}$ such that $\mathrm{dist}(\mathbb{S}^1 \cap C_+, C_- \cup (-C_-)) > 0$, and that $\varphi_*'(t) \in C_*$ with $|\varphi_*'(t)| \geq \tilde{\varepsilon}_* > 0$ for a.e. t. According to the last part of the proof of Theorem 2.22, this kind of condition is also necessary. Indeed, in this proof the information from Proposition 2.8(d) can be easily replaced by the bilipschitz condition on Φ without changing the argument, which also tells the following. At almost each t does $\varphi_*'(t)$ belong to the intersection of all cones of the type $\{y\ ;\ \sigma_s\langle y, \mathrm{i}\varphi_*'(s)\rangle \geq c|y|\}$, where $\sigma_s \in \{+1,-1\}$, $c = c_{bilip(\Phi)} > 0$ and s runs through almost all points in $(-\varepsilon, \varepsilon)$.*

Proof of Proposition 2.24 As Φ^{-1} is bilipschitz, it is clear that f is indeed a well-defined Lipschitz function. Moreover, if we compute the gradient of the map $\tilde{\Phi} : t \to \mathrm{i}(\varphi_-(t_2) - \varphi_+(t_1))$ then we obtain

$$\nabla\tilde{\Phi}(t) = \mathrm{i}(-\varphi_+'(t_1), \varphi_-'(t_2)) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \nabla\Phi(t) \cdot \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

As $\nabla\tilde{\Phi}(t) = \nabla f(\Phi(t)) \circ \nabla\Phi(t)$ and since the matrix $A_t = \nabla\Phi(t)$ is regular for almost every $t \in \mathrm{dom}(\Phi)$, we see that $\nabla f(\Phi(t)) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} A_t \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} A_t^{-1}$. Consequently, $\det(\nabla f(x)) = -1$ a.e. in Ω. Moreover, the following calculation shows that $\nabla f(x) \in \mathbb{M}^{2\times 2}_{sym}$ a.e. and finishes our proof. Indeed, Cramer's rule tells us that for $A = A_t$

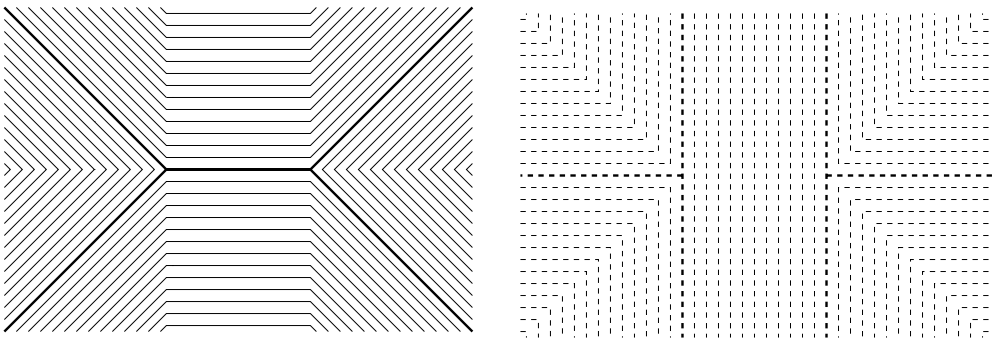
$$\det(A) \cdot A^{-1} = \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} A^\top \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and we obtain that $\det(A)\nabla f(x)$ equals to the symmetric matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} A \left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) A^\top \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} A \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A^\top \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

□

EXAMPLE 2.26. From the following pictures in Figure 3 it is obvious, that for a fixed domain Ω and a fixed given Lipschitz constant for f we can have at least two branch points arbitrarily close to each other. The same holds true for a larger number of branch points. Of course the Lipschitz constant, which is essentially given by the angles between l_+- and l_--paths via Proposition 2.8(d) will become larger with the cardinality but not depend on the distance between these branch points. The pictures show the level sets of f_+ and f_- and the crucial branch level sets are depicted using a greater line width. It is quite easy to see that in this situation the rectangular domain decomposes into four parts such that on each of them the assumptions of Proposition 2.24 are satisfied. So f given by this proposition is a Lipschitz function with $\nabla f \in \mathcal{H}_1$ almost everywhere on each of these pieces. As it is continuous across the crucial level set, it is also globally Lipschitz.



a) levelsets of f_+ *b) levelsets of f_-*

FIGURE 3. Two simple branch points

EXAMPLE 2.27. The Figure 4 presents the example of a Lipschitz map f with $\nabla f \in \mathcal{H}_1$ a.e. but such that $\mathrm{im}(f_+)$ has not only branch points but even selfsintersects. Because of Lemma 2.9 this is of course a kind of "global" effect, but it also shows that one has to be careful to choose suitable assumptions for Lemma 2.21.

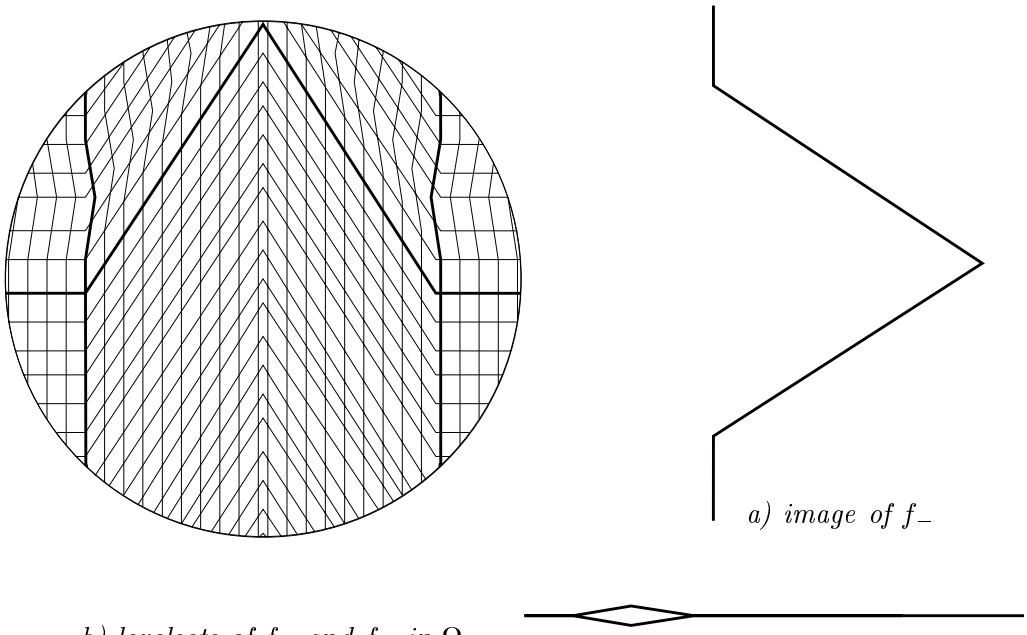


FIGURE 4. Selfintersection

THEOREM 2.28. *Let $\mathcal{A} \subset \mathcal{H}_{-1}$ be an arbitrary set and $A \in \mathcal{H}_{-1}$. Suppose $\Omega \subset \mathbb{R}^2$ is a domain and $f : \Omega \to \mathbb{R}^2$ is a Lipschitz function such that $\nabla f(x) \in \mathcal{A} \cup \{A\}$ for almost every $x \in \Omega$ and assume that at least one of the following conditions is true.*

(i) *The set $\mathcal{A} \cup \{A\}$ does not contain any rank-one connection.*
(ii) *The set $\{B - A\ ;\ B \in \mathcal{A}\}$ contains no rank-one matrix and $|\{x \in \Omega\ ;\ \nabla f(x) = A\}| > 0$.*
(iii) *The set $\{B - A\ ;\ B \in \mathcal{A}\}$ contains no rank-one matrix and for some $* \in \{+, -\}$ there exists a nontrivial segment $\sigma \subset \Omega$ such that $\nabla f(x) = A$ for $\mathscr{H}^1$-a.e. $x \in \sigma$ and $f_{*|\sigma}$ is constant.*

Then f is affine.

Proof. Assume that f is not affine, as this is a local result, we can of course assume that Ω is a disk. We define $\tilde{\mathcal{A}}$ to be the set of all $x \in \Omega$ for which $\nabla f(x) \in \mathcal{A} \cup \{A\}$ and denote by S the set of all $x \in \Omega$ such that

(a) $x \notin \tilde{\mathcal{A}}$ or,
(b) $\mathscr{H}^1(l_{*,i}(x) \setminus \tilde{\mathcal{A}}) > 0$ for some $(*, i) \in \{-, +\} \times \{1, 2\}$ or,
(c) there is some $(*, i) \in \{-, +\} \times \{1, 2\}$ and $z \in f_*(l_{*,i}(x))$ such that for each line l trough the origin there is a $\delta > 0$ such that there are points $y \in \mathrm{im}(f_*)$ arbitrarily closed to z with $\mathrm{dist}(y - z, l) \geq \delta |y - z|$.

We claim that $|S|$=0. Indeed, this is clear for condition (a). Concerning (b), we note that again due to Theorem 2.10.25 in [**35**] for any $(*, i)$ and almost every $t \in \mathbb{R}$ $\mathscr{H}^1((\Omega \setminus \tilde{\mathcal{A}}) \cap f_{*,i}^{-1}(t)) = 0$. As for (c), it is well known that at $\mathscr{H}^1$-almost each $z \in \mathrm{im}(f_*) = f_*(\partial\Omega)$ a classical tangent to $\mathrm{im}(f_*)$ exists, see e.g. Corollary 3.7 in [**34**]. So given the $i = 1, 2$, for almost every $t \in \mathbb{R}$ this tangent exists at each point $z \in \mathrm{im}(f_*)$ for which $z_i = t$. To conclude that the set S itself has measure zero, it is enough to notice that it is mapped under the map $f_{*,i}$ into the zero set of these exceptional t's. But the coarea formula together with the fact that $\nabla f_{*,i}$ vanishes only on a set of measure zero, see Proposition 2.8(b), show that $|f_{*,i}^{-1}(N)| = 0$ if $\mathscr{L}^1(N) = 0$.

First, we show that in any case the condition (iii) is fulfilled. Indeed, if (i) holds then we can of course replace our given A by any matrix from $\mathcal{A}\cup\{A\}$. We take an arbitrary $x \in \Omega\setminus S$ and a regular path $\varphi_x : (c,d) \to \Omega \cap l_+(x)$ which is provided by Lemma 2.15 and passes from $\partial\Omega$ trough x to $\partial\Omega$. Consider any point $y \in \tilde{\mathcal{A}} \cap \operatorname{im}(\varphi_x)$. As the tangent line l to $\operatorname{im}(f_+)$ in $f_+(y) = f_+(x)$ exists, we see that $\operatorname{im}(\nabla f_+(y)) \subset l$. Because $\operatorname{rank}(\nabla f_+(y) - \nabla f_+(x)) \neq 1$, we conclude that $\nabla f_+(y) = \nabla f_+(x)$. Hence, for the matrix $A_x = \nabla f(x) \in \mathcal{A} \cup \{A\}$ we have $\nabla f_+(y) = A_x$ whenever $y \in \tilde{\mathcal{A}} \cap \operatorname{im}(\varphi_+)$. On the other hand, whenever $\varphi_x'(t)$ exists for some $t \in (c,d) \cap \varphi_x^{-1}(\tilde{\mathcal{A}})$ then $A_x(\varphi_x'(t)) = 0$. If $x \notin S$, this happens due to the regularity of φ_x and since condition (b) is violated for almost every $t \in (c,d)$. Consequently, for each such x the path $\operatorname{im}(\varphi_x)$ is a segment $[a_x, b_x]$ with direction $d_x \in \operatorname{Ker}(A_x + \mathrm{i})$. This shows that (iii) is true. Moreover, precisely the same reasoning applies if condition (ii) is satisfied, since we now can find $x \in \Omega \setminus S$ with $\nabla f(x) = A$.

Now, we assume (iii) to hold and denote by M the set of those $y \in \Omega$ such that $\{z \in \Omega\ ;\ (A * \mathrm{i})(y-z) = 0\} \subset l_*(y)$. By the argument in the paragraph above we have $y \in M$ provided $\nabla f(y) = A$ and $y \in \Omega \setminus S$. Conversely, if $y \in M$ and $\nabla f(y) \in \mathcal{A} \cup \{A\}$, then $\operatorname{Ker}(\nabla f(x) * \mathrm{i}) \| \operatorname{Ker}(A * \mathrm{i})$ and so $\nabla f(x) = A$.

The set M is obviously closed in Ω, if we can show that it is nonvoid and open then $M = \Omega$ and hence $\nabla f = A$ a.e. in Ω which finishes the proof. For this purpose, we fix $y \in \operatorname{int}_{\mathrm{rel}}(\sigma)$ and $\varepsilon > 0$ such that $B(y,\varepsilon) \subset \Omega$ does not contain any branch points. By Theorem 2.22.2), see also Lemma 2.21.(b) there is a $\delta \in (0,\varepsilon)$ such that for all $z \in B(y,\delta)$ the inclusion $B(y,\delta) \cap (\sigma + (z-y)) \subset l_*(z)$ holds. As f_* is differentiable almost everywhere in $B(y,\delta)$, we conclude again that $\operatorname{Ker} \nabla f_* \| \operatorname{dir}(\sigma) \| \operatorname{Ker}(A * \mathrm{i})$ and hence $\nabla f(z) = A$ for a.e. $z \in B(y,\delta)$. By what was told above during the derivation of condition (iii) and since M is closed, we see that $B(y,\delta) \subset M$. In particular, $M \neq \emptyset$ and the argument just developed shows also that each non-branch point $y \in M$ produces a whole open "strip" in direction $\operatorname{Ker}(A * \mathrm{i})$ which is contained in M. Since each branch point is isolated, we see that M is as required an open set. □

6. On the rank-one cone

In this section we investigate the structure of solutions to the degenerate hyperbolic Monge-Ampère equation

$$(\nabla f) = (\nabla f)^T, \det(\nabla f) = 0$$

without the appriori convexity assumption often made in the literature (see e.g. [**17**] and [**98**]). Inspired by the idea of developable surfaces we use the following

Definition 2.29. *We say that a map f is developable in Ω if $\Omega \subset \mathbb{R}^n$ is a convex domain, f is defined on Ω and for each $x \in \Omega$ there is a direction $d(x) \in \mathbb{S}^1$ such that*

(i) *f is constant on the segment $s_x = \{y \in \Omega\ ;\ (x-y) \| d(x)\}$,*
(ii) *if $s_x \cap s_y \neq \emptyset$ then $s_x = s_y$.*

Proposition 2.30. *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain and let $f : \Omega \to \mathbb{R}^2$ be a Lipschitz map such that $\nabla f(x) \in \mathcal{H}_0 = \{M \in \mathbb{M}^{2\times 2}_{sym}\ ;\ \det(M) = 0\}$ almost everywhere in Ω. We denote by*

$$Const_f = \{x \in \Omega\ ;\ f_{|B(x,\varepsilon)} \equiv const \text{ for some } \varepsilon > 0\}$$

the open set of local constancy of f.

Then

(i) *For each $x \in \Omega \setminus Const_f$ there is a unique $d(x) \in \mathbb{S}^1$ (at least after identifying $\pm d(x)$) such that Definition 2.29.(i) is satisfied and that $s_x \cap Const_f = \emptyset$. Moreover, all these segments are disjoint and $d(x)$ is locally lipschitz.*
(ii) *Let U be any of the (at most countably many) connected compoments of $Const_f$. Then U is convex and $\operatorname{clos}(U) = \operatorname{conv}(\operatorname{clos}(U) \cap \partial\Omega)$.*

Moreover, f is developable in $\tilde{\Omega} \subset \Omega$ provided there are two segments s_1,$s_2 \subset \bar{\Omega}$ such that $\tilde{\Omega} \subset \mathrm{conv}(s_1 \cup s_2)$ and $f_{|s_j}$ is constant for both $j = 1$ and $j = 2$. In particular, given any $x \in \Omega$ the map f is developable on $B(x, \varepsilon)$ for all $\varepsilon > 0$ sufficiently small.

Proof. We proceed in a similar way as in the proof of Theorem 2.28. Recall that we may assume Proposition 2.8(a) to hold (so local constancy of f and $f_{,1}$ is the same) and that we also have Proposition 2.12 at our disposal. First we claim that there is a set R of full measure in $\mathbb{R}$ such that

- $\mathscr{H}^1(f_{,1}^{-1}(t)) < \infty$ if $t \in R$,
- each $t \in R$ is a regular value for $f_{,1}$,
- $\mathrm{im}(f)$ has a classical tangent T_z at each point $z \in \mathrm{im}(f)$ satisfying $\langle e_1, z\rangle \in R$, and
- if $t \in R$ then $\nabla f(y)$ is a singular symmetric matrix for $\mathscr{H}^1$-almost every $y \in f_{,1}^{-1}(t)$.
- $\mathscr{H}^1(f_{,1}^{-1}(t) \cap M) = 0$ whenever $t \in R$, where $M = \{x \,;\, \nabla f_{,1}(x) = 0\}$

Indeed, the first two statements follow as in the first part of the proof of Theorem 2.16 and repeating the reasoning for the statements (b) and (c) in the proof of Theorem 2.28 we obtain the next two statements. For the last claim we use instead of the generalized Fubini-Eilenberg estimate the more precise coarea formula, see Theorem 3.2.11 in [**35**]. Thus we obtain

$$\int_{\mathbb{R}} \mathscr{H}^1(f_{,1}^{-1}(t) \cap M)\, dt = \int_M \mathrm{Jac}_1(\nabla f_{,1}(x))\, dx = \int_M |\nabla f_{,1}(x)|\, dx = 0.$$

Now, we take any $t \in R$ and an arbitrary $x \in f_{,1}^{-1}(t)$. Since t is a regular value that we can find two different points $y_1, y_2 \in \partial\Omega \cap \mathrm{CC}(l(x), x)$ and Lemma 2.3 ensures that there are regular paths φ_j in $\mathrm{CC}(l(x), x)$ from $x = \varphi_j(0)$ to $y_j = \varphi_j(b_j)$. Note that if for some $r \in (0, b_j)$ both $\varphi_j'(r)$ exists and $\nabla f(\varphi_j(r)) \in \mathcal{H}_0 \setminus \{0\}$ we see that $\nabla f(\varphi_j(r)) = \lambda_r a_r \otimes a_r$ which implies $a_r \| T_{f(x)}$ and therefore also $\varphi_j'(r) \perp T_{f(x)}$. Since for almost every $r \in (0, b_j)$ both these assumptions are fulfilled, we conclude $\varphi_j'(r) \perp T_{f(x)}$ for almost all r. This shows $\mathrm{im}(\varphi_1), \mathrm{im}(\varphi_2) \perp T_{f(x)}$ and, since Ω is convex, we can choose $s_x = (x + \mathrm{i}\, T_{f(x)}) \cap \Omega$. This argument in the same moment proves that any such segment s_x can never (inside Ω) intersect any other segment s' along which f is also constant. Because of these strong uniqueness properties we will refer to these $s_x \subset f_{,1}^{-1}(R)$ as regular level lines.

Now (i) easily follows since $x \in \Omega \setminus \mathrm{Const}_f$ implies the existence of $x_k \to x$ with $f_{,1}(x_k) \in R$. Hence, the corresponding regular level lines s_{x_k} (which do not intersect Const_f due to the fifth condition), converge to the required s_x. Clearly s_x is again disjoint from the open set Const_f and different s_x, s_y for $x, y \in \Omega \setminus \mathrm{Const}_f$ can not intersect as otherwise some of the approximating regular level lines would do so - leading to a contradiction. Similarly, we see that s_x is the only segment through x along which f is constant. So $d(x)$ is unique as a direction and the disjointness of the $\{s_x\}$ gives an upper bound on its lipschitz constant of order $\mathrm{dist}(x, \partial\Omega)^{-1}$.

We turn to (ii), convexity of $\mathrm{clos}(U)$ follows once we establish the convexity of U. However, if $x, y \in U$ then each point in the closed segment $[x, y]$ must belong to Const_f since otherwise the regular level lines passing sufficiently close to z must have x and y on different sides and would disconnect U. On the other hand, since the segment $[x, y]$ is connected and in Const_f it also a part of U. Therefore, (ii) is a consequence of the Krain-Milman theorem if we rule out the existence of a convex extreme point $x \in \Omega$ of $\mathrm{clos}(U)$. But such a point certainly does not belong to Const_f, so again we have a sequence s_k of regular level lines converging to a level segment s trough x. Let Ω_1 and Ω_2 be the two connected components of $\Omega \setminus s$ and consider an arbitrary x' in the open segment s. It is now easy to see that if $i = 1$ or 2 is fixed and $B(x', \varepsilon) \cap \Omega_i \setminus \mathrm{Const}_f \neq \emptyset$ for all $\varepsilon > 0$ then $U \cap \Omega_i = \emptyset$ - indeed the regular segments $s_{x_k'} \to s$ (for $x_k' \to x'$) ensure that no connected subset of Const_f intersecting Ω_i can reach up to x. Consequently, there exists precisely one $i \in 1, 2$ such that for each compact segment $\tilde{s} \subset s$ and some ε positive $B(\tilde{s}, \varepsilon) \cap \Omega_i \subset \mathrm{Const}_f$. This implies of

course also $B(\tilde{s},\varepsilon)\cap\Omega_i\subset U$ and $s\subset \mathrm{clos}(U)$. We infer therefore, that the inner point x of the segment s can not be a convex extreme point and just verified (ii).

To finish the proof, we can of course suppose $\emptyset\neq\tilde{\Omega}=\mathrm{int}(\mathrm{conv}(s_1\cup s_2))$, and due to (i) the set $\tilde{\Omega}\setminus\mathrm{clos}(\mathrm{Const}_f)$ is already uniquely fibrated by level lines of f. So we establish that f is developable on $\tilde{\Omega}$ by showing that each connected component U of $\mathrm{Const}_f(\tilde{\Omega})$ can be fibrated by segments with endpoints in $\partial\tilde{\Omega}$. The basic observation here is that if $\mathrm{clos}(U)$ intersects s_i then $\mathrm{clos}(U)=\mathrm{conv}(\mathrm{clos}(s_i\cup U))$. Indeed, let $x\in\mathrm{int}(\mathrm{conv}(s_i\cup U))\setminus\mathrm{Const}_f$ then we find $x_k\to x$ with $f(x_k)_{,1}\in R$ but all values different. The corresponding regular level lines s_{x_k} can not intersect U, and thus finally have to intersect s_i since $U\cup s_i$ is connected. Consequently these values $f(x_k)$ do agree, contradiction. The same reasoning shows that $\tilde{\Omega}=\mathrm{Const}_f(\tilde{\Omega})$ if $s_1\cap s_2\neq\emptyset$, so we can exclude this situation. Anyhow, independent whether $\mathrm{clos}(\tilde{\Omega})$ has 3 or 4 convex extreme points analyzing a few cases it turns out that each time $\mathrm{clos}(U)=\mathrm{conv}(s\cup s')$ where $s,s'\subset\partial\tilde{\Omega}$ are closed segments, one perhaps degenerated to a point. Obviously, such $U=\mathrm{int}(\mathrm{clos}(U))$ can be fibrated in the (not always unique) required way.

The local developability easily follows now, as we obviously need only to consider $x\in\partial\mathrm{Const}_f$ together with its level line $s_x\subset\partial\mathrm{Const}_f$. If "regular" points and their corresponding level lines approach x from both sides of s_x then we are done by our just establish criterion. Both otherwise f is constant on a whole half disk centred at x with diametr in s_x. Thus we again find the required second segment along which f is constant and which together with a regular line convexly generates a whole neighbourhood of x. Hence, also in this case our criterion finishes the proof. □

COROLLARY 2.31. *Let $f:\mathbb{R}^2\to\mathbb{R}^2$ be a locally lipschitz map such that*

$$\nabla f\in\mathcal{H}_0=\{M\in\mathbb{M}^{2\times 2}_{sym}\ ;\ \det(M)=0\}$$

almost everywhere. Then f is cylindrical, i.e. there exist $g:\mathbb{R}\to\mathbb{R}$ and $d\in\mathbb{S}^1$ such that

$$f(x)=g(\langle x,d\rangle)\cdot d+const.$$

Obviously, the same holds true if $f:[0,1]^2\to\mathbb{R}^2$ is lipschitz, $\nabla f\in H_0$ a.e. und f has $\mathbb{Z}^2$-periodic boundary data – i.e. $f(x)=f(y)$ provided $x-y\in\{e_1,e_2\}$.

Proof. The set Const_f can of course be defined as before, and then Proposition 2.30.(i) implies that for each $x\notin\mathrm{Const}_f$ there is at least locally a unique segment s_x along which f is constant. On the other hand, if there were $x,y\notin\mathrm{Const}_f$ such that s_x,s_y fail to be parallel then we can go to a large disk inside which this two "extended" level lines (still existing by our result) will have to intersect - contradiction. So there is a $d\in\mathbb{S}^1$ normal to all s_x, and hence $\partial\mathrm{Const}_f$ consists of lines also normal to d. Thus $f(x)=G(\langle x,d\rangle)$ and G is locally lipschitz. Therefore $\nabla f(x)=G'(\langle x,d\rangle)\otimes d$, implying that $G'(t)\|d$ for almost all t and giving the claim. Finally, the f with periodic boundary data obviously leads to a $\mathbb{Z}^2$-periodic extension to which we apply part one. □

The following result shows that the properties established in Proposition 2.30 are essentially characterizing mappings with gradients in $\mathcal{H}_0$. Indeed, it shows among others that all convex sets satisfying the condition (ii) in Proposition 2.30 can indeed occur as the set Const_f - and in particular f might not be developable on its whole domain. On the other hand, statement (i) gives some severe, but not completely transparent constraints when we try to built Const_f consisting of more than one component.

PROPOSITION 2.32. *Let $r:I=[t_{min},t_{max}]\to\mathbb{R}$ be a Lipschitz map and let $0<\Delta<1/\mathrm{lip}(r)$. Fix any $\lambda:I\to R$ bounded and measurable. Then the function*

$$\Phi:(t,s)\to\begin{pmatrix}t+s\,r(t)\\ s\end{pmatrix}$$

is a bilipschitz function from the rectangle $R = I \times (-\Delta, \Delta)$ *onto the trapezium* $T = \Phi(I \times (-\Delta, \Delta))$ *and by*

$$f(\Phi(t,s)) = \int_{t_{min}}^{t} \lambda(\tau) \begin{pmatrix} 1 \\ -r(\tau) \end{pmatrix} d\tau$$

we define a Lipschitz map from T *into the plane which fulfills* $\nabla f \in \mathcal{H}_0$ *almost everywhere. Obviously, the direction of the level sets introduced in Proposition 2.30 satisfies* $d(\Phi(t,s)) \| (r(t), 1)$.

Proof. The map Φ is obviously Lipschitz, and we claim that Φ^{-1} exists and is has lipschitz constant at most $c = 1 + (2(m+2)/(1 - \Delta \operatorname{lip}(r)))$, where $m = \max(r(I))$. Indeed, for $(t_1, s_1), (t_2, s_2) \in I \times (-\Delta, \Delta)$ we estimate $|\Phi(t_1, s_1) - \Phi(t_2, s_2)| \geq |s_1 - s_2|$ and

$$|\Phi(t_1, s_1) - \Phi(t_2, s_2)| \geq |t_1 - t_2 + s_1(r(t_1) - r(t_2))| - |s_1 - s_2||r(t_2)| \geq |t_1 - t_2|(1 - \Delta \operatorname{lip}(r)) - m|s_1 - s_2|.$$

So, if $|s_1 - s_2| \geq (1 - \Delta \operatorname{lip}(r))|t_1 - t_2|/2(m+1)$ then $c|s_1 - s_2| \geq |t_1 - t_2| + |s_1 - s_2|$ and hence $|\Phi(t_1, s_1) - \Phi(t_2, s_2)| \geq (|t_1 - t_2| + |s_1 - s_2|)/c$. Else, we have $|s_1 - s_2| \leq |t_1 - t_2|$ and hence

$$\frac{|t_1 - t_2| + |s_1 - s_2|}{c} \leq \frac{m+2}{c}|t_1 - t_2| \leq \frac{1}{2}(1 - \Delta \operatorname{lip}(r))|t_1 - t_2| \leq |\Phi(t_1, s_1) - \Phi(t_2, s_2)|.$$

Since Φ is bilipschitz, f is well defined and easily checked to be Lipschitz on T. Moreover, it is clear that for almost every $(t,s) \in R$ the vector $(r(t), 1) = D_2\Phi(t,s)$ belongs to $\operatorname{Ker}(\nabla f(\Phi(t,s)))$. Hence, we can write $\nabla f(\Phi(t,s)) = b \times a$ with $a = a(t,s) = (1, -r(t))$. On the other hand, $\operatorname{im}(\nabla f(\Phi(t,s))$ is then generated by $D_1(f \circ \Phi)(t,s) = \lambda(t)(1, -r(t)) = \lambda(t)a(t,s)$, at least if we ignore (t,s) from another set of plane measure zero. This shows that $\nabla f(\Phi(t,s)) = \tilde{\lambda}(t,s) a \otimes a$ is a symmetric singular matrix for a.e. $(t,s) \in R$. As Φ preserves the class of measure zero sets, the proposition is proved. □

The foregoing result showed that in the degenerate case $D = 0$ we can in fact not prove the existence of rank-one connections inside the uncountable range of a gradient. In fact, it gives a way to construct even nonaffine C^1 functions without rank-one connections among their gradients. (Just choose the function r occurring in the construction to be injective). On the other hand, in this case we can draw some conclusions also if a gradient is attained in a single point only.

COROLLARY 2.33. *Let* $\mathcal{A} \subset \mathcal{H}_0$ *be a countable set,* $\Omega \subset \mathbb{R}^2$ *a domain and let* $f : \Omega \to \mathbb{R}^2$ *be a Lipschitz function such that* $\nabla f(x) \in \mathcal{A}$ *for almost every* $x \in \Omega$.

(a) *If* $0 \notin \mathcal{A}$, *then* $d(x)$ *is constant, i.e. all level sets of* f *are parallel segments.*
(b) *Let* $\mathcal{A} = \{A\} \cup \mathcal{A}_0$ *and assume that one of the following conditions hold.*
  (i) *The set* $\mathcal{A}$ *does not contain any rank-one connection.*
  (ii) *The set* $\{B - A \ ; \ B \in \mathcal{A}_0\}$ *contains no rank-one matrix and* $|\{x \in \Omega \ ; \ \nabla f(x) = A\}| > 0$.
  (iii) *The set* $\{B - A \ ; \ B \in \mathcal{A}_0\}$ *contains no rank-one matrix and there exists at least one point* $x \in \Omega$ *such that the Frechet derivative exists and satisfies* $\nabla f(x) = A$.

Then f *is affine.*

Proof. To verify (a), set $M = \{v \in \mathbb{S}^1 \ ; \ A \cdot v = 0 \text{ for some } A \in \mathcal{A}\}$. As $0 \notin \mathcal{A}$ we see that M is an at most countable set. We infer from Proposition 2.30 that for almost every $x \in \Omega$ $d(x) \in \operatorname{Ker}(\nabla f(x))$, so in fact $d(x) \in M$ a.e. If d would not be constant, then we find $x, y \in \Omega$ such that $d(x) \neq d(y)$ but such that $d(z) \in M$ $\mathscr{H}^1$-a.e. on $[x,y]$. As d is Lipschitz on the whole segment $[x,y]$, we see that $\mathscr{H}^1(d([x,y])) = 0$. Because $d([x,y])$ is also connected, it has to be a singleton, contradiction.

Now we turn to (b). Obviously, if $0 \in \mathcal{A}$ then $\mathcal{A} = \{A\} = \{0\}$ and hence $\nabla f \equiv 0$, so f is affine. Therefore, we suppose $0 \notin \mathcal{A}$. Similar like in the proof of Theorem 2.28 we show that we can assume (iii) to hold. But, due to part (a) we see that $\operatorname{Ker}(A) \| d(x) \| d(y)$ for all $y \in \Omega$. Therefore,

for almost each $y \in \Omega$ it is true that both $\nabla f(y) \in \mathcal{A}_0 \cup \{A\}$ as well as $\operatorname{Ker}(\nabla f(y)) \| d(y) \| \operatorname{Ker}(A)$ and hence $\nabla f(y) = A$. Consequently, f is affine and our proof finished. □

7. Comparing approaches

THEOREM 2.34. *Let $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ be Lipschitz and $D = 1$. For a given point $x^0 \in \Omega$ the following is equivalent*

i) *We have $\nabla f \in \mathcal{H}_{-D}$ a.e. near x^0 and x^0 is not a branch point of f.*
ii) *There is a suitable orthogonal coordinate system and some $\varepsilon > 0$, $\sigma \in \{-1, +1\}$ such that on the disk $B(x^0, \varepsilon)$ both $\nabla f \in \mathcal{H}_{-D}$ and $\sigma(\partial f_1/\partial x_1) > \varepsilon$ almost everywhere hold.*
iii) *After a suitable exchange and transform of coordinates we obtain the solution of the plane wave equation. In other words, there are $\varepsilon > 0$ and real-valued Lipschitz maps g and h defined on a neighbourghood of $f_1(x^0) + x_2^0$ and $f_1(x^0) - x_2^0$, respectively, such that for each $x \in B(x^0, \varepsilon)$*

$$x_1 = g(f_1(x) + x_2) + h(f_1(x) - x_2), \text{ and}$$

$$f_2(x) = -g(f_1(x) + x_2) + h(f_1(x) - x_2).$$

Proof. First, we show how i) implies ii). Due to Theorem 2.22 we know there exists $\varepsilon > 0$ with $B(x^0, 2\varepsilon) \subset \Omega$ and two regular paths $\varphi_-, \varphi_+ : (-\varepsilon, \varepsilon) \to \mathbb{R}^2$ such that $\varphi_+(0) = \varphi_-(0) = 0$ and

a) $\Phi : t \to \varphi_+(t_1) + \varphi_-(t_2)$ bilipschitzly maps $(-\varepsilon, \varepsilon)^2$ onto a neighbourhood of the origin. Hence, compare e.g. with Remark 2.25, we can also assume the existence of two closed cones $C_+ = \{y \ ; \ \langle y, d_+ \rangle \geq |y|\}$ and $C_- = \{y \ ; \ \langle y, d_- \rangle \geq |y|\}$, $d_+, d_- \in \mathbb{R}^2$ such that $\varphi_*'(t) \in C_*$ for a.e. $t \in (-\varepsilon, \varepsilon)$, $* \in \{+, -\}$ and that $\operatorname{dist}(\mathbb{S}^1 \cap C_+, C_- \cup (-C_-)) > 0$.
b) $f(x^0 + \Phi(t)) = f(x^0) + \mathrm{i}(\varphi_-(t_2) - \varphi_+(t_1))$ for all $t \in (-\varepsilon, \varepsilon)^2$.

Now it is clear that we find a halfspace strictly containing both of these cones, i.e. there is a $d_0 \in \mathbb{S}^1$ and some constant, which we again might suppose to be ε, such that $z \in C_+ \cup C_-$ implies $\langle z, d_0 \rangle \geq \varepsilon |z|$. We use $-\mathrm{i}d_0$, d_0 as an orthogonal basis and get for the function Φ, expressed with respect to the new coordinates on its target space, that $\partial \Phi_2/\partial t_1$, $\partial \Phi_2/\partial t_2 \geq \varepsilon$ a.e. on $(-\varepsilon, \varepsilon)^2$. Moreover, since no $x \in C_+ \cap \mathbb{S}^1$ and $y \in C_- \cap \mathbb{S}^1$ are colinear, we conclude that $\det(x \otimes e_1 + y \otimes e_2)$ does not change sign if (x, y) runs trough the connected set $(C_+ \cap \mathbb{S}^1) \times (C_- \cap \mathbb{S}^1)$. Therefore, equation (5) gives that in this new coordinates

$$\nabla f(\Phi(t)) = \begin{pmatrix} -2\big(\Phi_{2,1}\Phi_{2,2}\big)(t) & \big(\Phi_{1,1}\Phi_{2,2} + \Phi_{1,2}\Phi_{2,1}\big)(t) \\ \big(\Phi_{1,1}\Phi_{2,2} + \Phi_{1,2}\Phi_{2,1}\big)(t) & -2\big(\Phi_{1,1}\Phi_{1,2}\big)(t) \end{pmatrix} (\det \nabla \Phi(t))^{-1},$$

and so we find the $\sigma \in \{-1, +1\}$ with $\sigma f_{1,1} \geq 2\varepsilon^2$ almost everywhere on $\Phi((-\varepsilon, \varepsilon)^2)$.

Next, we use the method mentioned already in Section 1 to interchange one dependent and one independent variable and to conclude iii) from ii). Because the same method works also in the degenerate case when the gradient is in the rank-one cone and the elliptic case (determinant equals plus one), we consider all these 3 cases together. So, we know that $\det(\nabla f) = -D$ and the matrix ∇f is symmetric almost everywhere. Considering f as a map from the (x_1, x_2)-plane into the (y_1, y_2)-plane, we infer from these two conditions that the differential forms $\omega_1 = D dx_1 \wedge dx_2 + dy_1 \wedge dy_2$ and $\omega_2 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ vanish (almost everywhere) along the graph $\operatorname{Gr}(f)$ of the function f. Since we assumed that $y_1 = f_1(x_1, x_2)$ is a bilipschitz function of x_1, we can also define the Lipschitz function $w : \{(x_2, f_1(x)) \ ; \ x \in B(x^0, \varepsilon)\} \to \mathbb{R}^2$ by $w(x_2, f_1(x)) = (x_1, f_2(x))$. Obviously, w has the same graph as f, which is the image of an open subset of the (x_2, y_1)-plane under the map $\hat{w} = \operatorname{Id} \otimes w$. Considering the pushforward of the standard orientation of the domain we get that $\langle \Lambda_2 \nabla \hat{w}(p)(dx_2 \wedge dy_1), \omega_j \rangle = 0$ for almost all points p and $j = 1, 2$, and evaluating that

$$\Lambda_2 \nabla \hat{w}(p)(dx_2 \wedge dy_1) = (dx_2 + w_{1,1}(p) dx_1 + w_{2,1}(p) dy_2) \wedge (dy_1 + w_{1,2}(p) dx_1 + w_{2,2}(p) dy_2),$$

we derive

$$(Dw_{1,2} + w_{2,1})(p) = 0 \text{ and } (w_{1,1} + w_{2,2})(p) = 0 \text{ for almost all } p \in \mathrm{dom}(w). \tag{2.13}$$

Because $\mathrm{dom}(w)$ is simply connected, we infer the existence of $u : \mathrm{dom}(w) \to \mathbb{R}$ such that $\nabla u(p) = (-Dw_1(p), w_2(p))$, and in particular u is a $\mathcal{C}^{1,1}$ function satisfying (at least in the distributional sense)

$$(\frac{\partial^2}{\partial x_2^2} - D\frac{\partial^2}{\partial y_1^2})u \equiv 0 \text{ on } \mathrm{dom}(u).$$

After this general calculations, we go back to our "hyperbolic" case, so $D = 1$. Hence, the function u solves the wave equation on a plane domain, consequently, we find $\eta > 0$ and functions $G : (f_1(x_0) + x_2^0 - \eta, f_1(x_0) + x_2^0 + \eta) \to \mathbb{R}$, $H : (f_1(x_0) - x_2^0 - \eta, f_1(x_0) - x_2^0 + \eta) \to \mathbb{R}$ such that $|x_2 - x_2^0| + |y_1 - f_1(x_0)| < \eta$ implies $(x_2, y_1) \in \mathrm{dom}(u)$ and $u(x_2, y_1) = G(y_1 + x_2) + H(y_1 - x_2)$. Varying just one of the variables x_2 or y_1 we see that $G, H \in \mathcal{C}^{1,1}$ and differentiating this equation gives

$$x_1 = w_1 = -u_{,1} = g(y_1 + x_2) + h(y_1 - x_2) \text{ and } y_2 = w_2 = u_{,2} = -g(y_1 + x_2) + h(y_1 - x_2),$$

where $g = -G'$ and $h = H'$ are Lipschitz. Because $x \to (x_2, f_1(x))$ is certainly continuous, we can transform these formulae back onto a neighbourhood of x^0 and obtain iii).

Finally, we will verify that i) is a consequence of iii). For this purpose we could to a large extend revert the calculations just made. However, we prefer to give a straightforward reasoning much more in the spirit of our geometric approach from the first sections of this chapter. Because it recycles some of the arguments already used there, this argument also illustrates how much simpler everything becomes if we are in the "wave equation" situation. Let $U = B(x^0, \varepsilon)$, then the two equations from iii) give $f_2(x) + x_1 = \frac{1}{2}h(f_1(x) - x_2)$ and $f_2(x) - x_1 = \frac{-1}{2}g(f_1(x) + x_2)$ for all $x \in U$. Using again the (complex) notation $f_\pm = f(x) \pm \mathrm{i}x$, we obtain in particular that $f_+(U) = \mathrm{Gr}(\frac{1}{2}h_{|(f_1 - \langle\cdot, e_2\rangle)(U)})$ and $f_-(U) = \mathrm{Gr}(-\frac{1}{2}g_{|(f_1 + \langle\cdot, e_2\rangle)(U)})$, so both this images are graphs of Lipschitz functions over the x_1-axis. Our first conclusion is that $\nabla f(x) \in \mathcal{H}_1$ almost everywhere in U. Indeed, denoting by R the rotation given by complex multiplication with i, we infer from $\det(\nabla f_-(x)) = \det(\nabla f_+(x)) = 0$ a.e. that for a.e $x \in U$

$$\det(\nabla f(x)) \pm \langle \nabla f(x), \mathrm{cof}\, R\rangle + \det(R) = 0,$$

so $\langle \nabla f(x), \mathrm{cof}\, R\rangle = 0$, which shows that $\nabla f(x)$ is symmetric, and $\det(\nabla f(x)) = -\det(R) = -1$. Secondly, since $f_{\bar{*}}$ acts for $* \in \{-, +\}$ on the level sets of f_* again like a shifted rotation, the graph structure of $f_+(U)$ and $f_-(U)$ excludes the existence of branch points in U.

At this moment we would actually be done, but we will continue and show that in this easy case both the statement ii) and the full structure statement of Theorem 2.22 are obtained very easily. In fact, iii) gives us also for any $(t^1, s), (t^2, s) \in U$ that

$$\begin{aligned}|t^1 - t^2| &= |g(f_1(t^1, s) + s) - g(f_1(t^2, s) + s) + h(f_1(t^1, s) - s) - h(f_1(t^2, s) - s)| \\ &\leq (\mathrm{lip}(g) + \mathrm{lip}(h))|f_1(t^1, s) - f_1(t^2, s)|.\end{aligned}$$

As this excludes the existence of local extrema for the function $t \to f_1(t, s)$ and due to continuity in s, we see that we can find $\sigma \in \{-1, +1\}$ such that $\sigma f_{1,1} \geq (\mathrm{lip}(g) + \mathrm{lip}(h))^{-1}$ a.e in U. So, we just got ii). Moreover, the implicit function theorem now tells us that the level sets of $x \to f_1(x) \pm x_2$ are really the entire Lipschitz graphs over the x_2-axis (perhaps after replacing U by an open interval) corresponding to the rotated copies of $\mathrm{im}(f_+)$ and $\mathrm{im}(f_-)$.

We denote by φ^+ a regular path mapping $(-\varepsilon, \varepsilon)$ into $\{z \ ; \ z + x^0 \in U \text{ and } f_1(z + x^0) - (z + x^0)_2 = f_1(x^0) - x_2^0\}$ and analogous for φ^- mapping into $(f_-)_1^{-1}((f_-(x^0))_1) - x^0$. Now, we claim that for each x sufficiently close to x^0 and $* \in \{-, +\}$ there are unique $x^* = x^0 + \varphi^*(t^*)$ such that $(f_{\bar{*}}(x^*))_2 = (f_{\bar{*}}(x))_2$. Indeed, it is clear that x^+ is completely described by its second coordinate as the solution of $f_1(x^+) + x_2^+ = f_1(x) + x_2$ and $f_1(x^+) - x_2^+ = f_1(x^0) - x_2^0$. Analogous,

$f_1(x^-)-x_2^+ = f_1(x)-x_2$ and $f_1(x^-)+x_2^- = f_1(x^0)+x_2^0$, which gives $x_2^+ = \frac{1}{2}(f_1(x)-f_1(x^0)+x_2+x_2^0)$ and $x_2^- = \frac{1}{2}(f_1(x^0)-f_1(x)+x_2+x_2^0)$. Hence x^+, x^- and t^+, t^- do exist and Lipschitzly depend on x. Again using the fact that f_* affinely acts on the level sets of f_* (which are the level sets of $x \to f_1(x) * x_2)$), we have

$$2\mathfrak{i}(x^0 - x^-) = f_+(x^0) - f_+(x^-) = f_+(x^+) - f_+(x) = 2\mathfrak{i}(x^+ - x),$$

so $x^+ + x^- = x + x^0$ which means $x = x^0 + \varphi^+(t^+) + \varphi^-(t^-)$. This just gives statement a) of Theorem 2.22. The part b) follows now from the calculation

$$\begin{aligned} f(x) &= f_-(x) + \mathfrak{i}x = f_-(x^+) + 2\mathfrak{i}x^+ + \mathfrak{i}(x - 2x^+) = f_+(x^+) + \mathfrak{i}((x^- - x^0) - x^+) \\ &= f_+(x^0) - \mathfrak{i}x^0 + \mathfrak{i}((x^- - x^0) - (x^+ - x^0)) = f(x^0) + \mathfrak{i}(\varphi^-(t^-) - \varphi^+(t^+)). \square \end{aligned}$$

Note that in the last part of this proof there is a certain unsymmetry between $(f_*)_2$ and $(f_*)_1$, which is present already in iii). In fact, it is not clear that g and h are injective, and so level sets of $(f_*)_2$ could be much larger then those of $(f_*)_1$ which we actually worked with.

The example $(x_1, x_2) \to (x_1 + x_2, x_1)$ corresponds to $g \equiv 0$ and $h(t) = t$ and shows that such a degeneracy might occur indeed, since $(f_-)_2 = y_2 - x_1$ is constant. However, in our geometrical approach developed at the begin of this chapter we treat $(f_*)_1$ and $(f_*)_2$ completely symmetrically! This seeming contradiction is resolved if we notice that our basic assumptions as given in Proposition 2.8.a) rules out the degenerate situation when $\mathrm{im}(f_-)$ or $\mathrm{im}(f_+)$ contains a vertical or horizontal segment. But precisely this might show up for an unfortunate choice of g and h. Nevertheless, an arbitrarily small rotation of the coordinate system gets us into the regular setting when $(f_*)_1$ and $(f_*)_2$ have due to Proposition 2.8.c) the same connected components of their level sets. Because this rotation changes the functions g and h just a bit, we conclude that both of them are at least (weakly) monotone.

Finally, we would like to understand the possibility of such an exchange of dependent and independent coordinates in the case when we consider maps with gradients in $\mathcal{H}_0$. As the following example shows, the situation can be much worse than in the hyperbolic case $\mathcal{H}_{-1}$.

EXAMPLE 2.35. *There is a Lipschitz map $f : [-1,1]^2 \to \mathbb{R}^2$ satisfying $\nabla f(x) \in \mathcal{H}_0$ almost everywhere on $[-1,1]^2$ and such that there is* no *nonempty open subset U of $[-1,1]^2$ with the following property.*

(2.14)

There exists a coordinate system, an $\varepsilon > 0$ and $\sigma \in \{-1,+1\}$ with $\sigma(\partial f_1/\partial x_1) > \varepsilon$ a.e. in U.

Proof (by construction). Indeed, if (2.14) fails on a ball $U \subset [-1,1]^2$ then the calculation carried out in the proof of Theorem 2.34, in particular (2.13) with $D = 0$, imply $\partial y/\partial x^2 = w_{2,1} = -Dw_{1,2} \equiv 0$. In other words $f_2(x) = w_2(x_2, f_1(x)) = \tilde{w}(f_1(x))$ on each open $V \subset U$ such that the image of V under the bilipschitz map $x \to (x_2, f_1(x))$ is convex. This implies that the set $f(V)$ is the graph of a function (in a suitable coordinate system). We will use the construction given in Proposition 2.32 to obtain an $f : [-1,1]^2 \to \mathbb{R}^2$, $\nabla f \in H_0$ which violates precisely this graph condition.

Adopting the notation from that construction, we set $I = [-2,2]$, $\Delta = 1$ and $r(t) = -t/2$. Note that $\Phi(t,s) = (t - \frac{1}{2}st, s)$ satisfies $\Phi(I \times [-\Delta, \Delta]) \supset [-1,1]^2$. Thus, our function f is given by

$$f\left(\begin{pmatrix} t - \frac{st}{2} \\ s \end{pmatrix}\right) = \int_{-2}^{t} \lambda(\tau) v(\tau)\, d\tau, \text{ for } (t,s) \in I \times [-1,1],$$

where $v(t) = e_1 + (t/2)e_2$. The choice of the function λ is the crucial point here - for this more subtle purpose we utilize an idea from classical real analysis.

First we note that for any $\eta > 0$ there is a compact nowhere dense set $C_\eta \subset [-1/2, 0]$ such that $|C_\eta| > 1/2 - \eta$. Indeed, enumerating by $\{q_k\}_{k=1}^\infty$ all rationals in $[-1/2, 0]$, the set $C_\eta = [-1/2, 0] \setminus \bigcup_{k=1}^\infty (q_k - \eta 2^{-k-3}, q_k + \eta 2^{-k-3})$ obviously does the job.

Now we construct the set A in the following way inductively. We order all dyadic intervals contained in I in an injective sequence $\{I_k\}_{k=1}^{\infty}$ with $|I_k| \geq |I_{k+1}|$ and set $A_0 = \emptyset$, $\eta_0 = 1/4$. If for $k \geq 1$ the condition $I_k \cap A_{k-1} \neq \emptyset$ holds, then we set $A_k = A_{k-1}$ and $\eta_k = \eta_{k-1}/2$. Else, we choose $\eta_k = \eta_{k-1}|I_k|/4$ and set

$$A_k = A_{k-1} \cup \tilde{A}_k, \text{ where } \tilde{A}_k = (|I_k|\eta_{k-1} \cdot C_{\eta_k} + \frac{1}{2}(\min(I_k) + \max(I_k)) \subset I_k.$$

In this way we define $\{A_k\}_{k=0}^{\infty}$ and finally put $A = \bigcup_k A_k$ and $\lambda = 2\chi_A - 1$.

It remains to show that $\mathrm{im}(f)$ is nowhere a graph, or precisely that for any $d \in \mathbb{S}^1$ and $a < b$ in I there are $a' < b'$ in (a, b) such that

$$\int_{a'}^{b'} \lambda(\tau)v(\tau)\, d\tau \in d^{\perp} \setminus \{0\}. \tag{2.15}$$

For this purpose, we observe first that given the direction d, we can assume, by switching to a subinterval if necessary, the existence of $\delta > 0$ and $\sigma \in \{-1, 1\}$ such that $\sigma\langle d, v(t)\rangle > \delta$ for $t \in [a, b]$. It is also clear that we find arbitrary large k with $\emptyset \neq \tilde{A}_k \subset I_k \subset (a, b)$. We set $c_k = (\max(I_k) + \min(I_k))/2$, $a_k = c_k - \eta_{k-1}|I_k|/2$ and $b_k = c_k + \eta_{k-1}|I_k|/2$ and notice that $|[a_k, c_k] \setminus A| \leq \eta_k|c_k - a_k|$ and that

$$|[c_k, b_k] \cap A| \leq \sum_{l>k} |\tilde{A}_l \cap I_k| \leq \sum_{l>k} \eta_{l-1}|I_l|/2 \leq |I_k|\eta_k \leq |I_k||b_k - c_k|/2.$$

Because $\eta_k, |I_k| \to 0$, we see that such k sufficiently large

$$\frac{1}{|c_k - a_k|}\int_{a_k}^{c_k} (2\chi_A - 1)(\tau)\langle \sigma d, v(\tau)\rangle\, d\tau, \frac{1}{|b_k - c_k|}\int_{c_k}^{b_k} (1 - 2\chi_A)(\tau)\langle \sigma d, v(\tau)\rangle\, d\tau \geq \frac{\delta}{2}.$$

Therefore, if we denoting $\tilde{f}(t) = f(t, 0)$ we see that $\sigma\langle d, \tilde{f}(c_k) - \tilde{f}(a_k)\rangle, \sigma\langle d, \tilde{f}(c_k) - \tilde{f}(b_k)\rangle > 0$. Moreover, if we put $M = \tilde{f}((a_k, c_k)) \cap \tilde{f}((c_k, b_k))$, then clearly $\mathscr{H}^1(M) = 0$ since these two arcs have their tangents in the disjoint (double) cones generated by $v((a_k, c_k))$ and $v((c_k, b_k))$. We write $\pi_d(x) = \langle d, x\rangle$ and infer that the set

$$\pi_d(\tilde{f}((a_k, c_k)))\cap\pi_d(\tilde{f}((c_k, b_k)))\setminus\pi_d(M) \supset \{\pi_d(\tilde{f}(a_k)), \pi_d(\tilde{f}(c_k))\}^{co}\cap\{\pi_d(\tilde{f}(b_k)), \pi_d(\tilde{f}(c_k))\}^{co}\setminus\pi_d(M)$$

is nonvoid. But it is clear that each element in this set leads to a desired pair $a' \in (a_k, c_k)$ and $b' \in (c_k, b_k)$ satisfying (2.15). So, we are done. □

CHAPTER 3

Solving partial differential inclusions

The goal of this chapter is rather easily described. We would like to understand when does a partial differential inclusion $\nabla f \in \mathcal{K} \subset \mathbb{M}^{n\times m}$ have a solution. We will consider bounded sets $\mathcal{K}$ only and therefore focus on Lipschitz solutions, and we are in particular interested which boundary values these solutions can have. Of course, also the large class of inhomogeneous partial differential inclusions $\nabla f(x) \in \mathcal{K}(x, f(x))$ will be studied.

The case of (ordinary) differential inclusions, i.e. $m = 1$ and f defined on an interval, was investigated already for a very long time in a systematic manner and is indeed well understood, for a survey see e.g. [**3**]. As already mentioned, partial differential inclusions also occured quite some time ago - in particular in connection with famous geometrical problems concerning the existence of non-regular isometric embeddings, see [**54**] and [**72**]. A systematic treatment was given first in [**38**], or see the more recent [**84**] for some of the aspects. However both books focus on the existence of $\mathcal{C}^1$-solutions for partial differential inclusions.

This class of maps is of course too narrow for the general nonconvex variational problems we are interested in and also for a description of the experimental data coming from the related observations in material sciences. A very simple but typical example is the situation when our prefered set $\mathcal{K}$ of gradients is disconnected, $\mathcal{C}^1$-solutions could of course use only one component of it. Therefore, in the 90's these questions were considered and quite sucessfully attacked in the framework of Lipschitz mappings. One method due to B. Dacorogna and P. Marcellini extends the previous Baire category approach to ordinary differential inclusions, which started with (see [**19**]) and was mostly developped by de Blasi and Pianigiani (see [**27**] and [**28**]), but compare also to the later work by Bressan and Flores [**14**]. The other way was paved by S. Müller and V. Šverák, and more recent contributions of M. Sychev, which closer followed the notational concept by Gromov - in particular the way the approximate solutions of the partial differential inclusion are gradually improved. At a first glance it might seem that these two approaches are quite different in nature. But we are rather hopeful that after developing some of our results we can in a discussion at the end of Section 3 convince the reader that these methods can be unified to a large extent. In any case, already a short look at the work [**24**] or [**25**] on the one side and the papers [**65**] or [**66**] on the other, gives an understanding of these the two approaches which makes clear that the main difference consists in the limiting step. This is the way the exact solution is obtained from better and better approximate ones. Up to this final operation, both methods follow roughly the same scheme, which is to some extent also present in our work. Therefore, it deserves a short explanation. Suppose we are given a domain $\Omega \subset \mathbb{R}^m$, boundary data $g : \partial\Omega \to \mathbb{R}^n$ and a partial differential inclusion $\nabla f \in \mathcal{K}$ with $\mathcal{K} \subset \mathbb{M}^{n\times m}$ bounded and closed. In general, we are hardly able to write down the exact solution fulfilling

$$f : \Omega \to \mathbb{R}^n \text{ Lipschitz }, \nabla f(x) \in \mathcal{K} \text{ a.e. in } \Omega \text{ and } f(x) = g(x) \text{ for all } x \in \partial\Omega$$

in one strike. Therefore, we start by defining an auxiliar "working space" where we try to build these solutions gradually. For this purpose we choose another set $\mathcal{U} \subset \mathbb{M}^{n\times m}$ of matrices such that

a) we can realize the boundary data g with a "simple map" from Ω into $\mathbb{R}^n$ having gradients in $\mathcal{U}$ almost everywhere.

b) "simple maps" with a gradient in $\mathcal{U}$ allow a gradual and local modification of their gradient distribution, which moves this distribution inside $\mathcal{U}$ towards $\mathcal{K}$.

Therefore, we will often use the expression *"universum $\mathcal{U}$"*, since we usually work inside $\mathcal{U}$ all the time. Once the universum $\mathcal{U}$ is given we denote by *"admissible"* those boundary values g which can be realized in the sense of statement a) from above. Here, the phrase "simple map" is a bit vague and slightly differs in a) and b). Usually in b) it means an affine map and in a) it refers to a (countably) piecewise affine, or equivalently, almost everywhere locally affine* Lipschitz, map. Note that then working on each of the affine pieces of such a map from a) separately, we can utilize assumption b).

Because b) immediately implies that any $M \in \mathcal{U}$ can be approximated by Lipschitz φ_k with $\operatorname{dist}(\nabla\varphi_k, \mathcal{K}) \to 0$ in measure, it is clear that $\mathcal{U}$ has to be contained in $\mathcal{K}^{qc}$ as introduced in Definition 1.8. However, in the definition of $\mathcal{K}^{qc}$ we do not care where the $\nabla\varphi_k$ exactly sits nor whether the next and better approximation φ_{k+1} is in some sense an improving modification of φ_k. The universum $\mathcal{U}$ will usually be the maximal set of linear maps which can by our basic method, i.e. the local modification of gradients, be pushed towards $\mathcal{K}$. Hence, it is clear that in our construction process the gradient is never ever allowed to leave $\mathcal{U}$, not even on a very small set. Otherwise, we would not know how to move it later on this little set towards $\mathcal{K}$ and lose all hope to end up with an exact solution. This difference between the existence of exact and very good approximate might seem to originate from our basic method of local modification. But as indicated already shortly after stating the N-Gradient problem in Section 1 of Chapter 1 and rigorously shown by our results in Section 2 of Chapter 4, the difference exists independent of the concrete construction method and is to some extent the heart of the matter. This also explains why we introduce in Definition 3.19 our notion $\mathscr{PL}(\mathcal{U})$ of prelaminates which apriori do not only live but are also generated inside the universum $\mathcal{U}$.

The key idea of this chapter is presented in Section 3. It is the very simple but apparently in this context completely overlooked fact that in our situation the gradient map $f \in (W^{1,\infty}, \|\cdot\|_{L^\infty}) \to \nabla f \in L^p$, $p < \infty$, has in the framework of Baire category a rather surprising continuity property. Because the continuity looks quite counterintuitive from the point of view of the linear theory, it reflects the underlying nonlinearity in our situation, which is essentially caused by the boundedness of $\mathcal{U}$. So continuity of the map $f \to \nabla f$ occurs only where the usually very easy pertubations of ∇f is constrained by $\partial\mathcal{U}$. A more abstract version which however gives the full strength of this observation is stated in Proposition 3.17. Lemma 3.20 presents a more managable and in particular more geometrical, even if formally weaker, variant of our abstract result. In any case, the message is the following. Instead of working very hard to construct better and better approximating solutions and to enforce their convergence to an exact one, we can use the Baire category method to simulate a kind of random walk of the gradient inside all of $\mathcal{U}$. This will lead to a function f whose gradient is, roughly spoken, only in those parts of $\mathcal{U}$ (or $\bar{\mathcal{U}}$) where it got stuck, i.e. where it is suitably stable. If we know, that we can easily move all gradients in $\mathcal{U}$ to a set $\mathcal{K}$, then the gradient must actually already be in $\mathcal{K}$. So the construction task is reduced to the understanding of a stability question. Another, completely new kind of result we easily obtain from this new argument is the solvability of $\nabla f \in \mathcal{K}$ a.e. for $\mathcal{K}$ not necessarily closed, but rather defined in the natural terms of gradient extreme points of $\bar{\mathcal{U}}$. As already noticed in the literature, such $\mathcal{K}$ will in fact be the smallest set for which such a solution can be expected to exist. Because $\mathcal{K}$ is not closed, our arguments become more subtle. In order to prove that ∇f_0 can not stay outside $\mathcal{K}$ we have to use more carefully choosen approximations f_k of f_0 which would disprove the continuity of ∇f in f_0. This is possible only for a better geometry of the universum $\mathcal{U}$. Nevertheless, since the case of

*as a matter of fact, this second descriptions seems mathematically more precise - however, the first one is the conventional notion

convex $\mathcal{U}$ has already attracted some interest in the literature (see Section 4 of [**71**]), we state in Theorem 3.22 the optimal result valid for such more special sets $\mathcal{U}$.

In the next Section 4 we shortly recall the Banch-Mazur game, the most powerful tool for studying Baire category problems which originally was also used to obtain our continuity property for the gradient map. As an example of its power, we show how easily it allows us to transfer the currently sharpest existence results, derived by the convex integration approach and given in Theorem 1.3 in [**71**], into the Baire category framework.

Finally, in the last Section 5 we consider inhomogeneous partial differential inclusions $\nabla f(x) \in \mathcal{K}(x, f(x))$ a.e. as in [**25**] or [**71**]. We can establish results similar to what what we obtained in the homogeneous case in Section 3, even if the technical complications grow considerably.

The first two sections have a rather introductory character. Section 1 presents techniques how to modify distributions of gradients locally, i.e. without changing the boundary values of the functions involved, and also how to respect additional linear or minor constraints in this process. Since we already mentioned that the smallest set $\mathcal{K}$ for which we can hope to find solutions of $\nabla f \in \mathcal{K}$ is related to certain sets of extreme points of the universum $\mathcal{U}$, we study this kind of question in more detail in Section 2. The results obtained there will be particularly useful when we consider nonhomogeneous partial differential inclusions and nonclosed sets $\mathcal{K}$.

1. Modifying gradient distributions

In this section we give a survey on the different known techniques how to change the distribution of the gradient of a Lipschitz map without changing their boundary values. As a matter of fact, the methods usually work the best if the boundary value is an affine function. But since this extends immediately also to piecewise affine functions, we can use the approximation results Lemma 3.3 and Proposition 3.6 to handle also other, sufficiently smooth boundary data. The material presented here is by no means an original contribution of the author. In particular the modification of an unconstrained gradient can be found in many places in the literature, we refer to [**64**] for a introductory text. Because of later applications in the construction of piecewise affine solutions to certain special partial differential inclusions studied in Section 4.4, we prefer the particular kind of the construction as given in Lemma 3.2. Concerning modifications respecting nonlinear minor constraints, we simply quote the results from [**68**]. From our point of view, in particular having in mind our solution of the five gradient problem in Section 4.3, the most important tool is the modification procedure that respects the symmetry of the gradients. Even if we consider our results Lemma 3.3 and Proposition 3.4 dealing with this problem rather as folklore and in particular as influenced by the presentation choosen in [**68**], we could not find them in this short and easily applicable way in the literature. But first, we recall one of the basic observations in the theory of partial differential inclusions, which among others says that in the homogeneous case and for given affine boundary data the concrete choice of our domain does not matter at all.

Construction 3.1. (The basic rescaling and exhaustion argument) The following argument is used over and over in situations when we are interested in the distribution of the gradients of mappings rather than in the distribution of their values. The argument essentially says that once we have found on some open bounded set U, satisfying the regularity condition $|\partial U| = 0$, a Lipschitz map $f : \bar{U} \to \mathbb{R}^n$ which agrees on ∂U with an affine map A and has an interesting distribution of its gradient, we can reproduce this situation elsewhere. In fact, we do not just get on any other open set $\tilde{U}$ a mapping $\tilde{f}$ with the same gradient distribution and the same affine boundary data A, but we can even make $\|\tilde{f} - A\|_\infty$ as small as we wish.

The transformation is done as follows. First we observe that for any choice of the new open set $\tilde{U}$ there is a sequence $(x_i, r_i) \in \mathbb{R}^m \times (0, \infty)$, $i \geq 1$, such that

a) $U_i = x_i + r_i U$ is contained in $\tilde{U}$ for all i,

b) $U_i \cap U_j = \emptyset$ if $i \neq j$,
c) $|\tilde{U} \setminus \bigcup_i U_i| = 0$.

In the literature about calculus of variations this fact is quite often called the Vitali covering principle, but I can not share this point of view. Indeed, the observation just made is a straightforward consequence of the very simple fact that any open set is the nonoverlapping union of dyadic cubes. This allows us to cover with a disjoint and finite family $\mathcal{C}$ of copies of U a fixed portion of $\tilde{U}$. Then we just reiterate this procedure on $\tilde{U} \setminus \overline{\bigcup \mathcal{C}}$ and in the limit we have covered almost all of $\tilde{U}$. On the contrary, Vitali's covering result is a much more sophisticated and powerful statement as it is able to select such a disjoint cover from a system much smaller than the family of all dyadic cubes.

Therefore, we will refer to the observation above as well as to the whole construction to be described here as the exhaustion argument or the construction by exhaustion.

Once the (x_i, r_i) are given, we can simply define the map $\tilde{f}$ on $\mathrm{clos}(\tilde{U})$ by

$$\tilde{f}(x) = \begin{cases} r_i f(\frac{x-x_i}{r_i}) & \text{if } x \in U_i \text{ for some } i, \\ A(x) & \text{else.} \end{cases}$$

From this it is clear that $\tilde{f}(x) = A(x)$ if $x \in \partial U_i$ for some i or $x \in \partial \tilde{U}$. Because $\tilde{f}(x) = r_i(f - A)((x - x_i)/r_i) + A(x)$ for $x \in U_i$, we conclude that $\tilde{f}$ is Lipschitz on $\overline{U_i}$, due to the first expression for $\tilde{f}$ with Lipschitz constant $\mathrm{lip}(f)$ which is not smaller than $\mathrm{lip}(A)$. Since we always can imagine the new function $\tilde{f}$ to agree outside U with A, we infer that $\mathrm{lip}(\tilde{f}) = \mathrm{lip}(f)$. Because $\nabla \tilde{f}(x) = \nabla f((x - x_i)/r_i)$ for a.e. $x \in U_i$, it is also clear that in case $|\tilde{U}| < \infty$

$$\frac{1}{|\tilde{U}|}|\{x \in \tilde{U} \; ; \; \nabla \tilde{f}(x) \in M\}| = \frac{1}{|U|}|\{x \in U \; ; \; \nabla f(x) \in M\}| \text{ for all } M \subset \mathbb{M}^{2\times 2} \text{ (Borel)},$$

in other words, both gradients have the same distribution. In case that $|\tilde{U}| = \infty$, we must of course understand these distributions in a proper way. But in any case it will be true that ∇f and $\nabla \tilde{f}$ have the same essential range.

Finally, because $\tilde{f} - A$ is certainly $(2\,\mathrm{lip}(f))$-Lipschitz and vanishes on $\bigcup_i \partial U_i$, we can estimate $\|\tilde{f} - A\|_\infty \leq \mathrm{lip}(f)(\sup_i r_i)\mathrm{diam}(U)$. As nothing prevents us from choosing all r_i smaller than any fixed positive number, we can make $\|\tilde{f} - A\|_\infty$ as small as needed. □

There are several possibilities to modify an affine function keeping its boundary data and ensuring that the new gradient distribution lives on a finite set of matrices but uses up to an arbitrarily small error the two desired matrices only. Inspired by Lemma 3.1 from [**71**] we formulate the result as follows.

LEMMA 3.2. *Let $A, B \in \mathbb{M}^{n\times m}$ fulfill $\mathrm{rank}(A - B) = 1$ and let $C = \lambda A + (1 - \lambda)B$ with $\lambda \in (0, 1)$. We write $A - B$ as $a \otimes b$ and suppose we are given more vectors $b_3, \ldots, b_k$ such that $0 \in \mathrm{int}_{\mathbb{R}^m}(\mathrm{conv}(\{b, -b, b_3, \ldots, b_k\}))$. Then, for every domain $U \subset \mathbb{R}^m$ and $\varepsilon > 0$ there is a piecewise affine function $f : U \to \mathbb{R}^n$ such that*

a) *$f(x) = Cx$ if $x \in \partial U$ and $\|f - C\|_{L^\infty(U)} < \varepsilon$,*
b) *$\nabla f(x) \in \{A, B, C + a \otimes b_3, \ldots, C + a \otimes b_k\}$ a.e. in U, and*
c) *$|\{x \in U \; ; \; \nabla f(x) = A\}| > (1 - \varepsilon)\lambda|U|$ and $|\{x \in U \; ; \; \nabla f(x) = B\}| > (1 - \varepsilon)(1 - \lambda)|U|$.*

Proof. We follow some ideas from the proof of Lemma 3.1 in [**71**], but make the construction more explicit. It is quite clear that we can without loss of generality assume that $C = 0$, and hence $A = (1 - \lambda)a \otimes b$ and $B = -\lambda a \otimes b$. Moreover, it is enough to prove the result only in the situation when $n = 1$ and $a = 1$ as in the general case the desired function is obtained by multiplying the scalar solution with the vector a.

Let P be the set of all $x \in \mathbb{R}^m$ such that $\langle x, b_i\rangle > -1$ for all $i \leq k$, here $b_1 = b$, and $b_2 = -b$. Obviously, P is a convex polyhedron containing the origin in its interior. In addition, P is bounded

since it is contained in the polar of the convex hull of the $-b_i$'s, which is a neighbourhood of the origin. We use the auxilary function $h : \mathbb{R} \to \mathbb{R}$ which is 1-periodic and fulfills $h(0) = 0$, $h'(t) = (1-\lambda)$ if $t \in (0,\lambda)$, and $h'(t) = -\lambda$ for $t \in (\lambda, 1)$. For $l \geq 1$ set

$$f_l(x) = \min(\min_{i=3,\dots,k}(1 + \langle x, b_i\rangle), \frac{1}{l}h(l\langle x, b\rangle).$$

As $h \geq 0$ we see that $f_l \geq 0$ on P. It is also straightforward to check that $f_l \equiv 0$ on ∂P, as in all boundary points either the first minimum vanishes or $|\langle x, b\rangle| = 1$ which implies $h(l\langle x, b\rangle) = 0$. As f is affine on a finite decomposition of P into convex pieces, is also quite easy to see that $\nabla f_l \in \{(1-\lambda)b, -\lambda b, b_3, \dots, b_k\}$ almost everywhere in P. Moreover, as $\min_{i\geq 3} 1 + \langle x, b_i\rangle \geq s$ on $(1-s)P$, $s > 0$, we see that c) is fulfilled provided $l = l_0$ is sufficiently large.

Now, it is clear that using the construction by exhaustion 3.1, we can replace $\mathrm{int}(P)$ by any other open set. This finishes the proof. □

In the remaining part of this section we will discuss how the gradient distribution can be modified respecting an additional linear or nonlinear (i.e. minor) constraint. So again we split a Dirac mass, corresponding to the gradient distribution of an affine function, into two atoms sitting at the endpoints of a rank-one segment which contains the barycenter of the Dirac. However, in this situation it is much more complicated to control the location of the additional gradient distribution produced in the process of interpolating with the original boundary condition. During the construction we need to approximate smooth functions with piecewise affine ones respecting the additional constraint. That this is possible for the symmetricity constraint is the content of the next

LEMMA 3.3. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $f \in \mathcal{C}^1(\Omega, \mathbb{R}^n)$ such that $\nabla f \in \mathbb{M}^{n\times n}_{sym}$ everywhere. For any lowersemicontinuous function $\varepsilon : \Omega \to (0,\infty)$ we can find a piecewise affine $g \in \mathcal{C}^1$ with symmetric gradient satisfying $|\nabla f(x) - \nabla g(x)| + |f(x) - g(x)| < \varepsilon(x)$ for all $x \in \Omega$ (in particular, we can achieve $\lim_{x\to\partial\Omega} |\nabla f(x) - \nabla g(x)| + |g(x) - f(x)| = 0$).*

Note: The condition $g \in \mathcal{C}^1$ and piecewise affine is quite unusual, in particular it implies that the underlying affine decomposition of the domain of g is not locally finite.

Proof. The statement is obviously a local one (look at a locally finite decomposition of Ω into sufficiently small simplexes). Therefore, we can in addition suppose that $f \in \mathcal{C}^1(\bar\Omega)$ and $\varepsilon = const > 0$, but will definitely require that $g \equiv f$, $\nabla g \equiv \nabla f$ on $\partial\Omega$. Moreover, it suffices to prove for each $\varepsilon > 0$ the existence of an $\tilde f \in \mathcal{C}^1(\bar\Omega)$ with symmetric gradient and a G open such that $\bar G \subset \Omega$, $\tilde f_{|G}$ is locally affine, $|\partial G| = 0$ but $|G| > |\Omega|/2^{n+1}$ and that $\|\nabla\tilde f - \nabla f\|_\infty + \|\tilde f - f\|_\infty < \varepsilon$ with $f \equiv \tilde f, \nabla f \equiv \nabla\tilde f$ on $\partial\Omega$. Indeed, once this approximation result is established, we easily find a sequence $f_k \in \mathcal{C}^1(\bar\Omega)$ which is Cauchy in the $\mathcal{C}^1$-norm and open sets $G_k \subset G_{k+1} \subset \Omega$ such that

- $f \equiv f_k$, $\nabla f \equiv \nabla f_k$ on $\partial\Omega$,
- $\|\nabla f_k - \nabla f\|_\infty + \|f_k - f\|_\infty \leq \varepsilon/2$,
- $|\Omega \setminus G_k| < (1 - 2^{-n-1})^k|\Omega|$, $|\partial G_k| = 0$, and $\bar G_k \subset \Omega$,
- $f_{k+l} \equiv f_k$ on G_k for $l > 0$ and is locally affine there, $\nabla f_k \in \mathbb{M}^{n\times n}_{sym}$ everywhere.

Then $g = \lim f_k \in \mathcal{C}^1(\bar\Omega)$ has a symmetric gradient everywhere, the right boundary behaviour and is locally affine on $\bigcup_k G_k$. As this set is of full measure, g is piecewise affine in Ω.

To carry out this basic construction, we fix a $\mathcal{C}^\infty$-function $\psi : \mathbb{R}^n \to [0,1]$ vanishing on $B(0, 3/4)$ and with $\psi(x) = 1$ if $|x| > 4/5$. We set $c_1 = \|D^2\psi\|_\infty + \|D\psi\|_\infty > 1$. Given the admissible error $\varepsilon > 0$ we choose finitely many disjoint closed balls $B_i = B(x_i, r_i)$, $i = 1, \dots, N$ in Ω such that $\mathrm{osc}_{B_i} \nabla f < \varepsilon/16c_1$, $r_i < 1$ and $|\bigcup_i B_i| > |\Omega|/2$. For a fixed $i \leq N$ we set $A_i = \nabla f(x_i) \in \mathbb{M}^{n\times n}_{sym}$ and as $\mathrm{curl}(\nabla f - A_i) \equiv 0$, we find $F_i \in \mathcal{C}^2(B(x_i, r_i + \delta)) \subset \Omega$ for some $\delta > 0$ with $\nabla F_i = f - A_i - f(x_i) + A_i \cdot x_i$ and $F(x_i) = 0$. We put $\tilde F_i = \psi((x - x_i)/r)F_i(x)$. Using $\|F_i\|_{L^\infty(B_i)} \leq r_i\|\nabla F_i\|_{L^\infty(B_i)}$

and $\|\nabla F_i\|_{L^\infty(B_i)} \le r_i \|\nabla f - A_i\|_{L^\infty(B_i)}$, we easily estimate

$$\|D^2\tilde{F}_i\|_{L^\infty(B_i)} \le c_1 r_i^{-2}\|F_i\|_{L^\infty(B_i)} + 2c_1 r_i^{-1}\|\nabla F_i\|_{L^\infty(B_i)} + \|\nabla f - A_i\|_{L^\infty(B_i)} \le 4c_1 \operatorname{osc}_{B_i} \nabla f.$$

Hence, $\tilde{f}_i = \nabla\tilde{F}_i + A_i + f(x_i) - A_i \cdot x_i \in \mathcal{C}^1$ satisfies $\tilde{f}(x) = f(x)$ if $|x - x_i| > 4r_i/5$. Moreover, $\|\nabla\tilde{f} - A_i\|_{L^\infty(B_i)} \le \|D^2\tilde{F}_i\|_{L^\infty(B_i)} \le 4c_1 \operatorname{osc}_{B_i} \nabla f < \varepsilon/4$ which in turn implies $\|\nabla\tilde{f}_i - \nabla f\|_\infty \le \varepsilon/2$. Obviously, we also have $\|f_i - \tilde{f}_i\|_\infty < r_i\|\nabla\tilde{f}_i - \nabla f\|_\infty < \varepsilon/2$. Hence, if we set $G = \bigcup_{i=1}^N \operatorname{int} B_i$ and $\tilde{f}(x) = \tilde{f}_i(x)$ if $x \in B_i$, $\tilde{f}(x) = f(x)$ else, then this functions does what is requested. □

PROPOSITION 3.4. *Let $A, B \in \mathbb{M}^{n\times n}_{sym}$ fulfill* $\operatorname{rank}(A - B) = 1$ *and let $C = \lambda A + (1-\lambda)B$ with $\lambda \in (0,1)$. Then, for every domain $U \subset \mathbb{R}^n$ and $\varepsilon > 0$ there is a piecewise affine function $f : U \to \mathbb{R}^n$ such that*

a) *$f(x) = Cx$ if $x \in \partial U$ and $\|f - C\|_{L^\infty(U)} < \varepsilon$,*
b) *$\nabla f(x) \in \mathbb{M}^{n\times n}_{sym} \cap B([A,B],\varepsilon)$ a.e. in U, and*
c) *$|\{x \in U\ ;\ \nabla f(x) = A\}| > (1-\varepsilon)\lambda|U|$ and $|\{x \in U\ ;\ \nabla f(x) = B\}| > (1-\varepsilon)(1-\lambda)|U|$.*

Proof. The construction is similar to the one used to prove Lemma 3.2. However, in order to get all gradients symmetric, we have to work with the potentials of our Lipschitz mappings. Again, without loss of generality, we can suppose $C = 0$, $A = (1-\lambda)a\otimes a$ and $B = \lambda a \otimes a$ for some $a \in \mathbb{R}^n$ of unit length.

First, we consider the closed cylinder $P = \{x + ta\ ;\ t \in [0,1] \text{ and } x \in a^\perp \cap \bar{B}(0,1)\}$. Moreover, given the $\varepsilon > 0$, we fix $r \in (0,1)$ with $1 - r^{n-1} < \varepsilon/2$ and a $\mathcal{C}^\infty$-function $\varphi : \mathbb{R} \to [0,1]$ such that $\varphi(s) = 1$ if $|s| \le (2r+1)/3$ and that $\varphi(s) = 0$ if $|s| \ge (2+r)/3$. Finally, we choose the auxilar Lipschitz functions $H, h : \mathbb{R} \to \mathbb{R}$ such that $H(0) = 0$, $H' = h$ and h is 1-periodic with $h'(t) = 1 - \lambda - \chi_{[\lambda/2, 1-\lambda/2]}(t)$ for $t \in [0,1]$. Note that $h(1/2) = 0$ and that $h'(1/2 + t) = h'(1/2 - t)$ if $|t| \le 1/2$ and hence, $h(1/2 + t) = -h(1/2 - t)$ for the same t. This shows that $\int_0^1 h = 0$, and therefore H is 1-periodic as well.

Now, we pick a large integer k and define the $\mathcal{C}^{1,1}$-function

$$F_P(x) = \frac{1}{k^2} H(k\langle x, a\rangle)\varphi(|x - a\langle x, a\rangle|),$$

and the Lipschitz map

$$f_P(x) = \nabla F_P(x) = \frac{1}{k} h(k\langle x,a\rangle)\varphi(|x - a\langle x,a\rangle|)\cdot a + \frac{1}{k^2} H(k\langle x,a\rangle)\varphi'(|x - a\langle x,a\rangle|)\frac{x - a\langle x,a\rangle}{|x - a\langle x,a\rangle|}. \tag{3.1}$$

Note, that $f_P(x) = a \cdot h(k\langle x,a\rangle)/k$ if $x \in P$ and $|x - a\langle x,a\rangle| \le r$. So, we conclude $\nabla f_P(x) = h'(k\langle x,a\rangle)a\otimes a \in \{A, B\}$ for almost every such x. Due to our choice of r, this proves c) in the case $U = P$. It is also clear, that $f_P(x) = 0$ if $x \in P$ and $|x - a\langle x,a\rangle| > (2+r)/3$. Because $H(k) = h(k) = H(0) = h(0) = 0$, we infer that $f_P(x) = 0$ if $x \in \partial P$, hence a) is established. Obviously, ∇f_P exists and agrees with the second distributional gradient of F_P almost everywhere, and therefore is a symmetric matrix. It remains to verify that $\nabla f_P(x) \in B([A,B],\varepsilon)$ a.e. if $k \ge k_\varepsilon$. But indeed, as $\varphi'(|x - a\langle x,a\rangle|)(x - a\langle x,a\rangle)/|x - a\langle x,a\rangle|$ and $H(k\langle a,x\rangle)/k$ are Lipschitz uniformly in k, it is clear that for k large enough, the second summand in (3.1) gets an arbitrarily small Lipschitz constant. Hence, it contributes at most $\varepsilon/2$ to ∇f_P. Similarly, if we differentiate the first summand in (3.1), then the term containing the (uniformly bounded) gradient of φ is multiplied with the arbitrarily small factor $h(k\langle a,x\rangle)/k$ and contributes again only $\varepsilon/2$. Hence, ∇f_P is up to an error of size ε equal to $h(k\langle x,a\rangle)\varphi(|x - a\langle x,a\rangle|)a\otimes a \in [A,B]$, which establishes b).

Finally, we have to correct the fact, that f_P is not piecewise affine. But it is easy to see that this map is $\mathcal{C}^1$ on each of the k-subcylinders $P_i = \{x \in P\ ;\ k\cdot\langle x,a\rangle \in [j-1, j]\}$, $j = 1,\dots,k$. Hence, applying Lemma 3.3 to each of the pieces $P_j \cap \{x\ ;\ |x - a\langle a,x\rangle| > r\}$ in the end gives the required Lipschitz map $\tilde{f}_P$. As before, we transfer this function onto a general domain U by filling

U with countably many mutually disjoint affine copies of P and defining f on each of them as the affine rescaling of $\tilde{f}_P$. $\square$

For later use, we also quote the following result by Müller and Šverák which allows one to modify the gradient of an affine map respecting a nonlinear (minor) condition. As the proof is much more involved than those given above, and since the result is used only in one place, we refer the reader to Theorem 6.1 in [**68**] for the details. In any case, this result is also obtained in the same two natural steps, in fact our presentation was influenced by this paper. Having in mind our forecoming application, we separately quote the statement (which is Lemma 6.3 in [**68**]) describing the step when the smooth function with the new gradient is approximated by a piecewise affine one.

THEOREM 3.5. *Let $A, B \in \mathbb{M}^{n\times m}$ fulfill $\operatorname{rank}(A-B)=1$ and let $C=\lambda A+(1-\lambda)B$ with $\lambda\in(0,1)$. Suppose moreover, that $M:\mathbb{M}^{n\times m}\to\mathbb{R}$ is a minor of order $r\geq 2$ and that $M(A)=M(B)\neq 0$. Then, for every domain $\Omega\subset\mathbb{R}^m$ and $\varepsilon>0$ there is a piecewise affine map $f:\Omega\to\mathbb{R}^n$ such that*

a) *$f(x)=Cx$ if $x\in\partial\Omega$ and $\|f-C\|_{L^\infty(\Omega)}<\varepsilon$,*
b) *$\nabla f(x)\in\{X\in\mathbb{M}^{n\times m}\ ;\ M(X)=M(A)$ and $\operatorname{dist}(X,[A,B])\leq\varepsilon\}$ a.e. in Ω, and*
c) *$|\{x\in U\ ;\ \nabla f(x)=A\}|>(1-\varepsilon)\lambda|\Omega|$ and $|\{x\in U\ ;\ \nabla f(x)=B\}|>(1-\varepsilon)(1-\lambda)|\Omega|$.*

PROPOSITION 3.6. *Let $M:\mathbb{M}^{n\times m}\to\mathbb{R}$ be a minor of order $r\geq 2$ and $\alpha>0$. If $\Omega\subset\mathbb{R}^m$ is open and $u\in\mathcal{C}^{2,\alpha}_{loc}(\Omega,\mathbb{R}^n)\cap\operatorname{Lip}(\Omega,\mathbb{R}^n)$ with $M(\nabla u)=1$ everywhere in Ω then for any $\varepsilon>0$ there is a piecewise affine map $v:\Omega\to\mathbb{R}^n$ such that*

a) *$M(\nabla v)=1$ a.e. in Ω,*
b) *$\|\nabla u-\nabla v\|_{L^\infty}<\varepsilon$, and*
c) *$u(x)=v(x)$ for all $x\in\partial\Omega$.*

Moreover, if $r=m$, then $\mathcal{C}^{2,\alpha}_{loc}$ can be replaced by $\mathcal{C}^{1,\alpha}_{loc}$.

2. Convexity notions and extreme points

Here we introduce some of the notions related to different kinds of convexity and the corresponding extreme points. If we want to solve inhomogenous partial differential inclusions, we also have to understand how these sets of extreme points depend on the universum, i.e the basic set under consideration. This result is given in Lemma 3.9, a special case of it is Lemma 4.2 in [**71**]. Then we derive in Lemma 3.10 and Corollary 3.11 two easy stability properties related to quasiconvex extreme points which will be used in Section 5. We conclude the section with a short and almost selfcontained proof of the fact, scattered around in several places in the literature, that for convex sets all these different notions of extreme points conincide.

DEFINITION 3.7. *Following* [**29**] *we say that a multivalued map, i.e. a map $\Phi:X\to 2^Y$, where both X and Y are topological spaces is*

- *upper semicontinuous if $\{x\in X\ ;\ \Phi(x)\subset U\}$ is open for any $U\subset Y$ open,*
- *lower semicontinuous if $\{x\in X\ ;\ \Phi(x)\cap U\neq\emptyset\}$ is open for any $U\subset Y$ open.*

Moreover, we define in addition that Φ is

- *strongly lower semicontinuous if $\{x\in X\ ;\ \Phi(x)\supset C\}$ is open for any $C\subset Y$ compact.*

DEFINITION 3.8. *Let $\mathcal{K}\subset\mathbb{M}^{n\times m}$ be compact and let $\square\in\{rc,qc,pc,co\}$. We say that a compact subset $\mathcal{F}\subset\mathcal{K}$ is a $\square$-face of $\mathcal{K}$ if for any measure $\mu\in\mathscr{M}_\square$ with $\operatorname{spt}(\mu)\subset\mathcal{K}$ and $\bar{\mu}\in\mathcal{F}$ the support of μ is necessarily contained in $\mathcal{F}$. If the $\square$-face is a singleton, we say that it is a $\square$-extreme point of $\mathcal{K}$. By $\operatorname{extr}_\square(\mathcal{K})$ we denote the set of all such points.*

Obviously, any compact union of $\square$-faces of $\mathcal{K}$, so in particular any compact subset of $\mathrm{extr}_\square(\mathcal{K})$, is a $\square$-face of $\mathcal{K}$ again. The following two results do not rely on the particular convexity notion used. In fact they work equally well whenever we consider a concept of convexity which fits into the theory of Choquet boundaries, for the general approach see [**1**] or for a closer look on notions relevant for the Calculus of Variations compare to [**53**]. Lemma 3.9 presumably is folklore in this theory, but we could not find a reference for this statement - even if the notion of a "stable", which precisely corresponds to that of our faces, can already be found on page 46 of [**1**].

For the Corollary 3.11, which is crucial for further applications, the convexity notion used, however, has to be at least as strong as quasiconvexity.

LEMMA 3.9. *The map $\mathcal{K} \to extr_\square(\mathcal{K})$ is a lower semicontinuous multifunction from the space of all compact subsets of $\mathbb{M}^{n\times m}$ equipped with the Hausdorff distance into the G_δ-subsets of $\mathbb{M}^{n\times m}$.*

Proof. As $A \in \mathcal{K} \setminus \mathrm{extr}_\square(\mathcal{K})$ if and only if there is a measure $\mu \in \mathscr{M}_\square(\mathcal{K})$ with $\bar{\mu} = A$ and $\int |A - X|\, d\mu(X) \geq 1/k$ for some $k \in \mathbb{N}$, we easily see that $\mathcal{K} \setminus \mathrm{extr}_\square(\mathcal{K})$ is an F_σ-set and hence $\mathrm{extr}_\square(\mathcal{K})$ is of type G_δ.

To check lower semicontinuity, we choose any open $\mathcal{U} \subset \mathbb{M}^{n\times m}$ and a compact $\mathcal{K} \subset \mathbb{M}^{n\times m}$ such that $A \in \mathrm{extr}_\square(\mathcal{K}) \cap U$ exists. Now suppose there is a sequence $\mathcal{K}_k \to \mathcal{K}$ in Hausdorff distance with $\mathrm{extr}_\square(\mathcal{K}_k) \cap \mathcal{U} = \emptyset$ for all k. As $A \in \mathcal{K}$ we find a sequence $A_k \in \mathcal{K}_k$ with $A_k \to A$. We use the very intuitive fact, for a formal statement see e.g. Theorem I.5.19, that for each k we can find $\mu_k \in \mathscr{M}_\square(\mathcal{K}_k)$ with $\mathrm{spt}(\mu_k) \subset \mathrm{clos}(\mathrm{extr}_\square(\mathcal{K}_k))$ and $\overline{\mu_k} = A_k$. Since all $\mathcal{K}_k$ stay in a fixed compact set, passing to a subsequence if necessary we can assume $\mu_k \rightharpoonup^* \mu$. Then obviously $\mu \in \mathscr{M}_\square(\mathcal{K})$ and $\bar{\mu} = A$. As $A \in \mathrm{extr}_\square(\mathcal{K})$, we infer $\mu = \delta_A$. This shows, modifying the integrand far away from the $\mathcal{K}_k$'s into a compactly supported function, that $\lim_k \int |A - X|\, d\mu_k(X) = \int |A - X|\, d\delta_A(X) = 0$. Hence, $\mathrm{dist}(A, \mathrm{extr}_\square(\mathcal{K}_k)) \to 0$ which contradicts $\mathrm{extr}_\square(\mathcal{K}_k) \cap \mathcal{U} = \emptyset$. $\square$

LEMMA 3.10. *Let $\mathcal{F}$ be a $\square$-face of $\mathcal{K}$, then for any positive ε there is a $\delta > 0$ such that each $\mu \in \mathscr{M}_\square(B(\mathcal{K}, 1/\varepsilon))$ which fulfills $\mu(B(\mathcal{K}, \delta)) > 1 - \delta$ and $\bar{\mu} \in B(\mathcal{F}, \delta)$ also satisfies $\mu(B(\mathcal{F}, \varepsilon)) > 1 - \varepsilon$.*

Proof. The proof is straightforward. If the statement fails then we find a sequence $\{\mu_k\}_{k=1}^\infty$ of counterexamples for $\delta_k = {}^1\!/\!_k$. Since the supports of all these measures stay in a fixed compact, we find a clusterpoint μ of this sequence in the vague topology. Obviously, $\mu \in \mathscr{M}_\square(B(\mathcal{K}, 1/\varepsilon))$, moreover $\mathrm{spt}(\mu) \subset \mathcal{K}$ and $\bar{\mu} \in \mathcal{F}$. On the other hand, $\mu(\mathbb{M}^{n\times m} \setminus B(\mathcal{F}, \varepsilon)) \geq \varepsilon$, contradiction. $\square$

COROLLARY 3.11. *Let $\mathcal{K} \subset \mathbb{M}^{n\times m}$ be compact, $\mathcal{F}$ a qc-face of $\mathcal{K}$. Then for each positive ε there is a $\delta > 0$ such that for each bounded domain $\Omega \subset \mathbb{R}^m$ there exists a positive η with the following property. If $f : \Omega \to \mathbb{R}^n$ Lipschitz satisfies*

- *$dist(\nabla f(x), \mathcal{K}) \leq \delta$ for almost each $x \in \Omega$,*
- *there exists an affine map A with $\nabla A \in B(\mathcal{F}, \delta)$ and $|f(x) - A(x)| < \eta$ for all $x \in \partial\Omega$*

then the estimate $|\{x \in \Omega\ ;\ \mathrm{dist}(\nabla f(x), \mathcal{F}) \geq 2\varepsilon\}| \leq 2\varepsilon|\Omega|$ holds.

Proof. Set $c = \sup\{|M|\ ;\ M \in \mathcal{K}\}$. We can of course assume that $\varepsilon \in (0, 1)$ is small enough to ensure that $B(0, 5c + 5) \subset B(\mathcal{K}, 1/\varepsilon)$. We choose a positive $\delta < \varepsilon$ according to Lemma 3.10. Given Ω we pick the η such that $|\Omega \setminus \Omega_{2\eta}| < \delta|\Omega|$, where $\Omega_t = \{x\ ;\ B(x, t) \subset \Omega\}$ for $t > 0$ - so η depends only on the geometry of Ω.

Now, suppose f fulfills the assumptions. We can check that $g : (\partial\Omega \cup \partial\Omega_\eta) \to \mathbb{R}^n$, defined by $g(x) = f(x)$ if $x \in \partial\Omega_\eta$ and $g(x) = A \cdot x$ else, is $(5c + 5)$-Lipschitz. Hence, we find using Kirszbraun's Theorem a Lipschitz map $\tilde{f} : \bar{\Omega} \to \mathbb{R}^n$ such that $\tilde{f}_{|\partial\Omega} \equiv A$, $f(x) = \tilde{f}(x)$ if $x \in \overline{\Omega_\eta}$ and that $\mathrm{lip}(\tilde{f}_{|\bar{\Omega}\setminus\Omega_\eta}) \leq (5c + 1)$. We define the probability measure $\mu = \nabla\tilde{f}_\#((\mathscr{H}^n \llcorner \Omega)/|\Omega|)$, in other words $\mu(\mathcal{S}) = |\{x \in \Omega\ ;\ \nabla\tilde{f} \in \mathcal{S}\}|/|\Omega|$ for $\mathcal{S} \subset \mathbb{M}^{n\times m}$. Since this is the distribution of the

gradient of a map with affine boundary data, it is a Gradient Young measure which obviously lives in $B(\mathcal{K}, 1/\varepsilon)$. Moreover, $\bar{\mu} = \nabla A \in B(\mathcal{F}, \delta)$ and $\mu(B(\mathcal{K}, \delta)) \geq |\Omega_\eta|/|\Omega| \geq 1 - \delta$. By Lemma 3.10 we infer $\mu(B(F, \varepsilon)) \geq 1 - \varepsilon$, and therefore

$$|\{x \in \Omega \; ; \; \mathrm{dist}(\nabla f(x), F) > \varepsilon\}| \leq |\{x \in \Omega \; ; \; \mathrm{dist}(\nabla \tilde{f}(x), F) > \varepsilon\}| + |\Omega \setminus \Omega_\eta| \leq 2\varepsilon|\Omega|.$$

□

Finally, we consider the situation (which is studied in detail in the subsection 5) when K is a convex body. We want to understand to which extent these different notions of extreme points in this case agree.

Definition 3.12. *We say that $[A, B]$ is a δ-rank-one segment if* $\mathrm{rank}(A - B) = 1$ *and* $|A - B| \geq \delta$. *For any set $\mathcal{K} \subset \mathbb{M}^{n\times m}$ we denote by* $\mathrm{extr}_{\mathrm{gr}}(\mathcal{K})$ *the set of those matrices $A \in \mathcal{K}$ for which there is no nontrivial rank-one segment in $\mathcal{K}$ which has centre A. Moreover,* $\mathrm{extr}_{\mathrm{prelam}}(\mathcal{K})$ *stands for the set of all $A \in \mathcal{K}$ such that there are not even $B, C \in \mathcal{K}$ with $A \in (B, C)$ and* $\mathrm{rank}(B - C) = 1$.

We have to admit that these notations break a little bit out of our scheme, since they are not defined using some of the usual classes $\mathscr{M}_\square$ of measures. However, there is a perfect duality with the class $\mathscr{PL}(\mathcal{K})$ in Definition 3.19 and its modification in Remark 3.21. The following results comparing different kind of extreme points are scattered around in the literature, our proof follows in particular some ideas from [**107**] and [**30**].

Lemma 3.13. *If $K \subset \mathbb{M}^{n\times m}$ is compact and convex then* $\mathrm{extr}_{\mathrm{gr}}(\mathcal{K}) = \mathrm{extr}_{\mathrm{rc}}(\mathcal{K}) = \mathrm{extr}_{\mathrm{qc}}(\mathcal{K})$ *and this set agrees also with the set* $\mathrm{gr\ extr}(\mathcal{K})$ *defined in* [**71**]. *Moreover, if* $\min(n, m) \leq 2$ *then also* $\mathrm{extr}_{\mathrm{gr}}(\mathcal{K}) = \mathrm{extr}_{\mathrm{pc}}(\mathcal{K})$.

Proof. First recall that $\mathrm{gr\ extr}(\mathcal{K})$ was in [**71**] defined to be the set of all $A \in \mathcal{K}$ for which there exists a chain of affine spaces $\mathcal{S}_0 = \mathbb{M}^{n\times m} \supset \mathcal{S}_1 \supset \ldots \supset \mathcal{S}_k$ such that $\mathcal{S}_{j+1}$ supports $\mathcal{K} \cap \mathcal{S}_j$ in A and such that $\mathcal{S}_k$ does not contain any rank-one line. Let $A \in \mathrm{gr\ extr}(\mathcal{K})$. It is quite easy to see that for any segment fulfilling $A \in (B, C) \subset \mathcal{K}$ we inductively obtain $[B, C] \subset \mathcal{S}_j$, $j = 1, 2, \ldots, k$. So if $A \in \mathcal{K} \setminus \mathrm{extr}_{\mathrm{gr}}(K)$, we infer $A \notin \mathrm{gr\ extr}(\mathcal{K})$. Conversely, let $A \in \mathrm{extr}_{\mathrm{gr}}(\mathcal{K})$ and choose such a descending chain of maximal length, i.e. for which $\mathcal{S}_k$ has minimal possible affine dimension. If $\dim(\mathcal{S}_k) = 0$ then it obviously does not contain a rank-one line. Else, the minimality of $\mathcal{S}_k$ together with the geometric version of the Hahn-Banach theorem ensure that $A \in \mathrm{int}_{\mathcal{S}_k}(\mathcal{K} \cap \mathcal{S}_k)$. This shows that $\mathrm{extr}_{\mathrm{gr}}(\mathcal{K}) \subset \mathrm{gr\ extr}(\mathcal{K})$.

Now we consider the remaining inclusions. For $\mathrm{rank}(A - B) = 1$ and any $\lambda \in [0, 1]$ the measure $\lambda\delta_A + (1 - \lambda)\delta_B$ is in $\mathscr{M}_{\mathrm{rc}} \subset \mathscr{M}_{\mathrm{qc}} \subset \mathscr{M}_{\mathrm{pc}}$. Hence, we see that $\mathrm{extr}_{\mathrm{gr}}(\mathcal{K})$ of course contains any of the other sets. For the converse, consider any $A \in \mathcal{K} \setminus \mathrm{extr}_{\mathrm{qc}}(\mathcal{K})$ and fix a $\mu \in \mathscr{M}_{\mathrm{qc}}(\mathcal{K})$ with $\mu \neq \delta_A$ and $\bar{\mu} = A$. Again we consider the "minimal" end S_k of a supporting chain, so we have $A \in \mathrm{int}_{S_k}(K \cap S_k)$ with $S_k = \{A\}$ if $\dim(S_k) = 0$. As before, we can gradually prove $\mathrm{spt}(\mu) \subset \mathcal{S}_j$, for $j = 1, 2, \ldots, k$ (This obviously holds for any measure μ). Assuming now that $A \in \mathrm{extr}_{\mathrm{gr}}(\mathcal{K})$ we use the argument appearing in [**11**] to show that $\mathrm{spt}(\mu) = \{A\}$, contradiction. As the case $\mathcal{S}_k = \{A\}$ is trivial, we consider $\mathcal{V} = \mathcal{S}_k - A$ and let π be the orthogonal projection from $\mathbb{M}^{n\times m}$ onto $\mathcal{V}^\perp$. Since our assumption implies $\mathcal{V} \cap \{X \; ; \; \mathrm{rank}(X) = 1\} = \emptyset$, it ensures also the existence of an ε positive such that $|\pi(X)| > \varepsilon$ whenever $|X| = 1$ and $\mathrm{rank}(X) = 1$. Consequently, the quadratic form φ corresponding to the bilinear form $\Phi(X, Y) = \langle \pi(X), \pi(Y)\rangle - \varepsilon\langle X, Y\rangle/2$ fulfills $\varphi(X) \geq \varepsilon|X|^2/2$ for all X of rank one. In particular, φ is rank-one convex and therefore, see e.g. Theorem 1.7 in S ection 4.1.2.2 of [**23**], also quasiconvex. As $\lambda \in \mathscr{M}_{\mathrm{qc}}(\mathcal{V})$ for the measure defined by $\lambda(M) = \mu(M + A)$ we obtain $0 \leq \varphi(\bar{\lambda}) \leq \int_{\mathcal{V}} \varphi(X)\, d\lambda(X)$. This together with $\varphi(X) < 0$ if $X \in \mathcal{V} \setminus \{0\}$ implies $\mathrm{spt}(\lambda) = \{0\}$ as required.

In case $\min(n, m) \leq 2$ the rank-one convexity of φ also implies that φ is polyconvex, see [**83**]. So we derive in this situation even that $A \notin \mathrm{extr}_{\mathrm{pc}}(\mathcal{K})$ implies $A \notin \mathrm{extr}_{\mathrm{gr}}(\mathcal{K})$. □

3. Points of continuity of the gradient map

After the more or less introductory first two sections we are now going to present the central theme of the whole chapter. As we already mentioned, it consists in a very simple observation about the Baire class of the gradient map together with the proper understanding when the counterintuitive continuity of the gradient map might occur. In this way we obtain a very simple argument which unifies and explains most of the known results about the existence of Lipschitz solutions to partial differential inclusions.

As defined before, we say that a typical function (in a certain function space $\mathscr{X}$) has property $(\mathcal{P})$ if the set of all $f \in \mathscr{X}$ satisfying $(\mathcal{P})$ is residual in $\mathscr{X}$. Because we will always work in some spaces of Lipschitz functions equipped with the L^∞-norm, no ambiguity should arise from the phrase "typical gradient" refering to the gradient of a typical function. Our argument, whose core can be found in Lemma 3.15 and Proposition 3.17, immediately gives a description of the essential range of a typical gradient in terms of stability of certain affine maps inside the class of all admissible maps. However, we do not rush for the quickest possible interpretation of this stability notion, but rather give a more geometrical and at least necessary criterion for it. This criterion will be used several times in the applications we present in the sequel - however, some of these applications (in particular the in-approximation result as given in Theorem 3.2 in [**66**]) require a deeper understanding of the geometry of rank-one convexity which will be discussed only in Section 1 below.

Next, we consider the situation when the universum $\mathcal{U}$ for our gradients is a particularly simple set, i.e. a convex or starshaped body, and we are able to show that then the essential values of typical gradients are in fact contained in the set of all "laminational-extreme"-points. Note that this set might be considerably smaller than its closure which is the set studied in the literature before. These results are nicely supplemented by the last statement saying that a typical gradient not only uses exclusively those parts of the universe $\mathcal{U}$ where it stabilizes but it also uses "all" of these parts. Most of these arguments can be generalized to the case of nonhomogeneus differential inclusions $\nabla f(x) \in K(x, f(x))$, but to keep the presentation simple in the begining we postpone these more technical results to the Section 5. The section is concluded with several comments comparing our results to related ones from the literature.

Let us start with a precise specification of the function spaces we have in mind.

Definition 3.14. *Given $\Omega \subset \mathbb{R}^m$ bounded and open, $\mathcal{U} \subset \mathbb{M}^{n\times m}$ bounded but not necessarily open, we put*

$$\mathscr{P} = \mathscr{P}(\Omega,\mathcal{U}) = \{f \in \mathrm{Lip}(\bar{\Omega}, \mathbb{R}^n)\ ;\ f \text{ is piecewise affine and } \nabla f(x) \in \mathcal{U} \text{ a.e. in } \Omega\}.$$

If there is in addition a map g from a superset of $\partial\Omega$ into $\mathbb{R}^n$ given, then consider also the space

$$\mathscr{P}(\Omega,\mathcal{U},g) = \{f \in \mathscr{P}(\Omega,\mathcal{U})\ ;\ f(x) = g(x) \text{ for all } x \in \partial\Omega\}.$$

Lemma 3.15. *Let $\Omega \subset \mathbb{R}^m$ be a bounded open set and let $\mathscr{X} \subset Lip(\Omega, \mathbb{R}^n)$ equipped with the supremum norm $\|\cdot\|_\infty$ be complete. Then for any $p < \infty$ the map $\nabla : f \to \nabla f$ is a Baire class one map from $(\mathscr{X}, \|\cdot\|_\infty)$ into $L^p(\Omega, \mathbb{M}^{n\times m})$, i.e. is the pointwise limit of a sequence $\Delta_k : (\mathscr{X}, \|\cdot\|_\infty) \to L^p(\Omega, \mathbb{M}^{n\times m})$ of continuous maps.*

In particular, the typical f (i.e. all $f \in \mathscr{X}$ except those from a set of first Baire category in $\mathscr{X}$) is a point of continuity of the map ∇.

Proof. For $k \geq 1$ and $f \in \mathscr{X}$ we define, denoting by e_j the j-th canonical unit vector,

$$(\Delta_k(f))(x)_{i,j} = \begin{cases} k(f_i(x + \frac{e_j}{k}) - f_i(x)) & \text{if } \mathrm{dist}(x, \mathbb{R}^m \setminus \Omega) > \frac{1}{k} \\ 0 & \text{else.} \end{cases}$$

Obviously, $\|\Delta_k(f)\|_\infty \le \mathrm{lip}(f)$ for all f and k, moreover, if f is differentiable at x then $\Delta_k(f)(x) \to \nabla f(x)$ as $k \to \infty$. By Rademacher's theorem and Lebesgue's dominated convergence theorem, we have $\lim_k \Delta_k(f) = \nabla f$ in $L^p(\Omega, \mathbb{M}^{n\times m})$ as $k \to \infty$. On the other hand, each Δ_k is a linear operator which has norm at most $2k|\Omega|^{1/p}$ when acting from $(\mathscr{X}, \|\cdot\|_\infty)$ into $L^p(\Omega, \mathbb{M}^{n\times m})$. Thus ∇ is a Baire-one map.

It is well-known, see e.g. Chapter 7 of [**75**] for a proof which works in the desired generality, that any Baire-one function from a complete metric space into any metric space has a dense G_δ-set of points of continuity. For the convenience of the reader, we reproduce here a simple argument which actually shows many similarities with the usual way the Banach-Steinhaus Theorem is proved.

Fix any $\varepsilon > 0$ and for $k \ge 1$ let M_k be the set of all $f \in \mathscr{X}$ such that $\|\Delta_l(f) - \nabla f\|_p \le \varepsilon$ whenever $l \ge k$. Obviously, M_k is closed and moreover $\bigcup_k M_k = \mathscr{X}$. From Baire's category theorem we conclude that the set $U_\varepsilon = \bigcup_k \mathrm{int}(M_k)$ is dense in $\mathscr{X}$. It is clear that whenever $f \in U_\varepsilon$ then there is a $\delta > 0$ such that $\|\nabla f - \nabla g\|_p < 3\varepsilon$ if $\|f - g\|_\infty < \delta$, of course δ is choosen to fulfill $B(f,\delta) \subset M_k$ for some k. Consequently, each f in the residual (i.e. comeager) set $\bigcap_k U_{1/k}$ is a point of continuity of ∇. □

Definition 3.16. *Let $\mathcal{U}, \mathcal{K} \subset \mathbb{R}^{n\times m}$ be given. We say that* gradients in $\mathcal{U}$ are stable only near $\mathcal{K}$ *if $\mathcal{U}$ is bounded, $\mathcal{K}$ is closed and for each $\varepsilon > 0$ there is a $\delta = \delta_\varepsilon > 0$ such that for all $A \in \mathcal{U}$ with* $\mathrm{dist}(A, \mathcal{K}) > \varepsilon$ *there exists a piecewise affine $\varphi \in \mathrm{Lip}(\mathbb{R}^m, \mathbb{R}^n)$ with bounded support which satisfies*

- *$A + \nabla\varphi(x) \in \mathcal{M}_A$ for a.e. $x \in \Omega$, where $\mathcal{M}_A \subset \mathcal{U}$*
- *$\int |\nabla\varphi(x)| > \delta|\,\mathrm{spt}(\varphi)|$.*

Similarly, we say that gradients in $\mathcal{U}$ are stable with respect to compact pertubations only near $\mathcal{K}$ *if the same as above holds and in addition $\mathcal{M}_A$ can be choosen compact.*

Proposition 3.17. *Suppose that gradients in $\mathcal{U}$ are stable only near $\mathcal{K}$. Given any $A \in \mathcal{U}$ and $\Omega \subset \mathbb{R}^m$ bounded open, let $\mathscr{P}$ be the space of all piecewise affine Lipschitz $f : \Omega \to \mathbb{R}^n$ with $f_{|\partial\Omega} \equiv A$ and $\nabla f(x) \in \mathcal{U}$ almost everywhere.*

Then the typical $f \in \overline{\mathscr{P}}^\infty$ satisfies $\nabla f(x) \in \mathcal{K}$ a.e. in Ω and consequently, for any $\varepsilon > 0$ we can choose δ_ε appearing in the Definition 3.16 to be equal $\varepsilon/2$.

Similarly, if gradients in $\mathcal{U}$ are stable with respect to compact pertubations only near $\mathcal{K}$ then the analogous statement holds true for the space $\mathscr{P}_c$ of all $f \in \mathscr{P}$ with $\nabla f(x) \in \mathcal{M}$ a.e. and $\mathcal{M} \subset \mathcal{U}$ compact. The conclusion concerning δ_ε remains unchanged.

Proof. Let $\mathscr{X} = \mathscr{P}$ or $\mathscr{X} = \mathscr{P}_c$ in case stability with respect to compact pertubations is considered. Applying Lemma 3.15 it is enough to show $\nabla f(x) \in \mathcal{K}$ a.e. if f is a point of continuity of ∇ restricted to $\overline{\mathscr{X}}^\infty$. Suppose for a contradiction that f is a point of continuity but $\nabla f \notin \mathcal{K}$ on a set of positive measure. Use Lusin's theorem we can find a compact set $\tilde{C} \subset \Omega$ and an $\varepsilon > 0$ such that

- $|\tilde{C}| > \varepsilon$, $\nabla f_{|\tilde{C}}$ is continuous, and
- $\mathrm{dist}(\nabla f(x), \mathcal{K}) > \varepsilon$ for all $x \in \tilde{C}$.

We pick an $\eta > 0$ such that $\|\nabla f - \nabla g\|_{L^1} < \varepsilon \cdot \delta_\varepsilon/4$ whenever $g \in B_\infty(f,\eta) \cap \overline{\mathscr{X}}^\infty$. By definition of $\overline{\mathscr{X}}^\infty$ there is a sequence of $f_k \in \mathscr{X}$ with $f_k \rightrightarrows f$. Hence, due to the choice of f we have, switching to a subsequence of necessary, that $\nabla f_k(x) \to \nabla f(x)$ almost everywhere. Consequently, we find $k_0 \in \mathbb{N}$ and another compact $\hat{C} \subset \Omega$ with $|\hat{C}| > \varepsilon$ and $\mathrm{dist}(\nabla f_{k_0}(x), \mathcal{K}) > \varepsilon$ for all $x \in \hat{C}$. Of course, we can also assume that $\|f - f_{k_0}\|_\infty < \eta/2$. As f_{k_0} is piecewise affine, there are disjoint open subsets $\{G_i\}_1^\infty$ of Ω such that $|\Omega \setminus \bigcup G_i| = 0$ and $f_{k_0|G_i}$ is affine with gradient $A_i \in U$. We choose $I \subset \mathbb{N}$ finite such that $\hat{C} \cap G_i \ne \emptyset$ for each $i \in I$ and that $\sum_{i\in I} |G_i| > \varepsilon$. Due to the definition of δ_ε and using Construction 3.1, we find for each $i \in I$ a piecewise affine Lipschitz map $\varphi_i : G_i \to B(0, \eta/2; \mathbb{R}^n)$ with $\int_{G_i} |\nabla\varphi_i(x)|dx > \delta_\varepsilon|G_i|$, $\varphi_{i|\partial G_i} \equiv 0$ and $A_i + \nabla\varphi_i(x) \in M_i$ a.e. in G_i,

where $M_i \subset U$ and is compact in case $X = \mathscr{P}_c$ is considered. Then it is clear that $g = f_{k_0} + \sum_{i \in I} \varphi_i$ belongs to $B_\infty(f, \eta) \cap \mathscr{X}$ and the same is true for f_{k_0} itself. We have, however, $\|\nabla f_{k_0} - \nabla g\|_{L^1} \geq \varepsilon \delta_\varepsilon$, and this contradiction to the choice of η shows that $\nabla f(x) \in \mathcal{K}$ a.e. indeed. Since f is a point of continuity, it is clear that for $\delta = \varepsilon/2$ a suitable φ is obtained by choosing any element $\tilde{\varphi}$ of $\mathscr{X}$ sufficiently close to f and setting $\varphi = \tilde{\varphi} - A$. □

COROLLARY 3.18. *As before, suppose that gradients in $\mathcal{U}$ are stable only near $\mathcal{K}$. Given any $\Omega \subset \mathbb{R}^m$ open and bounded, the typical $f \in \operatorname{clos}_\infty(\mathscr{P}(\Omega, \mathcal{U}))$ satisfies $\nabla f \in \mathcal{K}$ a.e. in Ω. If we have in addition $g : \partial\Omega \to \mathbb{R}^n$ given, then the same holds true for the typical $f \in \operatorname{clos}_\infty(\mathscr{P}(\Omega, \mathcal{U}, g))$.*

Since this statement can be shown repeating the proof of the foregoing Proposition 3.17 (perhaps using the better bound on δ_ε), and because full details will be given for the nonhomogeneous case in the proof of Theorem 3.29, we skip them here. However, it should be noted that obviously $\mathscr{P}(\Omega, \mathcal{U}, g)$ is nonvoid iff $g = \hat{g}_{|\partial\Omega}$ for some $\hat{g} \in \mathscr{P}(\Omega, \mathcal{U})$. To decide if the boundary values of a given function $\tilde{g}$ are of this kind, one can try to use Lemma 3.3 and Proposition 3.6 in the cases when $\mathcal{U}$ is open relative to a constraint. If $\mathcal{U}$ is open in the full $\mathbb{M}^{n \times m}$ then $\tilde{g} \in \mathcal{C}^1(\Omega) \cap \operatorname{Lip}(\bar{\Omega})$ with $\nabla \tilde{g} \in \mathcal{U}$ on Ω is clearly sufficient.

Now, we would like to become a little bit more specific about the question when gradients are stable. Of course, they fail to be stable in those M through which sufficiently long rank-one segments in $\mathcal{U}$ pass. Sometimes, however, and as a particularly important example might serve Section 3 of Chapter 4 below the segments themself can be rather short but iterating our modification procedure we can still show that the gradients are not stable in certain regions of $\mathcal{U}$. Motivated by this fact, we introduce a slightly more general tool.

DEFINITION 3.19. *Let $\mathcal{V} = \mathbb{M}^{n \times m}$ or $\mathcal{V} = \mathbb{M}^{n \times n}_{sym}$ or $\mathcal{V} = \{X \in \mathbb{M}^{n \times m} \ ; \ M(X) = 1\}$ where $M : \mathbb{M}^{n \times m} \to \mathbb{R}$ is a minor of rank at least two. Suppose we are given a set $\mathcal{U} \subset \mathcal{V}$ open in $\mathcal{V}$. We denote by $\mathscr{PL}(\mathcal{U})$ the prelaminates generated in $\mathcal{U}$, which means that their entire generating splitting sequence is contained in $\mathcal{U}$. In other words, this is the smallest class of probability measures containing all Dirac measures δ_X for $X \in \mathcal{U}$ and such that $\mu = \sum_{i=1}^k \lambda_i \delta_{X_i} \in \mathscr{PL}(\mathcal{U})$ (X_i's not necessarily distinct) implies that also $\tilde{\mu} = \lambda_1(\lambda \delta_{\tilde{X}_0} + (1-\lambda)\delta_{\tilde{X}_1}) + \sum_{i=2}^k \lambda_i \delta_{X_i} \in \mathscr{PL}(\mathcal{U})$ if $[\tilde{X}_0, \tilde{X}_1] \subset \mathcal{U}$ is a rank-one segment, $\lambda \in [0,1]$ and $X_1 = \lambda \tilde{X}_0 + (1-\lambda)\tilde{X}_1$. Note, that all measures in such a generating splitting sequence keep the same barycentre.*

Now we define, similar to [**13**], [**14**], *the function $\Phi_{\mathcal{U}} : \mathcal{U} \to \mathbb{R}$ by*

$$\Phi_{\mathcal{U}}(X) = \sup\Big\{ \int |Y - X|^2 \, d\mu(Y) \ ; \ \mu \in \mathscr{PL}(\mathcal{U}) \text{ and } \bar{\mu} = X \Big\}.$$

LEMMA 3.20. *The function $\Phi_{\mathcal{U}}$ has the following properties.*

a) *For all matrices $A \in \mathcal{U}$ and for all $c < \Phi_{\mathcal{U}}(A)/diam(\mathcal{U})$ there exists a piecewise affine $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ with bounded support such that $(\nabla\varphi)(x) + A$ is for almost all x compactly in a fixed compact subset of $\mathcal{U}$ and $\int |\nabla\varphi| \geq c|\operatorname{spt}(\varphi)|$. So, given the additional set $\mathcal{K}$ from Definition 3.16, we see that gradients in $\mathcal{U}$ are stable with respect to compact pertubations only near $\mathcal{K}$ provided that $\Phi_{\mathcal{U}}(X_k) \to 0$ implies $\operatorname{dist}(X_k, \mathcal{K}) \to 0$.*
b) *$\Phi_{\mathcal{U}}$ is rank-one concave, i.e. concave on any rank-one segment contained in $\mathcal{U}$. Moreover, we have the following lower bounds: $\Phi_{\mathcal{U}}(P) \geq \varepsilon^2$ if $\mathcal{U}$ contains a 2ε-rank-one segment centred in P.*

Proof. Since our assumption a) implies the existence of $\mu \in \mathscr{PL}(\mathcal{U})$ with $\bar{\mu} = A$ and $\int |Y - A| \, d\mu(Y) > c$, the conclusion follows by induction with respect to the length of the generating

splitting sequence for μ using Lemma 3.2, Proposition 3.4 or Theorem 3.5 depending on the choice of $\mathcal{V}$.

The lower bound $\Phi_{\mathcal{U}}(P) \geq \varepsilon^2$ is obvious under the assumptions in b), and the rank-one concavity is derived in a rather standard way. Indeed, if $\hat{X}_0 \in [\hat{X}_1, \hat{X}_2] \subset \mathcal{U}$ is a rank-one segment and any $\varepsilon > 0$ is given, then we find $\mu_i \in \mathscr{PL}(\mathcal{U})$ with $\bar{\mu}_i = \hat{X}_i$ and $\Phi_{\mathcal{U}}(\hat{X}_i) - \varepsilon < \int |Y - \hat{X}_i|^2 \, d\mu_i(Y) = \int |Y|^2 \, d\mu_i(Y) - |\hat{X}_i|^2$ for $i = 1, 2$. Choosing $\lambda \in [0, 1]$ such that $\hat{X}_0 = \lambda \hat{X}_1 + (1 - \lambda)\hat{X}_2$, the measure $\mu_0 = \lambda\mu_1 + (1 - \lambda)\mu_2$ belongs to $\mathscr{PL}(\mathcal{U})$, has barycentre $\hat{X}_0$ and shows that

$$\begin{aligned}
\Phi_{\mathcal{U}}(\hat{X}_0) &\geq \int |Y|^2 \, d\mu_0(Y) - |\hat{X}_0|^2 = \int |Y|^2 \, d(\lambda\mu_1 + (1 - \lambda)\mu_2)(Y) - |\lambda\hat{X}_1 + (1 - \lambda)\hat{X}_2|^2 \\
&\geq \lambda \int |Y|^2 \, d\mu_1(Y) + (1 - \lambda) \int |Y|^2 \, d\mu_2(Y) - \lambda|\hat{X}_1|^2 - (1 - \lambda)|\hat{X}_2|^2 \\
&\geq \lambda\Phi_{\mathcal{U}}(\hat{X}_1) + (1 - \lambda)\Phi_{\mathcal{U}}(\hat{X}_2) - \varepsilon.
\end{aligned}$$

Because $\varepsilon > 0$ was arbitrary, rank-one concavity follows. □

REMARK 3.21. *If there is no constraint on $\mathcal{V}$ we can also work with the modified class $\mathscr{PL}'(\mathcal{U})$ defined by requiring in the spliting procedure only that $\tilde{X}_0, \tilde{X}_1 \in \mathcal{U}$. Moreover, in case that we consider $\mathcal{U}$ which is not open it would still be enough to require in the definition that all "intermediate" generating points X_j are in* $\operatorname{int}(\mathcal{U})$ *- for the final ones is $X_i \in \mathcal{U}$ a sufficiently strong requirement. Indeed, Lemma 3.2 allows to keep the whole gradient of the new modification in $\{\tilde{X}_0, \tilde{X}_1\}$ and a small neighbourhood of X_1. Therefore, in the unconstrained case we can carry out all the arguments from the proof of Lemma 3.20.a) above also under the weaker assumptions on the generating spliting sequences.*

The results developed by now would already allow one to handle some of the known and applications of the theory of partial differential inclusions. One of them concerns the two-well problem, however we will cover it by considering its inhomogeneous version in Section 5. The most interesting applications will be presented in Chapter 4. First, we there reprove the rank-one convex in-approximation result from [**66**]. Then, in Section 4.3 we essentially rely on the fact that we only need to check stabilty of gradients and do not need to construct the approximate solutions (as in [**71**]) or the explicite in-approximations.

Thus, we postpone these applications and rather continue with a completely new kind of result. It tells that under suitable geometrical conditions on the set of gradients used by our (piecewise affine) approximating functions, we can ensure that the gradients of (typical) limit functions take values only in the set of gradient (also called laminate) extreme points, as introduced in Section 2 - even if this set is not necessarily closed. This problem is reminiscent of the question if measures representing a given point x in a compact convex set K as their barycentres can live on the (G_δ)-set of extreme points of K rather then just on its closure, compare to Choquet's Theorem in Section I.4 of [**1**]. Moreover, this result will also be the starting point for the similar statement in the inhomogeneous case, see Theorem 3.34. A result of a similar nature was mentioned to me by M. Sychev [**94**]. It should also be noticed, that in the case when $n = 1$ and $\mathcal{U}$ is convex the result was obtained in [**14**] using the function $\Phi_{\mathcal{U}}$ defined in Definition 3.19.

THEOREM 3.22. *Let $\mathcal{V}$ be one of the spaces $\mathbb{M}^{n\times m}$ or $\mathbb{M}^{n\times n}_{sym}$, where we have $m = n$ in the latter case. Assume $\mathcal{K} \subset \mathcal{V}$ is compact and regularly starshaped, which means that there exists $C_0 \in \mathcal{K}$ such that $[C_0, A) \subset \operatorname{int}_{\mathcal{V}}(\mathcal{K})$ for all $A \in \mathcal{K}$.*

Given an open bounded set $\Omega \subset \mathbb{R}^m$ we consider again the spaces of functions $\mathscr{P} = \mathscr{P}(\Omega, \operatorname{int}_{\mathcal{V}}(\mathcal{K}))$ or $\mathscr{P} = \mathscr{P}(\Omega, \operatorname{int}_{\mathcal{V}}(\mathcal{K}), g)$ and the classes of extreme points as in Definition 3.12. Then the typical $f \in \operatorname{clos}_\infty(\mathscr{P}(\Omega, \operatorname{int}_{\mathcal{V}}(\mathcal{K})))$ is a Lipschitz function with $\nabla f(x) \in \operatorname{extr}_{\mathrm{gr}}(\mathcal{K})$ for a.e. $x \in \Omega$.

Moreover,

(a) *If* $\mathcal{V} = \mathbb{M}^{n\times m}$ *then we have even* $\nabla f(x) \in \mathrm{extr}_{\mathrm{prelam}}(\mathcal{K})$ *almost everywhere in* Ω.
(b) *Let* Ω *be regularly starshaped domain. Given any affine boundary data* g *with* $\nabla g \in \mathrm{int}_{\mathcal{V}}(\mathcal{K})$, *we also have for typical* $f \in \mathrm{clos}_\infty(\mathscr{P}(\Omega, \mathrm{int}_{\mathcal{V}}(\mathcal{K}), g))$ *that* $\nabla f(x) \in \mathrm{extr}_{\mathrm{gr}}(\mathcal{K})$ *a.e. in* Ω. *Also in this situation the statement analogous to (a) holds.*

Proof. As all statements remain unchanged when a suitable affine function is added, we can suppose that the origin is the centre C_0 of $\mathcal{K}$. We first prove the main inclusion. By the foregoing Lemma 3.15 it is enough to show that $\nabla f \in \mathrm{extr}_{\mathrm{gr}}(\mathcal{K})$ a.e. whenever f is a point of continuity of the gradient map into $L^1(\Omega)$.

So, suppose the contrary to hold for such a fixed f. It is clear from the continuity of ∇ and the closedness of $\mathcal{K}$ that $\nabla f(x) \in \mathcal{K}$ almost everywhere. Thus, it remains to show that the gradient is in the extreme points. For $\varepsilon > 0$ let $\mathcal{K}_\varepsilon$ be the set of all A which are centre of some ε-rank-one segment contained in $\mathcal{K}$. We already noticed in the proof of Lemma 3.9 that each such $\mathcal{K}_\varepsilon$ is compact. If $\nabla f(x) \notin \mathrm{extr}_{\mathrm{gr}}(\mathcal{K})$ on a set of positive measure, then Lusin's theorem ensures the existence of $\varepsilon > 0$ and of a compact $C \subset \Omega$ such that

- $\nabla f : C \to \mathbb{M}^{n\times m}$ is well defined and continuous,
- $\nabla f(C) \subset \mathcal{K}_{2\varepsilon}$.

We choose $\eta \in (0, \varepsilon|C|/4)$ and $\delta > 0$ such that $\tilde{f} \in \overline{\mathscr{P}}^\infty \cap B_\infty(f, 3\delta)$ implies $\|\nabla f - \nabla \tilde{f}\|_{L^1} < \eta$. Next, we fix $\gamma > 0$ small such that $\hat{f} = (1-\gamma)f$ satisfies $\hat{f} \in B_\infty(f, \delta)$. Then it is clear that for each $x \in C$ there exists a ε-rank-one segment inside $\mathrm{int}_{\mathcal{V}}(\mathcal{K})$ with centre $\nabla \hat{f}(x)$. Because $\nabla \hat{f} : C \to \mathcal{K}$ is continuous, a simple compactness argument even shows that there is a $\beta > 0$ such that for all $x \in C$ we find

$$D_x \in \{D\ ;\ \mathrm{rank}(D) = 1 \text{ and } |D| = 1\} \text{ fulfilling } B(tD_x + \nabla \hat{f}(x), \beta) \cap \mathcal{V} \subset \mathrm{int}_{\mathcal{V}}(\mathcal{K}) \tag{3.2}$$

whenever $t \in [-\varepsilon, \varepsilon]$.

By definition we know that there exists a sequence $f_k \in \mathscr{P}$ with $f_k \rightrightarrows f$. Hence, due to the choice of f also $\nabla f_k(x) \to \nabla f(x)$ almost everywhere in Ω. Obviously, for k sufficiently large we have $\|(1-\gamma)f_k - f\|_\infty < 2\delta$ and the set C_k of all $x \in C$ with $|(1-\gamma)\nabla f_k(x) - \nabla \hat{f}(x)| < \eta$ satisfies the inequality $|C_k| > {}^3\!/_4|C|$. In other words, the piecewise affine map $\tilde{f} = (1-\gamma)f_k$ has the properties

- $\tilde{f} \in \overline{\mathscr{P}}^\infty \cap B_\infty(f, 2\delta)$
- there exist pairwise disjoint domains $G_j \subset \Omega$ such that $\nabla \tilde{f}_{|G_j} \equiv A_j$ is the centre of an ε-rank-one segment in $\mathrm{int}_{\mathcal{V}}(\mathcal{K})$ and $|\bigcup_j G_j| > {}^3\!/_4|C|$.

Now we can apply Lemma 3.2 for $\mathcal{V} = \mathbb{M}^{n\times m}$ or Proposition 3.4 if $\mathcal{V} = \mathbb{M}^{n\times n}_{sym}$ on each piece G_j and get a piecewise affine map $h \in B_\infty(\tilde{f}, \delta)$ such that $\nabla h(x) = \nabla \tilde{f}(x)$ for almost every $x \notin \bigcup_j G_j$ and that $\nabla h(x) \in \mathrm{int}_{\mathcal{V}}(\mathcal{K})$ a.e in G_j but $|\nabla h(x) - A_j| = \varepsilon$ for x in three quarters of the volume of G_j. This shows $\|\nabla h - \nabla \tilde{f}\|_{L^1} \geq 3\varepsilon|\bigcup_j G_j|/4 > \varepsilon|C|/2 > 2\eta$ which of course contradicts $\nabla h, \nabla \tilde{f} \in B_{L^1}(\nabla f, \eta)$. Thus $\nabla f \in \mathrm{extr}_{\mathrm{gr}}(\mathcal{K})$ a.e.

Now, as concerns the additional part (a) everything works the same except that we require in (3.2) the inclusion to be true only for $t \in \{T_x^-, 0, T_x^+\}$ for some $-T_x^-, T_x^+ \geq \varepsilon$ and modify in the obvious way second property of $\tilde{f}$.

The proof of part (b) follows again the same pattern but requires a little bit of extra effort to interpolate between the rescaled function and the original boundary data. So, without loss of generality we can suppose that 0 is a centre for the regularly starshaped Ω and we fix $r_0 > 0$ such that $B(A, r_0) \cap \mathcal{V} \subset \mathrm{int}_{\mathcal{V}}\,\mathcal{K}$ where $A = \nabla g$. As before, we consider an arbitrary continuity point f of the gradient map, which is, however, this time defined only on $\overline{\mathscr{P}}^\infty = \mathrm{clos}_\infty(\mathscr{P}(\Omega, \mathrm{int}_{\mathcal{V}}(\mathcal{K}), g))$. First of all, after adding a constant to f if necessary, we can extend f outside Ω to agree with the linear map A. For $\alpha, \beta \in (0,1)$ let $f_{\alpha,\beta}(x) = (1-\beta)(1-\alpha)f(x/(1-\alpha))$. If f does not fulfill the required condition $\nabla f(x) \in \mathrm{extr}_{\mathrm{gr}}(\mathcal{K})$ a.e. we find again a compact set $C \subset \Omega$ and $\varepsilon > 0$ such

that ∇f continuously maps C into $\mathcal{K}_{2\varepsilon}$. As before we choose $\eta \in (0, \varepsilon|C|/5)$ and $\delta > 0$ such that $g \in \overline{\mathscr{P}}^{\infty} \cap B_\infty(f, 3\delta)$ implies $\|\nabla f - \nabla g\|_{L^1} < \eta$. Now, we pick $\alpha > 0$ such that $\|f_{\alpha,0} - f\|_\infty < \delta$. Obviously, $f_{\alpha,0}(x) = Ax$ for all $x \in \Omega \setminus (1-\alpha)\Omega$. We claim that for $\beta > 0$ sufficiently small there is a piecewise affine $\tilde{g}_\beta : \mathrm{clos}(\Omega \setminus (1-\alpha)\Omega) \to \mathbb{R}^n$ such that $\tilde{g}_{\beta|\partial(1-\alpha)\Omega} = -\beta A$, $\tilde{g}_{\beta|\partial\Omega} = 0$ and $\nabla \tilde{g}_\beta \in B(0, r_0/2) \cap \mathcal{V}$ almost everywhere in the domain of $\tilde{g}_\beta$. Indeed, fix a $\mathcal{C}^\infty$-function φ identical to 1 near $(1-\alpha)\Omega$ and vanishing near $\mathbb{R}^m \setminus \Omega$. In case that $\mathcal{V} = \mathbb{M}^{n\times n}_{sym}$ we consider $\hat{g}_\beta(x) = \nabla(-\beta\varphi(x)\langle Ax, x\rangle/2)$, else set $\hat{g}_\beta(x) = -\beta\varphi(x)Ax$. This gives a $\mathcal{C}^\infty$-function with the right boundary data and the gradient in the requested (relatively) open set. Now, if $\mathcal{V} = \mathbb{M}^{n\times n}_{sym}$ then Lemma 3.3 provides us with the required function $\tilde{g}_\beta$. Otherwise, $\tilde{g}_\beta$ can be obtained by the obvious (locally finite) triangulation procedure. So, for β positive but small enough it is clear that

$$\tilde{g} = \begin{cases} f_{\alpha,\beta}(x) & \text{if } x \in (1-\alpha)\Omega \\ Ax + \tilde{g}_\beta(x) & \text{if } x \in \Omega \setminus (1-\alpha)\Omega \end{cases}$$

belongs to $\overline{\mathscr{P}}^{\infty} \cap B_\infty(f, \delta)$. Since we can repeat the argument from the proof of the main inclusion in the subdomain $(1-\alpha)\Omega$ and because the measure of the "bad set" C was decreased only by an arbitrary small amount determined by α, we obtain the same contradiction as before. This finishes our proof. □

Of course, all the additional effort needed to handle the affine boundary data in part b) of the foregoing theorem is not necessary if ∇g itself is in the centre of the starshaped set $\mathcal{K}$, i.e if ∇g could play the role of C_0 in the assumption of Theorem 3.22. This is in particular true if $\mathcal{K}$ is convex. In this case we also have the following more natural description of our space $\overline{\mathscr{P}}^{\infty}$. The main argument behind this lemma, i.e. the approximation by smooth functions, is rather folklore and can be found e.g. in [**39**]; a similar idea appears in the proof of Lemma 4.1 in [**71**]. In the work [**99**] a nonhomogeneous differential inclusion is considered - this is a straightforward modification, but provides a convenient way to treat also nonaffine (but $\mathcal{C}^1$) boundary data.

LEMMA 3.23. *Let Ω and $\mathcal{V}$ be as in Theorem 3.22 and let $\mathcal{K}$ be a convex body, i.e. compact with $\mathcal{K} = \mathrm{clos}(\mathrm{int}_{\mathcal{V}}(\mathcal{K}))$. Then*

$$\mathrm{clos}_\infty(\mathscr{P}(\Omega, \mathrm{int}_{\mathcal{V}}(\mathcal{K}))) = \{f \in Lip(\Omega, \mathbb{R}^n)\ ;\ \nabla f(x) \in \mathcal{K} \text{ a.e. in } \Omega\}.$$

The same statement holds if we consider subspaces of functions fulfilling a given affine boundary condition with gradient in $\mathrm{int}_{\mathcal{V}}(\mathcal{K})$.

Proof. Of course, we can again assume that $0 \in \mathrm{int}_{\mathcal{V}}(\mathcal{K})$. The fact that indeed any $f \in \mathrm{clos}_\infty(\mathscr{P}(\Omega, \mathrm{int}_{\mathcal{V}}(\mathcal{K})))$ satifies $\nabla f(x) \in \mathcal{K}$ a.e. is rather obvious using seperating affine functions, or see Lemma 3.33 for a more general statement. The way how to prove the converse inclusion is quite straightforward. We show that any $f \in \mathrm{Lip}(\Omega)$ with $\nabla f \in \mathcal{K}$ a.e. can be uniformly approximated by $g \in \mathcal{C}^\infty(\Omega) \cap \mathrm{Lip}(\Omega)$ with $\nabla g(x) \in \mathrm{int}_{\mathcal{V}}(\mathcal{K})$ for all $x \in \Omega$ and $g \equiv (1-\gamma)f$ on $\partial\Omega$ for some small positive γ. The piecewise affine approximations are then given either by the usual triangulation procedure or using Lemma 3.3 in case $\mathcal{V} = \mathbb{M}^{n\times n}_{sym}$. It seems quite natural to use for the $\mathcal{C}^\infty$-approximation mollifiers with an x-dependent radius , compare also to [**63**].

So, we fix a $\mathcal{C}^\infty$-function $\varepsilon : \mathbb{R}^m \to [0,1]$ such that $\mathrm{dist}(x, \mathbb{R}^m \setminus \Omega) > 2\varepsilon(x) > 0$ if $x \in \Omega$ and $\varepsilon(x) = 0$ else. For $\gamma, \delta \in (0,1)$ we consider the function

$$g(x) = \frac{1-\gamma}{(\delta\varepsilon(x))^n} \int_\Omega \varphi\left(\frac{x-y}{\delta\varepsilon(x)}\right) f(y)\, dy, \text{ for } x \in \Omega,$$

where $\varphi : \mathbb{R}^m \to [0,\infty)$ is the standard $\mathcal{C}^\infty$-mollifier with $\mathrm{spt}(\varphi) \subset B(0,1)$ and $\int \varphi = 1$. Obviously, as ε is a positive function, g is $\mathcal{C}^\infty$ in Ω. On the other hand, we have the representation

$$g(x) = (1-\gamma)\int_{B(0,1)} \varphi(y) f(x - \delta\varepsilon(x)y)\,dy,$$

and hence

$$\begin{aligned}
\nabla g(x) &= (1-\gamma)\int_{B(0,1)} \varphi(y)\nabla_x f(x-\delta\varepsilon(x)y)\,dy \\
&= (1-\gamma)(\int_B \varphi(y)(\nabla f)(x-\delta\varepsilon(x)y)\,dy - \int_B \varphi(y)(\nabla f)(x-\delta\varepsilon(x)y)\cdot(y\otimes\delta\nabla\varepsilon(x))\,dy) \\
&\in B((1-\gamma)\mathcal{K}, (1-\gamma)\delta\,\mathrm{lip}(f)\,\mathrm{lip}(\varepsilon)).
\end{aligned}$$

As f is a Lipschitz map, it is also clear that $g(x)-(1-\gamma)f(x) \to 0$ when $\varepsilon(x) \to 0$, which shows that $g \equiv (1-\gamma)f$ on $\partial\Omega$. Moreover, we see that choosing γ such that $\|(1-\gamma)f - f\|_\infty$ is sufficiently small and sending δ to zero, we obtain g arbitrarily close to $(1-\gamma)f$ such that $\nabla g \in (1-\gamma/2)\mathcal{K} \subset \mathrm{int}_{\mathcal{V}}(\mathcal{K})$. Hence, g is the required $\mathcal{C}^\infty$-approximation. In case that $f(x) = Ax$ on $\partial\Omega$, we can of course replace A by 0 (and $\mathcal{K}$ by $\mathcal{K} - A$), and hence $g = (1-\gamma)f \equiv 0 \equiv A$ on $\partial\Omega$ as needed. This finishes the proof. □

The previous statements showed that typical functions, independent of the particular choice of the admissible boundary value, have their gradients only near those points where "the gradient stabilizes" in the sense of Definition 3.16. The results in Section 2 indicate that these points are characterized as some kind of extreme points, at least up to conditions on the freedom to modify the gradient inside our "universum" $\mathcal{U}$. It is also known from work by Kewei Zhang, see [**105**],[**106**] and the remarks in Section 4 of [**71**], that for convex $\mathcal{U}$ the closure of the set of all gradient extreme points is the smallest closed set $\mathcal{K}$ which allows solutions of $\nabla f \in \mathcal{K}$ for each affine boundary data from $\mathcal{U}$. It turns out that the typical function in $\overline{\mathscr{P}}^\infty$ illustrates this fact in a very convincing way. Indeed, as shown by the next result, the gradient of such a single function already uses all "parts" of the set of such extreme points in an essential way.

PROPOSITION 3.24. *Let $\mathcal{U}$ a bounded domain in $\mathbb{M}^{n\times m}$ or in $\mathbb{M}^{n\times n}_{sym}$. Then for the typical $f \in \mathrm{clos}_\infty(\mathscr{P}(\Omega,\bar{\mathcal{U}}))$, any $z \in \mathrm{extr}_{\mathrm{qc}}(\bar{\mathcal{U}})$ and any $\varepsilon > 0$ does the set $\{x \in \Omega\ ;\ \nabla f(x) \in B(z,\varepsilon)\}$ have positive measure in any subdomain of Ω.*

Given an admissible boundary datum g, the same holds for the typical f in the subspace $\mathrm{clos}_\infty(\mathscr{P}(\Omega,\bar{\mathcal{U}},g))$.

Proof. Using the fact that both $\bar{\mathcal{U}}$ and Ω have a countable basis for their topologies, the theorem follows easily once we established that for all nonvoid open $\mathcal{V} \subset \Omega$ and any open $\mathcal{W}$ with $\mathcal{W} \cap \mathrm{extr}_{\mathrm{qc}}(\bar{G}) \neq \emptyset$ is the set

$$M_{\mathcal{V},\mathcal{W}} = \mathrm{int}(\{f \in \overline{\mathscr{P}}^\infty\ ;\ \mathcal{V} \cap \{x\ ;\ \nabla f(x) \in \mathcal{W}\}) > 0\})$$

dense in $\overline{\mathscr{P}}^\infty$.

For this purpose, we first consider arbitrary $A, B \in \mathcal{U}$ and a nonvoid open $\tilde{\mathcal{V}} \subset \mathbb{R}^m$. Given $u : \tilde{\mathcal{V}} \to \mathbb{R}^n$ affine with gradient A and a positive ε we claim there is a $\tilde{u} : \tilde{U} \to \mathbb{R}^n$ piecewise affine such that $\nabla\tilde{u} \in \mathcal{U}$ a.e., $\|u - \tilde{u}\| < \varepsilon$ and $|\{x \in \tilde{U}\ ;\ \nabla\tilde{u}(x) = B\}| > 0$. Indeed, as $\mathcal{U}$ is pathwise connected, there is a $\delta > 0$ and a chain $\{A_i\}_{i=1}^k$ of matrices in $\mathcal{U}$ such that $A_0 = A$, $A_k = B$, $\mathrm{rank}(A_{i+1} - A_i) = 1$, $|A_i - A_{i+1}| < \delta$ and $B(A_i, 2\delta) \subset \mathcal{U}$. Hence, repeatedly using the lamination technique (see Lemma 3.2 or Proposition 3.4) we obtain a sequence $\{u_i\}_{i=1}^k$ of piecewise affine functions on $\tilde{\mathcal{V}}$ with $u_0 = u$, $\|u_{i+1} - u_i\|_\infty < \varepsilon/k$ and $|\{x\ ;\ \nabla u_i(x) = A_i\}| \geq |\tilde{U}|/3^i$. Obviously, $\tilde{u} = u_k$ does the job.

Combining this fact with Corollary 3.11, the proof of the proposition is easily completed. As it is easy to see that we modified the functions during the whole proof only in the interior of the domain, it is clear that the proof works as well in case admissible boundary data are prescribed. □

We conclude the section by comparing our method with the two other approaches already mentioned, the Baire category method according to B.Dacorogna and P.Marcellini (originally presented in [**24**]) and the convex integration technique as developed by S.Müller, V.Šverák and later by M. Sychev. As already explained, each of these methods consists of three building blocks

0) a technique for a local 1-step modification of the gradient (distribution), at present this is done always along rank-one lines, perhaps respecting additional constraints.
1) a strategy how to use point 0) to move the gradient distribution inside the universum $\mathcal{U}(\cup\mathcal{K})$ towards $\mathcal{K}$. This provides better and better approximate solutions.
2) an argument ensuring that these approximate solutions lead to an exact solution.

Item 0) itself, of course, does not particularly belong to any of the methods under consideration. Techniques for perturbing unconstrained gradients were commonly known for more than one century. The main achievement concerning this kind of question is certainly the result in [**68**] which allows one to respect an additional minor constraint in the modification.

If we consider item 1), then the paper [**24**] presents a fairly abstract condition, formulated in terms of a localized quasiconvexification, which is on this abstract level equivalent to the existence of such a desired strategy. Based on this abstract condition, the only examples handled in [**24**] had either a one-step strategy or a strategy which consisted of finitely many steps and uses convexity of the universum $\mathcal{U}$. The first really infinite strategy was presented in [**65**] and utilized Gromov's idea of an in-approximation. This indeed seems to be a very flexible tool in a situation when the set $\mathcal{K}$ itself is* rank-one connected only to boundary points of $\mathcal{U}$ and not directly to inner points of $\mathcal{U}$. In their most recent work [**25**] Dacorogna and Marcellini formulated another condition, the so called *relaxation property*, see Section 6.2.1 there. Translated into our "first order" terminology, $\mathcal{K}$ has the relaxation property with respect to $\mathcal{U}$ if every admissible boundary data (i.e. one coming from a function with gradients in $\mathcal{U}$) allows piecewise affine modifications (arbitrarily small in the L^∞-norm), preserving the boundary value, keeping the gradient in $\mathcal{U}$ and moving it arbitrarily close to $\mathcal{K}$. It is clear that this condition is again equivalent to the existence of a strategy we need for item 1), but still on the same level of abstractness. The same condition appeared also in M.Sychev's work [**93**], in the later paper [**71**] (see Definition 1.1 there) the authors use the phrase "$\mathcal{U}$ can be reduced to $\mathcal{K}$" for it. The weakest assumption developed by Müller and Šverák, which is of course formally stronger than the relaxation property but considerably easier to handle, asks for the existence of a rank-one convex in-approximation, we will discuss this in more detail in Corollary 4.13. In our approach, we can replace the search for a strategy as in point 1) by a search for an understanding in which parts of $\mathcal{U}$ the gradient is stable in the sense of Definition 3.16. On the abstract level it is clear that whenever the relaxation property or the existence of an (rc)-in-approximation for a certain set $\mathcal{K}$ is established, then we know that gradients can be stable only near $\mathcal{K}$, so our assumption is certainly implied by each of the other two. As will be clear from our solution of the five gradient problem, in situations with a more complicated geometry of $\mathcal{U}$ it can be really helpful if one needs to check only that the gradient can be perturbed at least once, instead of describing the complete route to move it towards $\mathcal{K}$. Another nice feature of this approach is, that given the universum $\mathcal{U}$, the best possible set $\mathcal{K}$ is uniquely and in a natural way determined. Indeed, since $\mathcal{U}$ is relatively compact, it is not difficult to check the following. If we consider the family $\mathscr{K}$ of all closed sets $\mathcal{K}$ such that gradients in $\mathcal{U}$ are stable only near $\mathcal{K}$ then $\bigcap \mathscr{K} \in \mathscr{K}$ and hence is certainly the smallest suitable set.

*or at least appears to be

Finally, the main differences between the methods show up, when we consider item 2), i.e. the limiting process. The Baire category method as given in [**25**] needs an additional condition here, the set $\mathcal{K}$ must be a *weakly extreme set* in their terminology, see Definition 6.5 there. To verify this rather abstract condition, the authors usually involve quasiconvexity, see the remark below Definition 6.5 there. This makes it difficult to handle sets $\mathcal{K}$ which are not described in terms of zero sets of quasiconvex functions. In principle it is clear that each set contains some maximal weakly extreme compact subset. However, at least in the case of a nonconvex universum $\mathcal{U}$ the real question is, whether this subset still has the relaxation property.

As we already mentioned, restricting ourself to the class of examples where weakly extreme sets are at hand is not necessary at all if one exploits the full strength of the Baire category and the Baire classification, see e.g. Proposition 3.17. The argument used in the convex integration approach ensures via a "controlled L^∞-convergence" the pointwise convergence of the gradients of the approximate solutions to the gradient of a Lipschitz map, see Lemma 2.1 in [**71**] or Lemma 3.27 in our work. This argument is very flexible and, as shown in Section 4 using a more sophisticated tool from the Baire category theory, is strong enough to give the categorial result without additional assumption involving quasiconvexity as needed in [**24**] and [**25**]. However, it is quite amusing to notice, that the pointwise convergence of the gradient comes from our argument typically for free, as it holds at all continuity points of the gradient map. Another advantage of the categorial result is that once established, it instantly allows to combine several properties - like e.g. the conclusion from Proposition 3.17 and Proposition 3.24.

A last few words about the function $\Phi_{\mathcal{U}}$ as introduced in Definition 3.19. It appears quite naturally if one looks for a way how to evaluate the possible distortions of a gradient under prelaminates in $\mathscr{PL}(\mathcal{U})$ and if one wants moreover some good property, like some concavity, to bound it locally away from zero. Nevertheless it is just a very convenient tool for understanding where "gradients in $\mathcal{U}$" are stable, but not really indispensible for us. On the other hand, it first showed up in the work of Bressan, see [**13**], for much more severe reasons. As a matter of fact, it was used there to evaluate the Hausdorff measure of noncompactness in L^2 of families of solutions of ordinary differential inclusions. There, Bressan used $\Phi_{\mathcal{U}}$ to show that the solution of such a differential inclusion "most likely" stays as much inside the universum as possible. It turned out in paper [**14**], that in the framework of Baire category the same function can be used to prove the opposite result - the typical solution to a (scalar valued) partial differential inclusion has its gradient in the (convex) extreme points of the universe. Applying this result to the vectorial case would certainly require quite some additional work, as the lower semicontinuity property coming with the integrand $\Phi_{\mathcal{U}}$ would have to be established first.

4. The Banach-Mazur game

In this section we present the Banach-Mazur game, which is a tool to establish the topological prevalence of sets under consideration. More precisely, it gives a necessary and sufficient condition for residuality of a subset of a topological space. The necessary condition, which is in fact quite obvious, was together with the concept of such a criterion introduced around 1928 by Stanislaw Mazur. He conjectured also the sufficiency of his condition, which was proved to be true by Stefan Banach. As Banach never published this work, the main reference is [**74**] where the results are also presented in larger generality. A simplified presentation can be found in [**75**] or Chapter 10 of the more recent book [**15**].

The Banach-Mazur game is quite often applied to prove Baire category results when they are not easily accesible using just the plain definitions. Unfortunately, the game, which considerably refines these definitions, is not known to the extent it deserves. Therefore, it happens quite often that results which could be easily derived using the Banach-Mazur game are presented in a cumbersome way giving complicated explicite constructions of the open dense sets occuring in the definition of

residuality. This does of course not increase the attractivity of these topological approaches and in turn discourages even more people to become familiar with the more sophisticated arguments in this field - like e.g. the Banach-Mazur game. Because most of the results presented in the foregoing sections (and definitely those in the following one) were originally obtained using the Banach-Mazur game, I believe it is worth taking the time to give a short introduction to this approach and an example demonstrating its power.

So, let us first revise the content of the Baire category theorem. The theorem, or rather its proof, tells us how to select from a given sequence of open dense sets U_n a sequence of points $x_n \in U_n$ whose limit is in $\bigcap_n U_n$, at least if the underlying space is complete. All we conclude from this is the fact that residual sets in a complete metric space are dense. Apriori, this does not imply that they are indeed large sets. The real power of this conclusion comes from the observation, that the class of residual sets is essentially by definition closed under countable intersections. Hence, these sets must be much larger than just dense as they can not avoid each other.

The Banach-Mazur game goes the other way round, it does primarily not care about the size of residual sets. But it says that a set A is residual if and only iff we are able to find "sufficiently many" sequences with limit in A, or at least not in the complement of A if we go out of the class of complete spaces. In fact, then residual sets might even be empty - nevertheless, the Banach-Mazur game still yields a (at least formally stronger) criterium for residuality. We will not go to such a generality, but can restrict our attention to the so-called class of Baire spaces. These are precisely the spaces where the Baire category theorem holds, i.e. residual sets are dense. The class contains all G_δ-subsets of complete metric spaces (see Chapter 12 of [**75**]) and it is easily checked that a space is Baire provided it contains a dense set which is as a topological subspace Baire. We could also always imagine to live in a complete world since up to measurabilty problems irrelevant for us, any metric Baire space is residual in its completion. Alltogether this shows why spaces of Lipschitz functions with gradient almost everywhere in a compact subset $\mathcal{K}$ of $\mathbb{M}^{n\times m}$ do always fit very well in the categorial framework. Even if such a space for general $\mathcal{K}$ certainly fails to be complete, our results show that it is residual in its L^∞-closure and hence Baire in itself.

The last issue we want to discuss in this rather informal way is the meaning of the phrase "sufficiently many" selected sequences. What we really have in mind is the following

> Given the already selected first n-members $x_1, \dots x_n$ of our sequence, we can choose a neighbourhood $V_n = V_n(x_1, \dots, x_n)$ of x_n such that we are able to find our x_{n+1} in any arbitrarily small but nonvoid and open subset of V_n.

The proof of the Baire category theorem is able to do this choice if the decisive condition on the x_{n+1} is essentially to stay inside the open dense set U_{n+1} which is independent of the already choosen $x_1, \dots, x_n$. There are of course other situations when a suitable choice of the x_{n+1} is still possible but has to be done much more carefully. This happens in particular if the decisive property talks about the limit object only and has to be ensured by an along each individual sequence gradually adjusted controll of the choice of the next element, see for example the proof of Theorem 3.28 below.

Now, we can give the formal statement, where we will for the sake of generality replace the little neighbourhoods of our x_n's by more general open sets and forget about their centres x_n, but after all our comments it should be easy to specify the problem back to the situations we have in mind for our applications.

Definition 3.25. *Suppose we are given a topological space (X, τ), two (abstract) players $\mathcal{A}, \mathcal{B}$ and a subset B of X. A* play *of the Banach-Mazur game is nothing but an infinite decreasing sequence $U_1 \supset U_2 \dots$ of nonvoid open sets in X, where $\mathcal{A}$ started by choosing U_1, $\mathcal{B}$ replied by choosing a subset U_2 of U_1, then $\mathcal{A}$ picks $U_3 \subset U_2$ and so on. The outcome of the play is now defined using the set B. We say that player $\mathcal{B}$* won the play according to $(\mathcal{R})$-rules *if $\cap U_n \subset B$,*

else $\mathcal{A}$ is the $(\mathcal{R})$-winner. Another way to evaluate the outcome, is to say that player $\mathcal{B}$ is the $(\mathcal{S})$-winner if $\bigcap U_n \cap B \neq \emptyset$, else $\mathcal{A}$ wins.

Finally, we say that player $\mathcal{B}$ has a winning strategy *for the $(*)$-rules, $* = \mathcal{R}$ or $* = \mathcal{S}$, if there exists a sequence of mappings*

$$\Phi_k : \{U \in (\tau \setminus \{\emptyset\})^{2k-1} \; ; \; U_i \supset U_{i+1} \text{ for } 1 \leq i \leq 2k-2\} \to \tau \setminus \{\emptyset\}, \quad k = 1, 2, \ldots$$

such that always $\Phi_k(U) \subset U_{2k-1}$ and that player $\mathcal{B}$ is the $(*)$-winner of each play he played according to this $\{\Phi_k\}_k$-strategy. i.e for every play $U_1 \supset U_2 \supset \ldots$ satisfying $U_{2k} = \Phi_k((U_1, \ldots, U_{2k-1}))$ for all $k \geq 1$.

THEOREM 3.26. *Let (X, τ) be any topological space and $B \subset X$ arbitrary.*

a) *Player $\mathcal{B}$ has a winning strategy for the $(\mathcal{R})$-rules if and only if B is residual in X.*
b) *If player $\mathcal{B}$ has a winning strategy for the $(\mathcal{S})$-rules then $B \cap U$ is of second category, so in particular not empty, whenever $U \neq \emptyset$ is open in X.*

For a proof we refer to [**74**]. It should be noted, that statement b) is a little bit different in spirit. First of all, it really talks about the set B being non-neglectible, secondly it is a very easy sufficient condition and, perhaps most interesting, without additional "measurability assumptions" on B its converse is not true. We will not discuss these quite interesting counterexamples, essentially constructed by transfinite induction and therefore not really natural in our context. Indeed, because we just want to show that certain sets are large, we are interested in sufficient conditions only.

After these explanations of the concept of the Banach-Mazur game, it is certainly time to see it at work. To demonstrate its large flexibilty we will show how easily it allows us to translate the main result of [**71**] (see Theorem 1.3 there) into the categorial framework. Even if we consider in the sequel Ω bounded, it is clear that the arguments work for unbounded domains as well. Indeed, all we need to do is to replace Lebesgue measure by a finite one with an everywhere positive density.

But first, we give our version of the "controlled L^∞-convergence implies L^1-convergence of the gradients"-principle, a similar one can be found in [**71**]. This principle is also a crucial idea in many recent refinements of classical differentiability results for Lipschitz functions, see [**77**] or [**56**]. The conclusion of our lemma might look a bit surprising, but if we start from a sequence $\{f_i\}_i$ with non-converging gradients, then the intersection considered in the Lemma will become empty.

LEMMA 3.27. *Let $\Omega \subset \mathbb{R}^m$ be bounded and open. For a Lipschitz function $f : \Omega \to \mathbb{R}^n$ and $k \in \mathbb{N}$ let $r(f, k)$ be the supremum of all $r > 0$ such that there is a compact $K \subset \Omega$ with $|\Omega \setminus K| < 2^{-k}$ and*

$$|f(x+y) - f(x) - \langle \nabla f(x), y\rangle| \leq \frac{1}{k}|y| \text{ if } x \in K \text{ and } |y| \leq k \cdot r.$$

Obviously, $r(f, k) > 0$ by Rademacher's theorem. Consider a sequence $f_k : \Omega \to \mathbb{R}^n$ of uniformly Lipschitz mappings and suppose $0 < r_k < \min(1/k^2, r(f_k, k))$ for all k. If $f \in \bigcap_k B_\infty(f_k, r_k)$ then $\lim_{k\to\infty} \nabla f_k(x) = \nabla f(x)$ for almost every $x \in \Omega$.

Proof. As $f_k \rightrightarrows f$, this function is again Lipschitz on Ω. Therefore, it suffies to prove $\nabla f_k(x) \to \nabla f(x)$ provided $\nabla f(x)$ exists and $x \in \bigcup_{l=1}^\infty \bigcap_{k=l}^\infty K_k$, where K_k was choosen as in the definition of $r(f_k, k)$. But indeed, for any such $x \in \bigcap_{k=l}^\infty K_k$ and $k \geq l$ we can estimate

$$\begin{aligned} kr_k\|\nabla f_k(x) - \nabla f(x)\| &= \sup_{|y|=kr_k} |\langle \nabla f_k(x) - \nabla f(x), y\rangle| \\ &\leq \sup_{|y|=kr_k} \big(|f(x+y) - f(x) - \langle \nabla f(x), y\rangle| + 2r_k + |f_k(x+y) - f_k(x) - \langle \nabla f_k(x), y\rangle|\big) \\ &\leq \mathcal{O}(kr_k) + 3r_k \end{aligned}$$

and hence $\|\nabla f_k(x) - \nabla f(x)\| \leq (3/k) + (\mathcal{O}(kr_k)/kr_k) \to 0$. □

After this preparation, we transfer the result from [**71**] itself into the realm of Baire's category. For this purpose, we could to a large extent recycle its proof as given in [**71**], but to keep a certain line in our presentation we make some more (cosmetical) changes to that presentation choosen. The only substantial change is when we invoke the Banach-Mazur game, but the argument is rather straightforward.

THEOREM 3.28. *Let $\mathcal{U}, \mathcal{K}$ be multivalued maps from $\Omega \times \mathbb{R}^m$ to the subsets of $\mathbb{R}^{n\times m}$ with equibounded values. Let also*

$$d : \{(x, u, v) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{n\times m} \ ; \ v \in (\mathcal{U}(x, u) \cup \mathcal{K}(x, u))\} \to [0, M]$$

be a function such that

- *$d(\cdot, \cdot, v)$ is upper semicontinuous on its domain for each fixed $v \in \mathbb{R}^{n\times m}$.*
- *Whenever $(x, u) \in \Omega \times \mathbb{R}^n$ then $\mathcal{K}(x, u) = \{v \in (\mathcal{U}(x, u) \cup \mathcal{K}(x, u)) \ ; \ d(x, u, v) = 0\}$ and $\mathcal{K}(x, u)$ is compact.*
- *Given any $(x, u) \in \Omega \times \mathbb{R}^n$ we have $d(x, u, v_k) \to 0$ if and only if $\operatorname{dist}(v_k, \mathcal{K}(x, u)) \to 0$ (and d is defined in the (x, u, v_k)'s).*

Assume that for each $(x_0, u_0) \in \Omega \times \mathbb{R}^m$, each $v_0 \in \mathcal{U}(x_0, u_0)$ and each $\varepsilon > 0$ there exists a piecewise affine function $\Phi \in \operatorname{Lip}_0(\bar{B}(0, 1; \mathbb{R}^m), \mathbb{R}^n)$ with

- *$\int_{B(0,1)} d(x_0, u_0, v_0 + \nabla\Phi(y))\, dy < \varepsilon$, and*
- *if (x, u) is sufficiently close to (x_0, u_0) then $v_0 + \nabla\Phi(y) \in \mathcal{U}(x, u)$ for a.e. $y \in B(0, 1)$.*

Given a piecewise affine $g \in \operatorname{Lip}(\Omega, \mathbb{R}^n)$ satisfying $\nabla f(x) \in \mathcal{U}(x, f(x))$ almost everywhere in Ω, we consider the space $\mathscr{P}$ of all piecewise affine $f \in \operatorname{Lip}(\Omega, \mathbb{R}^n)$ such that $\nabla f(x) \in (\mathcal{K}(x, f(x)) \cup \mathcal{U}(x, f(x)))$ a.e. in Ω and $(f - g)_{|\partial\Omega} \equiv 0$.

Then the typical $f \in \overline{\mathscr{P}}^{\infty}$ satisfies $\nabla f(x) \in \mathcal{K}(x, f(x))$ almost everywhere.

Proof. First we verify that under these assumptions for each $\varepsilon > 0$ the set $\mathscr{P}_\varepsilon$ of all $f \in \mathscr{P}$ with $\int_\Omega d(x, f(x), \nabla f(x))\, dx < \varepsilon|\Omega|$, is dense in $\mathscr{P}$. For this purpose, we prefer to follow more the lines of the proof of Theorem 3.29 below, even if precisely the fact we are going to show is proved in [**71**].

So, given $f \in \mathscr{P}$ and $\eta > 0$ let G_i be open disjoint sets such that $|\Omega \setminus \bigcup_i G_i| = 0$ and that $f_{|G_i}$ is affine with gradient A_i. We choose a compact set $C \subset \bigcup_i G$ with $|\Omega \setminus C| < \varepsilon|\Omega|/2M$ and such that $A_i \in \mathcal{U}(x, f(x))$ for all $x \in G_i \cap C$. Hence, our assumptions then allow us to find for each $x \in C \cap G_i$ an $r_x \in (0, \eta)$, $\mathcal{M}_x \subset \mathcal{U}(x, f(x))$ and a piecewise affine $\varphi_x \in \operatorname{Lip}_0(\bar{B}(0, 1), \mathbb{R}^n)$ such that

- $\nabla\varphi_x(y) + A_i \in \mathcal{M}_x$ for almost every $y \in B(0, 1)$,
- $\int_{B(0,1)} d(x, f(x), A_i + \nabla\varphi_x(y))\, dy < \varepsilon|B(0, 1)|/3$,
- $B(x, 2r_x) \subset G_i$ and $\mathcal{M}_x \subset \mathcal{U}(y, u)$ if $|x - y| + |u - f(y)| < 2r_x$.

Now we apply the idea from the proof in [**71**] how to ensure that for suitable $\tilde{f}$ sufficiently close to f the integral $\int d(y, \tilde{f}(y), \nabla\tilde{f}(y))$ is not much larger then the integral with frozen $(x, f(x))$ which was estimated above. As the φ_x are piecewise affine, we find for each $x \in G_i$ a finite set $\mathcal{N}_x \subset \mathcal{M}_x$ with $|\{y \in B(0, 1) \ ; \ A_i + \nabla\varphi_x(y) \notin \mathcal{N}_x\}| < \varepsilon|B(0, 1)|/12M$. Then, we can also fix a new sizebound $r'_x \in (0, r_x)$ such that $d(y, u, A) < d(x, f(x), A) + \varepsilon/12$ if $A \in \mathcal{N}_x$ and (y, u, A) is in the domain of d and satisfies $|x - y| + |u - f(y)| < 2r'_x$.

Of course, due to compactness we find $\{x_j\}_{j=1}^N \subset C$ and disjoint open sets Q_j with $Q_j \subset B(x_j, r'_{x_j})$ and $|C \setminus \bigcup_j Q_j| = 0$. We choose any $r'_0 \in (0, \min_j r'_{x_j})$ and based on the exhaustion argument in Construction 3.1 we find for each $j \le N$ a piecewise affine (multiple) copy $\varphi_j \in \operatorname{Lip}_0(Q_j, B(0, r'_0; \mathbb{R}^n))$ of φ_{x_j}. Hence, we see that $\nabla(\varphi_j + f)(y) \in \mathcal{M}_{x_j}$ a.e. in Q_j and so, extending φ_j by setting it zero outside Q_j, we obtain that $f + \sum_j \varphi_j(y) \in \mathscr{P} \cap B_\infty(f, r'_0)$. Moreover, we can

estimate

$$\begin{aligned}\int_{Q_j} & d(y,(f+\varphi_j)(y),\nabla(f+\varphi_j)(y))\,dy \\ &< \int_{Q_j}(d(x_j,(f+\varphi_j)(x_j),\nabla(f+\varphi_j)(y))+\frac{\varepsilon}{12})\,dy + M|\{y\in Q_j\,;\,\nabla(f+\varphi_j)(y)\notin N_{x_j}\}| \\ &\le \frac{\varepsilon}{3}|Q_j|+\frac{\varepsilon}{12}|Q_j|+M\frac{\varepsilon}{12M}|Q_j|\le\frac{\varepsilon}{2}|Q_j|.\end{aligned}$$

Summing up over all Q_j we get

$$\int_C d(y,f(y)+\sum_{j=1}^N\varphi_j(y),\nabla(f+\sum_{j=1}^N\varphi_j)(y))\,dy<\frac{\varepsilon}{2}|\Omega|,$$

and hence

$$\int_\Omega d(y,f(y)+\sum_{j=1}^N\varphi_j(y),\nabla(f+\sum_{j=1}^N\varphi_j)(y))\,dy<\frac{\varepsilon}{2}|\Omega|+M|\Omega\setminus C|<\varepsilon|\Omega|.$$

This established the required density result for $\mathscr{P}_\varepsilon$. After this preparation, the proof itself is straightforward. Suppose we are given by Player $\mathcal{A}$ a ball $B_\infty(f_{2k-1},r_{2k-1})$ in $\overline{\mathscr{P}}^\infty$, where $k\ge 1$. We do the following. Since this ball intersects also the set $\mathscr{P}$ itself, we use what was just shown to pick $f_{2k}\in B_\infty(f_{2k-1},r_{2k-1}/2)\cap\mathscr{P}$ with $\int_\Omega d(x,f_{2k}(x),\nabla f_{2k}(x))\,dx<2^{-k}$. According to Lemma 3.27 we select $R_{2k}=r(f_{2k},k)>0$. Our new radius will be a positive number $r_{2k}\in(0,R_{2k})\cap(0,r_{2k-1}/3)$ which is moreover sufficiently small such that the following holds. The set M_k of all $x\in\Omega$ which fulfill $d(x,u,\nabla f_{2k}(x))\le d(x,f_{2k}(x),\nabla f_{2k}(x))+2^{-k}$ whenever $u\in B(f_{2k}(x),r_{2k})$, satisfies $|\Omega\setminus M_k|<2^{-k}$. Indeed, since $x\to\nabla f_k(x)$ is piecewise constant, we see that $(x,u)\to d(x,u,\nabla f_k(x))$ is "piecewise upper semicontinuous" without any further measurablity assumptions on d. This of course implies the existence of the positive r_{2k}. Now, we return $B_\infty(f_{2k},r_{2k})$ which is obviously included in the ball we were given by Player A. Since R_{2k} and hence also r_{2k} tend to zero, it remains to show that $f=\lim f_k$ satisfies $\nabla f(x)\in\mathcal{K}(x,f(x))$ almost everywhere. Indeed, to Lemma 3.27 implies that $\lim_{k\to\infty}\nabla f_{2k}(x)=\nabla f(x)$ almost everywhere. Moreover, since $|\Omega\setminus\cup_{l=k}^\infty M_l|<2^{-k+1}$ and because $d(x,f(x),\nabla f_{2k}(x))\le d(x,f_{2k}(x),\nabla f_{2k}(x))+2^{-k}$ on M_k, we see that $\operatorname{dist}(\nabla f_{2k}(x),\mathcal{K}(x,f(x)))\to 0$ for almost every $x\in\Omega$. Therefore, we are done. □

5. The inhomogeneous case and minimal differential inclusions

We now show how the results for homogeneous partial differential inclusions $\nabla f(x\in\mathcal{K}$ can be extended to the inhomogeneous case $\nabla f(x)\in\mathcal{K}(x,f(x))$. Naturally, this is technically more demanding, but it also shows the strength of the approach since the extension can be done essentially without additional assumptions. Of course, our Theorem 3.29 below is, at least if we consider possible applications, very close to Theorem 1.3. in [**71**]. A completely new result is obtained in Theorem 3.34, in particular if it is combined with Corollary 3.36. It is a full generalization of Theorem 3.22 to the nonhomogeneous case.

THEOREM 3.29. *Let $\mathcal{U},\mathcal{K}$ be multivalued functions from $\Omega\times\mathbb{R}^n$ to the subsets of $\mathbb{R}^{n\times m}$ such that*

- *$\mathcal{U}$ is strongly lower semicontinuous with values contained in a fixed bounded set $\mathcal{U}_0$, and*
- *$\mathcal{K}$ has compact values.*

Assume that for each $(x,u)\in\Omega\times\mathbb{R}^n$ gradients in $\mathcal{U}(x,u)$ are stable with respect to compact pertubation only near $\mathcal{K}(x,u)$. Consider, similar to Definition 3.14, the spaces

$$\mathscr{P}=\mathscr{P}(\Omega,\mathcal{U})=\{f\in\operatorname{Lip}(\Omega,\mathbb{R}^n);f\ \textit{piecewise affine}\ ,\nabla f(x)\in\mathcal{U}(x,f(x))\ \textit{a.e.}\ \},$$

or

$$\mathscr{P} = \mathscr{P}(\Omega, \mathcal{U}, g) = \{f \in \mathrm{Lip}(\Omega, \mathbb{R}^n); f \textit{ piecewise affine }, \nabla f(x) \in \mathcal{U}(x, f(x)) \textit{ a.e., and } f_{|\partial\Omega} \equiv g\},$$

and the functional

$$\mathcal{F}(f) = \int_\Omega \mathrm{dist}(\mathcal{K}(x, f(x)), \nabla f(x))\, dx, \quad f \in \mathrm{Lip}(\Omega, \mathbb{R}^n).$$

Suppose that

$$\textit{for typical } f \in \overline{\mathscr{P}}^\infty : \textit{ either } \mathcal{F}(f) = 0 \textit{ or } \inf \mathcal{F}(B(f, \eta) \cap \overline{\mathscr{P}}^\infty) > 0 \textit{ for some } \eta > 0. \tag{3.3}$$

Then for typical $f \in \overline{\mathscr{P}}^\infty$ we have $\nabla f(x) \in K(x, f(x))$ almost everywhere.

Note that the assumption (3.3) is fulfilled for instance if either

- *$\mathcal{K}(x, \cdot)$ is lower semicontinuous for (almost) every $x \in \Omega$, or*
- *$\mathcal{K}(x, \cdot)$ is upper semicontinuous for (almost) every $x \in \Omega$.*

Proof. Due to Lemma 3.15 the main conclusion follows once we have shown that $\inf \mathcal{F}(B(f, \eta) \cap \overline{\mathscr{P}}^\infty) = 0$ for each $\eta > 0$ provided f is a point of continuity of ∇ restricted to $\overline{\mathscr{P}}^\infty$. The proof closely follows the one of Proposition 3.17.

So let f be a counterexample to this claim. We proceed as in the proof of Proposition 3.17 and using that $\mathcal{F}$ is bounded away from zero be obtain the following. There is a compact set $C \subset \Omega$ and ε, η positive and a function $f_{k_0} \in B_\infty(f, \eta/2) \cap \mathscr{P}$ such that

- $g \in B_\infty(f, \eta) \cap \overline{\mathscr{P}}^\infty$ implies $\|\nabla f - \nabla g\|_{L^1} < \varepsilon^2/4$
- $|C| > \varepsilon$, $\nabla f_{k_0|C}$ is continuous,
- $\mathrm{dist}(\nabla f_{k_0}(x), \mathcal{K}(x, f_{k_0}(x)) > \varepsilon$ for all $x \in C$.

As f_{k_0} is piecewise affine, there are disjoint open subsets $\{G_i\}_1^\infty$ of Ω such that $|\Omega \setminus \bigcup G_i| = 0$ and $f_{k_0|G_i}$ is affine with gradient $A_i \in B(x, f_{k_0}(x))$ whenever $x \in G_i$ ((after possible removal of closed null sets G_i). Slightly decreasing the measure of C we can suppose that $C \subset \bigcup_i G_i$.

From this and Proposition 3.17 we easily infer for each $x \in C$ the existence of $r_x \in (0, \eta/2)$, $M_x \subset B(x, f_{k_0}(x))$ compact and a piecewise affine $\varphi_x \in \mathrm{Lip}(\mathbb{R}^m, \mathbb{R}^n)$ with bounded support such that for $x \in G_i \cap C$ the following holds

- $\nabla \varphi_x(y) + A_i \in M_x$ for almost every y,
- $\int |\nabla \varphi_x| > \varepsilon |\operatorname{spt}(\varphi_x)|/2$,
- $B(x, 2r_x) \subset G_i$ and $M_x \subset B(y, u)$ if $|x - y| + |u - f_{k_0}(y)| < 2r_x$.

Of course, we then find $\{x_j\}_{j=1}^N \subset C$ and disjoint open sets Q_j with $x_j \in Q_j \subset B(x_j, r_{x_j})$ and $|C \setminus \bigcup_j Q_j| = 0$. Based on the exhaustion argument in Construction 3.1 we pick piecewise affine $\varphi_j \in \mathrm{Lip}_0(Q_j, \mathbb{R}^n)$ with $\|\varphi_j\|_\infty < r_{x_j}$, $\nabla(\varphi_j + f_{k_0}) \in M_{x_j}$ a.e. and $\int |\nabla \varphi_j| > \varepsilon |Q_j|/2$. Since both f_{k_0} and $f_{k_0} + \sum_1^N \varphi_j$ belong to $B_\infty(f, \eta) \cap \mathscr{P}$ and $\|\nabla(\sum_j \varphi_j)\|_{L^1} > \varepsilon^2/2$ we derive the desired contradiction.

It remains to verify that both upper and lower semicontinuity of $\mathcal{K}$ imply (3.3). The case of upper semicontinuous $\mathcal{K}$ is clear, as then $\mathrm{dist}(v, \mathcal{K}(x, u)) \le \liminf_k \mathrm{dist}(v_k, \mathcal{K}(x, u_k))$ whenever $u_k \to u$, $v_k \to v$ and hence Fatou's Lemma implies $\liminf \mathcal{F}(f_k) \ge \mathcal{F}(f)$ if $f, f_k \in \overline{\mathscr{P}}^\infty$, $f_k \rightrightarrows f$ and f is a point of continuity of ∇. So, we make the (more natural, see e.g. Lemma 3.9) assumption that $\mathcal{K}(x, \cdot)$ is lower semicontinuous for almost every $x \in \Omega$. As in the proof of Lemma 3.15 we introduce the discrete difference quotients, i.e. for $k \ge 1$ and $f \in X$ we define, denoting by e_j the j-th canonical unit vector,

$$(\Delta_k(f))(x)_{i,j} = \begin{cases} k(f_i(x + \frac{e_j}{k}) - f_i(x)) & \text{if } \mathrm{dist}(x, \mathbb{R}^m \setminus \Omega) > \frac{1}{k} \\ 0 & \text{else.} \end{cases}$$

and set

$$\mathcal{F}_k(f) = \int_{\{x\ ;\ B(x,1/k)\subset\Omega\}} \operatorname{dist}(\mathcal{K}(x,f(x)),\Delta_k(f)(x))\,dx.$$

Note that $f_l \rightrightarrows f$ and $\mathcal{K}(x,\cdot)$ lower semicontinuous implies that $\operatorname{dist}(\mathcal{K}(x,f(x)),\Delta_k(f)(x)) \geq \limsup_l \operatorname{dist}(\mathcal{K}(x,f_l(x)),\Delta_k(f_l)(x))$. Therefore, we conclude from Fatou's Lemma that $f_l \rightrightarrows f$ implies $\mathcal{F}_k(f) \geq \limsup_l \mathcal{F}_k(f_l)$ and consequently, for any $\varepsilon > 0$ and $k \in \mathbb{N}$ is the set $\{f\ ;\ \mathcal{F}_k(f) \geq \varepsilon\}$ closed in the supremum metric. Since $\mathcal{F}(f) = \lim_k \mathcal{F}_k(f)$ for all Lipschitz f, it is easy to see that $\{f\ ;\ \mathcal{F}(f) > 0\} = \bigcup_{l\geq 1} M_l$ where each $M_l = \bigcap_{k\geq l}\{f, \mathcal{F}_k(f) \geq 1/l\}$ is closed. So, also in this case, the condition (3.3) is valid since $M_l \setminus \operatorname{int}(M_l)$ is always nowhere dense and because $f \in \operatorname{int}(M_l)$ implies that $\mathcal{F} \geq 1/l$ in a whole neighbourhood of f. □

Remark 3.30. The last part of the foregoing proof yields as a byproduct (which was indeed the reason to go once more back to the difference quotients) the fact, mentioned to me by David Preiss for K fixed, that the set $\{\mathcal{F} = 0\}$ of exact solutions of our differential inclusion is in case of lsc K a G_δ-set. This explains why results stating that solutions are L^∞-dense in the space of admissible Lipschitz functions necessarily lead to the conclusion that these solutions are prevalent in the topological sense.

As a first application, we want to show that Theorem 3.29 allows us to find solutions to the inhomogeneous 2-well problem.

Corollary 3.31. *We use the notation from Paragraph 1.11 in Chapter 1. As in Lemma 1.12, the open set $\mathcal{W}$ consists of all $(A,B) \in (\mathbb{M}^{2\times 2})^2$ with $\det(A), \det(B) > 0$ such that the singular values fulfill $\lambda_1(BA^{-1}) < 1 < \lambda_2(BA^{-1})$. Let $\Omega \subset \mathbb{R}^2$ be bounded and open, let $(A,B) : \Omega\times\mathbb{R}^2 \to \mathcal{W}$ be continuous and such that either*

$$\det(A(x,u)) \neq \det(B(x,u)) \text{ for all } (x,u) \in \Omega\times\mathbb{R}^2$$

or

$$\det(A(x,u)) = \det(B(x,u)) = d_0 \text{ for all } (x,u) \in \Omega\times\mathbb{R}^2.$$

If we set

$$\mathcal{K}(x,u) = \mathrm{SO}(2)A(x,u) \cup \mathrm{SO}(2)B(x,u)$$

and $\mathcal{U}(x,u) = \operatorname{int}_{rel}(\mathcal{K}(x,u)^{pc})$, then the typical f in both of the spaces $\operatorname{clos}_\infty(\mathscr{P}(\Omega,\mathcal{U}))$ or $\operatorname{clos}_\infty(\mathscr{P}(\Omega,\mathcal{U},g))$ satisfies

$$\nabla f(x) \in \mathcal{K}(x,f(x)) \text{ almost everywhere in } \Omega.$$

Proof. Since residuality of a set is implied by its residuality in all bounded subsets of the space under consideration, we can assume that the setvalued maps $\mathcal{K}$ and $\mathcal{U}$ have uniformly bounded values, of course $\mathcal{K}$ being compactly valued. The strong lower semicontinuity of $\mathcal{U}$ follows now from our assumptions and Lemma 1.12, statements ii) and iii)*. Clearly, $(x,u) \to \mathcal{K}(x,u)$ is both upper and lower semicontinuous. In view of Theorem 3.29 it only remains to show that gradients in $\mathcal{U}(x,u)$ are stable with respect to compact pertubations only near $\mathcal{K}(x,u)$. To verify this we may drop the dependence on x and u. We recall the function

$$\Phi_{\mathcal{U}}(X) = \sup\{\int |Y-X|^2\,d\mu(Y)\ ;\ \mu \in \mathscr{PL}(\mathcal{U}) \text{ and } \bar{\mu} = X\}$$

introduced immediately before Lemma 3.20. In view of Lemma 2.10.a) it suffices to show that $\Phi_{\mathcal{U}}(X) \geq \operatorname{dist}(X,\mathcal{K})^2$ for $X \in \mathcal{U}$. For this purpose, we use the in-approximation approach from [**65**]. We consider $\mathcal{K} = \mathcal{K}_{A,B}$ and fix $X \in \mathcal{U}$. Then due to Lemma 1.12.iv) for any given $\varepsilon >$

*On should note that in general strong lower semicontinuity of $\mathcal{U}$ does not follow from continuity in the Hausdorff distance. It is this fact which prevents us from combining the situation with different and with equal determinants in a straightforward way - however, for some more special maps A, B it is possible to do more.

0 there are $A', B' \in \mathcal{U}$ with $\mathrm{dist}(A', \mathrm{SO}(2)A), \mathrm{dist}(B', \mathrm{SO}(2)B) < \varepsilon$. Hence, $\mathcal{K}^{pc}_{A',B} \subset \mathrm{int}_{rel}(\mathcal{U})$ and for ε sufficiently small we have $X \in \mathcal{K}^{pc}_{A',B'} = L_3(\mathcal{K}_{A',B'})$. Using the (at most 8th order) prelaminate generated in $\mathcal{K}^{pc}_{A',B'}$ with barycentre X and supported in $\mathcal{K}_{A',B'}$, we estimate that $\Phi_{\mathcal{U}}(X) \geq \mathrm{dist}(X, \mathcal{K}_{A',B'})^2 \geq (\mathrm{dist}(X, \mathcal{K}) - \varepsilon)^2$. □

Now we will obtain the nonhomogeneous version of Theorem 3.22, at least for the case of convex universa. Nevertheless, even under this simplification the proof is certainly the most involved in the whole chapter. We are motivated to carry it out by at least three reasons. First of all, as noticed in Section 4 of [**71**] the result we will derive would be the best possible which means that in general we can not hope to find solution to "smaller" differential inclusions. Secondly, the proof essentially uses the Baire category method allowing us to reduce the main statement to a weaker one. And last, but not least, even to prove this weaker, approximate statement we need one more tool which was prepared already in Section 2 - we use the stability property for quasiconvex extreme points there derived. In this way, we keep the gradient in each of the inividual open sets approximating $\mathrm{extr}_{\mathrm{gr}}$ from above. This also shows the present limitation of our technique if we try to enlarge the class of universa under consideration. Indeed, we need to be able to move the gradient inside the universum close to the quasiconvex extreme points, so we would need $\mathrm{extr}_{\mathrm{rc}} = \mathrm{extr}_{\mathrm{qc}}$ since we are not able to move the gradient in a controlled manner inside the universum $\mathcal{U}$ away from a rank-one convex extreme point. On the other hand, due to the lack of examples distuingishing quasi- and rank-one convexity, all these questions seem rather abstract.

NOTATION 3.32. As in Theorem 3.22, we assume that $\mathcal{V} = \mathbb{M}^{n\times m}$ or $\mathcal{V} = \mathbb{M}^{n\times n}_{sym}$ and $n = m$. We denote by $\mathscr{C}\mathscr{C}(\mathcal{V})$ the family of all compact and convex subsets of $\mathcal{V}$, as usually equipped with the Hausdorff metric. In the remaining part of this section, Ω will be a bounded open set in $\mathbb{R}^m$, Φ will be a multivalued mapping $\Phi : \Omega \times \mathbb{R}^n \to \mathscr{C}\mathscr{C}(\mathcal{V})$ which is continuous and has equibounded values. Similar to Theorem 3.29 we consider the following spaces of functions

$$\mathscr{P}(\Omega, \Phi(, g)) = \{f : \Omega \to \mathbb{R}^n \ ; \ f \text{ is piecewise affine }, \nabla f(x) \in \mathrm{int}\,\Phi(x, f(x)) \text{ a.e. in } \Omega \ (, f_{|\partial\Omega} \equiv g)\},$$

or shorter $\mathscr{P}$ and its closure $\overline{\mathscr{P}}^{\infty}$.

As already noticed, the map $f \to \nabla f$ is a Baire-one map from $\overline{\mathscr{P}}^{\infty}$ into $L^1(\Omega)$. We denote by $\widetilde{\mathscr{P}}$ the set of points of continuity of this map, therefore $\widetilde{\mathscr{P}}$ is a dense G_δ-subset of $\overline{\mathscr{P}}^{\infty}$. In particular, just assuming upper semicontinuity of a general compactly valued Φ, we already infer that $\nabla f(x) \in \Phi(x, f(x))$ a.e. in Ω provided $f \in \widetilde{\mathscr{P}}$.

However, if we consider the values of Φ that are quasiconvex, we obtain the following stronger result which perhaps deserves to be stated in this precise way, even if the argument is presumably folklore.

LEMMA 3.33. *Consider a general upper semicontinuous compactly valued Φ which has equibounded values. For each $f \in \overline{\mathscr{P}(\Omega, \Phi)}^{\infty}$ and any $x \in \Omega$ we have $\nabla f(x) \in \Phi(x, f(x))$ whenever $\Phi(x, f(x))$ is quasiconvex and f is differentiable at x.*

Proof. On contrary, suppose $A = \nabla f(x) \notin \Phi(x, f(x))$ quasiconvex. Then there exists $F : \mathbb{M}^{n\times m} \to [0, \infty)$ quasiconvex and $\varepsilon > 0$ such that $F(B(\Phi(x, f(x)), \varepsilon)) = \{0\}$ and $F(A) = 1$. Consequently, there is a positive radius r such that $y \in B(x, r)$, $g \in B(f, r)$ and $C \in \Phi(y, g(y))$ then $F(C) = 0$ as well.

We define for k sufficiently large on $B(0, 1)$ the functions $f_k(z) = k(f(x + (z/k)) - f(x))$. Then the f_k are uniformly Lipschitz and $f_k \rightrightarrows A \cdot z$. So $\int_{B(0,1)} F(A) \leq \liminf_k \int_{B(0,1)} F(\nabla f_k(z))\, dz$ and in particular we find $k_0 > 1/r$ with $\int_{B(0,1)} F(\nabla f_{k_0}(z))\, dz > 0$. By definition of $\overline{\mathscr{P}}^{\infty}$ we find also $g_l \in \mathscr{P}$ uniformly converging to f and we consider the sequence of uniformly Lipschitz maps on $B(0, 1)$ given by $h_l(z) = k_0(g_l(x + (z/k_0)) - g_l(x)) \rightrightarrows f_{k_0}$. Hence we find again a member of the

sequence, say h_{l_0} such that $\int_{B(0,1)} F(\nabla h_{l_0}(z)\,dz > 0$. On the other hand, if $g_{l_0} \in B(f,r)$ then for all $z \in B(0,1)$ we have $\nabla h_{l_0}(z) = \nabla g_{l_0}(x + (z/k_0)) \in F^{-1}(0)$. This contradiction finishes the proof.

THEOREM 3.34. *In the setting of Notation 3.32, the typical mapping $f \in \overline{\mathscr{P}}^{\infty}$ satisfies $\nabla f(x) \in \mathrm{extr}_{\mathrm{gr}}(\Phi(x, f(x)))$ for almost every $x \in \Omega$.*

Proof. We fix an $N > 1$ and consider the set U_N which is the $\widetilde{\mathscr{P}}$-interior of the set

$$(3.4)\{f \in \widetilde{\mathscr{P}}\,;\ |\{x\,;\ \nabla f(x) \text{ is the center of a } \frac{1}{N}\text{–rank one segment in } \Phi(x, f(x))\}| < \frac{1}{N}\}.$$

We will prove that $\mathrm{clos}_\infty(U_N) \supset \widetilde{\mathscr{P}}$ and then we are done, since $\bigcap_N U_N$ is residual in $\widetilde{\mathscr{P}}$ and hence also in $\overline{\mathscr{P}}^{\infty}$ and contains of course only good functions.

To prove density we proceed in two steps. Fix $f \in \mathscr{P}$ and a radius $r > 0$. Using Theorem 3.22, i.e. the homogeneus version of the result we wish to prove, and the usual exhaustion argument from Construction 3.1 we first construct a piecewise affine map $g \in B(f,r)$ such that $\nabla g \in \mathrm{int}(\Phi(x, g(x))$ and, for most $x \in \Omega$, the gradient $\nabla g(x)$ is close to a qc-face of $\Phi(x, g(x))$. In particular, for most x the gradient $\nabla g(x)$ is not the centre of an $\frac{1}{N}$-rank one segement. In the second step we exploit the stability results for such faces stated in Corollary 3.11 to show that this property of g persists for all $h \in \widetilde{\mathscr{P}}$ which are L^∞-close to g. More precisely, we prove that $B(g,\eta) \cap \widetilde{\mathscr{P}} \subset U_N \cap B(f, 2r)$ for sufficiently small $\eta > 0$.

To begin with step one, we use that f is piecewise affinein order to choose a decomposition (up to a Lebesgue zero set) $\{G_i\}_{i=1}^\infty$ of Ω into open sets such that $f(x) = A_i x + b_i$ whenever $x \in G_i$. We choose any i, $x_0 \in G_i$ and set $y_0 = f(x_0)$. From Theorem 3.22 we infer the existence of a function $\tilde{f} = \tilde{f}_{x_0} : B_{\mathbb{R}^m}(0,1) \to B_{\mathbb{R}^n}(0,1/2)$ which belongs to $\mathscr{X} = \mathrm{clos}_\infty(\mathscr{P}(B(0,1), \Phi(x_0,y_0) - A_i, 0))$ and fulfills

- $\nabla \tilde{f} \in \mathrm{extr}_{\mathrm{gr}}(\Phi(x_0,y_0) - A_i)$ for almost each $x \in \Omega$.
- $\tilde{f}$ is a point of continuity of $\nabla : \|\cdot\|_{L^\infty} \to \|\cdot\|_{L^1}$ on $\mathscr{X}$.

Using Lusin's theorem we find a compact set $K_{x_0} \subset B(0,1)$ such that $|B(0,1)\backslash K_{x_0}| < |B(0,1)|/10N$, $\nabla \tilde{f}_{|K}$ is continuous and $\tilde{K}_{x_0} = \nabla \tilde{f}(K_{x_0})$ is contained in $\mathrm{extr}_{\mathrm{gr}}(\Phi(x_0,y_0) - A_i)$ which is due to the first remark after Definition 3.8 a *qc*-face of $\Phi(x_0,y_0) - A_i$. An easy compactness argument yields the existence of $\varepsilon = \varepsilon_{x_0} \in (0, 1/10N)$ such that

$$(3.5)\qquad \text{there is no } \frac{1}{N}\text{–rank-one segment in } B(\Phi(x_0,y_0) - A_i, \varepsilon) \text{ centred in } B(\tilde{K}_{x_0}, 3\varepsilon).$$

Using Corollary 3.11, we choose for $(\Phi(x_0,y_0), \tilde{K}_{x_0} + A_i, \varepsilon)$ the correspoding $\delta = \delta_{x_0} < \varepsilon_{x_0}$.

By definition, we find a sequence $u_k \in \mathscr{P}(B(0,1), \Phi(x_0,y_0) - A_i, 0)$ with $u_k \rightrightarrows \tilde{f}$. Since $\tilde{f}$ is a point of continuity of the gradient map we can also assume that $\nabla u_k \to \nabla \tilde{f}$ pointwise almost everywhere and that $\|u_k\|_{L^\infty} < 1$. So we select k_0 such that the set of those $x \in B(0,1)$ for which the condition $|\nabla u_{k_0}(x) - \nabla \tilde{f}(x)| < \delta/2$ fails is in measure smaller than $|B(0,1)|/10N$. Next, we choose $c_{x_0} < 1$ but sufficiently close to 1 such that for $\tilde{u}_{x_0} = c_{x_0} \cdot u_{k_0}$ the set

$$(3.6)\quad S_{x_0} = \{x \in B(0,1)\,;\ |\nabla \tilde{u}_{x_0}(x) - \nabla \tilde{f}(x)| < \delta\} \text{ satisfies } |B(0,1) \setminus S_{x_0}| < \frac{|B(0,1)|}{10N}.$$

We extend $\tilde{u}_{x_0}$ outside $B(0,1)$ with value identically zero. Finally, we find $r_{x_0} > 0$ such that

$$(3.7)\qquad c_{x_0}(\Phi(x_0,y_0) - A_i) + A_i \subset \mathrm{int}(\Phi(x,y)) \text{ if } |x - x_0| + |y - y_0| < r_{x_0}$$

and

$$(3.8)\qquad \Phi(x,y) \subset B(\Phi(x_0,y_0), \delta) \text{ if } |x - x_0| + |y - y_0| < r_{x_0}.$$

After all these preparations, we choose positive radii $\hat{r}_{x_0} < \min\{\mathrm{dist}(x_0, \mathbb{R}^m \setminus G_i), r_{x_0}/(4 + 2|A_i|), r)$ and introduce the family of balls $\mathcal{B}^i_{x_0} = \{B(x_0,r)\,;\ r \in (0, \hat{r}_{x_0})\}$. We obtain similar families

for any other $x \in G_i$ and repeat the procedure for all other i. Applying Vitali's covering theorem (essentially on each G_i separately) we get new families of balls $\mathcal{B}^i = \{B^i_{x_l}\ ;\ l = 1, 2, \dots\}, i = 1, 2, \dots$ such that $B^i_{x_l} \in \mathcal{B}^i_{x_l}$ and $\mathcal{B}^i$ is a disjoint decomposition of almost all of G_i. We define

$$g(x) = \begin{cases} f(x) & \text{if } x \notin \bigcup_{i,l} B^i_{x_l} \\ f(x) + r\tilde{u}_{x_l}\big(\frac{x-x_l}{r'}\big) & \text{if } x \in B(x_l, r') = B^i_{x_l} \end{cases}$$

As the $\tilde{u}_x$ are uniformly Lipschitz, we have $g \in \mathrm{Lip}(\Omega, \mathbb{R}^n)$. Obviously, $g \in B(f, r)$ and g is piecewise affine.

Moreover, for all i, l and almost each $x \in B^i_{x_l} = B(x_l, r')$ we have $\nabla g(x) = \nabla f(x) + \nabla \tilde{u}_{x_l}((x - x_l)/r')$ and therefore $\nabla g(x) \in A_i + c_{x_l}(\Phi(x_l, f(x_l)) - A_i)$. Since $|x - x_l| + |g(x) - f(x_l)| \le \hat{r}_{x_l} + \hat{r}_{x_l} + |f(x) - f(x_l)| \le \hat{r}_{x_l}(2 + |A_i|) < r_{x_l}$ we infer from (3.7) that $\nabla g(x) \in \mathrm{int}(\Phi(x, g(x))$, so $g \in \mathscr{P}(\Omega, \Phi, f)$.

This finished step one and we will be done if we show that for $\eta > 0$ but sufficiently small

$$B(g, \eta) \cap \widetilde{\mathscr{P}} \subset U_N \cap B(f, 2r). \tag{3.9}$$

Only the inclusion in U_N is nontrivial, to prove it we will use the fact that g is affine on a triangulation $\{\hat{G}_j\}_j$ refining the Vitali cover $\{B^i_{x_l}\}_{i,l}$. Since we can localize the measure estimate on countably many pieces with summable measure, it is enough to show that

$$\begin{gathered} \text{for all } B = B^i_{x_l} \text{ there is an } \eta > 0 \text{ such that } h \in B(g, \eta) \cap \widetilde{\mathscr{P}} \text{ implies} \\ |\{x \in B\ ;\ \nabla h(x) \text{ centre of a } \tfrac{1}{N}\text{–rank-one segment in } \Phi(x, h(x))\}| < \tfrac{|B|}{2N}. \end{gathered} \tag{3.10}$$

Given the $B = B(x_l, r')$, we first denote $\tilde{S} = \{x \in B\ ;\ \frac{1}{r'}(x - x_l) \in S_{x_l} \cap K_{x_l}\} \cap \bigcup_j \hat{G}_j$, so $|B \setminus \tilde{S}| < \frac{1}{10N} + \frac{1}{10N}$ due to the choice of K_{x_l} and (3.6). We first consider a "good" $\hat{G}_j \subset B$, i.e. such one that there is an $\tilde{x} \in \hat{G}_j \cap \tilde{S}$. Notice that due to (3.6)

$$(\nabla g)_{|\hat{G}_j} \equiv \nabla g(\tilde{x}) = A_i + \nabla \tilde{u}_{x_l}\big(\frac{x - x_l}{r'}\big) \in B(\tilde{K}_{x_l} + A_i, \delta_{x_k}), \tag{3.11}$$

and that

$$h_{|\hat{G}_j} \text{ is a Lipschitz map satisfying } \nabla h(x) \in \Phi(x, h(x)) \subset B(\Phi(x_l, f(x_l), \delta_{x_l}) \text{ if } x \in \hat{G}_j, \tag{3.12}$$

due to (3.8) since $|x - x_l| + |h(x) - h(x_l)| < \hat{r}_{x_l} + \eta + |f(x) - f(x_l)| \le \hat{r}_{x_l}(2 + |A_i|) < r_{x_l}$, provided $\eta < r_{x_l}$. So we conclude that for $\hat{G}_j \subset B$ intersecting $\tilde{S}$ the Corollary 3.11 ensures the existence of $\tilde{\eta} = \eta_{G_j} < r_{x_l}$ such that $\|g - h\|_{L^\infty(G_j)} < \eta$ and $g \in \mathscr{P}$ implies

$$|\{x \in \hat{G}_j\ ;\ \mathrm{dist}(\nabla h(x), \tilde{K}_{x_l} + A_i) \ge 2\varepsilon_{x_l}\}| \le 2\varepsilon_{x_l}|\hat{G}_j|$$

since by (3.12) also $\Phi(x, h(x)) \subset B(\Phi(x_l, f(x_l)), \varepsilon_{x_l})$. Next, we infer from (3.5) that the set of all $x \in \hat{G}_j$ for which $\nabla h(x)$ is the centre of a $\frac{1}{N}$–rank-one segment contained in $\Phi(x, h(x))$ is at most of measure $2\varepsilon_{x_l}|\hat{G}_j| \le \frac{1}{5N}|\hat{G}_j|$. Those $\hat{G}_j \subset B$ which do not intersect $\tilde{S}$ will be called "bad" $\hat{G}_j$'s. Their union is of measure at most $|B \setminus \tilde{S}|$. Summarizing, we obtain that for all $\gamma > 0$ (telling us how many of the $\hat{G}_j$ have to be taken into consideration) there is a $\eta_\gamma > 0$ such that for all $\eta \in (0, \eta_\gamma)$

the following estimate holds.

$$\begin{aligned}
&|\{x \in B \quad ; \quad \nabla h(x) \text{ is the centre of a } \frac{1}{N}\text{–rank-one segment in } \Phi(x,h(x))\}| \\
&\leq \frac{1}{5N}|\bigcup_{\text{good } \hat{G}_j \subset B} \hat{G}_j| + |\bigcup_{\text{bad } \hat{G}_j \subset B} \hat{G}_j| + \gamma \\
&\leq \frac{|B|}{5N} + |B \setminus \tilde{S}| + \gamma \leq \big(\frac{1}{5N} + \frac{1}{10N} + \frac{1}{10N}\big)|B| + \gamma \\
&\leq \frac{2}{5N}|B| \leq \frac{1}{2N}|B| \quad \text{for } \gamma \text{ small enough ,}
\end{aligned}$$

as was required. □

The Theorem 3.34 just proved is nicely complemented by the following result. It says that a typical function not only uses no points different from rank-one extreme points as values of its gradients, but that it in fact uses a dense subset of these extrem points in every portion of its domain.

PROPOSITION 3.35. *In the setting of Notation 3.32 the following holds. Let $U \subset \Omega$ and let $\mathcal{W}$ be a relatively open subset of $\mathbb{M}^{n\times m}$ or $\mathbb{M}^{n\times n}_{sym}$. Then for the typical $f \in \overline{\mathscr{P}}^{\infty}$ the following implication hold.*

(3.13) *If there is $x \in U$ such that $\mathcal{W} \cap \mathrm{extr}_{\mathrm{qc}}(\Phi(x,f(x)) \neq \emptyset$ then $|\{y \in U \ ; \ \nabla f(y) \in \mathcal{W}\}| > 0$.*

Proof. As $\mathcal{K} \to \mathrm{extr}_{\mathrm{qc}}\mathcal{K}$ is lower semicontinous (see Lemma 3.9 or [**71**]), we see that the set $\mathscr{Y}$ of all $f \in \overline{\mathscr{P}}^{\infty}$ for which the assumption in (3.13) holds is open. We denote by $\mathscr{X}$ the $f \in \overline{\mathscr{P}}^{\infty}$ for which the whole implication (3.13) is true, and we have to show that $\overline{\mathscr{P}}^{\infty} \setminus \mathscr{X}$ is first category in $\overline{\mathscr{P}}^{\infty}$.

For this purpose, fix any $f \in \overline{\mathscr{P}}^{\infty}$ and $r > 0$. If $B(f,r) \cap \mathscr{Y} = \emptyset$, we infer $B(f,r) \subset \mathscr{X}$ and are happy. Else we choose $g \in \mathscr{P} \cap B(f,r) \cap \mathscr{Y}$. We finish the proof by showing the existence of a $h \in \mathscr{P}$ nearby g and a small positive η such that $B(h,\eta) \subset B(f,r)$ and, more importantly, that the conclusion of (3.13) holds for all $\tilde{h} \in \widetilde{\mathscr{P}} \cap B(h,\eta)$. This shows that $\widetilde{\mathscr{P}} \setminus \mathscr{X}$ is nowhere dense in $\widetilde{\mathscr{P}}$ and hence in $\overline{\mathscr{P}}^{\infty}$, therefore $\overline{\mathscr{P}}^{\infty} \setminus \mathscr{X} \subset (\overline{\mathscr{P}}^{\infty} \setminus \widetilde{\mathscr{P}}) \cup (\widetilde{\mathscr{P}} \setminus \mathscr{X})$ is of first category.

The construction of h is very similar to the one in the proof of Theorem 3.34, but starts from the different homogeneous result Proposition 3.24. The nonvoid set $M = \{x \ ; \ \mathcal{W} \cap \mathrm{extr}_{\mathrm{qc}}(\Phi(x,f(x)) \neq \emptyset\}$ is due to the lower semicontinuity result open. Thus we can find $x_0 \in M \cap T_0$, where T_0 is open and $g_{|T_0}$ is affine, with $A = \nabla g(x_0) \in \mathrm{int}(\Phi(x_0,g(x_0)))$. We fix $F \in \mathrm{extr}_{\mathrm{qc}}(\Phi(x_0,g(x_0))) \cap \mathcal{W}$ and $\varepsilon \in (0, 1/4)$ such that $B(F,3\varepsilon) \subset \mathcal{W}$. By Proposition 3.24 we can select $\tilde{f} \in \mathrm{clos}(\mathscr{P}(B(0,1),\Phi(x_0,g(x_0))-A,0))$ with $\|\tilde{f}\| < 1/2$ which is a point of continuity of the gradient map from this space into $L^1(B(0,1))$ and such that $|\{x \in B(0,1) \ ; \ \nabla\tilde{f}(x) \in B(F-A,\tilde{\delta})\}| > 0$ for all $\tilde{\delta} > 0$. We select $\delta > 0$ according to Corollary 3.11 for $(\Phi(x_0,g(x_0)),F,\varepsilon)$. By the choice of $\tilde{f}$, there is a sequence $u_k \rightrightarrows \tilde{f}$, $u_k \in \mathscr{P}(B(0,1),\Phi(x_0,g(x_0)) - A,0)$ and a $c < 1$, $k \in \mathbb{N}$ such that for $u = c \cdot u_k$ both $\|u\|_{L^\infty} < 1$ and $|\{x \ ; \ \nabla u(x) \in B(F-A,\delta)\}| > 0$. Hence, we also find $x_1 \in T_1 \subset B(0,1)$ open such that $u_{|T_1}$ is affine and $\nabla u(x_1) \in B(F-A,\delta)$.

Finally, we select $r_1 > 0$ such that for $r_2 = r_1(2+|A|)$ the conditions $2r_1 < r - \|f-g\|$, $B(x_0,r_1) \subset T_0 \cap U$ and

$$c \cdot (\Phi(x_0,g(x_0)) - A) + A \subset \Phi(x,y) \text{ if } |x-x_0| + |y-g(x_0)| < 2r_2, \text{ and} \tag{3.14}$$

$$\Phi(x,y) \subset B(\Phi(x_0,g(x_0)),\delta) \text{ if } |x-x_0| + |y-g(x_0)| < 2r_2. \tag{3.15}$$

are met. So we can define

$$h(x) = g(x) + r_1 u\big(\frac{x-x_0}{r_1}\big), \text{ where } u \equiv 0 \text{ outside } B(0,1).$$

We see that $\|h-g\|_\infty \le r_1$, and hence $B(h,r_1) \subset B(f,r)$, that h is piecewise affine and $g=h$ on $\partial\Omega$.

Now, for almost each $x \notin B(x_0,r_1)$ one has $\nabla h(x) = \nabla g(x)$ and $h(x) = g(x)$ and hence $\nabla h(x) \in \Phi(x,h(x))$. On the other hand, if $x \in B(x_0,r_1)$ then $|x-x_|| + |h(x)-g(x_0)| < r_1 + r_1 + |g(x_0)-g(x)| \le r_2$. Therefore, for almost all such x the inclusion $\nabla h(x) = A + \nabla u((x-x_0)/r_1) \in A + c\cdot(\Phi(x_0,g(x_0)) - A) \subset \Phi(x,h(x))$ holds due to (3.14). This shows $h \in \mathscr{P}$

Consider $T_2 = r_1 \cdot T_1 + x_0 \subset T_0 \cap U$, and choose $\eta < r_2$ in Corollary 3.11 for $\Phi(x_0,g(x_0)), F, \varepsilon, \delta$ and the domain $\Omega' = T_2$. As $h_{|T_2}$ is affine with gradient in $B(F,\delta)$ and since for each $\tilde{h} \in B(h,\eta) \cap \widetilde{\mathscr{P}}$ according to the Lemma 3.33 for a.e. $x \in T_2$ the inclusion $\nabla\tilde{h}(x) \in \Phi(x,\tilde{h}(x))$ holds, we infer from (3.15) that also $\nabla\tilde{h}(x) \in B(\Phi(x_0,g(x_0),\delta)$ almost everywhere in T_2. Now Corollary 3.11 ensures that $|\{x \in T_2\ ;\ \mathrm{dist}(\nabla\tilde{h}(x),F) \le 2\varepsilon\}| \ge (1-2\varepsilon)|T_2|$. This surely implies that for each such $\tilde{h}$ the conclusion in (3.13) is valid. □

COROLLARY 3.36. *In the setting of Notation 3.32, for the typical $f \in \overline{\mathscr{P}}^\infty$ the following is true. If for some $x \in \Omega$ $A \in \mathbb{M}^{n\times m}$ the inclusion $A \in \mathrm{extr}_{\mathrm{qc}}(\Phi(x,f(x))$ holds, then for each ε positive is the set $\{y \in B(x,\varepsilon)\ ;\ \nabla f(y) \in B(A,\varepsilon)\}$ of positive Lebesgue measure.*

Proof. Since both Ω and $\mathbb{M}^{n\times m}$ have a countable basis of their topologies, this is a simple consequence of the foregoing Proposition 3.35.

CHAPTER 4

Deformations with finitely many gradients

In this chapter we will harvest the results of our previous efforts to understand regularity and existence of solutions to partial differential inclusions. In partiular, we will construct new kinds of deformations with finitely many gradients. For this purpose we recall

DEFINITION 4.1. *A set $\mathcal{A} \subset \mathbb{M}^{n\times m}$ is called* rigid *if any locally lipschitz $f : \Omega \to \mathbb{R}^n$, where Ω is a domain in $\mathbb{R}^m$, which satisfies $\nabla f \in \mathcal{A}$ almost everywhere in Ω is necessarily affine. Moreover, we say that $\mathcal{A}$ is* properly non-rigid *if $\mathcal{A}$ is not rigid and* $\operatorname{rank}(A - B) \neq 1$ *for all $A, B \in \mathcal{A}$.*

As already noted, if a set $\mathcal{A}$ contains a rank-one connection then $\mathcal{A}$ of course fails to be rigid. But as we are not really interested in these classical and easy examples we introduced the notion of proper non-rigidity. The existence of infinite properly non-rigid sets is well known, an example being the range of the gradient for a non-affine holomorphic function. If we require that f has affine boundary data, the task becomes more interesting. However, $\mathcal{C}^1$- and also smooth examples in the optimal dimension $\mathbb{M}^{3\times 2}$ were given in [**5**]. There also the question, due to J.M. Ball and R.D. James, about the existence of finite properly non-rigid sets was raised explicitely. In fact, also countable properly non-rigid sets are found quite easily but for a systematic understanding of the solvability of partial differential inclusions it is clearly a natural task to find properly non-rigid sets of minimal size.

Here we will describe a general construction of finite properly non-rigid sets. We also give a sharp estimate of the minimal cardinality of such sets. Finally, we provide examples of some properly non-rigid finite sets $\mathcal{K}$ which allow particular simple, i.e. piecewise affine, solutions of $\nabla f \in \mathcal{K}$.

In the first section we choose a geometric approach to the study of rank-one convex sets and in this way we derive stability properties for rank-one convex hulls. During the last years, more and more attention in the nonconvex calculus of variations and related areas has focused on such questions concerning the continuity of naturally occuring generalized convex hull operators. This interest seems obvious if one takes into account that the precise or numerical calculation of such hulls is at least very complex (e.g. in the case of rank-one convex or polyconvex hulls) or even quite impossible - like for genuine quasiconvex hulls. It turns out, however that these continuity problems appear not only in connection with the question of well-posedness as such. In fact, continuity also allows to find and solve new kinds of partial differential inclusions and leads in this way to the construction of surprising counterexamples, see [**69**] or the mappings obtained below.

This stability question as a main topic of independent interest was considered for the first time by K.Zhang in [**108**] on different levels of generality, however, focused on quasiconvex hulls. In that paper he gave the following

DEFINITION 4.2. *(see Definition 1.1 in* [**108**]*) We consider the space $\mathscr{C}(\mathbb{M}^{n\times m})$ of all compact subset of the space $\mathbb{M}^{n\times m}$ of matrices equipped with the Hausdorff metric. We say that for a compact $\mathcal{K}_0 \subset \mathbb{M}^{n\times m}$ the quasiconvex hull is strongly stable if the map $\mathcal{K} \to \mathcal{K}^{qc}$ mapping this space $\mathscr{C}(\mathbb{M}^{n\times m})$ into itself is continuous at $\mathcal{K}_0$.*

Since quasiconvex hulls, as well as rank-one convex or polyconvex ones, will turn out to be sublevelsets of coercive generalized convex functions, see e.g. Lemma 4.6 below, it is quite natural to expect that the maps $\mathcal{K} \to \mathcal{K}^{qc}$, $\mathcal{K} \to \mathcal{K}^{rc}$ and $\mathcal{K} \to \mathcal{K}^{pc}$ are upper semicontinuous, e.g. in the sense of Definition 3.7. A proof of this is given in Theorem 3.2 in [**108**]. It seems more striking, however, that for the lamination convex hull operator $\mathcal{K} \to \mathcal{K}^{lc}$ this upper semicontinuity fails eg. in the 3×3-diagonal matrices, as shown in [**48**]. Returning to the paper [**108**], it turned out that questions concerning the continuity of the quasiconvex hull operator are rather ambitious, in particular no nontrivial example of a set with a strongly stable quasiconvex hull was found. One of the central theme of the first section is the fact that rank-one convexity is of a much more geometric nature than quasi- or polyconvexity. We use this to derive stability properties of certain rank-one convex hulls and to give also nontrivial examples of sets with a strongly stable quasiconvex hull, i.e. of sets where the quasiconvex hull operator is lower semicontinuous. For constructions of lipschitz mappings we anyhow need results on the stability of rank-one convex hull, because this notion fits in the framework of Chapter 3 as well as in the framework of convex integration used in [**69**]. To be more precise, the geometric nature of rank-one convexity allows to generalize in-approximation results from the laminational convex to the rank-one convex case. This was done in [**66**],[**68**] and [**69**] and we present the argument at the end of Subsection 1.1. It also allows to obtain stability results for rank-one convex hulls which fail for lamination convex hulls since no Krein-Milman type results do hold for the latter notion. A more detailed discussion can be found in the starting example in Subsection 1.2. This distinguishion in terms of stability properties enables us to do the following. We can reduce the full boundary of an open convex set $\mathcal{U} \subset \mathbb{M}^{n\times m}$ to a finite set $\mathcal{K}$ with a trivial lamination convex hull but such that the rank-one convex hull reduces just a bit and is still almost as large as $\mathcal{U} \subset (\partial\mathcal{U})^{rc}$. Therefore, $\mathcal{K}^{rc}$ has an interior sufficiently large to find solutions of $\nabla f \in \mathcal{K}$ and to conclude that $\mathcal{K}$ is the desired properly non-rigid set. Of course, there are many other questions related to a geometric understanding of rank-one convexity. We try to treat in the first section a few more of them, in particular connectedness and localization of hulls and characterization of extreme points.

In the next two sections we will present results obtained in joint work with Miroslav Chlebík and David Preiss (see [**22**] and [**46**]) which precisely tell how large properly non-rigid sets have to be. It turns out that there is a small but definite difference between rigidity for exact solutions of $\nabla f \in \mathcal{K}$ and rigidity for approximate solutions in terms of Gradient Young measures $\mathscr{M}_{\mathrm{qc}}(\mathcal{K})$. Indeed, the already mentioned "Tartar square" provides a set $\mathcal{K}$ consisting of four matrices without rank-one connections but such that $\mathscr{M}_{\mathrm{qc}}(\mathcal{K})$ is nontrivial, i.e. contains not only Dirac measures. It was also known that 2- and 3-point configurations without rank-one connections are rigid both for exact solutions, see [**104**], and for approximate Gradient Young measure solutions, see [**87**] or [**88**]. These rigidity results can be obtained in the most convenient way using quasiregular maps, also denoted as mappings of bounded distortion. We recall a very simple argument giving rigidity for exact solutions and notice that this argument extends also to approximate solutions if one uses an observation by J.M. Ball and R.D. James on volume fractions of inhomogeneous Gradient Young measures ([**9**]). So, consider 3 mutually not rank-one connected matrices $A_1, A_2, A_3 \in \mathbb{M}^{2\times 2}$ and $f : B(0,1) \to \mathbb{R}^2$ lipschitz with $\nabla f(x) \in \{A_1, A_2, A_3\}$ a.e. Since $A_i - A_j$ is regular for all $i \neq j$, we can, after postmultiplication if necessary, assume that at least two of these differences have a positive determinant and, perhaps after a renumberation only, that $\det(A_i - A_1) > 0$ for $i = 2, 3$. Therefore, the map $\tilde{f}(x) = f(x) - A_1 \cdot x$ satisfies for some $\varepsilon > 0$ and almost every x either $\nabla \tilde{f}(x) = 0$ or $\det(\nabla \tilde{f}(x)) > \varepsilon$. Since $\tilde{f}$ is also lipschitz, we conclude that it is a map of bounded distortion. Therefore, the famous dichotomy for such maps (see e.g. Theorem 6.3 and Theorem 6.7 in §6.3 together with Corollary 1 and 2 in §10.1 of [**79**] for a modern proof working in any dimensions) tells us that either $\tilde{f}$ is constant or $\nabla \tilde{f} \neq 0$ a.e. In the latter case we are therefore in the two-gradient situation which was handled much earlier e.g. in [**6**].

It turns out that mappings of bounded distortion are also the decisive tool to give a complete answer to the four-gradient problem, which is stated e.g. in Subsection 2.5 in [**64**] and asks for the existence of properly non-rigid sets of four matrices. Indeed, a much more careful reduction using both linear algebra and deep results about the elliptic Monge-Ampère equation due to Šverák and Caffarelli show that the heart of the matter is to decide whether there is a properly non-rigid four point configuration $\{A_1, A_2, A_3, A_4\}$ in the hyperboloid* $\mathcal{H}_{-1} = \mathbb{M}^{2\times 2}_{sym} \cap \{\det(\cdot) = -1\}$. In this way we reduce the problem to a situation studied in Chapter 2. There the openess of nonconstant mappings of bounded distortion is the key to derive that the modifications $f_\pm : f(z) \to z \pm \mathrm{i}z$ of our exact solution of $\nabla f \in \{A_1, A_2, A_3, A_4\}$ have a lipschitzly parametrized one-dimensional image. In that chapter this observation is the first but crucial step to obtain a complete local description of exact solutions of the more general partial differential inclusion $\nabla f \in \mathcal{H}_{-1}$. In [**22**] it is shown that after this first step there is a considerably shorter way to establish the partial result that lipschitz maps with at most countably many not rank-one connected gradients all contained in $\mathcal{H}_{-1}$ are necessarily locally affine. If we next consider 5 point configurations, it soon becomes clear that there is not enough freedom to transform the problem to a situation where mappings of bounded distortion can be exploited in a similar way. Fortunately, we do not get stuck at this point but together with D.Preiss (see [**46**]) we found a five point properly non-rigid set which we present in Section 3. This was certainly a pleasant surprise on its own, but it was even more striking that this example lives in the proper affine subspace $\mathbb{M}^{2\times 2}_{sym}$ of $\mathbb{M}^{2\times 2}$. Indeed, before it seemed quite plausible that the difference between the four and the five gradient problem would come from the fact that in the latter situation the matrices do not need to satisfy any linear constraint.

Finally, in the last section of this chapter we present some examples of partial differential inclusions for which we can find solutions without going through all the machinery developped in Chapter 3. We consider the situation when starting from an arbitrary matrix A in the universum $\mathcal{U}$ we can reach via splitting in rank-one directions the set $\mathcal{K}$ within a finite number of steps, without loosing too much of the (gradient) mass distribution. Moreover, the one-step modifications have to change the gradient distribution in the same way as in Lemma 3.2. This means, the small mistakes produced in the splitting process should stay near the midpoint we started from and not close to the endpoints of the rank-one interval used. Indeed, these endpoints will be in some moment precisely in $\mathcal{K}$ and hence not anymore in the interior of $\mathcal{U}$. So even a tiny bit of gradient produced near but not in these endpoints might destroy all hope for an exact solution since we will never be able to move it back towards $\mathcal{K}$. But if these two conditions (reach $\mathcal{K}$ from the interior of $\mathcal{U}$ with rank-one segments, and e.g. $\mathcal{U}$ is open in the full unconstrained $\mathbb{M}^{n\times m}$) are satisfied, then we get much simpler solutions of the partial differential inclusion. In fact, near almost each point we need in our construction to carry out the splitting process only a finite number of times. Hence, we do not need to worry about (pointwise) convergence of the gradient of our approximate solutions and we obtain in this way exact solutions of $\nabla f \in \mathcal{K}$ that are still piecewise affine. On contrary, solutions constructed using convex integration or a categorial argument have a highly oscillatory gradient which is, in particular, not constant on any nontrivial open set. The natural question comes up how simple such solutions might actually be, we briefly discuss this issue at the end of that section. In the paper [**20**] piecewise affine solutions to a partial differential inclusionalready appeared. Note that the solutions f constructed there in a quite explicite and rather technical manner are even more simple than those we obtain. Indeed, any compact subset of the open unit ball, which is there choosen to be the domain of f, is almost covered by only finitely many pieces on each of which f is affine. On the other hand, such simple functions can of course not solve $\nabla f \in \mathcal{A}$ for properly non-rigid $\mathcal{A}$. Morover, for the partial differential inclusion $\nabla f \in \mathcal{R} \subset \mathrm{O}(3)$ studied in [**20**] does the universum have a relatively simple structure, in particular it is convex. Our Proposition 4.42,

*consult Section 2 for this geometric interpretation

however, straightforwardly provides piecewise affine solutions also to the following more general problems which are the main applications considered in [**24**]:

- prescribed gradient values, see §3 in [**24**],
- scalar (x, u)-independent Hamilton-Jacobi equations, see §4 in [**24**],
- prescribed singular values, see §5 in [**24**].

The task to find properly non-rigid sets which allow piecewise affine solutions seems more interesting. The solution presented in Corollary 4.46 was found together with David Preiss ([**46**]). We conclude this section by constructing in this way piecewise affine solutions even for the two-well problem with different determinants. It is quite amusing to notice that the (modification of Gromov's) convex integration method developed in [**65**] in order to solve this two-well problem is therefore not needed. On the other hand, this fact was realized only in the final stage of our investigations on non-rigid sets, which we would hardly had started not being armed with the first convex integration results.

1. Geometry and stability of rank-one convex hulls

During this section we will work at different levels of generality, reflecting the more or less special character of the properties considered. The local character of the generation of rank-one convex hulls is a very general feature, just reflecting that rank-one convexity of a function is decided locally. Therefore, it does in principal not depend on the dimension of the spaces considered, nor on the particualr schoice of directions alonf which we require convexity of our functions. Hence, we follow the approach by Matoušek and Plecháč to consider these questions in a unifying framework, applications to partial differential inclusions, however, use rank-one convexity only. We try to keep this general line where possible also in the second subsection where we study the geometric nature of extreme points. But the sufficiency of our criterium is much more sensitive to the situation we are in, we need, for instance, that the cone of convexity directions disconnects the space or that nontrivial functions affine in all convexity directions do exists. Finally, in the last subsection where we investigate stability issues, we work in light of the application we aim at only in the setting of rank-one convexity. It should be noticed, however, that then our main results hold equally well in all dimensions.

1.1. Rank-one convexity and locality. As already explained, following the approach of [**58**] and [**59**] we develop our locality results in the full generality. However, since they are of a topological nature, we need that functions, which are convex in our sense are necessarily continuous. Quite obviously, this requires that our directions of convexity allow to connect any point to any other in the space. Indeed, it is well known that then each functions convex in all those directions is even locally lipschitz in the interior of its domain, see e.g. page 112 in [**61**] or Observation 2.3 in [**58**] for the general setting. Now, the following notations come as a natural generalization of rank-one convexity defined in Subsection 1.2.2.

Definition 4.3. *We consider a general n-dimensional linear space, $n > 1$, for simplicity always being identified with $\mathbb{R}^n$ but not necessarily using its euclidean geometry. We suppose that a cone $\mathcal{D}$ spanning the whole space is given, i.e. $\mathcal{D} \subset \mathbb{R}^n$, $\operatorname{span}(\mathcal{D}) = \mathbb{R}^n$ and $\lambda x \in \mathcal{D}$ for all $\lambda \in \mathbb{R}$ and $x \in \mathcal{D}$. Then we say that $f : M \subset \mathbb{R}^n \to \mathbb{R}$ is $\mathcal{D}$-convex if f is convex on any line segment in M parallel to a direction in $\mathcal{D}$, and that a compact set $C \subset \mathbb{R}^n$ is $\mathcal{D}$-convex if for every $x \notin C$ there is a $\mathcal{D}$-convex function $f : \mathbb{R}^n \to \mathbb{R}$ such that $f(x) > \sup f(C)$. Given any (bounded) set K we define the $\mathcal{D}$-convex hull of $K^{\mathcal{D}c}$ to be the smallest $\mathcal{D}$-convex set containing K, as usual this is the intersection of all (compact) $\mathcal{D}$-convex supersets of K.*

In analogy with Definition 1.3 we define the class of $\mathcal{D}$-laminates $\mathscr{M}_{\mathcal{D}c}$ to be the set of all compactly supported probability measures satisfying $f(\bar{\mu}) \leq \int f d\mu$ for all $\mathcal{D}$-convex $f : \mathbb{R}^n \to \mathbb{R}$. Again we localize this class of measures defining $\mathscr{M}_{\mathcal{D}c}(K) = \{\mu \in \mathscr{M}_{\mathcal{D}c}\ ;\ \operatorname{spt}(\mu) \subset K\}$. We note

that consequently $\bar{\mu} \in \mathrm{spt}(\mu)^{\mathcal{D}c}$, *and will derive in Corollary 4.11 that also in this situation the converse is true, i.e.*

$$K^{\mathcal{D}c} = \{\bar{\mu}\ ;\ \mu \in \mathscr{M}_{\mathcal{D}c}(K)\}.$$

REMARK 4.4. In principle, we consider closed sets $\mathcal{D}$ only. But if the domain M is open, then due to the continuity properties of $\mathcal{D}$-convex functions discussed above, $\mathcal{D}$ could be always replaced with its closure.

The following Lemma shows that the "local"-maximum of $\mathcal{D}$-convex functions is $\mathcal{D}$-convex again. Based on the fact that a function is convex if and only if it is convex near each point of its (convex) domain, the argument is a standard trick in convex analysis (see the remark in the proof of Theorem 3.1 in [**59**]). In the context of convexity notions from the calculus of variations it was applied first time in [**85**].

LEMMA 4.5. *Let* $f, g : \mathbb{R}^n \to \mathbb{R}$ *be* $\mathcal{D}$*-convex,* $X \subset \mathbb{R}^n$ *an arbitrary set such that*

a) $f \geq g$ *on* X
b) $f = g$ *on* ∂X.

Then

$$h(x) = \begin{cases} f(x) & x \in X \\ g(x) & x \notin X \end{cases}$$

is a $\mathcal{D}$*-convex function on* $\mathbb{R}^n$. *The same holds true if* f *is defined and* $\mathcal{D}$*-convex on* $\bar{X}$ *only.*

Proof. Fixing arbitrary $x \in \mathbb{R}^n$, $y \in \mathcal{D}$, all we need to show is that $t \in [0,1] \to \tilde{h}(t) = h(x+ty)$ is convex. Adding a suitable affine function to both f and g (and hence to h), we can assume $\tilde{h}(0) = \tilde{h}(1) = 0$ and will prove the inequality $\tilde{h}(t^0) \leq 0$ for any $t^0 \in (0,1)$.

Since $h \geq g$ on $\mathbb{R}^n$, we see that $g(x+ty) \leq 0$ for all $t \in [0,1]$, so we restrict to the case $x_0 = x + t^0 y \in X \setminus \partial X$. For

$$t^+ = \inf(\{1\} \cup \{t \in (t^0, 1]\ ;\ x + ty \in \partial X\}), t^- = \sup(\{0\} \cup \{t \in [0, t^0)\ ;\ x + ty \in \partial X\}),$$

we have $f(x + t^{\pm} y) \leq 0$ and $\tilde{h}(t) = f(x+ty)$ whenever $t \in [t^-, t^+]$. Consequently, $\tilde{h}(t^0) = f(x + t^0 y) \leq 0$ follows. □

The definition of $\mathcal{D}$-convex sets suggests that each such set is the sublevelset of some $\mathcal{D}$-convex function. This fact is made more precise in following statement which is Theorem 3.1 of [**59**], but see also Lemma 2.3 in [**69**] or (for the case of a minor constraint) Lemma 3.4 in [**68**]. It turns out that the required function is found in a natural way. This result has a corresponding one in convex analysis - however it requires more work since the distance function from a $\mathcal{D}$-convex set is in general not $\mathcal{D}$-convex, see e.g. [**107**] or [**30**].

LEMMA 4.6. *Let* $K \subset \mathbb{R}^n$ *be compact, then* $K^{\mathcal{D}c}$ *is precisely the zero set of the* $\mathcal{D}$*-convexification* $\mathrm{dist}_K^{\mathcal{D}c}$ *of the distance function to the set* K *- again* $\mathrm{dist}_K^{\mathcal{D}c}$ *is the pointwise supremum of all* $\mathcal{D}$*-convex* $f : \mathbb{R}^n \to \mathbb{R}$ *which satisfy* $f(x) \leq \mathrm{dist}(x, K)$ *for all* $x \in \mathbb{R}^n$. *Moreover, it is easy to verify that* $\mathrm{dist}_K^{\mathcal{D}c}$ *is also* 1*-Lipschitz.*

Proof. It is clear that the $\mathcal{D}$-convexification of any function is always $\mathcal{D}$-convex and, since the function vanishing identically is $\mathcal{D}$-convex, that $\mathrm{dist}_K^{\mathcal{D}c}$ vanishes on $K^{\mathcal{D}c}$. Conversely, let $x \notin K^{\mathcal{D}c}$ be given. Then we find a $\mathcal{D}$-convex $f : \mathbb{M}^{n\times m} \to \mathbb{R}$ such that $f(x) > 0 \geq \sup(f(K))$. We fix $R > 0$ satisfying $K \subset B(0,R)$ and $\delta > 0$ such that $\delta f(y) < R$ if $|y| = 2R$ and that $\delta f_{|B(0,2R)}$ has Lipschitz constant at most one half. Next, we set $X = \{y \in B(0,2R)\ ;\ \delta f(y) \geq |y| - R\}$ and our choice of δ ensures that $\bar{X} \subset \mathrm{int}(B(0,2R))$ and hence $\delta f(y) = |y| - R$ on ∂X. Due to the foregoing Lemma 4.5 we know that $g : \mathbb{M}^{n\times m} \to \mathbb{R}$, defined by $g(y) = \delta f(y)$ if $y \in X$ and $g(y) = |y| - R$ else, is $\mathcal{D}$-convex. It is also easily checked that $g \leq \mathrm{dist}(\cdot, K)$ and, since only the case $x \in B(0,2R)$ is interesting, that $\mathrm{dist}_K^{\mathcal{D}c}(x) \geq g(x) \geq \delta f(x) > 0$. □

Now we are prepared to prove a precise description how the generation of the $\mathcal{D}$-convex hulls localizes to portions of the whole $\mathbb{R}^n$. It says that the part of the hull of a given set K contained in some set B is exactly generated by the part of K in B and the part of $K^{\mathcal{D}c}$ on the boundary of B.

THEOREM 4.7. *Let B be bounded and K be compact in $\mathbb{R}^n$, then*

$$K^{\mathcal{D}c} \cap B = [(B \cap K) \cup (\partial B \cap K^{\mathcal{D}c})]^{\mathcal{D}c} \cap B.$$

Proof. Since $(K^{\mathcal{D}c})^{\mathcal{D}c} = K^{\mathcal{D}c}$, the right side is obviously contained in the left one. To prove the converse inclusion, fix any $x_0 \in B \setminus [(B \cap K) \cup (\partial B \cap K^{\mathcal{D}c})]^{\mathcal{D}c}$. Due to Lemma 4.6, there are $\mathcal{D}$-convex $f, g : \mathbb{R}^n \to [0, \infty)$ with

$$\{f = 0\} = [(B \cap K) \cup (\partial B \cap K^{\mathcal{D}c})]^{\mathcal{D}c} \text{ and } \{g = 0\} = K^{\mathcal{D}c}.$$

We define the $\mathcal{D}$-convex function $f_1(x) = f(x) - (f(x_0)/2)$ and the compact set $C = \partial B \setminus \{f_1 < 0\}$. Since $\partial B \cap K^{\mathcal{D}c} \subset \{f_1 < 0\}$, we have $K^{\mathcal{D}c} \cap C = \emptyset$ and $\delta = \min g(C) > 0$. Putting

$$f_2(x) = \frac{\delta f_1(x)}{2(1 + \max\{|f_1(x)| \ ; \ x \in \partial B\})},$$

we see that $f_2 < g$ on ∂B. Indeed, if $x \in C$ then $f_2(x) \le \delta/2 < \delta \le g(x)$. Else $x \in \partial B \setminus C$, so $f_2(x) < 0 \le g(x)$. Therefore, defining $X = \{f_2 > g\} \cap B$, we see that $\bar{X} \cap \partial B = \emptyset$ and $\bar{X} \subset \bar{B}$, so $\bar{X} \subset \mathrm{int}(B)$. Consequently, $\partial X \subset \{f_2 = g\} \cap \mathrm{int}(B)$, and Lemma 4.5 ensures that

$$h(x) = \begin{cases} f_2(x) & x \in X \\ g(x) & x \notin X \end{cases}$$

is $\mathcal{D}$-convex. Moreover, it has the desired property to seperate x_0 from $K^{\mathcal{D}c}$.

Indeed, since $h \ge f_2$ on B, we conclude $h(x_0) \ge f_2(x_0) > 0$. On the other hand, for any $y \in K$ we will show $h(y) \le 0$ and be done. In fact, if $y \in B$ then $y \in B \cap K$ which implies that $g(y) = f(y) = 0$, therefore $f_1(y), f_2(y)$ are negative and $h(y)$ at most zero. Otherwise, $y \notin B$ implies $y \notin X$, so $h(y) = g(y) = 0$. □

COROLLARY 4.8. *Let $C_1, \ldots, C_k$ be mutually disjoint compact subsets of $\mathbb{R}^n$. If a further compact A satisfies $A^{\mathcal{D}c} \subset \bigcup_{i=1}^k C_i$ then*

$$A^{\mathcal{D}c} \cap C_i = (A \cap C_i)^{\mathcal{D}c} \textit{ for all } i \le k.$$

Proof. First, we select mutually disjoint compacts B_i with $\mathrm{int}(B_i) \supset C_i$. Now Theorem 4.7 applied to $K = A \cap C_i$ and $B = \mathbb{R}^n \setminus B_i$ shows that $(A \cap C_i)^{\mathcal{D}c} \subset B_i$. Another application of this Theorem, this time for $K = A$, $B = B_i$ together with the observations that $A^{\mathcal{D}c} \cap C_i = A^{\mathcal{D}c} \cap B_i$ yield the conclusion. □

This foregoing corollary was originally obtained in [**58**] and [**59**] and initiated to some extend our interest in the local structure of $\mathcal{D}$-convex hulls. We can go on and easily refine the just obtained theorem in order to get a better insight in the building blocks of the $\mathcal{D}$-convex hull. In this way we give a more precise formulation and transparent proof of the so called "Structure Theorem" from [**76**].

THEOREM 4.9. *Suppose $\mu \in \mathscr{M}_{\mathcal{D}c}(\mathbb{R}^n)$. Then $C = \mathrm{spt}(\mu)^{\mathcal{D}c}$ is a connected set.*

Proof. Obviuosly, C is contained in the convex hull of $\mathrm{spt}(\mu)$ and hence compact. If C is not connected, then it is the disjoint union of two nonvoid compacts C_1, C_2. Due to Corollary 4.8 we have $C_i = C_i^{\mathcal{D}c}$ for both $i = 1, 2$. Since the barycenter $x_0 = \bar{\mu}$ belongs to C, we can assume $x_0 \in C_1$.

We put $B = \{x \ ; \ \mathrm{dist}(x, C_1) \le \mathrm{dist}(x, C_2)\}$ and $f(x) = \mathrm{dist}(x, C)$, then Lemma 4.6 tells us that $C = \{f^{\mathcal{D}c} = 0\}$, where $f^{\mathcal{D}c}(x) = \sup\{g(x) \ ; \ g : \mathbb{R}^n \to \mathbb{R} \ \ \mathcal{D}\text{-convex}, g \le f\}$ is again the $\mathcal{D}$-convexification of f. Obviously, $C_1 \subset \mathrm{int}(B)$ and $\partial B \cap C = \emptyset$. Since $f(x) \ge |x - x_0| - \mathrm{diam}(C)$ and the left hand side is a convex function, this left hand side minorizes also $f^{\mathcal{D}c}$. Therefore, we

conclude that $\varepsilon = \min f^{\mathcal{D}c}(\partial B)$ is a positive number even if ∂B is unbounded. We define U to be the connected component of $\{f^{\mathcal{D}c} < \varepsilon\}$ containing x_0. Moreover, since $U = (U \cap \mathrm{int}(B)) \cup (U \setminus B)$ is connected, we infer that $U \subset B$, and hence $\bar{U} \cap C_2 = \emptyset$. Because $f^{\mathcal{D}c} \equiv \varepsilon$ on ∂U, Lemma 4.5 ensures that

$$g(x) = \begin{cases} \varepsilon & x \in U \\ f^{\mathcal{D}c}(x) & x \notin U \end{cases}$$

is $\mathcal{D}$-convex. Observe that $g \leq \varepsilon$ on $\mathrm{spt}(\mu) \subset C$ and $g \equiv 0 < \varepsilon$ on C_2. Since μ is a $\mathcal{D}$-laminate and $x_0 \in U$, Jensen's inequality gives

$$g(x_0) = \varepsilon \leq \int_{C_1 \cup C_2} g(x)\, d\mu(x).$$

Therefore, $\mu(C_2) = 0$ follows, implying that $\mathrm{spt}(\mu) \subset C_1$ and $C = \mathrm{spt}(\mu)^{\mathcal{D}c} \subset C_1$. This contradiction to our choice $C_2 \neq \emptyset$ finishes the proof. □

Finally, we would like to emphasise once more that the analogies to Theorem 4.7, Corollary 4.8 and Theorem 4.9 fail if we consider poly- or quasiconvexity. Indeed, for polyconvexity we may consider the following well-known example in 2×2-diagonal matrices. Let $\mathcal{K} = \{\mathrm{diag}(-1,-1), \mathrm{diag}(1/2, 2), \mathrm{diag}(2, 1/2)\}$, then we easily find that $X \in \mathcal{K}^{pc}$ if and only if $\det(X) = 1$ and $X \in \mathcal{K}^{co}$. This shows $\mathcal{K}^{pc} = \{\mathrm{diag}(-1,1)\} \cup \{\mathrm{diag}(t, 1/t)\ ;\ t \in [1/2, 2]\}$. For $\tilde{\mathcal{K}} = \{\mathrm{diag}(1/2, 2), \mathrm{diag}(2, 1/2)\}$ we infer from [**90**] and $\det(X - Y) < 0$ if $X, Y \in \tilde{\mathcal{K}}$ different that $\mathscr{M}_{\mathrm{pc}}(\tilde{\mathcal{K}})$ consists of dirac measures only and hence $\tilde{\mathcal{K}}^{pc} = \tilde{\mathcal{K}}$. Because $\mathcal{K}^{pc}$ has two connected components, all similarities to the rank-one convex hull fail. The counterexample for quasiconvexity is much more sophisticated. Since it talks about a difference to rank-one convexity it is heavily based on Šverák's counterexample from [**86**]. In fact, it is just the complexification of the construction in [**86**], for the necessary details see Section 4.7 in [**64**]. Anyhow, it provides a compact set $\mathcal{K} \subset \mathbb{M}^{6\times 2}$ such that $\mathcal{K}^{pc} = \mathcal{K}^{qc} = \mathcal{K} \cup \{0\} \neq \mathcal{K}$, which obviously again violates all connectedness properties we look for. Because results sufficiently similar to Lemma 4.6 hold both in the poly- and the quasiconvex situation, we also conclude that the only other ingredience in the proof of Theorem 4.7, i.e. Lemma 4.5 fails for these two convexity notions. This shows, that poly- and quasiconvexity of a function is not determined locally, a much deeper analysis of this fact can be found in the papers [**50**] and [**51**].

Based on these locality results, one can also derive the following connection between $\mathcal{D}$-laminates and $\mathcal{D}$-prelaminates entirely living in a small neighbourhood of the $\mathcal{D}$-convex hull of the support of the $\mathcal{D}$-laminate. This approximation result was obtained in [**68**] where also the in-approximation result for rank-one convexity in the context of convex integration was proved. Having this approximation result, our preparations from Section 3.3 immediately yield the corresponding in-approximation result in our categorial framework. The statement about approximation is based on an extension result, again we give both of them in full generality, the application to partial differential inclusions of course considers rank-one convexity only.

LEMMA 4.10. *(see Lemma 3.5 in* [**68**] *and Lemma 2.3 in* [**69**]*) Let $K \subset \mathbb{R}^n$ be a compact set and let U be an open set containing $K^{\mathcal{D}c}$. If $f : U \to \mathbb{R}$ is $\mathcal{D}$-convex, then there exist a globally lipschitz $\mathcal{D}$-convex $F : \mathbb{R}^n \to \mathbb{R}$ which coincides with f on $K^{\mathcal{D}c}$.*

Proof. Using Lemma 4.6 we first pick a $\mathcal{D}$-convex $g : \mathbb{R}^n \to [0, \infty)$ with $K^{\mathcal{D}c} = \{g = 0\}$ and continue similar to the proof of that Lemma 4.6. So, we choose $\varepsilon > 0$ such that $B(K^{\mathcal{D}c}, 2\varepsilon) \subset U$ and put $\delta = \inf g(\partial B(K^{\mathcal{D}c}, \varepsilon)) > 0$. If we consider the $\mathcal{D}$-convex function

$$G(x) = \min f(K^{\mathcal{D}c}) + \frac{2}{\delta}(\max f(\bar{B}(K^{\mathcal{D}c}, \varepsilon)) + 1 - \min f(K^{\mathcal{D}c})) \cdot g(x),$$

and the set $X = \{G \leq f\} \cap B(K^{\mathcal{D}c}, \varepsilon) \supset K^{\mathcal{D}c}$ then $\bar{X} \subset B(K^{\mathcal{D}c}, \varepsilon)$ and hence $G(x) = f(x)$ if $x \in \partial X$. So, Lemma 4.5 ensures that the required extension F can be defined by putting $F(x) = f(x)$ if $x \in X$ and $F(x) = G(x)$ else. □

Note that there are examples of Lipschitz functions defined on rank-one convex set that can not be extended to a rank-one convex function on any neighbourhood of that set. As a first application of the foregoing lemma we establish the equivalence of the two definition of the hull of a set, after this we prove a dual statement about measures.

COROLLARY 4.11. *For any compact set K is $K^{\mathcal{D}c} = \{\bar{\mu}\ ;\ \mu \in \mathscr{M}_{\square}(K)\}$.*

Proof. Indeed, the left hand side is by definition included in the right one. For the converse implication, we fix an arbitrary $x \in K^{\mathcal{D}c}$ and consider the function $f(y) = \mathrm{dist}(K, y)^2$. We know that its envelope $g = f^{\mathcal{D}c}$ is $\mathcal{D}$-convex and, therefore, vanishes in x. Obviously again $g(y) = \inf\{\langle f, \mu\rangle\ ;\ \mu \in \mathscr{PL}_{\mathcal{D}c}$ and $\bar{\mu} = x\}$, see the statement of Theorem 4.12 for the natural definition of these De-prelaminates. Hence, we find a sequence $\{\mu_k\}_{k=1}^{\infty}$ with $\bar{\mu}_k = x$, $\langle f, \mu_k\rangle < 1/k$ and, up to taking a subsequence, $\mu_k \rightharpoonup^* \mu$. Since f grows superlinearly, it is clear that μ is a probablity with barycentre x and support in $\mathcal{K}$. We finish by showing that it is also a laminate. For this purpose we can use Lemma 4.10 which tells us that any rank-one convex test function h can be changed by a modification far away from the convex hull of $\mathcal{K}$ into a globally Lipschitz rank-one map. Since we keep the second momenta of our μ_k bounded, we can go to the limit in Jensen's inequality for this Lipschitz modification which then gives the result for h itself.. □

THEOREM 4.12. *(see Theorem 3.1 in* [**68**]*) In addition to the notions introduced in Definition 4.3, we also introduce similar to Definition 3.19 the class $\mathscr{PL}_{\mathcal{D}c}(M)$ of $\mathcal{D}$-prelaminates generated in a given set M. This is the smallest class of probabilities which contains all Dirac masses in M and which together with $\mu = c\delta_x + \tilde{\mu}$, $\tilde{\mu} \geq 0$ contains also $c(\lambda\delta_{x'} + (1-\lambda)\delta_{x''}) + \tilde{\mu}$ provided $x = \lambda x' + (1-\lambda)x'' \in [x', x''] \subset M$, and $(x'' - x') \in \mathcal{D}$.*

Now let $K \subset \mathbb{R}^n$ be a compact set and let $\mu \in \mathscr{M}_{\mathcal{D}c}(K)$ be given. Then for any ε positive there is a sequence $\{\mu_k\}_{k=1}^{\infty} \subset \mathscr{PL}_{\mathcal{D}c}(B(K^{\mathcal{D}c}, \varepsilon))$ which weakly* converges to μ and satisfies $\bar{\mu}_k = \bar{\mu}$ for all $k \geq 1$.

Proof. Suppose the conclusion to fail. As all the sets $M_x = \{\lambda \in \mathscr{PL}_{\mathcal{D}c}(B(K^{\mathcal{D}c}, \varepsilon))\ ;\ \bar{\lambda} = x\}$, where $x \in B(K^{\mathcal{D}c}, \varepsilon)$, are easily checked to be convex, we infer from the definition of weak* convergence the existence of $f \in \mathcal{C}_c(\mathbb{R}^n)$ such that $\int f\, d\mu < c = \inf\{\int f\, d\lambda\ ;\ \lambda \in M_{\bar{\mu}}\}$. If we define $\tilde{f}(x) = \inf(\{\int f\, d\lambda\ ;\ \lambda \in M_x\})$, then $\tilde{f}$ is a $\mathcal{D}$-convex function on $B(K^{\mathcal{D}c}, \varepsilon)$ and $\tilde{f}(\bar{\mu}) > \int f\, d\mu$. But since Lemma 4.10 ensures the existence of a $\mathcal{D}$-convex $F : \mathbb{R}^n \to \mathbb{R}$ with $F = \tilde{f} \leq f$ on $K^{\mathcal{D}c}$, we conclude that

$$\tilde{f}(\bar{\mu}) > \int f\, d\mu \geq \int F\, d\mu \geq F(\bar{\mu}) = \tilde{f}(\bar{\mu}),$$

because obviously $\bar{\mu} \in K^{\mathcal{D}c}$. This contradiction finishes the proof. □

So we get following result, just recovering the unconstrained version of Theorem 1.3 in [**68**].

COROLLARY 4.13. *Let us be given a compact set $\mathcal{K} \subset \mathbb{M}^{n\times m}$ together with a rank-one convex in-approximation $\{\mathcal{U}_i\}_{i=0}^{\infty}$. This means*

a) *The $\mathcal{U}_i$'s are open such that $\sup_{X\in\mathcal{U}_i} \mathrm{dist}(X, \mathcal{K}) \to 0$ as i tends to infinity.*
b) *Each $X \in \mathcal{U}_i$ is the barycentre of a laminate $\mu \in \mathscr{M}_{\mathrm{rc}}$ compactly supported in $\mathcal{U}_{i+1}$.*

Let us be given any $\Omega \subset \mathbb{R}^m$ open and bounded. We define the universum as $\mathcal{U} = \bigcup\{\mathcal{C}^{rc}\ ;\ \mathcal{C} \subset \mathcal{U}_i$ for some i and $\mathcal{C}$ compact$\}$ and as in Definition 3.14 we consider the space $\mathscr{P} = \mathscr{P}(\Omega, \mathcal{U}(, g))$ of all piecewise affine functions from Ω into $\mathbb{R}^n$ having their gradient almost everywhere in $\mathcal{U}$ (and respecting the boundary data given by the function g). Then a typical f in $\mathrm{clos}_\infty(\mathscr{P})$ satisfies $\nabla f(x) \in \mathcal{K}$ for a.e. $x \in \Omega$.

Proof. Since we intend to use Proposition 3.17, it only remains to check that gradients in $\mathcal{U}$ are stable only near $\mathcal{K}$. First of all we notice that $\mathcal{U}$ is open in $\mathbb{M}^{n\times m}$ due to the fact that taking the rank-one convex hull commutes with translations and that sufficiently small shifts of any $\mathcal{C}_i$ appearing in the definition of $\mathcal{U}$ are still in the same $\mathcal{U}_i$. Next we observe for any such $\mathcal{C} \subset \mathcal{U}_i$ compact and $\mu \in \mathscr{M}_{\mathrm{rc}}(\mathcal{C})$ and any $\delta > 0$ there is a preliminate $\tilde{\mu}$ supported in $\mathcal{U}$ such $\bar{\tilde{\mu}}=\bar{\mu}$ and $\tilde{\mu}(\mathcal{U}_i) > 1-\delta$. Indeed, this is an immediate consequence of Theorem 4.12. Using induction, we conclude from this and condition b) in the definition of an in-approximation that for any $X \in \mathcal{U}$ and any $i \in \mathbb{N}$, $\delta > 0$ there is a prelaminate $\mu \in \mathscr{PL}(\mathcal{U})$, such that $X = \bar{\mu}$ and $\mu(\mathcal{U}_i) > 1-\delta$. Due to condition a) above we infer $\Phi_{\mathcal{U}}(X) \geq \mathrm{dist}(X, \mathcal{K})$, and so Lemma 3.20.a) finishes the proof. □

An interesting question is whether this kind of in-approximation statement holds true also in case of minor or symmetry constraints. The minor constraint was handled in [**68**], our treatment of the symmetric case follows these lines.

LEMMA 4.14. *(see Lemma 3.6 in* [**68**]*) Let* $\mathcal{V} = \mathbb{M}^{n\times n}_{sym}$ *or* $\mathcal{V} = \{X \in \mathbb{M}^{n\times m}\ ;\ M(X) = t\}$ *with* $t \neq 0$*, where* $M : \mathbb{M}^{n\times m} \to \mathbb{R}$ *is a minor of rank at least two. If* $\mathcal{K} \subset \mathcal{V}$ *is compact,* $\mathcal{U} \subset \mathcal{V}$ *is relatively open and* $\mathcal{K}^{rc} \subset \mathcal{U}$ *and* $f : \mathcal{U} \to \mathbb{R}$ *is rank-one convex then for all* $\varepsilon > 0$ *there is a* $g : \mathbb{M}^{n\times m} \to \mathbb{R}$ *rank-one convex and satisfying* $|f(X) - g(X)| < \varepsilon$ *for all* $X \in K$*.*

Proof. Let $\mathcal{V} = \mathbb{M}^{n\times n}_{sym}$. Using mollifications inside $\mathcal{V}$ if necessary, we can of course assume that f is $\mathcal{C}^\infty$ on a relatively open domain $\mathcal{U}$ containing $\mathcal{K}^{rc}$. As in [**85**], we show that for any $\varepsilon > 0$ and $c > c_\varepsilon$ is the function

$$g_{\varepsilon,c}(X) = f(\frac{1}{2}(A + A^T)) + \varepsilon|A + A^T|^2 + c|A - A^T|^2$$

rank-one convex on some set $\tilde{\mathcal{U}}$ open in $\mathbb{M}^{n\times n}$ and containing $\mathcal{K}^{rc}$. Choosing ε small enough, Lemma 4.10 will the finish the proof. To check our claim about $g_{\varepsilon,c}$ we notice first that it is the sum of 3 functions smooth in a fixed neighbourhood of $\mathcal{K}^{rc}$. Hence, if the claim fails, we send c to infinity and conclude from compactness the existence of $A \in \mathcal{K}^{rc}$ and of a rank-one direction $B \in \mathcal{V} \setminus \{0\}$ such that $D^2h(A)(B, B) \leq 0$, where $h(X) = f(\frac{1}{2}(X + X^T)) + \varepsilon|X + X^T|^2$. This is of course impossible.

The slightly more complicated case $\mathcal{V} = \{X \in \mathbb{M}^{n\times m}\ ;\ M(X) = t\}$ with $t \neq 0$ was handled in [**68**] in the same way, the condition $t \neq 0$ is necessary for the existence of a smooth projection onto (compact parts of) $\mathcal{V}$. □

It is easy to check that Lemma 4.10 is the key ingredience in Theorem 4.12. Since now Lemma 4.14 provides us with the analogous statements in the constrained situations, we can recycle the proof of Theorem 4.12 and, due to the possible choices for $\mathcal{V}$ in Definition 3.19, also the proof of Corollary 4.13. In this way we obtain the following result, the more complicated nonlinear case in it was originally established in Theorem 1.3 and Theorem 3.1 in [**68**].

COROLLARY 4.15. *Let* $\mathcal{V} = \mathbb{M}^{n\times n}_{sym}$ *or* $\mathcal{V} = \{X \in \mathbb{M}^{n\times m}\ ;\ M(X) = t\}$ *with* $t \neq 0$*.*

i) *If* $\mathcal{K} \subset \mathcal{V}$ *is compact and* $\mu \in \mathscr{M}_{\mathrm{rc}}(\mathcal{K})$*, then for any* $\varepsilon > 0$ *there is a sequence* $\{\mu_k\}_{k=1}^\infty \subset \mathscr{PL}(B(\mathcal{K}^{rc}, \varepsilon) \cap \mathcal{V})$ *which weakly* converges to* μ *and satisfies* $\bar{\mu} = \bar{\mu}_k$ *for all* $k \geq 1$*.*

ii) *Given* $\mathcal{K} \subset \mathcal{V}$ *together with a rank-one convex in-approximation* $\{\mathcal{U}_i\}_{i=0}^\infty$ *in* $\mathcal{V}$*. This means*

   a) *The* $\mathcal{U}_i$*'s are relatively open subsets of* $\mathcal{V}$ *such that* $sup_{X\in\mathcal{U}_i} \mathrm{dist}(X, \mathcal{K}) \to 0$ *as* i *tends to infinity.*

   b) *Each* $X \in \mathcal{U}_i$ *is the barycentre of a laminate* $\mu \in \mathscr{M}_{\mathrm{rc}}$ *compactly supported in* $\mathcal{U}_{i+1}$*.*

   Let $\Omega \subset \mathbb{R}^m$ *be open and bounded and let the universum be given by* $\mathcal{U} = \bigcup\{\mathcal{C}^{rc}\ ;\ \mathcal{C} \subset \mathcal{U}_i$ *for some* i *and* $\mathcal{C}$ *compact*$\}$*. As in Definition 3.14 we consider the space* $\mathscr{P} = \mathscr{P}(\Omega, \mathcal{U}, g)$*, or* $\mathscr{P} = \mathscr{P}(\Omega, \mathcal{U}, g)$*, of all piecewise affine functions from* Ω *into* $\mathbb{R}^n$ *having their gradient almost everywhere in* $\mathcal{U}$ *(and respecting the boundary data given by the function* g*, respectively). Then a typical* f *in* $\mathrm{clos}_\infty(\mathscr{P})$ *satisfies* $\nabla f(x) \in \mathcal{K}$ *for a.e.* $x \in \Omega$*.*

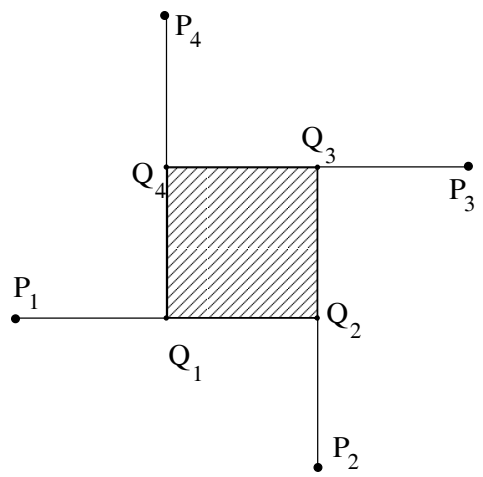
1.2. Towards a geometric understanding of rank-one extreme points. The notion of extreme points is very important in convex analysis, mainly due to the Krein-Milman theorem which states that each compact convex set C (in a locally convex vector space) is the convex hull of its extreme points. Hence, these points form in a very natural way the minimal set which generates all of C as its convex hull. If we consider generalized convexity notions, in particular those related to the moredimensional calculus of variations, then we can choose a proper notion of extreme points which ensures that the Krein-Milman argument still works. This is done e.g. in [**1**] in full generality, the more specific situations we are interested in are treated in [**106**] and [**53**], and yields the natural but quite abstract characterization of the minimal generator of a given hull.

Using for $\square \in \{rc, qc, pc, co\}$ the notion of $\square$-extreme points as given in Definition 3.8 and making for $\square = \mathcal{D}c$ the obvious modification, we have the following

LEMMA 4.16. *(Krein-Milman theorem) Let $\square \in \{\mathcal{D}c, rc, qc, pc, co\}$ and let $\mathcal{K}$ be a compact and $\square$-convex set. Then $\mathcal{A} = \operatorname{clos}(extr_\square(\mathcal{K}))$ is the minimal closed set that contains all of $\mathcal{K}$ in its $\square$-convex hull. This means*

(a) $\mathcal{K} = \mathcal{A}^\square$ *and*
(b) $\mathcal{K} \setminus \mathcal{M}^\square \neq \emptyset$ *if $\mathcal{M}$ is a proper closed subset of $\mathcal{A}$.*

The fact that many problems in convex geometry can be attacked by surprisingly efficient algorithms, however, is due to a simple characterization of convex extreme points in geometric terms and not just in an abstract way. In the paper [**58**] (and also [**59**]) where the authors aim at computational methods to determine rank-one and separately convex hulls, this more geometrical approach was choosen. Indeed, the authors essentially used the notion of an extreme point in the sense of lamination convexity, which is of course very similar to ordinary convexity and therefore completely geometric. In other words, given a cone of directions $\mathcal{D}$ they defined that $x \in A$ is "$\mathcal{D}$-extreme" if there is no segment $s \subset A$ parallel to a direction in $\mathcal{D}$ and containing x as its interior point. Comparing with the "Krein-Milman"-kind of extreme points introduced e.g. in Definition 3.8, $\mathcal{D}$-extremality is just extremality with respect to $\mathcal{D}$-prelaminates . Because all these measures are also $\mathcal{D}$-laminates, we see that $\mathcal{D}$-extremality in the sense of Matoušek and Plecháč necessary for being a $\mathcal{D}c$-extreme point in the sense of Definition 3.8. The following nice application of $\mathcal{D}$-extremality in the Krein-Milman context was given in [**58**] and demonstrates how this notion can be helpful in a quick determination of rank-one convex hulls.



We consider the classical Tartar square, which is the finite set $\mathcal{K} = \{P_1, P_2, P_3, P_4\}$ where $P_1 = -P_3 = \operatorname{diag}(-3/2, -1/2)$ and $P_2 = -P_4 = \operatorname{diag}(1/2, -3/2)$. Identifying $(x, y) \in \mathbb{R}^2$ with the matrix $\operatorname{diag}(x, y) \in \mathbb{M}^{2\times 2}$ we can visualize the situation in the picture on the right. Note that now a segment runs in a rank-one direction if and only if it is horizontal or vertical. Using rank-one convex functions of the type $(x, y) \to (\sigma_1(x - x_0))^+(\sigma_2(y - y_0))^+$, $\sigma_1, \sigma_2 \in \{-1, +1\}$, we easily find that $\mathcal{K}^{rc} \subset \tilde{\mathcal{K}}$, where $\tilde{\mathcal{K}}$ to is the union of the inner closed square $Q_1Q_2Q_3Q_4$ and the four arms $[Q_iP_i]$. The crucial step is now to show that all of $\tilde{\mathcal{K}}$ is in the rank-one convex hull of $\mathcal{K}$.

But for $\mathcal{D} = \langle e_1\rangle \cup \langle e_2\rangle$ it is easily checked that the P_i's are the only $\mathcal{D}$-extreme points of $\tilde{\mathcal{K}}$ (since

any of these points is also a convex extreme point of $\tilde{\mathcal{K}}$, they are also $\mathcal{D}c$-extreme points). So we conclude from the Krein-Milman theorem that indeed $\tilde{\mathcal{K}} \subset (\mathcal{K})^{rc}$. This examples confirms also that the Krein-Milman theorem does not hold for the lamination convex hull, as was to be expected since this hull is not defined in terms of separation properties of a certain cone of functions.

This example and some more thoughts about the algorithmic determination of $\mathcal{D}c$-hulls in [**58**] suggest the following approach to compute rank-one or general $\mathcal{D}c$-hulls of a finite set $\mathcal{K}$. First one gets an upper bound for this hull by removing as many points as possible (from $\mathcal{K}^{co}$) using some "standart" seperating functions like the x^+y^+-function above. Then consider the remaining points forming the set $\tilde{\mathcal{K}}$. Due to our locality arguments, if $x \in \tilde{\mathcal{K}}$ is in $\mathcal{K}^{\mathcal{D}c}$ then x is also generated by $\mathcal{K}^{\mathcal{D}c}$ intersected with a little sphere around x. In other words, such an x is not a $\mathcal{D}c$-extreme point of $\tilde{\mathcal{K}}$ nor $\tilde{\mathcal{K}} \cap B(x, \varepsilon)$. Conversely, if $x \in \tilde{\mathcal{K}} \setminus \mathcal{K}$ is a $\mathcal{D}c$-extreme point of $\tilde{\mathcal{K}} \cap B(x, \varepsilon)$ then $x \notin \mathcal{K}^{\mathcal{D}c}$ and we can continue to reduce $\tilde{\mathcal{K}}$, in principle until we are left with $\mathcal{K}^{\mathcal{D}c}$. In light of these considerations a simple and geometrical (and hence local) characterization of $\mathcal{D}c$-extreme points is desirable.

Our first example shows that the simple lamination-extremality in the sense of Matoušek and Plecháč does not characterize rank-one convex extreme points. It was also discovered by S.Müller and V.Šverák ([**70**]) during their work on generalized Tartar configurations used to contruct counterexamples to elliptic regularity of systems ([**69**]). In fact, it is a degenerate limit of a generalized Tartar square, but seems in spirit even more close to the example given in [**73**]. The idea of this construction works as well for general $\mathcal{D}$-convexity, see Corollary 4.19 below. In this way we find a new necessary condition, if x is a $\mathcal{D}c$-extreme point of $\mathcal{K}$ then x can not be in the convex hull of the $\mathcal{D}$-rays starting from x into K. This is a natural generalization of the condition not to be in the middle of a $\mathcal{D}$-segment, but it is not clear whether it is also close to a sufficient condition. In particular, do $\mathcal{D}$-segment or $\mathcal{D}$-rays play at all a role in such a characterization? M.Chlebík asked a more modest question which points as well at our basic problem to understand the geometry of rank-one convex hulls: Does any such nontrivial hull contain rank-one segments, or at least does such a ray start at each of the non-extreme points? The main results of this subsection is Theorem 4.20 which shows that in the case of rank-one convexity in $\mathbb{M}^{2\times 2}$ there are quite many rank-one rays starting from each $A \in \mathcal{K}^{rc} \setminus \mathcal{K}$ into the set $\mathcal{K}^{rc}$. In fact, these rays are not even contained in any proper (one sided) cone with tip in A - so in this sense they "almost" contain A in their convex hull. The question of a full characterization, however, remains open in this situation. If we consider $\mathcal{D}$-convexity in the plane, which is the case of main interest in [**58**], then Proposition 4.21 shows that the necessary condition from Corollary 4.19 is also sufficient. We conclude this subsection with some comments about limits of possible generalizations.

In our first example we prefer to compute the quasiconvex hull instead of $\mathcal{K}^{rc}$ since the effort is precisely the same. We just need to quote the result ensuring that the "standard x^+y^+-functions" separate even from the quasiconvex hull.

THEOREM 4.17. *For $X \in \mathbb{M}^{2\times 2}_{sym}$ let* $\operatorname{index}(X)$ *be the number of negative eigenvalues counting multiplicities. For $k \in \{0, 1, 2\}$ we define $g_k : \mathbb{M}^{2\times 2}_{sym} \to \mathbb{R}$ by*

$$g_k(X) = \begin{cases} |\det(X)| & \text{if } \operatorname{index}(X) = k \\ 0 & \text{else.} \end{cases}$$

Then all the g_k satisfy Jensen's inequality for all Gradient Young Measures living in $\mathbb{M}^{2\times 2}_{sym}$, in other words each g_k is quasiconvex on its domain.

For the proof see [**87**].

EXAMPLE 4.18. *Let* $\mathcal{K} = \{Y_1, Y_2, Y_3, Y_4\}$ *where*

$$Y_1 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, Y_2 = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, Y_3 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, Y_4 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then $\mathcal{K}^{rc} = \mathcal{K}^{qc} = S := \{t \cdot Y_j \ ; \ t \in [0,1] \ \text{and} \ j \leq 4\}$.

Proof. First, we show that $S \subset \{Y_1, Y_2, Y_3, Y_4\}^{rc}$. For this it obviously suffies to prove $0 \in \{Y_1, Y_2, Y_3, Y_4\}^{rc}$. Otherwise, there exists a rank-one convex $f : \mathbb{M}^{2\times 2} \to [0, \infty)$ such that $f(0) = 1$ and $f(Y_j) = 0$ for all j. Observe that $0 \in \mathcal{K}^{co}$, which is of course necessary for $0 \in \mathcal{K}^{rc}$, and that $\mathcal{K} \subset \{\text{rank}(\cdot) = 1\}$. Hence we find $\lambda_1 = \lambda_2 = 1/3$ and $\lambda_3 = \lambda_4 = 1/6$ such that $0 = \sum_1^4 \lambda_i Y_i$. For $\varepsilon > 0$ and $i \leq 4$ we set $X_i^\varepsilon = \sum_{j=1}^i \varepsilon \lambda_i Y_i$ and $Y_i^\varepsilon = X_i^\varepsilon + Y_i$. Then $X_i^\varepsilon \to 0$ as $\varepsilon \to 0$ and, since $X_4^\varepsilon = 0$, we can take i modulo 4 and have $X_{i+1}^\varepsilon \in [X_i^\varepsilon, Y_{i+1}^\varepsilon]$ with $(Y_{i+1}^\varepsilon - X_i^\varepsilon) \| Y_{i+1}$. So we see again that for $\mathcal{C} = \bigcup_{i=1}^4 [X_i^\varepsilon, Y_{i+1}^\varepsilon]$ only the Y_{i+1}^ε's, $i = 1, \ldots, 4$ are rank-one extreme points. Hence, $X_i^\varepsilon \in \{Y_i^\varepsilon \ ; \ i = 1, \ldots, 4\}^{rc}$. In particular, $f(X_i^\varepsilon) \leq \max f(Y_i^\varepsilon)$ which gives for $\varepsilon \to 0_+$ a contradiction to the continuity of f.

It remains to prove that $\{Y_1, \ldots, Y_4\}^{qc} \subset S$. The proof consists in several suitable applications of Theorem 4.17. So, assume there is an $Y_0 \in \{Y_1, \ldots, Y_4\}^{qc} \setminus S$. Of course,

$$Y_0 \in \mathbb{M}^{2\times 2}_{sym} \cap \{Y \ ; \ \det(Y) = 0\} \cap \{Y_1, \ldots, Y_4\}^c$$

in particular,

$$Y_0 = \begin{pmatrix} a & c \\ c & b \end{pmatrix} \text{ with } |a|, |b|, |c| \leq 1 \text{ and } ab = c^2.$$

First we consider the case $c = 0$, so either a or b vanishes. Since $Y_0 \notin S$, we have $a = 0$ and $b \in (0, 1]$ or $b = 0$ and $a \in (0, 1]$. In the first subcase we define the quasiconvex function

$$f(X) = g_0 \left(X + \begin{pmatrix} b/4 & 0 \\ 0 & -b/2 \end{pmatrix} \right).$$

Then $f(Y_0) = \det \begin{pmatrix} b/4 & 0 \\ 0 & b/2 \end{pmatrix} > 0$. But $f(Y_1) = f(Y_2) = 0$ since $X + \begin{pmatrix} b/4 & 0 \\ 0 & -b/2 \end{pmatrix}$ has negative entries on the diagonal if $X = Y_1, Y_2$. Also $f(Y_3) = f(Y_4) = 0$ because for $X \in \{Y_3, Y_4\}$

$$\det(X + \begin{pmatrix} b/4 & 0 \\ 0 & -b/2 \end{pmatrix}) = (1 + b/4)(1 - b/2) - 1 = -\frac{b}{4} - \frac{b^2}{8} < 0.$$

This shows $Y_0 \notin \{Y_1, \ldots, Y_4\}^{qc}$ – contradiction. In the remaining subcase $a \in (0, 1], b = 0$ we argue similarly using

$$f(X) = g_0 \left(X + \begin{pmatrix} -a/2 & 0 \\ 0 & a/4 \end{pmatrix} \right).$$

Consequently, we can assume $c \neq 0$, $ab > 0$. If $a, b < 0$, we put

$$f(X) = g_2 \left(X + \begin{pmatrix} 0 & -c \\ -c & 0 \end{pmatrix} \right).$$

Then $f(Y_0) = ab > 0$, $f(Y_i) = 0$ for $i = 1, 2$ since then $\det(Y_i + \begin{pmatrix} 0 & -c \\ -c & 0 \end{pmatrix}) < 0$ and $f(Y_3) = f(Y_4) = 0$ because for $X \in \{Y_3, Y_4\}$ the matrix $X + \begin{pmatrix} 0 & -c \\ -c & 0 \end{pmatrix}$ has positive entries on the main diagonal. This shows that only positive a, b remain interesting. If $a = b \in (0, 1]$ then $c = \pm a$, hence $Y_0 \in S$ - contradiction. Therefore $a \neq b$ and we set $\Delta = (a - b)/2$. The seperating function is this time defined to be

$$f(X) = g_0 \left(X + \begin{pmatrix} -\Delta & 0 \\ 0 & \Delta \end{pmatrix} \right).$$

Since $|\Delta| < 1/2$, we see that for $i = 1, 2$ the matrix $Y_i + \begin{pmatrix} \Delta & 0 \\ 0 & -\Delta \end{pmatrix}$ has a negative entry on the main diagonal and hence $f(Y_i) = 0$. We also check that $\det(Y_i + \begin{pmatrix} \Delta & 0 \\ 0 & -\Delta \end{pmatrix}) = (1+\Delta)(1-\Delta) - 1 < 0$, so $f(Y_i) = 0$ if $i = 3, 4$. Finally, because $Y_0 + \begin{pmatrix} \Delta & 0 \\ 0 & -\Delta \end{pmatrix} = \begin{pmatrix} (a+b)/2 & c \\ c & (a+b)/2 \end{pmatrix}$, we conclude $f(Y_0) = (a+b)^2/4 - ab = (a-b)^2/4 > 0$ and are done. □

Just for curiosity we note that in the foregoing example $\mathcal{K}^{pc} = \mathcal{K}^{co} \cap \{X \; ; \; \det(X) = 0\}$, which is obviously strictly larger then $\mathcal{K}^{qc}$. A more important observation is that the argument from the begin of the foregoing proof immediately generalizes and gives the following.

COROLLARY 4.19. *For any n, m and $\mathcal{D}$-convexity the following holds. If $X \in Y^{co}$ and $Y \subset (X + \mathcal{D})$ then $X \in Y^{\mathcal{D}c}$.*

Now we ask how far is the foregoing criterium from giving also a necessary condition for not being an rank-one extreme point. Our next result shows that it is quite close, at least if we consider rank-one convexity in two dimensions. In fact, given a non-extreme point X of a set $\mathcal{K}^{rc}$ and considering the rank-one segments starting from X into $\mathcal{K}^{rc}$ then their convex hull almost contains X itself - at least this hull is not included in a proper (one sided) cone.

THEOREM 4.20. *Let $\mathcal{K} \subset \mathbb{M}^{2\times 2}$ be compact and $X_0 \in \mathcal{K}^{rc} \setminus \mathcal{K}$. Then for all $Y \neq 0$ and for all $\varepsilon > 0$ there exists*

$$X \in \mathcal{K}^{rc} \setminus \{X_0\} \text{ such that } \det(X - X_0) = 0 \text{ and } \langle X - X_0, Y\rangle < \varepsilon |X - X_0|.$$

Proof. Since the generation of the rank-one convex hull commutates with translations, we can assume $X_0 = 0$. We choose $R \in (0, \operatorname{dist}(0, \mathcal{K}))$ and $\mathcal{C} = \mathcal{K}^{rc} \cap \partial B(0, R)$. We know from Theorem 4.7 that $0 \in \mathcal{C}^{rc}$. We also suppose the conclusion to fail, hence we find

$$\varepsilon \in (0,1) \text{ and } Y \in \partial B(0,1) \text{ such that } \langle X, Y\rangle \geq \varepsilon |X| \text{ if } X \in \mathcal{C}^{rc} \text{ and } \det(X) = 0.$$

For $j \in \{-1, +1\}$ let

$$\mathcal{S}_j = \mathcal{C} \cap \{X \; ; \; \langle X, Y\rangle \leq \frac{\varepsilon}{2}|X|\} \cap \{X \; ; \; j \cdot \det(X) > 0\},$$

then by compactness of the $\mathcal{S}_j$'s we know that

$$\text{there exists } \delta > 0 \text{ such that } j \cdot \det(X) > \delta \text{ whenever } X \in \mathcal{S}_j \text{ and } j \in \{-1, +1\}.$$

Now we claim that for $\eta = \varepsilon\delta/8R$

$$\mathcal{C}^{rc} \cap B(0, \eta) \subset \{X \; ; \; \langle X, Y\rangle \geq \frac{\varepsilon}{4}|X|\}. \tag{4.1}$$

Since $\mathcal{C}^{rc} \cap B(0,\eta) \setminus \{\det \neq 0\} \subset \{X \; ; \; \langle X, Y\rangle \geq \varepsilon|X|\}$, the claim follows from Theorem 4.7 if we prove that for $j \in \{-1, +1\}$

$$(\mathcal{C}^{rc} \cap \partial(B(0,R) \cap \{X, j \cdot \det(X) > 0\}))^{rc} \cap B(0,\eta) \subset \{X \; ; \; \langle X, Y\rangle \geq \frac{\varepsilon}{4}|X|\}.$$

For this purpose we fix an arbitrary laminate μ whose support is contained in

$$\mathcal{C}^{rc} \cap \partial(B(0,R) \cap \{X, j \cdot \det(X) > 0\}) = (\mathcal{C} \cap \{X \; ; \; j \cdot \det(X) > 0\}) \cup (\mathcal{C}^{rc} \cap \{X \; ; \; \det(X) = 0\}),$$

note that $\mathcal{C}^{rc} \cap \partial B(0,R) = \mathcal{C}$ due to the strict convexity of the euclidean balls.

Since $j \cdot \det(\cdot)$ is nonnegative μ-a.e. and fulfills the Jensen inequality for this laminate, we conclude

$$\mu(\mathcal{S}_j) \leq \frac{j \cdot \det(\bar{\mu})}{\delta} \leq \frac{|\bar{\mu}|^2}{\delta}.$$

Consequently,

$$\begin{aligned}\langle\bar\mu, X\rangle &= \int \langle X,Y\rangle\, d\mu(X) \geq \frac{\varepsilon}{2}\int |X|\, d\mu(X) + \int (\langle X,Y\rangle - \frac{\varepsilon}{2}|X|)\, d\mu(X)\\ &\geq \frac{\varepsilon}{2}|\bar\mu| + \int_{\{X\ ;\ \langle X,Y\rangle < \frac{\varepsilon}{2}|X|\}} (\langle X,Y\rangle - \frac{\varepsilon}{2}|X|)\, d\mu(X)\\ &\geq \frac{\varepsilon}{2}|\bar\mu| - 2R\mu(\mathcal{S}_j) \geq |\bar\mu|(\frac{\varepsilon}{2} - 2R\frac{|\bar\mu|}{\delta})\\ &\geq \frac{\varepsilon}{4}|\bar\mu| \text{ if } |\bar\mu| \leq \eta.\end{aligned}$$

After having established (4.1), we can once more apply Theorem 4.7 to conclude that

$$0 \in (\mathcal{C}^{rc} \cap \partial B(0, \frac{\eta}{2}))^{rc} \subset \{X\ ;\ \langle X,Y\rangle \geq \frac{\varepsilon}{8}|\eta|\}^{co},$$

which is obviously impossible. □

PROPOSITION 4.21. *Suppose $\mathbb{S}^1 \cap \mathcal{D}$ is finite but not contained in a single line. If $C \subset \mathbb{R}^2$ is compact and $0 \in C^{\mathcal{D}c} \setminus C$ then*

$$0 \in (\mathcal{D} \cap C^{\mathcal{D}c} \setminus \{0\})^{co}.$$

Proof We denote by $\mathcal{D}_0$ the intersection of $\mathcal{D}$ with $\mathbb{S}^1$. In the first step of our proof we show the following

> *Claim* Let $v_1, v_2 \in \mathcal{D}_0$ be such that an open subarc $\mathrm{arc}(v_1, v_2)$ of $\mathbb{S}^1$ from v_1 to v_2 does not intersect $\mathcal{D}_0$, and set $U = (0,\infty)\cdot \mathrm{arc}(v_1, v_2)$ to be the open $\mathcal{D}$-sector generated by this arc. If K is compact, $0 \in K^{\mathcal{D}c} \setminus K$ and $(0,\infty)\cdot v_i \cap K^{\mathcal{D}c} = \emptyset$ for $i = 1, 2$ then $K^{\mathcal{D}c} \cap \bar U = ((K \cap U) \cup \{0\})^{\mathcal{D}c} = (K \cap U)^{\mathcal{D}c} \cup \{0\}$.

Indeed, the first two sets are by Theorem 4.7 equal and obviously contain the third one, so it remains to check the converse inclusion. We put $\tilde K = K \cap U$, by our assumption this is a compact set. We find ε positive such that $B(\tilde K, \varepsilon) \subset U$ and hence $B(0,\varepsilon) \cap \tilde K = \emptyset$. Due to our assumption concerning $\mathrm{arc}(v_1, v_2)$ we find a quadratic form $Q : \mathbb{R}^2 \to \mathbb{R}$ such that $Q(v) \geq 0$ for all $v \in \mathcal{D}$ but $Q(x) < 0$ whenever $x \in U$. Observe that by the first condition Q is a $\mathcal{D}$-convex function. Now, consider any $x \in U \cap \partial B(0,\varepsilon)$. It is easy to see that the $\mathcal{D}$-convex function $Q_x : z \to Q(z - x)$ has value 0 in x but $Q_{x|\tilde K\cup\{0\}} < -\delta < 0$, so $x \notin (\{0\} \cup \tilde K)^{\mathcal{D}c}$. Hence, $\bar U \cap \partial B(0,\varepsilon) \cap (\tilde K \cup \{0\})^{\mathcal{D}c} = \emptyset$. Now we apply Theorem 4.7 and obtain, since $(\tilde K \cup \{0\})^{\mathcal{D}c} \subset K^{\mathcal{D}c}$ that

$$(\tilde K \cup \{0\})^{\mathcal{D}c} \cap (\bar U \cap B(0,\varepsilon)) = (\{0\})^{\mathcal{D}c} \cap (\bar U \cap B(0,\varepsilon)) = \{0\},$$

$$(\bar U \setminus B(0,\varepsilon)) \cap (\tilde K \cup \{0\})^{\mathcal{D}c} = ((\bar U \setminus B(0,\varepsilon)) \cap (\tilde K \cup \{0\}))^{\mathcal{D}c} \cap (\bar U \setminus B(0,\varepsilon)) \subset (U \cap K)^{\mathcal{D}c}.$$

So the Claim follows, and we can return to our proof.

Denote by R the union of all rays in $\mathcal{D}$ which intersect $C^{\mathcal{D}c} \setminus \{0\}$. If the conclusion of the Proposition fails, then we find $a \in \mathbb{S}^1$ and ε positive such that

$$\langle a, v\rangle \geq \varepsilon|v| \text{ whenever } v \in R.$$

Choosing the ε sufficiently small, we derive from the Claim just proven that for any open $\mathcal{D}$-sector S contained in the halfplane $\{\langle a,\cdot\rangle < 0\}$ the equations

$$C^{\mathcal{D}c} \cap \bar S \cap B(0,\varepsilon) = \{0\} \text{ and } C \cap B(0,\varepsilon) = \emptyset$$

hold. We set S_0 to be the union of the closures of all such sectors, it is easy to see that the sector S_0 intersects any line with a direction in $\mathcal{D}$. We can find a quadratic form Q such that $Q(x) > 0$ if and only if $\{x, -x\} \cap \mathrm{int}(S_0) \neq \emptyset$, so Q is $\mathcal{D}$-convex. Lemma 4.5 ensures that $f = \chi_{S_0}\cdot Q$ is $\mathcal{D}$-convex as well. By compactness, we infer the existence of $\delta > 0$ such that $B(S_0, 2\delta) \cap C^{\mathcal{D}c} \cap \partial B(0,\varepsilon) = \emptyset$

and we fix any point $y \in \operatorname{int}(S_0) \cap \partial B(0,\delta)$. Then $f_y : x \to f(x+y)$ is a $\mathcal{D}$-convex function which vanishes on $C^{\mathcal{D}c} \cap \partial B(0,\varepsilon)$ but is positive in 0. Hence $0 \notin (C^{\mathcal{D}c} \cap \partial B(0,\varepsilon))^{\mathcal{D}c}$ which implies by Theorem 4.7 that $0 \notin C^{\mathcal{D}c}$. By this contradiction, the proof is finished. □

COROLLARY 4.22. *Let the cone $\mathcal{D} \subset \mathbb{R}^2$ consists of two lines only and let $K \subset \mathbb{R}^2$ be compact. Then a point $x \in K^{rc}$ is $\mathcal{D}c$-extreme if and only if it is $\mathcal{D}$-extreme in the sense of Matoušek and Plecháč.*

Finally, let us notice that Theorem 4.20 does not straightforwardly generalize into higher dimensions which might indicate that this kind of topological methods give the best results only if the rank-one cone is the common boundary of proper subsets of the full space. In fact, in [**47**] an example of a set $\mathcal{K}$ consisting of three segements in 3×3-diagonal matrices was found such that $0 \in \mathcal{K}^{rc}$ but that all matrices $X \in \mathcal{K}^{rc}$ with $\operatorname{rank}(X) = 1$ satisfy $\operatorname{tr}(X) < -|X|/3$. There are, nevertheless, still three such rays in $\mathcal{K}^{rc}$ starting in the origin, so the following two question naturally appear

Q1 If $\mathcal{K} \subset \mathbb{M}^{2\times 2}$ is compact and $0 \in \mathcal{K}^{rc} \setminus \mathcal{K}$, is then the origin necessarily in the (nonclosed) convex set
$$(\{X \ ; \ \det(X) = 0\} \cap (\mathcal{K}^{rc} \setminus \{0\}))^{co}.$$

Q2 If $\mathcal{K} \subset \mathbb{M}^{m\times n}$ is compact and $0 \in \mathcal{K}^{rc} \setminus \mathcal{K}$, does then necessarily a rank-one ray from the origin pass into $\mathcal{K}^{rc}$? Or does K^{rc} necessarily contain at least one nontrivial rank-one segment? What about the same questions if we consider general $\mathcal{D}$-convexity?

1.3. Stability of quasiconvex hulls and finite non-rigid sets. This subsection contains in Theorem 4.24 the already mentioned stability result for rank-one convex hulls. Because we deal with rank-one convexity, we can reason in a very geometric manner. Nevertheless, if we apply this result to more special sets we can also answer two questions concerning the stability of the "nongeometric" quasiconvex hulls, see Corollary 4.25.

But stability of the rank-one convex hull is even more interesting for our purposes. In fact, the outcome of Chapter 3 is that such hulls allow to find solutions of partial differential inclusions if the hulls provide enough space to modify gradients. Our stability results now enable us to keep the rank-one convex hulls large in this sense and, in the same moment, to perturb away rank-one connections between the rank-one extreme points. In this way we obtain the first examples of non-rigid sets without rank-one connections, see Theorem 4.27. It follows also that such sets can be found by a kind of random choice.

Our first argument provides some lamination convex hulls with nonvoid interior.

LEMMA 4.23. *Let $X \in \mathbb{M}^{n\times m}$ be of rank one. Then for any ε positive there is a set $\mathcal{M}_X \subset B(X,\varepsilon)$ consisting of at most $4nm$ points such that $X \in \operatorname{int}(\mathcal{M}_X^c)$ and*

$$\mathcal{M}_X \subset \{Y \ ; \ \operatorname{rank}(Y) = 1\}. \tag{4.2}$$

Moreover, (4.2) implies that

$$(\{0\} \cup \mathcal{M}_X)^{co} = (\{0\} \cup \mathcal{M}_X^{co})^{co} = (\{0\} \cup \mathcal{M}_X^{co})^{lc} \tag{4.3}$$

and that for each $f : \mathbb{M}^{n\times m} \to \mathbb{R}$ rank-one convex

$$f(Y) < \max f(\mathcal{M}_X^c) \text{ if } f(0) < f(Y) \text{ and } Y \in (\{0\} \cup \mathcal{M}_X)^{co} \setminus \mathcal{M}_X^{co}. \tag{4.4}$$

Proof. After a suitable transformation using pre- and postmultiplication we can reduce the problem to the case when $X = e_1 \otimes e_1$. Let the $\varepsilon > 0$ be given, we fix a positive $\delta < \min(1, \varepsilon/4)$. Our set $\mathcal{M}_X$ will consist of the following points. Gradually, we take all $(i,j) \in \{1,\dots,n\} \times \{1,\dots,m\}$, and in case that $i = 1$ or $j = 1$ we add $M_{i,j}^k = X + (-1)^k \delta^2 e_i \otimes e_j$ for $k = 1,2$ to our set. Else we put for $k = 1,2$ and $l = 1,2$ $M_{i,j}^{k,l} = X + (-1)^k \delta^2 e_i \otimes e_j + (-1)^l \delta (e_1 \otimes e_j + (-1)^k e_i \otimes e_1)$ into our

set and finally, we obtain $\mathcal{M}_X$. It is clear that $\mathcal{M}_X \subset B(X,\varepsilon) \cap \{Y\ ;\ \mathrm{rank}(Y)=1\}$ and as for any (i,j) both points $X \pm \delta^2 e_i \otimes e_j$ belong to $\mathcal{M}_X^c$, we also have $X \in \mathrm{int}(\mathcal{M}_X^c) \subset B(X,\varepsilon)$.

After this, we have to verify that (4.2) implies (4.3) and (4.4). For this purpose we proceed by induction with respect to the length of the convex combination involved. It is clear that, for any $Z \in \bigcup_{t\in[0,1]} t\mathcal{M}_X^{co} = (\{0\} \cup \mathcal{M}_X)^{co}$ we can choose a shortest representation of the kind

$$Z = \sum_{P\in\mathcal{M}_Z} \lambda_P P \text{ where } \mathcal{M}_Z \subset \mathcal{M}_X, \lambda_P > 0 \text{ and } \sum_{\mathcal{M}_Z} \lambda_P \le 1.$$

Arguing by contradiction, we choose among all $Z \in (\{0\}\cup\mathcal{M}_X)^{co}$ such that either $Z \notin (\{0\}\cup\mathcal{M}_X^{co})^{lc}$ or that (1) fails one matrix, denoted by Z_0, which minimizes the cardinality of $\mathcal{M}_Z$. Obviously, $\mathrm{card}(\mathcal{M}_{Z_0}) > 0$ since $Z_0 \neq 0$. Hence, we can find $P_0 \in \mathcal{M}_{Z_0}$ and set $\mu = \lambda_{P_0}/(1-\sum_{P\neq P_0}\lambda_P)$. It is easy to check that

$$Z_0 = \mu Z_1 + (1-\mu) Z_2 \text{ with } Z_1 = (1-\sum_{P\neq P_0}\lambda_P)P_0 + \sum_{P\neq P_0}\lambda_P P \text{ and } Z_2 = \sum_{P\neq P_0}\lambda_P P.$$

Checking the sum of the weights one easily verifies that Z_1 is contained in $\mathcal{M}_X^{co}$. By minimality of $\mathrm{card}(\mathcal{M}_{Z_0})$ we know $Z_2 \in (\{0\}\cup(\mathcal{M}_X^{co}))^{lc}$ and $f(Z_2) < \max f(\mathcal{M}_X^{co})$ or $f(Z_2) \le f(0)$. Since Z_1, Z_2 differ only by the rank-one matrix $(1-\sum_{P\neq P_0}\lambda_P)P_0$, we conclude that their convex combination Z_0 belongs to $(\{0\}\cup\mathcal{M}_X^{co})^{lc}$ as well. Because $Z_1 - Z_0, Z_0 - Z_2 \neq 0$, rank-one convexity of f implies that $f(Z_0) < \max f(\mathcal{M}_X^c)$ if $f(Z_0) > f(0)$. This contradiction shows that we are done. □

It is a well known fact that open covers of compact spaces can not just be reduced to finite ones but do also cover the space in a uniformly robust way. The open sets from the foregoing Lemma correspond to a cover of a certain rank-one convex hull. Therefore, given a compact set $\mathcal{K}$ in that hull, we can reduce the rc-generator of $\mathcal{K}$ to a finite set and the robustness of the corresponding cover yields then the required stability result.

THEOREM 4.24. *Let $U \subset \mathbb{M}^{n\times m}$ be open and bounded. Then for any compact set C in U there is a positive ε such that $C \subset M^{rc}$ whenever the set M fulfills $\partial U \subset B(M,\varepsilon)$.*

Proof. Obviously, the result follows once we know that it is true for C being any closed ball $\bar{B}(X_0,R) \subset U$. For later use in the proof of Theorem 4.27, we will more specifically show the existence of an $\varepsilon > 0$ such that for each set M satisfying $\partial U \subset B(M,\varepsilon)$ there is $S \subset M$ of the following kind. If $X \in S$, then there exists a set $\mathcal{M}_X$ such that

(i) $\mathcal{M}_X - X \subset \{Y\ ;\ \mathrm{rank}(Y)=1\}$,
(ii) $\mathcal{M}_X$ is a subset of $B(X_0,R)$ of cardinality at most $4nm$
(iii) $(\{X\}\cup\mathcal{M}_X)^{co} = (\{X\}\cup\mathcal{M}_X^{co})^{lc}$, and
(iv) $\bigcup_{X\in S}\mathrm{int}(\{X\}\cup\mathcal{M}_X)^c \supset \partial\bar{B}(X_0,R)$.

For this purpose, we fix any $Y \in \partial B(X_0,R)$. Since $Y - X_0$ is the sum of rank-one matrices, we find D_Y of rank one such that $\langle Y - X_0, D_Y\rangle > 0$. We define $Y_t = Y + t\cdot D_Y$, then $Y_t \notin \bar{B}(X_0,R)$ whenever t is positive. We choose $t_0 > 0$ such that $X_Y = Y_{t_0} \in \partial U$ and select $P_Y = Y_{t_1} \in B(X_0,R)$ for some t_1 negative but sufficiently close to zero. Finally, we fix $r_Y > 0$ such that $\bar{B}(P_Y,3r_Y) \subset B(X_0,R)$. Now, Lemma 4.23 ensures the existence of $\delta_Y \in (0,r_Y)$ and $\mathcal{M}_{X_Y} \subset B(P_Y,r_Y)$ fulfilling the properties (i),(ii), (iii) from above and such that $B(Y,2\delta_Y) \subset (\{X_Y\}\cup\mathcal{M}_{X_Y}^c)^{lc}$. By compactness of $\partial B(X_0,R)$ we find $Y_1,\dots,Y_N \in \partial B(X_0,R)$ such that $\bigcup_{i=1}^N B(Y_i,\delta_{Y_i}) \supset \partial B(X_0,R)$. We also select the desired positive $\varepsilon < \mathrm{dist}(\bar{B}(X_0,R),\mathbb{M}^{n\times m}\setminus U), \min_i \delta_{Y_i}$.

Now we take any set M satisfying $B(M,\varepsilon) \supset \partial U$. Then for each $i \le N$ we find $X_i \in M$ such that $X_{Y_i} \in B(X_i,\varepsilon)$ and set $S = \{X_i\ ;\ i \le N\}$. We claim that $\bar{B}(X,R) \subset S^{rc}$ and, moreover, that puting $\mathcal{M}_{X_i} = \mathcal{M}_{X_{Y_i}} + (X_i - X_{Y_i})$ we obtain the sets fulfilling (i),...,(iv) from above.

Indeed, (i),(ii), (iii) are clearly satisfied and, as concerns (iv) it is easily verified using the way the Y_i's were choosen and the fact that for each $i \leq N$

$$B(Y_i, \delta_{Y_i}) \subset B(Y_i, 2\delta_{Y_i}) + (X_i - X_{Y_i}) \subset \mathrm{int}(\{X_{Y_i}\} \cup \mathcal{M}^c_{X_{Y_i}})^{lc} + (X_i - X_{Y_i}) = \mathrm{int}(\{X_i\} \cup \mathcal{M}^c_{X_i})^{lc}.$$

Finally, if $\bar{B}(X_0, R)$ would not be contained in S^{rc} then there would be a rank-one convex function $f : \mathbb{M}^{n\times m} \to [0, \infty)$ vanishing on S but attaining value 1 in some $Z_0 \in \bar{B}(X_0, R)$. We can of course even assume $1 = \max(f(\bar{B}(X_0, R)))$ and that $Z_0 \in \partial B(X_0, R)$. Hence due to (iv), we find $i \leq N$ such that $Z_0 \in \mathrm{int}(\{X_i\} \cup \mathcal{M}^c_{X_i})^{lc}$. Moreover, because $\mathcal{M}^c_{X_i} \subset B(P_{Y_i}, 2r_{Y_i}) \subset B(X_0, R - r_{Y_i})$, we have $Z_0 \notin \mathcal{M}^c_{X_i}$. Die to the second part of Lemma 4.23 we know that (i) enforces (4.4) to be true. Since $f(Z_0) > f(X_i)$ we get that $f(Z_0) < \max f(\mathcal{M}^c_{X_i}) \leq \max(f(\bar{B}(X_0, R)))$. This contradiction finishes our proof. □

Corollary 4.25.

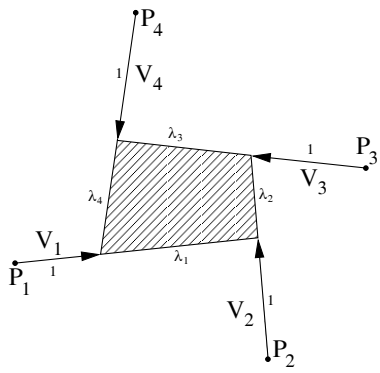
a) *If $C \subset \mathbb{M}^{n\times m}$ is a compact set then its boundary $\partial(C^{qc})$ has a strongly stable quasiconvex hull.*
b) *Let $X, Y \in \mathbb{M}^{n\times m}$ with $\mathrm{rank}(X - Y) = 1$ and let $\varepsilon > 0$. Then the quasiconvex hull of set $\bar{B}(\{X, Y\}, \varepsilon)$ is strongly stable.*

This settles two questions from [**108**], one of them concerning the existence of compact sets of matrices that have strongly stable quasiconvex hulls but are themself not quasiconvex. In [**108**], see the remark after Question 3.1, the author refers to the stability of the classical Tartar square in $\mathbb{R}^2$ under perturbations in the plane. This was presumably a starting point for his considerations and discussions with other mathematician, which led e.g to Example 3.6 in [**58**] where the construction of a kind of stable set in $\mathbb{M}^{2\times 2}_{sym}$ was given. Since this contruction uses only a very small cone $\mathcal{D}$ of three rank-one directions, it is rather involved. In fact, the authors in [**58**] conjecture that using the full cone of rank-one directions could give simpler examples of smaller stable sets. It turns out, that this is true - in fact, we show that the classical Tartar square is stable even if we allow small pertubations in the full space $\mathbb{M}^{2\times 2}$! (Jan Kolář informed me, [**49**], that he also observed this property of the classical Tartar square.) As will be proved in the next section, any four point configuration, however, is always too small to be a nontrivial non-rigid set. In [**69**], a more involved consideration leads to a similar example which lives in $\mathbb{M}^{4\times 2}$ and is crucial for the construction of counterexamples to elliptic regularity.

Proposition 4.26. *The classical Tartar square in the diagonal 2×2-matrices is, as a subset of $\mathbb{M}^{2\times 2}$, has a strongly stable quasiconvex hull.*

Proof. As strong stability is preserved under pre- and postmultiplication with regular matrices as well as under translations, we might assume that our configuration is $\mathcal{K}^0 = \{P^0_1, P^0_2, P^0_3, P^0_4\}$ where $P^0_1 = -P^0_3 = \mathrm{diag}(-3/2, -1/2)$ and $P^0_2 = -P^0_4 = \mathrm{diag}(1/2, -3/2)$. Again it is enough to check lowersemicontinuity of the quasiconvex hull operator in this set $\mathcal{K}^0$. Using the well known fact that the quasi- and rank-one convex hull of $\mathcal{K}^0$ agree, we are done if we show that for each $\varepsilon > 0$ there is a $\delta > 0$ such that $(\mathcal{K}^0)^{rc} \subset B(\mathcal{K}^{rc}, \varepsilon)$ provided $\mathcal{K} = \{P_1, P_2, P_3, P_4\}$ with $|P_i - P^0_i| < \delta$ for all i.

We introduce the explicite notation V for the rank-one "arms" of the Tartar square, i.e. the vector from the outer point to the corresponding inner one, so we have $V_1^0 = -V_3^0 = e_1 \otimes e_1$ and $V_2^0 = -V_4^0 = e_2 \otimes e_2$. Moreover, by $\lambda_i > 0$ we denote the proportion between the length of the side of the "generalized" (i.e. skew) Tartar square to the length of the corresponding arm, hence we have $\lambda_i^0 = 1$ for $i \leq 4$. Now it is clear, that using this kind of parametrization generalized Tartar squares are given by the zeros of the function



$$\Phi : (\mathbb{M}^{2\times 2})^4 \times (\mathbb{M}^{2\times 2})^4 \times (0,\infty)^4 \to (\mathbb{M}^{2\times 2})^4 \times \mathbb{R}^4$$

defined via

$$\Phi(P,V,\lambda) = (\{(1+\lambda_{i-1})V_{i_1} + P_{i-1} - V_i - P_i\}_{i=1}^4, \{\det(V_i)\}_{i=1}^4), \text{ where } i = 0 \simeq i = 4.$$

Since Φ is $\mathcal{C}^\infty$, it is clear that if we show that in our starting point the derivative $D_{V,\lambda}\Phi(P^0,V^0,\lambda^0) : (\mathbb{M}^{2\times 2})^4 \times \mathbb{R}^4 \to (\mathbb{M}^{2\times 2})^4 \times \mathbb{R}^4$ is regular, then the implicit function theorem implies for $P - P^0$ small the existence of V close to V^0 and λ close to λ^0 such that the new triple (P,V,λ) provides a generalized Tartar square. An additional argument will then show that the "square" becomes filled, so the rank-one convex hull consists not only of the outer frame but approximates all of $(\mathcal{K}^0)^{rc}$. First, we compute the derivative of Φ in the direction of test vectors δV and $\delta\lambda$. Then the condition $D_{V,\lambda}(P^0,V^0,\lambda^0)(\delta V,\delta\lambda) = 0$ turns out to be equivalent to

$$0 = 2\delta V_{i-1} - \delta V_i + \delta\lambda_{i-1}V_{i-1} \text{ for } i = 1,\dots,4 \tag{4.5}$$

$$0 = \delta V_i(j_i,j_i) \text{ if } i = 1,\dots,4, i_j \in \{1,2\}, \text{ and } i + j_i \text{ odd.} \tag{4.6}$$

Now, (4.5) and (4.6) give for all i that $0 = 2\delta V_i(j_i,j_i) - \delta V_{i+1}(j_i,j_i) + \delta\lambda_i V_i(j_i,j_i) = -\delta V_{i+1}(j_i,j_i) + \delta\lambda_i \cdot 0$ as $V_i(j_i,j_i) = 0$. Hence, together with the original (4.6) we actually have $\delta V_i(j_k,j_k) = 0$ for all i,k. Therefore (4.5) implies also that $\delta\lambda_{i-1}V_{i-1}(j_k,j_k) = 0$, so $\delta\lambda_{i-1} = 0$ for $k = i = 1,\dots,4$. This shows that (4.5) yields $2\delta V_{i-1} = \delta V_i$, and after four substitutions we end up with $\delta V_i = 16\delta V_i$, in other words $\delta V_i = 0$ for all i.

As already noticed, $\Phi(P,V,\lambda) = 0$ ensures only that $\{P_1,P_2,P_3,P_4\}^{rc}$ contains the outer frame $\bigcup_{i=1}^4 [P_i, P_i + (1+\lambda_i)V_i]$ of the generalized Tartar square. We will not show that the whole "interior" of this configuration, whatever this might be, becomes filled. However, we establish stability in the sense of Definition 4.2. For this purpose, we fix any of the "inner" points $X^0 = \mathrm{diag}(x_1^0, x_2^0)$ of $(\mathcal{K}^0)^{rc}$, hence $|x_i^0| < 1/2$. We consider the endpoints $X_+^0 = \mathrm{diag}(1/2, x_2^0)$ and $X_-^0 = \mathrm{diag}(-1/2, x_2^0)$ of a rank-one segment connecting the outer frame and passing trough X^0. We have $X_+^0 = P_2^0 + (3/2 + x_2^0)\cdot V_2^0$ and $X_-^0 = P_4^0 + (3/2 - x_2^0)\cdot V_4^0$. As

$$\frac{d}{d\lambda}\det(X_+^0 - (X_-^0 + \lambda V_4^0))\Big|_{\lambda=0} = \frac{d}{d\lambda}\det(e_1\otimes e_1 + \lambda e_2 \otimes e_2)\Big|_{\lambda=0} = 1,$$

again the implicit function theorem tells us that for (X_+, X_-, V_4) close to (X_+^0, X_-^0, V_4^0) we find $\lambda = \lambda(X_+, X_-, V_4)$ close to zero such that $\det(X_+ - (X_- + \lambda V_4)) = 0$. We freeze the two coefficients $(3/2 + x_2^0)$ and $(3/2 - x_2^0)$, so given perturbed vectors P and V of matrices, we again denote $X_+ = P_2 + (3/2 + x_2^0)\cdot V_2$ and $X_- = P_4 + (3/2 - x_2^0)\cdot V_4$. Our foregoing result tells that for P sufficiently close to P^0 we find a corresponding (V,λ) close to (V^0,λ^0) creating the outer frame inside the rank-one convex hull of $\{P_1,P_2,P_3,P_4\}$. Moreover, since $|X_+ - X_+^0| + |X^- X_-^0|$ is small if $|P - P^0| + |V - V^0|$ is so, we find the small $\lambda = \lambda(X_+, X_-, V_4^0)$ and see that both $1 < 3/2 + \lambda - x_2^0 < 1 + \lambda_4$ and $1 < 3/2 + x_2^0 < 1 + \lambda_2$. Hence, the points $X_+ = P_2 + (3/2 + x_2^0)V_2$ and $X_- = P_4 + (3/2 + \lambda - x_2^0)V_4$ are in $\{P_1,P_2,P_3,P_4\}^{rc}$

and close to their corresponding X_+^0 and X_-^0. Consequently, $X = X_- + (x_1^0 + 1/2)(X_+ - X_-)$ is also in $\{P_1, P_2, P_3, P_4\}^{rc}$ and close to our X^0. □

Finally, we apply our main stability result to construct finite non-rigid sets of matrices without rank-one connections. The heart of the matter is of course the construction of the appropriate set of matrices. The fact that this set is non-rigid, i.e. that is allows nontrivial solutions of the corresponding partial differential inclusion could be shown in several ways. Even if we prefer a proof using our point of continuity argument from Chapter 3, it is just to notice that our whole search for stability results concerning rank-one convex hulls was motivated by their potential applications via the rank-one convex in-approximation results already established in [**66**].

THEOREM 4.27. *For any $n, m \geq 2$ we can find a finite number of matrices $A_1, \dots, A_N \in \mathbb{M}^{n\times m}$ and an $\varepsilon > 0$ such that the following holds. If $B_i \in B(A_i, \varepsilon)$ for each $i \leq N$ then*

i) $\operatorname{rank}(B_i - B_j) = \min(m, n)$ *for* $i \neq j$,
ii) *there exists a lipschitz map $f : (0,1)^n \to \mathbb{R}^m$ which is nonaffine (but has affine boundary data) and fulfills*

$$\nabla f(x) \in \{B_1, \dots, B_N\} \text{ a.e. in } (0,1)^n.$$

Proof. We consider the open unit ball $B(0,1)$ in $\mathbb{M}^{n\times m}$ and the compact subball $C = \bar{B}(0, 1/2)$. In this situation, we fix an $\varepsilon_1 > 0$ whose existence is stated in the begin of the proof of Theorem 4.24.

Now we start by choosing any finite set $\mathcal{A}_0 \subset \partial B(0,1)$ with $B(\mathcal{A}_0, \varepsilon_1/2) \supset \partial B(0,1)$. As for any minor M of order $\min(n,m)$ and any $X \in \mathbb{M}^{n\times m}$ the set $\{Y \ ; \ M(X - Y) \neq 0\}$ is open and dense in $\mathbb{M}^{n\times m}$, we can certainly find a set $\mathcal{A}_1$ with $\operatorname{card}(\mathcal{A}_1) = \operatorname{card}(\mathcal{A}_0)$, $\mathcal{A}_0 \subset B(\mathcal{A}_1, \varepsilon_1/8)$ and $\operatorname{rank}(A_i - A_j) = \min(m,n)$ if $A_i, A_j \in \mathcal{A}_1$ are different. Since we have to consider only finitely many pairs, $\operatorname{rank}(B_i - B_j) = \min(m,n)$ is preserved provided $\varepsilon_2 > 0$ is small enough and $B_i \in B(A_i, \varepsilon_2)$ for all i. Finally, we claim that $(\mathcal{A}_1, \varepsilon)$, where $\varepsilon = \min(\varepsilon_2, \varepsilon_1/8)$ is the promised pair.

Indeed, if we fix arbitrary $B_i \in B(A_i, \varepsilon)$ then (i) was already verified. Put $M = \{B_i \ ; \ i \leq \operatorname{card}(\mathcal{A}_1)\}$, then $\partial B(0,1) \subset B(\mathcal{A}_1, 5\varepsilon_1/8) \subset B(M, \varepsilon_1)$. So, by the choice of ε_1 we find $S \subset M$ which has for $B(X_0, R) = B(0, 1/2)$ all properties (i),...,(iv) stated in the proof of Theorem 2. We set $U = B(0,1/2) \cup \bigcup_{X\in S} \operatorname{int}(\mathcal{C}_X)$, where $\mathcal{C}_X = (\{X\} \cup \mathcal{M}_X^c)^{lc} = (\{X\} \cup \mathcal{M}_X)^c$, $K = S$ and finish the proof by showing that the assumptions of Proposition 3.17 are satisfied. As $0 \in U$ we would then of course obtain f with $\nabla f(x) \in K$ a.e. but with zero boundary data and hence nonaffine. For this purpose it is enough to verify that gradients in $\mathcal{U}$ are stable only near $\mathcal{K}$. in other words we will check the existence of a $\delta_0 > 0$ such that for each $A \in U \setminus \bar{B}(K, \varepsilon)$ there is a rank-one matrix $D \in \mathbb{M}^{n\times m}$ with $|D| \geq \min(\delta_0, \varepsilon/4nm)$ and $A + tD \in U$ for $|t| \leq 1/2$.

As $\bar{B}(0, 1/2) \subset \operatorname{int}(U)$, we find δ_0 with $\bar{B}(0, 1/2 + \delta_0) \subset U$ and so we need only to consider $A \in \operatorname{int}(\mathcal{C}_X)$ for some $X \in K$. But then $A - X = \sum_{Y\in\mathcal{M}_X} \lambda_Y (Y - X)$ and as $\operatorname{card}(\mathcal{M}_X) \leq 4nm$ we find $Y_A \in \mathcal{M}_X$ with $\lambda_{Y_A}|Y_A - X| \geq |A - X|/4nm > \varepsilon/4nm$. Obviously, $D = \min(1, \delta_0/\lambda_{Y_A}|Y_A - X|)\lambda_{Y_A}(Y_A - X)$ is a sufficiently large rank-one matrix. Moreover, the following geometric observation ensures that $[A - D/2, A + D/2] \subset U$. It says that if $A - X = \sum_{P\in\mathcal{M}_A} \lambda_P (P - X)$ with $\mathcal{M}_A \subset \mathcal{M}_X$, $\sum \lambda_P \leq 1$ and if $A \in \operatorname{int}(\mathcal{C}_X)$ then $\tilde{B} \in U$ whenever $\tilde{B} - X = \sum_{P\in\mathcal{M}_A} \tilde{\lambda}_P (P - X)$ with $\sum \tilde{\lambda}_P \leq 1$ and $\tilde{\lambda}_P > 0$ for all P. Indeed, we can of course suppose $\tilde{B} \notin \mathcal{M}_X^c \subset U$ and hence $\sum \tilde{\lambda}_P < 1$. We fix $\eta > 0$ such that $B(A, \eta) \subset \mathcal{C}_X$ and $\kappa \in (0, 1/2(1 - \sum \tilde{\lambda}_P))$ with $\kappa \max_P \lambda_P < \min_P \tilde{\lambda}_P$. Then we can represent each $C \in B(\tilde{B}, \kappa\eta)$ in the form

$$C - X = \kappa(A + \frac{C - \tilde{B}}{\kappa} - X) + ((\tilde{B} - X) - \kappa(A - X)) \in \kappa(\mathcal{C}_X - X) + ((\tilde{B} - X) - \kappa(A - X)) \subset \mathcal{C}_X - X$$

where the last inclusion is checked easily. Therefore, we are done. □

COROLLARY 4.28. *Let $\mathcal{U} \subset \mathbb{M}^{n\times m}$ be an open bounded set. If we choose countably many points in $\mathcal{U}$ randomly with respect to the (normalized) Lebesgue measure on $\mathcal{U}$, then almost surely a finite non-rigid set without rank-one connections can be found among the choosen points. The same holds true if the countably set is typical in the sense of Baire category, e.g. considering the Hausdorff distance.*

Proof. The fact that such "generic" countable sets do not contain rank-one connections is fairly obvious. Our stability result in Theorem 4.27 say that counterexamples to our statement have to avoid some open set of finite subconfigurations. Of course, a generic and infinite set will violate such a restriction. □

2. The four gradient problem

In this section we show that four point configurations can never form a properly non-rigid set. After a dimension reduction argument in Lemma 4.29 we state in Lemma 4.30 some facts from linear algebra which allow a further reduction of the cases necessary to be considered. In Lemma 4.31 we treat the degenerate case, the argument is presumably folklore as the rigidity of the classical Tartar square was well known among experts - see e.g. Subsection 2.5 in [**64**]. Next we recall a very important result about regularity properties of solutions to the elliptic Monge-Ampère equation without appriori convexity assumptions. This allows us a final reduction to the case of the hyperbolic Monge-Ampère equation treated in Chapter 2.

LEMMA 4.29. *Let $\tilde{\mathcal{A}} \subset \mathbb{M}^{n\times m}$ be finite and without rank-one connections. Suppose there exist a nonaffine lipschitz map on a domain $\tilde{f} : \tilde{\Omega} \subset \mathbb{R}^m \to \mathbb{R}^n$ such that $\nabla \tilde{f} \in \tilde{\mathcal{A}}$ almost everywhere. Then there are $g \in \mathbb{M}^{2\times m}$ and $h \in \mathbb{M}^{n\times 2}$ and a lipschitz map f from $[0,1]^2$ into $\mathbb{R}^2$ such that*

- *$\mathcal{A} = \{h \circ A \circ g \ ; \ A \in \tilde{\mathcal{A}}\}$ does not contain any rank-one connection*
- *$\nabla f \in \mathcal{A}$ almost everywhere and f is nonaffine.*

Proof. First we note that for any fixed $M \in \mathbb{M}^{n\times m}$ of rank at least two, the set of those $(g,h) \in \mathbb{M}^{2\times m} \times \mathbb{M}^{n\times 2}$ satisfying $\operatorname{rank}(h \circ M \circ g) = 2$ is an open and dense subset of $\mathbb{M}^{2\times m} \times \mathbb{M}^{n\times 2}$. Indeed, we consider the polynomial mapping $\Phi : (h,g) \to \det(h \circ M \circ g)$. This map does obviously not vanish identically, so it is different from zero on an open dense set in $\mathbb{M}^{2\times m} \times \mathbb{M}^{n\times 2}$, as was claimed.

Next, we observe that $\tilde{f}$ nonaffine implies the existence of $y, z \in B(x,r) \subset B(x,3r) \subset \tilde{\Omega}$ with $\tilde{f}(y+z-x) + \tilde{f}(x) \neq \tilde{f}(y) + \tilde{f}(z)$. We choose $g_0 \in \mathbb{M}^{n\times 2}$ and $h_0 \in \mathbb{M}^{2\times m}$ such that

$$g_0(e_1) = y - x, g_0(e_2) = z - x \text{ and } h_0(\tilde{f}(y+z-x) + \tilde{f}(x) - \tilde{f}(y) - \tilde{f}(z)) \neq 0.$$

As we have only finitely many pairs in $\tilde{\mathcal{A}} \times \tilde{\mathcal{A}}$, we find arbitrarily close to (g_0, h_0) a pair (g,h) and an $\varepsilon > 0$ with $\operatorname{rank}(h \circ (A - B) \circ g) = 2$ for all $A, B \in \tilde{\mathcal{A}}$ different and such that still

$$h(\tilde{f}(g(e_1+e_2) + \hat{x}) + \tilde{f}(\hat{x})) \neq h(\tilde{f}(g(e_1) + \hat{x}) + \tilde{f}(g(e_2) + \hat{x})) \text{ if } |x - \hat{x}| < \varepsilon.$$

Fubini's theorem implies that for almost every translation $\hat{x} \in \mathbb{R}^m$ we have for a.e. $p \in \mathbb{R}^2$ the inclusion $\nabla \tilde{f}(g(p) + \hat{x}) \in \tilde{\mathcal{A}}$ provided $g(p) + \hat{x} \in \tilde{\Omega}$. We pick such a translation x_0 with $|x - x_0| < r, \varepsilon$ and see that the map $f : p \in [0,1]^2 \to h(\tilde{f}(g(p) + x_0))$ does the job. □

LEMMA 4.30. *Consider 2×2-matrices $A_1 = 0$, $A_2 = Id$, A_3 and A_4 with $\min_j \det(A_j) < 0$. Then one of the following two cases occurs.*

a) *There are $v, w \in \mathbb{S}^1$ such that $w \| A_j v$ for $j = 1, \ldots, 4$.*
b) *There exists a regular matrix $P \in \mathbb{M}^{2\times 2}$, a matrix $S \in \mathbb{M}^{2\times 2}_{sym}$ and a real D such that $PA_j - S$ is symmetric and $\det(PA_j - S) = D$ for $j = 1, \ldots, 4$.*

Proof. At the very beginning, we notice that condition a) remains unchanged if we pre- and postmultiply all A_j with the same fixed matrices. First, we try to find the regular matrix P making all PA_j symmetric. As $A_1 = 0$, we see that requiring symmetry of PA_j gives only three linear constraints and consequently there is a nonzero matrix P in the at least onedimensional "kernel" of these conditions. Since $P = PA_2$, it is itself symmetric. Now suppose that P is singular, after multiplication with a suitable real we can suppose $P = u \otimes u \neq 0$. We put $w = v = \mathrm{i} \cdot u$ in complex notation and claim that we are in case a) now. Indeed, take any $j \in \{3, 4\}$ and $y \in \mathbb{R}^2$. Since $(u \otimes u)A_j$ is symmetric, we see that $\langle y, ((u \otimes u)A_j)v\rangle = \langle (u \otimes u)A_j(y), v\rangle = \langle A_j(v), u \otimes u(v)\rangle = 0$. Hence $0 = ((u \otimes u)A_j)v = u\langle u, A_j v\rangle$ which shows that $A_j(v) \| v$.

Consequently, we can suppose $\det(P) \neq 0$ and set $\tilde{A}_j = PA_j$. Searching now for the suitable $S \in \mathbb{M}^{2\times 2}_{sym}$ fulfilling b) we obtain the equivalent conditions $\det(\tilde{A}_j - S) = \det(\tilde{A}_1 - S) = \det(S)$, i.e. $\langle \operatorname{cof} \tilde{A}_j, S\rangle = \det(\tilde{A}_j)$ for $j = 2, 3, 4$, and $\langle M, S\rangle = 0$ where $M \hat{=} \mathrm{i}$. Obviously, these four linear conditions have a simultaneous solution establishing case b), provided the conditions are linearly independent. Since all matrices $\operatorname{cof} \tilde{A}_j$, $j = 2, 3, 4$ are orthogonal to M, this can fail only if $\operatorname{cof} \tilde{A}_j$, $j = 2, 3, 4$ are linearly dependent.

It is clear that we then find a $\lambda \in \mathbb{R}^3 \setminus \{0\}$ with $\sum_{j=1}^{3} \lambda_j \tilde{A}_{j+1} = 0$. Due to our assumptions we find $j_0 \geq 2$ such that $\det(\tilde{A}_{j_0}) > 0$ and can even suppose $\tilde{A}_{j_0} > 0$. Replacing $\tilde{A}_j$ by $\sqrt{\tilde{A}_{j_0}^{-1}}\tilde{A}_j\sqrt{\tilde{A}_{j_0}^{-1}}$ and reshuffling the indices, we can in addition assume that $\tilde{A}_2 = Id$ again. Now, if $\lambda_3 = 0$ we see that $\tilde{A}_3$ is a multiple of the identity and hence that $w = v$ an eigenvector of $\tilde{A}_4$ brings us into situation a). Else we have $\tilde{A}_4 \in \operatorname{span}(\{\tilde{A}_2, \tilde{A}_3\})$ and hence any eigenvector of $\tilde{A}_3$ does what is needed. □

LEMMA 4.31. *Assume $\mathcal{A} \subset \mathbb{M}^{2\times 2}$ does not contain any rank-one connection and that there are $v, w \in \mathbb{S}^1$ with $w \| Av$ for all $A \in \mathcal{A}$. If the lipschitz map $f : [0, 1]^2 \to \mathbb{R}^2$ satisfies $\nabla f \in \mathcal{A}$ almost everywhere, then f is necessarily affine.*

Proof. We again use complex notations, e.g. identifying i with a 90-degree rotation, and set $M = \{x \in (0, 1)^2 \ ; \ \nabla f(x) \in \mathcal{A}\}$, so M is of full measure. Note that, since $\mathcal{A}$ does not contain any rank-one connection, we infer from $A, B \in \mathcal{A}$ and $(\mathrm{i}w)^\top \cdot A = (\mathrm{i}w)^\top \cdot B$ or $A \cdot v = B \cdot v$ that $A = B$.

Due to our assumption, the map $x \to \langle \mathrm{i}w, f(x)\rangle$ is constant in direction v and hence the same is true for its gradient $x \to \nabla\langle \mathrm{i}w, f(x)\rangle = (\mathrm{i}w)^\top \cdot \nabla f(x)$. By what we told in the beginning, we see that ∇f is constant along the intersection of lines in direction v with M. This means that f is affine along such lines where M has full measure and so Fubini's theorem together with the continuity shows that f is affine along all lines in direction v. Continuity now implies as well that the slope of f must be the same on nearby v-lines and hence by the remark used already once, ∇f must indeed be constant. For the reader prefering a more analytical argument we give a simple calculation using that the second distributional derivatives commute and yielding this way

$$D(Df \cdot v) \cdot d = D(Df \cdot d) \cdot v = (D(Df) \cdot v) \cdot d \equiv 0 \text{ for all } d \in \mathbb{S}^1,$$

hence $Df \cdot v = \nabla f \cdot v$ is constant and then the same must hold for ∇f itself almost everywhere. □

A very important ingredience in our solution of the four gradient problem is the following result from [**88**], to be more precise the first part is just Theorem 4 of [**88**] and as noted in that paper, the second statement follows then from regularity results for convex solutions of the Monge-Ampère equation (see [**18**]).

THEOREM 4.32. *Let $\Omega \subset \mathbb{R}^2$ be an open set and $F \in W^{2,\infty}(\Omega, \mathbb{R})$ be such that $\det D^2F(x) = u(x) \geq \varepsilon > 0$ for a.e. $x \in \Omega$. Then F is locally either convex or concave. If $u \in \mathcal{C}^\infty(\Omega)$ then $F \in \mathcal{C}^\infty(\Omega)$.*

THEOREM 4.33. *(see also Theorem 6 in* [**22**]*) Let us be given four matrices* $A_1, \dots, A_4 \in \mathbb{M}^{n\times m}$ *with* $\operatorname{rank}(A_i - A_j) \neq 1$ *for all* i, j*. If* $f : \Omega \to \mathbb{R}^n$*,* $\Omega \subset \mathbb{R}^m$ *a domain, is a lipschitz map with* $\nabla f(x) \in \{A_1, \dots, A_4\}$ *almost everywhere, then* f *is necessarily affine.*

Proof. We can of course assume that $n = m = 2$ as Lemma 4.29 tells us that any counterexample leads to one in this lowerdimensional situation. It is also clear that we can suppose for our f that $|\{x \in \Omega \ ; \ \nabla f(x) = A_1\}| > 0$. Adding the affine map $x \to -A_1 \cdot x$ to f and postmultiplying with the (well defined) matrix $(A_2 - A_1)^{-1}$ we can also request that $A_1 = 0$, $A_2 = Id$. Now, if $\det(A_3), \det(A_4) \geq 0$ then both determinants are in fact positive, and hence f is a mapping with bounded distortion which has gradient zero on a set of positive measure. By Corollary 2 in §10.1 in Chapter II of [**79**], f has to be affine and we are happy. This shows that we are in a position to apply Lemma 4.30, note that case a) there is taken care of by Lemma 4.31. Therefore, we forget the request $A_1 = 0$, $A_2 = Id$ and make the new assumption that

$$A_i \in \mathbb{M}^{2\times 2}_{sym} \text{ and } \det(A_i) = D \text{ for some } D \in \mathbb{R}.$$

If $D > 0$, then Theorem 4.32 implies that the potential F of f is $\mathcal{C}^\infty$ and hence $\nabla f = D^2 F$ can not jump between the four possible values. Consequently, we can suppose $D \leq 0$ and moreover, because nonaffinity is a local property, that f is defined on $B(0,1)$. But now Theorem 2.28 together with Corollary 2.33 ensure that f is affine. □

CONSTRUCTION 4.34. Even if we have just shown that all non-rigid 4 point configurations are trivial, our proof also indicates which of the non-trivial 4-point configurations might be the "most non-rigid". Indeed, configurations of the degenerate type discussed in Lemma 4.31 seem to be very rigid - independent of their cardinality. All other configurations are are on a hyperboloid of type $\mathcal{H}_D = \{M \in \mathbb{M}^{2\times 2}_{sym} \ ; \ \det(M) = D\}$ and here the case $D > 0$ seems to be more rigid than $D < 0$. In fact, all exact solutions for $D > 0$ are due to Theorem 4.32 $\mathcal{C}^\infty$-smooth, whereas the setting of Chapter 2 admits much rougher exact solutions, even having singularities. The difference can also be seen on the level of laminates or even polyconvex Young measures. Since $\det(X - Y) > 0$ whenever $X - Y \in \mathcal{H}_D$ different and $D > 0$, we know from [**90**] that all polyconvex Young measures supported in $\mathcal{H}_D$ are Diracs. On contrast, for $D < 0$ there are through each point in $\mathcal{H}_D$ two rank-one lines entirely contained in $\mathcal{H}$. Moreover, the special geometry of the one-sheeted hyperboloid $\mathcal{H}_D$ allows then the following improvement of the classical "Tartar square". In fact, four points in $\mathcal{H}_D$ can in the same moment generate four different squares.

To be more precise, we indentify via (restricted) conformal coordinates, compare with Paragraph 1.10 in Chapter 1, $\mathbb{M}^{2\times 2}$ with $\mathbb{R}^3$, so

$$M = \begin{pmatrix} z + x & y \\ y & z - x \end{pmatrix} = z_{\mathcal{H}} + (x + \mathrm{i}y)_{\bar{\mathcal{H}}} \sim p = (x, y, z) \in \mathbb{R}^3 \text{ and } \det(M) \sim \widehat{\det}(p) = z^2 - x^2 - y^2.$$

Now it is easily checked that given $p = (x, y, z) \in \mathcal{H}_{-1}$ the two lines in direction $(a, b, 1)$, where $(a + \mathrm{i}b) = (z \pm \mathrm{i})/(x - \mathrm{i}y)$, through p are rank-one lines staying inside the hyperboloid. We choose $P_1 = (x, y, z)$ and $P_2 = (-x, -y, z)$, $z > 1$ on $\mathcal{H}_{-1}$, then it is not difficult to verify that the line l_1^u through p_1 in direction $(\frac{z+\mathrm{i}}{x-\mathrm{i}y}, 1)$ and the line l_3^u through p_2 in direction $(\frac{-z+\mathrm{i}}{x-\mathrm{i}y}, 1)$ intersect in some point $R \in \mathcal{H}_{-1}$ with $\langle R, e_3\rangle < z$. Symmetry reasons ensure that also l_2^u through P_1 in direction $(\frac{z-\mathrm{i}}{x-\mathrm{i}y}, 1)$ and l_4^u through P_2 in direction $(\frac{-z-\mathrm{i}}{x-\mathrm{i}y}, 1)$ intersect in $R' \in \mathcal{H}_{-1}$. A simple calculation gives that $\langle R, e_3\rangle = \langle R', e_3\rangle = z - ((x^2 + y^2)/z) = -1/z > -z$. We say that the polygon P_1, R, P_2, R', P_1 forms the upper boundary, and repeat the same construction in the lower part of the hyperboloid. However, in order to avoid intersection of the lower and the upper boundary, we rotate the lower part by 90° in the xy-plane. So $P_3 = (-y, x, -z)$ and $P_4 = (y, -x, -z)$.

As the two pictures in Figure 1 show, do the upper and the lower boundary together enclose an relatively open strip in $\mathcal{H}_{-1}$. Since all the "boundary" segments are in rank-one directions ,

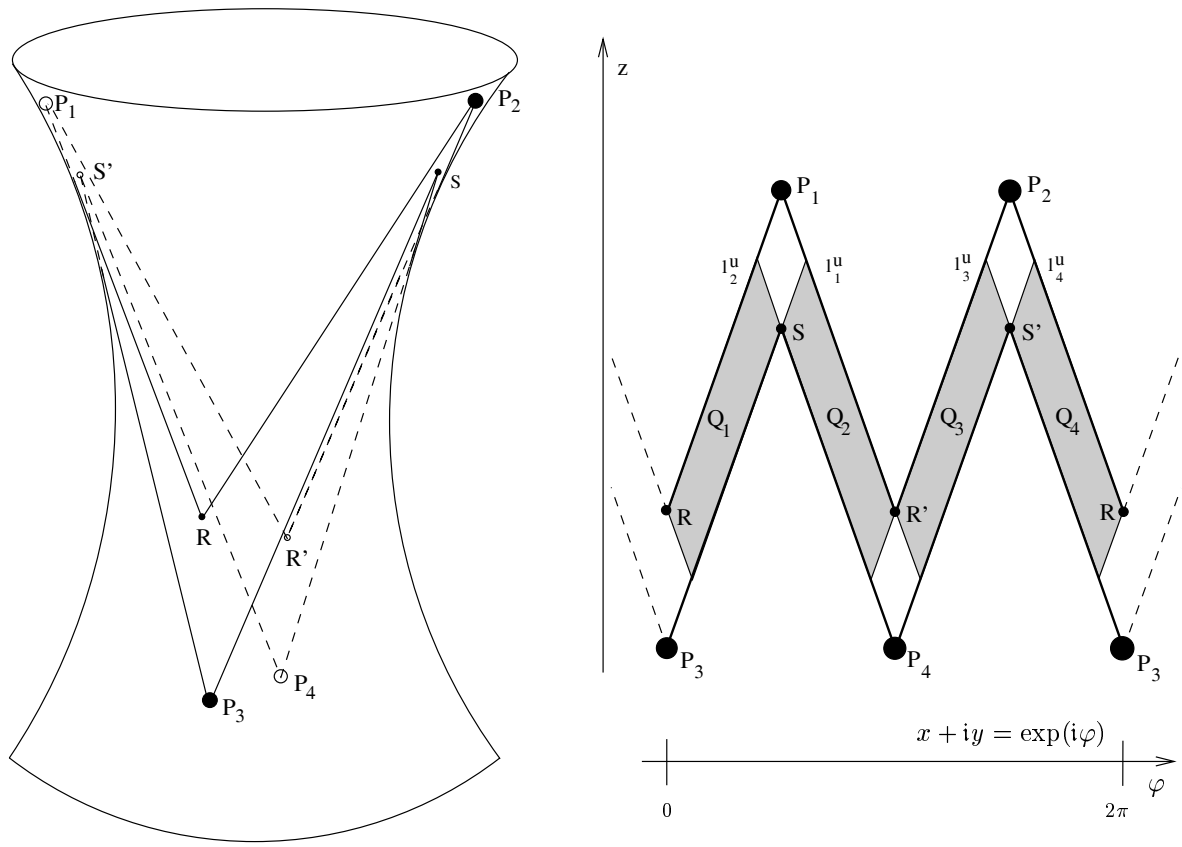


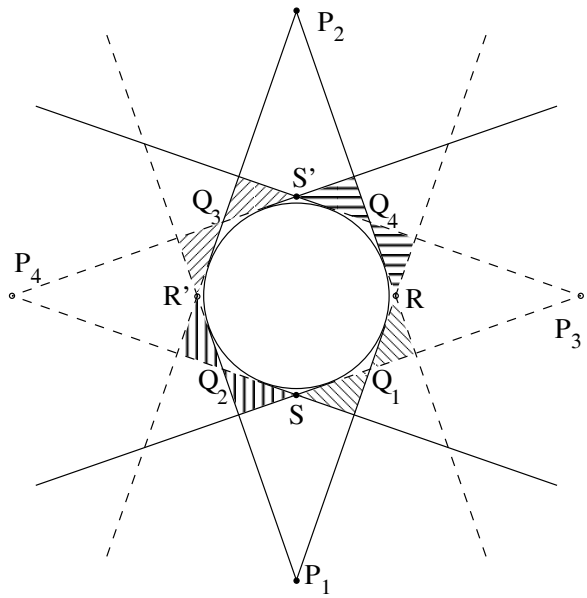
FIGURE 1. The fourfold Tartar configuration

we see that the concave corners R, R', S, S' fail to be rank-one convex extreme points of the closed strip. So, $P_1, \ldots, P_4$ are the only rank-one convex extreme points and hence due to Lemma 4.16 the whole closed strip is contained in (and in fact equal to) $\{P_1, \ldots, P_4\}^{rc}$. Another way to see this is to recognise the four Tartar squares $Q_1, \ldots, Q_4$ in the picture. As they belong into $\{P_1, \ldots, P_4\}^{rc}$, it is now easy to see that the whole strip does as well.

Concerning non-rigidity, this configuration is certainly more promissing than the original Tartar square, not just because it is nondegenerate in the sense of Lemma 4.31 or since it contains at once four squares. The main advantage is that the (relative) interior of the hull reaches up to the generating extreme points. This is in some sense the freedom necessary to modify gradients and to make sure that they are nowhere, except near the extreme points, stable. However, we know from Corollary 2.7 that openess relative to the symmetry together with the determinant

constraints does not give enough freedom to modify gradients locally. In the following section we will therefore show how adding a fifth point enables us to "pop up" this configuration into the full space of symmetric matrices, were openess already allows the splitting of gradients.

Following a suggestion by V.Šverák, we visualize in the picture on the right the fourfold Tartar configuration in a different way. We consider the orthogonal projection into the xy-plane, so the waist of the hyperboloid becomes the inner circle of the picture. Rank-one segments in the upper half of the hyperboloid $\mathcal{H}_{-1}$ are depictured as full lines, those in the lower part dashed. The four Tartar squares twist into four "butterflies".



3. The five gradient problem

Here we will present a non-rigid five point configuration in $\mathbb{M}^{2\times 2}_{sym}$ without rank-one connections which was obtained jointly with David Preiss, see [**46**]. Moreover, it turns out that these properties of our configuration are stable under small pertubations inside $\mathbb{M}^{2\times 2}_{sym}$. Throughout the whole section we again use the identification of $\mathbb{M}^{2\times 2}_{sym}$ and $\mathbb{R}^3$ via (restricted) conformal coordinates as given in Construction 4.34 or Paragraph 11.10. So we set

$$M = \begin{pmatrix} z+x & y \\ y & z-x \end{pmatrix} = z_{\mathcal{H}} + (x+\mathrm{i}y)_{\bar{\mathcal{H}}} \sim p = (x,y,z) \in \mathbb{R}^3 \text{ and } \det(M) \sim \widehat{\det}(p) = z^2 - x^2 - y^2.$$

Since our final set is a union of several segment, triangles and simplexes, we use the short and more intuitive notation $[A_1, \dots, A_k]$ for the convex hull of the set $\{A_1, \dots, A_k\}$.

Construction 4.35. We choose the following left, right, front and back points on the one-sheeted hyperboloid $\mathcal{HYP} = \mathcal{H}_{-1} = \{(x,y,z) \in \mathbb{R}^3 \ ; \ z^2 - x^2 - y^2 = -1\}$:

$$P_{L_0} = (0, -2, \sqrt{3}), P_{R_0} = (0, 2, \sqrt{3}), P_{F_0} = (2, 0, -\sqrt{3}), \text{ and } P_{B_0} = (-2, 0, -\sqrt{3}).$$

Suppose now that P_L, P_R, P_F and P_B are points on $\mathcal{HYP}$ sufficiently close to their corresponding original points just selected. We are going to define various further auxilary points, lines, planes and bodies. If not explicitely mentioned otherwise, these objects will continuously depend on the four points $P_L, \dots, P_B$ considered. Consequently, their existence, and certain further properties we are interested in, follow once they are established for the original quadrupel $(P_{L_0}, \dots, P_{B_0})$. All together, after adding a fifth point to our configuration we will obtain a figure like presented in the picture below. Therefore, we will refer to parts of the figure as left arm, front leg etc. For $X \in \{F, B\}$ and $Y \in \{R, L\}$ we select the rank-one lines d_{XY} in $\mathcal{HYP}$ trough P_X with direction close to $(d_1^X, d_2^Y, 2)$ where $d_1^F = -d_1^B = -\sqrt{3}$ and $d_2^R = -d_2^L = 1$. Similar, d_{YX} is the rank-one line in $\mathcal{HYP}$ trough P_Y with direction close to $(\tilde{d}_1^X, \tilde{d}_2^Y, -2)$, $\tilde{d}_1^F = -\tilde{d}_1^B = 1$ and $\tilde{d}_2^R = -\tilde{d}_2^L = -\sqrt{3}$. As all of these lines run in a 2-dimensional surface, we can in a continuous way define for c_1, c_2 the point $P_{c_1c_2}$ as the intersection of the line $d_{c_1c_2}$ with the line $d_{c_2\bar{c}_1}$, where $\{c_1, \bar{c}_1\} \in \{\{F, B\}, \{R, L\}\}$

and $c_2 \in \{F, B, L, R\} \setminus \{c_1, \bar{c}_1\}$. These are precisely the (four plus eight) points already considered in the fourfold Tartar-configuration presented in Section 2. Note that always $P_{\bar{c}_2 c_1} \in [P_{c_1}, P_{c_1 c_2}]$.

As our final configuration consists of five points, we have to break the symmetry kept until now. First, we choose a point $P_K \in \{p \in \mathbb{R}^3 \ ; \ \langle e_2, p\rangle = 0\}$ close to $P_{K_0} = (0, 0, 2 - \sqrt{3})$ such that $\operatorname{rank}(P_K - P_F) = \operatorname{rank}(P_K - P_B) = 1$. The required continuous dependence is an easy consequence of the implicit function theorem. Further, we define two points in $\mathcal{HYP}$ by

$$\{R\} = d_{FR} \cap d_{BR}, \text{ close to } (0, 2/\sqrt{3}, 1/\sqrt{3}) \text{ and } \{L\} = d_{FL} \cap d_{BL}, \text{ close to } (0, -2/\sqrt{3}, 1/\sqrt{3}).$$

Moreover, we set $\Delta = [R, L, P_K]$ - then P_F and P_B are stricly separated by the plane containing Δ. Hence, we can already define the two legs $\mathcal{L}_B = [P_B, R, L, P_K]$ and $\mathcal{L}_F = [P_F, R, L, P_K]$ with their feets P_B and P_F and their union $\mathcal{L} = \mathcal{L}_B \cup \mathcal{L}_F$. The description of the lower part of the figure is completed by setting for $X \in \{F, B\}$

- λ_X^∞ is the plane containing P_X, P_{XR} and P_{XL}, or equivalently containing P_X, R, and L.
- λ_X^- is the halfspace below λ_X^∞, so more formally $\lambda_X^- = \{p + te_3 \ ; \ p \in \lambda_X^\infty \text{ and } t < 0\}$.
- $\mathcal{L}_X^\infty = \bigcup_{s>0} P_X + s(\mathcal{L}_X - P_X)$ is the extension of the leg $\mathcal{L}_X$ to a cone.
- Note that $P_{\bar{X}} \in \lambda_X^-$ and also that the ray from P_X through P_K intersects $\lambda_{\bar{X}}^\infty$ in an inner point of $[P_{\bar{X}}, R, L]$.
- For later use we also observe that $P_{R\bar{X}}, P_{L\bar{X}} \notin \mathcal{L}_X^\infty$, since $\{P_{Y\bar{X}}\} = d_{Y\bar{X}} \cap d_{\bar{X}\bar{Y}}$ and $d_{\bar{X}\bar{Y}} \cap \mathcal{L}_X^\infty = \{\bar{Y}\}$.

Now, we define the upper torso. For $Y \in \{L, R\}$ we set

- $\alpha_Y = [P_Y, P_{YF}, P_{YB}]$ and α_Y^∞ is the plane spanned by this triangle
- α_Y^- is again the halfspace below α_Y^∞, α_Y^+ the one above.

Then, given $Y \in \{L, R\}$ and $X \in \{F, B\}$ we denote by $\mathcal{S}_{XY} = \alpha_Y^- \cap \mathcal{L}_X^\infty$ the shoulder with upper corner P_{XY}.

Next, let L' be the intersection of α_L with $[L, R]$ and similar $\{R'\} = \alpha_R \cap [L, R]$. Both points are inner points of the segment $[L, R]$ continuously dependent on P_L, P_R, P_F, P_B. L'' is now the unique point on $[LR]$ and near $(0, -2 + \sqrt{3} - (1/\sqrt{3}), 1/\sqrt{3})$ which is rank-one connected to P_L, similar $R'' \in [LR]$ near $(0, 2 - \sqrt{3} + (1/\sqrt{3}), 1/\sqrt{3})$ is rank-one connected to P_R. Now we define for $X \in \{F, B\}$, $Y \in \{L, R\}$ the sets $\mathcal{A}_{YX} = [P_Y, Y', Y'', P_{YX}]$ and then we obtain the left and right arm setting $\mathcal{A}_Y = \mathcal{A}_{YF} \cup \mathcal{A}_{YB}$.

Before finishing our construction by adding the fifth point and the head coming with it, we establish some further geometrical properties of the parts already produced.

Lemma 4.36. *We have*

a) $\operatorname{int}(\mathcal{L}) = \operatorname{int}(\mathcal{L}_F \cup \mathcal{L}_B) = (\operatorname{int}(\mathcal{L}_F^\infty) \cap \lambda_B^-) \cup (\operatorname{int}(\mathcal{L}_B^\infty) \cap \lambda_F^-)$.

b) *For all ε positive and $Y \in \{L, R\}$ there is $\delta > 0$ such that $p \in \lambda_F^- \cap \lambda_B^-$, $\operatorname{dist}(p, \mathcal{A}_Y) < \delta$ and $\operatorname{dist}(p, \{P_{YF}, P_{YB}\}) > \varepsilon$ implies $x \in \operatorname{int}(\mathcal{L})$.*

c) *For all ε positive there is $\delta > 0$ such that $X \in \{F, B\}$, $Y \in \{L, R\}$ and $p \in \alpha_Y^+$ with $\operatorname{dist}(p, \{P_{XY}, P_{YX}\}) > \varepsilon$ and $\operatorname{dist}(p, \mathcal{S}_{XY} \cap (\lambda_X^\infty \cup \lambda_{\bar{X}}^\infty \cup \lambda_{\bar{X}}^+)) < \delta$ implies $p \in \operatorname{int}(A_Y \cup \mathcal{L})$.*

Proof. Considering the plane Δ^∞ spanned by Δ, we see that P_F and P_B are on different sides of this plane. Therefore, it is easy to check that $\operatorname{int}(\mathcal{L}) = \operatorname{int}(\mathcal{L}_B) \cup \operatorname{int}(\mathcal{L}_F) \cup \operatorname{int}_{\Delta^\infty}(\Delta)$. Moreover, since $\operatorname{int}_{\Delta^\infty}(\Delta) \subset \lambda_F^- \cap \lambda_B^-$, we see that $\operatorname{int}(\mathcal{L}_X) \cup \operatorname{int}_{\Delta^\infty}(\Delta) \subset \operatorname{int}(\mathcal{L}_X^\infty) \cap \lambda_{\bar{X}}^-$ with $\{X, \bar{X}\} = \{F, B\}$. Concerning the inverse implication, let $p \in \operatorname{int}(\mathcal{L}_X^\infty) \cap \lambda_{\bar{X}}^- \setminus (\operatorname{int}(\mathcal{L}_X) \cup \operatorname{int}_{\Delta^\infty}(\Delta))$. Note that this set equals to $\lambda_{\bar{X}}^- \cap (\operatorname{int}(\mathcal{L}_X^\infty) \setminus \mathcal{L}_X)$ and hence, is in fact open. Using $[\dots]$ also as the notation for the convex hull of infinte sets, we see that $p \in [(\mathcal{L}_X^\infty \cap \lambda_{\bar{X}}) \cup \{P_X\}] \setminus \mathcal{L}_X$. This implies $p \in [(\mathcal{L}_X^\infty \cap \lambda_{\bar{X}}) \cup \Delta] \subset [(\mathcal{L}_{\bar{X}} \cap \lambda_{\bar{X}}) \cup \Delta] \subset \mathcal{L}_{\bar{X}}$. Because the set considered was open, this last inclusion holds also for all p' near p, and hence $p \in \operatorname{int}(\mathcal{L}_{\bar{X}})$, So, a) is shown.

Considering statement b), if it fails we find points $p_k \in \lambda_F^- \cap \lambda_B^-$ with $\operatorname{dist}(p_k, \mathcal{A}_y) < 1/k$ and $\operatorname{dist}(p_k, \{P_{YF}, P_{YB}\}) \geq \varepsilon$ but $p_k \notin \operatorname{int}(\mathcal{L})$. We can of course assume $p_k \to p_\infty$, then $p_\infty \in \mathcal{A}_Y \cap \operatorname{clos}(\lambda_F^- \cap \lambda_B^-) = [Y', Y'', P_{YF}] \cup [Y', Y'', P_{YB}]$. Because $\operatorname{dist}(p_\infty, \{P_{YF}, P_{YB}\}) \geq \varepsilon$, $p_\infty \in \operatorname{int}_{\mathrm{rel}}(\mathcal{L}_B^\infty \cap \lambda_B^\infty) \cup \operatorname{int}_{\mathrm{rel}}(\mathcal{L}_F^\infty \cap \lambda_F^\infty)$, so surely $p_k \in \operatorname{int}(\mathcal{L}_F) \cup \operatorname{int}(\mathcal{L}_B)$ for k sufficiently large. But due to a) this implies $p_k \in \operatorname{int}(\mathcal{L})$, contradiction.

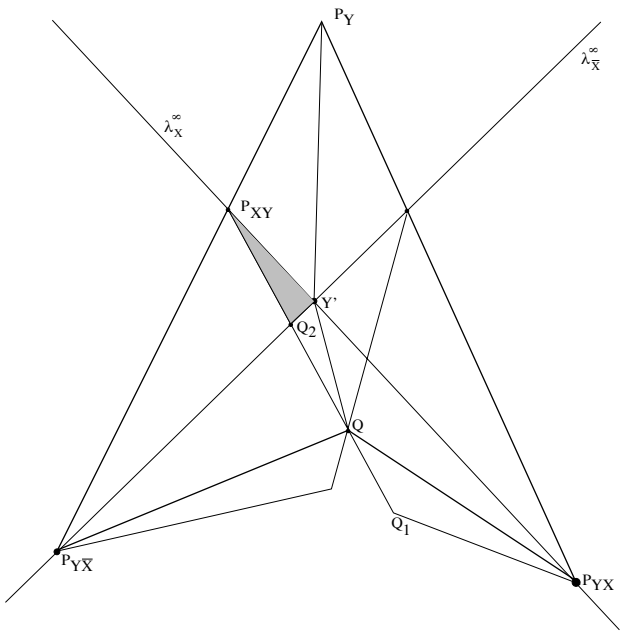
To verify c), we can similar to part b) suppose

$$p_k \in \alpha_Y^+ \setminus \operatorname{int}(\mathcal{A}_Y \cup \mathcal{L}) \to p_\infty \in \alpha_Y^\infty \cap (\lambda_X^\infty \cup \lambda_{\bar{X}}^\infty \cup \lambda_{\bar{X}}^+) \cap \overline{\mathcal{S}_{XY}} \setminus \{P_{XY}, P_{YX}\}$$

Because $Y'' \in \alpha_Y^+$ is contained in each of the convex sets $\mathcal{M} \in \mathscr{M} = \{\mathcal{A}_{YF}, \mathcal{A}_{YB}, \mathcal{L}_F, \mathcal{L}_B\}$, we see that any of the convex hulls $[(\mathcal{M} \cap \alpha_Y^\infty) \cup \{Y''\}]$ is contained in $\bigcup \mathscr{M} = \mathcal{L} \cup \mathcal{A}_Y$. Hence, we obtain the desired contradiction if we show that $p_\infty \in \operatorname{int}_{\alpha_Y^\infty}(\bigcup \mathscr{M} \cap \alpha_Y^\infty) = \operatorname{int}_{\alpha_Y^\infty}((\mathcal{L} \cup \mathcal{A}_Y) \cap \alpha_Y^\infty)$. It is clear that $(\mathcal{L} \cup \mathcal{A}_Y) \cap \alpha_Y^\infty$ contains the four triangles $[Y', P_Y, P_{YB}]$, $[Y', P_Y, P_{YF}]$,$[Y', Q, P_{YF}]$ and $[Y', Q, P_{YB}]$, where $[Q, Y'] = \Delta \cap \alpha_Y^\infty$.
To prove that p_∞ is indeed an interior point, we first have to obtain a better description of $\alpha_Y^\infty \cap \cap(\lambda_X^\infty \cup \lambda_{\bar{X}}^\infty \cup \lambda_{\bar{X}}^+) \cap \overline{\mathcal{S}_{XY}}$. Again by an initial calculation and a continuity argument, we find that the intersection of α_Y^∞ with the line through P_X and P_K takes place at the point $Q_1 \in (P_K, P_X) \subset \mathcal{L}^X$. Now it is easy to see that $\alpha_Y^\infty \cap \mathcal{L}_X^\infty = [P_{XY}, P_{YX}, Q_1]$. Since Q_1 and P_{XY} are on different sides of the plane Δ^∞ containing Δ and because $[Q_1, P_{XY}] \subset \partial\mathcal{L}_X^\infty \cap \alpha_Y^\infty$, it is also clear that $\Delta^\infty \cap [Q_1, P_{XY}] \subset \partial\delta \cap \alpha_Y^\infty$ and as $(Q_1, P_{XY}) \subset \lambda_X^- \subset \mathbb{R}^3 \setminus [L, R]$, we conclude $\{Q\} = (Q_1, P_{XY}) \cap \Delta$.

The intersection $\alpha_Y^+ \cap (\mathcal{L} \cup \mathcal{A}_Y)$ consists of the union of several triangles. This will make the argument slightly less comprehensible - however, a short look at the figure on the right should clarify the situation. Since $P_{XY} \in \lambda_{\bar{X}}^+$ and $Q \in \lambda_{\bar{X}}^-$, it is clear that $(P_{XY}, Q) \subset \lambda_X^-$ intersects the line trough $P_{Y\bar{X}}$ and Y' which equals $\alpha_Y^+ \cap \lambda_{\bar{X}}^\infty$. Because we already established at the very begin that $P_{Y\bar{X}}$ does not belong to the convex set $\mathcal{L}_X^\infty (\cap \alpha_Y^\infty)$, we infer that the intersection point Q_2 is in $(P_{Y\bar{X}}, Y')$. (In particular, the situation is properly depicted in the suggestive picure on the right). This shows that $[Q, P_{XY}, Y'] = [Q, Q_2, Y'] \cup [Q_2, P_{XY}, Y'] \subset [Q, P_{Y\bar{X}}, Y'] \cup [P_{Y\bar{X}}, P_Y, Y'] \subset \alpha_Y^+ \cap (\mathcal{L} \cup \mathcal{A}_Y)$.

Because $Q' \in \alpha_Y \cap \lambda_X^-$ and $P_Y \in \alpha_Y \cap \lambda_X^+$ are in different halfspaces of α_Y^∞ given by the line $\alpha_Y^\infty \cap \lambda_X = \operatorname{line}(P_{XY}, P_{YX})$, it is clear that $(P_{XY}, P_{YX}) \subset \operatorname{int}_{\alpha_Y^\infty}([P_{XY}, Q', P_{YX}, P_Y])$.



But now we have

$$\begin{aligned}
[P_{XY}, Q, P_{YX}, P_Y] &= [P_{XY}, P_{YX}, P_Y] \cup [P_{XY}, P_{YX}, Q] \\
&= [P_Y, P_{XY}, Y'] \cup [P_Y, Y', P_{YX}] \cup [P_{XY}, Y', Q] \cup [Y', P_{YX}, Q] \\
&\subset [P_Y, P_{Y\bar{X}}, Y'] \cup [P_Y, Y', P_{YX}] \cup [P_{XY}, Q, Y'] \cup [Q, Y', P_{YX}]
\end{aligned}$$

and, as already noticed, this last set is contained in $(\mathcal{L} \cup \mathcal{A}_Y) \cap \alpha_Y^\infty$.

It remains to show that also $\alpha_Y^\infty \cap (\lambda_{\bar{X}}^\infty \cup \lambda_{\bar{X}}^+)\overline{\mathcal{S}_{XY}} \setminus \{P_{XY}\} = [P_{XY}, Q_2, Y'] \setminus \{P_{XY}\}$ contains inner points only, but as this set is included in $\operatorname{int}_{\mathrm{rel}}([P_{Y\bar{X}}, Y', P_Y]) \cup [Y', Q_2]$ and since due to

$Q \in \lambda^-_{\bar{X}}$, $P_Y \in \lambda^+_{\bar{X}}$ the inclusion $[Y', Q_2] \subset \mathrm{int}_{\mathrm{rel}}([P_{Y\bar{X}}, Y', Q] \cup [P_{Y\bar{X}}, Y', P_Y])$ holds, we are done. □

COROLLARY 4.37.

a) *For each $\varepsilon > 0$ there is δ positive such that any $p \in \mathrm{int}(\mathcal{L})$ with $\mathrm{dist}(p, [L, R] \cup \{P_F, P_B\}) > \varepsilon$ is the centre of a rank-one segment of length δ contained in $\mathrm{int}(\mathcal{L})$.*

b) *For any $Y \in \{L, R\}$ and $\varepsilon > 0$ there is $\delta > 0$ such that each $p \in \mathrm{int}(\mathcal{A}_Y)$ satisfying $\mathrm{dist}(p, \{P_Y\} \cup [Y', Y'', P_{YF}] \cup [Y', Y'', P_{YB}]) > \varepsilon$ is the centre of a rank-one segment of length δ contained in $\mathrm{int}(\mathcal{A}_Y)$.*

Proof. If statement a) fails, we pick a sequence $p_k \in \mathrm{int}(\mathcal{L})$ disproving the existence of the required positive δ. Of course, we can assume $p_k \to p_\infty$ and hence $p_\infty \in \partial\mathcal{L} \setminus ([L, R] \cup \{P_F, P_B\})$.

In case that $p_\infty \in \partial\mathcal{L}_X \setminus \Delta$, we fix a radius $r > 0$ such that $\partial\mathcal{L}_X \cap B(p_\infty, 2r)$ is disjoint with Δ and contained in the union of at most two planes. Moreover, since all edges of $\mathcal{L}_X$ run in rank-one directions, we can find a rank-one direction d contained in all planes intersecting the ball. So, an r-segment parallel to d and centered in any $p_k \in B(p_\infty, r) \cap \mathrm{int}(\mathcal{L}_X)$ can not intersect $\partial\mathcal{L}_X$ and, therefore, entirely stays in $\mathrm{int}(\mathcal{L}_X)$. Hence, considering p_k for k large gives a contradiction.

Consequently, we can assume that $p_\infty \in \partial\mathcal{L} \cap \Delta$ and, switching to a subsequence if necessary, Lemma 4.36.a) ensures that all p_k are contained in $\mathrm{int}(\mathcal{L}^\infty_X)$ for some $X \in \{F, B\}$. Because $p_\infty \in \lambda^-_{\bar{X}}$, we conclude as before that for a suitable rank-one direction d and some $r > 0$ all p_k, k large enough, are centres of r-segments in direction d and contained in $\mathrm{int}(\mathcal{L}^\infty_X) \cap \lambda^-_{\bar{X}} \subset \mathrm{int}(\mathcal{L})$. This final contradiction finishes our proof.

We turn to part b). If the claim fails, we proceed as in part a) and use that the limiting point p_∞ is contained in those faces of $\partial\mathcal{A}_Y$ that are spanned by the rank-one edges of $\mathcal{A}_Y$. □

CONSTRUCTION 4.38 (continuation). We finish our construction by defining the head of the figure. Going back to our original configuration of points, we set $P_0 = (L'_0 + R'_0)/2 + \frac{1}{2}|L'_0 - R'_0|e_3$. Clearly, P_0 is rank-one connected to L'_0 and R'_0. Moreover, the rank-one connections from P_0 pass in L'_0 and R'_0 into $\mathrm{int}(\mathcal{L}_0)$. This shows that we can choose points P_{H_0} (near P_0) and $P_{1_0}, \dots, P_{4_0}$ such that $\widehat{\det}(P_{H_0} - P_{j_0}) = 0$ if $j = 1, \dots, 4$, that $[P_{1_0} \dots P_{4_0}] \subset \mathrm{int}(\mathcal{L}_0)$ and that $[L'_0 R'_0] \in \mathrm{int}[P_{H_0} P_{1_0} \dots P_{4_0}]$. Now for a five point configuration $\mathcal{K} = \{P_L, P_R, P_F, P_B, P_H\}$ sufficiently close to $\{P_{L_0}, P_{R_0}, P_{F_0}, P_{B_0}, P_{H_0}\}$ we know that $\mathcal{L}$ is sufficiently close to $\mathcal{L}_0$ and hence, we can again find $P_1, \dots, P_4$ rank-one connected to P_H and such that $[P_1, P_2, P_3, P_4] \subset \mathrm{int}(\mathcal{L})$ and $[L', R'] \subset \mathrm{int}([P_H, P_1, P_2, P_3, P_4])$. We denote by $\mathcal{H} = [P_H, P_1, P_2, P_3, P_4]$ the head of our figure.

After all this preparations, we will consider the open set

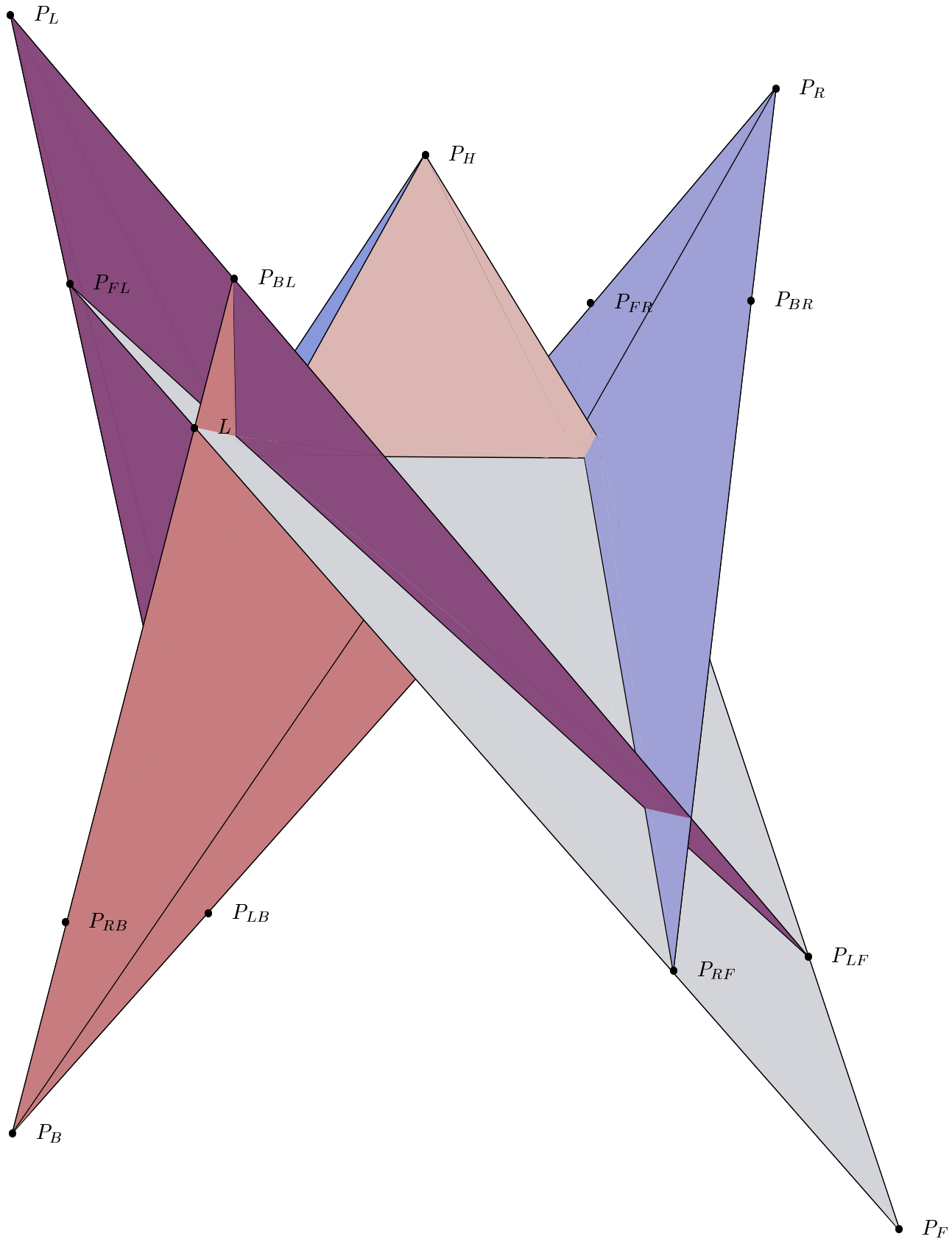
$$\mathcal{U} = \mathrm{int}(\mathcal{L} \cup \mathcal{A}_R \cup \mathcal{A}_L \cup \bigcup_{X=F,B} \bigcup_{Y=L,R} \mathcal{S}_{XY} \cup \mathcal{H})$$

together with our set $\mathcal{K}$ and show that symmetric gradients in $\mathcal{U}$ are stable only near $\mathcal{K}$. It can be checked that $\mathcal{U}$ is not starshaped with respect to any point, indeed a quite straightforward calculation gives that points above the bottoms of both legs are also above the tops of the two arms. Therefore to verify that gradients are stable only near $\mathcal{K}$, it is really helpful to use the function $\Phi_{\mathcal{U}}$ introduced in Definition 3.19.

THEOREM 4.39. *Consider a pair $\mathcal{U}, \mathcal{K}$ as obtained in the construction given above. Then for each $\varepsilon > 0$ there is a $\delta > 0$ such that $\Phi_{\mathcal{U}}(p) > \delta$ if $p \in \mathcal{U}$ and $\mathrm{dist}(p, \mathcal{K}) > \varepsilon$.*

Proof. First of all, we note that we need only to consider

$$p \in \mathrm{int}(\mathcal{L}) \cup \mathrm{int}(\mathcal{H}) \cup \bigcup_{X=F,B} \bigcup_{Y=L,R} \mathrm{int}(\mathcal{S}_{XY}) \cup \mathrm{int}(\mathcal{A}_{YX}).$$


P_L
P_R
P_H
P_{FL}
P_{BL}
P_{FR}
P_{BR}
L
P_{RB}
P_{LB}
P_{LF}
P_{RF}
P_B
P_F

Indeed, as $\Phi_{\mathcal{U}}$ is rank-one concave and hence also continuous on U, we can replace any $p \in \mathcal{U}$ by p' close to it increasing the value of $\Phi_{\mathcal{U}}$ as little as desired. Of course, we can suppose that $p' \in \mathcal{U}$ is not contained in the very small union of the boundaries of all the simplexes involved in building the figure. As it is still in the union of their closures, it has to be in the interior of one of them. Moreover, if the statement of the theorem fails,we can even assume that we have a whole sequence p_k staying in one of these interiors and away from $\mathcal{K}$ but such that $\Phi_{\mathcal{U}}(p_k) < 1/k$.

So, first we consider the case that all p_k are in $\mathcal{H}$. Applying the argument from the first part of the proof of Corollary 4.37.a), we see that a subsequence denoted in the same way satisfies $p_k \to p_\infty \in [P_1, P_2, P_3, P_4]$. Therefore, $p_\infty \in \operatorname{int}(\mathcal{L})$ which shows that $\lim_k \Phi_{\mathcal{U}}(p_k) = \Phi_{\mathcal{U}}(p_\infty) > 0$, contradiction.

Next, let all $p_k \in \operatorname{int}(\mathcal{A}_{YX})$, by Corollary 4.37 we have $p_k \in p_\infty \in [Y', Y'', P_{YX}]$. Moreover, since $[Y', Y''] \subset \operatorname{int}(\mathcal{H})$, we can suppose $p_\infty \notin [Y', Y'']$. This implies the existence of $r > 0$ such that $\partial\mathcal{A}_{YX} \cap B(p_\infty, 2r) \subset [Y', Y'', P_{YX}] \cup [P_Y, P_{YX}, Y'] \cup [P_Y, P_{YX}, Y'']$. In particular, a rank-one segment in direction $P_{YX} - P_Y$ through any of the inner points p_k, k large, can not intersect the second nor the third set of this union inside the ball $B(p_\infty, 2r)$. Note that $[Y', Y'', P_{YX}] \subset \lambda_Y^\infty$ and $p_\infty \in \lambda_Y^-$. In case that $p_\infty \neq P_{YX}$, Lemma 4.36.b) implies the existence of a $\delta > 0$ such that any $p_k \in U(p_\infty, r) \cap \operatorname{int}(\mathcal{A}_{YX}) \subset \lambda_Y^+$ is the centre of a segment in rank-one direction $P_{YX} - P_Y$, of length at least $\min(\delta, r)$ and contained in $\mathcal{U}$. So, we turn to the case $p_\infty = P_{YX}$. Then we can not ensure that a sufficiently large part of the ray starting at $p_k \in B(p_\infty, r)$ in direction $P_{YX} - P_Y$ stays in $\mathcal{U}$. However, each such "bad" ray intersects $[Y', Y'', P_{YX}]$ in a point different from P_{YX}, since P_{XY} in the unreachable part of $\partial\mathcal{A}_{YX}$. Obviously, the ray passes at this point from λ_X^+ into λ_X^- and in particular, again due to Lemma 4.36.b) into the set $\mathcal{L} \cap B(P_{YX}, 2r)$ which has for r sufficiently small a positive distance from $[L, R] \cup \{P_F, P_B\}$. Due to Corollary 4.37.a), $\Phi_{\mathcal{U}}$ is bounded away from zero on this set. As the ray of length r from p_k in the opposite direction $P_Y - P_{YX}$ does not intersect $\partial\mathcal{A}_{YX}$, it is entirely contained in $\mathcal{U}$. So $\Phi_{\mathcal{U}}$ has value at least $r^2/4$ at the middle of this segment. So, the rank-one concavity of $\Phi_{\mathcal{U}}$ bounds $\Phi_{\mathcal{U}}(p_k)$ away from zero for $k \to \infty$. This contradiction finishes the consideration of the arms.

We turn to the situation, when all p_k's belong to $\operatorname{int}(\mathcal{S}_{XY})$. As $\mathcal{S}_{XY} \subset \mathcal{L}_X^\infty$, we conclude as in the proof of Corollary 4.37 that $p_k \to p_\infty \in \alpha_Y$ and we know again that $p_\infty \neq Y'$. As before, we find a rank-one direction $d \in \{P_{XL} - P_X, P_{XL} - P_X, P_K - P_X\}$ and $r > 0$ such that $B(p_\infty, 2r) \cap \partial\mathcal{S}_{XY} \setminus \alpha_Y^\infty$ is included in (at most two) planes containing the direction d. Hence, segments in this direction can not intersect this part of the boundary. If $p_\infty \notin \{P_{YX}, P_{XY}\}$, then Lemma 4.36.c) ensures as in the considerations of the arms in the last paragraph that p_k, $k \geq k_0$ is the centre of an $\min(r, \delta)$-segment in $\mathcal{U}$ parallel to d. If $p_\infty = P_{YX}$, then $p_k \in \operatorname{int}(\mathcal{L}_X)$ for k large and $\operatorname{dist}(p_k, [L, R] \cup \{P_F, P_B\})$ bounded away from zero. Therefore, Corollary 4.37.a) gives what is needed. Finally, let $p_\infty = P_{XY}$. Now again, the p_k's are not centres of uniformly long rank-one segments in $\mathcal{U}$. But as for the arms, the backwards rays in direction $P_X - P_{XY}$ are sufficiently long. Since $P_{XY} \in \alpha_Y^\infty$ and $P_X \in \alpha_Y^-$, the forward rays in direction $P_{XY} - P_X$ lead by Lemma 4.36.c) into $B(P_{XY}, r) \cap \mathcal{A}_Y$. By Corollary 4.37.b), $\Phi_{\mathcal{U}}$ is bounded away from zero on this set. This gives again a uniform lower bound for $\Phi_{\mathcal{U}}$ on $\mathcal{S}_{XY}$.

Therefore, we are done with the shoulders and in the only remaining case we have to deal with all p_k's in $\operatorname{int}(\mathcal{L}_X)$. By Corollary 4.37.a) we infer that $p_k \to p_\infty \in [L, R]$. As $[L', R'] \subset \operatorname{int}(\mathcal{H})$, we see that $p_\infty \in [L, R] \setminus [L', R'] \subset \alpha_R^- \cup \alpha_L^-$. But if $p_\infty \in \alpha_Y^-$, then for k sufficently large $p_k \in \alpha_Y^-$ as well and therefore $p_k \in \operatorname{int}(S_{FY}) \cup \operatorname{int}(S_{BY})$. By what we have just shown, this implies $\liminf_k \Phi_{\mathcal{U}}(p_k) > 0$. So, our proof is finished. □

COROLLARY 4.40. *There is are five matrices $P_{F_0}, P_{B_0}, P_{L_0}, P_{R_0}, P_{H_0}$ in $\mathbb{M}^{2\times 2}_{sym}$ and an $\varepsilon > 0$ such that for arbitrary*

$$P_1 \in B(P_{F_0}, \varepsilon), P_2 \in B(P_{B_0}, \varepsilon), P_3 \in B(P_{L_0}, \varepsilon), P_4 \in B(P_{R_0}, \varepsilon) \text{ and } P_5 \in B(P_{H_0}, \varepsilon)$$

in $\mathbb{M}^{2\times 2}_{sym}$ *is* $\{P_1, \dots, P_5\}$ *a non-rigid set without rank-one connections.*

Proof. In fact, we can choose the just before constructed five points $P_{F_0}, P_{B_0}, P_{L_0}, P_{R_0}, P_{H_0}$ for our purpose. It is checked very easily that then the unperturbed set does not contain any rank-one connection. The same is true of course also for configurations sufficiently close to this one. It remains to show nonrigidity for all configurations close in $\mathbb{M}^{2\times 2}_{sym}$. First we observe that similar to the proof of Lemma 4.30 that $\det(P_j - S) = \det(P_1 - S)$, $j = 2, 3, 4$ and $S \in \mathbb{M}^{2\times 2}_{sym}$ is equivalent to $\langle \operatorname{cof}(P_1 - P_j), S\rangle = \det(P_1) - \det(P_j)$, $j = 2, 3, 4$. For the $(P_1, \dots, P_4)$ sufficiently close to $(P_{F_0}, \dots, P_{R_0})$ this system has a unique and also very small solution S since the linear dependence of $\operatorname{cof}(P_2 - P_1), \operatorname{cof}(P_3 - P_1)$ and $\operatorname{cof}(P_4 - P_1)$ would imply that the affine space containing $P_1, \dots, P_4$ is 2-dimensional. Consequently, $\tilde{P}_j = P_j - S$ will be very close to P_j for all $j \le 5$ and the common determinant of $\tilde{P}_1, \tilde{P}_2, \tilde{P}_3$ and $\tilde{P}_4$ will be very close to -1. Now a tiny rescaling making this determinant precisely -1 leads to a configuration very close to $(P_{F_0}, P_{B_0}, P_{L_0}, P_{R_0}, P_{H_0})$ and hence of the type as considered in Theorem 4.39. Consequently, this transformed and hence also the original configuration $P_1, \dots, P_5$ can not be rigid. □

Example 4.41. Finally, we will present a construction suggested by T.Iwaniec [**40**] which allows us to transform our exact solution for the just obtained non-rigid five point configuration into a function which is at almost each point of its domain holomorphic or antiholomorphic and even has vanishing gradient on a set of positive measure, but nevertheless ψ is not constant. More precisely, and using the concepts introduced in Notation 1.10, we will have ψ from a bounded domain $\Omega \subset \mathbb{C}$ into the plane which satisfies

$$\psi \in W^{1,p}(\Omega), p > 2, (\nabla\psi(z))_{\mathcal{H}} \cdot (\nabla\psi(z))_{\bar{\mathcal{H}}} = 0 \text{ a.e. in } \Omega, \text{ and } 0 < |\{z \in \Omega \,;\, \nabla\psi(z) = 0\}| < |\Omega|.$$

For the this purpose, we start from the unpertubed configuration $\mathcal{K}_0 = \{P_{F_0}, P_{B_0}, P_{R_0}, P_{L_0}, P_{H_0}\}$ and a non-affine lipschitz $\tilde{f} : [0,1]^2 \to \mathbb{R}^2$ with $\nabla\tilde{f} \in \mathcal{K}_0$ a.e. Due to Theorem 4.33, $\nabla\tilde{f}$ attains each $P \in \mathcal{K}_0$ on a set of positive measures. The crucial observation is that

$$\det(P_{B_0} - P_{F_0}) < 0 < \det(P_{L_0} - P_{F_0}), \det(P_{R_0} - P_{F_0}), \det(P_{H_0} - P_{F_0}).$$

So, after subtracting the linear map P_{F_0} and postmultiplication with a suitable matrix we have a lipschitz and non-affine

$$f : [0,1] \to \mathbb{C}, \nabla f \in \{\operatorname{diag}(1,-1), 0, A, B, C\} \text{ a.e. in } [0,1]^2, \text{ where } \det A, \det B, \det C > 0.$$

We consider the Borel sets $M_- = \{z \,;\, \nabla f(z) = \operatorname{diag}(1,-1)\}$, $M_0 = \{z \,;\, \nabla f(z) = 0\}$, and $M_+ = \{z \,;\, \nabla f(z) \in \{A, B, C\}\}$, which form a disjoint cover of almost all of $(0,1)^2$. As we intend to transform f via a quasiconformal map F, we define the Beltrami coefficients of F as follows.

$$\mu(z) = \begin{cases} (\nabla f(z))_{\bar{\mathcal{H}}} / (\nabla f(z))_{\mathcal{H}} & \text{if } z \in M_+ \\ 0 & \text{if } z \in \mathbb{C} \setminus M_+. \end{cases}$$

Because $\det(X) = |X_{\mathcal{H}}|^2 - |X_{\bar{\mathcal{H}}}|^2$ as stated in equation (1.3) in 1.10, we conclude that $\|\mu\|_\infty < 1 - \varepsilon$ for some $\varepsilon > 0$. Now, Bojarski's result [**12**] ensures the existence of global solution for this Beltrami equation, see also the representation in Chapter 10 of [**41**] which also covers a later improvement [**26**] of Bojarski's result by G.David. Anyhow, we know that there is a unique $F \in W^{2,p}_{loc}(\mathbb{C})$ which is a homeomorphism of $\mathbb{C}$ onto itself satisfying

$$(\nabla F(z))_{\bar{\mathcal{H}}} = \mu(z)(\nabla F(z))_{\mathcal{H}} \text{ almost everywhere in } \mathbb{C} \tag{4.7}$$

and the normalization condition $F(0) = 0$, $F(1) = 1$ and $\lim_{z\to\infty} F(z) - z = 0$. We introduce $\psi = f \circ F^{-1} : F((0,1)^2) \to \mathbb{C}$ and notice that

$$(\nabla f(z))_{\bar{\mathcal{H}}} = \mu(z)(\nabla f(z))_{\mathcal{H}} \text{ almost everywhere in } M_0 \cup M_+, \text{ and} \tag{4.8}$$

$$(\nabla f(z))_{\mathcal{H}} = (\nabla\psi(F(z)))_{\mathcal{H}} \cdot (\nabla F(z))_{\mathcal{H}} + (\nabla\psi(F(z)))_{\bar{\mathcal{H}}} \cdot \overline{((\nabla F(z))_{\bar{\mathcal{H}}})}, \tag{4.9}$$

$$(\nabla f(z))_{\bar{\mathcal{H}}} = (\nabla\psi(F(z)))_{\mathcal{H}} \cdot (\nabla F(z))_{\bar{\mathcal{H}}} + (\nabla\psi(F(z)))_{\bar{\mathcal{H}}} \cdot \overline{(\nabla F(z))_{\mathcal{H}}}, \tag{4.10}$$

due to the chain rule (1.4) in 1.10 applied to $f = \psi \circ F$. From (4.9) and (4.10) we conclude that due to (4.7)

$$\begin{aligned}(\nabla f)_{\bar{\mathcal{H}}} - \mu(\nabla f)_{\mathcal{H}} &= (\nabla\psi)_{\mathcal{H}}((\nabla F)_{\bar{\mathcal{H}}} - \mu(\nabla F)_{\mathcal{H}}) + (\nabla\psi)_{\bar{\mathcal{H}}}(\overline{(\nabla F)_{\mathcal{H}}} - \mu\overline{(\nabla F)_{\bar{\mathcal{H}}}}) \\ &= (\nabla\psi)_{\bar{\mathcal{H}}}\overline{(\nabla F)_{\mathcal{H}}(1-|\mu|^2)} \text{ almost everywhere in } [0,1]^2.\end{aligned}$$

Because $\|\mu\|_\infty < 1$, we infer from (4.8) that $(\nabla\psi)_{\bar{\mathcal{H}}} \equiv 0$ a.e. on $F(M_0 \cup M_+)$ - note that this set is also a Borel one, since F has the N-property. This means, it maps Lebesgue zero sets onto Lebesgue zero sets, the more complicated proof of this property for general quasiregular maps can be replaced by a much more simple one for functions in $W^{1,p}$, $p > n$, see [**57**]. On the other hand, for a.e. $z \in M_0 \cup M_-$ (4.7) implies that $(\nabla F(z))_{\bar{\mathcal{H}}} = 0$. Since $(\nabla F(z))_{\mathcal{H}} \neq 0$, again see e.g. Corollary 2 in §10.1 in Chapter II of [**79**], and since $(\nabla f(z))_{\mathcal{H}} = 0$, we conclude from (4.9) that $(\nabla\psi(F(z)))_{\mathcal{H}} = 0$. Hence, $(\nabla\psi)_{\mathcal{H}} \equiv 0$ on $F(M_- \cup M_0)$ and in particular $\nabla\psi \equiv 0$ on $F(M_0)$ which is as well as M_0 itself of positive Lebesgue measure.

4. Piecewise affine solutions

PROPOSITION 4.42. *Suppose we are given two bounded sets $\mathcal{U}, \mathcal{K} \subset \mathbb{M}^{n\times m}$ where $\mathcal{U}$ is open and such that there exists a $\delta > 0$ with the following property.*

> *For all $A \in \mathcal{U}$ we can find a prelaminate $\mu \in \mathscr{PL}(\mathcal{U})$ with barycentre $\bar{\mu} = A$ and matrices $X \in \mathcal{U}$, $\tilde{X} \in \mathcal{U}\cup\mathcal{K}$ and $Y_A \in \mathcal{K}$ such that: $\mu(\{X\}) > \delta$, $\operatorname{rank}(Y_A - \tilde{X}) = 1$ and $X = \lambda Y_A + (1-\lambda)\tilde{X}$ with $\lambda \in (\delta, 1]$.*

Then for each $A \in \mathcal{U}$ and any $\Omega \subset \mathbb{R}^m$ open there is a piecewise affine map $f : \bar{\Omega} \to \mathbb{R}^n$ such that $f_{|\partial\Omega} = A$ and that $\nabla f(x) \in K$ a.e. in Ω.

Proof. It suffices to show that for all $A \in \mathcal{U}$ and $\Omega \subset \mathbb{R}^m$ open there is $\tilde{f} : \bar{\Omega} \to \mathbb{R}^n$ piecewise affine such that $\tilde{f}(x) = A \cdot x$ if $x \in \partial\Omega$, $\nabla\tilde{f} \in \mathcal{K}\cup\mathcal{U}$ a.e. in Ω and $|\{x \in \Omega\ ;\ \nabla\tilde{f}(x) \in \mathcal{K}\}| > \delta^2|\Omega|/2$. Using this fact then iteratively on each open "piece" in Ω where the function is affine with a gradient in $\mathcal{U} \setminus \mathcal{K}$, we gradually obtain a sequence $\{f_i\}$ of maps of the following kind.

Starting with $f_0 \equiv A$ we construct piecewise affine maps $\{f_i\}_{i=1}^\infty : \bar{\Omega} \to \mathbb{R}^n$ such that the sets $U_i = \operatorname{int}(\{x \in \Omega\ ;\ \nabla f_i(x) \in \mathcal{K}\})$ satisfy:

i) $f_j(x) = f_i(x)$ if $j \geq i$ and $x \in U_i$, moreover $f_i(x) = A \cdot x$ if $x \in \partial A$.
ii) $|\Omega \setminus U_{i+1}| \leq (1-\delta^2/2)|\Omega \setminus U_i|$, note that $U_i \subset U_{i+1}$ by i).
iii) $\Omega \setminus U_i$ is up to a set of measure zero a countable union of open sets on each of which f_i is affine with gradient in $\mathcal{U}$.

Now, since $|\Omega \setminus U_i| \to 0$ and the f_i are uniformly lipschitz, we conclude from i) and ii) that $f_i \rightrightarrows f$ with $f_{|\partial\Omega} \equiv A$. Moreover, for each $x \in \bigcup_i U_i$, in particular for a.e. $x \in \Omega$, does f on a certain fixed neighbourhood of x agree with f_i for all i sufficiently large. Because also $x \in U_i$ for all large i, we conclude that $\nabla f(x) \in \mathcal{K}$ and that f is piecewise affine.

It remains to establish the existence of the $\tilde{f}$. Using the assumption of this proposition, we find $\tilde{X} \in \mathcal{U} \cup \mathcal{K}$, $X \in \mathcal{U}$, and $Y_A \in K$ together with a spliting sequence $\{\mu_i\}_{i=0}^l$ in $\mathscr{PL}(\mathcal{U})$ such that $\mu_0 = \delta_A$, $\mu_l(\{X\}) > \delta$, $\operatorname{rank}(Y_A - \tilde{X}) = 1$ and $X = \lambda Y_A + (1-\lambda)\tilde{X}$ for some $\lambda \in (\delta, 1]$. Fix any $\varepsilon > 0$, we claim that for all $i = 0, \ldots, l$ there are $\tilde{f}_i : \Omega \to \mathbb{R}^n$ piecewise affine with $\tilde{f}_i(x) = A \cdot x$ on

$\partial\Omega$, $\nabla\tilde{f}_i \in \mathcal{U}$ a.e. and $|\{x\ ;\ \nabla\tilde{f}_i(x) = Y\}| \geq (1-\varepsilon/(l+1-i))\mu_i(\{Y\})$ for all $Y \in \mathcal{U}$. Of course, the right side of the inequality becomes positive only for finitely many matrices. If $i = 0$ then $\tilde{f}_0(x) = A \cdot x$ obviously does the job, so let us be given $\tilde{f}_i$ for $i < l$. We know that $\mu_i = \kappa\delta_B + \tilde{\mu}$, $B \in \mathcal{U}$, $\kappa > 0$ and that $\mu_{i+1} = \kappa(\eta\delta_{B_1} + (1-\eta)\delta_{B_2}) + \tilde{\mu}$ where $B_1, B_2 \in \mathcal{U}$, $\operatorname{rank}(B_1 - B_2) = 1$ and $B = \eta B_1 + (1-\eta)B_2$ for some $\eta \in (0,1)$. Of course, due to the induction assumption we can find finitely many disjoint closed cubes $\{Q_j\}_{j=1}^T$ in Ω such that $\tilde{f}_i$ is on each of the Q_j's an affine function with gradient B and that $|\bigcup_1^T Q_j| = (1-\varepsilon/(l-i+1/2))\kappa$. We choose $r > 0$ with $B(B,r) \subset \mathcal{U}$ and now Lemma 3.2 ensures the existence of a piecewise affine map $\varphi : [0,1]^m \to \mathbb{R}^n$ such that $\varphi(x) = B \cdot x$ if $x \in \partial[0,1]^m$, $\nabla\varphi(x) \in \{B_1, B_2\} \cup B(B,r)$ and

$$|\{x \in [0,1]^m\ ;\ \nabla\varphi(x) = B_1\}| \geq \eta\big(1 - \frac{\varepsilon}{l-i}\big)\big/\big(1 - \frac{\varepsilon}{l-i+1/2}\big), \text{ and}$$

$$|\{x \in [0,1]^m\ ;\ \nabla\varphi(x) = B_2\}| \geq (1-\eta)\big(1 - \frac{\varepsilon}{l-i}\big)\big/\big(1 - \frac{\varepsilon}{l-i+1/2}\big).$$

Placing properly rescaled copies in the sense of the exhaustion argument from Construction 3.1 into each of the Q_j's and not changing $\tilde{f}_i$ otherwise, we obtain a new piecewise affine function $\tilde{f}_{i+1}$ with $\nabla\tilde{f}_{i+1} \in \mathcal{U}$ a.e. and the original behaviour on $\partial\Omega$. Since we had choosen the measure of $|\bigcup_1^T Q_j| < (1-\varepsilon/(l-i+1))\kappa$, it is clear that $|\{x \in \Omega \setminus \bigcup_1^T Q_j\ ;\ \nabla\tilde{f}_i(x) = Y\}| \geq (1-\varepsilon/(l-i+1))\tilde{\mu}(\{Y\})$ for all Y*. This quite easily implies that also the distribution of $\nabla\tilde{f}_{i+1}$ satisfies the required control from below in terms of μ_{i+1}, so the existence of $\tilde{f}_{i+1}$ is shown.

At the very end, and only in case that $X \notin \mathcal{K}$, we perform the construction once more, now replacing the summand $\mu_l(\{X\})\delta_X$ in μ_l by $\mu_l(\{X\})(\lambda\delta_{Y_A} + (1-\lambda)\delta_{\tilde{X}})$. It is clear that the new map $\tilde{f}$ fulfills all requirements since $|\{x \in \Omega\ ;\ \nabla\tilde{f}(x) = Y_A\}| \geq \mu_l(\{X\})(2\lambda/3)|Q| > \delta^2|Q|/2$ if ε is sufficiently small. □

In our first example we will have to make sure that among a quite large number of matrices no rank-one connections exist. Instead of checking each pair of matrices individually, the following pertubation result will be used.

LEMMA 4.43. *If $A, C \in \mathbb{M}^{n\times m}$ satisfy $A \neq 0$ and $\operatorname{rank}(C) \leq 1$ then for all ε positive there is $C' \in B(C,\varepsilon)$ such that $\operatorname{rank}(C') \leq 1$ and $\operatorname{rank}(A + C') \geq 2$.*

Proof. We can of course suppose $A + C = a \otimes b$, else put $C' = C$, and that $C = c \otimes d$. It is easy to check that $\operatorname{rank}(x \otimes y + u \otimes v) \leq 1$ is equivalent to $x \| u$ or $y \| v$. Consequently, if $\operatorname{rank}(A) = 1$ then $A + c' \otimes d'$ will have rank at least two for suitable $c' \approx c$ and $d' \approx d$. On the other hand, if $\operatorname{rank}(A) \geq 2$, then both a, c and b, d are non-colinear. Therefore, $\operatorname{rank}(A + (1+\varepsilon)C) \geq 2$ whenever $\varepsilon \neq 0$. □

PROPOSITION 4.44. *There exist linearly independent rank-one matrices $M_1, \ldots, M_4 \in \mathbb{M}^{2\times 2}$ such that for the "cube" $\mathcal{Q} = \{\sum_{i=1}^4 \lambda_i M_i\ ;\ 0 \leq \lambda_i \leq 1\}$ and any of its corners $C_I = \sum_{i\in I} M_i$, where $I \subset \{1, \ldots, 4\}$, there is a rank-one matrix B_I satisfying*

i) *$C_I - tB_I \in \operatorname{int}(\mathcal{Q})$ if $0 < t \leq 1$,*
ii) *$\mathcal{K} = \big\{C_I + B_I\ ;\ I \subset \{1, \ldots, 4\}\big\}$ does not contain any rank-one connection.*

Proof. Using the classical conformal coordinates, see Notation 1.10, we set

$$M_1 = (e_1, e_1), M_2 = (e_1, e_2), M_3 = (e_2, e_1) \text{ and } M_4 = (e_2, -e_2).$$

*This additional care is needed only for the case when $Y = B$ appears also in the support of $\tilde{\mu}$, which is the part of μ we did not alter.

Hence, $\sum_1^4 \lambda_i M_i = 0$ implies $\lambda_1 = -\lambda_2 = -\lambda_3$, $\lambda_2 = \lambda_4$ and $\lambda_3 = -\lambda_4$ and so $\lambda_i = 0$ for all $i \leq 4$. Moreover, we easily compute that

$$\begin{aligned}
\frac{1}{2}\det(\sum_{i=1}^{4} \lambda_i M_i) &= \frac{1}{2}\big((\lambda_1+\lambda_2)^2 + (\lambda_3+\lambda_4)^2 - (\lambda_1+\lambda_3)^2 - (\lambda_2-\lambda_4)^2\big) \\
&= \lambda_1\lambda_2 + \lambda_3\lambda_4 - \lambda_1\lambda_3 + \lambda_2\lambda_4 \\
&= (\lambda_1+\lambda_4)(\lambda_2+\lambda_3) - 2\lambda_1\lambda_3 \\
&= \lambda_1\lambda_3\big((1+\frac{\lambda_4}{\lambda_1})(1+\frac{\lambda_2}{\lambda_3}) - 2\big) \text{ for } \lambda_1\lambda_3 \neq 0.
\end{aligned}$$

Since it is easy to check that for any $\bar{\sigma}_1, \bar{\sigma}_2 \in \{-1,1\}$ there are $t_1, t_2 > 0$ such that $(1+\bar{\sigma}_1 t_1)(1+\bar{\sigma}_2 t_2) = 2$, we conclude that for any $\sigma \in \{-1,1\}^4$ there is $t \in (0,1)^4$ such that $\det(\sum_1^4(\sigma_i t_i)M_i) = 0$. It is also obvious that for each $I \subset \{1,\dots,4\}$ and σ^I defined as $\sigma_i^I = 2\chi_I(i) - 1$ the matrix $C_I - (\sum_{i=1}^4(\sigma_i^I t_i)M_i)$ is in $\mathrm{int}(\mathcal{Q})$ provided $t \in (0,1)$. This, together with the foregoing observation, ensures the existence of $\tilde{B}_I$ of rank one such that $C_I - \tilde{B}_I \in \mathrm{int}(\mathcal{Q})$ for all $I \subset \{1,\dots,4\}$, so condition i) is already satisfied. In case condition ii) is violated, we use that fact that $C_I + \tilde{B}_I \notin Q$ and hence $C_I + \tilde{B}_I - C_J \neq 0$ if $I \neq J$ to infer from Lemma 4.43 the existence of perturbed rank-one matrices B_I arbitrarily close to $\tilde{B}_I$ which now fulfill both (open) conditions i) and ii). This finishes our proof. □

LEMMA 4.45. *The open set* $\mathrm{int}(\mathcal{Q})$ *and the 16-point configuration* $\mathcal{K}$ *given in the foregoing Proposition 4.44 satisfy the assumptions of Proposition 4.42.*

Proof. Indeed, we will show that we can choose in the assumptions of Proposition 4.42 the constant δ to be $1/17$. To verify this claim, consider any $A = \sum_1^4 \lambda_i M_i \in \mathrm{int}(\mathcal{Q})$. We set $I = \{i \leq 4 \ ; \ \lambda_i \geq 1/2\}$, so C_I is the corner of $\mathcal{Q}$ closest to A. We define $\varphi : \{1,\dots,4\} \to \mathbb{R}$ by $B_I = \sum_1^4 \varphi(i)M_i$ and and choose $t \in (0, 1/2)$ such that $\kappa_i = (\chi_I - t\varphi)(i)$ is strictly between λ_i and $\chi_I(i)$ for all $i \leq 4$. Using the canonical parametrization Φ of $\mathcal{Q}$ by $[0,1]^4$, i.e. $\Phi(x) = \sum_{i=1}^4 x_i M_i$, we gradually obtain the following splitting sequences of prelaminates in $\mathscr{PL}(\mathrm{int}(\mathcal{Q}))$

$$\begin{aligned}
\mu_0 &= \delta_A = \delta_{\Phi((\lambda_1,\dots,\lambda_4))} \\
\mu_1 &= \frac{1}{2}\delta_{\Phi((\kappa_1,\lambda_2,\lambda_3,\lambda_4))} + \frac{1}{2}\delta_{\Phi((2\lambda_1-\kappa_1,\lambda_2,\lambda_3,\lambda_4))} \\
\mu_2 &= \frac{1}{4}\delta_{\Phi((\kappa_1,\kappa_2,\lambda_3,\lambda_4))} + \frac{1}{4}\delta_{\Phi((\kappa_1,2\lambda_2-\kappa_2,\lambda_3,\lambda_4))} + \frac{1}{2}\delta_{\Phi((2\lambda_1-\kappa_1,\lambda_2,\lambda_3,\lambda_4))} \\
\mu_3 &= \dots \\
\mu_4 &= \frac{1}{16}\delta_{\Phi((\kappa_1,\kappa_2,\kappa_3,\kappa_4))} + \dots \\
&= \frac{1}{16}\delta_{(C_I - tB_I)} + \dots,
\end{aligned}$$

as our definitions of I and the κ_i's obviously ensure that $2\lambda_i - \kappa_i \in (0,1)$.

Moreover, puting $X = C_I - tB_I$, $Y_A = C_I + B_I \in \mathcal{K}$ and $\tilde{X} = C_I - B_I \in \mathrm{int}(\mathcal{Q})$, we see that $X = \Phi(\kappa_1,\dots,\kappa_4)$ and that both $\mu_4(\{X\}) > 1/17$ and $X = \eta Y_A + (1-\eta)\tilde{X}$ with $\eta > 1/4$ holds. Hence, we are done. □

COROLLARY 4.46. *There exist* $A_1,\dots,A_{16} \in \mathbb{M}^{2\times 2}$ *without rank-one connections and an open nonvoid paralleliped* $\tilde{\mathcal{Q}}$ *in* $\mathbb{M}^{2\times 2}$ *such that for all* $B \in \tilde{\mathcal{Q}}$ *and any open bounded* $\Omega \subset \mathbb{R}^2$ *there is a lipschitz and piecewise affine* $f : \bar{\Omega} \to \mathbb{R}^2$ *with* $f(x) = B \cdot x$ *for each* $x \in \partial\Omega$ *and* $\nabla f(x) \in \{A_1,\dots,A_{16}\}$ *a.e. in* Ω.

Also the foregoing example was found in joint work [**46**] with D.Preiss. In the last part of this section we want to establish the existence of piecewise affine solutions also for the compatible two-well problem. The basic notions for this situation where already introduced in Paragraph 1.11. So we consider two SO(2) wells each of them with a different positive determinant and with stable rank-one connections between them, in other words each matrix is rank-one connected to two matrices on the other well. As already mentioned, the existence of lipschitz solutions to $\nabla f \in \mathcal{K}_{A,B} = \mathrm{SO}(2)A \cup \mathrm{SO}(2)B$ was shown e.g. in [**65**]. We find our piecewise affine solutions using Proposition 4.42, but also in our situation as well as in [**65**], a kind of in-approximation argument is at the heart of the matter.

THEOREM 4.47. *Let $(A,B) \in \mathcal{W}$ as defined in Lemma 1.12 and assume* $\det(A) < \det(B)$. *Then for each $F_0 \in \mathrm{int}_{(\mathbb{M}^{2\times 2})}(\mathcal{K}^{lc}_{A,B})$ and any open set $\Omega \subset \mathbb{R}^2$ there is a piecewise affine mapping $f : \bar{\Omega} \to \mathbb{R}^2$ such that $f(x) = F_0 \cdot x$ for all $x \in \partial\Omega$ and $\nabla f \in \mathcal{K}_{A,B}$ a.e. in Ω.*

Proof. We set $\mathcal{K} = \mathcal{K}_{A,B}$ and $\mathcal{U} = \mathrm{int}(\mathcal{K}^{lc}_{A,B})$, which is more explicitely given in (1.5) and Lemma 1.12.ii), and will show that for $\delta = 1/9$ the assumptions of Proposition 4.42 are satisfied. For this purpose, we recall that Lemma 1.12 ensures for each $\varepsilon > 0$ the existence of $A_\varepsilon, B_\varepsilon \in \mathcal{U}$ such that

a) $A_\varepsilon - A, B_\varepsilon - B$ are rank-one matrices of norm smaller than ε.
b) $A + 2(A_\varepsilon - A)$, $B + 2(B_\varepsilon - B)$ both belong to $\mathcal{U}$.

Now, from (1.5) and (1.7) we infer that for all ε sufficiently small $F_0 \in \mathrm{int}(\mathcal{K}^{lc}_{A_\varepsilon,B_\varepsilon}) = L_3(\mathcal{K}_{A_\varepsilon,B_\varepsilon})$. A simple induction on the order of the laminate hull shows that there is $\mathcal{K}'_\varepsilon \subset \mathcal{K}_{A_\varepsilon,B_\varepsilon}$ of cardinality at most 2^3 with $F_0 \in (\mathcal{K}'_\varepsilon)^{lc}$. In other words, there is a $\mu \in \mathscr{PL}((\mathcal{K}'_\varepsilon)^{lc}) \cap \mathscr{M}(\mathcal{K}'_\varepsilon)$ with $\bar{\mu} = F_0$. Obviously, we find $X \in \mathcal{K}'_\varepsilon$ such that $\mu(\{X\}) > 1/9$.

Because $\mathcal{K}_{A,B}$ is invariant under postmultiplication with $Q \in \mathrm{SO}(2)$, the same holds true for $\mathcal{K}^{lc}_{A,B}$ and hence also for $\mathcal{U}$. So, we find $\eta > 0$ with $B(\mathcal{K}_{A_\varepsilon,B_\varepsilon}, \eta) \subset \mathcal{U}$ and, since the generation of any hull of this kind commutes with translations, we infer that even $B(K^{lc}_{A_\varepsilon,B_\varepsilon}, \eta) \subset \mathcal{U}$ and $\mu \in \mathscr{PL}(\mathcal{U})$. Finally, if we rotate via multiplication from the left A_ε or B_ε to X, then we find due to b) two matrices $Y_{F_0} \in \mathcal{K}$ and $\tilde{X} \in \mathcal{U}$ with $\mathrm{rank}(Y_{F_0} - \tilde{X}) = 1$ and $X = (\tilde{X} + Y_{F_0})/2$. Therefore, we sucessfully checked all assumptions of Proposition 4.42 and are done. □

Of course, once we know that the set where the solutions to our partial differential inclusion are not affine can be made neglectable in Lebesgue measure it is natural to ask how small this set can actually be. Results obtained in [**31**] and extended in [**45**] show that locally finite $(m-1)$-dimensional Hausdorff measure is already a very restrictive condition. Indeed, such functions esentially behave as if they consist of finitely many affine pieces only. This motivates us to conclude this chapter with the follwing two open problems.

Q3 Consider the 16 matrices $A_1, \ldots, A_{16}$ from Corollary 4.46 and a gradient B from the open paralleliped $\tilde{\mathcal{Q}}$ introduced there. If f is a solution to the partial differential inclusion $\nabla f \in \{A_1, \ldots, A_{16}\}$ a.e. with affine bounadry data B, how small can the Hausdorff dimension of the closed set of those x near which f is not affine be.

Q4 Do there exist piecewise affine solutions to the two well problem for compatible SO(2)-wells with equal determinant?

I have to admit that I am rather convinced that Question 3 should allow dimensions strictly less than two, but in the moment an efficient bound seperating the optimal dimension away from 1 or from 2 lacks completely.

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