APPLICATIONS OF QUASILINEAR PDE TO ALGEBRAIC GEOMETRY AND ARITHMETIC LATTICES

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December 3, 1993

INTRODUCTION

In this survey we wish to describe some of our work on harmonic maps and rigidity questions in geometry. Since the subject is presently very active and since often different authors have produced overlapping results we also wish to present a more comprehensive and balanced overview of the topic than others available in the literature.

In order to outline the scope of our enterprise, we shall first state some theorems that highlight various aspects and facets of what follows:

Theorem A (Jost-Yau). *Let M be a compact Kähler manifold, $g : M \to S$ a continuous map to some compact curve of genus ≥ 2. Assume g is nontrivial on the second homology. Then there exists a nonsingular holomorphic curve C of genus ≥ 2 and a nontrivial holomorphic map $h : M \to C$. In particular, the universal cover of M admits a bounded holomorphic function.*

Theorem B (Margulis). *Let Γ be a lattice in an irreducible symmetric space G/K of noncompact type. If rank$(G/K) \geq 2$ then Γ is arithmetic.*

Theorem C. *Let G/K be as is in Theorem B, in particular rank$(G/K) \geq 2$. Let Γ be a cocompact lattice. Then every discrete homomorphism $f : \Gamma \to SO(n,1)$ or $SU(n,1)$ is finite.*

Theorem D (Manin, Grauert). *Let C be a nonsingular holomorphic curve, $\Sigma(t)$ a curve of genus $g \geq 2$ over the function field of C. If $\Sigma(t)$ is not isotrivial, then it has at most finitely many rational points.*

Theorem E (Arakelov, Parshin). *Let C be a nonsingular holomorphic curve, $S \subset C$ a finite subset. Then there exists at most finitely many nonisotrivial curves of given genus $g \geq 2$ over the function field of C with bad reductions at most over S.*

What do these apparently quite diverse results have in common? What makes them amenable to elliptic partial differential equations? The answer is that they are all concerned in some way or another with mappings into spaces of nonpositive curvature. Let us elaborate this point for the various results just quoted.

Research of the second author was in part supported by NSF DMS 92 06938

Abstract

We survey applications of harmonic mappings to various rigidity questions in algebraic and complex geometry, including lattices in linear algebraic groups, symmetric spaces, and algebraic varieties over function fields

AMS Class.; 58E20, 14M15, 22E55, 22E50, 32G05, 32G15, 53C35, 32M15, 53C55

In the situation of Theorem A, one equips S with some hyperbolic metric, i.e. a Riemannian metric of curvature -1. This at the same time makes S into a Riemann surface. However, S may have the wrong complex structure for admitting a holomorphic map from M. Holomorphic maps have to satisfy the Cauchy Riemann equations. These form an overdetermined system of PDE, and hence in general do not admit solutions. We therefore relax the system and look for so-called harmonic maps from M to S, in a similar way as in complex analysis one often constructs harmonic functions instead of holomorphic ones. Such a harmonic map turns out to exist, because S has nonpositive curvature, and while not being holomorphic itself, its generic level sets can be shown to be holomorphic varieties. We thus get some kind of holomorphic foliation on M, and the leaf space is the desired curve C.

Margulis deduced Theorem B from his superrigidity theorem. Here one considers homomorphisms $\rho : \Gamma \to H$ where H is a semisimple noncompact Lie group with trivial center, or the completion $G(K_v)$ with respect to a nonarchimedian valuation K_v of a simple algebraic group over a number field like $Sl(n, Q_p)$. Margulis then shows that such a homomorphism extends to a homomorphism from G onto H when $\rho(\Gamma)$ is Zariski dense, in the first case, or has its image contained in a compact subgroup in the second case. From our point of view, we start with a ρ-equivariant harmonic map $f = G/K \to H/L$ where H/L is the symmetric space associated with H in the first case, or $f : G/K \to \Delta(G(K_v))$ where $\Delta(G(K_v))$ is the Euclidean Tits building on which $G(K_v)$ naturally operates in the second case. Again, this harmonic map exists because the image has nonpositive curvature. One then shows, again strongly using nonpositive curvature that in case $\mathrm{rank}(G/K) \geq 2$, such a harmonic map has to be totally geodesic, i.e. maps geodesics to geodesics. Since geodesics correspond to 1 parameter subgroups of G etc., Margulis' superrigidity theorem readily follows.

Theorem C can likewise be reduced to considering harmonic maps into a locally symmetric space, which is the quotient of real or complex hyperbolic space. The foliation alluded to in the description of Theorem A also plays a role here.

Concerning Theorem E, we consider a curve over the function field of C as a holomorphic map $h : C\backslash S \to M_g$ into the moduli space of curves of genus g. After lifting to finite covers – a technicality we want to suppress for the sake of the discussion – M_g is smooth and carries a natural Kähler metric, the Weil-Petersson metric. This metric was shown to have negative curvature by Tromba, and using this geometric information, one may show that there exist at most finitely many such holomorphic maps. Similarly, in Theorem D, after passing to finite cover, one has a smooth universal curve M_g (alternatively, the moduli space M_g of once punctured curves of genus g) and a rational point, i.e. a section of the curve $\Sigma(t)$, is interpreted as holomorphic map

$$s : C\backslash S \to M_g \, .$$

Again, one has negative image curvature, and the proof becomes the same as the one for Theorem E.

This should serve as an outline, and we now embark upon a more detailed discussion where also precise references will be given. We have to start by defining harmonic maps, i.e. solutions of the quasilinear elliptic equations alluded to in the title. This will lead us into the realm of real Riemannian geometry as the natural setting for the concept of harmonic maps. Whenever we have additional structures (algebraic varieties, Kähler manifolds, symmetric spaces etc.), then we shall see that at least in the presence of negative curvature, harmonic maps tie in quite well with these structures and have additional properties that allow us to reach the desired conclusion.

1. HARMONIC MAPS AS A NATURAL CLASS OF MAPS BETWEEN RIEMANNIAN MANIFOLDS.

Before turning to more general situations, we look at two compact Riemannian manifolds M and N and set up some notation that will be maintained throughout this paper.

$x^\alpha, \alpha = 1, \ldots m = \dim M$, will be local coordinates on M.

We shall always assume that these coordinates are normal at the point under consideration, allowing us to freely lower indices in forms of notation.

The curvature tensor of M in our coordinates is $R_{\alpha\beta\gamma\delta}$, and the Ricci tensor

$$R_{\alpha\gamma} = R_{\alpha\beta\gamma\beta},$$

using here and in the sequel the standard Einstein summation convention. For a map $f : M \to N$, we put

$$f_\alpha := \frac{\partial}{\partial x^\alpha} f .$$

The Laplace-Beltrami operator on M is in on our coordinates

$$\Delta = \Sigma_\alpha \frac{\partial^2}{(\partial x^\alpha)^2} .$$

On N, we shall use invariant notation. We denote the pointwise product in the tangent bundle TN by $< \cdot, \cdot >$. The curvature tensor of N is $R(\cdot, \cdot)$. For a map $f : M \to N$, the metric in the pull back bundle $f^{-1}TN$, and the metric in $T^*M \otimes f^{-1}TN$ will both be denoted by $< \cdot, \cdot >$. The L^2 metric in $T^*M \otimes f^{-1}TN$ will be denoted by $(\cdot, \cdot)$. (All integrations are w.r.t. the volume form of M.) ∇ denotes the Levi Civita connection in $T^*M \otimes f^{-1}TN$. We put

$$\nabla_\alpha = \nabla_{\frac{\partial}{\partial x^\alpha}}$$

and

$$f_{\alpha\beta} := \nabla_\beta f_\alpha$$

$$e(f) = \frac{1}{2} < f_\alpha, f_\alpha > , \tag{1.1}$$

and the energy

$$E(f) = \int_M e(f) \tag{1.2}$$

Critical points of E are called harmonic. They solve the system of elliptic differential equations

$$\operatorname{trace} \nabla df = 0 \tag{1.3}$$

In the sequel, harmonic maps will mean solutions of this equation. Since, in general, regularity of critical points of variational integrals is a difficult issue, we shall make the additional assumption that N has nonpositive sectional curvature. Under this assumption, harmonic maps are known to be smooth. We have the existence result of Eells-Sampson [ES] and Alber [A1, A2]:

Theorem 1. *Let M, N be compact Riemannian manifolds, N of nonpositve sectional curvature. Then any map from M to N is homotopic to a harmonic one.*

Moreover, by Alber [A2] *and Hartman* [Ha], *any such harmonic map is essentially unique in its homotopy class, more precisely, any two homotopic harmonic maps are contained in a parallel and isometric family of harmonic maps. Thus, they are geometrically alike.*

Harmonic maps $f : S^1 \to N$ are simply closed geodesics, harmonic maps $f : M \to \mathbb{R}$ are harmonic functions. For later purposes, we need to generalize the setting somewhat. We let X and Y be complete, simply connected Riemannian manifolds, for example the universal covers of M and N respectively (The isometric groups are denoted by $I(X)$ and $I(Y)$, respectively. We shall assume that Y has nonpositive sectional curvature.

Let Γ be a discrete subgroup of $I(X)$ with compact quotient X/Γ, for example $\Gamma = \pi_1(M)$. Let

$$\rho : \Gamma \to I(Y)$$

be a homomorphism, for example $\rho(\Gamma) \subset \pi_1(N) \subset I(Y)$. A map

$$f : X \to Y$$

is called ρ-equivariant if for all $x \in X$ and $\gamma \in \Gamma$

$$f(\gamma x) = \rho(\gamma) f(x) .$$

After results of Diederich-Ohsawa [DO], Donaldson [Dn], and Corlette [C1] in special cases, the existence of ρ-equivariant harmonic maps was shown independently by Jost-Yau [JY5] and Labourie [L].

Theorem 2. *Let X, Y be as above, Y of nonpositive sectional curvature. Let $\Gamma \subset I(X)$ be discrete with compact quotient, and let $\rho : \Gamma \to I(Y)$ be a reductive homomorphism. Then there exists a ρ-equivariant harmonic map*

$$f : X \to Y.$$

A subgroup G of $I(Y)$ is called reductive, if whenever $(y_n)_{n \in \mathbb{N}} \subset Y$ is an unbounded sequence with

$$\text{dist}\,(y_n, gy_n) \leq \text{const}$$

for every $g \in G$ (the constant may depend on g, but not on n), then G stabilizes a totally geodesic flat subspace of Y. ρ is called reductive if $\rho(\Gamma)$ is. If Y is a symmetric space G/K, then our reductivity condition reduces to the standard one for reductive subgroups of linear algebraic groups.

The reductivity condition is necessary, because parabolic elements in $I(Y)$ do not admit axes. A simple picture is obtained by taking N as a hyperbolic Riemann surface with a cusp. Consider maps from S^1 into N with image a short loop about this cusp. This homology class, although being nontrivial, does not contain a closed geodesic. The corresponding homomorphism $\rho : \mathbb{Z} = \pi_1(S^1) \to I(Y)$ has parabolic image and is not reductive.

Theorem 2 contains Theorem 1, because the fundamental group of a compact quotient of Y is reductive. (Labouric's condition for existence was formulated somewhat differently, but it is essentially the same as reductivity in our sense.)

2. THE BOCHNER-MATSUSHIMA TECHNIQUE

Let M again be a compact Riemannian manifold, with first Betti number

$$p = b_1(M) .$$

Integrating harmonic 1-forms over cycles gives the Albanese map

$$h : M \to T^p$$

to a p-dimensional Euclidean torus, and h in fact is harmonic. From this point of view with some oversimplification, harmonic 1-forms on M can be interpreted as harmonic maps

$$f : M \to S^1$$

to the unit circle.

We now describe some vanishing results for harmonic 1-forms that will be generalized in subsequent sections to obtain restrictions for harmonic maps into general manifolds of nonpositive curvature.

A harmonic 1-form $f : M \to S^1$ satisfies the Bochner formula

$$1/2\Delta f_\alpha f_\alpha = f_{\alpha\beta} f_{\alpha\beta} + R_{\alpha\beta} f_\alpha f_\beta \ . \tag{2.1}$$

Since M is compact, the integral of the left hand side of (2.1) vanishes, and so then does the integral of the right hand side. If we now assume that M has nonnegative Ricci curvature, then both terms on the right hand side are pointwise nonnegative; therefore, they both have to vanish identically. In particular,

$$f_{\alpha\beta} \equiv 0 \quad \text{for all } \alpha, \beta \ . \tag{2.2}$$

Thus, the (covariant) second derivatives of f vanish identically and the harmonic 1-form is parallel.

If the Ricci tensor of M is even positive definite, then

$$R_{\alpha\beta} f_\alpha f_\beta \equiv 0 \tag{2.3}$$

implies

$$f_\alpha \equiv 0 \quad \text{for all } \alpha \ ,$$

i.e., f is a trivial 1-form.

In view of Hodge's theorem that every cohomology class can be represented by a harmonic form, we deduce Bochner's theorem

Theorem 3. *For a compact Riemannian manifold M of positive Ricci curvature*

$$b_1(M) = 0 \ .$$

This type of reasoning does not look too helpful for manifolds with negative Ricci curvature. Matsushima [M], however, discovered that sometimes the term involving the $f_{\alpha\beta}$ can balance a negative Ricci curvature term in such a way that one can still conclude $b_1(M) = 0$. He considered symmetric spaces G/K of noncompact type and quotients $M = \Gamma\backslash G/K$ by a compact lattice Γ. In order to describe his argument, we shall now present a generalization of his Bochner type formula obtained in [JY7]:

Theorem 4. *Let M be a compact Einstein manifold, and let $f : M \to S^1$ be harmonic. Then for any $\lambda \in \mathbb{R}$*

$$\begin{aligned} &\lambda \int_M f_{\alpha\beta} f_{\alpha\beta} + 2\int_M R_{\alpha\beta\gamma\delta} f_{\alpha\delta} f_{\beta\gamma} \\ &= -\lambda \int_M R_{\alpha\beta} f_\alpha f_\beta - \int_M R_{\alpha\beta\gamma\delta} R_{\eta\beta\gamma\delta} f_\alpha f_\eta \, . \end{aligned} \tag{2.4}$$

Note that the first terms on each side of (2.4) are just the ones that come from integrating (2.1).

In order to proceed, we shall now assume that $M = \Gamma\backslash G/K$ is a locally symmetric space of noncompact type. The aim then is to find a value for λ for which the left hand side of (2.4) is nonnegative, perhaps even positive unless all $f_{\alpha\beta} = 0$, whereas the right hand side is nonpositive. Thus we have to find λ with

$$2R_{\alpha\beta\gamma\delta} f_{\alpha\delta} f_{\beta\gamma} \geq \lambda f_{\alpha\beta} f_{\alpha\beta} \tag{2.5}$$

$$R_{\alpha\beta\gamma\delta} R_{\eta\beta\gamma\delta} f_\alpha f_\eta \geq -\lambda R_{\alpha\eta} f_\alpha f_\eta \, . \tag{2.6}$$

Now first of all

$$R_{\alpha\beta\gamma\delta} f_{\alpha\delta} f_{\beta\gamma} \geq \lambda_1 f_{\alpha\beta} f_{\alpha\beta} \tag{2.7}$$

where λ_1 was computed by Calabi-Vesentini [CV] and Borel [B] in the hermitian cases (see [M;§9] for a table), and by Kaneyuki-Nagano [KN] for the other cases.

We let $c^\lambda_{\alpha\beta}$ be the structure constants of the symmetric space G/K covering M, i.e.,

$$[X_\alpha, X_\beta] = c^\lambda_{\alpha\beta} X_\lambda \tag{2.8}$$

with associated Lie algebra decomposition

$$\mathfrak{G} = \mathfrak{k} \oplus \mathfrak{p}$$

and bases $X_1, \ldots X_m$ of $\mathfrak{p}$, $X_{m+1}, \ldots X_{m+k}$ of $\mathfrak{k}$ satisfying

$$B(X_\alpha, X_\beta) = \delta_{\alpha\beta} \, , \quad \alpha, \beta = 1, \ldots m$$

$$B(X_\lambda, X_\mu) = -\delta_{\lambda\mu} \, , \quad \lambda, \mu = m+1, \ldots m+k \, . \tag{2.9}$$

Here B is the Killing form of $\mathfrak{g}$. In the sequel, such a base will be called orthonormal for the Killing form.

$$c^\lambda_{\alpha\beta} c^\lambda_{\gamma\beta} = -\frac{1}{2}\delta_{\alpha\gamma} \tag{2.10}$$

and

$$R_{\alpha\beta\gamma\delta} = -c^{\lambda}_{\alpha\beta}c^{\lambda}_{\gamma\delta}\,. \tag{2.11}$$

Since $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$ in the previous formulae, λ only ranges from $m+1,\ldots m+k$, if $\alpha,\beta \in \{1,\ldots m\}$. In particular, we have for the Ricci tensor

$$R_{\alpha\beta} = -\frac{1}{2}\delta_{\alpha\beta}\,. \tag{2.12}$$

Let

$$\mathfrak{k} = \mathfrak{z} \oplus \mathfrak{k}_1 \oplus \ldots \oplus \mathfrak{k}_l$$

where $\mathfrak{z}$ is the center of $\mathfrak{k}$ and $\mathfrak{k}_1,\ldots\mathfrak{k}_l$ are the simple ideals. We choose our basis (X_λ) of $\mathfrak{k}$ to respect this decomposition, i.e., each X_λ is contained in one of the summands. Then

$$c^{\lambda}_{\alpha\beta}c^{\mu}_{\alpha\beta} = a_i\delta_{\lambda\mu} \quad \text{for} \quad X_\lambda \in \mathfrak{k}_i\,, \quad i = 1,\ldots l \textit{ with } 0 < a_i < 1 \tag{2.13}$$

$$c^{\lambda}_{\alpha\beta}c^{\mu}_{\alpha\beta} = \delta_{\lambda\mu} \quad \text{for} \quad X_\lambda \in \mathfrak{z}\,. \tag{2.14}$$

For unity of notation, we sometimes put $\mathfrak{k}_0 := \mathfrak{z}$; then $a_0 = 1$, in case $\mathfrak{z} \neq \{0\}$; if $\mathfrak{z}$ is trivial, we put $a_0 = 0$. We also put for $i = 0,\ldots l$

$$\sum_{X_\lambda \in \mathfrak{k}_i} c^{\lambda}_{\alpha\beta}c^{\lambda}_{\eta\beta} = b_i\delta_{\alpha\eta} \tag{2.15}$$

by (2.10),

$$\sum_{i=0}^{1} b_i = 1/2\,. \tag{2.16}$$

We compute

$$\begin{aligned} R_{\alpha\beta\gamma\eta}R_{\eta\beta\gamma\delta}f_\alpha f_\eta &= c^{\lambda}_{\alpha\beta}c^{\lambda}_{\gamma\delta}c^{\mu}_{\eta\beta}c^{\mu}_{\gamma\delta}f_\alpha f_\eta \\ \sum_{i=0}^{l} a_i b_i f_\alpha f_\alpha &= -2\sum_{i=0}^{l} a_i b_i R_{\alpha\eta}f_\alpha f_\eta\,. \end{aligned} \tag{2.17}$$

Moreover, if rank $(G/K) \geq 2$, we obtain that the terms in (2.6) have to vanish, and noting (2.12), we conclude

$$f_\alpha = 0 \quad \text{for all } \alpha\,,$$

i.e., the harmonic 1-form is trivial.

We thus have Matsushima's theorem [M].

Theorem 5. *Let $M = \Gamma\backslash G/K$ be a compact locally symmetric space of noncompact type. If rank $(G/K) \geq 2$, then*

$$b_1(M) = 0\,.$$

3. BOCHNER FORMULAE FOR HARMONIC MAPS

Since we have interpreted harmonic 1-forms as harmonic maps

$$f : M \to S^1 ,$$

it seems natural to try to extend the reasoning of the previous section from the linear to the nonlinear case by taking a general Riemannian manifold N instead of S^1 as target. We shall see that this is possible provided N has nonpositive or negative curvature in a suitable sense. Note that this is in accord with the existence theorem of §1.

The analogue of the Bochner formula (2.1) for harmonic maps was discovered by Eells-Sampson [ES] and Al'ber [A2]

$$\begin{aligned} 1/2\Delta\langle f_\alpha, f_\alpha\rangle &= \langle f_{\alpha\beta}, f_{\alpha\beta}\rangle + R_{\alpha\beta}\langle f_\alpha, f_\beta\rangle \\ &- \langle R(f_\alpha, f_\beta)f_\beta, f_\alpha\rangle \end{aligned} \tag{3.1}$$

(in the notation at the beginning of §1).

If M again is compact and has nonpositive Ricci curvature and if N now has nonpositive sectional curvature, then again all three terms on the right hand side have the same sign pointwise, and therefore, all of them have to vanish identically. In particular,

$$f_{\alpha\beta} \equiv 0 \quad \text{for all } \alpha, \beta , \tag{3.2}$$

meaning that f is totally geodesic (it maps geodesics to geodesics). If M has even positive Ricci curvature, then

$$f_\alpha \equiv 0 \quad \text{for all } \alpha ,$$

and f is constant.

This result implies that manifolds of positive Ricci curvature have topology quite different from those of nonpositive sectional curvature.

Again, however, if one wishes to apply this method to spaces of nonpositive curvature, one needs to balance the Ricci curvature term in (3.1) against the other ones. The corresponding analogue of Matsushima's formula that we now want to discuss was found in [JY7]. (A similar formula was independently obtained by Mok-Siu-Yeung [MSY].)

Theorem 6. *Let $f : M \to N$ be a harmonic map between Riemannian manifolds where M is compact and Einstein. Then for any $\lambda \in \mathbb{R}$*

$$\begin{aligned} &\lambda \int_M \langle f_{\alpha\beta}, f_{\alpha\beta}\rangle + 2\int_M R_{\alpha\beta\gamma\delta}\langle f_{\alpha\delta}, f_{\beta\gamma}\rangle \\ = \; & -\lambda \int_M R_{\alpha\beta}\langle f_\alpha, f_\beta\rangle - \int_M R_{\alpha\beta\gamma\delta}R_{\eta\beta\gamma\delta}\langle f_\alpha, f_\eta\rangle + \lambda \int_M \langle R(f_\alpha, f_\beta)f_\beta, f_\alpha\rangle \\ + \; & \int_M R_{\alpha\beta\gamma\delta}\langle R(f_\gamma, f_\delta)f_\beta, f_\alpha\rangle . \end{aligned} \tag{3.3}$$

With the help of this formula, one can show the following result.

Theorem 7. *Let $X = G/K$ be an irreducible symmetric space of noncompact type and of rank ≥ 2. Let Γ be a discrete cocompact subgroup of G.*

Let Y be a complete simply connected Riemannian manifold of nonpositive curvature operator, with isometry group $I(Y)$.

Let $\rho : \Gamma \to I(Y)$ be a homomorphism. Then any ρ-equivariant harmonic map

$$f : X \to Y$$

is totally geodesic.

Note that Γ may have fixed points. Theorem 6 generalizes to that situation by taking M as a fundamental region for Γ in G/K.

The applications of Theorem 7 to superrigidity will be discussed in the following section.

Theorem 7 was proved in [JY7]. (This work was essentially carried out in 1990, while Siu-Yeung [SY] pursued the foliation approach suggested by Gromov (see below) and obtained some partial results.) A similar result was announced by Y.T. Siu as joint work with Yeung at the International Symposium on Algebraic Geometry and Related Topics in Inchoen. The details were subsequently worked out and published in collaboration with N. Mok, see [MSY]. See also Siu's article in these Proceedings.

In fact, the curvature condition on N can be somewhat weakened, see [MSY]. We omit this point, however, because the main application of Theorem 7 is Margulis' superrigidity, and here Y is also a symmetric space of noncompact type and hence has nonpositive curvature operator.

The strategy of proof of Theorem 7, while technically much more involved, is conceptually similar to the one of Theorem 5. Thus, again one has to find a value for λ for which the left hand side of (3.3) is nonnegative, perhaps even positive unless all $f_{\alpha\beta} = 0$, whereas the right hand side is nonpositive. Thus, this time, we have to find λ with

$$2R_{\alpha\beta\gamma\delta}\langle f_{\alpha\delta}, f_{\beta\gamma}\rangle \geq -\lambda\langle f_{\alpha\beta}, f_{\alpha\beta}\rangle \tag{3.4}$$

$$R_{\alpha\beta\gamma\delta}R_{\eta\beta\gamma\delta}\langle f_\alpha, f_\eta\rangle \geq -\lambda\sum_{\alpha\eta} R_{\alpha\eta}\langle f_\alpha, f_\eta\rangle \tag{3.5}$$

$$R_{\alpha\beta\gamma\delta}\langle R(f_\gamma, f_\delta)f_\beta, f_\alpha\rangle \leq -\lambda\langle R(f_\alpha, f_\beta)f_\beta, f_\alpha\rangle\,. \tag{3.6}$$

As in §2, with the same notation,

$$R_{\alpha\beta\gamma\delta}\langle f_{\alpha\delta}, f_{\beta\gamma}\rangle \geq \lambda_1\sum_{\alpha\beta}\langle f_{\alpha\beta}, f_{\alpha\beta}\rangle \tag{3.7}$$

and

$$R_{\alpha\beta\gamma\delta}R_{\eta\beta\gamma\delta}\langle f_\alpha, f_\eta\rangle \geq -\mu R_{\alpha\eta}\langle f_\alpha, f_\eta\rangle\,. \tag{3.8}$$

The main technical accomplishment then is to show that (3.6) also holds for $\lambda = \mu$. This is obtained in [JY7] and likewise in [MSY] by a difficult case by case study. It would be interesting to have a proof of this result that does not rely on the classification of symmetric spaces. v We also note that for $G/K = Sp(p,1)/Sp(p) \times Sp(1)$ or the hyperbolic Cayley plane Theorem 7 reduces to Corlette's theorem [C2]. For Hamiltonian symmetric spaces, the result is due to Mok [Mk4].

If rank $(G/K) \geq 2$, we may choose as in §2 any λ with $-\lambda_1 < 2\lambda < 2\mu$ in (3.3) and conclude from (3.4), (3.5), and (3.6),

$$f_{\alpha\beta} \equiv 0 \quad \text{for all } \alpha, \beta. \tag{3.9}$$

and also

$$R_{\alpha\beta}\langle f_\alpha, f_\beta\rangle = \langle R(f_\alpha, f_\beta)f_\beta, f_\alpha\rangle\,. \tag{3.10}$$

Since we are employing geodesic normal coordinates, (3.9) is equivalent to f being totally geodesic.

4. GEOMETRIC RIGIDITY

The first rigidity theorems for lattices in symmetric spaces were obtained by Selberg [Sb]. Calabi-Vesentini [CV], Calabi [Ca] and Weil [W]. They showed that the cohomology groups arising from the theory of Kodaira-Spencer describing deformations of geometric structures vanish. This is infinitesimal rigidity, and it implies local rigidity by reasoning based on the implicit function theorem. For example, Calabi-Vesentini showed that compact quotients of irreducible Hermitian symmetric spaces of noncompact type other than $Sl(2,\mathbb{R})/SO(2)$ are locally rigid. In other words, any deformation of a compact lattice in such a space G/K is trivial, i.e., obtained by conjugation with elements of G.

Of course, this result does not hold for $Sl(2,\mathbb{R})/SO(2)$, because there exist nontrivial continuous families of compact quotients, namely (hyperbolic) Riemann surfaces of a given genus $g \geq 2$. This case, however, is the only exception.

Mostow [Ms] showed strong rigidity of compact quotients of irreducible symmetric spaces of noncompact type. This means that any two lattices, such as Γ, Γ', which are isomorphic as abstract groups, are in the same G and are isomorphic as subgroups of G. Geometrically this means that the quotients $\Gamma\backslash G/K$ and $\Gamma'\backslash G/K$ are isometric. Here, they carry the Riemannian metric induced from the symmetric metric on G/K. (For simplicity of the present discussion, we assume that the lattices operate without fixed points on G/K, i.e., no element in Γ other than the neutral element leaves any point fixed. Then the quotient spaces are locally symmetric Riemannian manifolds. This condition on the lattices is called torsion free. It is known that any lattice has a torsion free sublattice of finite index. Therefore, this assumption gives no serious restriction.) Margulis [Mg2] then showed superrigidity if rank $(G/K) \geq 2$. This means that any homomorphism $\rho : \Gamma \to H$ (Γ as above) extends to a homomorphism $G \to H$. This is true in either of the followoing cases: H (like G) is a simple noncompact algebraic group and if $\rho(\Gamma)$ is Zariski dense, or $\rho(\Gamma)$ is contained in a compact subgroup of H, if H is an algebraic subgroup of some $Sl(n, Q_p)$. (If H is defined over $\mathbb{C}$, then actually either of the two cases may occur but this does not interest us here.) Margulis' results and their proofs are nicely presented in Zimmer's book [Zi]. We also refer to the monograph of Margulis [Mg2] for more general results.

Superrigidity for quotients of quaterionic hyperbolic space and the hyperbolic Cayley plane was shown by Corlette [C2] in the Archimedean case and by Gromov-Schoen [GS] in the p-adic case, as will be discussed in more detail below. For real or complex hyperbolic space, however, superrigidity does not hold anymore.

Margulis also showed that superrigidity implies arithmeticity of a lattice. While the local rigidity theorems described above proceed by showing the vanishing of certain harmonic forms and might therefore be considered as precursors of a harmonic map approach as argued above, the method of Mostow and Margulis studies the action of a lattice at the sphere at infinity of G/K. They develop and use powerful results from ergodic theory to obtain rigidity of such action.

We now want to indicate how harmonic maps can extend and strengthen such results.

The first result in this direction was obtained by Siu [S1], taking up a suggestion of the second author. In a later reformulation [S2], he obtained the following Bochner type identity for a harmonic map

$$f : M \to N$$

between compact *Kähler* manifolds M and N

$$\begin{aligned} \int_M f^i_{\alpha\bar\alpha} f^{\bar i}_{\beta\bar\beta} &= \int f^i_{\alpha\bar\beta} f^{\bar i}_{\bar\alpha\beta} \\ &= \int R^N_{i\bar j k \bar \ell}(f^i_{\bar\alpha} f^{\bar j}_{\bar\beta} - f^i_{\beta} f^{\bar j}_{\alpha})(f^k_{\bar\beta} f^{\bar l}_{\bar\alpha} - f^k_{\bar\alpha} f^{\bar\ell}_{\bar\beta}) \,. \end{aligned} \tag{4.1}$$

Here, we use holomorphic normal coordinates on both M and N, and $R^N_{i\bar j k\bar l}$ is the curvature tensor of N. We do not get a Ricci contribution from M here, because the Kähler condition implies that for functions $f : M \to \mathbb{R}$

$$f_{\alpha\bar\alpha\bar\beta} = f_{\alpha\bar\beta\bar\alpha}$$

(we only get a curvature term if we commute a barred index with an unbarred one).

The curvature term in (4.1), however, is less symmetric than the sectional curvature expression, and so in order to exploit (4.1), one needs some stronger curvature condition. The precise statement of this curvature condition, however, will not be of interest to us, so suffice it to say that it is implied by a nonpositive curvature operator. Thus, whenever N has nonpositive curvature operator,

$$f_{\alpha\bar\beta} = 0 \quad \text{for all } \alpha, \beta \,, \tag{4.2}$$

meaning that f is *pluriharmonic*, i.e., the restriction of f to every complex submanifold is again harmonic. With the help of lengthy curvature computations (somewhat simplified in [Zh] and [S2]), Siu [S1] then showed

Theorem 8. *Let M be a compact Kähler manifold of $\dim_{\mathbb{C}} M \geq 2$, homotopically equivalent to a compact locally Hermitian symmetric space N of noncompact type. Then M is $\pm$ biholomorphically equivalent to N.*

The proof proceeds along a pattern already outlined: by Theorem 1, we obtain a *harmonic* homotopy equivalence $f : M \to N$. Curvature computations show that f then has to be $\pm$ holomorphic. As a homotopy equivalence, it then also has to be a diffeomorphism.

If we assume furthermore that M is also locally Hermitian symmetric, an application of Royden's version of Yau's Schwarz lemma ([R],[Y]) then shows that f is an isometry. This in particular implies Mostow's rigidity theorem in the Hermitian symmetric case.

Siu's reasoning, however, does not apply in the case when N is a hyperbolic Riemann surface. Although in that case, the image has negative curvature in the best possible sense, the conclusion about holomorphicity of harmonic maps cannot possibly hold because the complex structure of N can be deformed.

It was discovered by Jost-Yau [JY1] that although in that case, harmonic maps in general will not be holomorphic, such maps nevertheless exhibit some kind of holomorphic structure:

Theorem 9. *Let $f : M \to N$ be a harmonic map from a compact Kähler manifold M onto a hyperbolic Riemann surface. Then the generic preimage of $p \in N$ consists of holomorphic subvarieties of M.*

In a certain sense, this result again is a generalization of the linear case where harmonic 1-forms on M, i.e., the Albanese map of M to a torus, yield holomorphic 1-forms. Thus, we get some kind of holomorphic foliation on M. Considering the leaf space for this foliation, one easily obtains the following corollary:

Theorem 10. *Let M be a compact Kähler manifold. If M admits a continuous map onto some surface of genus ≥ 2 that is nontrivial on the second homology, then there exists a Riemann surface N of genus ≥ 2 and a nontrivial holomorphic map*

$$f : M \to N .$$

In particular, the universal cover of M admits a bounded holomorphic function.

This result is sometimes called Siu's theorem or the Siu-Beauville theorem, because Siu [S3] reformulated the argument of [JY1] and Beauville [Be] obtained a proof by algebro-geometric methods.

On the basis of Theorem 9, also the following result was obtained by Jost-Yau [JY2] for $n = 2$ and by Mok [Mk1], [Mk3] in the general case.

Theorem 11. *Let the compact Kähler manifold M be homotopically equivalent to a compact quotient $\Gamma\backslash H^n = N$, where $H = Sl(2,\mathbb{R})/SO^2$ is the upper half plane. Then M is $\pm$ biholomorphically equivalent to a quotient of H^n. If ($n \geq 2$ and) Γ acts irreducibly, M is $\pm$ biholomorphically equivalent to N itself.*

The holomorphic foliation result of [JY1] also applies to other cases of negative image curvature where the harmonic map again need not be holomorphic. Because of this generality, this holomorphic foliation has found many more applications, e.g., by Carlson-Toledo [CT1], Jost-Yau [JY5], Simpson [S1], Zuo [Z], Jost-Zuo [JZ1], [JZ2].

However, there are also situations where one obtains factorizations not through curves, but through higher dimensional varieties, see [Si2], [Z]. Most of these applications also use a Bochner type identity of Sampson [Sa] and its consequences. This identity is more versatile than Siu's one, and it holds for harmonic maps

$$f : M \to N$$

from a Kähler manifold M to a Riemannian manifold N, and we now describe it. One considers

$$f_{\alpha\beta} dz^\alpha \otimes dz^\beta := \langle f_\alpha, f_\beta \rangle dz^\alpha \otimes dz^\beta . \tag{4.3}$$

Differentiating and contracting, one obtains the (1,0) form

$$\xi_\alpha dz^d := \langle f_{\alpha\bar\beta}, f_\beta \rangle dz^\alpha \tag{4.4}$$

(using normal coordinates as usual) and computes

$$div(\xi_\alpha dz^\alpha) = \langle f_{\alpha\bar\beta}, f_{\alpha\bar\beta}\rangle - \langle R(f_\alpha, f_\beta) f_{\bar\beta}, f_{\bar\alpha}\rangle \ . \tag{4.5}$$

The curvature term is nonpositive if N has so-called Hermitian seminegative curvature. Sampson [Sa] thus obtained

Theorem 12. *Let M be a compact Kähler manifold, N a Riemannian manifold of Hermitian seminegative curvature, $F : M \to N$ harmonic. Then f is pluriharmonic, i.e., $f_{\alpha\bar\beta} \equiv 0$ for all α, β and also the curvature term in (4.5) vanishes identically.*

In particular, under the conditions of Theorem 12,

$$f_{\alpha\beta} dz^\alpha \otimes dz^\beta$$

is a holomorphic 2-form (if $\dim_{\mathbb{C}} M = 1$, this holds without a curvature condition on N and also for noncompact M).

The curvature tensor of Riemannian locally symmetric spaces G/K of noncompact or Euclidean type is Hermitian seminegative. We consider the corresponding Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

of the Lie algebras. $\mathfrak{g}$ is realized as an algebra of real matrices, and $\mathfrak{p}$ is identified from an analysis of the vanishing of the curvature expression in (4.5).

Theorem 13. *Under the assumptions of Theorem 12, for each $z \in M$, $f(T_z^{(1,0)}M)$ is an abelian subalgebra of $\mathfrak{p}^{\mathbb{C}}$, the complexification of $\mathfrak{p}(\cong T_{f(z)}N)$.*

Theorem 13 means that

$$[df(V), df(W)] = 0 \quad \text{for all } V, W \in T_z^{1,0}M \ .$$

¿From this point of view, the classification of harmonic maps is reduced to studying abelian subalgebras of $\mathfrak{p}^{\mathbb{C}}$. They can be of rather different types, namely nilpotent or semisimple, and their classification is not so easy, see [CT1], [CT2].

Sampson's result also yields an interpretation of harmonic maps in terms of Higgs bundles. This line of investigation was started by Hitchin [H] and further pursued by Simpson,Corlette, Zuo, Jost-Zuo, see e.g., [Si1], [Si2], [Si3], [C1], [Z], [JZ1]. Let us briefly describe the setting.

We consider a flat $Gl(n, \mathbb{C})$ bundle V over the Kähler manifold M. (For the following discussion, one may replace $Gl(n, \mathbb{C})$ by an arbitrary linear algebraic group $G^{\mathbb{C}}$.) V then carries a flat connection D. Introducing a metric g on V leads to a decomposition

$$D = D_g + \vartheta \tag{4.6}$$

corresponding to the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \quad (\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C}))$$

where D_g respects the metric.

The metric is harmonic if

$$D_g^* \vartheta = 0 . \tag{4.7}$$

A metric is given by a ρ-equivariant map f from the universal cover X of M,

$$f : X \to Gl(n, \mathbb{C})/U(n) ,$$

where

$$\rho : \pi_1(M) \to Gl(n, \mathbb{C})$$

is the representation defining V. The metric is harmonic if f is harmonic; namely, we have

$$df = \vartheta . \tag{4.8}$$

Thus

$$df \in H^0(\Omega^1(M) \otimes T(G/K)) = H^0(\Omega^1(M) \otimes \mathfrak{p}) .$$

Thus, we may consider $\vartheta = df$ as 1-form with values in the self adjoint endomorphism of V,

$$df \in H^0(\Omega^1(X) \otimes End\, V) .$$

Theorem 13 now says that

$$\vartheta \wedge \vartheta = 0 . \tag{4.9}$$

We now split

$$\begin{aligned} \vartheta &= \vartheta^{1,0} + \vartheta^{0,1} \\ D_g &= D' + D'' \end{aligned}$$

into components, and we obtain from Theorem 13

$$(D'')^2 = 0 .$$

$E = (V, D'')$ therefore is a holomorphic bundle. The complex structure on E, however, is different from the complex structure induced on V by the flat connection D, unless f is constant, i.e., ρ is a $U(n)$ representation.

A Higgs bundle over M now is a holomorphic bundle E together with

$$\vartheta^{1,0} : E \to \Omega^{1,0}(X) \otimes E$$

satisfying the integrability condition

$$\vartheta^{1,0} \wedge \vartheta^{1,0} = 0 . \tag{4.10}$$

By (4.9), from V and the harmonic map f, we thus obtain a Higgs bundle.

We now state some results that were obtained in similar form in [CT1] and [JY5]. Let $H^n_{\mathbb{R}}$ and $H^n_{\mathbb{C}}$ be real and complex hyperbolic space, resp., i.e., $H^n_{\mathbb{R}} = SO(n,1)/S(O(n) \times O(1)), H^n_{\mathbb{C}} = SU(n,1)/S(U(n) \times U(1))$.

Theorem 14. *Let M be a compact Kähler manifold, $\Gamma = \pi_1(M)$.*

a) *Let*

$$\rho : \Gamma \to SO(n,1)$$

be a reductive homomorphism with discrete image. Then the ρ-equivariant harmonic map

$$f : M \to \rho(\Gamma)\backslash H^n_{\mathbb{R}} =: N$$

factors holomorphically through a compact Riemann surface Σ, i.e.,

$$\begin{tikzcd} M \arrow[d, "\pi"'] \arrow[dr, "f"] & \\ \Sigma \arrow[r, "h"'] & N \end{tikzcd}$$

with holomorphic π and harmonic h.
Consequently, unless $\rho(\Gamma)$ is finite, there exists a nontrivial homomorphism $\sigma : \Gamma \to \mathbb{Z}$.

b) *For $SU(n,1)$ instead of $SO(n,1)$, the same result holds, unless*

$$f : M \to \rho(\Gamma)\backslash H^n_{\mathbb{C}}$$

is $\pm$ holomorphic.

Note that this factorization result includes the case where $f(M)$ is a closed geodesic in N. In the case of quaternionic hyperbolic space, Carlson-Toldedo [CT1] show that again the result corresponding to a) holds, unless f lifts to a variation of Hodge structure.

In [JY5], the result of Theorem 14 is extended to the case where the image is not necessarily symmetric, but still satisfies an appropriate negative curvature condition. Combining Theorem 14 with results of Mok [Mk2], in [JY7] also the following result was obtained

Theorem 15. *Let Γ be a cocompact lattice in G/K, a Hermitian-symmetric space of noncompact type of rank ≥ 2, and with $\Gamma\backslash G/K$ irreducible. Then every discrete homomorphism*

$$\rho : \Gamma \to SO(n,1) \quad or \quad SU(n,1)$$

has finite image.

in this regard, we also refer to results of Spatzier-Zimmer [SZ], obtained by a different method (without harmonic maps).

The next significant step was taken by Corlette [C2]. He showed that the product of the Hessian of a harmonic map with any parallel form on the domain vanishes if the image has nonpositive curvature. For quotients of quaternionic hyperbolic space and the hyperbolic Cayley plane this allowed him to conclude that the Hessian itself vanishes, i.e., that the map is totally geodesic. This gives his result stated above.

At this point, we should describe how the result of Theorem 7 implies superrigidity. Under the situation of Theorem 7, by Theorem 2, there exists a ρ-equivariant harmonic map

$$f : G/K \to Y\ ,$$

which by Theorem 7 then is totally geodesic. One then takes $Y = H/L$ to be a symmetric space as well. f maps geodesics in G/K to geodesics in H/L, hence 1-parameter subgroups of G to 1-parameter subgroups of H. Thus, we obtain a homomorphism

$$\bar{\rho} : G \to H$$

extending $\rho : \Gamma \to H$.

If H is an algebraic subgroup of $Sl(n, \mathbb{Q}_p)$, it operates on a Euclidean Tits building T. T, while not smooth, still is a metric space with nonpositive curvature in the generalized sense of Alexandrov. In the paper [GS] of Gromov-Schoen, a theory of harmonic maps into Euclidean Tits buildings was developed, with sufficiently strong regularity results to justify applications of Bochner type identities and to extend Corlette's result to representations into $Sl(n, \mathbb{Q}_p)$. Combining their results with the arguments leading to Theorem 7, in [JY7] the following general theorem was obtained:

Theorem 16. *Let $X = G/K$ be an irreducible symmetric space of noncompact type and rank ≥ 2, Γ a cocompact lattice. Let*

$$\rho : \Gamma \to Sl(n, \mathbb{Q}_p)$$

be a homomorphism, for some $n \in \mathbb{N}$ and some prime p. Then $\rho(\Gamma)$ is contained in a compact subgroup of $Sl(n, \mathbb{Q}_p)$.

As mentioned above, these results imply that cocompact lattices in G/K of rank ≥ 2 are arithmetic.

An alternative to the preceding argument was suggested by Gromov [G], namely to use foliations of arbitrary symmetric spaces by Hermitian symmetric ones and to use foliated versions of the Bochner identities of Siu and Sampson.

5. THE FINITE VOLUME CASE

The rigidity theorems of Mostow and Margulis continue to hold for lattices Γ in G/K that are not cocompact (Prasad [Pr] in the rank one case, Margulis [Mg2] for rank ≥ 2). This means that

$$\Gamma \backslash G/K$$

has finite volume, but is no longer compact.

In order to extend the harmonic map approach to that case, one needs to produce harmonic maps on $\Gamma\backslash G/K$ that are still amenable to Bochner type vanishing arguments. Since those arguments depend on the fact that the integral of the Laplacian Δf of a smooth function vanishes in the compact case, here one needs a good control at infinity over those functions to which Δ is applied, so that this integral still vanishes. It is not hard to see that the condition that f has finite energy,

$$E(f) < \infty$$

suffices for that purpose on domains of finite volume.

The task therefore is to construct ρ-equivariant maps of finite energy. In the Hermitian symmetric case, this problem was solved almost completely in [JY3], [JY4]. Let us quote the results.

Theorem 17. *Let N be a locally irreducible Hermitian locally symmetric space of noncompact type, rank 1 (i.e., a quotient of complex hyperbolic space), $\dim_{\mathbb{C}} N \geq 2$, and finite volume. Let M be a Kähler manifold, properly homotopically equivalent to N. Assume that there exists a smooth Kähler compactification $\bar{M}$ of M with $\bar{M}\setminus M$ a hypersurface with simple normal crossings. Then M is $\pm$ biholomorphic to N.*

The assumption on the existence of a compactifying divisor is necessary in order to exclude purely topological deformations of a divisor. The smoothness assumption can be satisfied at least in the quasiprojective case by Hironaka's theorem.

Theorem 18. *Let N be a locally irreducible Hermitian locally symmetric space of noncompact type, rank ≥ 2, and finite volume. Let M be a quasiprojective manifold, properly homotopically equivalent to N, with a projective compactification $\bar{M}$ with $\bar{M}\setminus M$ a subvariety of $\mathrm{codim}_{\mathbb{C}} \geq 3$ in $\bar{M}$. Then M is $\pm$ biholomorphically equivalent to N.*

Again, the existence of a suitable compactification is necessary. Whether the codimension requirement is necessary, is not clear. In any case, Theorems 17 and 18 together cover all cases of rigidity for the Hermitian symmetric spaces, except quotients of the Siegel upper half plane of degree 2.

In the situation of Theorem 17, a finite energy harmonic map can be constructed because N can be compactified by adding finitely many points. Therefore, short loops around $\bar{M}\setminus M$ can be mapped to short loops in N, and the energy density can be controlled.

In the situation of Theorem 18, because of the high codimension of $\bar{M}\setminus M$, the volume form of a suitable metric on M decays sufficiently fast towards $\bar{M}\setminus M$ to compensate a possible blow-up of the energy density.

In fact, locally symmetric spaces of finite volume in most cases admit compactifications of rather high codimension, and therefore, one should get finite energy harmonic maps. For quotients of quaternionic hyperbolic space and the hyperbolic Cayley plane, this was verified by Corlette [C2]. The situation becomes easier here, because the spaces are of rank 1 and therefore can be compactified by adding finitely many points.

Recent work of Saper [Sp] produces finite energy maps in almost all cases, but his result depends on certain properties of the compactification that hold for arithmetic lattices. Of course, the lattices under consideration are all arithmetic by Margulis' theorem, but the point of the present approach is to give an alternative proof of Margulis' results and this should not assume arithmeticity.

Other instances wherefinite harmonic maps from quasiprojective manifolds have been constructed and applied are [JZ1], [JZ2].

We would like to conclude by mentioning that in [JY4], a result analogous to Theorem 18 was proved for (finite covers of) the moduli space M_p of Riemann surfaces of genus $p \geq 2$, and by correcting an error there. Namely, in order to produce a proper harmonic homotopy equivalence, we had to exclude that M_p can be retracted onto its ideal boundary ∂M_p. The argument for this fact given in [JY4] is incorrect. That M_p cannot be retracted onto ∂M_p can be easily argued as follows, however: it is well known that the first Chern class $c_1(\Lambda)$ of the Hodge line bundle $\wedge$ over M_p cannot be supported on ∂M_p, and thus, there is a topological obstruction for the retraction.

6. ALGEBRAIC VARIETIES OVER FUNCTION FIELDS

Theorems D and E of the introduction are the solutions of the so-called Mordell and Shafarevitch problems over function fields. In geometric terms, one has an algebraic surface fibered by curves with generic fiber of genus g. The cases $g = 0, 1$ were treated by the Italian school and Kodaira. For $g \geq 2$ the so-called Mordell and Shafervitch conjectures for curves over function fields were solved by Manin [Mn] with other proofs offered by Grauert [Ga] and Arakelov [Ar], respectively (building on earlier work of Parshin [P]). Arakelov's theorem says that for a fixed base curve C, and $S \subset C$ consisting of finitely many points, and $g \geq 2$, there exist only finitely many nontrivial curves over C, of genus g over C/S. The theorem of Manin says that such a nontrivial curve over C admits at most finitely many sections, and even an isotrivial curve ('isotrivial' meaning constant over a finite cover of C) admits at most finitely many nontrivial sections.

Now a fibration as in Arakelov's theorem induces a holomorphic map from C into $\bar{M}_g$, the Deligne-Mumford compactification of the moduli space of curves of genus g. This map satisfies the technical condition of local liftability to the Teichmüller space level near the singularities of M_g, because it arises from a topological fiber space. Likewise, a section induces a locally liftable, holomorphic map into a compactification of the universal modular curve $\mathcal{M}_g$. For the sake of simplicity, we shall therefore neglect the question of the singularities of M_g and $\mathcal{M}_g$ in the subsequent discussion and pretend that they are smooth.

It was first suggested by Grauert-Reckziegel [GaR] to use the geometry of M_g and $\mathcal{M}_g$, in order to deduce the above results. In this setting, the proof consists of two steps. The first one is boundedness, namely to show that there are only finitely many homotopy classes of maps from C into $\bar{M}_g$ or $\bar{\mathcal{M}}_g$ with the image of $C\backslash S$ contained in the interior that contain holomorphic maps. The second one is rigidity, namely to show that each such homotopy class contains only finitely many holomorphic maps, in particular no nontrivial families of holomorphic maps.

A boundedness proof along these lines was obtained by Noguchi [N] who exploited the fact that M_g is a hyperbolic space in the sense of Kobayashi. Therefore, holomorphic maps into M_g are equicontinuous and can hence only be contained in finitely many homotopy classes. A rigidity proof complementing Noguchi's approach was found by Imayoshi-Shiga [IS].

Results of Tromba [T] (negativity of the curvature) and Wolpert [W] (existence of convex exhaustion functions) on the geometry of Teichmüller space equipped with its Weil-Petersson metric make it possible to use harmonic maps to prove the above theorems ([JY6]). Since the holomorphic sectional curvature of the Weil-Petersson metric is bounded from above, the Schwarz lemma ([Y],[R]) implies equicontinuity of holomorphic maps into M_g or $\mathcal{M}_g$, yielding boundedness as before. For the rigidity part, one looks at a possible limit of homotopic holomorphic maps from C to $\bar{M}_g$ or $\bar{\mathcal{M}}_g$, with $C\backslash S$ contained in the interior as before. This limit either maps $C\backslash S$ again in the interior, or it maps all of C into the boundary, and in both cases one can use uniqueness considerations from the theory of harmonic maps to derive a contradiction. This implies that no nontrivial sequence of homotopic holomorphic maps can exist in this setting, yielding rigidity.

Of course, this approach works as naturally if one has an arbitrary compact Kähler manifold N and a divisor A instead of a curve C and a finite collection S of points.

With the methods of [JY6], one then obtains the following version of Arakelov's theorem:

Theorem 19. *Let X be a compact Kähler manifold, $R \subset X$ a divisor with only simple normal crossings. Then there exist at most finitely many nontrivial fiber spaces over X with one-dimensional fibers that have genus $g \geq 2$ over $X \backslash R$. In algebraic terminology: there exist at most finitely many nontrivial holomorphic curves of genus $g \geq 2$ over the function field of X, with bad reduction at most over R.*

In the same way, one also obtains the following version of Manin's theorem:

Theorem 20. *Let $\pi : M \to X$ be a fiber space as in Theorem 19. If M is nonisotrivial, it admits at most finitely many sections $\sigma : X \to M$ ($\pi \circ \sigma = id$), i.e., rational points. If it is isotrivial, it admits at most finitely many nonconstant sections.*

A particular interesting example of algebraic surfaces fibred by curves of genus $g \geq 2$ is given by the Kodaira surfaces [Kd].

With these techniques, one can also study variations of the singular set. For example, one has

Theorem 21. *Let C be a compact algebraic curve, S_1, S_2 linearly equivalent divisors on C; assume that $C \backslash S_1$ and $C \backslash S_2$ are not conformally equivalent. Let $f_i : B_i \to C$, $i = 1, 2$, induce fiber spaces over $C \backslash S_i$ with fibers of genus $g \geq 2$. Suppose these fiber spaces are topologically equivalent. Let $L : CP^m \to Div(C)$ be a linear system, $L(z_i) = S_i$, $i = 1, 2$. Then there exists a holomorphic family of fiber spaces including the two given ones, described by a holomorphic map*

$$H : C \times CP^m \to \bar{M}_g$$

with

$$H(C \backslash L(z) \times \{z\}) \in M_g$$

for all $z \in CP^m$ and

$$H_{|C \backslash L(z_i)} = h_i \, ,$$

where $h_i : C \backslash S_i \to M_g$ is the map induced by f_i, $i = 1, 2$.

Again, this can be generalized to higher dimensions.

The preceding constructions also work for fibrations by algebraic varieties other than curves. The case of fibrations by Abelian varieties was solved by Faltings [F]. While boundedness holds as before, for rigidity one needs an additional condition, even for nontrivial families. Namely, any family of Abelian varieties can be multiplied by a constant Abelian variety, and as the latter can be deformed, also the product can be deformed without necessarily being trivial itself. A more sophisticated example is presented in [F].

The above techniques then apply to reprove these results (the boundedness part was also obtained in [N]). More generally, if one has a fibration of a Kähler manifold M by algebraic varieties, one obtains the period mapping of Griffiths [Gi]

$$\varphi : N\backslash A \to H\backslash G/\Gamma ,$$

where N is the base of the fibration, A a divisor, the homogeneous complex manifold is the period domain, and the discrete group Γ is the global monodromy group. In general, $H\backslash G$ does not have nonpositive curvature so that the harmonic map techniques do not apply directly, but there is a natural projection

$$\pi : H\backslash G \to K\backslash G$$

onto a Riemannian symmetric space of noncompact type. The composition $\pi \circ \varphi$, while in general not holomorphic, is still harmonic, and thus harmonic map techniques can be applied after all as $K\backslash G$ has nonpositive sectional curvature.

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