CLASSIFICATION OF SOLUTIONS OF A TODA SYSTEM IN $\mathbb{R}^2$

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ABSTRACT. In this paper, we consider solutions of the following (open) Toda system (Toda lattice) for $SU(N+1)$

$$-\frac{1}{2}\Delta u_i = \sum_{j=1}^{N} a_{ij} e^{u_j} \quad \text{in } \mathbb{R}^2,$$

for $i = 1, 2, \cdots, N$, where $K = (a_{ij})_{N\times N}$ is the Cartan matrix for $SU(N+1)$. We show that any solution $u = (u_1, u_2, \cdots, u_N)$ with

$$\int_{\mathbb{R}^2} e^{u_i} < \infty, \quad i = 1, 2, \cdots, N,$$

can be obtained from a rational curve in $\mathbb{C}P^N$.

1. INTRODUCTION

Let $N > 0$ be an integer. The 2-dimensional (open) Toda system (Toda lattice) for $SU(N+1)$ is the following system

$$-\frac{1}{2}\Delta u_i = \sum_{j=1}^{N} a_{ij} e^{u_j} \quad \text{in } \mathbb{R}^2, \tag{1.1}$$

for $i = 1, 2, \cdots, N$, where $K = (a_{ij})_{N\times N}$ is the Cartan matrix for $SU(N+1)$ given by

$$\begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdot & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & -1 & 2 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

(Here the factor $\frac{1}{2}$ comes from $\frac{1}{2}\Delta u = u_{z\bar{z}}$.) System (1.1) is a very natural generalization of the Liouville equation

$$-\Delta u = 2e^u, \tag{1.2}$$

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which is completely integrable, known from Liouville [26]. Roughly speaking, any solution of (1.2) in a simply connected domain arises from a holomorphic function. System (1.1) is also completely integrable. All solutions of (1.1) in a simply connected domain arise from N holomorphic functions, see [21, 23, 24, 14]. However, it is difficult to determine the precise form of these holomorphic functions, when we require additional (or boundary) conditions for (1.2) or (1.1).

Recently, Chen-Li [6] classified all solutions of (1.2) in $\mathbb{R}^2$ with

$$\int_{\mathbb{R}^2} e^u < \infty. \tag{1.3}$$

Their result is very useful for 2-dimensional problems, especially for the study of the Moser-Trudinger inequality and the mean field equation, see [12, 27]. To obtain their classification, they used an energy inequality of Ding [11] and the method of moving plane to show that any solution has a rotational symmetry. Other proofs of the classification result were given by Chou-Wan [10] by using the complete integrability mentioned above and complex analysis and by Chanillo-Kiessling [5] by using an isoperimetric inequality and a global Pohozaev identity. The latter was applied to classify solutions of a class of Liouville type systems with non-negative coefficients in [5]. See also the work of Chipot-Shafrir-Wolansky [9]. The methods used by Chen-Li and Chanillo-Kiessling rely on the maximum principle. Hence, it is difficult (or impossible) to apply their method to study the similar problem for System (1.1). We believe the method of Chou-Wan can be applied to study (1.1). In fact, we notice that a similar method was used by Bryant [3] in his study of pseudo-metrics. We believe that his method can be adapted to classify (1.1) by using the Nevanlinna theory for holomorphic curves into $\mathbb{C}P^N$ instead of that for holomorphic functions. For the Nevanlinna theory for holomorphic curves into $\mathbb{C}P^N$, see for example [16].

System (1.1) has a very close relationship with a few geometric objects, holomorphic curves into $\mathbb{C}P^N$, flat $SU(N+1)$ connections and harmonic sequences, see for instance, [4, 22, 18, 17, 1, 15, 8]. To classify solutions of (1.1), it is natural to seek the help of differential geometry. Here, with such a help, we classify system (1.1) with

$$\int_{\mathbb{R}^2} e^{u_i} < \infty, \quad i = 1, 2, \cdots, N. \tag{1.4}$$

Theorem 1.1. *Any C^2 solution $u = (u_1, u_2, \cdots, u_N)$ of (1.1) and (1.4) has the following form*

$$u_i(z) = \sum_{j=1}^{N} a_{ij} \log \|\Lambda_j(f)\|^2 \tag{1.5}$$

for some rational curve in $\mathbb{C}P^N$. For the definition of $\Lambda_k(f)$, see section 3 below.

Any rational curve in $\mathbb{C}P^N$ can be transformed to

$$\phi_0(z) = [1, z, \cdots, \sqrt{\binom{N}{k}} z^k, \cdots, z^N],$$

by a holomorphic isometry, which is an element of $PSL(N+1, \mathbb{C})$. Hence the space of solutions of (1.1) and (1.4) is equivalent to $PSL(N+1, \mathbb{C})/PSU(N+1)$. The dimension of the solution space is $N^2 + 2N$.

Theorem 1.1 can be restated in a geometric way as follows:

Theorem 1.2. *Any totally unramified holomorphic map ϕ from $\mathbb{C}$ to $\mathbb{C}P^N$ satisfying the finite energy condition (3.11) below can be compactified to a rational curve.*

Theorem 1.2 is a generalization of the following well-known result: Any totally unramified compact curve in $\mathbb{C}P^N$ is rational.

When $N = 1$, Theorem 1.1 is just the classification result of Chen-Li.

The Toda system is of great interest not only in geometry, but also in mathematical physics. One of our motivations to study this system is the non-abelian Chern-Simons Higgs model, in which non-topological solutions are solutions of a perturbed Toda system. See [14, 19, 33, 30, 28].

2. Analytic aspects of the Toda system

In this section, we analyze the asymptotic behavior of solutions of (1.1)-(1.4) and obtain a global Rellich-Pohozaev identity. Since some results were presented in our previous work [20], we only give an outline of ideas. Similar methods were used in [6, 7, 5].

Let $I = \{1, 2, \cdots, N\}$. First, we have

Lemma 2.1. *Let u be a solution of (1.1)-(1.4). Then*

$$u_i(z) = -\gamma_i \log|z| + a_i + O(|z|^{-1}) \quad \textit{for } |z| \textit{ near } \infty,$$

where $a_i \in \mathbb{R}$ are some constants and γ_i are given by

$$\gamma_i = \frac{1}{\pi} \sum_{j=1}^{N} a_{ij} \int_{\mathbb{R}^2} e^{u_j}.$$

Proof. First, one shows that

$$\max_{i\in I}\sup_{z\in\mathbb{R}^2} u_i(z) < \infty, \tag{2.1}$$

see [20]. Set

$$v_i(z) = \frac{1}{\pi}\int_{\mathbb{R}^2}(\log|x-y| - \log(|y|+1))\sum_{j=1}^N e^{u_j}(y)dy,$$

for $i \in I$. (Note again that (1.1) has a factor $\frac{1}{2}$.) The potential analysis implies

$$-\gamma_i \log|z| - C \le v_i(z) \le -\gamma_i \log|z| + C, \tag{2.2}$$

for some constant $C > 0$, see [7]. Clearly $u_i - v_i$ is a harmonic function. Hence (2.1) and (2.2) imply that $u_i - v_i = c_i$ for some constant c_i. That is, u has the following representation formula

$$u_i(z) = \frac{1}{\pi}\int_{\mathbb{R}^2}(\log|x-y| - \log(|y|+1))\sum_{j=1}^N e^{u_j}(y)dy + c_i.$$

The above results and (1.4) imply

$$\gamma_i > 2, \quad i \in I. \tag{2.3}$$

Furthermore, we can show that

$$u_i = \gamma_i \log|z| + a_i + O(|z|^{-1}).$$

See, for example, [31]. □

Next, we have a global Rellich-Pohozaev identity for our system (1.1). Such an identity was obtained in [5] for a Liouville type system with nonnegative entries. Similar arguments work for our case, see [20]. Here we give another proof that is similar to the spirit of proof of the Main Theorems.

Proposition 2.2. *Let u be a solution of (1.1) and (1.4). Then we have*

$$\sum_{j,k=1}^N a^{jk}(4\gamma_k - \gamma_j\gamma_k) = 0, \tag{2.4}$$

where the matrix (a^{ij}) is the inverse of the Cartan matrix (a_{ij}).

Proof. Set

$$f = \sum_{j,k=1}^N a^{jk}\{(u_k)_{zz} - \frac{1}{2}(u_j)_z \cdot (u_k)_z\}.$$

We can check that f is a holomorphic function as follows:

$$\begin{aligned} f_{\bar z} &= \tfrac{1}{2}\textstyle\sum_{j,k=1}^{N} a^{jk}\{(\Delta u_k)_z - (u_j)_z \cdot \Delta u_k\} \\ &= -\textstyle\sum_{j,k,l=1}^{N} a^{jk}\{a_{kl}e^{u_l}(u_l)_z - a_{kl}(u_j)_z e^{u_l}\} \\ &= -\textstyle\sum_{j=1}^{N}(e^{u_j}(u_j)_z - e^{u_j}(u_j)_z) \\ &= 0. \end{aligned}$$

In the first equality we have used the symmetry of the matrix (a^{ij}). Using Lemma 2.1, we have the following expansion of f near infinity

$$\frac{1}{8}\frac{1}{z^2}\sum_{j,k-1}^{N} a^{jk}(4\gamma_k - \gamma_j\gamma_k) + \frac{c_{-3}}{z^3} + \cdots .$$

Hence, f is a constant (zero, in fact) and

$$\sum_{j,k=1}^{N} a^{jk}(4\gamma_k - \gamma_j\gamma_k) = 0.$$

□

3. Geometric aspects of the Toda system

In this section, we recall some relations between the Toda system and various geometric objects, flat connections, holomorphic curves into $\mathbb{C}P^N$ and harmonic sequences. Furthermore, we relate the mild singularities of solutions of the Toda system with the holonomy of the corresponding flat connections.

3.1. **From solutions of Toda systems to flat connections.** Let Ω be a simply connected domain and $u = (u_1, u_2, \cdots, u_N)$ a solution of (1.1) on Ω. Define $w_0, w_1, w_2, \cdots, w_N$ by the following relations

$$u_i = 2w_i - 2w_0 \quad \text{for } i \in I \text{ and } \sum_{i=0}^{N} w_i = 0. \tag{3.1}$$

It is easy to check that $w_0, w_1, \cdots, w_N$ satisfies

$$\left\{\begin{array}{lcl} -\Delta w_0 = 2(w_0)_{z\bar z} &=& e^{w_1 - w_0} \\ -\Delta w_1 = 2(w_1)_{z\bar z} &=& -e^{w_1-w_0} + e^{w_2-w_1} \\ \cdots & \cdots & \cdots \\ -\Delta w_N = 2(w_N)_{z\bar z} &=& -e^{w_N - w_{N-1}}. \end{array}\right. \tag{3.2}$$

It is well-known that (3.2) is equivalent to an integrability condition

$$\mathcal{U}_{\bar z} - \mathcal{V}_z = [\mathcal{U}, \mathcal{V}] \tag{3.3}$$

of the following two equations

$$\phi^{-1} \cdot \phi_z = \mathcal{U} \tag{3.4}$$

and

$$\phi^{-1} \cdot \phi_{\bar z} = \mathcal{V}, \tag{3.5}$$

where

$$\mathcal{U} = \begin{pmatrix} (w_0)_z & & & \\ & (w_1)_z & & \\ & & \cdots & \\ & & & (w_N)_z \end{pmatrix} + \begin{pmatrix} 0 & e^{w_1 - w_0} & & \\ & 0 & & \\ & & \cdots & e^{w_N - w_{N-1}} \\ & & & 0 \end{pmatrix}$$

and

$$\mathcal{V} = -\begin{pmatrix} (w_0)_{\bar z} & & & \\ & (w_1)_{\bar z} & & \\ & & \cdots & \\ & & & (w_N)_{\bar z} \end{pmatrix} - \begin{pmatrix} 0 & & & \\ e^{w_1 - w_0} & 0 & & \\ & & \cdots & \\ & & e^{w_N - w_{N-1}} & 0 \end{pmatrix}$$

Hence, from a solution of (1.1) (or equivalently (3.2)) we first get a one-form $\alpha = \mathcal{U}dz + \mathcal{V}d\bar z$. Then, with the help of the Frobenius Theorem, we obtain a map $\phi : \Omega \to SU(N+1)$ such that

$$\alpha = \phi^{-1} \cdot d\phi.$$

It is clear that α (or $d+\alpha$) is a flat $SU(N+1)$ connection on the trivial bundle $\Omega \times \mathbb{C}^{N+1} \to \Omega$, i.e., α satisfies the Maurer-Cartan equation

$$d\alpha + \frac{1}{2}[\alpha, \alpha] = 0$$

which is equivalent to the integrability condition (3.3), hence (3.2).

Lemma 3.1. *The map ϕ is determined upto an element of $SU(N+1)$. That is, any two $\phi_1, \phi_2 : \Omega \to SU(N+1)$ with $\phi_1^{-1} d\phi_1 = \phi_2^{-1} d\phi_2 = \alpha$ satisfy*

$$\phi_1 = g \cdot \phi_2,$$

for some element $g \in SU(N+1)$.

We call ϕ and α a *Toda map* and *Toda form* respectively.

3.2. **Holonomy of a singular connection.** Now consider an $SU(N+1)$ connection α on the punctured disk D^*. We can define its holonomy as in [29]. When $\Omega = D^*$ is not simply connected, we cannot apply the Frobenius theorem directly and have to consider the *holonomy.* Let (r, θ) be the polar coordinates. Write α as $\alpha = \alpha_r dr + \alpha_\theta d\theta$. α_r and α_θ are $su(N+1)$-valued. For any given $r \in (0,1)$, the following initial value problem,

$$\frac{d\phi_r}{d\theta} + \alpha_\theta \phi_r = 0, \quad \phi_r(0) = Id,$$

has a unique solution $\phi_r(\theta) \in SU(N+1)$. Here Id is the identity matrix.

Lemma 3.2. *If α is a flat connection on D^*, then $\phi_r(2\pi)$ is independent of r.*

Let h_α denote $\phi_r(2\pi)$. h_α is called the *holonomy* of α.
Remark. Here, we use a slightly different definition of holonomy. The usual holonomy is defined by the conjugacy class of h_α, which is invariant under gauge transformations.

Proposition 3.3. *Let $u = (u_1, u_2, \cdots, u_N)$ be a solution of (2.1) with*

$$u_i(z) = -\mu_i \log|z| + O(1), \qquad \text{near } 0.$$

If $\mu_i < 2$ for $i \in I$, then the corresponding flat connection α has holonomy

$$h_\alpha = \begin{pmatrix} e^{2\pi i\beta_0} & & & \\ & e^{2\pi i\beta_1} & & \\ & & \cdots & \\ & & & e^{2\pi i\beta_N} \end{pmatrix},$$

where $\beta_0, \beta_1, \cdots, \beta_N$ are determined by

$$\beta_i - \beta_0 = \frac{1}{2}\mu_i \quad (i \in I) \qquad \text{and} \qquad \sum_{j=0}^{N} \beta_j = 0. \tag{3.6}$$

Proof. Define w_i by (3.1). From the assumption, we have

$$w_i = -\beta_i \log|z| + O(1), \quad \text{near } 0.$$

A direct computation shows that

$$\mathcal{U} = \frac{1}{2z}\begin{pmatrix} -\beta_0 & & & \\ & -\beta_1 & & \\ & & \cdots & \\ & & & -\beta_N \end{pmatrix} + o(\frac{1}{|z|}),$$

where $o(\frac{1}{|z|})$ means that a matrix (b_{ij}) with entries satisfying $|z|b_{ij} \to 0$ as $|z| \to 0$. Here, we have used the condition that $\mu_i < 2$ for any $i \in I$. Similarly,

$$\mathcal{V} = \frac{1}{2\bar{z}}\begin{pmatrix} \beta_0 & & & \\ & \beta_1 & & \\ & & \cdots & \\ & & & \beta_N \end{pmatrix} + o(\frac{1}{|z|}).$$

Hence,

$$\alpha_\theta = \sqrt{-1}\begin{pmatrix}\beta_0 & & & \\ & \beta_1 & & \\ & & \cdots & \\ & & & \beta_N\end{pmatrix} + o(1).$$

Now it is easy to compute the holonomy of α. □

3.3. **From solutions of (1.1) to holomorphic curves.** When we have a Toda map $\phi : \Omega \to SU(N+1)$ from a solution of the Toda system, we can get a harmonic sequence as follows. First, define $N+1$ $\mathbb{C}^{N+1}$-valued functions $\hat{f}_0, \hat{f}_1, \cdots \hat{f}_N$ by

$$(\hat{f}_0, \hat{f}_1, \cdots \hat{f}_N) = \phi \cdot \begin{pmatrix} e^{w_0} & & & \\ & e^{w_1} & & \\ & & \cdots & \\ & & & e^{w_N}\end{pmatrix}.$$

Let f_i denote the map into $\mathbb{C}P^N$ obtained from $\hat{f}_i$. It is easy to check that f_i is a harmonic map and satisfies

$$\begin{aligned} (\hat{f}_k)_z &= \hat{f} + a_k \hat{f}_k, \\ (\hat{f}_k)_{\bar{z}} &= b_k \hat{f}_{k-1}, \end{aligned} \tag{3.7}$$

where

$$a_k = (\log|\hat{f}_k|^2)_z = (e^{2w_k})_z \text{ and } b_{k-1} = -|\hat{f}_k|^2/|\hat{f}_{k-1}|^2 = -w^{2(w_k - w_{k-1})}.$$

Here we assume that $\hat{f}_{-1} = \hat{f}_{N+2} = 0$. Hence, f_0 is a holomorphic map and f_{N+1} is an anti-holomorphic map into $\mathbb{C}P^N$. In fact, (3.7) is the Frenet frame of the holomorphic map f_0, see [18] or below. Furthermore, f_0 is unramified in Ω. For the definition of the ramification index, see [18] or below.

3.4. **From a curve to a solution of the Toda system.** From a non-degenerate (i.e. not contained in a proper projective subspace of $\mathbb{C}P^N$) holomorphic curve f_0 into $\mathbb{C}P^N$, one can get a family of associated curves into various Grassmannians as follows. Lift f_0 locally to $\mathbb{C}^{N+1}$ and denote the lift by $v = (v_0, v_1, \cdots, v_N)$. Hence, $f_0 = [v_0, v_1, \cdots, v_N]$. The *k-th associated curve* of f_0 is defined by

$$\begin{aligned} f_k : \Omega &\to G(k+1, n+1) \subset \mathbb{C}P^{N_k} \\ f_k(z) &= [\Lambda_k], \end{aligned}$$

where

$$\Lambda_k = v(z) \wedge v'(z) \wedge \cdots \wedge v^{(k)}(z).$$

See for example [18]. Here $N_k = \binom{N+1}{k+1}$.

Let ω_k be the Fubini-Study metric on $\mathbb{C}P^{N_k}$. The well-known (infinitesimal) Plücker formula is

$$f_k^*(\omega_k) = \frac{\sqrt{-1}}{2}\frac{\|\Lambda_{k-1}\|^2 \cdot \|\Lambda_{k+1}\|^2}{\|\Lambda_k\|^4} dz \wedge \bar{z}, \tag{3.8}$$

which implies

$$\frac{\partial^2}{\partial z \partial \bar{z}} \log \|\Lambda_k\|^2 = \frac{\|\Lambda_{k-1}\|^2 \cdot \|\Lambda_{k+1}\|^2}{\|\Lambda_k\|^4}, \qquad \text{for } k = 1, \cdots, N, \tag{3.9}$$

where $\|\Lambda_0\|^2 = 1$ and $\|\Lambda_N\| = \det(f, f', \cdot, f^{(N)})$. By choosing the normalization $\|\Lambda_N\| = 1$ (we can do this when we lift f), we can identify (3.9) with the Toda system (1.1) as follows. By setting

$$v_k = \log \|\Lambda_k\|^2,$$

system (3.9) becomes

$$-\frac{1}{2}\Delta v_i = \exp\{\sum_{j=1}^{N} a_{ij} v_j\}. \tag{3.10}$$

Clearly, (3.10) is equivalent to (1.1) by setting

$$u_i = \sum_{j=1}^{N} a_{ij} v_j.$$

For any curve $f : \Omega \to \mathbb{C}P^n$, the *ramification index* $\beta(z_0)$ at $z_0 \in \Omega$ is defined by the unique real number such that

$$f^*\omega = \frac{\sqrt{-1}}{2} |z - z_0|^{2\beta(z_0)} \cdot h(z) \cdot dz \wedge d\bar{z}$$

with h C^∞ and non-zero at z_0, where ω is the Kähler form of the Fubini-Study metric on $\mathbb{C}P^n$. For other definitions, see [18]. f is *unramified* if for any $z_0 \in \Omega$ the ramification index $\beta(z_0)$ vanishes. Hence, a solution of the Toda system (1.1) corresponds to an unramified holomorphic curve.

Let $f : \mathbb{C} \to \mathbb{C}P^N$ be a holomorphic curve and f_k its k-th associated curves. The finite energy condition is defined by

$$\int_{\mathbb{R}^2} f^*(\omega_k) < \infty, \qquad \text{for any } k \in I. \tag{3.11}$$

(3.11) means that the area of the k-th associated curve is bounded.

4. Proof of Main Theorems

Now we start to prove our main Theorems.

Proof of Theorem 1.1. Let

$$v_i(z) = u_i(\frac{\bar{z}}{|z|^2}) - 4\log|z|, \quad i \in I.$$

$v = (v_1, v_2, \cdots, v_N)$ satisfies (1.1) on $\mathbb{R}^2/\{0\}$. Applying Lemma 2.1, we have

$$v_i(z) = (\gamma_i - 4)\log|z| + O(1) \quad \text{near } 0.$$

Hence, using (2.3) Proposition 3.3 implies that the holonomy of the corresponding Toda form of v is

$$h_\alpha = \begin{pmatrix} e^{2\pi i\beta_0} & & & \\ & e^{2\pi i\beta_1} & & \\ & & \cdots & \\ & & & e^{2\pi i\beta_N} \end{pmatrix},$$

where $\beta_0, \beta_1, \cdots, \beta_N$ are determined by

$$\beta_i - \beta_0 = \frac{1}{2}(\gamma_i - 4) \quad \text{and} \quad \sum_{j=0}^{N} \beta_j = 0.$$

Now we know that the holonomy is trivial, i.e., h_α is the identity matrix, which clearly implies

$$\beta_i = 1 \mod \mathbb{Z} \quad \text{for } i = 0, 1, \cdots, N.$$

Hence, we have

$$\gamma_i = 2 \mod \mathbb{Z} \quad \text{for any } i \in I,$$

which, together with (2.3), implies that $\gamma_i \geq 4$ for any $i \in I$. Thus, $4\gamma_k - \gamma_j\gamma_k \leq 0$ for any $j, k \in I$. On the other hand, one can check that the matrix (a^{ij}) admits only positive entries. In fact, a direct computation shows that

$$a^{ij} = \frac{i(N+2-j)}{N+2}, \quad \text{for } i, j \leq \{\frac{N+2}{2}\},$$

where $\{b\}$ means the least integer larger than or equal to b. Other entries are determined by an obvious symmetry. Altogether, we obtain

$$\sum_{j,k=1}^{N} a^{jk}(4\gamma_k - \gamma_j\gamma_k) \leq 0,$$

and the equality holds if and only if $\gamma_i = 4$ for any $i \in I$. Applying the global Rellich-Pohozaev identity (2.4), we have

$$\gamma_i = 4 \quad \text{for any } i \in I.$$

Hence, v_i is bounded near 0. The elliptic theory implies that v_i is smooth. From the discussions presented in Section 3, it follows that the corresponding holomorphic curve f can be viewed as an unramified map from $\mathbb{S}^2$ to $\mathbb{C}P^N$, hence this curve is a rational curve, namely

$$f = [1, z, \cdots, \sqrt{\binom{N}{k}} z^k, \cdots, z^N],$$

up to a holomorphic isometry, an element in $PSL(N+1, \mathbb{C})$. Now we can investigate any solution of (1.1) and (1.4) as in subsection 3.4. From a holomorphic curve f, we get the k-th associated curves f_k and Λ_k. The solution of (1.1) $u = (u_1, u_2, \cdots, u_N)$ is given by

$$u_i = \sum_{j=1}^{N} a_{ij} \log \|\Lambda_j\|^2.$$

This completes the proof.

Proof of Theorem 1.2. From such a holomorphic curve $f : \mathbb{C} \to \mathbb{C}P^N$, we get a solution u of (1.1). It is clear that the condition (3.11) implies that u satisfies (1.4). As in the proof of Theorem 1.1, f can be extended to a curve from $\mathbb{S}^2$ to $\mathbb{C}P^N$, which is totally unramified. Hence, it is a rational curve.

Corollary 4.1. *The space of solutions of (1.1) and (1.4) is* $PSL(N+1, \mathbb{C})/PSU(N+1)$.

Proof. It follows from Theorem 1.1 and Lemma 3.1. □

Corollary 4.2. *Any solution* $u = (u_1, u_2, \cdots, u_N)$ *of (1.1) and (1.4) satisfies*

$$\frac{1}{\pi} \sum_{j=1}^{N} a_{ij} \int_{\mathbb{R}^2} e^{u_j} = 4.$$

In particular, if $N = 2$, *then*

$$2 \int_{\mathbb{R}^2} e^{u_1} = 2 \int_{\mathbb{R}^2} e^{u_2} = 8\pi.$$

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