Reflexive Expectation Formation

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Abstract

How do economic agents form expectations regarding asset prices and the development of macroeconomic quantities, when there is a fundamental uncertainty about the market process, like after financial crises? How does the formation of expectations fold back to the realized economic process, and in particular, to the selection of one of multiple possible equilibria in the economic process? We present a game theoretical model in which these questions can be studied in a systematic way.

We argue that financial expectation formation (the anticipation of future supply and demand conditions) can be usefully studied as a game-like situation of mutual anticipation of economic agents, if the key uncertainties that drive the development are endogenous, that is, if there is uncertainty about the actions of other economic agents. For instance, after the financial crisis of 2008, there was uncertainty about how governments, banks, and firms would behave, and the anticipation of this behavior was key to the formation of financial expectations of individual agents.

In our model, individual agents entertain higher order beliefs regarding the expectations of other economic agents, which are the basis for their own expectation formation. These beliefs are updated in a learning process. The learning process is more complex as in REE models: Prices also contain endogenously generated information regarding the mutual expectations of different agents. Agents cannot fully distinguish if price movements are caused by a change in information about fundamentals, or a change in expectations of the other agents.

With our model, we also contribute to the literature on learning in games. In the literature, most models of learning in games study situations in which the actions of other agents can

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be observed. In our model, the agents cannot observe the actions of other agents, but they learn about the actions of others indirectly via the observation of aggregate quantities like prices.

1 Introduction

How do economic agents form expectations regarding asset prices and the development of macroeconomic quantities, when there is a fundamental uncertainty about the economic process, like after financial crises? How does this formation fold back to the realized economic process, and in particular, the selection of equilibria in the economic process? These questions are pressing, in particular after the financial crisis of 2008, and even today, as we do not yet know if we are in a stable economic trajectory.

In our view, a key shortcoming of the currently established modeling paradigms to study macroeconomic development, that is, Dynamic Stochastic General Equilibrium and Rational Expectations models, is that here price movements are assumed to be caused by events exogenous to the economy, but not by endogenously caused by changes in the higher order beliefs of the interacting agents. In many important contemporary market situations, the key uncertainties are generated endogenously, that is, agents are mutually uncertain about the actions of other agents that will determine future supply and demand conditions, and this results in mutual anticipation. Consequently, when forming expectations regarding asset prices, and macroeconomic quantities, agents need not only take into account their belief regarding fundamental values, but also regarding the expectations of other economic agents.

We hence suggest to study expectation formation as a game-like situation of mutual anticipation of economic agents. This thought has already been articulated by some macroeconomists (like in Morris and Shin, 2000). However, we argue that the analysis by Morris and Shin is not reaching far enough. In particular, Morris and Shin do not discuss how prices mediate the respective expectation formation and mutual anticipation of interacting agents.

Our contribution is to offer such a model. In our model, expectation formation is based on mutual anticipation of the agents and their actions, and prices take a central mediating role.

That prices mediate information is, of course, a commonplace. However, in REE models, the mediation by prices is oversimplified: By making the (rather strong) assumption of common knowledge of the price function, in these models, prices fully reveal the information of the participating agents. In reality, we argue, prices mediate expectations, but in far more complex ways. Prices reflect expectations of the interacting economic agents, but these expectations reflect if the market participants anticipate pessimistic or optimistic equilibria in the economic process, and how uncertain they are regarding the actions of others in the future. In some cases, some agents may even actively influence prices to shape (and even misguide) the expectation formation of other agents. In our model, prices hence become more informative than in REE models: Prices also contain endogenously generated information regarding the mutual expectations of different agents. The theoretical perspective that prices contain endogenous information, information regarding the mutual beliefs of interacting economic agents, is, to our knowledge, new to the literature.

We argue that this theoretical perspective promises to be helpful to analyze a range of pressing questions in a new way. Take the example of interest rates for government bonds of weak Euro countries like Ireland or Greece. The interest for these papers reflect the fear of default, but default is not an exogenous event. Default of government bonds is a complex outcome of actions of many participating agents. For instance, whether or not other Euro countries will support the weak countries heavily influences the possibility of default. But a decision whether to support or not is not an exogenous event, but an action from a deliberate government, who also anticipates the other participating agents. Hence, the interest rates for government bonds of weak countries are the outcome of mutual anticipation of different agents.

It is often remarked that severe depressions are *not* primarily caused by exogenous events, like natural catastrophes, but rather by collective uncertainty regarding the expectations of other economic agents (see, for instance, Akerlof and Shiller, 2009).

Furthermore, consider the following example of verbalized reasoning. As a plausible example how practitioners may think, consider the following quote from the newspaper "the Economist " in 2010: "Interest rates will stay low only if growth remains slow. But if economies grow slowly, then profits will not rise fast enough to justify current share prices and incomes will not rise far enough to justify the prevailing level of house prices. If, on the other hand, the markets are right about the prospects for economic growth, and the current recovery is sustained, then governments will react by cutting off the supply of cheap money later this year." ("Bubble warning", The economist, Jan 7, 2010)

The journalist considers conditions of future supply and demand, about the expectations of other economic agents ("the market"), and about possible strategic actions ("governments cutting off the supply of cheap money"). The journalist also wondered if the majority of other agents ("the market") is right in terms of its expectations on growth.

The journalists reasoning is clearly a kind of boundedly rational reasoning. However, in some sense, his reasoning (and the resulting expectations) actually involve more complex reasoning as would be assumed to be in rational expectation models. In particular, he is aware that all economic agents have to cope with only partial awareness about future supply and demand conditions, and that, in consequence, agents strategically co-ordinate their expectations.

What are the dynamics in an economy in which all agents strategically anticipate their peers, if we picture all agents reasoning like the journalist in the quote above? Below, we present a model to analyze this question.

In our model, agents form expectations on the basis of mutual anticipation of their actions. They entertain beliefs on beliefs of the other agents. But moreover, agents plan ahead how their own actions will alter the beliefs of other agents over time.

With our model, we offer a structure to study the following questions in a systematic way: 1. What can agents read, theoretically, from prices, when knowledge is incomplete? 2. We allow the beliefs of the agents to be a priori inconsistent. Will the expectations of the agents become consistent over time?

In the literature, there exist several sophisticated models how agents can model the expectation of the market, that is the aggregate expectations of the collective of other agents. In this context, in particular the effect of random perturbations or shocks has been studied, with mathematical tools from the theory of stochastic processes. In existing models in which forecasting the forecasts of others is a key problem (Townsend, [34],Romer [30], Lorenzoni, [23]), forecasting others boils down to a problem of learning about information diffusion. In those models, agents have to solve an optimal inference problem. The key point is that price movements, e.g., as a result of external perturbations, reveal more information than absolute prices only, in particular about fundamental information that others may possess. At equilibrium, the expectations of the agents are consistent with the market that they generate. However, the possibility for the agents to solve their optimal inference problem hinges upon a general agreement on the uncertainties that unfold while the economy evolves. We argue that this general agreement often does not exist. Consequently, the mutual anticipation of the agents may, or may not be resolved in a self-consistent way.

We are (in long term perspective) motivated by a search for an alternative theory of economic expectation building. In the recent discussion, macroeconomic models based on dynamic stochastic general equilibrium models (that are based on the idea of rational expectations) were subject to criticism (Krugman, 2009, [22]; Colander et al. 2008 [8]). However, few theoretical alternatives to rational expectation models have been proposed. The suggestions in [8] are to use agent-based modeling, or to follow an "engineering" approach to macroeconomics. The suggestion to use agent based simulations is valuable, but we see the danger that such modeling is no longer analytically tractable, and it might be difficult to develop a coherent theory based on agent based simulation. In an "engineering" approach to policy making, one would dispense itself of a micro-foundation of the actions of economic agents, and a potential risk is that little theoretical process is made.

To maintain analytical rigor, as in DGSE and REE models, we argue, the analysis of economic expectation formation should be based on game theoretical analysis. However, there are some theoretical assumptions from these models that we do not maintain in our own modeling. We will now discuss these deviations.

In rational expectation equilibrium models (compare e.g. Dubey (1987) for a review) prices are fully revealing of the information of the agents. Ben-Porath and Heifetz ([5]) show that prices may not fully reveal states of nature when agents entertain heterogeneous beliefs on the price function, even though rationality and market clearing are common knowledge among the market agents. We argue that heterogenous beliefs regarding the price function are the real-world condition we need to deal with. We regard Ben-Porath and Heifetz results as important and provocative, and argue that their paper raises further important questions: If prices are not necessarily revealing information, as a given price can be consistent with different combinations of fundamental values and higher order beliefs of the interacting agents, do the higher order beliefs of interacting agents become more consistent over time, due to a learning process of the involved agents? Does this then lead to the revelation of information via prices over time? Or can higher-order uncertainties lead to severe inefficiencies in markets, and may we need additional institutions in markets to ensure that prices reflect the information that individual agents posses?

We hence study an economy in which prices are, like in the Ben-Porath and Heifetz paper, consistent with different combinations of fundamentals and higher order beliefs. Our contribution is to study an economy in which the agents anticipate the (putative) evolution of beliefs of other agents, and base their own present decisions on this anticipation. This dynamic setting allows us to investigate under which conditions higher order beliefs get aligned (and if prices then become information revealing) or if in some cases higher order uncertainties lead to severe coordination failures.

We believe that this study is important, because it has consequences for policy-making. In particular, if prices are not necessarily fully revealing information this implies that economic coordination is not necessarily efficient.

In our own model, we maintain the assumption of full rationality. That is, in our model, rationality as defined in Savage (1954) of all agents is common knowledge among the agents. But as has been pointed out by other scholars already (compare e.g. Angeletos (2011), Ben-Porath and Heifetz (2010) even given common knowledge of rationality the expectation formation of economic agents can be complex, and lead to endogenous fluctuations in the economic process. Endogenous uncertainties and fluctuations can arise even given fully rational agents, and our paper is one contribution in the emerging literature that attempts to understand how these uncertainties and fluctuations can be tamed.

Before we proceed, we briefly discuss the key technical contributions of this paper.

With our model, we also contribute to the literature on learning in games. In the literature, most models of learning in games study situations in which the actions of other agents can be observed. In our model, the agents cannot observe the actions of other agents, but they learn about the actions of others indirectly via the observation of aggregate quantities like prices.

Furthermore, while market interaction and price building processes have already been studied as strategic market games, incomplete knowledge of the interacting agents has so far only been studied in the spirit of Harsanyi (1967). While Harsanyi's seminal contribution allowed huge steps toward analyzing market interaction given informational uncertainty, we argue that we need to move beyond Harsanyi to fulfill the promises we made above.

In Harsanyi's 'Bayesian games', *the structure of the uncertainty that is to be resolved* is assumed to be common knowledge. Even though agents are allowed to entertain different degrees of uncertainty, in Harsanyi (1967) each agent's uncertainty needs to be expressible in form of a partition of a state space, *and the agents need to entertain identical prior belief about the 'objective' probability distribution over the singleton elements in the state space.* By making this assumption, agents' beliefs are *by construction* consistent with the process that they generate. We think that a characterizing element of real economies is that the agents would not be able to agree on a prior distribution over fundamental values.

The assumptions of Harsanyi have already been critically addressed in the literature, for instance, in Kadane ([20]). In our approach, agents do share some knowledge about the economy (the equations of the model), but the disagreement on the probability distributions over fundamentals makes the expectation formation problem of the agents more complex. In particular, they need to plan ahead in their assumptions regarding the evolution of the beliefs of the other agents. In our model, the agents can potentially be 'autistic' at time 0, that is, their beliefs about the beliefs of other agents may mismatch the objective beliefs of the other agents. However, as time unfolds, some adjustment of beliefs will take place, as a simple matter of the common use of Bayes' rule and the equations of the model. This implies that some agreement in the agents' beliefs is found over time (see below). But this is a qualitatively different situation as starting from a common prior distribution over singletons in the states of the world. A look at current economic situations suggests that once fundamental uncertainties regarding one issue are about to be resolved, another fundamental uncertainty emerges. For instance, by May 2011, we are about to grasp the consequences of the financial crisis of 2008, but now already again, we are uncertain regarding the implications of the Euro crisis, and regarding the catastrophe in Japan. Hence we need to model how fundamental uncertainty, that may well involve disagreement among the agents, resolves over time.

In the context of the simple model sketched in this paper, agents may entertain heterogenous prior beliefs about the production abilities and endowments of the other agents.

Our model offers a structure to study the effects of heterogenous prior beliefs on the evolution of expectations over time. In particular, our model helps to study if the prices mediate information in such a way that the beliefs of the agents become mutually consistent over time or not.

The paper will continue as follows. In the next section, we will present our model. We will then discuss our model, and finally conclude the paper.

2 A Model of Reflexive Expectation Formation

We study reflexive expectation formation in a stylized market situation. The modeling approach regarding the expectation formation of the agents that we advocate can be applied to other market situations as well, but here we take a stylized market situation as a starting point.

We study a simple market for one scarce durable good. We assume a finite number of agents, and trade takes place at discrete time intervals.

Each agent i ($i = 1, \ldots, I$) acquires an income a_i money that he earns in every time period. Agent's primary utility comes from money. However, the endowment (possession better word?) of the durable good of an agent also contributes to his utility, but only after some saturation point of money is reached. For illustration, we may think that the durable good is some luxury good (like real estate at the seaside) that only contributes to utility when agents have enough money to satisfy their basic needs. Hence agents whose income is above the saturation point have an incentive to buy the durable good, if the price of this good is attractive. Agents whose income is below the saturation point have an incentive to sell their durable good, but they may consider *when* to sell, by considering when the highest price of the durable good can be achieved. Agents with an income above the saturation point may also consider to sell their durable goods, if they anticipate an attractive price.

The agents do not know how many potential buyers and potential sellers of the durable good are in the market. They make use of the prices to update their beliefs in this regard, but prices are ambiguous, as one price can be consistent with different combinations of fundamentals and higher order beliefs of the interacting agents.

The market prices are publicly observed. All agents possess the same utility function, and this is common knowledge. Agents have different incomes a_i's, but they do not know the incomes of the other agents. They entertain a prior belief regarding the distribution of income of the other agents. The durable good is automatically inventoried, money is always 'consumed' immediately. That is, money cannot be kept over time, but the durable goods are kept and continue to contribute to utility. The agents also have different initial endowments of the durable good, that is, different b_i^0's. b_i^t is agent t's endowment with the durable good at time t. However, the other agents do not know the initial endowments of the other agents, they again only entertain a prior probability distribution over the b_i^0's.

The demand/supply at time t of agent i for the durable good is β_i^t, with a positive (negative) sign for demand (supply), and his possession of the durable good is b_i^t. Naturally, the endowments with durable goods change with the trade, that is

$$b_i^t = b_i^{t-1} + \beta_i^t$$

.

We write $a = (a_1, a_2, \ldots a_I)$ and $b^t = (b_1^t, b_2^t, \ldots b_I^t)$ for vectors of income and current endowments with the durable good respectively. There are I agents in the economy.

The utility function of agent i at time t is

$$u_i^t = \begin{cases} a_i - \beta_i^t p_h^t \text{ if} a_i^t - \beta_i^t p_h < 1 \\ a_i - \beta_i^t p_h^t + b_i^t \text{ if } a_i^t - \beta_i^t p_h \geq 1 \end{cases}$$

$a_i^t - \beta_i^t p_h^t$ is essentially the amount of money that the agents have after trading in the market. If they do not trade at all, $a_i - \beta_i^t p_h^t$ is simply a_i.

Utility in this money is linear. If the saturation point of income, that we here assume to be 1, is not reached (first case) then only the amount of money the agents have after trading contributes to utility. If the saturation point of income 1 is reached, then the endowment of the durable good b_i^t also contributes to utility. For simplicity, we assume that utility in the durable good is also linear, and that units of money and units of the durable good are of equal utility if the saturation point of money is reached. The assumption of this specific utility function is by no means critical to our approach. Indeed, the core of our model is the expectation dynamic that will be specified below, and the assumptions made here are simply useful to illustrate our concepts in the sequel.

The market mechanism by which prices come about is a stylized and simplified version of a strategic market game (compare Dubey et al., 1987). That is, trade is discrete and agents commit to offer and demand vectors in the next time period, before actual trade takes place. In strategic market games, a move at a time point would be the specification of an arbitrary feasible offer and demand vector. However, as we mainly which to study the expectation dynamics in this game, we simplify and allow agents only three options.

Concretely, each agent i chooses at t among the following options for their behavior in the market for the durable good at $t+1$:

1. Do not participate in the market for the durable good: $\beta_i^{t+1} = 0$

2. Sell all the durable goods in their possession: $\beta_i^{t+1} = -b_i^t$. S^{t+1} is the set of agents at $t+1$ selling durable goods.

3. Buy durable goods from production income. If agent i chooses this option, he commits to buy the durable good with all his income above 1. K^{t+1} is the set of agents at $t+1$ wanting to buy the durable good.

Moves of agent i at time t are denoted with x_i^t. Correspondingly, the vector of moves of all agents at time t is denoted with x^t.

The market for the durable good clears:

$$\sum_i \beta_i^\tau = 0$$

The supply for money to buy the durable good s^t:

$$s^t = \sum_{i \in K^t} \begin{cases} a_i - 1 \text{ if } a_i^t > 1 \\ 0 \text{ otherwise} \end{cases}$$

The supply of durable goods is

$$d^t = \sum_{i \in S^t} \beta_i^t$$

The price for durable goods at t :

$$p_h^t = \frac{s^t}{d^t}$$

By convention, if there are either no buyers or no sellers or both, then no trade and no price comes about.

Every player i can observe the price p^t during the course of play (and his own payoff), but i doesn't observe the actions of all players, just the aggregate consequences of these actions. All equations that we specified above are common knowledge, hence agents would be able to calculate a payoff for this game, if they knew all relevant parameters, that is, if they knew the vector of income a, and the initial endowments b^0.

We study a simple case of this game in the following. There are two time periods, and four agents. Agents maximize their utility over both time periods. The payoffs in the game are:

$$\pi_i = u_i^1 + u_i^2$$

The agents may update their beliefs using Bayes' rule, making use of the fact that given the observation of a particular price, only certain fundamental values (the a, b^t's) and combinations of moves of the other agents are consistent with the observation, given knowledge of the equations of the model. As the equations of the model are common knowledge, agents can exploit this to update their beliefs.

Hence, given a prior over (a, b^t, x^t) and data p^t, agents can compute a posterior.

Concretely,

$$P(a, b^t, x^t|p^t) = \frac{P(p^t|a, b^t, x^t)P(a, b^t, x^t)}{\int_{a,b^t,x^t} P(p^t|a, b^t, x^t)P(a, b^t, x^t)\mathrm{d}a, \mathrm{d}b^t, \mathrm{d}x^t} \tag{1}$$

We solve the specified game using Bernheim's ([6]) concept of rationalizability. That is, we assume that rationality of the agents in the sense of Savage ((**??**)) is common knowledge. Concretely, we assume that it is common knowledge that the agents play best response, given beliefs about moves of other agents.

However, we do not require a priori consistency of agents' first and higher order beliefs, as in Harsanyi's 'Bayesian games'. Rather, each agent entertains subjective beliefs and plans his moves through the game using these subjective beliefs.

To express this 'subjective planning according to subjective beliefs', we distinguish between two different types of variables in our model: *objectively existing* quantities (small letters), and *putative* quantities (capital letters). Capital letters by convention denote probability distributions over values of the corresponding quantities in small letters. For instance, X^1 is a probability distribution over moves at time 1.

Beliefs of an agent i at a given time point t are denoted by m_i^t. m_i^0 are the prior beliefs of agent i at time zero. These beliefs have an important effect on the market process that unfolds. For all agents i, m_i^0 contains first order beliefs (a probability distribution over a and b^0), and can potentially contain second order, and third and higher order beliefs, but we do not require that agents *necessarily* have prior higher order beliefs. Common knowledge of rationality may force the agents to form higher order beliefs over time. In the sequel, we understand an agent's first order beliefs to be his beliefs regarding fundamentals, that is, regarding the a's and b^ts. We understand an agent's second order belief to be his belief regarding the first order beliefs of all other agents. We understand an agent's third order belief as his belief regarding the second order belief of all other agents, and so on ad infinitum.

Importantly, the agents use their beliefs to plan ahead their moves, but they also update their beliefs using observations of prices. While equation ((1)) describes how the agents update their beliefs on fundamentals and move that may have been taken by the other agents, this belief update indirectly also enforces that the agents change their higher order beliefs. Certain move combinations may not be consistent with a given observation of a price. Certain higher order beliefs may then turn out to be inconsistent with the observation of the prices as well (we will elaborate). Hence an agent can, for instance, form posteriors at time one regarding the prior beliefs at time zero of other agents. These dynamics of the evolution of the putative beliefs of other agents will take center stage in the dynamics of our model.

In our notation, and expression of the form (left quantities) (right quantities) means that the putative 'left quantities' are consistent with the quantities specified in the right bracket.

For instance, $M_j^{t-1}(m_i^t)$ is what agent i believes at time t about the belief of agent j at the earlier time $t-1$. Consistency means here consistency with common knowledge of rationality and common knowledge of the observation of the market price (we will elaborate).

We first point out how the agents in the model form expectations of moves of other agents. Given expectations of moves, agents simply play best response, given their expectations of payoffs in the game, which follow from their first order beliefs. However, we will see that the solution concept of rationalizabilty will only constrain possible expectations of the agents, and will not determine them. We see this as an advantage of our model: It does not predict a single economic process, but specifies sets of possible sets of economic evolutions. We see this as an advantage: We may wish to study which sets of initial conditions lead to a set of desirable possible economic processes. At some point, we may use our model to study how we can design policies that rule out undesirable economic processes. (This could, in the real world, then be the task of a policy maker.)

Rationality implies that every agent i's moves are best responses, given his expectations of moves of the other players. Common knowledge of rationality implies that all moves that i expects from j are best responses, respective to i's beliefs regarding j's expectations of $-j$'s moves.

These two requirements put constraints on the moves which each single agent i can expect from the other agents. His expectations have to be consistent with his higher order beliefs, his beliefs regarding what the other agents believe.

Formally, in $(X_j^1)(m_i^0)$, every x_j^1 that has support in (X_j^1) is a putative best response of agent

j and is hence consistent with

$$(x_j^1, x_j^2) \in BR((\Pi_{x^1,x^2})(A, B^0)(M_j^0)(m_i^0), (X_{-j}^1, X_{-j}^2)) \tag{2}$$

Π_{x^1,x^2} denotes a payoff matrix, that encodes the payoffs of all agents for all combination of moves x^1, x^2. $BR(v, d)$ denotes 'best response' given a probability distribution 'v' over games, given a probability distribution 'd' over the actions of the other players in the game 'v'.

$(X_j^2)(m_i^0)$, i's expectation regarding the moves of agent j at time two, comes about by more complex means. Every agents a priori believes that his beliefs are 'correct' and that the putative beliefs of the other agents may be incorrect, if his higher order beliefs are different from his first order beliefs. Hence every agent plans ahead how, in his subjective perspective, how the other agents will learn at time one, and how this will affect their behavior at time two. In other words, agent i forms a prior regarding the posterior of agent j at time one that is the basis of agent j's move at time two.

i has a prior at time zero regarding the 'data', the durable good price and the production amount that j will observe at time $t = 1$, conditional on a move that he executes at time one.

Hence $(P^1)(A, B^0, X^1)(m_i^0, x_i^1)$ is i's prior regarding the data that some agent j will observe at time $t = 1$, if i executes x_i^1.

Using (1), i can compute $(M_j^1)(m_i^0)$, his prior regarding the posterior of agent j. j's prior regarding the posterior of agent j puts constrains on i's expectations of moves of j at time two, that is, puts restraints on $(X_j^2)(m_i^0)$.

Every x_j^2 that has support in $(X_j^2)(m_i^0)$ needs to be a putative best response, given i's prior beliefs regarding the putative posterior beliefs of some agent j at time one, conditional on an action x_i^1 that he plans to execute at time one. That is,

$$(x_j^2) \in BR((\Pi_{x^1,x^2})(A, B^0, B^1, X^1)(M_j^1)(m_i^0), (X_{-j}^2(B_{-j}^1)(B_j^1)(b_i^0, x_i^1)) \tag{3}$$

At time one, the agents update their beliefs, using (1).

They entertain a prior probability distribution over $(x^1, b_0, a, \text{higher order beliefs})$. In their posterior distribution, some specific combinations $(x^1, b_0, a, \text{higher order beliefs})$ will loose support, as a simple matter of the fact that only certain combinations (x^1, b_0, a, p^1) are consistent with the equations of the model. All other combinations (x^1, b_0, a), that are consistent with the price maintain support, and their probabilities are 'blown up' proportionally, as a matter of fact that probabilities are normalized to sum up to one (compare (1)).

Now also only certain ($x^1, b_0, a,$ higher order beliefs) maintain support in the posterior of some agent i. Recall that according to (2), in a specific ($x^1, b_0, a,$ higher order beliefs), x^1 needs to be consistent with the higher order beliefs. In the posterior, those ($x^1, b_0, a,$ higher order beliefs) for which (x^1, b_0, a, p^1) is inconsistent loose support. It may hence be that certain higher order beliefs completely loose support, because they are do not appear in any ($x^1, b_0, a,$ higher order beliefs) that maintained support.

In short, the moves needs to be ex post rationalizable given these posterior beliefs over the prior beliefs of other agents. To be precise, only those higher order beliefs of an agent have support in his posterior beliefs, which are consistent (according to (2)) with the posterior beliefs over moves at time one of the agent, which in turn are formed according to (1).

Finally, posteriors beliefs regarding the posterior beliefs of the other agents can be derived. This is simply done by taking the posterior beliefs of an agent i at time one regarding the prior beliefs at time zero, and then to compute how these putative beliefs are updated at time one, given the commonly observed price p^1. This computation is simply done using (1).

Given posterior beliefs regarding the posterior beliefs of some other agent j, agent i's expectations regarding j moves at time two have to be again rationalizable given these beliefs.

To summarize, we model the interplay between past and future in the reasoning of the interacting agents. The agents have prior assumptions regarding what other agents believe. These priors are updated to posterior higher order beliefs, using the price as a source of new data. The updated higher order beliefs are then used to form expectations regarding the moves of the other agents in the future.

2.1 An Illustrative Example

To illustrate our solution concept, we consider the following concrete case of beliefs of the four players.

We assume that all agents are endowed with one durable good at time zero: $b^0_1 = b^0_2 = b^0_3 = b^0_4 = 1$ We assume that $\alpha_1 = \alpha_2 = \frac{1}{2}$ and $\alpha_3 = \alpha_4 = \frac{3}{2}$. We assume that all agents' first order beliefs are, in expectation, correct, and that all agents entertain uniform distributions on some interval.

We assume that agent 3 believes that agent 4 expects with probability 0.9 only buyers at time 1: $(P(X^1_{-4} = (\text{buy, buy, buy})) = 0.9)(M^0_4)(m^0_3)$. As we assume common knowledge of the equations of the model, this implies that agent 3 believes that 4's higher order beliefs are consistent with this beliefs about moves.

We assume that agent 4 in fact beliefs that, with probability 0.9, there will be only buyers in the market: $(P(X^1_{-4} = (\text{buy, buy, buy})) = 0.9)(m^0_4)$.

We assume that agents 1 and 2 have higher order beliefs that attribute probability 0.9 to the 'objective' higher order beliefs, the higher order beliefs of agent 3 and 4 that we just specified.

It is common belief that b^0_i's are uniformly distributed on $[0, 2]$.

Note that it is very hard to write down higher order beliefs explicitly, as the space of which higher order beliefs is a big space. For instance, second order beliefs are probability distributions over probability distribution over a, b^0. However, note that if we instead specify move expectations of the agents, our common knowledge of rationality and the equations of the model requires us to study the prior higher order beliefs of the agents that are consistent with these move expectations. And the specific prior higher order beliefs of some agent i matter in determining his posterior beliefs at time one, after observing the price. While prior probabilities of combinations $(x^1, b_0, a, \text{higher order beliefs})$ may loose support due to observations of prices, the probabilities of the remaining $(x^1, b_0, a, \text{higher order beliefs})$ that still have support come about by 'blowing up' their prior probabilities.

In the concrete case described above, it hence matters a lot how agent 4 distributes 'the remaining probability 0.1' to other move combinations, as these probabilities will be 'blown up' with factor 10, once the agent observes that there were sellers in the market, because a price could be observed.

What follows are some observations in the concrete illustrative example.

Agent 3 knows that agent 4 may 'spoil the price': agent 3 and 4 both should buy, if the durable good price is below one, and another buyer is hence not desirable. Now agent 3 assumes that agent 4 will not buy at time one, as he expects many buyers and hence a high price for durable goods. So he plans to buy at time one himself. But he 'plans ahead' that agent 4 will learn at time one that there have been sellers (by the simple fact that a price exists).

Agents 1 and 2, given that they believe with 0.9 in the 'objective' beliefs of the other players, mentally follow the consideration of agents 3 and 4. So they expect that agent 3 will buy at time one, and agent 4 may buy at time 2. They hence are in a coordination game situation, in which they should try to sell when the other seller is not selling.

However, the whole situation crucially hinges upon the beliefs of agent 4 for the case that we observes that there are sellers. Recall that agent 4 at time zero believes $(P(X^1_{-4} = \text{(buy, buy, buy)}) = 0.9)(m^0_4)$. Let's say that $(P(X^1_{-4} = \text{at least one seller}) = 0.09)(m^0_4)$. There are still many different possible higher order beliefs that are consistent with $(P(X^1_{-4} = \text{at least one seller}) = 0.09)$. We need to specify then, as the case that there were only buyers at time one will loose support in the posterior of 4, and the higher order beliefs of 4 consistent with $X^1_{-4} = \text{at least one seller}$ will be 'blown up'.

If one specifies just first and second order beliefs, often this leads to dynamics in which many different moves are rationalizable. This is interesting, as in the real world, we often may well entertain second order beliefs, but rarely entertain third and fourth order beliefs. Can uncertainty regarding such higher order beliefs be regarded as a cause of uncertainty in the real world?

Let's assume that agent 1 sold his durable good at time one, and that agents 3 bought the durable good at time one.

Now agents 1 will not participate further in the market, as he just sold all his durable good, and further durable goods bought would not contribute to his utility, as their income is $\frac{1}{2}$.

For the agents 2, 3 and 4, we know that these agents observed the market price of $\frac{1}{2}$. However, this observation is consistent with different possible combinations $(x^t_i, b_i, a, \text{higher order beliefs})$ (moves, endowments, production amounts, higher order beliefs).

In particular, it is possible that there were two sellers, and two buyers. It is also possible that there was only one seller, and two agents did not participate in the trade. The agents can create a 'list' of corresponding consistent possibilities and their probabilities. This 'list' is then used in calculating expectations for $t = 2$.

2.2 What we hope to know once we solved the model analytically

Here is a list of questions which we plan to investigate using our model:

1. Under which circumstances does the learning dynamic enforce that the agents adequately learn about the production abilities and endowments of the other agents?

2. Fix a given set of first and second order beliefs. Which set of prior higher order beliefs

induces that desirable equilibria are reached over time? Note that while, in our model, sets of possible move combinations are rationalizable, and most often point predictions are not possible, we think that this indeterminacy in the model even realistically reflects a real world condition for policy making. For a policy maker, it is often of bigger interest to exclude certain 'undesirable' economic dynamics like harsh bubble bursts or depressions, but a policy maker may not need to 'pin down' economic predictions to design a policy.

In a similar spirit, we may wish to study how public announcements or institutions enforce an alignment of higher order beliefs of the agents, and we can study with help of our model how crucial the alignment of higher order beliefs is for the appearance of desirable economic dynamics.

3. The model mirrors a complexity reduction task of agents in the real world: Agents need to make decisions, even though they are often uncertain regarding the beliefs of other agents. What are mechanisms of this complexity reduction in the real world?

4. Which initial conditions will yield the selection of a desirable equilibrium? How may, in the real world, conditions be designed that trigger a specific equilibrium?

5. Will similar expectations of the agents emerge over the time of the play? We may suspect that similar expectations of the agents emerge over time, as all agents make use of equation (**??**) to update their beliefs. However, these expectations may not coincide with expectations that the agents would form, if the fundamentals were common knowledge. It remains to be studied how emergent expectations deviate from the benchmark case of 'expectations under common knowledge of the fundamentals'. In other words, do prices help to align the higher order expectations of economic agents over time?

6. Do prices reveal the information of the agents, which are relevant for particular agents, through time? We may suspect that all information available to single agents is not necessarily revealed as time unfolds. First, even though agents observe concrete prices and production quantities, in general, these quantities can be consistent with various fundamentals (a, b) and moves x^t. However, we need to check how much information gets revealed in longer periods of play through multiple time periods. Of course, due to the symmetry of the agents, agents may never learn about specific (a, b)'s of specific agents. The fundamental

values for two agents j and k may be exchanged, and there is no way for some agent i to learn about this. But the more important point is that we suspect that there are situations in which the agents may not even be able to learn about a distribution of (a, b)'s.

7. Will we see 'speculation dynamics' and 'durable good price bubbles' in the specified game? Are these endogenous phenomena of the game we specified?

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3 Discussion and Conclusion

How do economic agents form expectations regarding asset prices and the development of macroeconomic quantities, when there is a fundamental uncertainty about the market process, like after financial crises? How does the formation of expectations fold back to the realized economic process, and in particular, to the selection of one of multiple possible equilibria in the economic process?

The main contribution of this paper is to present a model in which these questions can be addressed in a systematic way. We provide a structure for conceptualizing and understanding how the fundamental economic world of achieved trades and production and the "cognitive" world of the (mutual) expectations of the economic agents interact in a dynamical manner.

In our model, individual agents entertain higher order beliefs regarding the expectations of other economic agents, which are the basis for their own expectation formation.

The agents can observe prices as the aggregate result of the collective activity of all agents, but they do not know the full details about other agents. The agents hence learn indirectly about the expectations of the other agents by observing prices and their movements over time.

We argue that in context of global crisis such as the financial crisis of 2008, information about the likelihood that the economic process favors one equilibrium (for instance, an optimistic one) over another (for instance, a pessimistic one) is often more important than information about fundamentals. This type of information is endogenous, because it is basically information about the higher order beliefs of the other agents. With our model, we offer a structure to study how this type of information can be extracted from prices.

With our model, from a technical point of view, we also contribute to the literature on learning in games. In the literature, most models of learning in games study situations in which the actions of other agents can be observed. In our model, the agents cannot observe the actions of other agents, but they learn about the actions of others indirectly via the observation of aggregate quantities like prices.

In real life, we argue, beliefs of economic agents are often inconsistent. The model helps to study such situations, and if and how beliefs become consistent over time via the common observation of prices.

Uncertainty in higher order beliefs can cause inefficiencies in the coordination of the agents. Are such uncertainties one cause of inefficiencies after crises, when the market is 'confused'?

We see these questions as crucial for developing a more realistic understanding of current real world economic phenomena. We hope that our model can provide a basis for the study of these questions.

References

[1] Akerlof, G. 1970. The Market for "Lemons": Quality Uncertainty and the Market Mechanism. The Quarterly Journal of Economics, Vol. 84, No. 3, pp. 488-500

[2] Akerlof, G. Shiller, R. 2009. Animal Spirits. Princeton: Princeton University Press.

[3] Angeletos, G.M., Lao, J. 2011. Decentralization, Communication, and the Origins of Fluctuations. Working Paper. MIT.

[4] Arthur, B. 1994. Inductive reasoning and bounded rationality. American Economic Review, Vol.84, pp.406-411

[5] Ben-Porath, E., Heifetz, A. 2010. Common Knowledge of Rationality and Market Clearing in Economies with Asymmetric Information. Northwestern University: Discussion Paper.

[6] Bernheim, D. 1984. Rationalizable Strategic Behavior. Econometrica, Vol. 52, No. 4, pp. 1007-1028.

[7] Carlsson, H., van Damme, E. 1993. Global Games and Equilibrium Selection. Econometrica, Vol.61, pp.989-1018

[8] Colander, D., Howitt, P., Kirman, A., Leijonhufvud, A., and Mehrling, P. 2008. Beyond DSGE Models: Toward an Empirically Based Macroeconomics. American Economic Review: Papers & Proceedings 2008, 98:2, 236240

[9] Debreu, G. 1959. Theory of Value. Yale: Yale University Press.

[10] Diamond, P., Fudenberg, D. 1989. Rational Expectations Business Cycles in Search Equilibrium. The Journal of Political Economy, Vol. 97, No. 3, pp. 606-619.

[11] Dubey, P. Geanakoplos, J. and Shubik, M. 1987 The relevation of information in strategic market games. Journal of Mathematical Economics 16, pp 105-137.

[12] Ehrig, T. Jost, J. 2010a. Strategic Rational Expectations. Conference Paper. Midwest Theory Meeting, Northwestern University: Chicago.

[13] Ehrig, T. Jost, J. 2010b. A Game Theoretical Approach to Expectation Formation in Markets. Conference Paper. Symposium of the Spanish Economic Association, Madrid.

[14] Farmer, D. and Geanakoplos, J., The Virtues and Vices of Equilibrium and the Future of Financial Economics, Complexity, Vol. 14, no. 3, 2008

[15] Feinberg, Y. 2005. Games with Unawareness. Working Paper. Stanford: Stanford Business School.

[16] Foster, S. Young, P. 2003. Learning, Hypothesis Testing and Nash Equilbrium. Games and Economic Behavior Volume 45, Issue 1, Pages 73-96.

[17] Fudenberg, D. and Levine, D., 1998. The Theory of Learning in Games. MIT Press: Cambridge, MA.

[18] Geanakoplos, J. 2004. The Arrow-Debreu Model of General Equilibrium. Cowles Foundation Paper. Yale: Yale University.

[19] Harsanyi, J. 1967. Games with Incomplete Information Played by 'Bayesian' Players, Part I. The Basic Model. Management Science, 14(3), pp. 159-82.

[20] Kadane, J and Larkey, P. 1982. Subjective Probability and the Theory of Games. Management Science 28 (2): 113-120.

[21] Kawamura, E. 2005. Competitive equilibrium with unawareness in economies with production. Journal of Economic Theory 121, pages 167191

[22] Krugman, P. 2009. How did Economists get it so wrong?, New York Times, September 2, 2009

[23] Lorenzoni, G. A Theory of Demand Shocks. American Economic Review 2009, 99:5, 20502084

[24] Makowski, L., Ostroy, J. 1995. A revision of the first theorem of welfare economics, The American Economic Review, 85, 808-827

[25] Malevergne, Y., Sornette, D. 2006. Extreme financial risks, Springer

[26] Milgrom, P. Stokey, N. 1980. Information, Trade and Common Knowledge. Journal of Economic Theory, 26, 17-27, 1982

[27] Morris, S., Shin, H. 2000. Rethinking Multiple Equilibria in Macroeconomic Modeling. NBER Macroeconomics Annual, Vol. 15, pp. 139-161.

[28] Morris, S., Shin, H. 2006. Global Games: Theory and Applications. Chapter 3 (pp.57-114) in: Cambridge Collections Online, Cambr.Univ.Press

[29] Muth, J. 1961. Rational Expectations and the Theory of Price Movements. Econometrica, Vol. 29, No. 3 (Jul., 1961), pp. 315-335

[30] Romer, D. 2006, Advanced Macroeconomics, McGraw Hill

[31] Rosu, Ioanid. 2006. Asymetric Information: Walrasian Equilibria, and Rational Expectations Equilibria. Notes. Chicago: GSB.

[32] Savage, L. 1954. The foundations of statistics. John Wiley and Sons.

[33] Sornette, D. 2003. Why stock markets crash, Princeton Univ.Press

[34] Townsend, R. 1983. Forecasting the forecasts of others. Journal of Political Economy, Vol. 91, pp.546-588

[35] Yildiz, M. forthcoming. Walrasian Bargaining. Games and Economic Behavior