Minimal surfaces and Teichmüller theory

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1 Introduction These are the notes from a set of lectures I delivered at National Tsing-Hua University in Hsinchu, Taiwan, in the spring of 1992. In these notes, first the Plateau problem for orientable and nonorientable minimal surfaces of arbitrary genus is solved in Euclidean space and in Riemannian manifolds. We also present as applications some new examples of Plateau problems with infinitely many solutions. In the second chapter, Teichmüller theory for orientable Riemann surfaces is sketched. In particular, we introduce some kind of enlarged Teichmüller space consisting of both orientable and nonorientable surfaces. This space is stratified by genus. We hope that the constructions presented there will have further applications. In the third chapter, the Dirichlet energy functional is considered as a functional on this enlarged Teichmüller space. Estimates of the behaviour of this functional near lower dimensional strata are desired. These are the basis for a H. Morse-Conley theory for minimal surfaces of varying genus and orientability character in Riemannian manifolds, at least in those of nonpositive curvature. This will constitute an extension of my work with Struwe [JS]. In my lectures, however, I did not manage to cover this topic because of lack of time, and therefore, it is also not presented in the present lecture notes. I intend to write up the complete theory elsewhere. I thank Professor Shing-Tung Yau for arranging this stimulating visit at Hsinchu, and the department's chairman, Professor Heng-Lung Lai and his colleagues for making my stay in Taiwan most pleasant, professionally and personally.

1.1 Plateau's problem in Euclidean space. Let $\gamma$ be a closed Jordan curve in Euclidean space $\mathbb{R}^d$ of dimension $d \geq 2$. We want to find a minimal surface $C$ bounded by $\gamma$. A promising approach is to minimize area among all surfaces bounded by $\gamma$. If this approach is to be made rigorous, one first needs to define what a surface is, what a minimal one is, what is meant by the statement that a surface is bounded by $\gamma$, and what the area of a surface is. The difficulty involved here is that it is a priori not clear that the minimum of area is achieved by a smooth surface and the class of smooth surfaces is not closed under any type of convergence that naturally arises in the context of the area problem. Thus if one takes an area minimizing sequence in the class of smooth surfaces bounded by $\gamma$ and uses some compactness theorem in order to pass to some kind of weak limit, one may obtain a rather irregular object to which it may not even be easy to assign an area in a meaningful manner. Mathematicians have found various methods to resolve these difficulties. We present here the one that was first successfully applied to Plateau's problem (to find a minimal surface bounded by $\gamma$) and that will lead us to the theory with the richest mathematical content.
among all those applied to this problem. Let $D := \{x + iy \in \mathbb{C} : x^2 + y^2 \leq 1\}$ be the closed unit disk.

**Definition 1.1.1.** A minimal surface of disk type bounded by $\gamma$ is a continuous map

$$h : D \to \mathbb{E}^d$$

which is harmonic:

$$\Delta h = 0$$

and conformal:

$$(1.1.1) \quad h_x^2 - h_y^2 - 2ih_x \cdot h_y = 0$$

(we employ the Euclidean scalar product of $\mathbb{E}^d$) in the interior of $D$ and which maps $\partial D$ bijectively onto $\gamma$.

Let us discuss this definition:

1. $h$, since harmonic, is automatically smooth, even real analytic in the interior of $D$.
2. Since $h$ is smooth, we can define the area of $h$ as

$$(1.1.2) \quad A(h) = \int_D (h_x^2 - h_y^2 - (h_x \cdot h_y)^2)^{\frac{1}{2}} dxdy,$$

provided this expression is finite.

3. Since $h$ is conformal, the area of $h$ equals its Dirichlet integral:

$$(1.1.3) \quad A(h) = D(h) := \frac{1}{2} \int_D (h_x^2 + h_y^2),$$

while for a general (smooth) map $f : D \to \mathbb{E}^d$ one only has:

$$(1.1.4) \quad A(f) \leq D(f)$$

with equality precisely if $f$ is conformal.

4. One may show that the set of points in the interior of $D$ with $dh(x) = 0$ is isolated and that those points correspond to branch points of the surface $h(\hat{D})$. Otherwise, $h(\hat{D})$ is immersed.

5. At all points with $dh(x) \neq 0$, the mean curvature $H$ of the surface $h(\hat{D})$ vanishes. Therefore, except for the possibility of branch points, it is a minimal surface in the sense of differential geometry.

The following result represents the solution of Plateau's problem by Douglas [D1] and Rado [R]:

**Theorem 1.1.1.** Any closed Jordan curve $\gamma$ in $\mathbb{E}^d (d \geq 2)$ bounds a minimal surface of disk type.

The proof that we shall present goes back to Courant. An important tool is the so called Courant-Lebesgue Lemma which is based on an idea of Lebesgue and whose importance for minimal surface theory was fully recognized and exploited by Courant. It reads as follows:
Lemma 1.1.1. Let \( f \in H^{1,2}(D, E^s), D(f) \leq K, 0 < \delta < 1, z_0 \in D \). Then there exists some \( \rho \) with \( \delta < \rho < \sqrt{\delta} \) for which

\[
f|\partial B(z_0, \rho) \cap D (B(z_0, \rho) := \{ w \in C : |z_0 - w| \leq \rho \})
\]
is absolutely continuous and

\[
(1.1.5) \quad |f(z_1) - f(z_2)| \leq (8\pi K)^{\frac{1}{2}} \left( \log \frac{1}{\delta} \right)^{-1/2}
\]
for all \( z_1, z_2 \in \partial B(z_0, \rho) \cap D \).

Proof. We employ polar coordinates \((r, \varphi)\) with center \( z_0 \). Since \( f \in H^{1,2}, f|\partial B(z, \rho) \cap D \) is absolutely continuous for almost all \( \rho \), and for such \( \rho \) and \( z_1, z_2 \in \partial B(z_0, \rho) \cap D \),

\[
(1.1.6) \quad |f(z_1) - f(z_2)| \leq \int_{\partial B(z_0, \rho) \cap D} |f_\varphi (\rho e^{i\varphi})| \, d\varphi
\]

\[
\leq (2\pi)^{\frac{1}{2}} \left( \int_{\partial B(z_0, \rho) \cap D} |f_\varphi|^2 \, d\varphi \right)^{\frac{1}{2}}
\]

Since

\[
\int_{B(z_0, \rho) \cap D} \left( |f_r|^2 + \frac{1}{r^2} |f_\varphi|^2 \right) r \, dr \, d\varphi \leq 2D(f) \leq 2K
\]
there exists some \( \rho \) with \( \delta < \rho < \sqrt{\delta} \) and

\[
\int_{\partial B(z_0, \rho) \cap D} |f_\varphi|^2 \, d\varphi \leq \frac{2K}{\int_\delta^{\sqrt{\delta}} \frac{dx}{x}} = \frac{4K}{(\log \frac{1}{\delta})}
\]
Using this in (1.1.6) gives (1.1.5). \( \square \)

Proof of Theorem 1.1.1. We assume that there exists some map \( f : D \to E^s \) mapping \( \partial D \) bijectively onto \( \gamma \) and with \( D(f) < \infty \).

This is for example the case, if \( \gamma \) is rectifiable. Once the existence of a minimal surface is established for this case, the general case can be handled by an approximation argument. This is not very hard, but not of interest for us, and therefore omitted.

We then let \((f_n)_{n \in \mathbb{N}} \subset H^{1,2}(D, E^s)\) be a minimizing sequence for the Dirichlet integral \( D \) among all maps satisfying our boundary condition, i.e. mapping \( \partial D \) bijectively onto \( \gamma \). In order to get a better minimizing sequence, we replace \( f_n \) by the harmonic map \( h_n : D \to E^s \) with the same boundary values: \( \Delta h_n = 0, h_n|\partial D = f_n|\partial D \). Since \( h_n \) then minimizes Dirichlet's integral w. r. t. its own boundary values,

\[
D(h_n) \leq D(f_n)
\]
and therefore \((h_n)_{n \in \mathbb{N}}\) also is a minimizing sequence.
We would like to have a minimizing sequence which is equicontinuous so that we can get a good control over limits of subsequences. However, even a harmonic minimizing sequence need not be equicontinuous under our boundary condition. The paradigm for this phenomenon is the group of conformal automorphisms of the unit disk $D : \text{Conf}(D) := \{ k : D \to D \text{ bijective holomorphic} \} \cong PSU(1,1)$ is not compact, although all elements are harmonic and map $\partial D$ bijectively onto itself and minimize Dirichlet’s integral under this boundary condition.

This is an essential difficulty permeating the whole theory, but here we can turn this phenomenon to our advantage. We observe that for any two triples \( \{ e^{i\varphi_1}, e^{i\varphi_2}, e^{i\varphi_3} \} \subset \partial D, \{ e^{i\psi_1}, e^{i\psi_2}, e^{i\psi_3} \} \subset \partial D \) of distinct points lying on $\partial D$ in the same order (i.e. for example $\varphi_1 < \varphi_2 < \varphi_3, \psi_1 < \psi_2 < \psi_3 \mod 2\pi$), there exists $k \in \text{Conf}(D)$ with $k(e^{i\varphi_j}) = e^{i\psi_j}, j = 1, 2, 3$. We also observe that $\text{Conf}(D)$ leaves the Dirichlet integral invariant: for $f \in H^{1,2}(D, E^e), k \in \text{Conf}(D)$

\[
D(f \circ k) = D(f).
\]

We now select three distinct points $e^{i\varphi_1}, e^{i\varphi_2}, e^{i\varphi_3} \in \partial D$ and three distinct points $p_1, p_2, p_3 \in \gamma$. By possibly passing to a subsequence of $(h_n)$ and possibly renumbering the three points on $\gamma$, we conclude that for every $n$ there exists $k_n \in \text{Conf}(D)$ with

\[
h_n \circ k_n(e^{i\varphi_j}) = p_j \quad \text{for } j = 1, 2, 3.
\]

Because of (1.1.8), the maps $\tilde{h}_n := h_n \circ k_n$ likewise form a minimizing sequence of harmonic maps.

We now claim that $(\tilde{h}_n)_{n \in \mathbb{N}}$ is equicontinuous. For this, we let $\eta, 0 < \eta < 1$, be so small that each disk $B(z, \eta)(z \in D)$ contains at most one of the three points $e^{\varphi_j}$. For each $z_0 \in D$ and $0 < \delta < \eta^2$, Lemma 1.1.1 yields $r_n$ with $\delta < r_n < \sqrt{\delta}$ for which $\tilde{h}_n(\partial B(z_0, r_n) \cap D)$ is contained in a ball of radius $(8\pi K)^{\frac{1}{2}}(\log(\frac{1}{\delta}))^{\frac{1}{2}}$, where $K$ is a bound for the Dirichlet integrals:

\[
D(\tilde{h}_n) \leq K \quad \text{for all } n \in \mathbb{N}
\]

If $\bar{B}(z_0, r_n) \cap \partial D = \phi$, then the maximum principle for harmonic functions implies that $\tilde{h}_n$ maps $B(z_0, r_n)$ into the same ball. Given $\epsilon > 0$, we find $\delta$ with

\[
(8\pi K)^{\frac{1}{2}} \left( \log(\frac{1}{\delta}) \right)^{\frac{1}{2}} < \epsilon,
\]

hence control the modulus of continuity. If $\bar{B}(z_0, r_n) \cap \partial D \neq \phi$, then $\bar{B}(z_0, r_n) \cap \partial D$ consists of two points $w_1, w_2$. These two points divide $\partial D$ into two subarcs; a “small” one which contains at most one of the three points $e^{i\varphi_j}$ and a “large” one. Likewise $\tilde{h}_n(w_1)$ and $\tilde{h}_n(w_2)$ divide $\gamma$ into a “small” and a “large” subarc, here again the small one contains at most one of the points $p_j$. Since $\tilde{h}_n$ maps $\partial D$ bijectively onto $\gamma$ and satisfies $\tilde{h}_n(e^{i\varphi_j}) = p_j$, we conclude that $\tilde{h}_n$ maps the small subarc of $\partial D$ onto the small subarc of $\gamma$. Since $\gamma$ is a Jordan curve, there exists $\epsilon_o > 0$ with the property that for $0 < \epsilon < \epsilon_o$ there
exists $\beta > 0$ such that for all distinct $q_1, q_2 \in \gamma$ with $|q_1 - q_2| < \beta$, $|q_1 - q| < \frac{\epsilon}{2}$ for all $q$ contained in the small one of the subarcs into which $q_1$ and $q_2$ divide $\gamma$.

Given $\epsilon < \epsilon_0$, we choose $\delta$ with $0 < \delta < \eta^2$ and $(8\pi K)^{\frac{1}{2}}(\log(\frac{1}{\delta}))^{-\frac{1}{2}} < \beta(\leq \frac{\epsilon}{2})$.

We then conclude that $\tilde{h}_n$ maps $\partial(B(z_0, \epsilon) \cap D)$ into a ball of radius at most $\epsilon$. By the maximum principle for harmonic functions again, $B(z_0, \epsilon) \cap D$ is mapped into the same ball.

Since $\delta < \epsilon$, $\tilde{h}_n$ always maps $B(z_0, \delta) \cap D$ into a ball of radius at most $\epsilon$. Since $\delta$ is independent of $n$, this shows equicontinuity of $(\tilde{h}_n)_{n \in \mathbb{N}}$. After selection of a subsequence, $(\tilde{h}_n)$ then converges weakly in $H^{1,2}$ (because of (1.1.10)) as well as uniformly to some map $h : D \to E^4$ mapping $\partial D$ onto $\gamma$. Since a uniform limit of harmonic functions is again harmonic, it follows that $h$ is harmonic. It remains to show that $h$ is conformal and that it is bijective on $\partial D$. Conformality is more difficult, and we adress this first. $h$ maps $\partial D$ monotonically onto $\gamma$, and it minimizes Dirichlet's integral for the class of maps with monotonic boundary values, and by lower semicontinuity of $D$ under weak $H^{1,2}$ convergence, $D(h) \leq \liminf_{n \to \infty} D(\tilde{h}_n) = \inf\{D(f) : f \in H^{1,2}(D, \mathbb{R}^4), f \text{ maps } \partial D \text{ monotonically onto } \gamma\}$ and since $h$ also belongs to the class of which the inf is taken, we must have in fact equality. Consequently whenever:

$$\sigma : D \to D$$

is a diffeomorphism, we have:

$$D(h \circ \sigma) \geq D(h).$$

We now let

$$\sigma_t : D \to D$$

be a family of diffeomorphisms, depending differentiably on $t$, with $\sigma_0 = \text{id}$. Then

$$\frac{d}{dt} D(h \circ \sigma^{-1}_t)_{|t=0} = 0$$

(Our subsequent computations will show that this derivative exists.) We put

$$(1.1.11) \quad \sigma_t =: \xi + i\eta; \quad \frac{d\sigma_t}{dt}_{|t=0} =: \nu + i\omega$$

Then

$$D(h \circ \sigma^{-1}_t) = \frac{1}{2} \int_D (h_x x_\xi + h_y y_\xi)^2 + (h_x x_\eta + h_y y_\eta)^2 \, d\xi d\eta$$

$$= \frac{1}{2} \int_D ((h_x x_\xi + h_y y_\xi)^2 + (h_x x_\eta + h_y y_\eta)^2)(\xi \eta_\eta - \xi_\eta \eta) \, dxdy$$

At $t = 0$, $\xi + i\eta = x + iy$, hence $x_\xi = 1 = y_\eta$, $x_\eta = 0 = y_\xi$ for $t = 0$.

Thus, $a \cdot$ denoting a derivative w. r. t. $t$ at $t = 0$,

$$\frac{d}{dt} D(h \circ \sigma^{-1}_t)_{|t=0} = \frac{1}{2} \int_D (2h_x^2 x_\xi + 2h_y^2 y_\xi + 2h_x h_y (y_\xi + \dot{x}_\eta))$$

$$+ (h_x^2 + h_y^2)(\dot{\xi}_\eta - \xi_\eta) \, dxdy$$

$$= -\frac{1}{2} \int_D \{h_x^2 (\nu_x - \omega_y) + 2h_x h_y (\nu_x + \omega_y)\} \, dxdy$$
since at $t = 0$ $\dot{x} = -\dot{z} = -\nu$ etc.

Thus (1.1.12) implies

$$
(1.1.14) \quad \int_D \{ (h_x^2 - h_y^2)(\nu_x - \omega_y) + 2h_x \cdot h_y (\nu_y + \omega_z) \} dxdy = 0
$$

for any $\nu, \omega$ as above. With $\varphi := h_x^2 - h_y^2 - 2ih_x \cdot h_y$, (1.1.14) becomes

$$
(1.1.15) \quad \text{Re} \int \varphi (\nu + i\omega)dzdxdy = 0
$$

In particular,

$$
(1.1.16) \quad \varphi_z \equiv 0
$$

which, however, already follows from the fact that $h$ is harmonic. We now consider variations of the form

$$
(1.1.17) \quad \sigma_t(z) = ze^{it\alpha(z)}
$$

with real valued $\alpha$. Then

$$
(1.1.18) \quad \nu + i\omega = iz\alpha(z)
$$

If $\varphi$ is continuous on $\partial D$, integrating (1.1.15) by parts yields

$$
(1.1.19) \quad \text{Im} \int_{\partial D} \varphi (\nu + i\omega)dz = 0
$$

Since we do not know yet continuity of $\varphi$ on $\partial D$, we can only ascertain

$$
(1.1.20) \quad \lim_{\rho \to 1} \text{Im} \int_{\partial B(0,\rho)} \varphi (\nu + i\omega)dz = 0
$$

We write $z = re^{i\theta}$ in polar coordinates and obtain with $\nu + i\omega$ as in (1.1.18)

$$
(1.1.21) \quad 0 = \lim_{\rho \to 1} \text{Im} \int_{\partial B(0,\rho)} \alpha(z)z^2 \varphi(z)d\theta = \lim_{\rho \to 1} \int_{\partial B(0,\rho)} \alpha(z)\text{Im}(z^2 \varphi(z))d\theta
$$

We put $\eta(z) := \text{Im}(z^2 \varphi(z))$

$\eta$ then is harmonic.

For $w \in D$, $K_\rho(z, w) := \frac{e^{z-w}}{2\pi |z-w|}$ is the kernel of the Poisson representation formula on $B(0, \rho)$. We now put in (1.1.17) $\alpha(z) = K_\rho(z, w)\lambda(z)$, with the cut-off function:

$$
\lambda(z) := \begin{cases} 
1 & \text{for } |z| > \rho_0 \\
\frac{2|z|-|w|\rho_0}{\rho_0 - |w|} & \text{for } \rho_0 \geq |z| \geq \frac{\rho_0 + |w|}{2} \\
0 & \text{for } \frac{\rho_0 + |w|}{2} > |z|
\end{cases}
$$

and for $|w| < \rho_0 < 1$, $w \in D$ fixed, (1.1.21) becomes:

$$
(1.1.22) \quad 0 = \lim_{\rho \to 1} \text{Im} \int_{\partial B(0,\rho)} K_\rho(z, w)\eta(z)d\theta
$$
By the Poisson representation formula, since \( \eta \) is harmonic
\[
\int_{\partial B(0, \rho)} K_{\rho}(z, w) \eta(z) d\theta = \eta(w)
\]

Hence

(1.1.23) \[ \eta(w) = 0 \text{ for all } w \in D. \]

Since \( z^2 \varphi(z) =: \beta + i \eta \) is holomorphic, it then has to be constant. As it vanishes for \( z = 0 \), it vanishes identically, hence \( \varphi \equiv 0 \), and \( h \) is conformal.

As the uniform limit of the maps \( h_n \) with bijective boundary correspondence, \( h \) maps \( \partial D \) monotonically onto \( \gamma \). If \( h_{\partial D} \) is not injective, \( h \) as a monotonic map is constant on some subsegment of \( \partial D \). It can then be reflected locally across this subsegment as a harmonic and conformal map. The extended map would be constant on an arc interior to its domain of definition which could happen only if it were constant. Since \( h \) maps \( \partial D \) onto \( \gamma \), this is impossible. Thus, \( h \) maps \( \partial D \) bijectively onto \( \gamma \).

In our proof of Theorem 1.1.1, we have minimized Dirichlet's integral rather than area. So the question arises if, and if so, in which sense our minimal surface is area minimizing. The answer is given by

**Corollary 1.1.1.** The minimal surface \( h \) constructed in Theorem 1.1.1 minimizes area among all disk type surfaces with boundary \( \gamma \) in the sense that:

(1.1.24) \[ A(h) \leq A(g) \]

whenever \( g : D \to E^4 \) is smooth and maps \( \partial D \) bijectively onto \( \gamma \).

**Proof.** We assume first that 

\[ g : D \to E^4 \]

is an embedding. In this case, there exists a bijective conformal map:

\[ k : D \to g(D) \]

by a standard conformal representation theorem ( see e.g. [J3] ).

Since \( k \) is conformal,

(1.1.25) \[ D(k) = A(k) = A(g) \]

The 1st equality is (1.1.3), whereas the 2nd one expresses the fact that \( A \) is parametrization invariant: if \( \sigma : D \to D \) is a diffeomorphism,

(1.1.26) \[ A(g) = A(g \circ \sigma) \]

and we take \( \sigma = g^{-1} \circ k \). Since \( h \) minimizes Dirichlet's integral

(1.1.27) \[ D(h) \leq D(k) \]
Since also by (1.1.4)

\[(1.1.28)\]

\[A(h) \leq D(h),\]

(1.1.24) follows from (1.1.25), (1.1.27), (1.1.28) and in fact we must have equality in (1.1.28).

If \(g\) is not an embedding, we look at

\[g_\epsilon : \quad D \to E^{d+2}\]

\[(x, y) \mapsto (g(x, y), \epsilon x, \epsilon y)\]

\(g_\epsilon\) then is an embedding for \(\epsilon > 0\), and \(A(g_\epsilon) \to A(g), D(g_\epsilon) \to D(g)\) as \(\epsilon \to 0\).

We then apply the previous argument to \(g_\epsilon\), and passing to the limit \(\epsilon \to 0\) again gives (1.1.24).

The argument of Corollary 1.1.1 also gives an alternative proof of the fact that our minimizer \(h\) is conformal. By the same argument as in the proof of Corollary 1.1.1, we may assume \(h\) is an embedding. If \(h\) were not conformal,

\[A(h) < D(h),\]

and there exists a conformal map

\[k : D \to h(D)\]

satisfying

\[D(k) = A(k) = A(h) < D(h)\]

contradicting the minimizing property of \(h\).

The present proof of conformallity, however, is not really simpler or different from the previous one, because it depends on a conformal representation theorem, and in order to prove such a representation theorem, one needs the computations of the first conformallity proof.

1.2 Minimal surfaces of higher topological structure. We let \(\gamma = (\gamma_1, \ldots, \gamma_m)\) be a collection of disjoint closed Jordan curves in \(E^d (d \geq 2)\), and we want to find minimal surfaces bounded by \(\gamma\). Of course, by Thm. 1.1.1, there exist disk type minimal surfaces: \(h_k : D \to E^4\), with \(h_k\) mapping \(\partial D\) bijectively onto \(\gamma_k, k = 1, \ldots, m\). We ask whether there exist other ones, and in particular whether for suitable configurations \(\gamma\), we may find minimal surfaces of higher topological structure, but less area. Examples suggest that this should be the case for certain configurations \(\gamma\).

We first want to look for oriented minimal surfaces bounded by \(\gamma\). Therefore, we assume that \(\gamma_k, k = 1, \ldots, m\), is given an orientation. For \(m = 1\), this is irrelevant, but if we seek a connected minimal surface in case \(m \geq 2\), changing the orientation of some, but not of all the \(\gamma_k\) will change the problem.

We also assume that each \(\gamma_k\) is rectifiable so as to bound surfaces of finite area. One may successfully treat more general Jordan curves by approximation but this does not interest us here.

We let \(\Sigma\) be a connected Riemann surface with \(m\) boundary curves \(c_1, \ldots, c_m\) of genus \(p\). We say that a possibly disconnected Riemann surface \(\Sigma'\) with \(m\)
boundary curves and with components \((\Sigma_1', ... \Sigma_l')\) with genus \(\Sigma_j' = q_j (j = 1, ... l)\) has lower topological type than \(\Sigma\) if:

\[
(1.2.1) \quad \ell \geq 2 \text{ and } \Sigma_{j=1}^l q_j \leq p
\]

or if \(\ell = 1\) and

\[
(1.2.2) \quad q := q_1 < p
\]

We put \(d(\gamma, p) := \inf \{D(f, \Sigma) : \Sigma\ a Riemann surface of genus p with boundary curves } c_1, .. c_m, \ f : \Sigma \to E^d \text{ of class } C^0 \cap H^{1,2} \text{ mapping } c_j \text{ bijectively and with preserved orientation onto } \gamma_j, \ j = 1, ... m\} \text{ where } D(f, \Sigma) \text{ is Dirichlet's integral of } f \text{ on } \Sigma : \text{ in local conformal parameters } z = x + iy,

\[
(1.2.3) \quad D(f, \Sigma) = \frac{1}{2} \int_{\Sigma} (f_x^2 + f_y^2) dxdy
\]

Dirichlet's integral is invariant under conformal transformations of \(\Sigma\) and so the expression in (1.2.3) does not depend on the choice of local parameter.

We also put:
\(d^*(\gamma, p) := \inf \{D(f, \Sigma') : \Sigma' \ a Riemann surface of lower topological type than p, \text{ again with } m \text{ boundary curves, } f : \Sigma' \to E^d \text{ satisfying the same conditions as in the definition of } d(\gamma, p)\}\)

**Definition 1.2.1.** A minimal surface of genus \(p\) with boundary \(\gamma = (\gamma_1, ... \gamma_m)\), a collection of disjoint oriented closed Jordan curves in \(E^d\), is a map \(h : \Sigma \to E^d\) from a Riemann surface \(\Sigma\) of genus \(p\) with boundary curves \(c_1, .. c_m\) which is harmonic and conformal in the interior of \(\Sigma\) and maps \(c_j\) bijectively and with preserved orientation onto \(\gamma_j, j = 1, ... m\).

**Theorem 1.2.1.** Let \(\gamma = (\gamma_1, ... \gamma_m)\) be a collection of disjoint closed oriented rectifiable Jordan curves in \(E^d\). If

\[
(1.2.4) \quad d(\gamma, \rho) < d^*(\gamma, \rho)
\]

then \(\gamma\) bounds a connected minimal surface of genus \(p\).

Condition (1.2.4) is the Douglas condition, and Thm. 1.2.1 is commonly called the Douglas theorem. It was treated by Douglas [D2] in fierce competition with Courant [C], and the latter's student Shiffman [Sh1] (see also the extension of the Courant-Shiffman method by Luckhaus [L]). Interest in this result was revived by Tromba [T1]. A complete proof was given in [J1], see also [TT].

In order to develop some intuition for the problem, we shall first study the next simple case after the disc, namely that of an annulus, a surface of genus 0 with two boundary curves. Every such surface \(A\) is conformally equivalent to some circular cylinder:

\[
C_{s_o} := \{(s, \theta) : 0 \leq s \leq s_o, \theta \in S^1\}
\]

i.e. there exists a conformal bijection:

\[
k : C_{s_o} \to A
\]
Since Dirichlet's integral is conformally invariant, i.e.

(1.2.5) \[ D(f) = D(f \circ k) \]

for any \( f \in H^{1,2}(A, E^d) \), we may restrict ourselves to cylinders \( C_{s_o} \) as domains for our minimal surfaces.

We thus assume that we have two disjoint oriented closed Jordan curves \( \gamma_1, \gamma_2 \subset E^d \), rectifiable for simplicity again. We seek a minimal surface:

\[ h : C_{s_o} \to E^d \]

for some \( s_o, 0 < s_o < \infty \), mapping \( \partial C_{s_o} \) bijectively onto \( \gamma_1 \cup \gamma_2 \) and respecting the orientation on the boundary. We shall proceed in two steps:

1) We keep \( s_o \) fixed and minimize Dirichlet's integral \( D(f) \) over all maps:

\[ f : C_{s_o} \to E^d \]

satisfying the boundary condition.

As in Section 1.1, we may choose a harmonic minimizing sequence \( (h_n)_{n \in \mathbb{N}} \),

\[ h_n : C_{s_o} \to E^d \]

Let

(1.2.6) \[ D(h_n) \leq K. \]

We want to show that the boundary maps \( h_n|_{\partial C_{s_o}} \) are equicontinuous. Then a subsequence of the boundary values will converge uniformly and then so will the harmonic maps \( h_n \), with a harmonic limit \( h \). Again, we may also assume weak \( H^{1,2} \)-convergence, and therefore, as in section 1.1, \( h \) will minimize \( D \) among all maps with a monotonic boundary correspondence. In order to achieve equicontinuity of the boundary maps, we can no longer use a three point normalization as in section 1.1. Instead, we shall need the Douglas condition.

We observe that the Courant-Lebesgue Lemma 1.1.1 also holds for any Riemann surface instead of \( D \), with the same proof. In particular, we may apply it to \( C_{s_o} \), which for auxiliary proposes we equip with its standard Euclidean metric. For every \( z_0 \subset \partial C_{s_o} \), every \( n \in \mathbb{N} \), and every \( \delta, 0 < \delta < 1 \), there then exists \( r_n, \delta < r_n < \sqrt{\delta} \), with

(1.2.7) \[ |h_n(z_1) - h_n(z_2)| \leq (8\pi K)^{\frac{1}{2}} \left( \log \left( \frac{1}{\delta} \right) \right)^{-\frac{1}{2}} \]

whenever \( z_1, z_2 \in \partial B(z_o, r_n) \cap C_{s_o} \).

We denote the boundary component of \( C_{s_o} \) containing \( z_0 \) by \( c_1 \). \( \partial B(z_o, r_n) \cap c_1 \) then consists of two points \( w_1, w_2 \), dividing it into subarcs \( c'_n \) and \( c''_n \) of length \( 2r_n \) and \( 2\pi - 2r_n \), resp. \( h_n(w_1) \) and \( h_n(w_2) \) divide \( h_n(c_1) = \gamma_1 \) (for example) also into two subarcs \( \gamma'_n \) and \( \gamma''_n \). As in section 1.1, given \( \epsilon > 0 \), we can compute \( \delta_0 > 0 \) so that for \( \delta < \delta_0 \), the length of one of these subarcs, say \( \gamma'_n \) is smaller than \( \epsilon \). Here we use again the Jordan curve property of \( \gamma \).

Equicontinuity will result if for every sufficiently small \( \epsilon \), hence for small \( \delta \) and \( r_n \), \( c'_n \) is always mapped onto \( \gamma'_n \), and not onto \( \gamma''_n \). (As \( h_n \) is bijective on the boundary, these are the only possibilities.)
MINIMAL SURFACES AND TEICHMUELLER THEORY

Suppose on the contrary that after taking a subsequence of \((h_n)_{n \in \mathbb{N}}\), there exists a sequence \((\epsilon_n)_{n \in \mathbb{N}}, \epsilon_n \to 0\) as \(n \to \infty\) so that for \(\delta = \delta_n\), computed so that length \((\gamma'_n) < \epsilon_n, c'_n\) is mapped onto \(\gamma''_n\).

We cut \(C_{s_n}\) along \(\partial B(z_0, r_n) =: b_n\) and obtain two pieces \(C'_1\) and \(C'_2\) with \(C'_1\) being homeomorphic to a disc.

\[
\partial C'_1 = c'_n \cup b_n, \quad \partial C'_2 = c''_n \cup b_n \cup c_2 \quad \text{where} \quad \partial C_{s_n} = c_1 \cup c_2
\]

and with \(\beta_n := h_n(b_n)\),

\[
h_n(\partial C'_1) = \beta_n \cup \gamma''_n, \quad h_n(\partial C'_2) = \beta_n \cup \gamma'_n \cup \gamma_2
\]

By the proof of Lemma 1.1.1, we may assume that in polar coordinates \((r, \varphi)\) centered at \(z_0\),

\[
\int_{b_n} \left| \frac{\partial h_n}{\partial \varphi} \right|^2 \, d\varphi \to 0
\]

Therefore, it is not hard to construct maps \(h'_n : C'_1 \to E^d\) mapping \(\partial C'_1\) bijectively onto \(\gamma_1\) (i.e. we map \(b_n\) onto \(\gamma'_n\) instead of onto \(\beta_n\)) with

\[
D(h'_n) \leq D(h_n|_{C'_1}) + \alpha'_n \quad \text{with} \quad \alpha'_n \to 0 \quad \text{as} \quad n \to \infty
\]

Likewise, we may construct maps \(h''_n : C'_2 \to E^d\) which map \(c'_2 \cup b_n\) onto a point and satisfy \(h''_n|_{c_2} = h_n|_{c_2}\) and with

\[
D(h''_n) \leq D(h_n|_{c'_2}) + \alpha''_n \quad \text{with} \quad \alpha''_n \to 0 \quad \text{as} \quad n \to \infty
\]

The construction of \(h''_n\) depends on the following "logarithmic cut-off lemma":

**Lemma 1.2.1.** Let \(\rho < 1,\)

\[
\eta(r) := \begin{cases} 
0 & \text{for} \ r \leq \rho \\
1 + \log((\sqrt{\rho}/r)(\log(\sqrt{\rho}))^{-1} & \text{for} \ \rho \leq r \leq \sqrt{\rho} \\
1 & \text{for} \ r \leq \sqrt{\rho}
\end{cases}
\]

Let \(p \in E^d, f \in H^{1,2} (\Sigma, E^d)\) for some Riemann surface \(\Sigma\) and put

\[
f''(z) := \begin{cases} 
f(z) & \text{if} \ |f(z) - p| \geq \sqrt{\rho} \\
p + \eta(|f(z) - p|)(f(z) - p) & \text{otherwise}
\end{cases}
\]

Then

\[
D(f'') \to D(f) \quad \text{as} \quad \rho \to 0
\]

Since \(h''_n(c'' \cup b_n) = \gamma'_n \cup \beta_n\) converges to a point \(p\) as \(n \to \infty\), Lemma 1.2.1 indeed gives the maps \(h''_n\) with the desired properties.

We then attach a disk to \(c'' \cup b_n\) and thus close \(C'_2\) off to obtain a disk type surface \(D'_2\), conformally equivalent to \(D\) by a standard conformal representation.
theorem, see e.g. [J3]. We extend \( h'' \) to \( D' \) by mapping the inserted disk to \( p \). Thus
\[
D(h''|_{D'_2}) = D(h''|_{C'_2})
\]
We thus have constructed two maps
\[
h'_n : C'_1 \to E^d \text{ mapping } \partial C'_1 \text{ bijectively onto } \gamma_1
\]
and
\[
h''_n : D'_2 \to E^d \text{ mapping } \partial D'_2 \text{ bijectively onto } \gamma_2,
\]
both with disk type domains and
\[
(1.2.11) \quad D(h'_n) + D(h''_n) \leq D(h_n) + \alpha_n \text{ with } \alpha_n \to 0 \text{ as } n \to \infty
\]
We put \( d(\gamma, C_{s_o}) := \inf\{D(f) : f \in H^{1,2}(C_{s_o}, E^d) \text{ mapping } \partial C_{s_o} \text{ bijectively and with preserved orientation onto } \gamma\} \)
Thus (1.2.11) implies
\[
(1.2.12) \quad d^*(\gamma, 0) \leq d(\gamma, C_{s_o})
\]
Since
\[
d(\gamma, 0) = \inf_{s_o > 0} d(\gamma, C_{s_o})
\]
the Douglas condition implies that (1.2.12) cannot hold for all \( s_o > 0 \). Therefore, for certain \( s_o > 0 \), we must have equicontinuity of \( (h_n|_{\partial C_{s_o}})_{n \in \mathbb{N}} \) as otherwise we derive (1.2.12).
We now assume that \( s_o \) satisfies this equicontinuity property. As already observed, in this case we get a harmonic map:
\[
h : C_{s_o} \to E^d
\]
mapping \( \partial C_{s_o} \) monotonically and with preserved orientation onto \( \gamma \), which minimizes Dirichlet's integral among all such maps. In the case where the domain was a disk, we could conclude from this that \( h \) was conformal. In the present case this is no longer necessarily so, and this is related to the fact that the conformal structure of a cylinder is not rigid in contrast to the one of the disk.
In order to proceed, we establish the following:

**Lemma 1.2.2.** Let \( \Sigma \) be a compact Riemann surface with smooth boundary \( \partial \Sigma \), \( h \in H^{1,2}(\Sigma, E^d) \) and suppose:
\[
(1.2.13) \quad \frac{d}{dt} D(h \circ \sigma_t)|_{t=0} = 0
\]
for all smooth families of diffeomorphisms \( \sigma_t : \Sigma \to \Sigma \) with \( \sigma_o = id \).
Then, with \( z = x + iy \) a local conformal parameter on \( \Sigma \),
\[
(1.2.14) \quad \varphi(z)dz^2 := h_x^2dx^2 = \frac{1}{4}(h_x^2 - h_y^2 - 2ih_x \cdot h_y)(dx^2 - dy^2 + 2idxdy)
\]
is a holomorphic quadratic differential on \( \Sigma \) which is real on \( \partial \Sigma \).
That \( \varphi dz^2 \) is real on \( \partial \Sigma \), means the following: If we choose our local conformal parameter \( x + iy \) near \( \partial \Sigma \) in such a manner that \( \partial \Sigma \) locally is given by \( y = 0 \), then along \( \partial \Sigma \), \( dy = 0 \), hence if \( \varphi dz^2 \) is real on \( \partial \Sigma \),

\[
0 = \text{Im}(\varphi dz^2) = -\frac{1}{2} h_x \cdot h_y dx^2
\]

i.e. \( h_x \) and \( h_y \) are orthogonal along \( \partial \Sigma \).

**Proof.** As in Section 1.1, we write \( \sigma_t = \xi + i\eta, \quad \frac{d\sigma_t}{dt} \bigg|_{t=0} = \nu + i\omega \) and obtain (1.1.15), namely

\[
\text{Re} \int \frac{\varphi(\nu + i\omega)_{\xi}}{\xi} = 0
\]

Since this holds in particular for all \( \nu, \omega \) with compact support in the interior of \( \Sigma \), we conclude first of all from Weyl's lemma:

\[
\varphi_{\xi} \equiv 0 \quad \text{i.e. \( \varphi \) is holomorphic.}
\]

In order to show that \( \varphi dz^2 \) is real on \( \partial \Sigma \), for any boundary component of \( \Sigma \), we represent a neighborhood conformally as an annulus \( A \) in the plane, with outer boundary \( \{x^2 + y^2 = 1\} \) corresponding to the given boundary curve of \( \Sigma \).

We use polar coordinates on \( A \),

\[
x + iy = re^{i\theta}
\]

and choose, with real valued \( \beta \),

\[
\nu + i\omega = -\beta(\theta)ie^{i\theta} \quad \text{near} \quad \{x^2 + y^2 = 1\}
\]

and \( \nu + i\omega = 0 \) near the other boundary component of \( A \).

As in (1.1.20)

\[
0 = -\lim_{\rho \to 1} \text{Im} \int_{\{z| = \rho\}} \varphi(\nu + i\omega)dz
\]

\[
= \lim_{\rho \to 1} \text{Im} \int_{\{z| = \rho\}} \varphi \beta(\theta)ie^{i\theta} dz \quad \text{with} \quad \nu + i\omega \text{ as in (1.2.18)}
\]

\[
= \lim_{\rho \to 1} \text{Im} \int_{\{z| = \rho\}} \varphi(z)\beta(\theta)z^2 d\theta
\]

For suitable \( a, b \in \mathbb{C} \), there exists an analytic function \( \psi \) on \( A \) with

\[
(1.2.20) \quad \psi''(z) = \varphi(z) + \frac{a}{z} + \frac{b}{z^2}
\]

Since \( \varphi \in L^1(A) \) because \( h \) has finite Dirichlet integral, \( \psi \in H^{2,1} \), and since \( \psi \) is also holomorphic, it is continuous on \( A \) by Sobolev's embedding theorem.
(1.2.19) and (1.2.20) give,

\[(1.2.21) \quad \lim_{\rho \to 1} \int_{\{|z|=\rho\}} \left( \psi''(z) - \frac{a}{z} - \frac{b}{z^2} \right) \beta(\theta) z^2 d\theta = 0 \]

Since \( \varphi \in L^1(A) \), \( \psi' \) has boundary values in \( L^1(\partial A) \). Therefore, we integrate in (1.2.21) by parts and may put \( \rho = 1 \) afterwards. We obtain,

\[-\text{Im} \int_{\{|z|=1\}} \beta_\theta i e^{i\theta} d\theta - \text{Im} \int_{\{|z|=1\}} \beta_\theta i b \log(e^{i\theta}) d\theta = \text{Im} \int_{\{|z|=1\}} \psi'(e^{i\theta})(i e^{i\theta} \beta_\theta - e^{i\theta}) d\theta \]

\[= \text{Im} \int_{\{|z|=1\}} \psi'(e^{i\theta})i e^{i\theta} \beta_\theta d\theta \]

\[-\text{Im} \int_{\{|z|=1\}} \psi'(e^{i\lambda})e^{i\lambda} \{\beta(0) + \int_0^\lambda \beta_\theta(\theta) d\theta\} \]

\[= \text{Im} \int_0^\lambda \beta_\theta \{i e^{i\theta} \psi'(e^{i\theta}) - i \psi(e^{i\theta})\} d\theta \]

\[-\text{Im} \left( i \beta(0) \int_0^{2\pi} \frac{d}{d\lambda} \psi(e^{i\lambda}) d\lambda \right) \]

The last integral vanishes. Since \( \beta(\theta) \) was arbitrary, we conclude,

\[(1.2.22) \quad \text{Im}(iz \psi'(z) - i \psi(z) + iaz + ib \log z) = \text{const on } \{|z| = 1\} \]

Therefore, \( iz \psi'(z) - i \psi(z) \) is analytic along \( \{|z| = 1\} \), and hence \( \psi \) is smooth there. We differentiate (1.2.22) w.r.t. \( \theta \) and obtain:

\[(1.2.23) \quad 0 = \text{Im} \left( iz \left( \frac{d}{dz} (iz \psi' - i \psi + iaz + i b \log z) \right) \right) \]

\[= \text{Im}( -z^2 \varphi(z) ) \text{ on } \{|z| = 1\} \]

Therefore, \( z^2 \varphi(z) \) can be analytically continued across \( \{|z| = 1\} \) as a holomorphic function. Consequently, \( \varphi \) is smooth up to the boundary, and (1.2.23) implies that \( \varphi dz^2 \) is real on the boundary.

Lemma 1.2.2 also explains why in the disk case we could produce a minimal surface by just minimizing Dirichlet's integral under the Plateau boundary condition. Namely on the unit disk \( D \), every holomorphic quadratic differential which is real on \( \partial D \) vanishes identically, for example because such a quadratic differential can be reflected across \( \partial D \) as holomorphic quadratic differential on \( S^2 \), and all those vanish by Liouville's theorem.
On the other hand, on a cylinder $C_{s_n}$, there are nontrivial holomorphic quadratic differentials real on $\partial C_{s_n}$, namely those of the form

$$\text{const.} \ (ds + id\theta)^2$$

Actually, these are the only ones, as such differentials can be reflected to ones on a torus, and those have to be constant, again by Liouville’s theorem applied to the lift to the universal cover $\mathbb{C}$. The constant has to be real here, because it is real on the boundary.

We now minimize $d(\gamma, C_{s_n})$, w.r.t. $s_n$. Let $(s_n)_{n \in \mathbb{N}} \subset \mathbb{R}^+$ be a minimizing sequence. By the discussion after (1.2.12) we may assume because of the Douglas condition that on $C_{s_n}$ there exists a minimizing map $u_n$; in particular, $u_n$ is harmonic, satisfies the boundary condition, and

$$\varphi_n \ dz^2 = \left( \frac{\partial u_n}{\partial z} \right)^2 \ dz^2$$

is a holomorphic quadratic differential which is real on $\partial C_{s_n}$. We want to show that $(s_n)_{n \in \mathbb{N}}$ stays bounded away from 0 and $\infty$. Then a subsequence will converge to some $s_\circ$, and we shall see that the corresponding map $h$ will be a solution of our problem.

1. $(s_n)_{n \in \mathbb{N}}$ stays bounded away from 0 Otherwise, after selection of a subsequence, $s_n \to 0$ as $n \to \infty$

We have,

$$(1.2.24) \quad D(u_n, C_{s_n}) = \frac{1}{2} \int_{\theta = 0}^{2\pi} \int_{s = 0}^{s_n} \left( \left| \frac{\partial u_n}{\partial s} \right|^2 + \left| \frac{\partial u_n}{\partial \theta} \right|^2 \right) ds d\theta$$

Let

$$d := \text{dist}(\gamma_1, \gamma_2) > 0$$

Since $u_n$ maps $\{s = 0\}$ onto $\gamma_1$, $\{s = s_n\}$ onto $\gamma_2$ (or conversely),

$$|u_n(0, \theta) - u_n(s_n, \theta)| \geq d \text{ for all } \theta.$$  Hence, for every $\theta$,

$$(1.2.25) \quad d \leq \int_{0}^{s_n} \left| \frac{\partial u_n}{\partial \theta} \right| ds \leq s_n^{1/2} \left( \int_{0}^{s_n} \left| \frac{\partial u_n}{\partial \theta} \right|^2 \right)^{1/2}$$

(1.2.25) implies

$$(1.2.26) \quad D(u_n, C_{s_n}) \leq \frac{\pi d^2}{s_n} \to \infty \text{ as } n \to \infty,$$

hence $(s_n)_{n \in \mathbb{N}}$ cannot be a minimizing sequence.

2. $(s_n)_{n \in \mathbb{N}}$ stays bounded away from $\infty$. Otherwise, after selection of a subsequence, $s_n \to \infty$ as $n \to \infty$. Since $(s_n)_{n \in \mathbb{N}}$ is a minimizing sequence, we may assume,

$$(1.2.27) \quad D(u_n, C_{s_n}) \leq K \text{ for some constant } K.$$

Recalling (1.2.24), we conclude from the mean value theorem that there exists some $\sigma_n \in (\frac{1}{4}s_n, \frac{3}{4}s_n)$ for which $u_n|_{\{s=\sigma_n\}}$ is absolutely continuous and satisfies

$$(1.2.28) \quad \frac{1}{2} \int |\frac{\partial u_n(\sigma_n, \theta)}{\partial \theta}|^2 \ d\theta \leq \frac{2K}{s_n}$$
We now quote the following elementary lemma which can be proved by Fourier expansion.

**Lemma 1.2.3.** Let \( g : \partial D \to E^d \) be absolutely continuous with

\[
\frac{1}{2} \int_0^{2\pi} |g\varphi|^2 d\theta < \infty
\]

Then the harmonic extension \( f \) of \( g \), \( f : D \to E^d \), \( \Delta f = 0 \) in \( \hat{D} \), \( f|_{\partial D} = g \), satisfies

\[
D(f) \leq \frac{1}{2} \int_0^{2\pi} |g\varphi|^2 d\theta
\]

We now cut \( C_s \) along \( \{ s = \sigma_n \} \), obtain two annular pieces \( C' \), \( C'' \) and close each of them off to a disk type surface by inserting a copy of \( D \) along a copy of the cut curve \( \{ s = \sigma_n \} \) (identifying \( \theta \), the parameter on \( \{ s = \sigma_n \} \), with \( \varphi \), the parameter on \( \partial D \) given by polar coordinates.) The new surfaces are called \( D' \), \( D'' \).

We extend \( u_n \) from \( C' \) and \( C'' \) to \( D' \) and \( D'' \) by choosing the harmonic extension of the boundary values \( u_n|_{\{ s = \sigma_n \}} \) on each inserted disk. We get maps \( u'_n : D' \to E^d \), \( u''_n : D'' \to E^d \) satisfying our Plateau boundary condition for \( \gamma_1 \) and \( \gamma_2 \) respectively. Also, by Lemma 1.2.3 and (1.2.28)

\[
D(u'_n) + D(u''_n) \leq D(u_n) + 2 \cdot \frac{2K}{s_n}.
\]

Thus, the Douglas condition is violated, namely,

\[
d^*(\gamma, 0) \leq \lim_{n \to \infty} \inf D(u'_n) + D(u''_n)) \leq \lim_{n \to \infty} \left(D(u_n) + \frac{4K}{s_n}\right) \to d(\gamma, 0),
\]

since \( (s_n)_{n \in \mathbb{N}} \) was a minimizing sequence and we assume \( s_n \to \infty \).

In consequence, we show that \( (s_n)_{n \in \mathbb{N}} \) stays bounded away from both 0 and \( \infty \). After taking a subsequence, \( s_n \to s_o \), \( 0 < s_o < \infty \), and \( h = \lim_{n \to \infty} u_n \) (weakly in \( H^{1,2} \) as well as uniformly as before) satisfies by lower semicontinuity \( D(h, C_{s_o}) = d(\gamma, 0) \).

\( h \) is monotonic on the boundary, hence bijective as in the proof of Theorem 1.1.1, once conformallity is shown. Also, it of course preserves the orientation on the boundary because all the \( u_n \) do.

In order to show conformallity, one may again quote a conformal representation theorem as in the argument after Corollary 1.1.1. A more direct argument again uses stationarity w.r.t. variations of the independent variables. We let

\[
\sigma_t : C_{s_t} \to C_{s_o}
\]

be a smooth family of diffeomorphisms, with \( \sigma_o = id \), with \( s_t \) depending differentiably on \( t \), e.g.

\[
s_t = s_o(1 + t).
\]
We put again
\[
\frac{d\sigma_t}{dt}|_{t=0} = \nu + i\omega
\]
Since $h$ minimizes $D(f, C_{s_0})$ also w.r.t. variations of the domain $C_{s_t}$, we get again
\[
0 = \frac{d}{dt} D(h \circ \sigma_t^{-1}, C_{s_t})|_{t=0}
\]
(1.2.29)
With our notation: $\varphi = h_s^2 - h_s^0 - 2i h_s \cdot h_\theta$, this gives again, with $z = s + i\theta$,
\[
0 = \text{Re} \int_{C_{s_0}} \varphi(\nu + i\omega)_{\bar{s}} ds d\theta.
\]
(1.2.30)
By Lemma 1.2.2, $\varphi$ is a holomorphic quadratic differential on $C_{s_0}$ which is real on $\partial C_{s_0}$, hence equal to a constant $c_0 \in \mathbb{R}$.
Therefore, we just put $\sigma_t(s, \theta) = ((1 + t)s, \theta)$, hence $\nu + i\omega = s$, and (1.2.30) gives $0 = \text{Re} \int_{C_{s_0}} c_0 \; ds d\theta$, hence $c_0 = 0$, so that $\varphi \equiv 0$, and $h$ is conformal as desired.
As in the proof of Corollary 1.1.1, we see that $h : C_{s_0} \to E^d$ minimizes area among smooth annulus type surfaces with boundary $\gamma$.
In order to prove Theorem 1.2.1 for $p \geq 1$ or $m \geq 3$, we need some background from hyperbolic geometry.
We shall have to deal with compact oriented Riemann surfaces $\Sigma$ with smooth boundary. For such a surface $\Sigma$, we form its Schottky double $\Sigma$ as follows: We let $\Sigma'$ be an isometric copy of $\Sigma$, with isometry $i : \Sigma \to \Sigma'$. We then identify each $p \in \partial \Sigma$ with $i(p) \in \partial \Sigma'$, thus obtaining a closed Riemann surface $\overline{\Sigma}$. $\overline{\Sigma}$ carries an orientation reversing conformal involution, also denoted by
\[
i : \overline{\Sigma} \to \overline{\Sigma}
\]
If $\Sigma$ is the unit disk, its Schottky double is $S^2$, if $\Sigma$ is an annulus, the Schottky double is a torus, and in all other cases, it is a surface of genus at least 2. If $\Sigma$ has genus $p$ and $m$ boundary curves, then $\Sigma$ has genus $2p + m - 1$. We now quote the uniformization Theorem:

**Theorem.** Each Riemann surface $S$ is conformally equivalent to either one of the following:
(i) $\mathbb{C}$
(ii) $\mathbb{C} \setminus \{0\}$
(iii) $S^2(= \mathbb{C} \cup \{\infty\})$
(iv) $\mathbb{C}/\Lambda$, where $\Lambda$ is a lattice
(v) $H/\Lambda$, where $H := \{z = x + iy, y > 0\}$ is the upper half plane and $\Lambda$ is a fixpoint free, properly discontinuous group of conformal automorphisms of $H$.

Compact surfaces only occur in cases (iii), (iv), (v) may also be noncompact.
From the uniformization theorem, each Riemann surface naturally inherits a metric of constant curvature. The value of the curvature is 0 for cases (i), (ii),
(iv), 1 for (iii) and -1 for (v). If $\Sigma$ is a compact Riemann surface with boundary, its Schottky double $\Sigma$ is uniformized by (iii), (iv), or (v). The involution $i: \Sigma \to \Sigma$ becomes an isometry for the constant curvature metric, and the boundary curves of $\Sigma$ then are closed geodesics because they are fixed by an isometric involution.

We note that Dirichlet’s integral is conformally invariant and therefore does not depend on the choice of a metric on the underlying Riemann surface. Nevertheless, for auxiliary purposes, the constant negative curvature metric in case (v) will be very useful for us. In particular, we shall make use of the following two results which were first used in [SY] in connection with minimal surfaces. The first one is Mumford’s compactness theorem.

**Lemma 1.2.4.** Let $(\Lambda_n)_{n \in \mathbb{N}}$ be a sequence of cocompact lattices in $\mathbb{H}$ (here “cocompact” means that $\mathbb{H}/\Lambda_n$ is compact) which are isomorphic as abstract groups, i.e. the Riemann surfaces $\mathbb{H}/\Lambda_n$ are homeomorphic. Suppose that the lengths of simple ($=\text{nonselfintersecting}$) closed geodesics (w.r.t. the hyperbolic metric) are uniformly bounded away from 0. Then after selection of a subsequence, $(\Lambda_n)_{n \in \mathbb{N}}$ converges to a compact lattice $\Lambda$ representing a surface $\mathbb{H}/\Lambda$ of the same genus as the $\mathbb{H}/\Lambda_n$. The convergence can be interpreted as the convergence of (suitably normalized) fundamental regions.

For a proof, see e.g. [J3]. Lemma 1.2.4 is a very special case of much more general results concerning precompactness of classes of Riemannian manifolds of fixed dimension with uniform bounds on the curvature and on the diameter and a uniform positive lower bound for the injectivity radius. The latter in turn follows from the curvature and diameter bounds plus a lower volume bound. Precompactness of the class of manifolds satisfying those bounds is the Gromov compactness theorem. A complete proof was given in [P].

The next result is the collar lemma of Keen [Ke], Matelski [Mt], Halpern [Hp], and Randol [Rl]:

**Lemma 1.2.5.** Let $\Sigma = \mathbb{H}/\Lambda$ be a compact Riemann surface with a simple closed geodesic $\gamma$ of length 1. Then $\Sigma$ contains an annular region $A$, the “collar” about $\gamma$ whose boundary curves are at a distance of $\text{arc sinh}(1/\sinh(1/2))$ from $\gamma$. (Thus $A$ is isometric to $\{z = re^{i\varphi} \in H : 1 \leq r \leq e^{1}, \text{arc tan sinh}(1/2) \leq \varphi \leq \pi - \text{arc tan sinh}(1/2)\}$, with $\{r = 1\}$ and $\{r = e^{1}\}$ identified via $z \mapsto e^{i}z$. $\gamma$ corresponds to $\{re^{i\varphi}/2 : 1 \leq r \leq e^{1}\}$. The area of $A$ is $l/\sinh(1/2)$.)

We now start with the proof of Theorem 1.2.1 for the cases not yet treated, i.e. $p \geq 1 \text{ or } m \geq 3$. In these cases, all occurring Riemann surfaces $\Sigma$ have hyperbolic Schottky doubles $\Sigma$.

We want to adopt the scheme of proof of the annulus case. Thus, for each fixed Riemann surface $\Sigma$ of genus $p$ with $m$ boundary curves $c_1, \ldots, c_m$, we put

$$d(\gamma, \Sigma) := \inf\{D(f) : f \in H^{1,2} \cap C^0(\Sigma, E^d)\},$$

$f$ maps $c_j$ bijectively and with preserved orientation onto $\gamma_j (j = 1, \ldots, m)$. As before, we proceed in two steps:

(1) We minimize $D(f, \Sigma)$ over $f$ satisfying the Plateau boundary condition while keeping $\Sigma$ fixed. The issue again is to show that for a minimizing $(h_n)_{n \in \mathbb{N}},$
the boundary values \( h_n|_{\partial \Sigma} \) are equicontinuous. We can use the same argument as in the annulus case, because that argument was local near the boundary, to show that this is the case if

\[
d(\gamma, \Sigma) < d^*(\gamma, p).
\]

Since \( d(\gamma, p) = \inf\{d(\gamma, \Sigma) : \Sigma \text{ of genus } p\} \), (1.2.31) by the Douglas condition has to be satisfied for some \( \Sigma \), and we need only consider those \( \Sigma \). For any such \( \Sigma \), we obtain a harmonic map,

\[
h : \Sigma \to E^d
\]
satisfying the boundary condition for which \( h_z^2 dz^2 \) is real on \( \partial \Sigma \) by Lemma 1.2.2.

(2) We now minimize \( d(\gamma, \Sigma) \) over \( \Sigma \). Let \( (\Sigma_n)_{n \in \mathbb{N}} \) be a minimizing sequence, with corresponding harmonic maps \( u_n : \Sigma_n \to E^d \). From Lemma 1.2.4, the strategy of proof is clear: We have to exclude that the length \( l_n \) of the shortest closed geodesic \( c_n \) on the Schottky double \( \Sigma_n \) goes to \( O \) as \( n \to \infty \). In order to exclude this, we shall make crucial use of Lemma 1.2.5, and the Douglas condition, of course. We shall proceed by contradiction and assume,

\[
l_n \to 0 \text{ as } n \to \infty
\]

It will be convenient to extend \( u_n \) from \( \Sigma_n \) to the Schottky double as \( u_n : \tilde{\Sigma}_n \to E^d \) by the requirement \( u_n(z) = u_n(i(z)) \), where \( i : \tilde{\Sigma}_n \to \tilde{\Sigma}_n \) is the isometric involution fixing \( \partial \tilde{\Sigma}_n \). Then

\[
D(u_n, \Sigma_n) = 2D(u_n, \Sigma_n).
\]

By Lemma 1.2.5, \( \Sigma_n \) contains a collar of width

\[
\arcsinh(1/\sinh(l_n/2))
\]
on each side of \( C_n \).

We parameterize the collar conformally by a cylinder,

\[
Z_n := \{(r, \varphi) : -R_n \leq r \leq R_n, 0 \leq \varphi \leq l_n\}
\]

with \( \{r = 0\} \) corresponding to the geodesic \( c_n \). One also computes \( R_n \to \infty \), as \( l_n \to 0 \).

Let us assume first

\[
\text{length } (u_n(r, 0)) \geq \sigma > 0
\]

for all \( n \in \mathbb{N} \) and all \( r \) with \( -R_n \leq r \leq R_n \). (1.2.33) means that the lengths of the images of all parallel curves of \( c_n \) in the collar are bounded away from 0. Thus

\[
\sigma \leq \int_0^{l_n} \left| \frac{\partial u_n}{\partial \varphi}(r, \varphi) \right| d\varphi \text{ for all such } r,
\]
hence

\[ \sigma^2 \leq l_n \int_0^{l_n} \left| \frac{\partial u_n}{\partial \varphi}(r, \varphi) \right|^2 \, d\varphi \text{ for all such } r. \]

We integrate this w.r.t. \( r \in [-R, R] \) and obtain

\[
(1.2.34) \quad \frac{2R\sigma^2}{l_n} \leq \int_{-R}^{R} \int_0^{l_n} \left| \frac{\partial u_n}{\partial \varphi}(r, \varphi) \right|^2 \, d\varphi \, dr \leq 2D(u_n, \Sigma_n) = 4D(u_n, \Sigma_n)
\]

by (1.2.32). Here, we may take any \( R > R_n \) in particular, for small \( l_n \), we may take \( R = 1 \). (1.2.33) then implies,

\[ d(\gamma, \Sigma_n) = D(u_n, \Sigma_n) \to \infty \text{ as } n \to \infty \]

which is incompatible with \( (\Sigma_n)_{n \in \mathbb{N}} \) being a minimizing sequence. Therefore, (1.2.33) is not possible, if length \( (c_n) \to 0 \). This implies already that the lengths of all closed geodesics in \( \Sigma_n \) intersecting at least two of the boundary curves of \( \Sigma_n \) are bounded away from 0. Namely, in this case the image curves have length \( \geq \min_{j \neq k} \text{dist}(\gamma_j, \gamma_k) > 0 \).

We now want to exclude that the lengths \( l_n \) of closed geodesics \( c_n \) which are interior to \( \Sigma_n \) go to 0 as \( n \to \infty \). Again, there is a collar about \( c_n \) parametrized by geodesic parallel coordinates \((r, \varphi)\) in the same manner as before. We also parameterize the unit disk \( D \) by polar coordinates \((\rho, \theta)\). In the sequel, we shall determine a suitable value \( r_n \) and then map \( \partial D = \{(1, \theta)\} \) onto a curve parallel to \( c_n \) via,

\[
(1.2.35) \quad (1, \theta) \mapsto \left( \gamma_n, \frac{l_n}{2\pi} \theta \right).
\]

\[ v_n(1, \theta) = u_n \left( r_n, \frac{l_n}{2\pi} \theta \right) \]

then satisfies

\[
(1.2.36) \quad \int_0^{2\pi} \left| \frac{\partial}{\partial \theta} v_n(1, \theta) \right|^2 \, d\theta = \frac{l_n}{2\pi} \int_{\varphi=0}^{l_n} \left| \frac{\partial}{\partial \varphi} u_n(r_n, \varphi) \right|^2 \, d\varphi
\]

Now as in (1.2.34),

\[
(1.2.37) \quad \int_{-R}^{R} \int_0^{l_n} \left| \frac{\partial}{\partial \varphi} u_n(r, \varphi) \right|^2 \, d\varphi \, dr \leq 2D(u_n, \Sigma_n)
\]

Again, we may choose \( R = 1 \) for small \( l_n \). (1.2.37) implies that there exists some \( r_n \in [-R, R] = [-1, 1] \) with

\[
(1.2.38) \quad \int_0^{l_n} \left| \frac{\partial}{\partial \varphi} u_n(r_n, \varphi) \right|^2 \, d\varphi \leq D(u_n, \Sigma_n).
\]
We choose this value of $\tau_n$ in (1.2.35) and obtain
\[
\int_0^{2\pi} \left| \frac{\partial}{\partial \theta} v_n(1, \theta) \right|^2 d\theta \leq \frac{l_n}{2\pi} K,
\]
where
\[
D(u_n, \Sigma_n) \leq K
\]
may be assumed.

By Lemma 1.2.3, there exists a continuous (harmonic) extension of $v_n$ to $D$,
\[
v_n : D \to E^d
\]
with
\[
D(v_n) \leq \frac{l_n}{2\pi} K.
\]

We now cut $\Sigma_n$ along the curve $\{ (\tau_n, \varphi) : 0 \leq \varphi \leq l_n \}$ (parallel to $c_n$). Via (1.2.35), we identify each of the two resulting curves with $\partial D$, attach a copy of $D$ to each of them and extend $u_n$ to the two copies of $D$ as the maps $v_n$ just constructed. We thus have cut a handle of $\Sigma_n$ and obtained a surface $\Sigma'_n$ of lower topological type together with a map $u'_n : \Sigma'_n \to E^d$ satisfying the boundary condition and,
\[
D(u'_n, \Sigma'_n) \leq D(u_n, \Sigma_n) + 2D(v_n, D) \leq D(u_n, \Sigma_n) + \frac{K}{2\pi} l_n
\]
Since
\[
D(u'_n, \Sigma'_n) \leq d^*(\gamma, p),
\]
we see that $l_n \to 0$ is not compatible with the Douglas condition.

In the same way it is excluded that the lengths of boundary curves of $\Sigma_n$ go to 0 as $n \to \infty$ (Recall that the boundary curves are geodesics w.r.t. the hyperbolic metric of $\Sigma_n$).

The only remaining case to be treated is that of closed geodesics on $\Sigma_n$ intersecting precisely one of the boundary curves of $\Sigma_n$. This boundary curve, which we denote by $c$ for simplicity, then has to be intersected (at least) twice.

To deal with this case, we extend $u_n$ to $\Sigma_n$ as before and again parametrize a collar about such a geodesic $c_n$ by a cylinder $Z_n = \{ (r, \varphi) : -R_n \leq r \leq R_n, 0 \leq \varphi \leq l_n \}$.

As before, the length of $u_n(r, 0)$ is estimated by $l(u_n(r, 0)) \leq \int_0^{l_n} |\frac{\partial}{\partial \varphi} u_n(r, \varphi)| d\varphi$. As before, we integrate w.r.t. $r \in [-R, R]$, may assume $R = 1$ again and obtain that there exists some $r_n \in [-1, 1]$ with
\[
l(u_n(r_n, \cdot)) \leq (2l_n K)^{1/2}
\]
and
\[
\int_0^{l_n} \left| \frac{\partial}{\partial \varphi} u_n(r_n, \varphi) \right|^2 d\varphi \leq 2l_n K
\]
Let $w_n^1$ and $w_n^2$ be two points of intersection of $c_n$ with $c$. Then we have from (1.2.40),

$$|u_n(w_n^1) - u_n(w_n^2)| \leq (2l_n K)^{1/2}. \tag{1.2.42}$$

The image of $c$ under $u_n$ is one of our Jordan curves $\gamma_k$, say $\gamma_1$. $u_n(w_n^1)$ and $u_n(w_n^2)$ then divide $\gamma_1$ into two pieces $\gamma'$, $\gamma''$ with,

$$\text{length } (\gamma') \to 0 \text{ as } n \to \infty$$

by (1.2.42) and the Jordan curve property of $\gamma_1$. We have seen similar arguments in the proof of Theorem 1.1.1 as well as in the proof of equicontinuity of the boundary values for the annulus case. As there, we may cut $\Sigma_n$ along $(r_n, \cdot) \cap \Sigma_n$, close one of the two new boundary curves resulting from $c$ and $(r_n, \cdot) \cap \Sigma_n$ off by a disc, use harmonic condition, controlled by Lemma 1.2.3, inside this disc, while modifying the boundary values on the other new boundary curve slightly so as to restore our Plateau boundary continuation. Again the contribution to Dirichlet's integral required for this can be controlled by Lemma 1.2.3 and goes to 0 as $n \to \infty$. In this manner, one again reaches a contradiction to the Douglas condition.

In conclusion, we have shown that the lengths of closed geodesics on the surfaces $\Sigma_n$ stay uniformly bounded away from 0 as $n \to \infty$. Therefore, Lemma 1.2.4 implies that after selection of a subsequence, $(\Sigma_n)_{n \in \mathbb{N}}$ converges to a Riemann surface $\Sigma$ of the same topological type, and the maps $u_n$ then converge to a harmonic map,

$$h : \Sigma \to E^d$$

satisfying our boundary condition (again, the step from monotonic to bijective boundary values can be easily achieved by a reflection argument, once $h$ is shown to be conformal).

In order to show conformallity of $h$, we use our previous method adapted to higher genus surfaces. We assume \( \Sigma = H/\Lambda \).

We let $\sigma_t := H \to H$ be a smooth a family of diffeomorphisms, with $\sigma_0 = id$. We also assume that for every $t$, there exists a compact lattice $\Lambda^t$ in $H$, with $H/\Lambda^t$ of the same genus as $\Sigma = H/\Lambda$, with the property that for every $\Gamma \in \Lambda$, there exists $\Gamma^t \in \Lambda^t$, with

$$\sigma_t \circ \Gamma = \Gamma^t \circ \sigma_t \tag{1.2.43}$$

Finally, we assume that for every $t$ there exists an isometric involution $i^t : H \to H$, realizing $H/\Lambda^t$ as the Schottky double of a surface $\Sigma^t$, with,

$$\sigma_t \circ i = i^t \circ \sigma_t \tag{1.2.44}$$

where $i : H \to H$ is the involution of $\Sigma$ as the Schottky double of $\Sigma$. We again have

$$\frac{d}{dt}D(h \circ \sigma_t^{-1}, H/\Lambda^t)|_{t=0} = 0, \tag{1.2.45}$$
where we have extended $h$ to $\Sigma$ by $h \circ i(z) = h(z)$ as before. With
\[
\varphi = h_2^2 dz^2 \\
\frac{d\sigma_t}{dt}|_{t=0} = \nu + i\omega
\]
we get as before from (1.2.45)
\[
(1.2.46) \quad 0 = \text{Re} \int_{H/\Lambda} \varphi(\nu + i\omega) dx dy = 0 \quad (z = x + iy).
\]
$\varphi$ again is a holomorphic quadratic differential and as a quadratic differential, it transforms via
\[
(1.2.47) \quad \varphi(z) = \varphi(\Gamma z)(\Gamma_z(z))^2 \text{ for } \Gamma \in \Lambda
\]
Then
\[
(1.2.48) \quad \mu(z) := \varphi(z)y^2
\]
transforms via
\[
(1.2.49) \quad \mu(\Gamma z)\Gamma_z(z) = \mu(z)\Gamma_z(z)
\]
We then solve the Beltrami equation $(\sigma_t : H \to H)$
\[
(1.2.50) \quad \frac{\partial}{\partial z}\sigma_t = t\mu \frac{\partial}{\partial z}\sigma_t
\]
with the normalization
\[
\sigma_t(0) = 0, \sigma_t(1) = 1, \sigma_t(\infty) = \infty \text{ for all } t.
\]
This can be done if $|t|$ is so small that $|t\mu(z)| < 1$ for all $z \in H$. The normalization implies $\sigma_0 = id$, and $\sigma_t$ depends smoothly on $t$. A reference is e.g. [J3 (chapter 3)].

In our above notations
\[
(1.2.51) \quad (\nu + i\omega)_z = \mu
\]
The transformation behaviour (1.2.49) implies that $\sigma_t \circ \Gamma$ also solves (1.2.50), for any $\Gamma \in \Lambda$. By uniqueness, two solutions of (1.2.50) differ only by a conformal automorphism of $H$. Thus,
\[
\sigma_t \circ \Gamma = \Gamma^t \circ \sigma_t,
\]
for some $\Gamma^t$. The automorphisms $\Gamma^t$ form a lattice which we denote by $\Lambda^t$. Thus, (1.2.43) is satisfied. (1.2.44) is verified in the same manner.

We may thus use (1.2.51) in (1.2.46) to obtain
\[
0 = \text{Re} \int_{H/\Lambda} \varphi\bar{\varphi}y^2 dx dy
\]
i.e. \( \varphi \equiv 0 \), and thus conformallity of \( h \).

This completes the proof of Thm. 1.2.1. \( \square \)

We now want to solve the Plateau Douglas problem for nonorientable minimal surfaces. Again, we shall need a Douglas type condition, in order to present an analogue of Thm. 1.2.1. We shall give the proof of F. Bernatzki [Be].

We first recall some elementary facts about the topology of nonorientable surfaces.

Let \( S \) first be a compact nonorientable surface without boundary. We let \( H \) be the subgroup of \( \pi_1(S, x_0) \) of index 2 containing the orientation preserving loops. \( H \) then defines an oriented surface \( \tilde{S} \) which is a double cover of \( S \). \( \tilde{S} \) carries an orientation reversing involution \( j : \tilde{S} \to \tilde{S} \) without fixed points. We define the genus of \( S \) to be \( p \), if \( \tilde{S} \) has genus \( 2p-1 \). This is different from the customary definition of genus; in particular, with our definition, the genus of nonorientable surfaces takes half integer values. We shall see the advantage of the present definition when we have to make transitions between orientable and nonorientable surfaces below.

If \( S \) is a compact nonorientable surface with smooth boundary, we may define its Schottky double as in the oriented case. For example, the Schottky double of a Möbius strip is a Kleinbottle.

We say that a nonorientable surface \( \Sigma \) has the structure of a Riemann surface if it has an atlas of coordinate charts with conformal or anticonformal transition functions. This is the same as requiring that \( \tilde{\Sigma} \) has the structure of a Riemann surface which is carried into the Riemann surface structure with opposite orientation by the involution \( j \).

Let \( S \) be a compact connected nonorientable surface of genus \( p \) with boundary consisting of \( m \) smooth curves. We say that a compact surface \( S' \) with \( m \) boundary curves has lower topological type than \( S \) if it has smaller genus or if it has more than one component and the sum of the genera of the components does not exceed \( p \). \( S' \) may be orientable or nonorientable here.

Let \( \gamma = (\gamma_1, \ldots, \gamma_m) \) be a collection of disjoint, closed, rectifiable Jordan curves in \( \mathbb{R}^d \). We define \( d_-(\gamma, p) := \inf \{ D(f, \Sigma) : \Sigma \text{ a nonorientable connected Riemann surface of genus } p \text{ with } m \text{ boundary curves } c_1, \ldots, c_m, f \in C^\infty \cap H^{1,2}(\Sigma, \mathbb{R}^d), \text{ mapping } c_k \text{ bijectively onto } \gamma_k \text{ for } k = 1, \ldots, m \} \) \( d_+^*(\gamma, p) := \inf \{ D(f, \Sigma') : \Sigma' \text{ a Riemann surface of lower topological type than } p \text{ with } m \text{ boundary curves } c_1, \ldots, c_m, f \text{ as before} \}

The corresponding Douglas theorem then is (cf. [D2], [Be]).

**Theorem 1.2.2.** Let \( \gamma = (\gamma_1, \ldots, \gamma_m) \) be a collection of disjoint, closed, rectifiable Jordan curves in \( \mathbb{R}^d \). If

\[
(1.2.52) \quad d_-(\gamma, p) < d_+^*(\gamma, p)
\]

then there exists a connected nonorientable minimal surface of genus \( p \) with boundary \( \gamma \).

**Proof.** The proof can proceed in a manner analogous to that of Theorem 1.2.1. Some technical differences stem from the fact that one now also may have to cut a surface along an orientation reversing curve.
As in the proof of Theorem 1.2.1, one finds harmonic maps \( h_n : \Sigma_n \to E^d \), satisfying the boundary conditions, with an associated holomorphic quadratic differential on the oriented cover \( \Sigma_n \) which is real on \( \partial \Sigma_n \), for a minimizing sequence \((\Sigma_n)_{n \in \mathbb{N}}\), because of the Douglas condition.

Let us again look at the simplest case first, namely that of a Möbius strip. The oriented cover of a Möbius strip is an annulus; as before, we represent annuli as cylinders. Thus, for a minimizing sequence \((\Sigma_n)_{n \in \mathbb{N}}\) of Möbius strips, we get a family \((\tilde{C}_{s_n})_{n \in \mathbb{N}}\) of cylinders

\[
\tilde{C}_{s_n} := \{(s, \theta) : 0 \leq s \leq 2s_n, \theta \in S^1\}.
\]

In order to assure that a subsequence converges to a cylinder which then covers a minimizing Möbius strip, we again have to exclude that \((s_n)\) may converge to 0 or \(\infty\), after selection of a subsequence.

The cylinders \(\tilde{C}_{s_n}\) are parametrized in such a manner that the curve \(\{s = s_n\}\) is a two-fold cover of the orientation reversing core curve of the corresponding Möbius strip, while the curves \(\{s = \sigma\}\) for \(\sigma \neq s_n\) are mapped injectively onto orientation preserving curves of the Möbius strip. Each such curve is the boundary of a smaller Möbius strip. We also may pull our harmonic maps \(h_n\) back to maps

\[
\tilde{h}_n : \tilde{C}_{s_n} \to E^d,
\]

with

\[
D(\tilde{h}_n, \tilde{C}_{s_n}) = 2D(h_n, \Sigma_n) \leq K
\]

If \((s_n)\) should converge to \(\infty\) for \(n \to \infty\), we may find a curve \(\{s = \sigma\}, \frac{1}{2}s_n < \sigma < s_n\), with

\[
\int_0^{2\pi} \left| \frac{\partial}{\partial \theta} h_n^2(\sigma, \theta) \right|^2 d\theta \leq \frac{2K}{s_n}
\]

as in the annulus case in the proof of Theorem 1.2.1. We then cut \(\tilde{C}_{s_n}\) along \(\{s = \sigma\}\) and insert a disk. The lower part \(\tilde{C}_{\sigma}\) of our original cylinder together with the inserted disk forms a disk type surface \(D'\). On the inserted disk, we again extend the map as the harmonic extension of its boundary values. The contribution to Dirichlet's integral from this again goes to 0 by (1.2.53) and Lemma 1.2.3. Also, the map \(u'_n : D' \to E^d\) thus constructed satisfies the boundary condition on \(\partial D' = \{s = 0\}\), and so we reach a contradiction to the Douglas condition as before.

If \(s_n\) should go to 0, we proceed as follows.

For any \(\theta \in (0, 2\pi)\)

\[
\int_0^{2s_n} \left| \frac{\partial}{\partial s} \tilde{h}_n(s, \theta) \right| ds \leq (2s_n)^{1/2} \left( \int_0^{2s_n} \left| \frac{\partial}{\partial s} \tilde{h}_n(s, \theta) \right|^2 ds \right)^{1/2}
\]

Hence, there exists \(\theta_n\) with,

\[
\int_0^{2s_n} \left| \frac{\partial}{\partial s} \tilde{h}_n(s, \theta_n) \right| ds \leq \left( \frac{2Ks_n}{\pi} \right)^{1/2}
\]
Hence, 
\[ |\tilde{h}_n(0, \theta_n) - \tilde{h}_n(s_n, \theta_n)| \leq \left( \frac{2Ks_n}{\pi} \right)^{1/2} \]
\( \tilde{h}_n(0, \theta_n) \) and \( \tilde{h}_n(2s_n, \theta_n) \) then divide our boundary curve \( \gamma = \gamma_1 \) into two pieces \( \gamma', \gamma'' \), with

\[ \text{length}(\gamma') \to 0 \text{ as } n \to \infty. \]

The curves \((s, \theta_n)\) and \((s, \theta_n + \pi)\) (addition mod \(2\pi\)) correspond to the same curve on our Möbius strip. We then cut \( \tilde{C}_{s_n} \) along these curves. The two resulting pieces both correspond to the Möbius strip. If we do not identify \((s, \theta_n)\) and \((s, \theta_n + \pi)\) anymore, however, we obtain a disk type surface \( D' \). It has four boundary arcs, namely \((s, \theta_n), (s, \theta_n + \pi)\), with \(0 \leq s \leq 2s_n\), and \((0, \theta), (2s_n, \theta)\) with \(\theta_n \leq l \leq \theta_n + \pi\).

W. l. o. g. \( \tilde{h}_n(\{(0, \theta), \theta_n \leq l \leq \theta_n + \pi\}) = \gamma' \). Therefore, the images of the first three boundary arcs converge to a point on \( \gamma \), as \( n \to \infty \). With the help of Lemma 1.2.1, 1.2.3, one again perturbs the values of \( \tilde{h}_n \) on \( \partial D' \) to obtain a map \( h'_n : D' \to E^4 \), mapping \( \partial D' \) bijectively onto \( \gamma \), with

\[ D(h'_n) \leq D(\tilde{h}_n|_{D'}) + \alpha_n, \text{ with } \alpha_n \to 0 \text{ as } n \to \infty. \]

This again violates the Douglas condition.

These arguments prove Theorem 1.2.2 for the case of the Möbius strip. For the general case, one has to combine the arguments for the Möbius strip with those for higher genus surfaces in the proof of Thm.1.2.1. We omit the details.

We now draw some consequences of Theorems 1.2.1 and 1.2.2.

**Corollary 1.2.1.** Let \( \gamma = (\gamma_1, \ldots, \gamma_m) \) be a collection of disjoint oriented rectifiable closed Jordan curves in \( E^d \). Suppose that \( d^*(\gamma, p) \) is realized by a nonembedded minimal surface \( h : \Sigma' \to E^3 \) for which either \( \Sigma' \) has only one component (in which case genus (\( \Sigma' \)) < \( p \)) or that the images of the various components of \( \Sigma' \) are not all disjoint from each other. Then \( \gamma \) bounds an oriented minimal surface of genus \( p \).

**Proof.** By regularity results for minimal surfaces (see [J3]), a nonembedded minimal surface \( S' = h(\Sigma') \) contains an open Jordan arc \( \beta \) along which locally two (or more) embedded sheets \( S_1, S_2 \) of the surface meet transversally. We take (sufficiently small) closed smooth Jordan curves \( \beta_1, \beta_2 \) on \( S_1 \) and \( S_2 \) resp., with

\[ \beta_1 \cap \beta = \beta_2 \cap \beta = \beta_1 \cap \beta_2 = \{p_1, p_2\} \quad (p_1 \neq p_2). \]

\( p_1 \) and \( p_2 \) divide \( \beta_i \) into two subarcs \( \beta^1_i, \beta^2_i, i = 1, 2 \) and \( \beta \) and \( \beta^j_i \) bound a region \( S^j_i \) on \( S_i(i, j = 1, 2) \). Then \( \beta^1_1 \cup \beta^2_2, \text{ e.g., is a rectifiable closed Jordan curve} \)

and bounds an area minimizing disk \( S^1_{12} \) by the solution of Plateau's problem (Theorem 1.1.1 and Corollary 1.1.1). Since \( S^1_1 \cup S^1_2 \) is not smooth along \( \beta \), it cannot be area minimizing by the regularity of area minimizing disks. (Suppose on the contrary \( S^1_1 \cup S^1_2 = h(D) \), with \( h \) harmonic and conformal; then \( dh \) must
vanish along \( \beta \), since \( h \) is harmonic and smooth; however, \( dh \) can have at most isolated zeros, which gives a contradiction). Hence

\[
\text{Area}(S_{12}^{11}) < \text{Area}(S_1 \cup S_2);
\]

and similarly for \( \beta_1 \cap \beta_2^1, \beta_1^2 \cap \beta_2 \). We thus can decrease the area of \( S' \) by replacing \( S_1 \cup S_2 \cup S_2^1 \cup S_2^2 \) by either \( S_{12}^{11} \cup S_{12}^{22} \) or \( S_{12}^{12} \cup S_{12}^{21} \). We thus have replaced the union of two disks by a handle. If \( \Sigma' \) is connected, this increases the genus by 1.

If the two replaced disks come from different components of \( \Sigma' \), we have decreased the number of components and left the genus unchanged. If \( \Sigma' \) is connected, one of the new surface is orientable, the other one nonorientable. If we have joined two components of \( \Sigma' \) by our procedure, then both new surfaces are orientable, but we may have to change the orientation of one of the components, hence also the orientation of the corresponding boundary curve(s) in order to get an orientation. In any case, we can find an oriented surface \( \Sigma \) and a map

\[
f : \Sigma \to E^d
\]

mapping \( \partial \Sigma \) bijectively and with the same orientation as before onto \( \gamma \), and satisfying

\[
A(f) < A(h) = D(h) \quad \text{(since } h \text{ is conformal)}.
\]

By a conformal representation theorem, we may assume that \( f \) is conformal, hence

\[
D(f) = A(f) < D(h) = d^*(\gamma, p)
\]

Therefore, since \( h : \Sigma' \to E^d \) realizes \( d^*(\gamma, p) \), \( \Sigma \) must be connected of genus \( p \), and hence the Douglas condition

\[
d(\gamma, p) < d^*(\gamma, p)
\]

is satisfied, and we may apply Theorem 1.2.1.

REMARK. Reasoning in a somewhat more careful manner, one can also argue directly with Dirichlet’s integral instead of the area integral and thus dispense with the application of conformal representation theorems.

The argument of the proof of Corollary 1.2.1. also shows that if \( \Sigma' \) is connected, one may also decrease the area, hence Dirichlet’s integral by conformal representation again, of a nonembedded minimal surface by adding a nonorientable handle. We therefore also obtain by using Theorem 1.2.2.

**Corollary 1.2.2.** Let \( \gamma \) be a finite collection of disjoined, rectifiable, closed Jordan curves in \( E^3 \). Suppose \( d^*(\gamma, p) \) is realized by a nonembedded minimal surface \( h : \Sigma' \to E^d \) with connected \( \Sigma' \). (Here, for the definition of \( d^*(\gamma, p) \), we do not specify any orientation on the boundary.) Then \( \gamma \) bounds a nonorientable minimal surface.

We note that of course \( \Sigma' \) automatically has to be connected if \( \gamma \) has only one component. A modification of the argument of Corollary 1.2.1. shows that if \( \Sigma' \) is connected again, with nonembedded image, then one may decrease area
already by inserting a Möbius strip for the union of two disks. Hence we also obtain:

Corollary 1.2.3. Let \( \gamma \) be a finite collection of disjoined, rectifiable, closed Jordan curves in \( E^3 \). Suppose \( d^*(\gamma, p) \) is realized by a nonembedded minimal surface \( h : \Sigma' \to E^d \) with connected \( \Sigma' \). Then \( \gamma \) bounds a nonorientable minimal surface of genus \( p \).

A further consequence of Corollary 1.2.1. is

Corollary 1.2.4. Let \( \gamma = (\gamma_1, \ldots, \gamma_m) \) be a collection of disjoint, oriented, rectifiable, closed Jordan curves in \( E^3 \). Then either \( \gamma \) bounds an (oriented) embedded minimal surface of finite genus, or it bounds infinitely many (oriented) minimal surfaces.

Proof. We first find area minimizing disks \( h_i : D \to E^d \) with boundary \( \gamma_i \), \( i = 1, \ldots, m \). If the resulting surface is not embedded, by the argument of the proof of Cor.1.2.1, a Douglas type condition is satisfied, and we get a minimal surface of higher topological type which minimizes Dirichlet's integral in its topological class. Again, it either is embedded, or a Douglas type condition is satisfied, producing another minimal surface bounded by \( \gamma \). The argument can be iterated until an embedded minimal surface is found. \( \square \)

Remark. If \( \gamma \) is smooth (of class \( C^3,0 \)), then Hardt-Simon [HS] have actually shown that \( \gamma \) always bounds an embedded minimal surface (of finite genus) which minimizes area over surfaces of all possible topological types with boundary \( \gamma \). We shall see in a moment that the smoothness assumption is necessary for this result.

We define the genus of a closed Jordan curve \( \gamma \) in \( E^3 \) as the smallest \( p \) for which \( \gamma \) bounds an embedded surface of genus \( p \). (One may then specify in addition whether the surface is orientable or not.) If there is no such finite \( p \), we define the genus to be \( \infty \). There exist rectifiable Jordan curves of genus \( \infty \) which are smooth except at one point. To construct such a curve, one starts with any Jordan curve of positive genus, i.e. a nontrivial knot, and adds infinitely many copies of this curve (in the sense of addition of knots), but scales them down successively in length so that the final curve has finite length.

Corollary 1.2.5. Any rectifiable Jordan curve of infinite genus bounds infinitely many (orientable and nonorientable) minimal surface.

Proof. By the argument of the proof Corollary 1.2.1, 1.2.2 the Douglas conditions are satisfied for any \( p \). \( \square \)

Remark. The first to suggest an example of a rectifiable Jordan curve which is smooth except at one point and which bounds infinitely many minimal surfaces were R. Courant and P. Levy. To make their example rigorous, one needs the bridge theorem which was only proved much later by Meeks-Yau [MY]. Our above example is similar in spirit but does not need the bridge theorem.
1.3 Harmonic maps and minimal surfaces in Riemannian manifolds. In this §, we want to generalize the previous existence theorems for minimal surfaces (Theorems 1.1.1., 1.2.1., 1.2.2.) to Riemannian manifolds as target spaces instead of just Euclidean space. In the proofs of those theorems, an important tool were harmonic maps. Therefore, we first need to establish existence theorems for harmonic maps with values in Riemannian manifolds. Background material may be found in [J2].

Let $\Sigma$ be a Riemann surface, possibly with boundary, with local conformal parameter $z = x + iy$. Let $N$ be a Riemannian manifold of dimension $d$ with metric $\langle ., . \rangle$, $(g_{ij})_{i,j=1,\ldots,d}$ in local coordinates, and Levi-Civita connection $\nabla$. In local coordinates $(x^1, \ldots, x^d)$

\[
\nabla_{\frac{\partial}{\partial z^i}} \frac{\partial}{\partial x^j} = \Gamma_{ij}^k \frac{\partial}{\partial x^k},
\]

where

\[
\Gamma_{ij}^k = \frac{1}{2} g^{kl} (g_{li,j} + g_{lj,i} - g_{ij,l})
\]

are the Christoffel symbols of $N$,

\[(g^{ij}) = (g_{ij})^{-1}, g_{ij,k} = \frac{\partial}{\partial x^k}(g_{ij}),\]

and the standard summation convention is employed). The norm associated with $\langle ., . \rangle$ is denoted by $\| . \|$. Finally, $d(\cdot, \cdot)$ is the distance function of $N$, and for $p \in N$, $\rho > 0$,

\[B(p, \rho) = \{ q \in N : d(p, q) \leq \rho \}\]

For a $C^1$-map

\[f : \Sigma \rightarrow N\]

we define its energy to be

\[(1.3.1) \quad E(f) = \frac{1}{2} \int_{\Sigma} g_{jk}(f(z))(f_j^i f_k^i + f_j^i f_k^i) dx dy \]

\[= \int_{\Sigma} g_{jk}(f(z)) f_j^i f_k^i \frac{dxdy}{i}\]

Here, subscripts denote partial derivatives. In invariant notation,

\[E(f) = \frac{1}{2} \int_{\Sigma} \|df(z)\|^2 dx dy\]

We note that the energy is conformally invariant, hence independent of the choice of local conformal parameter as well as independent of the choice of a metric on $\Sigma$. If $N = \mathbb{E}^d$, then the energy of $f$ coincides with Dirichlet’s integral $D(f)$.

The energy can be defined for a more general class of maps than $C^1(\Sigma, N)$, namely for a Sobolev space $H^{1,2}(\Sigma, N)$. This space may be either defined through an isometric embedding of $N$ into some Euclidean space (which is possible by Nash’s theorem) or locally through approximability by smooth mappings w.r.t. the “(semi)norm” defined by $E$. 

**Definition 1.3.1.** A smooth map \( f : \Sigma \to N \) is called harmonic if it is a critical point of \( E \).

**Remark.** It has recently been proved by Helein [HI] that any critical point of \( E \) is smooth. We shall not need the result here as the map that we shall construct will automatically be regular.

A harmonic map \( h : \Sigma \to N \) has to satisfy the Euler-Lagrange equations for \( E \); in local coordinates, these take the form

\[
(1.3.2) \quad \Delta h^i + \Gamma^i_{jk}(h_z^j h_z^k + h_z^k h_z^l) = 0 \quad i = 1, \ldots, d.
\]

Here, \( \Gamma^i_{jk} \) is evaluated at \( h(z) \), and

\[
\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4 \frac{\partial^2}{\partial x \partial \bar{z}}
\]

is the Laplace operator of \( \Sigma \). In complex notation, (1.3.2) becomes

\[
(1.3.3) \quad h_z^i + \Gamma^i_{jk} h_z^j h_z^k = 0 \quad i = 1, \ldots, d.
\]

In invariant notation, (1.3.2) becomes

\[
(1.3.4) \quad \text{trace} \ \nabla^h dh = 0
\]

where \( \nabla^h \) is the covariant derivative in \( T^*\Sigma \otimes h^{-1}TN \) induced by \( \nabla \) (in \( T^*\Sigma \), we just take the ordinary derivative).

**Lemma 1.3.1.** If \( h : \Sigma \to N \) is harmonic, then

\[
\varphi dz^2 = (h_z, h_{\bar{z}}) dz^2
\]

is a holomorphic quadratic differential on \( \Sigma \).

The proof is a direct consequence of the Euler-Lagrange equations satisfied by \( h \).

For the existence of harmonic maps with values in \( N \) we have to impose some conditions on \( N \). We shall require here that \( N \) is a Riemannian manifold of bounded geometry, meaning that the sectional curvature is bounded in absolute value and the injectivity radius is positive. (We remark here that these conditions can be considerably relaxed. For example, in forthcoming work with I. Nikolaev, it will be shown that it suffices that \( N \) is a metric space with an upper curvature bound in the sense of Alexandrov.) We shall prove

**Theorem 1.3.1.** Let \( \Sigma \) be a compact Riemann surface with nonempty smooth boundary, \( N \) a complete Riemannian manifold of bounded geometry, \( g : \Sigma \to N \) a continuous map of finite energy. Then there exists a harmonic map \( h : \Sigma \to N \), with

\[
h_{|\partial \Sigma} = g_{|\partial \Sigma}
\]

If \( \pi_2(N) = 0 \), then \( h \) is homotopic to \( g \) with fixed boundary values.

**Remarks.** (1) By an approximation argument, the finite energy condition for \( g \) can be dispensed with.
(2) A similar result holds for $\Sigma$ a compact surface without boundary. In this case, however, one has to require that $N$ is compact.

We shall now state some auxiliary results.

The first one is the following version of the Courant-Lebesgue Lemma:

**Lemma 1.3.2.** Let $\Sigma$ be a Riemann surface, possibly with (smooth) boundary, equipped with some metric. All metric notations on $\Sigma$ will refer to this metric. Let $N$ be a Riemannian manifold with distance function $d(\cdot, \cdot)$. Let $z_0 \in \Sigma$, $0 < \delta < 1$, with $\partial B(z_0, \delta) \cap \Sigma$ connected for $\delta < r < \sqrt{\delta}$. Suppose $f \in H^{1,2}(\Sigma, N)$, with $E(f) \leq K$. Then there exists some $r$, $\delta < r < \sqrt{\delta}$, for which $f|_{\partial B(z_0, r) \cap \Sigma}$ is absolutely continuous and satisfies

$$d(f(z_1), f(z_2)) \leq (8\pi K)^{1/2}(\log(1/\delta))^{-1/2} \quad (1.3.5)$$

for all $z_1, z_2 \in \partial B(z_0, r) \cap \Sigma$.

The proof is the same as that of Lemma 1.1.1.

The next result, a local existence result, already follows from Morrey’s work [M]; a simple proof can be found in [J3].

**Lemma 1.3.3.** Let $\Omega$ be a Riemann surface with nonempty boundary of Lipschitz class. Let $N$ be a complete Riemannian manifolds with sectional curvature $\leq \kappa$. Let $p \in N$ with injectivity radius $i(p)$. Let

$$0 < \rho < \min(i(p), \pi/2\sqrt{\kappa}) \quad (1.3.6)$$

Let $g : \Omega \to B(p, \rho) \subset N$ be continuous of finite energy. Then there exists a harmonic map

$$h : \Omega \to B(p, \rho)$$

with $h|_{\partial \Omega} = g|_{\partial \Omega}$, and $h$ minimizes energy among all such maps. Conversely, any such energy minimizing map is harmonic. The modulus of continuity of $h$ can be estimated in terms of $\rho$, $\kappa$, $E(g)$, the modulus of continuity of $g|_{\partial \Omega}$, and the geometry of $\Omega$.

We also have ([J3]):

**Lemma 1.3.4.** If under the same assumptions of Lemma 1.3.3, we have

$$0 < \rho < \frac{1}{2} \min(i(p), \pi/2\sqrt{\kappa}),$$

then the harmonic map $h$ of Lemma 1.3.3 minimizes energy among all maps

$$f : \Omega \to N$$

with

$$f|_{\partial \Omega} = g|_{\partial \Omega}$$

(and not only among maps with image in $B(p, \rho)$). As a consequence, if $u : \Omega \to N$ is energy minimizing w.r.t. its own boundary values, and $u(\partial \Omega) \subset B(p, \rho)$, then also $u(\Omega) \subset B(p, \rho)$.

Finally, we shall need the following regularity result, due to Ladyzhenskaya-Ural’tseva:
Lemma 1.3.5. Let $h$ be a continuous weak solution of (1.3.2), i.e.,
\[
\int_{\Sigma} (h_{x}^{i} \varphi_{x}^{i} + h_{y}^{i} \varphi_{y}^{i}) = \int_{\Gamma} (h_{x}^{i} h_{x}^{k} + h_{y}^{i} h_{y}^{k}) \varphi^{i}
\]
for all $\varphi \in H_{0}^{1,2}(\Sigma)$. Then $h$ is a smooth map.

Proof of Theorem 1.3.1. We equip $\Sigma$ with some Riemannian metric, compatible with the conformal structure. All metric notions on $\Sigma$ occuring in the sequel will refer to this metric. We put
\[
\rho = \frac{1}{3} \min(i(N), \pi/2\sqrt{\kappa})
\]
where $i(N)$ is the injectivity radius of $N$, and $\kappa \geq 0$ is an upper bound for the sectional curvature of $N$. We choose $\delta > 0$ with

\[
(8\pi E(g) \left( \log \frac{1}{\delta} \right)^{1/2}) \leq \rho
\]

and
\[
d(g(y_{1}), g(y_{2})) \leq \rho
\]
for all $y_{1}, y_{2} \in \partial \Sigma$ with $d(y_{1}, y_{2}) \leq \sqrt{\delta}$. This choice of $\delta$ is possible because $g|_{\partial \Sigma}$ is continuous and $\partial \Sigma$ is compact. We choose points $x_{1}, \ldots, x_{m}$, $m = m(\delta)$, with

\[
\Sigma \subset \bigcup_{i=1}^{m} B \left( x_{i}, \frac{\delta}{2} \right)
\]

We put $u_{0} := g$. Having iteratively constructed $u_{j}$, $j = 0, \ldots, m - 1$, with

\[
E(u_{j}) \leq E(g)
\]

we obtain $u_{j+1}$ as follows: On account of (1.3.11) and (1.3.13), by Lemma 1.3.2, there exists $r_{j}^{0}$, $\delta < r_{j}^{0} < \sqrt{\delta}$, with

\[
u_{j} \left( \partial B(x_{j}, r_{j}^{0}) \right) \subset B(p, \rho)
\]
for some $p \in N$; here,
\[
\partial B(x_{j}, r) = \{ x \in \Sigma : d(x, x_{j}) = r \} \cup \{ x \in \partial \Sigma : d(x, x_{j}) \leq r \}
\]
We let
\[
h_{j} : B(x_{j}, r_{j}^{0}) \rightarrow B(p, \rho)
\]
be the harmonic map with
\[
h_{j} \left( \partial B(x_{j}, r_{j}^{0}) \right) = u_{j} \left( \partial B(x_{j}, r_{j}^{0}) \right)
\]
By Lemma 1.3.4, noting (1.3.10), we get

\[
E(h_{j}) \leq E(u_{j} |_{\partial B(x_{j}, r_{j}^{0})})
\]
We define

\[ u_{j+1}(x) := \begin{cases} 
  h_j(x) & \text{for } x \in B(x_j, r_j^n) \\
  u_j(x) & \text{for } x \in \Sigma \setminus B(x_j, r_j^n)
\end{cases} \]

By (1.3.15), (1.3.13)

\[ (1.3.16) \quad E(u_{j+1}) \leq E(u_j) \leq E(g) \]

We put \( v_0^1 := u_m \). Having iteratively constructed \( v_j^n \), \( j = 0, \ldots, m - 1, n \in \mathbb{N} \), with

\[ E(v_j^n) \leq E(g), \]

we obtain \( u_{j+1}^n \) from \( v_j^n \) in the same manner that \( u_{j+1} \) was obtained from \( u_j \), i.e. by energy minimizing replacement on a disk \( B(x_j, r_j^n) \), where \( r_j^n \) was chosen with the help of Lemma 1.3.2 to satisfy \( \delta < r_j^n < \sqrt{\delta} \), and

\[ v_j^n(\partial B(x_j, r_j^n)) \subset B(p, \rho) \]

for some \( p \in \mathbb{N} \). Then

\[ (1.3.17) \quad E(v_{j+1}^n) \leq E(v_j^n). \]

We put iteratively

\[ v_0^{n+1} := v_m^n. \]

Then

\[ (1.3.18) \quad E(v_{j+1}^{n+1}) \leq E(v_j^{n+1}) \leq \ldots \leq E(v_1^{n+1}) \leq E(v_m^n) \leq \ldots E(v_j^n) \leq \ldots E(g) \]

We now look at

\[ (v_1^n)_{n \in \mathbb{N}} \]

Since for all \( n \in \mathbb{N} \)

\[ (1.3.19) \quad E(v_1^n) \leq E(g) \]

and \( v_1^n |_{B(x_1, \delta)} \) is energy minimizing w.r.t. its own boundary values, the Courant-Lebesgue Lemma 1.3.2 and the maximum principle Lemma 1.3.4 easily imply that \( (v_1^n |_{B(x_1, \delta)}) \) is equicontinuous. Namely, given \( \epsilon > 0 (\epsilon \leq \rho) \), we choose \( \eta \) satisfying \( \eta < (\frac{\delta}{2})^2 \) and

\[ \left( \frac{8 \pi E(g)}{\log 1/\eta} \right)^{1/2} \leq \epsilon, \quad d(g(x_1), g(x_2)) \leq \frac{\epsilon}{2} \]

for all \( x_1, x_2 \in \partial \Sigma \) with \( d(x_1, x_2) \leq \sqrt{\eta} \). For every \( x \in B(x_1, \frac{\delta}{2}) \), we then find \( r \) with \( \eta < r < \sqrt{\eta} \), and

\[ v_j^n(\partial B(x, r)) \subset B(p, \epsilon) \]
for some \( p \in N \), by Lemma 1.3.2. By Lemma 1.3.4, we then also have
\[
v^n_1(B(x, r)) \subset B(p, \epsilon),
\]
hence in particular
\[
v^n_1(B(x, \eta)) \subset B(p, \epsilon).
\]
Since \( \eta \) does not depend on \( n \), this proves equicontinuity of \((v^n_1|_{B(x_1, \frac{\delta}{2})})\). Since \((v^n_1|_{B(x_1, \frac{\delta}{2})})_{n \in N}\) also has uniformly bounded energy ((1.3.19)), after selection of a subsequence, \((v^n_1|_{B(x_1, \frac{\delta}{2})})\) converges weakly in \( H^{1,2} \) as well as uniformly to a harmonic map
\[
v : B(x_1, \frac{\delta}{2}) \to N.
\]
We now want to show that \((v^n_1|_{B(x_2, \frac{\delta}{2})})_{n \in N}\) also converges to a harmonic map.
Since by (1.3.18)
\[
E(v^n_1) \leq E(v^n_2) \leq E(v^{n-1}_2),
\]
and
\[
v^n_2|_{\Sigma \setminus B(x_2, r^n_2)} = v^n_1|_{\Sigma \setminus B(x_2, r^n_2)}
\]
we conclude
\[
\lim_{n \to \infty} (E(v^n_1|_{B(x_2, r^n_2)}) - E(v^n_2|_{B(x_2, r^n_2)})) = 0
\]
Since \(v^n_2|_{B(x_2, r^n_2)}\) is energy minimizing w.r.t. its own boundary values and
\[
v^n_1|_{\partial B(x_2, r^n_2)} = v^n_2|_{\partial B(x_2, r^n_2)},
\]
we conclude that \(v^n_1|_{B(x_2, r^n_2)}\) converges strongly in \( H^{1,2} \), after passing to a subsequence, and
\[
(1.3.20) \quad \|DE(v^n_1|_{B(x_2, r^n_2)})\| = 0
\]
\((DE)\) is the first variation of energy; in local coordinates
\[
DE(v)(\varphi) := \frac{d}{dt} E(v + t\varphi)|_{t=0},
\]
for \( v : \Omega \to R^d(\text{a local coordinate chart for } N), \varphi \in H^{1,2}_0(\Omega, R^d)\).

Since \(v^n_1|_{B(x_2, \frac{\delta}{2})}\) is obtained from the harmonic map \(v^{n-1}_2|_{B(x_2, \frac{\delta}{2})}\) by a finite number of energy minimizing replacements, we also get equicontinuity from the estimate of Lemma 1.3.3, the Courant-Lebesgue Lemma and the maximum principle, hence again uniform convergence after selection of a subsequence. By (1.3.20), the limit then has to be a critical point of \(E\), i.e. harmonic, because continuous critical points of \(E\) are smooth (Lemma 1.3.5).

Iteratively, we see that (after selecting subsequences), \((v^n)_{n \in N}\) converges to a harmonic map \(h\) on every disk \(B(x_i, \frac{\delta}{2})\), \(i = 1, \ldots, m\), hence on all of \(\Sigma\) by (1.3.12). We finally observe that in case \(\pi_2(N) = 0\), any two maps from a disk into \(N\) with the same boundary values are homotopic. Hence in this case, the successive harmonic replacements on disks never change the homotopy class,
and all maps $\nu^n_j$ are homotopic to $g$, and then so is the uniform limit $h$ of a subsequence.

From the proof of Theorem 1.3.1, we get

**Corollary 1.3.1.** Let $g: \Sigma \to N$ be continuous of finite energy, where $\Sigma$ and $N$ are as in Theorem 1.3.1. If $\pi_2(N) = 0$, then there exists a harmonic map $h: \Sigma \to N$ with

$$h|_{\partial \Sigma} = g|_{\partial \Sigma}$$

minimizing the energy among all maps homotopic to $g$ and with the same boundary values. Without the assumption $\pi_2(N) = 0$, $h$ may be taken to minimize energy among all maps with the same boundary values as $g$ and inducing the same maps on fundamental groups, $g_{\#} : \pi_1(\Sigma, x_0) \to \pi_1(N, p_0)$. Consequently, there also exists a harmonic map $h: \Sigma \to N$ minimizing energy among all maps with the same boundary values as $g$, without any further restriction.

**Proof.** Let $(g_n)_{n \in N}$ be an energy minimizing sequence among maps with the same boundary values as $g$ and the desired topological restriction. We then produce a harmonic sequence $(h_n)_{n \in N}$ from $(g_n)_{n \in N}$ by the method of proof of Theorem 1.3.1. We then get

$$E(h_n) \leq E(g_n) \quad (1.3.21)$$

Hence $(h_n)_{n \in N}$ also is an energy minimizing sequence. As before, by the Courant-Lebesgue Lemma and the maximum principle, we get uniform convergence of a subsequence. The limit $h$ then is again harmonic. Because of (1.3.21), we may assume that the convergence is weak $H^{1,2}$ convergence as well. Hence, by lower semicontinuity of $E$ w.r.t. weak $H^{1,2}$ convergence,

$$E(h) \leq \lim_{n \to \infty} E(h_n),$$

and $h$ is energy minimizing as desired. □

Theorem 1.3.1 and Corollary 1.3.1 are due to Lemaire [Le]. The method of Sacks-Uhlenbeck [SU] may also be applied, although they only treat closed surfaces. The present proof is adapted from [J3]. It is somewhat more complicated than the proof of [J3] but has the advantage that this method when suitably refined may also be used to produce unstable harmonic maps by saddle point constructions, see [J3] for details. The last statement of Corollary 1.3.1 was already proved much earlier by Morrey [M].

We also mention the following result, due to Al'ber [Al11], [Al12], Hartman [Ht], Hamilton [Hm], without proof.

**Theorem 1.3.2.** If, under the assumptions of Theorem 1.3.1, $N$ has non-positive sectional curvature, then the harmonic $h$ of Corollary 1.3.1 is unique in its homotopy class.

Theorem 1.3.1 allows to obtain existence results for minimal surfaces in Riemannian manifolds by essentially the same method as in 1.1, 1.2. Let $\Sigma$ and $N$ be as before, and let

$$\alpha : \pi_1(\Sigma, x_0) \to \pi_1(N, p_0)$$
be a homomorphism of fundamental groups. If \( \pi_2(N) = 0 \), we may take \( \alpha \) as a homotopy class of maps from \( \Sigma \) to \( N \).

Let \( \gamma = (\gamma_1, \ldots, \gamma_m) \) be a collection of disjoint closed rectifiable Jordan curves with a specified orientation in case we seek oriented minimal surfaces.

Let \( \Sigma \) be a compact Riemann surface with boundary curves \( c_1, \ldots, c_m \), and let

\[
f \in C^0 \cap H^{1,2}(\Sigma, N)
\]

map \( c_i \) bijectively onto \( \gamma_i \), with preserved orientation in the oriented case \( (i = 1, \ldots, m) \), and

\[
f_# = \alpha
\]

where \( f_# : \pi_1(\Sigma, z_0) \to \pi_1(N, p_0) \) is the induced map on fundamental groups.

We define a primary reduction of \( (\Sigma, \alpha) \) to be one of the following procedures:

(i) If \( \beta \in \pi_1(\Sigma, z_0) \) is an oriented element representable by a simple closed curve \( c_\beta \) in the interior of \( \Sigma \), with

\[
f_#(\beta) = 0 \in \pi_1(N, p_0),
\]

we cut \( \Sigma \) along \( c_\beta \) and contract the one or two resulting curves to points or equivalently glue in disks.

(If the cut yields two curves, the process means that we either cut a handle and thereby reduce the genus by 1 or that we disconnect the surface. If we only have one cut curve, we replace a Möbius strip by a disk, thereby reducing the genus by 1/2.)

(ii) Let \( b \in \Sigma \) be a Jordan arc with endpoints on one of the boundary curves \( c_i \) of \( \Sigma \), and with \( f(b) \) homotopic to a subarc of \( \gamma_i \). We cut \( \Sigma \) along \( b \), again possibly disconnecting \( \Sigma \). One possibility is that the resulting surface has two new boundary curves \( c', c'' \), each consisting of a part of \( c_i \) and of a copy of \( b \) after the cut. After a homotopy, one of the curves \( c', c'' \) is mapped onto \( \gamma_i \) and one of the curves \( c', c'' \) has homotopically trivial image. We contract this curve to a point. In the case where the cut leads to a disconnected surface with one component of disk type, we assume that the new boundary curve of the other component has homotopically trivial image, and that boundary curve is contracted to a point.

The second possibility which can only occur in the nonorientable case, is that the resulting surface has a boundary curve consisting of \( c_i \) and two copies of \( b \). These two copies then are contracted to points so that we may identify the new boundary curve again with \( c_i \).

A primary reduction always reduces the topological type of \( \Sigma \). A reduction of \( (\Sigma, \alpha) \) is obtained by applying a finite number of primary reductions.

We now define in the oriented case

\[
d(\gamma, p, \alpha) := \inf \{ E(f, \Sigma) : \Sigma \text{ a connected oriented Riemann surface of genus } p \text{ with boundary curves } c_1, \ldots, c_m, f \in H^{1,2} \cap C^0(\Sigma, N) \text{ mapping } c_i \text{ bijectively and with preserved orientation onto } \gamma_i, i = 1, \ldots, m, \text{ and } f_# = \alpha \}
\]

d^*(\gamma, p, \alpha) \text{ then is defined similarly by taking the infimum of } E(f, \Sigma') \text{ over primary reductions of } (\Sigma, \alpha). \text{ If there are no primary reductions, i.e. if } \alpha \text{ is injective, we put } d^*(\gamma, p, \alpha) = \infty. \text{ It is not difficult to see that allowing arbitrary reductions would not alter the value of } d^*(\gamma, p, \alpha).
In the nonorientable case, we similarly define numbers \( d_-(\gamma, p, \alpha) \) and \( d_+^*(\gamma, p, \alpha) \).

The following analogue of Theorem 1.2.1 in Riemannian manifolds was proved in [J1].

**Theorem 1.3.3.** Let \( N \) be a Riemannian manifold of bounded geometry, \( \gamma = (\gamma_1, \ldots, \gamma_m) \) a system of disjoint, closed, oriented rectifiable Jordan curves in \( N \), \( \Sigma \) a compact oriented Riemann surface of genus \( p \) with boundary curves \( c_1, \ldots, c_m \),

\[
\alpha : \pi_1(\Sigma, z_0) \to \pi_1(N, p_0)
\]
a homomorphism, realized by some continuous map

\[
f : \Sigma \to N
\]

mapping \( c_i \) bijectively and with preserved orientation onto \( \gamma_i \), \( i = 1, \ldots, m \). If

\[
d(\gamma, p, \alpha) < d_+^*(\gamma, p, \alpha)
\]

there exists an oriented minimal surface of genus \( p \) with boundary \( \gamma \) and induced map \( \alpha \) on fundamental groups.

The corresponding result for nonorientable minimal surfaces was proved by F. Bernatzki [Be]:

**Theorem 1.3.4.** Modify the assumption of Theorem 1.3.2 by letting the curve \( \gamma_i \), \( i = 1, \ldots, m \), be unoriented, and \( \Sigma \) a nonorientable Riemann surface of genus \( p \). If

\[
d_-(\gamma, p, \alpha) < d_+^*(\gamma, p, \alpha)
\]

there exists a nonorientable minimal surface of genus \( p \) with boundary \( \gamma \), inducing again the map \( \alpha \) of fundamental groups.

The proofs of Theorems 1.3.3 and 1.3.4 are analogous to those of Theorems 1.2.1, 1.2.2, by using the existence result of Corollary 1.3.1 for harmonic maps. We only observe that if \( c_n \) is a simple closed geodesic on a hyperbolic surface \( \Sigma_n \), coming from a minimizing sequence \( (\Sigma_n) \), and if \( \alpha(c_n) \neq 0 \), \( f_n : \Sigma_n \to N \) a map with the required boundary conditions and \( f_{n^*} = \alpha \), then the lengths of \( f_n(c') \), \( c' \) homotopic to \( c_n \), are bounded away from 0. As explained in the proof of Theorem 1.2.1, this excludes degeneration of \( c_n \) in the limit \( n \to \infty \). For this reason, it suffices to consider primary reductions of \( \Sigma_n \).

**Remark.** If \( N \) is compact, one has similar existence results for closed minimal surfaces with prescribed action on the fundamental group, or prescribed homotopy class in case \( \pi_2(N) = 0 \).
2 Teichmüller theory

2.1 Basic definitions. We first consider compact Riemann surfaces without boundary. Noncompact ones and ones with boundary will be treated later.

Definition 2.1.1. Let $\Sigma_1$ and $\Sigma_2$ be (orientable or nonorientable) compact Riemann surfaces. $\Sigma_1$ and $\Sigma_2$ are called equivalent if there exists a conformal diffeomorphism,

$$ k : \Sigma_1 \to \Sigma_2 $$

Here, "conformal" means that $k$ is bijective and that its representations in local coordinates are holomorphic or antiholomorphic. The space of equivalence classes is called the moduli space,

$$ \mathcal{M}_p \left( \mathcal{M}_{p, -} \right) $$

if $\Sigma_1$ and $\Sigma_2$ are orientable (nonorientable) of genus $p$.

Remark. The space $\mathcal{M}_p$ defined here is usually called reduced moduli space in the literature. For the usual moduli space, one only allows holomorphic $k$.

Let now $\Sigma_0$ be a Riemann surface of the given topological type and let $s_i : \Sigma_0 \to \Sigma_i$, $i = 1, 2$, be diffeomorphisms. We denote by $[s]$ the homotopy class of a diffeomorphism.

Definition 2.1.2. $(\Sigma_1, [s_1])$ and $(\Sigma_2, [s_2])$ are equivalent, if $s_2 \circ s_1^{-1} : \Sigma_1 \to \Sigma_2$, is homotopic to a conformal diffeomorphism. The space of equivalence classes is called Teichmüller space $\mathcal{T}_p$ (or $\mathcal{T}_{p,-}$ in the nonorientable case).

If $s : \Sigma_0 \to \Sigma_0$ is any diffeomorphism, we obtain a new Riemann surface $\Sigma_0^s$ by pulling the conformal structure of $\Sigma$ back via $s^{-1}$. Thus, if $\Omega$ open in $\Sigma_0^s$, then $h : \Omega^s \to C$ is holomorphic, iff $hs : s^{-1}(\Omega^s) \to C$ is holomorphic on $s^{-1}(\Omega^s) \subset \Sigma_0$. Then, however, $(\Sigma_0, [id])$ and $(\Sigma_0^s, [s])$ are equivalent because $s : (\Sigma_0, [id]) \to (\Sigma_0^s, [s])$ is a conformal diffeomorphism in the required homotopy class. Vice versa, any "marked" Riemann surface, i.e. each pair $(\Sigma, [s])$, equivalent to $(\Sigma_0, [id])$ can be written as $(\Sigma_0^k, [k])$, where $k : \Sigma_0 \to \Sigma_0^k$ is a conformal diffeomorphism homotopic to $s$.

We shall study only the case of Riemann surfaces $\Sigma$ of genus $p > 1$. In this case, the universal cover of $\Sigma$ is the upper half plane $H$ and $\Sigma$ thus inherits a hyperbolic metric, i.e. $\Sigma = H/\Lambda$, where $\Lambda$ is a discrete group of automorphisms of $H$ without fixed points. We put $\Sigma_0 = H/\Lambda_0$

Given $(\Sigma, [s]), s : \Sigma_0 \to \Sigma$ can be lifted to

$$ \sigma : H \to H $$

with

$$ \Lambda = \sigma \circ \Lambda_0 \circ \sigma^{-1} $$

i.e. for every $\gamma \in \Lambda_0$, there exists $\gamma' \in \Lambda$ with

$$ \sigma \circ \gamma = \gamma' \circ \sigma $$
Conversely, if (2.1.1) holds, then $\sigma : H \to H$ induces a diffeomorphism $s : H/\Lambda_\sigma \to H/\Lambda$.

Thus $\Lambda$ is considered as a deformation of $\Lambda_\sigma$ if (2.1.1) is satisfied, and the deformation is considered to be trivial if $\sigma$ in (2.1.2) can be chosen to be holomorphic. This transforms our equivalence relation for marked Riemann surfaces $(\Sigma, [\sigma])$ into an equivalence relation for the corresponding lattices $\Lambda$.

If $s : \Sigma_\sigma \to \Sigma_0$ is a diffeomorphism, then $(\Sigma_\sigma, [id])$ is equivalent to $(\Sigma_0^s, [s])$, and there exists a commutative diagram with conformal $\sigma$,

\[
\begin{array}{ccc}
H & \xrightarrow{\sigma} & H \\
\downarrow & & \downarrow \\
H/\Lambda_0 = \Sigma_0 & \xrightarrow{s} & \Sigma_0^s = H/\Lambda_\sigma
\end{array}
\]

with lattices $\Lambda_\sigma$, $\Lambda_\sigma$ related by

\[\Lambda_\sigma = \sigma \circ \Lambda_0 \circ \sigma^{-1}\]

Since $\sigma$ is conformal, this means that $\Lambda_\sigma$ and $\Lambda_\sigma$ are equivalent. Vice versa, if $\Lambda_\sigma$ is equivalent to $\Lambda_0$, then $(H/\Lambda_\sigma, [s])$ is equivalent to $(H/\Lambda_0, [id])$.

In other words: $\Lambda_0$ is equivalent to $\Lambda_\sigma$, if and only if after conjugation by a conformal diffeomorphism $k$ of $H$, $\Lambda_\sigma$ is the same lattice as $\Lambda_0$, i.e.

\[
(2.1.3) \quad \Lambda_\sigma = (\kappa^{-1} \circ \sigma)\Lambda_\sigma(\kappa^{-1} \circ \kappa) = (\kappa^{-1} \circ \Lambda_\sigma \circ \kappa).
\]

and $\kappa^{-1} \circ \sigma$ thus is a $\Lambda_\sigma$-equivariant diffeomorphism of $H$. Thus, $\sigma$ leads to a trivial deformation of $\Lambda_0$ if it is $\Lambda_\sigma$-equivariant up to a conformal automorphism of $H$. This is now easy to understand: We may consider a deformation of $\Lambda_\sigma$ as a deformation of a fundamental region for $H/\Lambda_\sigma$, and the deformation is trivial if the new fundamental region is conformally equivalent to the original one (with appropriate boundary identifications).

We now turn to infinitesimal considerations.

Let $\sigma_t : H \to H$ be a smooth family of diffeomorphisms, with $\sigma_0 = id$, satisfying

\[
(2.1.4) \quad \sigma_t \circ \gamma = \gamma^t \circ \sigma_t \text{ for } \gamma \in \Lambda_\sigma, \gamma^t \in \Lambda_t.
\]

With

\[
\dot{\sigma} := \frac{d}{dt}\sigma_t|_{t=0}
\]

we get

\[
(2.1.5) \quad \dot{\sigma} \circ \gamma = \gamma \gamma_2 \dot{\sigma} + \dot{\gamma}
\]

If we have a family of $\Lambda_\sigma$-equivariant diffeomorphisms, i.e. $k_t \circ \gamma = \gamma \circ k_t$, we get

\[
(2.1.6) \quad \dot{k}(\gamma) = \gamma_2 k,
\]

and $\dot{k}$ thus defines an infinitesimal diffeomorphism, i.e. a vector field of the surface $H/\Lambda_\sigma$. 


Since we are interested in variations only up to holomorphic automorphisms of $H$, we take derivatives w.r.t. $\bar{z}$ of the preceding expressions. Thus we put
\[
\alpha := \dot{\gamma}_z
\]
Then from (2.1.5)
\[
(2.1.7) \quad \alpha(\gamma) \gamma_z = \gamma_z \alpha
\]
since $\gamma$ is holomorphic.

We then consider $\alpha$ satisfying (2.1.7) as an infinitesimal variation of $H/\Lambda_o$. On the space of infinitesimal variations, we have a natural $L^2$-product:
\[
(\alpha, \beta) = \text{Re} \int_{H/\Lambda_o} \alpha(z) \overline{\beta(z)} \frac{1}{y^2} \frac{i}{2} dz \wedge d\bar{z}
\]
where $\frac{1}{y^2} dz \otimes d\bar{z}$ is the hyperbolic metric of $H (z = x + iy)$ and where the integration may be performed over a fundamental region. The integral is well defined, i.e. independent of the choice of fundamental region, because $\alpha \beta$ transforms as a function if both $\alpha$ and $\beta$ transform according to (2.1.7).

We are interested in those infinitesimal variations that are orthogonal to the infinitesimal diffeomorphisms of $H/\Lambda_o$. $\alpha$ is orthogonal to those if
\[
(2.1.8) \quad \int_{H/\Lambda_o} \alpha(z) \left( \frac{\partial}{\partial \bar{z}} \frac{\bar{k}}{1} \right) \frac{1}{y^2} \frac{i}{2} dz \wedge d\bar{z}
\]
\[
= \int_{H/\Lambda_o} \dot{\gamma}_z \overline{\bar{k}_z} \frac{1}{y^2} \frac{i}{2} dz \wedge d\bar{z}
\]
\[
= 0
\]
Because of the transformation behaviour of $\bar{k}$ expressed by (2.1.6), or in other words because $\bar{k}$ is a vector field on $H/\Lambda_o$, we may integrate by parts in (2.1.8) and obtain,
\[
(2.1.9) \quad 0 = \int_{H/\Lambda_o} \frac{\partial}{\partial \bar{z}} \left( \alpha(z) \frac{1}{y^2} \right) \frac{i}{2} dz \wedge d\bar{z}
\]
If this holds for all vector fields $\bar{k}$ on $H/\Lambda_o$, we obtain
\[
(2.1.10) \quad \frac{\partial}{\partial \bar{z}} \left( \alpha(z) \frac{1}{y^2} \right) = 0
\]

**Definition 2.1.3.** $\alpha$ is called a Beltrami differential for $H/\Lambda_o$ if it has transformation behaviour

\[
\alpha(\gamma(z)) = \alpha(z) \frac{\gamma_{\bar{z}}}{\overline{\gamma_{\bar{z}}}} \text{ for all } \gamma \in \Lambda_o
\]
and it is called a harmonic Beltrami differential if it satisfies in addition

$$\frac{\partial}{\partial z} \left( \alpha(z) \frac{1}{y^2} \right) = 0.$$ 

$\mathcal{H}(\Sigma) = \mathcal{H}(\Lambda_\circ)$ is defined to be the vector space of harmonic Beltrami differentials on $\Sigma = H/\Lambda_\circ$.

The transformation behaviour of a Beltrami differential is the one of $\frac{\partial}{\partial \bar{z}} \otimes d\bar{z}$.

**Lemma 2.1.1.** $\mathcal{H}(\Lambda_\circ)$ is the orthogonal complement of the space of vector fields on $H/\Lambda_\circ$.

**Proof.** We need to show that the intersection of $\mathcal{H}(\Lambda_\circ)$ with the space of vector fields on $H/\Lambda_\circ$ is $\{0\}$. If $v$ is a vector field for which $v_z$ is a harmonic Beltrami differential then as in the derivation of (2.1.9),

$$\int_{H/\Lambda_\circ} v_z \bar{v}_z \frac{1}{y^2} \cdot i \frac{dz}{2} \wedge d\bar{z} = \int \frac{\partial}{\partial z} \left( v_z \frac{1}{y^2} \right) \bar{v} \frac{i}{2} dz \wedge d\bar{z} = 0,$$

hence $v_z \equiv 0$. Thus $v$ is a holomorphic vector field on $H/\Lambda_\circ$. Since $H/\Lambda_\circ$ is hyperbolic, this implies $v \equiv 0$. □

The following observation will be useful

**Lemma 2.1.2.** If $\alpha$ is a harmonic Beltrami differential on $\Sigma_\circ = H/\Lambda_\circ$, then

$$\varphi(z) := \frac{1}{y^2} \bar{\alpha}(z)$$

is a holomorphic quadratic differential on $H/\Lambda_\circ$.

**Proof.** First the transformation behaviour: the hyperbolic metric $\frac{1}{y^2}$ transforms, like every metric, via $dz \otimes d\bar{z}$, hence $\varphi$ transforms via $dz^2$, or in terms of $\gamma$

$$\varphi(\gamma(z)) = \varphi(z) \frac{1}{(\gamma_z(z))^2}$$

Thus, $\varphi$ is a quadratic differential. If $\alpha$ is harmonic, i.e. (2.1.12) holds, then $\frac{\partial}{\partial \bar{z}} \varphi = 0$, and $\varphi$ is holomorphic. □

**Definition 2.1.4.** $Q(\Sigma) = Q(\Lambda_\circ)$ is the space of holomorphic quadratic differentials on $\Sigma = H/\Lambda_\circ$.

**Corollary 2.1.1.**

$$\dim \mathbb{C}\mathcal{H}(\Lambda_\circ) = \dim \mathbb{C}Q(\Lambda_\circ) = 3p - 3,$$
where \( p \) is the genus of \( H/\Lambda_0 \).

Proof. \( \dim_{\mathbb{C}} Q(\Lambda_0) = 3p - 3 \) follows from the Riemann Roch theorem and Lemma 2.1.1 implies \( \dim \mathcal{H}(\Lambda_0) = \dim Q(\Lambda_0) \).

\[ \square \]

\textbf{Theorem 2.1.1.} \( \mathcal{T}_p \) carries the structure of a differentiable manifold with tangent space \( \mathcal{H}(\Lambda_0) \) at the point corresponding to \( H/\Lambda_0 \).

Let us outline the

Proof. We first define a topology on \( \mathcal{T}_p \). We say that a sequence of lattices \((\Lambda_n)_{n \in \mathbb{N}}\) converges to \( \Lambda \) if there exists a sequence \((\sigma_n)_{n \in \mathbb{N}}\) of diffeomorphisms \( \sigma_n : H \to H \) with

\[ \Lambda_n = \sigma_n \circ \Lambda \circ \sigma_n^{-1} \]

and \( \sigma_n \to \text{id} \) in some \( H^{m,2} \) or \( C^{m,\alpha} \) topology, for some large \( m \) and some \( 0 < \alpha < 1 \).

We next define a map from \( \mathcal{H}_1 := \{ \alpha \in \mathcal{H}(\Lambda_0), \sup_{z \in H/\Lambda_0} |\alpha(z)| < 1 \} \) into \( \mathcal{T}_p \) by assigning to \( \alpha \) the unique diffeomorphism,

\[ \sigma : H \to H \]

with

\[ \sigma_z = \alpha \sigma_z \]

and

\[ \sigma(0) = 0, \sigma(1) = 1, \sigma(\infty) = \infty. \]

The assumption \( \sup |\alpha| < 1 \) is needed for the solvability of (2.1.12); it guarantees that (2.1.12) is an elliptic equation.

Since harmonic Beltrami differentials are smooth (even real analytic) as a consequence of Lemma 2.1.1, so are the solutions of (2.1.12) by elliptic regularity theory. The unique solvability of (2.1.12) under the normalization (2.1.13) together with elliptic regularity theory also implies continuity of our map from \( \mathcal{H}_1 \) into \( \mathcal{T}_p \).

\[ \square \]

\textbf{Corollary 2.1.2.} \( \mathcal{T}_p \) becomes a Riemannian manifold when equipped with the \( L^2 \)-inner product on the tangent space \( \mathcal{H}(\Lambda_0) \).

\[ (\alpha, \beta) = \text{Re} \int_{H/\Lambda_0} \overline{\alpha(z)\beta(z)} \frac{1}{y^2} \frac{i}{2} dz \wedge d\bar{z} \]

\[ \text{(2.1.14)} \]

\textbf{Definition 2.1.5.} The metric defined by (2.1.14) is called Weil-Petersson metric on \( \mathcal{T}_p \).
Corollary 2.1.3. The Weil-Petersson metric identifies \( Q(\Lambda_0) \) with the cotangent space of \( T_p \) at the point corresponding to \( H/\Lambda_0 \).

Proof. The Weil-Petersson metric defines a pairing between \( \mathcal{H}(\Lambda_0) \) and \( Q(\Lambda_0) \) via

\[
((\alpha, \varphi)) = \text{Re} \int_{\mathcal{H}/\Lambda_0} \alpha(z) \bar{\varphi}(z) \frac{dz}{2} \wedge d\bar{z}
\]

for \( \alpha \in \mathcal{H}(\Lambda_0), \varphi \in Q(\Lambda_0) \). Because of

\[
\left( \alpha, \frac{1}{y^2} \bar{\alpha} \right) = (\alpha, \alpha)
\]

this establishes the required duality between \( \mathcal{H}(\Lambda_0) \) and \( Q(\Lambda_0) \).

Let us also briefly look at the case of nonorientable surfaces. Again we only consider the case of genus \( > 1 \) so that each surface may again be represented as \( H/\Lambda_0 \); \( \Lambda_0 \) however, now also contains orientation reversing automorphisms.

If \( \rho \in \Lambda_0 \) is orientation reversing, it is antiholomorphic and so we get in analogy to (2.1.5)

\[
(2.1.15) \quad \dot{\sigma} \circ \rho = \rho \dot{\sigma} + \dot{\rho}, \text{ or equivalently } \dot{\sigma} \circ \rho = \bar{\rho} \dot{\sigma} + \dot{\bar{\rho}} \text{ (note } \dot{\bar{\sigma}} = \dot{\sigma})
\]

hence

\[
(2.1.16) \quad \dot{\sigma}_z \rho \bar{z} = \bar{\rho} \dot{\sigma} \bar{z}
\]

i.e. for \( \alpha = \sigma_z \)

\[
(2.1.17) \quad \bar{\sigma}(\rho(z)) = \alpha(z) \frac{\bar{\rho}_z}{\rho_z}
\]

\[
(2.1.18) \quad \alpha(\rho(z)) = \bar{\sigma}(z) \frac{\rho_z}{\bar{\rho}_z}
\]

Definition 2.1.6. \( \alpha \) is called a Beltrami differential for \( \Lambda_0 \) if for orientation preserving \( \gamma \in \Lambda_0 \)

\[
(2.1.19) \quad \alpha(\gamma(z)) = \alpha(z) \frac{\gamma_z(z)}{\overline{\gamma_z(z)}}
\]

and for orientation reversing \( \rho \in \Lambda_0 \)

\[
(2.1.20) \quad \alpha(\rho(z)) = \bar{\alpha}(z) \frac{\rho_z(z)}{\bar{\rho}_z(z)}
\]

It is called harmonic if it always satisfies either

\[
\frac{\partial}{\partial z} \left( \alpha(z) \frac{1}{y^2} \right) = 0
\]
or
\[ \frac{\partial}{\partial \bar{z}} \left( \alpha(z) \frac{1}{y^2} \right) = 0 \]
\( \mathcal{H}(\Sigma) = \mathcal{H}(\Lambda_\alpha) \) is defined to be the vector space of harmonic Beltrami differentials for \( \Sigma = H/\Lambda_\alpha \), after identifying \( \alpha \) and \( \bar{\alpha} \) as Beltrami differentials for \( \Lambda_\alpha \).

Of course, in the oriented case, \( \mathcal{H}(\Lambda_\alpha) \) is isomorphic through complex conjugation to the space of “antiharmonic” Beltrami differentials, i.e. those satisfying
\[ \frac{\partial}{\partial \bar{z}} \left( \beta(z) \frac{1}{y^2} \right) = 0 \]
and thus also here we may perform the identification between \( \alpha \) and \( \bar{\alpha} \).

We may then define the Teichmüller space \( \mathcal{T}_{p,-} \) for a nonorientable Riemann surface of genus \( p \) in the same way as in the oriented case and results analogous to the preceding ones (Theorem 2.1.1. Corollary 2.1.2, Corollary 2.1.3) hold.

Finally, we let \( \mathcal{T}_{p,k} \) and \( \mathcal{T}_{p,k,-} \) be the Teichmüller space for Riemann surfaces of genus \( p \) with \( k \) boundary curves. In this case, the Teichmüller space is most easily described by the space of marked conformal structures on the Schottky double with an appropriate involution. Tangent elements then are given by harmonic Beltrami differentials invariant under this involution. Since this involution leaves the boundary of the original surface pointwise fixed, such harmonic Beltrami differentials have to be real on the boundary.

\( \mathcal{T}_{p,k} \) and \( \mathcal{T}_{p,k,-} \) then are differentiable manifolds of dimension \( 6p - 6 + 3k \).

For later purposes, we need to treat a still more general situation, namely that of a compact surface of genus \( p \), orientable or not, with \( k \) boundary curves and \( m \) distinguished points. The corresponding Teichmüller spaces \( \mathcal{T}_{p,k,m} \) are again differentiable manifolds. Their real dimension is
\[ 6p + 3k + 2m - 6 \]
because we get two real coordinates for the position of each distinguished point on the surface. In order to describe the marking, if \( p_1^0, \ldots, p_m^0 \) are the distinguished points on \( \Sigma_0 \) and \( p_1, \ldots, p_m \) those on \( \Sigma \), we require for each diffeomorphism
\[ s : \Sigma_0 \to \Sigma \]
that \( s(p_j^0) = p_j, j = 1, \ldots, m \). In particular, \( s \) yields a diffeomorphism between \( \Sigma_0 \setminus \{p_1^0, \ldots, p_m^0\} \) and \( \Sigma \setminus \{p_1, \ldots, p_m\} \). (\( \Sigma_1, [s_1], (p_1, \ldots, p_m) \)) and (\( \Sigma_2, [s_2], (p_1^0, \ldots, p_m^0) \)) are equivalent if \( s_2 \circ s_1^{-1} \) is homotopic to a conformal map \( k \), via a homotopy fixing all distinguished points. The tangent space of \( \mathcal{T}_{p,k,m} \) at \( \Sigma \setminus \{p_1, \ldots, p_m\} \) now is the space of all \( L^2 \)-Beltrami differentials \( \alpha \) orthogonal to all diffeomorphisms of \( \Sigma \) fixing \( p_1, \ldots, p_m \). Therefore, \( \alpha \) is harmonic on \( \Sigma \setminus \{p_1^0, \ldots, p_m^0\} \). Likewise, \( \varphi(z) = \frac{1}{y^2} \bar{\alpha}(z) \) then is a holomorphic quadratic differential on \( \Sigma \setminus \{p_1^0, \ldots, p_m^0\} \). Since we also require that it is of class \( L^2 \), it may have at most simple poles at the distinguished points. In particular, it becomes meromorphic on \( \Sigma \).

The \( L^2 \)-requirement is justified as follows. For simplicity, we assume \( k = 0 \). Each surface \( \Sigma \setminus \{p_1^0, \ldots, p_m^0\} \) of genus \( p \) with \( 2p + m > 2 \) may be equipped with
a complete hyperbolic metric of finite volume, and conversely each such metric on a noncompact surface corresponds to a compact surface with distinguished points. Therefore, we require that each deformation preserves the finiteness of volume, and this gives the $L^2$-requirement.

The preceding approach to Teichmüller theory is essentially due to Ahlfors and Bers. A global analytic approach is developed in [T3].

2.2 Global aspects and compactifications. We start by discussing Fenchel-Nielsen coordinates on Teichmüller space, again first for the oriented case.

Thus, let $\Sigma_o$ be an oriented Riemann surface of genus $p$. We select a maximal collection of mutually nonhomotopic and disjoint simple closed curves on $\Sigma_o$, denoted $\gamma_1^0, ..., \gamma_{3p-3}^0$, as shown in the figure.

![Diagram of a surface with curves and regions](image)

If $(\Sigma, [s])$ represents an element of $T_p$, we let $\gamma_i$, $i = 1, \ldots, 3p - 3$, be the closed geodesic (w.r.t. the hyperbolic metric) on $\Sigma$ homotopic to $s(\gamma_i^0)$. The curves $\gamma_1, \ldots, \gamma_{3p-3}$ then are again simple and mutually disjoint.

Cutting $\Sigma$ along the curves $\gamma_1, ..., \gamma_{3p-3}$ yields $2p-2$ regions $A_1, ..., A_{2p-2}$ each diffeomorphic to a disc with two interior subdisks removed. They are each equipped with a hyperbolic metric (induced from the one on $\Sigma$) for which all three boundary curves are geodesic. We call any region with these properties a three-circle-domain.

It is an easy lemma in hyperbolic geometry that given any three numbers $\ell_1, \ell_2, \ell_3 > 0$, there exists precisely one conformal class of three-circle-domains with boundary curves $c_1$, $c_2$, $c_3$ of lengths $\ell_1, \ell_2, \ell_3$.

The cutting process may be reversed by assembling a compact Riemann surface of genus $p$ from $2p-2$ three-circle-domains by pairwise identification via arclength of boundary curves, provided identified curves are of the same length. The surface will be orientable if curves are always identified with opposite orientation. The Riemann surface, however, will depend on the way the identification is performed. In order to clarify that dependance, we denote the boundary curves of the three-circle domain $A_\lambda$, $\lambda = 1, \ldots, 2p-2$, by $c_{1\lambda}^1, c_{1\lambda}^2, c_{1\lambda}^3$, and we let $c_{1\lambda}^1$ be the shortest geodesic arc from $c_{1\lambda}^1$ to $c_{1\lambda}^2$. We let $p_1^1, p_2^1, p_3^1$ be the initial points of $c_{12}^1, c_{23}^1, c_{31}^1$, resp. Thus $p_i^1 \in c_{1\lambda}^i$, $i = 1, 2, 3$.

If the curves $c_{1\lambda}^1$ and $c_{1\mu}^1$ are identified, we let $\varphi_{\lambda, \mu}$ be the oriented angle between $p_1^1$ and $p_2^1$. Here, we identify $c_{1\lambda}^1$, and hence also $c_{1\mu}^1$, with the unit circle.
$S^1$, parameterized by $[0, 2\pi]$, proportionally to arclength. The orientation is given by the orientation of $C^1$ coming from the one on $A^p$.

Identifications with different angles $\varphi_{\lambda \mu}$ lead to conformally different Riemann surfaces.

So far, we have not taken the marking into account. For that, we start with the following observation: If we cut $\Sigma$ along any of the curves $\gamma_j$, $j \in \{1, \ldots, 3p - 3\}$, rotate one copy of the cut curve by an angle $2\pi m$, $m \in \mathbb{Z}$, and reglue, we obtain a diffeomorphism $s_m : \Sigma \to \Sigma$, a so called Dehn twist. For $m \neq 0$, $s_m$ is not homotopic to the identity. Consequently, in order to describe a marked surface through a gluing pattern, we have to count the angles $\varphi_{\lambda \mu}$ in $\mathbb{R}$ and not only $\mod 2\pi$. Fixing the zero point for $\varphi_{\lambda \mu}$ then depends on a topological choice.

In this manner, we obtain a map

$$F : \mathcal{T}_p \to (\mathbb{R}^+)^{3p-3} \times \mathbb{R}^{3p-3}$$

mapping $(\Sigma, [s])$ onto the hyperbolic lengths $\ell_1, \ldots, \ell_{3p-3}$ of the curves $\gamma_1, \ldots, \gamma_{3p-3}$ and the corresponding gluing parameters $\varphi_1, \ldots, \varphi_{3p-3}$.

**Definition 2.2.1.** The parameters $(\ell_1, \ldots, \ell_{3p-3}, \varphi_1, \ldots, \varphi_{3p-3}) = F(\Sigma, [s])$ are called Fenchel-Nielsen coordinates of $(\Sigma, [s])$.

**Theorem 2.2.1.** $F : \mathcal{T}_p \to (\mathbb{R}^+)^{3p-3} \times \mathbb{R}^{3p-3}$ is a diffeomorphism.

The main point of the proof is to show injectivity of $F$. This is not hard; it requires a careful examination of the topological aspects of the cutting and gluing processes. Surjectivity is obvious and differentiability likewise is not difficult. Actually, from the preceding discussion one may deduce local injectivity, and global injectivity then follows because $\mathbb{R}^{6p-6}$ is simply connected.

**Corollary 2.2.1.** $\mathcal{T}_p$ is diffeomorphic to $\mathbb{R}^{6p-6}$.

Let us now describe the same process for nonorientable compact Riemann surfaces without boundary. If $\Sigma$ is such a surface, of genus $p$, we again want to cut it into subdomains. If $p \in \mathbb{N}$, then $\Sigma$ consists of $p$ handles of which $p - 1$ can be chosen to be orientable. In this case, we may again cut $\Sigma$ into $2p$ 3-circle-domains, choosing one of the cut curves, say $\gamma_1$, to be an orientable core curve of the nonorientable handle. Everything then is the same as before, except that when reassembling the surface from the 3-circle-domains the two boundary curves corresponding to $\gamma_0$ have to be identified with the same orientation, and not with opposite ones.

If $p = q + \frac{1}{2}$, with $q \in \mathbb{N}$, then $\Sigma$ consists of $q$ handles and one Möbius strip. We then cut $\Sigma$ along $3q-1$ mutually disjoint simple closed hyperbolic geodesics, $3q-2$ orientable and the remaining one not, into $2q-1$ 3-circle-domains; the nonorientable curve just opens a Möbius strip along the core. Consequently, for that curve, we do not get a gluing parameter. Therefore, we only get $3q-2$ gluing parameters. Therefore, we now get a map

$$F : \mathcal{T}_{p-1} \to (\mathbb{R}^+)^{3p-5/2} \times \mathbb{R}^{3p-7/2}$$

which again is diffeomorphic to $\mathbb{R}^{6p-6}$.
If $\Sigma$ is a compact oriented Riemann surface with $k$ boundary curves, of genus $p$, again, we choose our collection of curves to include the boundary curves. We then get $3p - 3 + 2k$ curves, and cutting of course is necessary and possible only along the $3p - 3 + k$ of them which are not boundary curves. We then get a map

$$ F : \mathcal{T}_{p,k} \rightarrow (\mathbb{R}^+)^{3p-3+2k} \times \mathbb{R}^{3p-3+k} $$

a space which is diffeomorphic to $\mathbb{R}^{6p-6+3k}$.

We want to discuss what happens if one of the length parameters, say $\ell_1$, tends to 0.

The collar lemma (Lemma 1.2.5) implies that in this case the corresponding surfaces degenerate in the sense that their diameter becomes unbounded. In the limit, we obtain a complete, noncompact hyperbolic surface with one or two punctures. We get two of them if there exists a gluing parameter corresponding to $\ell_1$, one otherwise. This surface is of lower topological type as it either has smaller genus, or it has more than one component, or one boundary curve less, and the same genus in the last two cases. It is conformally equivalent to a compact surface with one or two points removed. Therefore, the limit surface represents an element of some Teichmüller space $\mathcal{T}_{q,k,m}(\text{or } \mathcal{T}_{q,k,m,\ldots})$, $m = 1, 2$, or of a product of two such spaces. Actually, for elements in $\mathcal{T}_{p,k}$, the ordering of the distinguished points was relevant by our above definition. We obtain a natural ordering of the punctures on our limiting surfaces by numbering the boundary curves of the 3-circle-domains into which the genus $p$ surfaces were dissected.

A possible gluing parameter $\varphi_1$ becomes undetermined in the limit. Therefore, if we put $\ell_1 = 0$ for the limit surface, we have the same picture as when compactifying the punctured unit disk $D \setminus \{0\}$ represented by polar coordinates $(r, \varphi)$, $0 < r < 1$, $0 \leq \varphi \leq 2\pi$, by adding the origin corresponding to $r = 0$ and undetermined $\varphi$.

We therefore obtain a partial compactification $\overline{\mathcal{T}}_p$ of Teichmüller space by allowing one or several of the lengths parameters to become 0. This partial compactification depends on the choice of homotopy classes of curves defining the Fenchel-Nielsen-coordinates. A different choice leads to a different partial compactification. We may therefore add the boundary components corresponding to all possible choices. If we project to moduli space, then we obtain a compact space by Mumford’s compactness lemma (Lemma 1.2.4). Teichmüller space itself, however, cannot be compactified in this manner. This may also be explained as follows. Moduli space $\mathcal{M}_p$ is obtained by dividing the space of all conformal structures on a given surface $S$ by the diffeomorphism group $\mathcal{D}(S)$, the group of diffeomorphisms homotopic to the identity. Therefore,

$$ \mathcal{M}_p = \overline{\mathcal{T}}_p / G_p $$

where $G_p := \mathcal{D}(S)/\mathcal{D}_0(S)$ is the so-called mapping class group. It is the group of homotopy classes of diffeomorphisms of a surface of genus $p$. It is an infinite discrete group, generated for example by Dehn twists about certain curves on $S$. The preimage in $\overline{\mathcal{T}}_p$ of each point in $\mathcal{M}_p$ therefore is an infinite discrete set bijective to $G_p$. All these preimage points are conformally equivalent (but the conformal maps are not homotopic to the identity), and therefore also
the corresponding hyperbolic metrics are isometric, and hence the lengths of
godesics are the same. Thus, no analogous compactness theorem can hold at
the Teichmüller space level.

Once allowing a length parameter, say $\ell_1$ to become $O$, we may go even
further, at least if there exists a corresponding gluing parameter $\varphi_1$, by also
allowing $\ell_1$ to become negative. We define this to correspond to identifying the
same two boundary curves of length $\ell_1$ as before, but not with different ori-
tenation as before. In this manner, we may for example convert an orientable
handle into a nonorientable one, and vice versa. We are thus connecting the
Teichmüller spaces $T_p$ and $T_{p-}$ along a common boundary component. Since
we let the gluing parameter $\varphi_1$ become undetermined for $\ell_1 = 0$, this bound-
ary component has codimension 2 and therefore the resulting space acquires a
singularity along this component.

Performing these connections for all length parameters for all possible choices
of curves yielding Fenchel-Nielsen coordinates, we obtain a kind of enlarged
Teichmüller space $T_p$ including both orientable and nonorientable surfaces of
genus $p$. This space can be constructed as follows:

Given a marked surface with Fenchel-Nielsen coordinates

$$(\ell_1, \ldots, \ell_{3p-3}, \varphi_1, \ldots, \varphi_{3p-3}) \in \mathbb{R}^{6p-66},$$

also the lengths and twist parameters for all other simple closed geodesics are
determined. When one of the length parameters, say $\ell_1$, changes from positive
to negative values, the corresponding twist parameter $\varphi_1$ is only determined
mod $2\pi$ across $\ell_1 = 0$, since we can identify curves intersecting the geodesic $\gamma_1$
corresponding to $\ell_1$ for $\ell_1 > 0$ and $\ell_1 < 0$ only up to Dehn twists about $\gamma_1$.
Therefore our coordinate description across $\ell_1 = 0$ involves a topological choice.
We also say that $\gamma_1$ changes its sign from $+1$ to $-1$ if $\ell_1$ changes from positive to
negative values. Finally, each other simple closed geodesic $\gamma$ with intersection
number with $\gamma_1$, $i(\gamma, \gamma_1) = 1 \mod 2$, changes its orientability character when $\ell_1$
changes its sign. This does not occur if $i(\gamma, \gamma_1) = 0 \mod 2$.

In order to describe the enlarged Teichmüller space $T_p$, we start with any par-
tition of our base surface $\Sigma_0$ into 3-circle-domains and corresponding Fenchel-
Nielsen coordinates $(\ell_1, \ldots, \ell_{3p-3}, \varphi_1, \ldots, \varphi_{3p-3})$ and then allow the length pa-
rameters to assume also negative values. Each coordinate tuple determines a
Riemann surface of genus $p$, possibly degenerate (if some of the $\ell_j$ vanish). It
Carries a compact or complete hyperbolic metric, and the lengths and twist pa-
rameters for all simple closed geodesics are then determined, not only of those
involved in the chosen decomposition, as well as their orientability character
and sign (if $\ell_1$, say, changes from positive to negative values this does not affect
the sign of any geodesic other than $\gamma_1$).

If we start with a different 3-circle-decomposition, we obtain different Fenchel-
Nielsen coordinates $(\ell'_1, \ldots, \ell'_{3p-3}, \varphi'_1, \ldots, \varphi'_{3p-3})$. We identify

$$(\ell_1, \ldots, \ell_{3p-3}, \varphi_1, \ldots, \varphi_{3p-3}) \text{ and } (\ell'_1, \ldots, \ell'_{3p-3}, \varphi'_1, \ldots, \varphi'_{3p-3})$$

if the lengths, twist parameters, signs and orientability character of all simple
closed geodesics on the represented surfaces coincide.

If we also want to include the pinching of nonorientable geodesics as boundary
components, we need a different type of coordinates. A nonorientable handle
is topologically a Klein bottle, and it therefore is a sum of two Möbius strips, and it therefore contains two non homotopic and non intersecting simple closed geodesics. Their lengths give us two parameters $\lambda_1$, $\lambda_2$. The both have positive values, and we may allow them to become zero to describe the pinching of these geodesics. Topologically, this corresponds to replacing a Möbius strip by a disk by opening the strip along its core and gluing in a disk. Since nonorientable geodesics do not carry a twist parameter, there is no meaningful way of letting their length parameters become negative. We may change our coordinates by replacing the coordinates for the orientable geodesic, say $\gamma_1$, in our nonorientable handle, the length $\ell_1$ and the twist $\varphi_1$, by $\lambda_1$ and $\lambda_2$. This may be done for any nonorientable handle, and this procedure gives the additional coordinate systems. As before, we identify coordinate descriptions of surfaces with the same geometric data.

In this manner, we obtain our enlarged Teichmüller space $\tilde{T}_p$ containing all marked orientable and nonorientable surfaces of genus $\leq p$, with additional punctures if the genus is less than $p$.

We may also divide out the action of the individual mapping class groups and identify all conformally equivalent surfaces to obtain an enlarge moduli space $\mathcal{M}_p'$ consisting of orientable and nonorientable Riemann surfaces of genus $p$ and ones of lower topological types with punctures.

We now want to study the behaviour of the Weil-Petersson metric near the boundary components $\{ \ell_j = 0 \}$. The following estimates are due to H. Masur [Ma].

For these estimates, coordinates different form the Fenchel-Nielsen coordinates will be convenient. In order to describe these, let $\Sigma_0$ be a Riemann surface with two punctures $a, b$. Let $U, V$ be disjoint neighborhoods of $a$ and $b$, resp, with local coordinates $z : U \to D$, $w : V \to D$ and $z(a) = 0$, $w(b) = 0$.

For $s \in \mathbb{C}$, $|s| < 1$, we then form a Riemann surface $\Sigma_s$ by identifying $z$ with $s/w$ and removing the preimages of $\{|z| \leq |s|^{1/2}\}$ and $\{|w| \leq |s|^{1/2}\}$ from $\Sigma_0$.

In order to compute the Beltrami differential associated to a variation of $s$, we look at the family of diffeomorphisms ($t \in \mathbb{C}$)

$$\sigma_t : \Sigma_0 \to \Sigma_{se^{2t}}$$

with

$$\sigma_t(z) = z \exp \left( t \log \frac{|z|}{\log \rho} \right) \text{ with } \rho = |s|^{1/2}$$

$$\sigma_t(w) = w \exp \left( \frac{t \log |w|}{\log \rho} \right) \text{ for } |s|^{1/2} \leq |z|, |w| \leq 1 \text{ and } \sigma_t(p) = p \text{ for other } p \in \Sigma_s.$$ 

Then

$$\beta_t(z)_{|t=0} = z \frac{\log |z|}{\log \rho}$$

and with $\alpha(z) = (\frac{\partial}{\partial z} \sigma_0(z))$

$$\alpha(z) = \frac{1}{2 \log \rho} \frac{z}{\bar{z}}$$

Since

$$\frac{\partial}{\partial t}(se^{2t})_{|t=0} = 2s$$
we conclude that the variation of $\Sigma_s$ is represented by the Beltrami differential

\[
\frac{1}{2s \log |s|} \frac{z \, dz}{\bar{z} \, dz} \quad \text{on} \quad |s|^{1/2} \leq |z| \leq 1
\]

\[
\frac{1}{2s \log |s|} \frac{w \, dw}{\bar{w} \, dw} \quad \text{on} \quad |s|^{1/2} \leq |w| \leq 1
\]

and 0 on the rest of $\Sigma_s$.

We recall that $s = 0$ corresponds to the surface $\Sigma_o$ with punctures $a$ and $b$. We identify the punctures and call the result a node.

If more than one of the Fenchel-Nielsen length parameters vanishes, we get a surface with more than one node. We describe the situation by complex parameters $s_1, \ldots, s_{3p-3}$, and we denote the corresponding Riemann surface by $\Sigma(s_1, \ldots, s_{3p-3})$.

Let

\[
\Sigma_o = \Sigma(0, \ldots, 0, s_0, \ldots, s_{3p-3}, s_{m+1}, \ldots, s_{3p-3} \neq 0)
\]

be a surface with $m$ nodes.

Masur then constructs meromorphic quadratic differentials on $\Sigma(s_1, \ldots, s_{3p-3})$, $\varphi_1(\cdot, s), \ldots, \varphi_{3p-3}(\cdot, s)$, satisfying

(i) for $j \leq m$,

\[
\varphi_j(z_k, s) = \frac{1}{z_k^2}(\delta_j + \alpha_j(z_k, s) + \beta_j(z_k, s))
\]

where $z_k$ is a coordinate corresponding to the $k$-th node (identifying punctures $a_k, b_k$) as above, and $\alpha_j(z_k, s)$ and $\beta_j(1/z_k, s)$ are holomorphic in $z_k$ and $s$ and vanish for $s = 0$.

(ii) for $m < j \leq 3p - 3$, $\varphi_j(\cdot, 0)$ is holomorphic with at most simple poles at the nodes and

\[
\varphi_j(z_k, s) = \frac{1}{z_k^2}(\alpha_j(z_k, s) + \beta_j(z_k, s))
\]

where $\alpha_j$ and $\beta_j$ have the same properties as in (i).

The $\varphi_j$, $j = m + 1, \ldots, 3p-3$, then are cotangent vectors for the Teichmüller space of $\Sigma_o$, i.e. marked surfaces diffeomorphic to $\Sigma_o$ and with $2m$ punctures.

The $\varphi_j$, $j = 1, \ldots, m$, are supposed to be tangent to directions in $\mathcal{T}_p$ transversal to the Teichmüller space of $\Sigma_o$.

We compute for the pairing between Beltrami differentials and quadratic differentials

\[
\left(\frac{\partial}{\partial s_j}, \varphi_k\right) = \int_{|s_j|^{1/2} \leq |z| \leq 1} \frac{1}{2s_j \log |s_j|} \frac{z \, \varphi_k(x) \, dz}{\bar{z}} \wedge \bar{dz}
\]

\[
= -\frac{\pi}{s_j} \delta_{jk} \quad \text{for} \quad j = 1, \ldots, m; \quad k = 1, \ldots, 3p - 3
\]

We put

\[
\psi_j := -\frac{s_j}{\pi} \varphi_j \quad \text{for} \quad j = 1, \ldots, m
\]
Then
\[
\left(\left(\frac{\partial}{\partial s_j}, \psi_k\right)\right) = \delta_{jk} \text{ for } j = 1, \ldots, m; k = 1, \ldots, m
\]

Applying a change of base holomorphic in \( s_1, \ldots, s_{3p-3} \), we may also convert the \( \varphi_j, j = m+1, \ldots, 3p-3 \) into quadratic differentials \( \psi_j \) with the properties of (ii) so that
\[
\left(\left(\frac{\partial}{\partial s_j}, \psi_k\right)\right) = \delta_{jk} \text{ for } j, k = 1, \ldots, 3p-3
\]

Therefore, the meromorphic quadratic differentials \( \psi_k \) are dual to the variation \( \frac{\partial}{\partial s_j} \), and computing the Weil-Petersson metric w. r. t. the coordinates \( s_j \) then amounts to evaluating the integrals
\[
\int_{\Sigma_s} \psi_j \bar{\psi}_k \frac{1}{\lambda^2} \frac{i}{2} dz \wedge d\bar{z}
\]
where \( \frac{1}{\lambda^2} dz \otimes d\bar{z} \) is the hyperbolic metric on \( \Sigma_s(s = (s_1, \ldots, s_{3p-3})) \).

The hyperbolic metric on the annuli \( \{ |s_j|^{1/2} \leq |z_j| \leq 1 \} \) is controlled by the Poincaré metric on the punctured unit disk:

For any \( \epsilon > 0 \), there exist constants \( c_1 \) and \( c_2 \) with the property that for all \( s_j \neq 0 \), \( |s_j|^{1/2} \leq |z_j| \leq 1 - \epsilon \)
\[
c_1 |z_j|^{-2} (\log |z_j|)^{-2} \leq \lambda^2 \frac{2}{s_j}(z_j) \leq c_2 |z_j|^{-2} (\log |z_j|)^{-2}
\]
where \( \lambda^2 dz_j \otimes d\bar{z}_j \) is the hyperbolic metric on \( \{ |s_j|^{1/2} \leq |z_j| \leq 1 \} \)

We put \( A_j := \{ |s_j|^{1/2} \leq |z_j| \leq 1 \} \)
Then for \( j = 1, \ldots, m \)
\[
\int_{A_j} \psi_j \bar{\psi}_j \frac{1}{\lambda^2} \frac{i}{2} dz \wedge d\bar{z} \sim |s_j|^2 \int_{r=|s_j|^{1/2}} \frac{(\log r)^2}{r} dr \sim |s_j|^2 |\log |s_j||^3
\]
and for any fixed compact \( K \subset \Sigma_s \) ("fixed" means independent of \( s \))
\[
\int_{K} \psi_j \bar{\psi}_j \frac{1}{\lambda^2} \frac{i}{2} dz \wedge d\bar{z} \sim |s_j|^2
\]
and
\[
\int_{\Sigma_s} \psi_j \bar{\psi}_j \frac{1}{\lambda^2} \frac{i}{2} dz \wedge d\bar{z} \sim |s_j||s_k| \text{ for } j \neq k, k = 1, \ldots, m
\]
and
\[
\int_{\Sigma_s} \psi_j \bar{\psi}_k \frac{1}{\lambda^2} \frac{i}{2} dz \wedge d\bar{z} \sim |s_j| \text{ for } k = m + 1, \ldots, 3p-3
\]
Finally
\[
\lim_{s \to s_0} \int_{\Sigma_s} \psi_k \bar{\psi}_\ell \frac{1}{\lambda^2} \frac{i}{2} dz \wedge d\bar{z} = \int_{\Sigma_s} \psi_k \bar{\psi}_\ell \frac{1}{\lambda^2} \frac{i}{2} dz \wedge d\bar{z} \text{ for } k, \ell > m
\]
Consequently
\[
\det \left( \int_{\Sigma_s} \psi \bar{\psi}_k \frac{1}{\lambda^2} \frac{i}{2} dz \wedge d\bar{z} \right)_{k,t=1,\ldots,3p-3} \sim \prod_{j=1}^{p} |s_j|^2 |\log |s_j||^3.
\]

Since the metric tensor for the Weil-Petersson metric is the inverse of the tensor \( \left( \int_{\Sigma_s} \psi \bar{\psi}_k \frac{1}{\lambda^2} \frac{i}{2} dz \wedge d\bar{z} \right)_{k,t=1,\ldots,3p-3} \) (as the latter represents the metric on the cotangent space) we get Masur's theorem ([Ma]):

**Theorem 2.2.2.** The asymptotic behaviour of the Weil-Petersson metric near a surface \( \Sigma_{(0,0,s_{m+1}^0,\ldots,s_{3p-3}^0)} \) with \( m \) nodes is given by

\[
g_{ij}(s) = \left( \frac{\partial}{\partial s_i}, \frac{\partial}{\partial s_j} \right), (s_1, \ldots, s_{3p-3}) \rightarrow (0, \ldots, 0, s_{m+1}, s_{3p-3})
\]

For \( j = 1, \ldots, m \)
\[
g_{jj}(s) \sim \frac{1}{|s_j|^2 |\log |s_j||^3}
\]

For \( j, k = 1, \ldots, m, j \neq k \)
\[
g_{jk}(s) \sim \frac{1}{|s_j||s_k||\log |s_j||^3 |\log |s_k||^3}
\]

For \( j = 1, \ldots, m, k = m+1, \ldots, 3p-3 \)
\[
g_{jk}(s) \sim \frac{1}{|s_j||\log |s_j||^3}
\]

For \( k, \ell = m+1, \ldots, 3p-3 \)
\[
g_{k\ell}(s) \rightarrow g_{k\ell}^N(s_{m+1}, \ldots, s_{3p-3})
\]

where \( g_{k\ell}^N \) is the metric on the Teichmüller space for surfaces of the type of \( \Sigma_{(0,0,s_{m+1}^0,\ldots,s_{3p-3}^0)} \), i.e. with \( m \) nodes.

### 3 Teichmüller theory and minimal surfaces

#### 3.1 Dirichlet’s integral as a function on Teichmüller space

In this chapter, we let \( N \) be a complete Riemannian manifold of nonpositive curvature, and we let
\[
\gamma = (\gamma_1, \ldots, \gamma_m)
\]
be a collection of smooth disjoint closed Jordan curves in \( N \).

We fix some genus \( p \) and let \( \mathcal{T}_{p,m} \) be the enlarged Teichmüller’s space of surfaces of genus \( \leq p \) with \( m \) boundary curves. We want to study the set of minimal surfaces of genus \( \leq p \) with boundary \( \gamma \) in \( N \).

Let \((\Sigma, [h]) \in \mathcal{T}_{p,m}\), and let \( c_1, \ldots, c_m \) be the boundary curves of \( \Sigma \). As in the construction of Fenchel-Nielsen coordinates we may select points \( p_i^* \in c_i, i = 1, \ldots, m \), depending continuously on \((\Sigma, [h]) \) (In order to make this continuity...
requirement precise, we may identify \((\Sigma_1, [h_1])\) and \((\Sigma_2, [h_2])\) via the unique harmonic diffeomorphism \(h : \Sigma_1 \to \Sigma_2\) homotopic to \(h_2 \circ h_1^{-1}\). We may then identify each boundary curve \(c_i\) with \(S^1\) proportionally to arclength with \(p_e^i\) corresponding to \(1 \in S^1\) and orienting \(c_i\) so that \(\Sigma\) always lies to the left. This identification then depends continuously on \((\Sigma, [h])\). Thus, \(\partial \Sigma\) may always be identified with \((S^1)^m\); we denote the corresponding diffeomorphisms by \(\beta_i(\Sigma) : c_i \to S^1, i = 1, \ldots, m\). Let \(s_i : S^1 \to \gamma_i\) be a smooth diffeomorphism, \(i = 1, \ldots, m\). Let

\[
R := \{\eta \in H^{1/2,2} \cap C^0(S^1, S^1) : \eta \text{ monotone}\}
\]

We want to study the functional

\[
E : R^m \times \hat{T}_{p,m} \to \mathbb{R}
\]

mapping \((\eta, \Sigma)\) onto

\[
E(\eta, \Sigma) = D(h_{\eta,\Sigma}, \Sigma) = 1/2 \int_{\Sigma} \|dh_{\eta,\Sigma}\|^2
\]

where \(h_{\eta,\Sigma} : \Sigma \to N\) is the unique harmonic mapping with boundary values

\[
h_{\eta,\Sigma}|_{c_i} = s_i \circ \eta_i \circ \beta_i(\Sigma)
\]

**Lemma 3.1.1.** \(E\) is differentiable on \(R^m \times \hat{T}_{p,m}\), and for \(z \in (H^{1/2,2} \cap C^0(\mathbb{R}, \mathbb{R}))^m\), with \(z(x) = \zeta(x + 2\pi)\) for all \(x \in \mathbb{R}\)

\[
(3.1.1) \quad \left\langle \frac{\partial}{\partial \eta_i} E(\eta, \Sigma), \zeta \right\rangle_{H^{1/2}} = \int_{c_i} \frac{\partial h}{\partial \eta} \cdot \frac{\partial s_i}{\partial \zeta} \circ \zeta, \quad (h = h_{\eta,\Sigma})
\]

identifying \(c_i\) and \(S^1\) via \(\beta_i\).

For \(\alpha \in \mathcal{H}(\Sigma)\), we have

\[
(3.1.2) \quad \frac{\partial}{\partial \Sigma} E(\eta, \Sigma)(\alpha) = -2Re \int_{\Sigma} \varphi^i \frac{1}{2} dz \wedge d\bar{z},
\]

with \(\varphi(z)dz^2 = \langle h_z, h_\Sigma \rangle dz^2 (h = h_{\eta,\Sigma})\) (see [T1]).

Consequently, the derivative of \(E\) w.r.t. variations of the boundary values vanishes if the holomorphic quadratic differential \(\langle h_z, h_\Sigma \rangle dz^2\) is real on \(\partial \Sigma\), and it is critical w.r.t. variations of the conformal structure of \(\Sigma\) if it is orthogonal to all holomorphic quadratic differentials real on \(\partial \Sigma\).

Therefore, \((\eta, \Sigma)\) yield a critical point of \(E\) if \(h_{\eta,\Sigma}\) is a minimal surface with boundary \(\gamma\). We note here that a suitable extension of Hildebrandt's boundary regularity theorem (see [J3]) implies that \(h\) is smooth if

\[
\langle h_z, h_\Sigma \rangle dz^2 \text{ is real on } \partial \Sigma.
\]

We now want to study the extension of \(E\) to \(\hat{T}_{p,m}\), i.e. to surfaces of lower topological type.

In order to achieve this, we estimate the derivatives of \(E(\eta, \Sigma)\) w.r.t. \(\Sigma\) near the boundary of \(\hat{T}_{p,m}\).
We use the coordinates of 2.2, and define again

$$\sigma_t : \Sigma_s \to \Sigma_{se^{2i}}$$

via

$$\sigma_t(z) = z \exp \left( t \frac{\log|z| - \log \rho}{\log \rho} \right) \text{ for } \rho \leq |z| \leq \rho^{1/2} \text{ with } \rho = |s|^{1/2}$$

$$\sigma_t(z) = z \text{ for } \rho^{1/2} \leq |z| \leq 1$$

and the corresponding formulae for $$\sigma_t(w)$$.

Then again

$$\frac{\partial}{\partial s} E(\eta, \Sigma_s) = \frac{1}{s \log|s|} \left( \int_{|s|^{1/2} \leq |z| \leq |s|^{1/4}} \langle h_z, h_z \rangle \frac{z}{\bar{z}} \frac{i}{2} dz \wedge d\bar{z} \right)$$

$$+ \int_{|s|^{1/2} \leq |w| \leq |s|^{1/4}} \langle h_w, h_w \rangle \frac{w}{\bar{w}} \frac{i}{2} dw \wedge d\bar{w}$$

We now need the following

**Lemma 3.1.2.** In polar coordinates $$(r, \varphi)$$ on $$\{|s|^{1/2} \leq |z| \leq 1\} =: A_{|s|^{1/2}}$$ if $h$ induces the trivial map on $$\pi_1(A_{|s|^{1/2}})$$

$$\|dh\|^2 (re^{i\varphi}) = \|hr\|^2 + \frac{1}{r^2} \|h\varphi\|^2 \leq \frac{c_1}{r^2 (\log r)^2}$$

for a constant $$c_1$$ depending on $$\sup_{z \in \partial \Sigma} d^2(\eta(z), p)$$ (for any $$p \in N$$).

We let

$$H_{|s|^{1/2}} = \{|s|^{1/2} \leq |z| \leq 1\} \cup \{|s|^{1/2} \leq |w| \leq 1\},$$

with identification $$z = s/w$$ as before.

We apply the conformal transformation

$$x = \log r, \theta = \varphi \text{ on } \{|s|^{1/2} \leq |z| \leq 1\}$$

$$x = \log |s| - \log r, \theta = \varphi_0 - \varphi, \text{ where } s = e^{i\varphi_0} |s| \text{ on } \{|s|^{1/2} \leq |w| \leq 1\}$$

mapping $$H_{|s|^{1/2}}$$ onto a cylinder $$C_{2x_0}$$ of height $$-2x_0 (x_0 = \frac{1}{2} \log |s|)$$

The universal cover of $$C_{2x_0}$$ is

$$\tilde{C}_{2x_0} = \{(x, y) \in \mathbb{R}^2, 2x_0 \leq x \leq 0\}$$

and from our topological assumption $$h$$ lifts to a map

$$\tilde{h} : \tilde{C}_{2x_0} \to \tilde{N}$$
with bounded image. Since \( N \) has nonpositive curvature, \( d^2(\hat{h}(\cdot), p) \) is subharmonic for any \( p \in \hat{N} \). Therefore, the maximum principle implies,

\[
d^2(\hat{h}(z), p) \leq \max_{w \in \partial \Sigma} d^2(\hat{h}(w), p) =: d_0 \quad \text{for all} \quad z \in \hat{C}_{2z_o}
\]

Since \( h \) is harmonic, we also have the gradient estimate (cf. [J2])

\[
\|d\hat{h}(z)\|^2 \leq \frac{c_2}{R^2} \max_{w \in B(z, R)} d^2(\hat{h}(w), p) \leq \frac{c_2}{R^2} d_0
\]

provided \( B(z, R) \subset \hat{C}_{2z_o} \).

Since

\[
\|h_r\|^2 + \frac{1}{r^2} \|h_\varphi\|^2 = \frac{1}{r^2} (\|h_z\|^2 + \|h_\eta\|^2) = \frac{1}{r^2} \|d\hat{h}\|^2
\]

and since we may choose

\[
R = |x| = |\log r| \quad \text{for} \quad x_0 = r \leq 0,
\]

(3.1.4) follows from (3.1.5).

**Lemma 3.1.3.** Let \( s \) correspond to shrinking an interior closed geodesic or a boundary curve of \( \Sigma_s \). If \( h \) maps \( \{z = s\} \) (in our above coordinates) onto a homotopically trivial curve, we have the estimate

\[
\frac{\partial}{\partial s} E(\eta, \Sigma_s) \leq \frac{c_3}{s \log |s|^2}
\]

where \( c_3 \) depends on \( \sup_{z \in \partial \Sigma} d^2(\eta(z), p) \) (for any \( p \in N \)).

**Proof.** We first treat the shrinking of an interior closed geodesic. We have to control

\[
\int_{|s| \leq |z| \leq 1} \|dh\|^2 \frac{z \bar{z}}{2} dz \wedge d\bar{z} \leq 4\pi c_1 \int_{|s|^{1/2} \leq r \leq |s|^{1/4}} \frac{1}{r(\log r)^2} dr = \frac{8\pi c_1}{|\log s|}
\]

by Lemma 3.1.2.

Inserting this estimate into (3.1.3) proves (3.1.6). \( \square \)

For a boundary curve, we may essentially apply the same reasoning. The only modification is that we now use the variation

\[
\sigma_t : \Sigma_s \rightarrow \Sigma_{se^{2t}}
\]

defined by

\[
\sigma_t(z) = \begin{cases} 
  ze^{2t} & \text{for} \quad \rho \leq |z| \leq \rho^\frac{1}{4} \\
  z \exp \left( t \frac{\log |z|-\log \rho}{\log \rho} \right) & \text{for} \quad \rho^\frac{1}{4} \leq |z| \leq \rho^\frac{1}{2} \\
  z & \text{for} \quad \rho^\frac{1}{2} \leq |z| \leq 1
\end{cases}
\]
because the curve \(|z| = \rho\) now is a boundary curve. The previous estimates then apply with obvious minor modifications. In fact, this variation could have been applied to the case of an interior geodesic as well.

**Remark.** If \(N\) is Euclidean space, a similar estimate was proved in [S3] by a different method, for the case of an interior geodesic only.

**Corollary 3.1.1.**

\[
\lim_{s \to 0} \frac{1}{\left\| \frac{\partial}{\partial s} \right\|} \left| \frac{\partial}{\partial s} E(\eta, \Sigma_s) \right| = 0
\]

where the norm \(\| \cdot \|\) refers to the Weil-Petersson metric.

**Proof.** By Theorem 2.2.2,

\[
\frac{1}{\left\| \frac{\partial}{\partial s} \right\|} \leq c_4(|s|^2|\log |s||^3)^{\frac{1}{2}}
\]

(3.1.6) and (3.1.9) imply

\[
\frac{1}{\left\| \frac{\partial}{\partial s} \right\|} \left| \frac{\partial}{\partial s} E(\eta, \Sigma_s) \right| \leq \frac{c_5}{|\log |s||^{1/2}}
\]

giving (3.1.8).

In a similar way, one sees

**Lemma 3.1.4.** For \(s_0 \neq 0\)

\[
\lim_{s \to s_0} \frac{\partial}{\partial s} E(\eta, \Sigma_s) = \frac{\partial}{\partial s} E(\eta, \Sigma_{s_0})
\]

uniformly in \(T_{p,m}\) and \(\eta\) with \(\|\eta\|_{H^{1/2}} \leq \text{const.}\)

**Theorem 3.1.1.** Let \(\eta_0 \in R^m\) be of class \(C^{1,\alpha}\) (\(0 < \alpha < 1\)), \(\Sigma_0 \in \partial T_{p,m} \subset \hat{T}_{p,m}\). Then \(\text{grad} E\) extends continuously to \((\eta_0, \Sigma_0) \in R^m \times \hat{T}_{p,m}\), and the extension is tangential to the boundary stratum containing \((\eta_0, \Sigma_0)\).

**Proof.** Let \((\Sigma_n)_{n\in\mathbb{N}} \subset T_{p,m}\) converge to \(\Sigma_0 \in \partial T_{p,m}\) and let \((\eta_n)_{n\in\mathbb{N}} \subset R^m\) converge in \(H^{1/2}\) to \(\eta_0 \in R^m\). We get harmonic maps \(h_n = h(\eta_n, \Sigma_n)\) with associated holomorphic quadratic differentials \(\varphi_n\) on \(\Sigma_n\).

Let \(\varepsilon > 0\).

Let \(U_\varepsilon\) be an \(\varepsilon\)-neighborhood of the punctures on \(\Sigma_0\), determined with the help of the compact hyperbolic metric \(\lambda^2(z)dzd\bar{z}\) on \(\Sigma_0\). For large enough \(n\) (depending on \(\varepsilon\)), \(\Sigma_0 \setminus U_\varepsilon\) can be considered as a subset of \(\Sigma_n\), and since the boundary values \(\eta_n\) converge, also \(h_n|\Sigma_0 \setminus U_\varepsilon\) converges in \(H^1\) to a harmonic map \(h_0\) for every \(\varepsilon > 0\). Since \(h_0\) has bounded image, it extends as a harmonic map to all of \(\Sigma_0\). Let \(\varphi_0\) be the associated holomorphic quadratic differential.
Since $\eta_0 \in C^{1,\alpha}$, so is $h_0$. Hence $\varphi_0 \in C^{\alpha}$, and $\varphi_0$ is in particular bounded. Consequently, for every meromorphic quadratic differential $\psi_0$ on $\Sigma_0$ with poles of at most second order, and real on $\partial \Sigma_0$,

$$\int_{\Sigma_0} \varphi_0 \overline{\psi_0} \frac{1}{\lambda^2(z)} \frac{i}{2} dz \wedge d\overline{z}$$

is finite.

If $\psi_0$ represents the opening up of an interior node, then by Corollary 3.1.1,

(3.1.11) $$\int_{\Sigma_0} \varphi_0 \overline{\psi_0} \frac{1}{\lambda^2(z)} \frac{i}{2} dz \wedge d\overline{z} = 0$$

If $\psi_0$ represents the opening up of a node at the boundary, we approximate this node by interior nodes, and in the limit we obtain from the formula (3.1.10) for interior nodes again

(3.1.12) $$\int_{\Sigma_0} \varphi_0 \overline{\psi_0} \frac{1}{\lambda^2(z)} \frac{i}{2} dz \wedge d\overline{z} = 0.$$

We represent the opening of a boundary node again by $\frac{\partial}{\partial \nu}$, and we then get corresponding holomorphic quadratic differentials $\psi_n$ on $\Sigma_n$, representing $\frac{\partial}{\partial \nu}$, converging to $\psi_0$ away from the node.

As observed above, we may write

$$\Sigma_n = \Sigma_0 \setminus U_\varepsilon \cup \Omega_{\varepsilon,n}.$$ 

Then

(3.1.13) $$\int_{\Sigma_n} \varphi_n \overline{\psi_n} = \int_{\Sigma_0 \setminus U_\varepsilon} \varphi_n \overline{\psi_n} + \int_{\Omega_{\varepsilon,n}} \varphi_n \overline{\psi_n}.$$ 

On $\Sigma_0 \setminus U_\varepsilon$, $\varphi_n \overline{\psi_n}$ converges uniformly to $\varphi_0 \overline{\psi_0}$, whereas on $\Omega_{\varepsilon,n}$ it is essentially controlled by $\frac{1}{|z|}$, if the node corresponds to $z = 0$, because the $\varphi_n$ are bounded and $\psi_n$ approaches a meromorphic quadratic differential with a second order pole.

These observations and (3.1.11) imply, letting $\varepsilon \to 0$,

(3.1.14) $$\lim_{n \to \infty} \int_{\Sigma_n} \varphi_n \overline{\psi_n} = 0.$$ 

Products with holomorphic quadratic differentials or meromorphic ones with first order poles, again real on $\partial \Sigma_0$, are easier to handle, and the conclusion follows readily.

\[ \square \]

**Remark.** The higher regularity assumption $\eta_0 \in C^{1,\alpha}$ was only needed in order to control the opening up of nodes at the boundary. Namely, such a process is represented by meromorphic quadratic differential $\psi_0$ with a second order
pole at the boundary, and therefore, one needs boundedness of the holomorphic quadratic differential \( \varphi_0 \) to ensure that

\[
\int_{\Sigma_0} \varphi_0 \bar{\varphi}_0 \frac{1}{\lambda^2(z)} \frac{i}{2} \, dz \wedge d\bar{z}
\]

exists. As we have seen, this integral then vanishes.

We may then stipulate that also for \( \varphi_0 \) associated to arbitrary \( \eta \in H^\frac{1}{2} \), and \( \psi_0 \) as above

\[
\int_{\Sigma_0} \varphi_0 \bar{\varphi}_0 \frac{1}{\lambda^2(z)} \frac{i}{2} \, dz \wedge d\bar{z} = 0.
\]

This will be sufficient for the construction of our gradient flow below as smooth boundary values are dense in \( \mathcal{R}^n \).

### 3.2 Critical point theory for minimal surfaces

For the oriented case, the results of this section were developed in [JS]. We shall need a slight modification of the definition of \( \hat{T}_{p,m} \) and \( \hat{\mathcal{M}}_{p,m} \). Whenever a boundary curve of a Riemann surface \( \Sigma \) is pinched to a point, we add a once punctured disk to the resulting limit surface. Since a once punctured disk can carry only one conformal structure, this does not affect the structure of our moduli space. From now on, \( \hat{T}_{p,m} \) and \( \hat{\mathcal{M}}_{p,m} \) will denote these modified spaces.

We also need to take a closer look at the topology of our space

\[
\mathcal{R}^n \times \hat{T}_{p,m}
\]

**Definition 3.2.1.**

(i) If \( (p,m) \neq (0,1) \), and if \( \Sigma_0 \) contains no disk or annulus component, we say that \( (\eta_n, (\Sigma_n, [g_n])) \), converges to \( (\eta_0, (\Sigma_0, [g_0])) \) if \( (\Sigma_n, [g_n]) \) converges to \( (\Sigma_0, [g_0]) \) in Fenchel-Nielsen coordinates and if \( \eta_n \) converges to \( \eta_0 \) in \( H^\frac{1}{2} \) and if \( E(\eta_n, (\Sigma_n, [g_n])) \rightarrow E(\eta_0, (\Sigma_0, [g_0])) \).

(ii) If \( \Sigma_0 \) contains an annulus component, but no disk, we define convergence of \( (\eta_n, (\Sigma_n, [g_n])) \) to \( (\eta_0, (\Sigma_0, [g_0])) \) if \( (\Sigma_n, [g_n]) \) converges to \( (\Sigma_0, [g_0]) \) and if there exists a sequence \( (\tau_n)_{n \in \mathbb{N}} \) of conformal diffeomorphisms of \( \Sigma_n \) homotopic to the identity for which \( \eta_n - \eta_0 \circ \tau_n \) converges to 0 in \( H^{1/2} \) and again \( E(\eta_n, (\Sigma_n, [g_n]))) \rightarrow E(\eta_0, (\Sigma_0, [g_0])) \).

(iii) If \( \Sigma_0 \) contains a disk component, we define convergence of \( (\eta_n, (\Sigma_n, [g_n])) \) to \( (\eta_0, (\Sigma_0, [g_0])) \) to be as in (ii) or of the following type:

We let \( \partial \Sigma_n = c_{n,1}, \ldots, c_{n,m} \)

We write \( \Sigma_0 = \Sigma' \cup D \), where \( D \) is the unit disk and \( \Sigma' \) has \( m \) boundary curves. We put \( c_1 = \partial D, \partial \Sigma' = c_2 \cup \ldots \cup c_m \) for normalization.

We also write \( \eta_n = (\eta_1^n, ..., \eta_m^n), \eta_0 = (\eta_1^0, ..., \eta_m^0) \)

We say that a disk splits off at the boundary in the limit of the sequence \( (\eta_n, (\Sigma_n, [g_n])) \) if

(a) \( (\Sigma_n, [g_n]) \rightarrow (\Sigma', [g_0]) \) as before, with \( c_1 = \lim c_{n,1} \)

(b) \( \eta_j^n \) converges to a constant weakly in \( H^{1/2} \), \( \eta_j^n \rightarrow \eta_j^0 \) strongly in \( H^{1/2} \) for \( j = 2, ..., m \)

(c) there exists \( p_n \in c_{n,1} \), a sequence of radii \( r_n \rightarrow 0 \), conformal diffeomorphisms

\[
\tau_n : D \rightarrow B(p_n, r_n) := \{ z \in \Sigma_n : d(z, z_n) \leq r_n \}
\]
for which

\[ \tilde{\eta}_n^1 := h_{\eta_n}, \Sigma_n, g_n|\partial B(p_n, r_n) \to \eta_0^1 \]

strongly in \( H^{1/2} \)

\[(d)\]

\[ E(\eta_n, (\Sigma_n, [g_n])) \to E(\eta_0, (\Sigma_o, [g_o])) \]

Similarly, one defines the splitting off of more than one disk.

The convergence to be defined then is the process of splitting off of one or several disks at the boundary.

The topological structure arising from Definition 3.2.1 (iii) is such that for example

\[ R^m \times \{D^m\} \]

(\( D^m \) being the surface consisting of \( m \) disjoint unit disks) is contained in the completion of

\[ R^m \times \hat{T}_{p,m} \]

Similarly as at the end of 3.1, grad \( E \) again is tangent to \( R^m \times \{D^m\} \). Namely, if \( (\eta_n^1)_{n \in \mathbb{N}} \in R \) converges to a constant weakly in \( H^{1/2} \), its \( H^{1/2} \)–product with any fixed tangent vector tends to 0.

In particular, its product with a tangent vector to \( R^m \), vanishing on all boundary curves except the first one, and representing a direction transversal to \( R^m \times \hat{T}_{p,m} \), vanishes. (Note that directions tangential to \( R^m \times \{D^m\} \) cannot be represented on approximating surfaces in \( R^m \times \hat{T}_{p,m} \) because a blow-up is required in the limit in Def. 3.2.1 (iii).) This observation shows that grad \( E \) – which is given by the \( H^{1/2} \) product, see Lemma 3.1.1 – is tangential to \( R^m \times \{D^m\} \).

We want to divide out the action of the modular group \( G_{p,m} \). It acts on \( R^m \times \hat{T}_{p,m} \) via \( (\eta, (\Sigma, [g])) \mapsto (\eta \circ f^{-1}, f(\Sigma, [g])) \).

Since \( f^{-1} : f(\Sigma, [g]) \to (\Sigma, [g]) \) is a conformal diffeomorphism, this action leaves \( E \) invariant. Therefore, \( E \) descends to a function on

\[ \hat{P}_{p,m} := R^m \times \hat{T}_{p,m}/G_{p,m} \]

We say that \( (\eta_n, \Sigma_n)_{n \in \mathbb{N}} \subset \hat{P}_{p,m} \) converges to \( (\eta_0, \Sigma_0) \) if there exist lifts to \( R^m \times \hat{T}_{p,m} \) which converge.

On \( \hat{P}_{p,m} \), a Palais-Smale condition for \( E \) can be verified:

**Lemma 3.2.1.** Suppose \( (y_n)_{n \in \mathbb{N}} \subset \hat{P}_{p,m} \) satisfies

\[ E(y_n) \to \beta \in \mathbb{R} \]

\[ \| \text{grad } E(y_n) \| \to 0. \]

Then after selection of a subsequence, \( (y_n) \) converges to a critical point \( y \) of \( E \), i.e. a minimal surface with boundary \( \gamma \), with \( E(y) = \beta \).

A map

\[ \psi : R^m \times \hat{T}_{p,m} \to R^m \times \hat{T}_{p,m} \]
is called equivariant if it commutes with the action of \( G_{p,m} \) and also \( \psi(\eta \circ \tau, \Sigma) = \psi(\eta, \Sigma) \circ \tau \) for any conformal diffeomorphism \( \tau : \Sigma \to \Sigma \) homotopic to the identity. Similarly, one defines equivariant neighborhoods.

We put

\[
K_\beta = \{ x \in \mathbb{R}^n \times \hat{G}_{p,m} : E(x) = \beta, \ \text{grad} \ E(x) = 0 \}
\]

This corresponds to the minimal surfaces of genus \( \leq p \) and energy (=area) \( \beta \) with boundary \( \gamma \).

**Lemma 3.2.2.** For any \( \beta \in \mathbb{R} \), any \( \epsilon_0 > 0 \), any equivariant neighborhood \( U \) of \( K_\beta \) in \( \mathbb{R}^n \times \hat{G}_{p,m} \), there exist a continuous equivariant "flow"

\[
\phi : \mathbb{R}^n \times \hat{G}_{p,m} \times [0, 1] \to \mathbb{R}^n \times \hat{G}_{p,m}
\]

and \( \epsilon, \ 0 < \epsilon < \epsilon_0 \), with:

(i) \( \phi(y, t) = y \), if \( t = 0 \) or \( \text{grad} \ E(y) = 0 \) or \( |E(y) - \beta| \geq \epsilon_0 \)
(ii) \( E(\phi(y, t)) \) is nonincreasing in \( t \).
(iii) If \( E(y) \leq \beta + \epsilon \) and either \( y \notin U \) or \( \phi(y, 1) \notin U \), then \( E(\phi(y, 1)) \leq \beta - \epsilon \).

With the help of Lemma 3.2.1, 3.2.2, general critical point construction from Lusternik-Schnirelman and Morse-Conley theory can be developed for the function \( E \) on \( \hat{G}_{p,m} \).

For example

**Lemma 3.2.3.** Suppose \( S_1, S_2 \in \hat{G}_{p,m} \) are relative minima of \( E \). Then there exists a minimal surface \( S \) different from \( S_1, S_2 \) with \( E(S) = \kappa = \inf_{p} \sup_{t \in [0, 1]} \{ E(p(t) : p(t) \text{ is a continuous path in } \hat{G}_{p,m} \text{ with } p(t) = (\eta(t), \Sigma(t)), \}

\[
p(0) = S_1, \ p(1) = S_2
\]

If \( \kappa > \max(E(S_1), E(S_2)) \), then there exists such an \( S \) which is not a relative minimum of \( E \).

**Theorem 3.2.1.** Let \( \gamma_1, \gamma_2 \) be smooth oriented disjoint closed Jordan curves in \( E^3 \) which are linked.

Assume that there exists an involution \( i \) (i.e. an isometry of order 2) of \( E^3 \) leaving \( \gamma_1 \) pointwise fixed and mapping \( \gamma_2 \) onto itself but reversing its orientation. Suppose also that \( \gamma_1 \) and \( \gamma_2 \) each bound only one minimal surface. Then the configuration \( \gamma = (\gamma_1, \gamma_2) \) bounds infinitely many nonorientable minimal surfaces.

The result may be easily derived from the general Theorem 3.2.2 below. It is also instructive to give a direct proof, based on a \( \mathbb{Z}_2 \) symmetry argument. A similar argument has been used earlier by J.Pitts [P] for showing that a configuration of three circles with such a symmetry bounds infinitely many oriented minimal surfaces.

**Proof.** By Theorem 1.1.1, \( \gamma_1 \) and \( \gamma_2 \) each bound a minimal disk. By assumption they are unique. Therefore, they are invariant under the involution \( i \). These two disks together then form a minimal surface \( S_0 \) bounded by \( \gamma \).
Since $\gamma_1$ and $\gamma_2$ are linked, the two disks have to intersect, and so $S_o$ is not embedded. By Corollary 1.2.1, $\gamma$ then also bounds a minimal annulus $S_1$. Since $S_1$ has only one component, it cannot be invariant under $i$, and $i(S_1)$ then is another minimal surface bounded by $\gamma$.

The conformal structures of $S_1$ and $i(S_1)$ can be connected in the class of nonorientable surfaces of genus 1 (Klein bottles with two holes), i.e. in $\mathcal{M}_{1,2}$. We now distinguish two cases

(i) $S_1$ and $i(S_1)$ are energy minimizing among surfaces with conformal structure in $\mathcal{M}_{1,2}$. We then consider

$$\kappa_1 := \inf \sup_{\ P \in [0,1]} \{E(p(t)): p(t), t \in [0,1] \text{ is a continuous path in } \hat{P}_{1,2} \text{ with } p(0) = S_1, p(1) = i(S_1)\}$$

$\kappa_1$ then is realized by a minimal surface $S_2$ different form $S_1$ and $i(S_1)$, by Lemma 3.2.3.

$S_2$ is also different form $S_o$. Namely, $S_2$ is of critical index at most 1, whereas the energy of $S_o$ may be decreased in two independent ways in $\hat{P}_{1,2}$ by inserting two disjoint handles along its line of self intersection as in the proof of Corollary 1.2.1. Thus no critical path $p$ can pass through $S_o$.

In particular, $S_2$ has only one component as $S_o$ is the unique minimal surface with two components bounded by $\gamma$. Therefore, $S_2$ is not invariant under $i$, $i(S_2)$ then is another minimal surface bounded by $\gamma$ and different from all the preceding ones.

(ii) If $S_1$ and $i(S_1)$ are not energy minimizing in $\hat{P}_{1,2}$, we let $S_2$ be the minimizing surface in this class. Again $S_2$ is different form $S_o$, and hence $i(S_2)$ is another minimal surface bounded by $\gamma$.

The process can be iterated with additional parameters and performing higher order saddle point constructions. Whenever the genus is increased this allows a new independent energy decreasing variation of $S_o$ by inserting a new handle. Therefore, all subsequently obtained surfaces are different from $S_o$, hence not invariant under $i$. Therefore, they always occur in pairs, and this makes the iteration possible.

So much for an outline of the proof of Theorem 3.2.1.

Let us also quote some general results:

**Theorem 3.2.2.** Suppose the configuration $\gamma = (\gamma_1, \ldots, \gamma_m) \subset N$ of disjoint closed smooth Jordan curve in the simply connected, complete, nonpositively curved Riemannian manifold $N$ bounds only finitely many minimal surfaces $S_1, \ldots, S_k$ of finite genus. Let $i_t(S_j)$ be the Conley index of $S_j$ in $\hat{P}_{p,m}$, where $p \geq \max_j (\text{genus } (S_j))$. Then

$$\sum_{j=1}^{k} i_t(S_j) = 1 + (1 + t)Q(t)$$

where $Q(t)$ is a polynomial with nonnegative integer coefficients. (Note that $i_t(S_j)$ is a formal power series in $t$).

**Corollary 3.2.1.** If under the assumption of Thm.3.2.2 all $S_j$ are nondegenerate critical points of $E$ on $\hat{P}_{p,m}$, and if $c_n$ denotes the number of $S_j$ of
Morse index $n$, then

$$\sum_{n}(-1)^n c_n = 1.$$

References


[S3] ———, *Minimal surfaces of higher genus and general critical type*, to appear in Proc Nankai Inst.


