# Noise delays onset of sustained firing in a minimal model of persistent activity

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## Abstract

We report a noise-induced delay of bifurcation leading to persistent synaptically sustained firing in a simple neural circuit. Two neural saddle-node oscillators are coupled with reciprocal excitatory synapses. Increasing synaptic current moves the circuit through a bifurcation into a bistable anti-phase regime. Applying a small random current to both neurons delays the onset of bistability. We analyse the phase plane for the various activity regimes and find that this noise induced mechansim can be related to the structure of asynchronous and synchronous activity states.

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## 1. Introduction

The effects of random currents on the firing behaviour of neurons have received a considerable amount of attention [1,6,8,10,11,14]. A number of investigators have considered what happens to single neurons and circuits of neurons when noise perturbs periodically modulated input signals. Experimental work has identified noise-induced signal amplification and resonance in a number of preparations e.g. [1]. Noise effects have also been studied in the context of indigenous oscillations in neural models, focusing on the so-called "autonomous stochastic resonance" [8]. Noise also induces coherence resonance in a Hodgkin–Huxley model, with pre-cursors of the sub-critical

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Hopf bifurcation revealed by the action of random currents [6]. In this sense, noise "advanced" the bifurcation. Similar effects have also been found in a generic saddle-node driven oscillator where noise advances the onset of oscillations and upregulates the mean frequency [12,13]

Similarly, the behaviour of pulse-coupled neural networks has received much recent attention [7,9,15]. In particular a number of studies addressed the possible effects of noise on networks that generate persistent foci of activity, or bump attractors. Modelling studies have found that random noise tends to destabilize the position of the activity bump and induce a difusion-like motion (Wang, personal communication). At the same time Laing et al. [5] looked at bumps of sustained activity destabilized to travelling waves due to spike frequency adaptation and reported that noise has a curious stabilizing effect: it reduces the velocity of the bump.

In view of these reports we decided to examine what random noise would do to a minimal neural circuit that is capable of synaptically evoked sustained activity. Here, we look in detail at the effects of noise on the onset of synaptically sustained firing in a circuit of excitable neurons where activity occurs purely due to the recurrent synaptic interactions. To our knowledge, almost all efforts to study the interplay of noise and neural oscillators report noise-induced increase in firing and advancement of bifurcations, e.g. [13]. In this light our finding is rather intriguing since we observe a noise-induced delay of bifurcation in a purely positively coupled circuit of neural oscillators. We conclude that the relative width of the attractor basins for the quiescent and persistent asynchronous firing states is the key factor in determining if the noise has a delaying or advancing effect on the bifurcation. We propose this as a noise-dependent mechanism that could stabilize sustained activity by making it more difficult to recruit neurons just at the edges of the sustained bump.

## 2. The model: $\theta$-neuron

The $\theta$-neuron model [2–4] is derived from the observation that a wide class of physiological models of cortical neurons show a saddle-node type bifurcation at a critical current input value. This parameter, the bias, determines the dynamical behaviour of the solutions of the corresponding system of ordinary differential equations. General dynamical systems theory tells us that the qualitative behaviour in some neighbourhood of the bifurcation point (which may be quite large as it extends up to the next bifurcation or other dynamic transition) is governed by the reduction of the system to the centre manifold. In the present case this leads to the following differential equation:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \lambda + x^2. \tag{1}$$

Here, the bifurcation parameter $\lambda$ is the input to the neuron while $x$ records its activity. A solution to this equation tends to infinity in finite time. This is considered as a spiking event, and the initial values are then reset to $-\infty$. To avoid such formal singularities, one introduces a phase variable $\theta$ that is $2\pi$-periodic via

$$x = \tan\left(\frac{\theta}{2}\right). \tag{2}$$

$\theta$ is then a variable with domain the unit circle $S^1$, and a spike now corresponds to a period of $\theta$. Spikes are no longer represented by transitions through infinity, but by $\theta$ traversing some neighbourhood around $\pi$.

The original differential equation is then transformed into

$$\frac{d\theta}{dt} = (1 - \cos(\theta)) + (1 + \cos(\theta))\lambda. \tag{3}$$

As before, the bifurcation occurs at $\lambda = 0$, and there is a single degenerate rest point, namely, $\theta = 0$. The sensitivity to the input $\lambda$ is highest at $\theta = 0$ and lowest at $\theta = \pi$ which is considered as the spike point. When $A\lambda$ is positive, $\lambda$ continually increases, and the neuron fires perpetually. When $\lambda$ is negative, there are two rest points; a stable one denoted by $\theta_r$ and an unstable one $\theta_t > \theta_r$. If $\theta$ is larger than $\theta_t$ it increases until it completes a period and comes to rest at $\theta_r + 2\pi$ which is identified with $\theta_r$. Thus, if the phase is above the threshold value $\theta_t$, a spike occurs and the neuron returns to rest. The input to the model can be decomposed as

$$\lambda = \beta + \sigma\eta + I(t), \tag{4}$$

where $\beta$ is a constant term, the so-called bias, $v(t)$ is a time-dependent input, while $\eta$ is (white) noise and $\sigma$ its intensity. Sufficiently strong noise can occasionally push the phase $\theta$ beyond the threshold value $\theta_t$ causing intermittent firing (Fig. 1(C)).

In this report we consider a small circuit of two $\theta$-neurons coupled reciprocally with excitatory synapses.

The total input to the cell $i$ is then

$$\lambda_i = \beta_i + \sigma_i\eta_i + gs_{ij}s_{ij} \tag{5}$$

$$ds_{ij}/dt = syn(x_i) * (1 - s_{ij}) - s_{ij}/\tau \tag{6}$$

$$syn(x_i) = a * \exp(-20 * (1 - \cos(x_i - th))), \tag{7}$$

where $s_{ij}$ gives the dynamics of the synaptic coupling from cell $i$ to cell $j$, and $x_i$ is the presynaptic state variable, which triggers the synapses when the presynaptic cell fires a spike. For the numerical simulation we picked the duration of the synapse to be comensurate with fast AMPA-type glutamatergic excitatory synapses. Threshold $th$ was set at 3, $a = 1$, $\tau = 2$. The maximal synaptic strength $gs_i$ is a free parameter.

## 3. Numercial results

Fig. 1(A), upper trace shows the firing patterns of two cells whose spiking behaviour results from mutual excitatory synapses. The cells are initially quiescent (they are not intrinsically spiking) and their activity results from an transient input to one cell. Activity can be terminated by small levels of noise (Fig. 1(A), middle trace), whilst increased noise levels cause intermittent firing (Fig. 1(A), lower trace).

Fig. 1(B) plots the probability ($M_1$) of observing firing in the last 200 ms of a 2000 ms run over an ensemble of 1000 sample paths as a function of the strength of the synaptic coupling ($g_s$). In the noise-free circuit, $g_s^*$ is the critical value of

(a)

Time (secs)



(b)

$g^*_s = 8.625$

$M_1$

$g_s$

=0
0.02
0.04
0.06
0.08
0.1
0.12
0.14



(c)

$g_s$
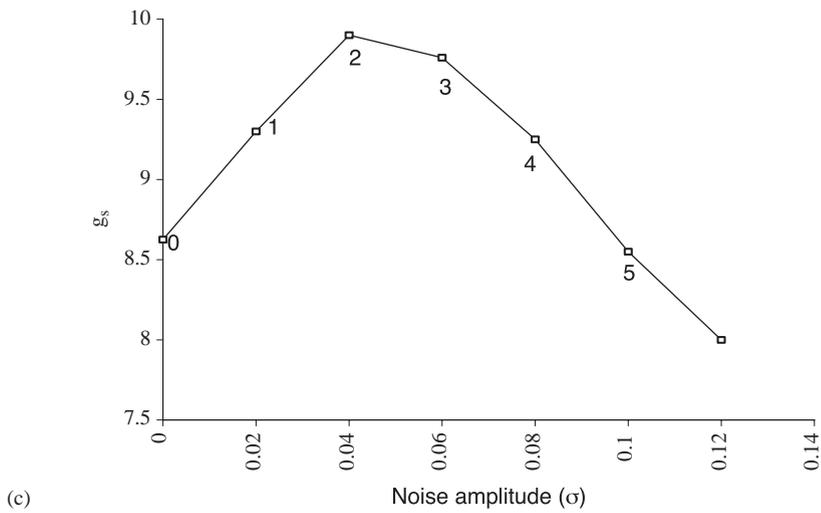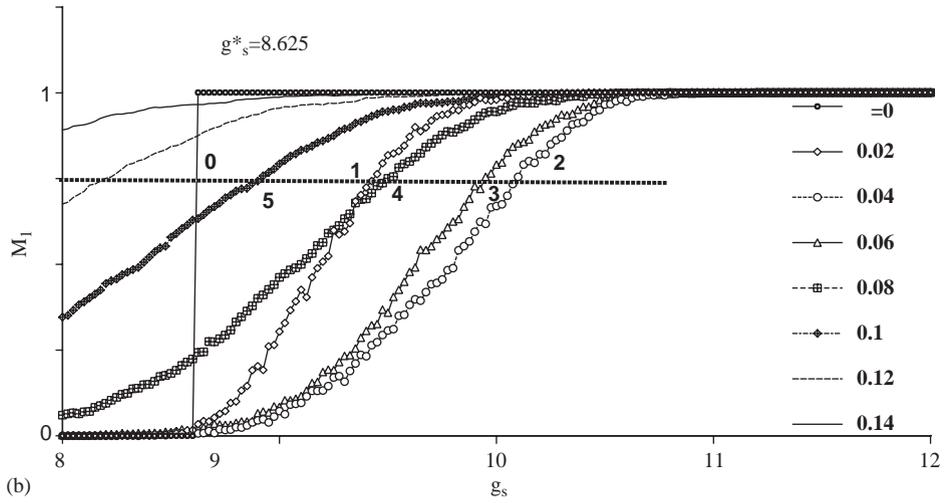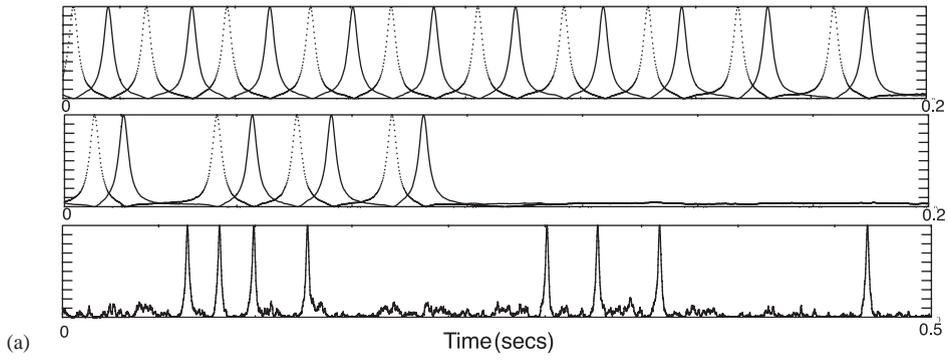
Noise amplitude ($\sigma$)

coupling above which sustained firing occurs (i.e. $M_1 = 0$ for $g_s < g_s^*$, $M_1 = 1$ for $g_s \geqslant g_s^*$). At small noise levels (Fig. 1(B), traces 1,2), increasing the noise amplitude progressively shifts the curves of $M_1$ to the right with respect to the noise-free case. This behaviour is surprising as addition of small amounts of noise for an autonomously spiking $\theta$-neuron (or a generic saddle-node oscillator) induces the opposite effect—noise advanced bifurcation (see [4,13]). Above a critical noise value, the onset of sustained firing appears to be shifted back to the left (Fig. 1(B), traces 3 and higher). This is in the sense that for high-noise levels we observe intermittent pre-cursors of the post-bifurcation behaviour. Thus we can state that the bifurcation is delayed for low-noise amplitudes and advanced with higher noise. Fig. 1(C) shows that there is a non-linear relationship between the amount of injected noise and the firing probability; if we consider the sustained firing as signal (perturbed by noise), this resembles in loose sense stochastic resonance. However, here the net effect of noise is "negative".

## 4. Analysis of coupled neurons

We now consider the situation where we have two neurons (distinguished by sub-scripts $i = 1, 2$). The dynamics then takes place on the product of two circles, i.e. on a two-dimensional torus $T$, represented by the square $[-\pi, \pi] \times [-\pi, \pi]$ in the plane, with periodic boundary identifications. We first consider the simple case of two uncoupled, noise-free neurons ($\sigma_1 = \sigma_2 = 0$) with the same bias $\beta$. Their dynamics are independent. In the phase plot (Fig. 2(i)) the diagonal is always an invariant curve, where the two neurons fire. If $\beta > 0$, both neurons continue to fire, although their phase difference, if not 0 initially, is not constant, due to non-linear effects. If $\beta = 0$, $(0,0)$ is a degenerate rest point (Fig. 2(ii)). The two curves, $\theta_1 = 0$ and $\theta_2 = 0$, are homoclinic orbits and all flow lines eventually terminate at this fixed point. One or both neurons will spike before returning to rest if their initial phase is between 0 and $\pi$.

If $\beta < 0$ (Fig. 2(iii)), we have four fixed points—the attractor ($\theta_1 = \theta_2 = \theta_r$), the repeller ($\theta_1 = \theta_2 = \theta_t$), and the two saddles where one of the neurons has its phase at $\theta_r$ (rest) and the other one at $\theta_t$ (threshold). Some special heteroclinic orbits are given by the straight lines where one of the two neurons stays at $\theta_t$ while the other one moves from the threshold to the rest value, spiking if its initial phase was above threshold. All other flow lines terminate at the attractor. We now add an interaction term $s_i j g_s$ to the input of neuron $i$. A precise equation for $s_i$ can be derived from

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Fig. 1. Asynchronous synaptically sustained oscillations in a positively coupled 2-cell circuit. (a) Upper trace: sustained firing in the noise-free circuit. Middle trace: sustained firing can be terminated by the action of small amplitude noise. Lower trace: larger amplitude noise induces an intermittent firing pattern. Noise injected into the two neurons is completely correlated, but the results are qualitatively identical for uncorrelated noises. (b) Increasing noise delays sustained firing for low noise levels (traces 1,2) and advances firing for higher noise levels (traces 3,4,5). The horizontal dashed line and the numbers mark the test points $g_s^{2/3}$. (c) Addition of noise has a non-linear effect on sustained firing in this coupled circuit. Here we plot the location of the test points $g_s^{2/3}$ See text for details.
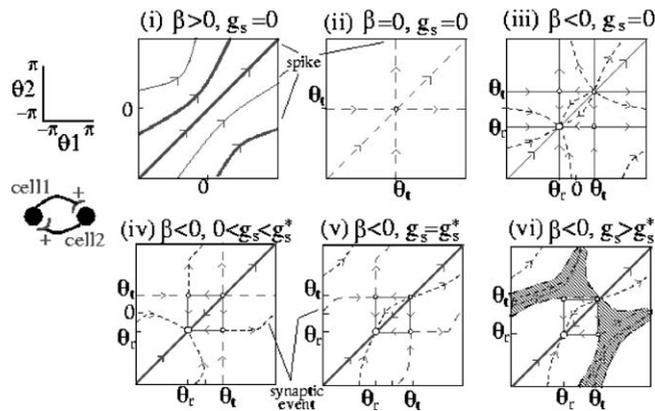
Fig. 2. Different states of the network for various values of the intrinsic excitability of the cells, $\beta$, and the coupling strength, $g_s$. Axes plot the phase $(\theta_1, \theta_2)$ of each cell. See text for details.

electrophysiological models. However, for our qualitative study we only need the characteristic features that it stays bounded between 0 and 1 and is peaked near the spike of neuron $j$. With this interaction term, the equation for neuron $i$ is then as described in Section 2. Once more, we first consider the situation without noise, i.e. $\sigma = 0$. We also assume that we are in the excitable region, i.e. $\beta < 0$. $g_s$ is positive (excitatory coupling), and so the coupling counteracts the effect of the bias to some extent. N.B. In contrast to the constant bias, the synaptic input to each neuron is time dependent. If $g_s$ is sufficiently small, the qualitative situation is similar to the case with zero coupling, i.e. $g_s = 0$. We still have a heteroclinic orbit from the saddle $(\theta_1 = \theta_t, \theta_2 = \theta_r)$ to the attractor $(\theta_r, \theta_r)$, although $\theta_2$ is no longer constant, but increases due to the input from neuron 1 before returning to the rest value. (Fig. 2(iv)). If $g_s$ reaches a critical value $g_s^*$, however, the heteroclinic orbit starting at $(\theta_t, \theta_r)$ no longer terminates at the attractor. The synaptic input advances the phase of neuron 2 to the other saddle point at $(\theta_r, \theta_t)$ (Fig. 2(v)). Besides two heteroclinic orbits that go from the repeller to the two saddles as before, all other orbits still terminate at the attractor $(\theta_r, \theta_r)$, for $g_s = g_s^*$. If $g_s$ is increased beyond $g_s^*$, however, the heteroclinic orbit between the two saddles mutates into a stable attractor (Fig. 2(vi)). This corresponds to sustained asynchronous firing of the two neurons. If the phase difference is too small, the dynamics converges to the double rest point (except in some region in the vicinity of the node), and firing stops as neither cell is sensitive enough to its synaptic input to maintain firing. Conversely, if they are out of synchrony, a single spike can induce the second neuron to fire at a time when the first cell is close to rest, and sensitive to synaptic input itself. If $g_s$ is only slightly above the critical value, the basin of attraction of that limit cycle will still be relatively small, but as $g_s$ is increased further, the basin grows in size until eventually it is larger than the basin of attraction of the double rest point. On the basis of the preceding analysis, it is now straightforward to predict the effect of noise. If $g_s$ is only slightly above the critical value $g_s^*$, a small amount of noise is more likely to kick the dynamics out of the narrow basin of attraction of the asynchronous

limit cycle and into the large basin of the double rest point than vice versa. In effect, a small noise level increases the critical parameter value required for the qualitative transition to sustained asynchronous firing. A larger amount of noise, however, may move the dynamics from the rest point into the basin of attraction of the asynchronous limit cycle. Once in that basin, the neurons will fire. Hence a larger value of noise will cause intermittent periods of sustained firing of the two neurons even at somewhat smaller values of $g_s$. In effect, it decreases the value of the critical parameter. Thus, we observe a genuinely non-linear effect of the noise level $\sigma$ (Fig. 1(E)). For values of the coupling $g_s$ that are substantially larger than the critical value $g_s^*$, even small amounts of noise have a good chance of perturbing the dynamics out of the attracting vicinity of the double rest point into the attracting region of the asynchronous limit cycle. This will further enhance the sustained asynchronous firing pattern of the two neurons.

## 5. Conclusions

In this work we report a new and unusual effect of noise in a simple neural circuit. When the sustained firing in the circuit is induced by recurrent excitatory coupling, small noise levels can exert a strong influence on the circuit dynamics, often abolishing the firing. The probability of observing sustained firing has been used to characterize the transition from quiescent to oscillatory behaviour. Fig. 1(B) clearly shows that in this system, noise delays this transition. Noise-induced delay of bifurcation can therefore occur in a completely positively coupled circuit. The same noise has the exact opposite effect of advancing the bifurcation when it is applied to a single autonomously firing neuron. The paradoxical effect of noise in this circuit can be understood by considering the structure of its phase plane. When the width of the attractor basin for asynchronous firing is small, weak noise can perturb the system into the larger basin of the stable quiescent state. Thus, as the coupling strength increases, the basin of attraction for the sustained firing solution grows at the expense of the quiescent state. The negative (bifurcation-delaying) effect of the noise is then eliminated. In this circuit, low levels of noise ensure that sustained firing can only take place above a critical coupling threshold. Transitions in the opposite direction from the rest-state to a sustained firing state can only occur when noise fluctuations reach a critical value. Above this value, transitions into the firing state begin to counteract transitions into the quiescent state.

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