Vanishing Theorems for L^2-Cohomology Groups

J. Jost
Max Planck Institute
for Mathematics
in the Sciences
Leipzig

and

Y. L. Xin
Institute
of Mathematics
Fudan University
Shanghai

August 2, 1999

1 Introduction

Let us start by discussing L^2-cohomology groups (see [17] for a useful overview). Let M be a complete Riemannian manifold. The reduced L^2-cohomology group $H^q(M)$, where L^2 stands for "square integrable", is defined as $Z^q/\overline{B^q}$, where

$$\begin{aligned} Z^q :=& \{u \ q\text{-form of class } L^2 \text{ with } du = 0\} \text{ (closed } L^2 \ q\text{-forms)} \\ B^q :=& \{u \ q\text{-form of class } L^2 \text{ with } u = dv \\ & \quad \text{for some } L^2 \ (q-1)\text{-form } v\} \text{ (exact } L^2 \ q\text{-forms)} \end{aligned} \tag{1.1}$$

and where $\overline{B^q}$ is the L^2-closure of B^q in Z^q. If M is compact, B^q is closed in Z^q, i.e. $\overline{B^q} = B^q$, and the L^2-cohomology is the same as the ordinary de Rham cohomology. In the compact case, in turn we have Hodge theory representing each de Rham cohomology class by a harmonic form. In this sense, L^2-cohomology is the appropriate extension to the noncompact case inasmuch as here every L^2-cohomology class can be represented by an L^2-harmonic form, see [2]. In the noncompact case, B^q need not be closed in Z^q, essentially because the spectrum of the Laplacian need not have a positive lower bound, or equivalently, the Poincaré inequality need not hold. An example is Euclidean space. For hyperbolic spaces, however, we do have such inequalities, and consequently B^q is closed. In any case, however, in order to have a uniform theory, one considers $\overline{B^q}$ in place of B^q. Besides offering the possibility to extend Hodge theory to the noncompact case, there is another, probably more compelling reason for considering L^2-cohomology. Namely, it can be used to obtain topological information about compact quotients of M, as we shall now explain, following Atiyah [2].

Let Γ be a discrete cocompact group of isometries acting freely on our noncompact manifold M. "Cocompact" means that the quotient M/Γ is compact. As the action of Γ is free, M/Γ thus is a compact Riemannian manifold. As Γ commutes with the Laplace operator, the Hilbert spaces $\mathcal{H}^q(M)$ of L^2-harmonic q-forms on M are Γ-moduli. On the basis of constructions in the theory of Von Neumann algebras, Atiyah defines real valued L^2-Betti numbers

$$B^q_\Gamma(M) := \dim_\Gamma H^q(M), \tag{1.2}$$

satisfying Poincaré duality, i.e. $B^q_\Gamma(M) = B^{\dim M - q}_\Gamma(M)$, and the corresponding L^2-Euler characteristic

$$\chi(M,\Gamma) := \sum_q (-1)^q B^q_\Gamma(M). \tag{1.3}$$

Atiyah shows that $\chi(M,\Gamma)$ equals the ordinary Euler characteristic of M/Γ which is an integer, and this is the basis of the relation between L^2-cohomology of M and the topology of M/Γ alluded to above.

More precisely, Hopf asked whether the sign of the sectional curvature determines the Euler characteristic of a compact Riemannian manifold. For example, if $\overline{M}^{2m}$ is a compact manifold of dimension $2m$ with negative sectional curvature, one should have

$$(-1)^m \chi(\overline{M}^{2m}) > 0. \tag{1.4}$$

Since the sign of the sectional curvature does not determine the sign of the Gauss-Bonnet integrand (see Geroch [13]), this cannot be deduced from algebraic considerations alone. Therefore, Dodziuk [10] and Singer [18] suggested to use L^2-cohomology to approach this problem as follows: Show

$$\mathcal{H}^q(M) = \{0\} \quad \text{for } q \neq m \tag{1.5}$$

– which implies $B^q_\Gamma(M) = 0$ for $q \neq m$ –
and

$$\mathcal{H}^m(M) \neq \{0\} \tag{1.6}$$

– which implies $B^m_\Gamma(M) \neq 0$ because $B^p_\Gamma \neq 0$ can be seen to be equivalent to $\mathcal{H}^p(M) \neq \{0\}$. However, Anderson [1] constructed simply connected complete negatively curved Riemannian manifolds on which this does not hold, thus indicating a certainly difficulty with this approach. This difficulty might be of a purely technical nature as Anderson's examples do not admit compact quotients. In a positive direction, Donnelly-Xavier [12] showed that (1.5) holds provided the sectional curvature of M satisfies a certain negative pinching condition, by using a certain integral identity for L^2-harmonic forms. However, as far as the Hopf problem is concerned, stronger results follow from the local analysis of the curvature tensor of Bourguignon-Karcher [7]. In the present paper, we take up the question of vanishing of L^2 harmonic forms by using a general integral identity for p-forms with values in a vector bundle based on the stress-energy tensor introduced by Baird-Eells [3] (all the material about the stress-energy tensor needed for our purposes is developed in the monograph [20]). Our results are stronger than the ones of Donnelly-Xavier inasmuch as we need a pinching condition that is strictly weaker than theirs, and, when applied to the Hopf problem, comparable to those of Bourguignon-Karcher.

In order to put this paper into the proper perspective, we should also discuss other cases where such results are known. In the case of Kähler manifolds, the vanishing result (1.5) was shown independently by M. Stern [19] if the sectional curvature is pinched between any two negative numbers and more generally by Gromov [14] for so-called Kähler hyperbolic manifolds. Gromov was also able to show the nonvanishing result (1.6), thus completely settling the Hopf problem in the Kähler case. Recently, Jost-Zuo [15] showed the vanishing result (1.5) in the Kähler case even for all metrics of nonpositive sectional curvature. Finally, Borel [6] showed the vanishing theorem for all symmetric spaces of noncompact type and rank 1, i.e. for those symmetric spaces that have negative sectional curvature. His analysis depends on the deep work of Harish-Chandra. Here, we shall indicate a more elementary approach to Borel's result by verifying that our pinching condition is general enough to include most Hermitian symmetric spaces. It seems that our method could be refined to handle all cases. On the other hand, vanishing of L^2-Betti numbers for Hermitian symmetric spaces also follows from Gromov's work mentioned above. Our point thus is that we have a method that does not need a Kähler form. As a final application we also show the vanishing of the L^2-Betti numbers of a semi-simple Lie group G in those cases where the corresponding symmetric space G/K (K a maximal compact subgroup of G) satisfies our preceding assumptions. The point is that our integral formula on G can be pushed down to G/K. The results can be considered as an L^2-version of Matsushima's vanishing theorem for the first Betti number [16].

After this outline of the results, we should also discuss how our techniques relate to the ones previously employed:
The well known Bochner identity (see e.g. [20]) computes the Laplacian of the squared norm of a harmonic differential form as the square of the norm of its derivative plus a curvature term. The integral of this Laplacian is zero, if we are on a compact manifold or some suitable decay near infinity permits an application of Stokes' theorem, and so then is the sum of the integrals of the two other terms. The first one is positive, and if the curvature involved is such that the second one is also positive or at least nonnegative, then both of them have to vanish. This is the basis for Bochner's vanishing theorem for the first Betti number of a compact manifold of positive Ricci curvature.

If the Ricci curvature is nonpositive such an approach seems invalid at first glance. However, Matsushima [16] exploited the fact that one may use the positive term to balance the other one in certain situations so as to make both of them still zero, and this is the basis of his vanishing theorem for the first Betti number of a locally symmetric space of noncompact type and rank bigger than one. Such a balancing strategy has been applied in the L^2-case first by Donnelly-Xavier [12]. They multiply the identity by some smooth function and integrate by parts which essentially converts the curvature term into a term involving the Hessian of the function utilized. In their particular application, they use the distance function from some point, and the curvature then reenters by applying a Hessian comparison theorem in order to estimate that Hessian through curvature terms. In a complex setting, the balancing method has been refined by Donnelly-Fefferman [11].

In our paper, we also use such a balancing strategy as pioneered by Matsushima and integrate an identity involving a harmonic form against a function and invoke a Hessian comparison theorem as did Donnelly-Xavier. Our approach is more refined (and more powerful as concerns applications as the latter one), however. In the first place, we do not use the standard identity of Bochner as a starting point, but rather a conservation law for harmonic forms as exhibited by the stress energy tensor discovered by Baird-Eells [3]. Of course, when rewritten, the fact that the stress-energy tensor when applied to a harmonic form is divergence free just becomes a rephrasing of the standard identity for harmonic forms, and in this sense the formula employed earlier by Donnelly-Xavier [12] can be considered as a special case for the one utilized here (the latter is more general, however, as, for example, it can also be used to obtain results about harmonic maps with potential where the occurring 1-form is not harmonic). The advantage of the stress energy tensor is that it guides us how to separate the diagonal and the off-diagonal terms in the Hessian of the distance function in the most efficient way. We work with the squared distance function, but again our argument is more subtle than the one in [12], as we need to balance integrals over balls of different radii and shift the center of the balls towards infinity.

Acknowledgements:
The first author thanks the DFG for support through its Leibniz program. The second author thanks the Max Planck Institute for Mathematics in the Sciences in Leipzig for providing good working conditions during the preparation of this paper, and also NNSFC and SFECC for support. We thank the referee for some useful comments.

2 Vanishing theorems for L^2-Betti numbers for negatively curved manifolds

Let M be a complete simply connected Riemannian manifold with nonpositive sectional curvature. Let ω be a differential p-form with values in a Riemannian vector bundle E over M. Its symmetric square $\omega \odot \omega$ is defined by

$$\omega \odot \omega(X, Y) = \langle i_X \omega, i_Y \omega \rangle,$$

where $X, Y \in \Gamma(TM), i_X\omega$ is the inner product of ω with the vector field X. The stress-energy tensor is defined by

$$S_\omega := \frac{1}{2}|\omega|^2 g - \omega \odot \omega, \tag{2.1}$$

where g is the Riemannian metric tensor of M. Take a compact domain $D \subseteq M$ whose boundary ∂D is a smooth hypersurface in M. Let $e_1, \ldots, e_m$ be a local orthonormal frame field along ∂D in M, such that $e_1, \ldots, e_m \in T(\partial D)$ and $e_m = n \in N(\partial D)$. The following integral identity then holds:

$$\begin{aligned} \int_{\partial D} \frac{1}{2}|\omega|^2 \langle X, n\rangle * 1 = &\int_D (\operatorname{div} S_\omega)(X) * 1 \\ &+ \int_D \langle S_\omega, \nabla X\rangle * 1 + \int_{\partial D} \langle i_X\omega, i_n\omega\rangle * 1, \end{aligned} \tag{2.2}$$

where X is a vector field in D; ∇X can be viewed as 1-form with values in T^*M defined by $\nabla X(e_i)(e_j) = \langle \nabla_{e_i} X, e_j\rangle$. If ω is harmonic, i.e. is closed and co-closed, $\operatorname{div} S_\omega \equiv 0$. The derivation of (2.2) can be found in [20], pp. 49-52.

Theorem 2.1. *Let M be a Cartan-Hadamard manifold of dimension $m > 2$ whose sectional curvature satisfies $-a^2 \leq K \leq 0$ and whose Ricci curvature is bounded from above by $-b^2$, where a, b are positive constants. If $b \geq 2\,p\,a$, then the L^2-Betti numbers $B^p_\Gamma(M)$ and $B^{m-p}_\Gamma(M)$ vanish, provided $p \neq \frac{m}{2}$.*

Proof. Choose $D = B_R(x_0)$, a geodesic ball of radius R with its center in $x_0 \in M$. Its boundary is a geodesic sphere $S_R(x_0)$. The square of the distance function r^2 in M from any point x_0 is smooth. Hence, $X = r\frac{\partial}{\partial r}$ is a smooth vector field in M, where $\frac{\partial}{\partial r}$ denotes the unit radius vector which is the unit normal vector field to the geodesic sphere $S_R(x_0)$. For any L^2-harmonic p-form ω, we have

$$\begin{aligned} \int_{\partial D} \frac{1}{2}|\omega|^2 \langle X, n\rangle * 1 - &\int_{\partial D} \langle i_X\omega, i_n\omega\rangle * 1 \\ &= \int_{S_R(x_0)} \frac{1}{2} R|\omega|^2 * 1 - \int_{S_R(x_0)} R\langle i_{\frac{\partial}{\partial r}}\omega, i_{\frac{\partial}{\partial r}}\omega\rangle * 1 \\ &\leq \frac{1}{2} R \int_{S_R(x_0)} |\omega|^2 * 1. \end{aligned} \tag{2.3}$$

On the other hand,

$$\begin{aligned} &\nabla_{\frac{\partial}{\partial r}} X = \frac{\partial}{\partial r},\ \nabla_{e_s} X = r \operatorname{Hess}(r)(e_s, e_t) e_t, \\ &\operatorname{div} X = 1 + r \operatorname{Hess}(r)(e_s, e_s), \end{aligned}$$

where $\{e_\alpha\} = \{e_s, \frac{\partial}{\partial r}\}$ is an orthonormal frame field in D. We agree that the range of the indices is as follows

$$\alpha, \beta = 1, \ldots, m; \ s, t = 1, \ldots, m-1.$$

We also use the summation convention. Therefore

$$\langle \omega \odot \omega, \nabla X \rangle = \left| i_{\frac{\partial}{\partial r}} \omega \right|^2 + \langle i_{e_s}\omega, i_{e_t}\omega \rangle r \,\mathrm{Hess}\,(r)(e_s, e_t).$$

and hence

$$\begin{aligned}(2.4) \quad \langle S_\omega, \nabla X \rangle = \frac{1}{2}|\omega|^2(1 + r\,\mathrm{Hess}(r)(e_s, e_s)) - \left| i_{\frac{\partial}{\partial r}} \omega \right|^2 \\ - \langle i_{e_s}\omega, i_{e_t}\omega \rangle r\,\mathrm{Hess}(r)(e_s, e_t).\end{aligned}$$

Noting

$$\begin{aligned}|\omega|^2 =& \frac{1}{p!}\langle \omega(e_{\alpha_1}, \ldots, e_{\alpha_p}), \omega(e_{\alpha_1}, \ldots, e_{\alpha_p}) \rangle \\ =& \frac{1}{p!}\langle \omega(\frac{\partial}{\partial r}, e_{s_2}, \ldots, e_{s_p}), \omega(\frac{\partial}{\partial r}, e_{s_2}, \ldots, e_{s_p}) \rangle \\ &+ \frac{1}{p!}\langle \omega(e_{s_1}, \ldots, e_{s_p}), \omega(e_{s_1}, \ldots, e_{s_p}) \rangle \\ =& \frac{(p-1)!}{p!} \sum_{s_2 < \ldots < s_p} \langle \omega(\frac{\partial}{\partial r}, e_{i_2}, \ldots, e_{i_p}), \omega(\frac{\partial}{\partial r}, e_{i_2}, \ldots, e_{i_p}) \rangle \\ &+ \frac{(p-1)!}{p!} \sum_s \sum_{s_2 < \ldots < s_p} \langle \omega(e_s, e_{s_2}, \ldots, e_{s_p}), \omega(e_s, e_{s_2}, \ldots, e_{s_p}) \rangle \\ =& \frac{1}{p}\langle i_{\frac{\partial}{\partial r}}\omega, i_{\frac{\partial}{\partial r}}\omega \rangle + \frac{1}{p}\langle i_{e_s}\omega, i_{e_s}\omega \rangle,\end{aligned}$$

(2.4) becomes

$$\begin{aligned}\langle S_\omega, \nabla X \rangle =& \frac{1}{2p}\left(\left| i_{\frac{\partial}{\partial r}} \omega \right|^2 + \sum_t \langle i_{e_t}\omega, i_{e_t}\omega \rangle \right) \left(1 + \sum_s r\,\mathrm{Hess}\,(r)(e_s, e_s) \right) \\ &- \left| i_{\frac{\partial}{\partial r}} \omega \right|^2 - \sum_{s,t} r\,\mathrm{Hess}\,(r)(e_s, e_t) \langle i_{e_s}\omega, i_{e_t}\omega \rangle \\ (2.5) \qquad =& \left(\frac{1}{2p}\sum_s r\,\mathrm{Hess}\,(r)(e_s, e_s) + \frac{1}{2p} - 1 \right) \left| i_{\frac{\partial}{\partial r}} \omega \right|^2 \\ &+ \sum_s \left(\frac{1}{2p} + \frac{1}{2p} r\Delta r \right) \langle i_{e_s}\omega, i_{e_s}\omega \rangle \\ &- \sum_{s,t} r\,\mathrm{Hess}\,(r)(e_s, e_t) \langle i_{e_s}\omega, i_{e_t}\omega \rangle.\end{aligned}$$

Since M is a Cartan-Hadamard manifold with Ricci curvature $\leq -b^2$, we have

$$\Delta r \geq b \coth(br).$$

(See, for example, Theorem 2.15 of [20].). We choose the e_s as principal curvature directions of $S_R(x_0)$, in order to diagonalize Hess(r). Since the sectional curvature satisfies $-a^2 \leq K \leq 0$, the Hessian comparison theorem yields

$$\frac{1}{r}\delta_{st} \leq \text{ Hess }(r)(e_s, e_t) \leq a \coth(ar)\delta_{st}.$$

Hence

$$\begin{aligned}(2.6) \quad \langle S_\omega, \nabla X\rangle \geq \frac{m-2p}{2p}\left|i_{\frac{\partial}{\partial r}}\omega\right|^2 \\ + \sum_s \left(\frac{1}{2p} + \frac{1}{2p} br \coth(br) - ar \coth ar\right)\langle i_{e_s}\omega, i_{e_s}\omega\rangle.\end{aligned}$$

Let $H(r) = r \coth r$ and $K(r) = \frac{1}{2p} + \frac{1}{2p}H(br) - H(ar)$. We have

$$H(0) = \lim_{x\to 0} x \coth x = 1$$
$$H'(x) = \frac{(\sinh x)(\cosh x) - x}{\sinh^2 x} \geq 0$$

and $H'(x) = 0$ only when $x = 0$. Differentiating again gives

$$H''(x) = \frac{2(x \coth x - 1)}{\sinh^2 x} \geq 0,$$

and $H''(x) = 0$ only when $x = 0$. Both $H(x)$ and $H'(x)$ are nondecreasing functions. Obviously, $K(0) = \frac{1-p}{p}$

$$K'(r) = \frac{b}{2p}H'(br) - aH'(ar).$$

Hence, in the case of $b \geq 2pa$, $K'(r) \geq 0$ and the equality occurs if and only if $r = 0$. Therefore

$$K(r) > \frac{1-p}{p} = -\delta_1 ,$$

and there exists $r_1(a, b, p)$ with $K(r) > \delta_2 > 0$ if $r \geq r_1$. We assume

$$\delta_2 < \frac{m-2p}{2p}.$$

Therefore,

$$\langle S_\omega, \nabla X\rangle \geq \begin{cases} -\delta_1|\omega|^2 & \text{for } r < r_1 \\ \delta_2|\omega|^2 & \text{for } r \geq r_1 \end{cases}$$

and for any $R > r_1$ and any $x_0 \in M$

$$\begin{aligned} \int_{B_R(x_0)} \langle S_\omega, \nabla X\rangle * 1 &\geq -\delta_1 \int_{B_{r_1}(x_0)} |\omega|^2 + \int_{B_R(x_0)\setminus B_{r_1}(x_0)} \delta_2|\omega|^2 \\ &= \delta_2 \int_{B_R(x_0)} |\omega|^2 - (\delta_1+\delta_2) \int_{B_{r_1}(x_0)} |\omega|^2. \end{aligned} \tag{2.7}$$

If ω is of class L^2,

$$\int_M |\omega|^2 * 1 =: c < \infty.$$

If $|\omega|^2 \not\equiv 0$, then $c > 0$. There exists r_0, such that

$$\int_{M\setminus B_{r_0}(x_0)} |\omega|^2 * 1 < \frac{\delta_2 c}{2(\delta_1+\delta_2)}.$$

Take $x \in M$ with

$$\operatorname{dist}(x, x_0) \geq r_0 + r_1$$

then

$$\int_{B_{r_1}(x)} |\omega|^2 * 1 < \int_{M\setminus B_{r_0}(x_0)} |\omega|^2 * 1 < \frac{\delta_2 c}{2(\delta_1+\delta_2)} \tag{2.8}$$

and for any $R > r_1$, (2.7) and (2.8) imply

$$\int_{B_R(x)} \langle S_\omega, \nabla X\rangle \geq \delta_2 \int_{B_R(x)} |\omega|^2 - \frac{\delta_2}{2}c.$$

Taking R sufficiently large, we have

$$\int_{B_R(x)} \langle S_\omega, \nabla X\rangle * 1 \geq \frac{\delta_2}{3}c. \tag{2.9}$$

For an L^2-harmonic p-form, $d\omega = \delta\omega = 0$, so $\operatorname{div} S_\omega \equiv 0$. Thus, (2.2), (2.3) and (2.9) give

$$\int_{S_R(x)} |\omega|^2 * 1 \geq \frac{2\delta_2 c}{3R}$$

and

$$\int_M |\omega|^2 * 1 \geq \int_{R_0}^{\infty} dR \int_{S_R(x)} |\omega|^2 * 1$$
$$\geq \frac{2}{3}\delta_2 c \int_{R_0}^{\infty} \frac{dR}{R} = \infty.$$

This is a contradiction. So c has to be zero, namely, $\omega \equiv 0$ in the case $m > 2p$. By the L^2-Hodge theorem, then $\mathcal{H}^p(M) = 0$. The same holds for $m < 2p$ by Poincaré duality.

q.e.d.

Remark. If M is Euclidean space $\mathbb{R}^m$ it is easily seen that (2.5) becomes

$$\begin{aligned}\langle S_\omega, \nabla X\rangle =&\frac{m-2p}{2p}\left(\left|i_{\frac{\partial}{\partial r}}\omega\right|^2 + \langle i_{e_s}\omega, i_{e_s}\omega\rangle\right)\\ =&\frac{m-2p}{2}|\omega|^2.\end{aligned}$$

Then, the conclusion follows similarly.

If $p = 1$ we can refine the result as follows.

Theorem 2.2. *Let M be as in Theorem 2.1. If $m > 2$ and $b \geq \sqrt{2}a$, then $B^1_\Gamma(M) = 0$.*

Proof. By L^2-Hodge theory it is only necessary to prove that any L^2-harmomic 1-form ω vanishes. In this case, (2.5) reduces to

$$\begin{aligned}\langle S_\omega, \nabla X\rangle =& \left(\frac{1}{2}\sum_s r\,\mathrm{Hess}\,(r)(e_s,e_s) - \frac{1}{2}\right)\left|i_{\frac{\partial}{\partial r}}\omega\right|^2\\ &+\sum_s\left(\frac{1}{2}+\frac{1}{2}\sum_t r\,\mathrm{Hess}\,(r)(e_t,e_t)\right)\langle i_{e_s}\omega, i_{e_s}\omega\rangle\\ &-\sum_{s,t} r\,\mathrm{Hess}\,(r)(e_s,e_t)\langle i_{e_s}\omega, i_{e_t}\omega\rangle.\end{aligned} \tag{2.10}$$

Choose a local orthonormal frame field $\{e_s\}$ near x in $S_r(x_0)$, such that $\mathrm{Hess}\,(r)$ is diagonalized at x. By parallel translating along the radial geodesics from x_0 we have a local orthonormal frame field in M. We have at x

$$\begin{aligned}\langle S_\omega, \nabla X\rangle =& \left(\frac{1}{2}\sum_s r\,\mathrm{Hess}\,(r)(e_s,e_s) - \frac{1}{2}\right)\left|i_{\frac{\partial}{\partial r}}\omega\right|^2\\ &+\sum_s\left(\frac{1}{2}+\frac{1}{2}\sum_t r\,\mathrm{Hess}\,(r)(e_t,e_t) - r\,\mathrm{Hess}\,(r)(e_s,e_s)\right)\langle i_{e_s}\omega, i_{e_s}\omega\rangle.\end{aligned} \tag{2.11}$$

First of all, by the Hessian comparison theorem

$$\frac{1}{2}\sum_s r \operatorname{Hess}(r)(e_s,e_s) - \frac{1}{2} \geq \frac{m-2}{2} > 0. \tag{2.12}$$

To estimate the coefficients of the second term of (2.11) let

$$A_s = \sum_t \operatorname{Hess}(r)(e_t,e_t) - 2\operatorname{Hess}(r)(e_s,e_s).$$

Since

$$\begin{aligned}\nabla_{\frac{\partial}{\partial r}} \operatorname{Hess}(r)(e_s,e_s) &= \langle \nabla_{\frac{\partial}{\partial r}} \nabla_{e_s}\frac{\partial}{\partial r}, e_s\rangle \\ &= -\langle R(\frac{\partial}{\partial r}, e_s)\frac{\partial}{\partial r}, e_s\rangle + \langle \nabla_{[\frac{\partial}{\partial r}, e_s]}\frac{\partial}{\partial r}, e_s\rangle \\ &= -\langle R(\frac{\partial}{\partial r}, e_s)\frac{\partial}{\partial r}, e_s\rangle - \langle \nabla_{e_s}\frac{\partial}{\partial r}, \nabla_{e_s}\frac{\partial}{\partial r}\rangle,\end{aligned}$$

which gives

$$\frac{d}{dr}(\Delta r) = -\operatorname{Ric}(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}) - |\operatorname{Hess}(r)|^2,$$

we have for $b \geq \sqrt{2}a$

$$\begin{aligned}&\frac{dA_s(r)}{dr} \\ &= (-\operatorname{Ric}(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}) - |\operatorname{Hess}(r)|^2) + 2\langle R(\frac{\partial}{\partial r}, e_s)\frac{\partial}{\partial r}, e_s\rangle + 2\langle \nabla_{e_s}\frac{\partial}{\partial r}, \nabla_{e_s}\frac{\partial}{\partial r}\rangle \\ &\geq b^2 - 2a^2 + 2\langle \nabla_{e_s}\frac{\partial}{\partial r}, \nabla_{e_s}\frac{\partial}{\partial r}\rangle - |\operatorname{Hess}(r)|^2 \\ &\geq 2\langle \nabla_{e_s}\frac{\partial}{\partial r}, \nabla_{e_s}\frac{\partial}{\partial r}\rangle - |\operatorname{Hess}(r)|^2 \\ &= 2\sum_t \operatorname{Hess}(r)(e_s,e_t)\operatorname{Hess}(r)(e_s,e_t) - \sum_{u,v}\operatorname{Hess}(r)(e_u,e_v)\operatorname{Hess}(r)(e_u,e_v) \\ &= (\operatorname{Hess}(r)(e_s,e_s))^2 - \sum_{\substack{u\neq s\\ v\neq s}}\operatorname{Hess}(r)(e_u,e_v)\operatorname{Hess}(r)(e_u,e_v).\end{aligned}$$

Noting that the sectional curvature of M is nonpositive and each $\text{Hess}\,(r)(e_t, e_t) \geq \frac{1}{r} > 0$,

$$
\begin{aligned}
\frac{dA_s(r)}{dr} &\geq (\,\text{Hess}\,(r)(e_s,e_s))^2 - \left(\sum_{t\neq s} \text{Hess}\,(r)(e_t,e_t)\right)^2 \\
&= \left(\text{Hess}\,(r)(e_s,e_s) - \sum_{t\neq s} \text{Hess}\,(r)(e_t,e_t)\right)\Delta r \qquad (2.13)\\
&= -A_s(r)\Delta r.
\end{aligned}
$$

Since $A_s(0) > 0$ because the dimension is at least 3, we may deduce from (2.13) that $A_s(r) > 0$ for all $r > 0$. Altogether, we conclude that

$$\langle S_\omega, \nabla X\rangle \geq \text{ const. } |\omega|^2$$

for a positive constant, and the proof is completed as the one of Thm. 2.1. q.e.d.

As for manifolds with sectional curvature pinched between two negative constants, we can improve a result in [12] as follows.

Theorem 2.3. *Let M be a complete simply connected Riemannian manifold of dimension m with sectional curvature $-a^2 \leq K \leq -b^2$, where a and b are positive constants. Then $B^p_\Gamma = 0 = B^{m-p}_\Gamma(M)$, when $p \neq \frac{m}{2}$ and $b \geq \frac{2p-1}{m-2}a$. Furthermore, let Γ be a discrete compact subgroup of the isometries of M. Then the Euler characteristic satisfies*

$$(-1)^{\frac{m}{2}}\chi(M/\Gamma) \geq 0, \qquad (2.14)$$

provided $\frac{b}{a} \geq \frac{m-3}{m-2}$ and m is even.

Proof. Noting that $\text{Hess}\,(r)(e_s, e_t)$ can be viewed as the second fundamental form of the geodesic sphere of $S_r(x_0)$, the Hessian comparison theorem

$$b\,\coth(br)(g - dr\otimes dr) \leq \text{ Hess}\,(r) \leq a\,\coth(ar)(g - dr\otimes dr) \qquad (2.15)$$

means that the principal curvatures of $S_r(x_0)$ lie in the interval $[b\coth(br), a\coth(ar)]$. Choose a local orthonormal frame field $\{e_s\}$ in $S_r(x_0)$, such that each e_s is a principle direction of $x \in S_r(x_0)$ and $\text{Hess}\,(r)$ is diagonal at x. So, (2.5) reduces to

$$
\begin{aligned}
\langle S_\omega, \nabla X\rangle &= \left(\frac{1}{2p}\sum_s r\Delta r + \frac{1}{2p} - 1\right)\left|i_{\frac{\partial}{\partial r}}\omega\right|^2 \qquad (2.16)\\
&\quad + \sum_s \left(\frac{1}{2p} + \frac{1}{2p}r\Delta r - r\,\text{Hess}\,(r)(e_s,e_s)\right)\langle i_{e_s}\omega, i_{e_s}\omega\rangle.
\end{aligned}
$$

Since

$$\begin{aligned}\frac{1}{2p}\Delta r - \text{ Hess }(r)(e_s, e_s) &= \frac{1}{2p}\sum_{t\neq s} \text{ Hess }(r)(e_t, e_t) - (1-\frac{1}{2p}) \text{ Hess }(r)(e_s, e_s)\\ &\geq \frac{1}{2p}(m-2)k_1 - (1-\frac{1}{2p})k_2\\ &\geq \frac{1}{2p}(m-2)b \coth(br) - (1-\frac{1}{2p})a \coth(ar),\end{aligned}$$

where k_1 and k_2 denote the minimal and maximal principal curvature at the concerned point, (2.16) reduces to

$$\begin{aligned}&\langle S_\omega, \nabla X\rangle\\ &\quad\geq \left(\frac{m-1}{2p}br\coth(br) + \frac{1}{2p} - 1\right)\left|i_{\frac{\partial}{\partial r}}\omega\right|^2\\ &\qquad + \left(\frac{1}{2p} + \frac{m-2}{2p}br\coth(br) - (1-\frac{1}{2p})ar\coth(ar)\right)\sum_s\langle i_{e_s}\omega, i_{e_s}\omega\rangle\\ &\quad\geq \frac{m-2p}{2p}\left|i_{\frac{\partial}{\partial r}}\omega\right|^2\\ &\qquad + \left(\frac{1}{2p} + \left[\frac{m-2}{2p}b - (1-\frac{1}{2p})a\right] r\coth(br)\right)\sum_s\langle i_{e_s}\omega, i_{e_s}\omega\rangle\\ &\quad\geq \delta|\omega|^2\end{aligned}$$

for a certain $\delta > 0$ in the case of $b \geq \frac{2p-1}{m-2}a$ and $2p < m$. (The second inequality above holds true because $\coth x$ is a nonincreasing function.) Then, a similar argument as in the proofs of the preceding theorems leads to $\omega \equiv 0$ for any L^2-harmonic p-form ω when $p \neq \frac{m}{2}$. This proves the first part of the theorem. The second part follows from the Atiyah L^2-index theorem [2] implying

$$\chi(M/\Gamma) = \sum(-1)^i B^i_\Gamma(M).$$

q.e.d.

Remark. In the present notation, the vanishing condition for L^2-Betti numbers in [12] is $b \geq \frac{2p}{m-1}a$, which is more restrictive than that of Theorem 2.3. Furthermore, the condition for (2.14) is comparable to that of [7] and better than the latter when $m \leq 6$.

When $m = 4$ and $-1 \leq K \leq -\frac{1}{4}$, Theorem 2.3 yields $B^1_\Gamma = 0$. It seems that the Dodziuk-Singer conjecture might be true under such a negative $\frac{1}{4}$-pinching condition.

3 Symmetric spaces of noncompact type

The following example may be instructive for understanding those parts of the geometry of symmetric spaces that are relevant for our analysis.

Let $\mathbb{R}^{m+n}_n$ be an $(m+n)$-dimensional pseudo-Euclidean space of index n, namely the vector space $\mathbb{R}^{m+n}$ endowed with the metric

$$ds^2 = (dx_1)^2 + \ldots + (dx_m)^2 - (dx_{m+1})^2 - \ldots - (dx_{m+n})^2.$$

The set of all m-dimensional space-like subspaces constitutes the pseudo-Grassmannian manifold $G^n_{m,n}$ which is the irreducible symmetric space $SO(m,n)/SO(m) \times SO(n)$.

Let $\{e_i, e_\alpha\}$ $(i,j = 1,\ldots,m; \alpha,\beta = m+1,\ldots,m+n; a,b = 1,\ldots,m+n)$ be a local Lorentzian orthonormal frame field in $\mathbb{R}^{m+n}_n$. Let $\{\omega_i, \omega_\alpha\}$ be its dual frame field, so that the metric $g = \sum_i \omega_i^2 - \sum_\alpha \omega_\alpha^2$. The Lorentzian connection forms ω_{ab} of $\mathbb{R}^{m+n}_n$ are uniquely determined by the equations

$$\begin{aligned} d\omega_i &= \omega_{ij} \wedge \omega_j - \omega_{i\alpha} \wedge \omega_\alpha, \\ d\omega_\alpha &= \omega_{\alpha j} \wedge \omega_j - \omega_{\alpha\beta} \wedge \omega_\beta, \\ d\omega_{ab} &= \varepsilon_c \omega_{ac} \wedge \omega_{cb}, \\ \omega_{ab} + \omega_{ba} &= 0, \end{aligned} \tag{3.1}$$

where $\varepsilon_i = 1$, $\varepsilon_\alpha = -1$. The canonical metric on $G^n_{m,n}$ is given by

$$ds^2 = \sum_{i,\alpha} (\omega_{\alpha i})^2. \tag{3.2}$$

From (3.1) and (3.2) it is easily seen that its curvature tensor is

$$R_{\alpha i\,\beta j\,\gamma k\,\delta l} = -\delta_{\alpha\beta}\delta_{\gamma\delta}\delta_{ik}\delta_{jl} - \delta_{\alpha\gamma}\delta_{\beta\delta}\delta_{ij}\delta_{kl} + \delta_{\alpha\beta}\delta_{\gamma\delta}\delta_{il}\delta_{kj} + \delta_{\alpha\delta}\delta_{\beta\gamma}\delta_{ij}\delta_{kl}, \tag{3.3}$$

and the Ricci tensor is

$$R_{\beta j \delta l} = -(m+n-2)\delta_{\beta\delta}\delta_{jl}. \tag{3.4}$$

We thus obtain an Einstein manifold with Ricci curvature $-(m+n-2)$. From (3.3) one can show that the range of the sectional curvature is [-2, 0].

Therefore, in this case, by Theorem 2.1 and Theorem 2.2, when $m+n \geq 6$ the first L^2-Betti number is zero. When $m+n \geq 8p^2+2, p \geq 2$ the p-th L^2-Betti number is zero.

When $n = 2$, $G^2_{m,2}$ belongs to the fourth type of bounded symmetric domains.

Remark. For the symmetric spaces $\mathrm{Sp}\,(m,n)/\,\mathrm{Sp}\,(m)\times\mathrm{Sp}\,(n)$ we can obtain a similar result.

The bounded symmetric domains were classified by E. Cartan. There are altogether 6 types, 4 classical types and two exceptional types.

To prove vanishing theorems for compact quotients of bounded symmetric domains, Calabi-Vesentini [8] defined a self-adjoint linear transformation Q as follows. Choose a local orthogonal Hermitian frame field $\{e_\alpha \bar{e}_\alpha\}$ in a Kähler manifold M, where $e_\alpha \in T^{1,0}M$. Define

$$Q(\xi_{\alpha\beta}) = R_{\gamma\bar{\alpha}\delta\bar{\beta}}\xi_{\alpha\beta}, \quad \xi_{\alpha\beta} = \xi_{\beta\alpha}, \tag{3.5}$$

where $R_{\gamma\bar{\alpha}\delta\bar{\beta}}$ are components of the Riemannian curvature tensor of the Kähler metric in M. It is a linear self-adjoint tranformation on symmetric tensors. All the eigenvalues of Q are real numbers. Calabi-Vesentini calculated all eigenvalues for the four classical types and A. Borel calculated those for the two exceptional types [5]. Let λ_1 be the minimum eigenvalue of Q. Suppose $Z = \xi^\alpha e_\alpha, \sum_\alpha |\xi^\alpha|^2 = 1$. Any holomorphic sectional curvature

$$\begin{aligned}\langle R(Z,\bar{Z})Z,\bar{Z}\rangle &= R_{\alpha\bar{\beta}\gamma\bar{\delta}}\xi^\alpha\bar{\xi}^\beta\xi^\gamma\bar{\xi}^\delta \\ &= \langle Q(\overline{\xi^\beta\xi^\delta}), \overline{\xi^\alpha\xi^\gamma}\rangle \\ &\geq \lambda_1 \sum |\xi^\alpha|^2 \sum |\xi^\beta|^2 = \lambda_1.\end{aligned}$$

For a Kähler manifold with nonpositive sectional curvature [4], the lower bound of the sectional curvature is attained on some holomorphic 2-plane. Hence, from table 1 in [8], we have

Table 3.1:

Type	$\mathbf{dim_R}$	**Sec. Curvature**	**Ric. Curvature**
I_{nm} $(\min(m,n) \geq 2)$	$2mn$	$-2 \leq K \leq 0$	$-(m+n)$
II_m $(m \geq 3)$	$m(m-1)$	$-2 \leq K \leq 0$	$-2(m-1)$
III_m $(m \geq 2)$	$m(m+1)$	$-4 \leq K \leq 0$	$-2(m+1)$
IV_m $(m \geq 2)$	$2m$	$-2 \leq K \leq 0$	$-m$
V	32	$-1 \leq K \leq 0$	-6
VI	54	$-1 \leq K \leq 0$	-9

Remark. For classical bounded domains one can also use the moving frame method to calculate their curvature tensors, as described at the beginning of this section. Note that the curvature table for classical bounded domains in [[20], § 2.4] uses a different normalization.

Using the theorems of the last section and the above table we can list the result as follows.

Table 3.2:

Type	$\mathbf{B}_\Gamma^{\mathbf{1}} = \mathbf{0}$	$\mathbf{B}_\Gamma^{\mathbf{p}} = \mathbf{0}, \mathbf{p} \geq \mathbf{2}$
I_{nm}	$m+n \geq 4$	$m+n \geq 8p^2$
II_m	$m \geq 3$	$m \geq 4p^2+1$
III_m	$m \geq 3$	$m \geq 8p^2-1$
IV_m	$m \geq 4$	$m \geq 8p^2$
V	O.K	
VI	O.K	

4 L²-Betti numbers of semi-simple Lie groups

Let G be a semi-simple Lie group, all of whose simple factors are non-compact. Let   be its Lie algebra of all left invariant vector fields on G and $K \subset G$ the Lie subgroup of G whose image in the adjoint group ad G is a maximal compact subgroup of ad G. Let   be the subalgebra of   corresponding to K and   the orthogonal complement of   in   with respect to the killing form $B(X,Y)$ of . Then

(4.1) $$= \ + \ , \quad [\ ,\ ] \subset \ , \quad [\ ,\ ] \subset \ , \quad [\ ,\ ] \subset \ .$$

It is known that the restriction of B to   (resp.) defines a positive (resp. negative) definite bilinear form on   (resp.). Hence we can choose a base $\{X, \ldots, X_r\}$ of   and a base $\{X_{r+1}, \ldots, X_n\}$ of   with

(4.2) $$\begin{aligned} B(X_i, X_j) &= \delta_{ij}, \\ B(X_\alpha, X_\beta) &= -\delta_{\alpha\beta}; \end{aligned}$$

here and in the sequel we employ the following range of indices

$$\begin{aligned} 1 \leq i, j, k, \ldots &\leq r \\ r+1 \leq \alpha, \beta, \gamma, \ldots &\leq n \\ 1 \leq a, b, c, \ldots &\leq n. \end{aligned}$$

Let

$$[X_a, X_b] = c_{ab}^c X_c.$$

By (4.1), among the structure constants c_{ab}^c, only $c_{\alpha\beta}^\gamma$, c_{ij}^α, $c_{j\alpha}^i$, $c_{\alpha j}^i$ can be $\neq 0$.

Let $B(X,Y)$ be the Killing form of . It is defined by

(4.3) $$B_{ab} = B(X_a, X_b) = \text{ trace}\,(\,\text{ad}\, X_a\, \text{ad}\, X_b) = c_{ae}^f c_{bf}^e.$$

Multiplying the Jacobi identity

$$c^e_{ab}c^f_{ce} + c^e_{ca}c^f_{be} + c^e_{bc}c^f_{ae} = 0$$

by c^c_{df} and summing over the index f, we have

$$c^c_{df}c^e_{ab}c^f_{ce} + c^c_{df}c^e_{ca}c^f_{be} + c^c_{df}c^e_{bc}c^f_{ae} = -c^e_{ab}B_{de} + c^c_{df}c^e_{ca}c^f_{be} + c^c_{df}c^e_{bc}c^f_{ae} = 0.$$

Denoting

$$c^e_{ab}B_{de} = c_{dab}, \tag{4.4}$$

we have

$$\begin{aligned} c_{dab} &= c^c_{df}c^e_{ca}c^f_{be} + c^c_{df}c^e_{bc}c^f_{ae} \\ &= c^f_{de}c^c_{fa}c^e_{bc} + c^c_{df}c^e_{bc}c^f_{ae} \\ &= c^e_{bc}(c^f_{de}c^c_{fa} + c^c_{df}c^f_{ae}), \end{aligned}$$

which is anti-symmetric in a, d. Hence c_{abc} is anti-symmetric in all indices.

(4.2), (4.3) and (4.4) give

$$\sum_{\alpha,k} c_{\alpha ik}c_{\alpha jk} = \frac{1}{2}\delta_{ij} \tag{4.5}$$

and

$$\sum_{i,j} c_{i\alpha j}c_{i\beta j} + \sum_{\gamma,\delta} c_{\gamma\alpha\delta}c_{\gamma\beta\delta} = \delta_{\alpha\beta}. \tag{4.6}$$

Now let $\{\omega^a\}$ be invariant forms dual to $\{X_a\}$. We have the Maurer-Cartan equations

$$d\omega^a = -\frac{1}{2}c^a_{be}\omega^b \wedge \omega^c. \tag{4.7}$$

We define the Riemannian metric ds^2 on G by

$$ds^2 = \sum(\omega^a)^2. \tag{4.8}$$

Let ω be a harmonic 1-form and

$$\omega = s_i\omega^i + s_\alpha\omega^\alpha. \tag{4.9}$$

From Section 3 in [16], we know that the s_α are constant.

We recall that we assume that ω is an L^2-harmonic 1-form. Since

$$|\omega|^2 = \sum_i s_i^2 + \sum_\alpha s_\alpha^2,$$

each constant s_γ should be zero. Furthermore,

$$\begin{aligned} X_\gamma \sum s_i^2 &= 2s_i X_\gamma s_i \\ &= 2s_i(X_i s_\gamma + c_{\gamma i}^k s_k) \\ &= 2c_{\gamma i}^k s_i s_k = 2c_{k\gamma i} s_i s_k = 0, \end{aligned}$$

noting that the c_{abc} are anti-symmetric in all indices.

In summary, we have shown that

$$\omega = s_i \omega^i$$

and

$$|\omega|^2 = \sum_i s_i^2$$

only depends on G/K.

There exists a G invariant metric on G/K, such that the quotient map $\pi : G \longrightarrow G/K$ is a Riemannian submersion with totally geodesic fibers. In this terminology, any L^2 1-form on G is a horizontal form and $|\omega|^2$ can be considered as a function on G/K.

Take any unit vector field n in G/K and any function b in G/K. We have the horizontal lift X of bn. X is the normal vector field of the fiber submanifold whose length is constant along the fibers. Choose an orthonormal frame field $\{\bar{e_i}\}$ in G/K, and call its horizontal lift $\{e_i\}$. $\{e_\alpha\}$ is an orthonormal frame on the fiber. Thus, $\{e_i, e_\alpha\}$ is an orthonormal frame field on G. Therefore $\langle \nabla_{e_\alpha} X, e_\alpha \rangle$ is a multiple of the mean curvature with respect to the normal direction n. It is zero since the fibers are totally geodesic. $\operatorname{div} X$ can be computed in the base manifold G/K. Since ω is horizontal,

$$\langle \omega \odot \omega, \nabla X \rangle = s_i s_j \langle \nabla_{e_i} X, e_j \rangle.$$

$\langle \nabla_{e_i} X, e_j \rangle$ also descends to G/K. Hence

$$\langle S_\omega, \nabla X \rangle = \frac{1}{2} |\omega|^2 \operatorname{div} X - \langle \omega \odot \omega, \nabla X \rangle$$

can be computed in G/K, provided X is of the above type.

We are now in a position to prove the following result.

Theorem 4.1. *Let G be a simple non-compact Lie group with center reduced to $\{e\}$, and K its maximal compact subgroup. Suppose that the center of K is not finite. Then $B^1_\Gamma(G)=0$, provided the dimension of G/K satisfies the corresponding conditions in Table 3.2.*

Proof. Choose $\bar{D}=\bar{B}_R(x_0)$, a geodesic ball in G/K of radius R with center in $x_0\in G/K$. Its boundary is a geodesic sphere $S_R(x_0)$ in G/K. Let $D=\pi^{-1}(\bar{D})\subset G$. ∂D is compact since K is compact. Let $\bar{X}=r\frac{\partial}{\partial r}$, which is a smooth vector field in G/K. Let X be the horizontal lift of $\bar{X}$. Since the fiber submanifold is orthogonal to the horizontal vector field, X is also a normal vector field on ∂D. Its length is equal to r. Thus, for any L^2-harmonic 1-form ω, we also have

$$\begin{aligned}\int_{\partial D}\frac{1}{2}|\omega|^2\langle X,n\rangle*1-\int_{\partial D}\langle i_X\omega,i_n\omega\rangle*1\\ =\int_{\partial D}\frac{1}{2}R|\omega|^2*1-\int_{\partial D}R\langle i_{\frac{\partial}{\partial r}}\omega,i_{\frac{\partial}{\partial r}}\omega\rangle\leq\frac{1}{2}R\int_{\partial D}|\omega|^2*1.\end{aligned}\tag{4.10}$$

From the previous discussion of this section, $\langle S_\omega,\nabla X\rangle$ can be computed in the base manifold G/K. On the other hand, G/K is a bounded symmetric domain and hence satisfies a curvature pinching condition. By the proof of Theorem 2.2, if $|\omega|\not\equiv 0$, then there exist $R_0>0$ and $c>0$, such that for $R>R_0$ and $D=\pi^{-1}(B_R(x_0))$,

$$\int_D\langle S_\omega,\nabla X\rangle*1\geq c.\tag{4.11}$$

From (2.2), (4.10) and (4.11), we obtain

$$\int_{\partial D}|\omega|^2*1\geq\frac{2c}{R}$$

and

$$\int_G|\omega|^2*1\geq\int_{R_0}^\infty dR\int_{\partial D}|\omega|^2*1=\infty$$

which contradicts the L^2-assumption. q.e.d.

Remark. The above result is not the most general one that can be obtained with our method. It just has been selected to demonstrate the typical features of our approach. In the same manner, we may obtain corresponding results for $SO(m,n)$ and $\mathrm{Sp}(m,n)$.

References

[1] M. T. Anderson. L^2 harmonic forms and a conjecture of Dodziuk-Singer. *Bull. A.M.S.*, 13(2):163–165, 1985.

[2] M. F. Atiyah. Elliptic operators, discrete groups and Von Neumann algebras. *Asterisque*, 32-33:43–72, 1976.

[3] P. Baird and J. Eells. A conservation law for harmonic maps. *Lect. Notes Math.* , 894 Springer:1–15, 1981.

[4] R. L. Bishop and S. I. Goldberg. Some implication of the generalized Gauss-Bonnet theorem. *Trans. A.M.S.*, 112:508–535, 1964.

[5] A. Borel. On the curvature tensor of the Hermitian symmetric manifolds. *Ann. Math.*, 71(3):508–521, 1960.

[6] A. Borel. The L^2-cohomology of negatively curved Riemannian symmetric spaces. *Ann. Acad. Sci. Fennicae*, 10:95–105, 1983.

[7] J. P. Bourguignon and H. Karcher. Curvature operators: Pinching estimates and geometric examples. *Ann. scient. Éc. Norm. Sup.* , 11:71–92, 1978.

[8] E. Calabi and E. Vesentini. On compact, locally symmetric Kähler manifolds. *Ann. Math.* , 71(3):412–507, 1960.

[9] J. Dodziuk. De Rham-Hodge theory for L^2 cohomology of infinite coverings. *Top.*, 16:157–165, 1977.

[10] J. Dodziuk. L^2 harmonic forms on rotationally symmetric Riemannian manifolds. *Proc. A.M.S.*, 11:395–400, 1979.

[11] H. Donnelly and C. Fefferman. L^2-cohomology and index theorems for Bergman metrics. *Ann. Math.*, 118:593–618, 1983.

[12] H. Donnelly and F. Xavier. On the differential form spectrum of negatively curved Riemannian manifolds. *Amer. J. Math*, 106:169–185, 1984.

[13] R. Geroch. Positive sectional curvature does not imply positive Gauss-Bonnet integrand. *Proc. A.M.S.*, 54:267–270, 1976.

[14] M. Gromov. Kähler hyperbolicity and L^2-Hodge theory. *J. Diff. Geom.*, 33:263–292, 1991.

[15] J. Jost and K. Zuo. Vanishing theorems for L^2-cohomology on infinite coverings of compact Kähler manifolds and applications in algebraic geometry. Preprint.

[16] Y. Matsushima. On the first Betti number of compact quotient spaces of higher-dimensional symmetric spaces. *Ann. Math.*, 25(2):312–330, 1962.

[17] P. Pansu. Introduction to L^2 Betti numbers.

[18] I. Singer. Some remarks on operator theory and index theory. *Lect. Notes Math.*, 575 Springer:128–138, 1977.

[19] M. Stern. L^2-cohomology of negatively curved Kähler manifolds, 1989. Preprint.

[20] Y. L. Xin. *Geometry of harmonic maps.* Birkhäuser, Boston-Basel-Berlin, 1996. PNLDE 23.