Variational Problems from Physics and Geometry

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1. Background and motivation

In the semiclassical approach (i.e. without field quantization) to particle physics, a particle is represented by a vector bundle V with certain geometric structures over a fixed space time (M, g), a Lorentzian manifold (we shall usually use here a Riemannian manifold (M, g), however, in order to arrive at elliptic variational problems). The geometry includes the field equations for sections φ of V, the (semiclassical) states of the particle. A prototype of such an equation is

$$(\Delta - m^2)\varphi = 0 \,, \tag{1}$$

where Δ is the Laplacian w.r.t. a fixed connection ∇^0 on V and m is the mass of the particle (we are ignoring physical constants like c, $\hbar$).

Typically, however, the particle is influenced by an external field, for example an electromagnetic one. Such an electromagnetic field is represented by a complex line bundle L over (M, g), equipped with a Hermitian metric h; we often write $h(v, w) =: \langle v, w\rangle$. States of the field are represented by Hermitian connections ∇ on L. We define the local potential A of the field by

$$\nabla\xi = d\xi - iqA\xi \tag{2}$$

for a section ξ in local coordinates (d is the exterior derivative, and q is the charge of the particle).

$$F = F_A = dA \tag{3}$$

is the electromagnetic field; we have

$$F = -\frac{i}{q}R^\nabla \quad (R^\nabla = \text{ curvature of } \nabla) \,. \tag{4}$$

The interacting particle bundle is then $L \otimes V$, and choosing a local unit section λ of L (i.e. $\langle\lambda, \lambda\rangle \equiv 1$), we identify a section φ of V with the section $\lambda \otimes \varphi$ of $L \otimes V$. On $L \otimes V$, we have the connection $D_A := \nabla \otimes \nabla^0$.

The Laplacian in (1) now has to be replaced by the Laplacian of the connection D_A. In addition, we also need a field equation for the connection. In fact, both field equations can be derived from a variational principle. We simply use the Lagrangian

$$\mathcal{L}(\varphi, A) := \int \big(\gamma_1 \left|F_A\right|^2 + \gamma_2(\left|D_A\varphi\right|^2 + m^2 \left|\varphi\right|^2)\big) d\,\mathrm{vol}(g) \,. \tag{5}$$

(F_A is a 2-form on M with values in $i\mathfrak{u}(1) \cong \mathbb{R}$, and the metric g on M therefore allows us to define $|F_A|^2$; on V, we also need a fiber metric $\langle \cdot, \cdot \rangle$ in order to define $|\varphi|^2$ and $|D_A\varphi|^2$, the latter one needing in addition the metric g.) γ_1 and γ_2 are (positive) constants. By normalization, we may put one of them $= 1$.

The Euler–Lagrange equations for $\mathcal{L}$ are

$$\Delta_A \varphi - m^2 \varphi = 0 \qquad (\Delta_A = -D_A^* D_A) \tag{6}$$

$$\gamma_1 D_A^* F_A = -\frac{1}{2} \gamma_2 \operatorname{Im} \langle D_A \varphi, \varphi \rangle \ . \tag{7}$$

The same procedure may be applied to nonabelian fields; one simply replaces the line bundle L by a bundle with fiber $\mathbb{C}^n$ ($n \geq 2$), again equipped with a Hermitian metric $\langle \cdot, \cdot \rangle$. F_A now becomes a 2-form on M with values in $i\mathfrak{u}(n)$; on $i\mathfrak{u}(n)$ we have the product $\phi_1 \cdot \phi_2 := tr \phi_1 \phi_2$, and so $|F_A|^2$ can be defined as before. The essential difference to the preceding situation, however, is that (7), while formally the same as before, becomes a nonlinear equation for A, the so-called coupled Yang–Mill equation.

Finally, in more realistic physical models, the particle φ is self interacting. This requires replacing the term $m^2 |\varphi|^2$ in (5) by a more general term $V(\varphi)$. An important example of a corresponding Lagrangian is the Yang–Mills–Higgs functional

$$\mathcal{L}(\varphi, A) := \int_M \left(\gamma_1 |F_A|^2 + \gamma_2 |D_A \varphi|^2 + \gamma_3 (1 - |\varphi|^2)^2 \right) d \operatorname{vol}(g) \ . \tag{8}$$

Functionals of this type also arise in Ginzburg–Landau type theories (e.g. in superconductivity) where φ represents a so-called order parameter describing the symmetry of a state. If φ is real valued, such functionals also serve to understand phase transitions along interfaces.

In (8), the potential term $V(\varphi)$ is a polynomial in $|\varphi|$ of order 4; in other examples, also polynomials of order 6 occur frequently. A good reference for the preceding is [De].

The geometry of connections and sections and the resulting variational problems have also found important mathematical applications. Best known are probably Donaldson's applications of antiselfdual Yang–Mills connections to the differential topology of four-manifolds, and the more recent Seiberg–Witten equations that are a special case of coupled abelian Yang–Mills equations and that will be stated in more detail below.

Other mathematical applications arise in connection with questions of algebraic geometry, and we would like to briefly indicate in some special cases how these lead to variational problems of the type considered above.

Let M be a (compact) Kähler manifold with Kähler metric g, and let

$$\rho : \pi_1(M) \to U(n)$$

be a unitary representation of the fundamental group of M. Such a representation ρ defines a flat unitary vector bundle E on M, i.e. a bundle with fiber $\mathbb{C}^n$, a Hermitian metric $\langle \cdot, \cdot \rangle$ and a unitary flat connection D (i.e. $d\langle v, w\rangle = \langle Dv, w\rangle + \langle v, Dw\rangle$ for all local sections and $F = D^2 = 0$). If D is irreducible, it defines a stable holomorphic structure on E. Conversely, if a topological vector bundle with vanishing first and second Chern classes admits a stable holomorphic structure ("stability" roughly means that for every subbundle F of E

$$\mu(F) < \mu(E) \, , \tag{9}$$

with

$$\mu(F) := \frac{1}{rk(F)} \int_M c_1(F) \wedge \omega^{n-1} \, , \quad \omega := \text{ Kähler form of } \quad M \, , \qquad n := \dim_{\mathbb{C}} M \;) \, , \tag{10}$$

then by the work of Narasimhan–Seshadri [NS], Donaldson [D1], [D2], and Uhlenbeck–Yau [UY], it admits a flat unitary connection, i.e. the PDE

$$F = 0 \tag{11}$$

can be solved. This is of great help for understanding the moduli spaces of stable bundles.

If one considers more generally homomorphisms $\rho : \pi_1(M) \to G(= G\ell(n, \mathbb{C})$ or some other linear algebraic group), one is led to equations of the type

$$\bar{\partial}_A \varphi = 0 \tag{12}$$

$$F_A + [\varphi, \varphi^*] = 0 \tag{13}$$

where D_A $(= \partial_1 + \bar{\partial}_1)$ now is a connection on a principal G-bundle (or equivalently, on a vector bundle E with structure group G) over M, and φ is a $(1, 0)$ form on M with values in $\mathfrak{g}^{\mathbb{C}}$, the complexification of the Lie algebra of G. φ is called Higgs field. These connections have been studied by Hitchin, Simpson, Corlette, and others ([Hi], [Si1], [Si2], [C1], [C2],..). These equations can also be arrived at from a somewhat different point of view. Let M be a (compact) Riemannian manifold, and let

$$\rho : \pi_1(M) \to G = G\ell(n, \mathbb{C})$$

be again a representation. ρ defines a flat vector bundle E on M with structure group G. Let D be the resulting flat connection. If h is a Hermitian metric on E, we obtain a decomposition

$$D = D_h + \vartheta$$

where D_h is a connection that is unitary w.r.t. h, and ϑ is a 1-form on M with values in the selfadjoint endomorphisms of E. Since D is flat, we have

$$F_h + \frac{1}{2}[\vartheta, \vartheta] = 0 \qquad (F_h = \text{ curvature of } D_h) \tag{14}$$

(15) $$D_h \vartheta = 0 \ .$$

In fact,

$$\vartheta = df \ ,$$

where $f : M \to G/K$, K = maximal compact subgroup of G, is a map from M into a symmetric space, defining the Hermitian metric h. This map f is a harmonic map iff

(16) $$D_h^* \vartheta = 0 \ .$$

A metric h satisfying this equation is called harmonic. This setting has been studied by Corlette, Simpson, Zuo, Jost–Zuo, and others ([C1], [C2], [Si1], [Si2], [Z1], [Z2], [JZ1], [JZ2],...).

The equations (11)–(16) are first order equations for the objects A, φ, ϑ, in contrast to (6), (7) that are second order. The antiselfduality equations used by Donaldson and the Seiberg–Witten equations are likewise first order equations. We now wish to explain that by appropriate choices of the parameters involved, variational problems of the type considered can lead to first order equations for the absolute minima.

As a simple example we first consider harmonic maps from a Riemann surface Σ to a Riemannian manifold (N, g). In general, for a map

$$f : (M, \gamma) \to (N, g)$$

between two Riemannian manifolds, the energy of f is defined as

(17) $$E(f) = \int_M \|df(x)\|^2 \, d\,\mathrm{vol}(\gamma)(x) \ ,$$

with

(18) $$\|df(x)\|^2 = \gamma^{\alpha\beta}(x) g_{ij}\big(f(x)\big) \frac{\partial f^i}{\partial x^\alpha} \frac{\partial f^j}{\partial x^\beta}$$

in local coordinates, in usual tensor notation, and

(19) $$d\,\mathrm{vol}(\gamma)(x) = \det(\gamma_{\alpha\beta})^{1/2} dx \ .$$

Smooth critical points are called harmonic and satisfy the nonlinear elliptic system

(20) $$\Delta_M f^i + \gamma^{\alpha\beta}(x) \Gamma^i_{jk}\big(f(x)\big) \frac{\partial f^j}{\partial x^\alpha} \frac{\partial f^k}{\partial x^\beta} = 0 \ , \qquad i = 1, \dots, \dim N \ ,$$

with Δ_M the Laplace–Beltrami operator of M and Γ^i_{jk} the Christoffel symbols of N. Harmonic maps were first studied by Al'ber [A1], [A2] and Eells–Sampson [ES]. A general reference is [J5].

2. Selfduality

If Σ is a Riemann surface with a conformal metric $\lambda^2(z)dz\,d\bar z$, we have

$$
\begin{aligned}
E(f) &= \int_\Sigma \lambda^{-2}(z) g_{ij}\big(f(z)\big)\left(f_z^i f_z^j + f_{\bar z}^i f_{\bar z}^j\right)\lambda^2(z)\frac{i}{2}dz\wedge d\bar z \\
&= \int_\Sigma g_{ij}\big(f(z)\big)\left(f_z^i f_z^j + f_{\bar z}^i f_{\bar z}^j\right)\frac{i}{2}dz\wedge d\bar z
\end{aligned} \tag{21}
$$

Thus, $E(f)$ is independent of the choice of a conformal metric on Σ. In other words, the energy is a conformally invariant functional for maps from a Riemann surface. We abbreviate the preceding expression as

$$
E(f) = \int_\Sigma \left(|\partial f|^2 + |\bar\partial f|^2\right)\frac{i}{2}dz\wedge d\bar z\ . \tag{22}
$$

If we have a map f of degree d between compact Riemann surfaces Σ, Σ', then

$$
\int_\Sigma \left(|\partial f|^2 - |\bar\partial f|^2\right)\frac{i}{2}dz\wedge d\bar z = \int J(f)\frac{i}{2}dz\wedge d\bar z = d\int_\Sigma \frac{i}{2}dz\wedge d\bar z \tag{23}
$$

is a topological invariant ($J(f) =$ Jacobian of f). Therefore

$$
\begin{aligned}
E(f) &= 2\int_\Sigma |\bar\partial f|^2\,\frac{i}{2}dz\wedge d\bar z + d\int_\Sigma \frac{i}{2}dz\wedge d\bar z, \\
&\ge d\int_\Sigma \frac{i}{2}dz\wedge d\bar z\ .
\end{aligned} \tag{24}
$$

This lower bound is achieved iff

$$
\bar\partial f = 0\ , \tag{25}
$$

i.e. if f is holomorphic. (25) is a first order equation, in contrast to (20). Since absolute minima are critical points, solutions of (25) have to be harmonic, i.e. have to solve here an equation of the form

$$
f_{z\bar z} + \Gamma\big(f(z)\big) f_z f_{\bar z} = 0\ .
$$

Of course, one sees also directly that (25), i.e. $f_{\bar z} = 0$, implies (26). (In case $d < 0$, the bound in (24) can never be achieved, since always $E(f) \ge 0$, and absolute minima then solve $f_z = \partial f = 0$ instead.)

Let D be a $\mathfrak{su}(n)$ connection in a vector bundle E over a Riemannian manifold (M,g) of dimension n with curvature $-iF = D^2$. Then F is a 2-form with values in $i\mathfrak{su}(n)$, hence $tr(F) = 0$, and the 2nd Chern class of E is given by

$$
c_2(E) = \frac{-1}{8\pi^2} tr(F\wedge F)\ . \tag{27}
$$

We decompose F into its selfdual and antiselfdual parts,

$$F = F^+ + F^- \ , \tag{28}$$

with

$$-iF^+ = *(-iF^+), -iF^- = - * (iF^-)$$

($* =$ Hodge star operator defined by the metric g, mapping 2-forms to 2-forms since $\dim M = 4$.)

Then

$$\begin{aligned} tr(F \wedge F) &= tr(F^+ \wedge F^+) + tr(F^- \wedge F^-) \\ &= tr(F^+ \wedge *F^+) - tr(F^- \wedge *F^-) \\ &= \left|F^+\right|^2 - \left|F^-\right|^2 \ , \end{aligned} \tag{29}$$

hence

$$c_2(E)[M] = \frac{1}{8\pi^2} \int_M \left(\left|F^-\right|^2 - \left|F^+\right|^2\right) d\,\mathrm{vol}(g) \ . \tag{30}$$

Thus, the Yang–Mills functional

$$YM(D) = \int_M |F|^2 \, d\,\mathrm{vol}(g) = \int_M \left(\left|F^+\right|^2 + \left|F^-\right|^2\right) d\,\mathrm{vol}(g) \tag{31}$$

is minimized, if

$$F^+ = 0 \quad \text{or} \quad F^- = 0$$

(depending on the sign of $c_2(E)[M]$), i.e. if

$$F = F^- \quad \text{or} \quad F = F^+$$

i.e. if

$$F = \pm * F \ , \tag{32}$$

i.e. if F is selfdual or antiselfdual which is again a 1st order PDE for D.

Let Σ be a Riemann surface with a conformal metric g, L a Hermitian line bundle over Σ, with metric $\langle \cdot, \cdot \rangle$. We recall the Ginzburg–Landau functional for a section φ of L and a unitary connection A on L,

$$\mathcal{L}(\varphi, A) = \int_\Sigma \left(|dA|^2 + |D_A \varphi|^2 + \frac{1}{4}(1 - |\varphi|^2)^2\right) d\,\mathrm{vol}(g) \ . \tag{33}$$

We decompose D_A into its $(1,0)$ and $(0,1)$ parts,

$$D_A = \partial_A + \bar{\partial}_A \ . \tag{34}$$

If $\Sigma = \mathbb{R}^2$, we may rewrite

$$\mathcal{L}(\varphi, A) = \int_{\mathbb{R}^2} \Big(2 \left| \bar{\partial}_A \varphi \right|^2 + \big(*F - \frac{1}{2}(1 - |\varphi|^2) \big)^2 \Big) dx + 2\pi d \,, \tag{35}$$

for an integer d, the so-called vortex number. In case $d \geq 0$, the absolute minimum therefore is realized by solutions of the first order equations

$$\bar{\partial}_A \varphi = 0 \tag{36}$$

$$*F = \frac{1}{2}(1 - |\varphi|^2) \,. \tag{37}$$

Taubes [T1] showed that for any collection of d points $x_j \in \mathbb{R}^2$, possibly with multiplicities, there exists a solution (φ, A) of these vortex equations with

$$\varphi(x_j) = 0 \qquad \text{for} \quad j = 1, \ldots, d \,.$$

This solution is unique up to gauge equivalence, i.e. up to transformations of the form

$$(\varphi, A) \to (\varphi e^{i\rho}, A + d\rho) \tag{38}$$

for a real valued function ρ.

Similarly, if Σ is a compact Riemann surface,

$$\mathcal{L}(\varphi, A) = \int_{\Sigma} \Big(2 \left| \bar{\partial}_A \varphi \right|^2 + \big(*F - \frac{1}{2}(1 - |\varphi|^2) \big)^2 \Big) d \operatorname{vol}(g) + 2\pi \deg L \,, \tag{39}$$

and the absolute minima satisfy the same first order equations as before.

In order to see the equivalence between (39) and (33), one computes

$$\begin{aligned} &\int \big(*F - \frac{1}{2}(1 - \langle \varphi, \varphi \rangle) \big)^2 d \operatorname{vol}(g) \\ &\qquad = \int \Big(|F|^2 + \frac{1}{2}\big(1 - |\varphi|^2\big)^2 - *F + *F \langle \varphi, \varphi \rangle \Big) d \operatorname{vol}(g) \end{aligned}$$

and observes

$$\int *F d \operatorname{vol}(g) = 2\pi \deg L$$

$$\begin{aligned} \int \langle *F\varphi, \varphi \rangle d \operatorname{vol}(g) &= \int \langle *(d - iA)(d - iA)\varphi, \varphi \rangle d \operatorname{vol}(g) \\ &= \int \langle *(\partial_A + \bar{\partial}_A)(\partial_A + \bar{\partial}_A)\varphi, \varphi \rangle d \operatorname{vol}(g) \\ &= \int \big(|\partial_A \varphi|^2 - \left| \bar{\partial}_A \varphi \right|^2 \big) d \operatorname{vol}(g) \,, \end{aligned}$$

using the commutation rules between $*$ and $\partial_A, \bar{\partial}_A$, and integrating by parts.

In general, a critical point of our Ginzburg–Landau functional satisfies the 2nd order equations

(40) $$\Delta_A \varphi = -\frac{1}{2}(1 - |\varphi|^2)\varphi \qquad (\Delta_A = -D_A^* D_A)$$

(41) $$d^* F = -Im\langle D_A \varphi, \varphi \rangle \ .$$

One further remark: the term $(1 - |\varphi|^2)$ may be replaced by $(\lambda - |\varphi|^2)$, for $\lambda > 0$, without changing the essential structure of the equations. This becomes a little different, however, if we take $\lambda = 0$, because then the zero section becomes a prefered state for the functional in contrast to the aim of creating solutions with a zero set that is as small as possible.

We finally wish to discuss selfduality for the Seiberg–Witten functional which we now introduce ([SW1], [SW2], [W]).

Let (M, g) be a four-dimensional, oriented, compact Riemannian manifold. We recall that the universal covering of $SO(4)$ is Spin(4) $\cong SU(2) \times SU(2)$, and if the second Stiefel-Whitney class $w_2(M) = 0$, the bundle of oriented orthonormal frames of M, an $SO(4)$ principal bundle, can be lifted to a Spin(4) principle bundle, a so-called spin structure. To such a spin structure, we may associate a complex vector bundle S of rank 4 with structure group Spin(4), with a decomposition

$$S = S^+ \oplus S^-$$

corresponding to the above decomposition of Spin(4). Tangent vectors of M act on S by Clifford multiplication:

$$\sigma : TM \otimes S^\pm \to S^\pm$$
$$x \otimes \varphi \mapsto x \cdot \varphi \ .$$

The Levi–Civita connection Δ of g induces a connection Δ^S on S, and the Dirac operator is defined as

$$\mathcal{D} = \sigma \circ \nabla^S : \Gamma(S) \xrightarrow{\nabla^S} \Gamma(T^*M \otimes S) \cong \Gamma(TM \otimes S) \xrightarrow{\sigma} \Gamma(S) \ ,$$

where the identification between $\Gamma(T^*M \otimes S)$ and $\Gamma(TM \otimes S)$ uses the Riemannian metric g.

In fact $\mathcal{D} = \sum_{i=1}^4 e_i \cdot \nabla^S$ for an ONB basis of TM. We have

$$\mathcal{D} : \Gamma(S^\pm) \to \Gamma(S^\mp) \ .$$

If $w_2(M) \neq 0$, we need to use a spin$^{\mathbb{C}}$ structure instead, where

$$\mathrm{Spin}^{\mathbb{C}}(4) = \mathrm{Spin}(4) \times U(1)/\mathbb{Z}_2 = \{(A, B) \in U(2) \times U(2) : \det A = \det B\} \ ,$$

with similar constructions, with objects like $S^\pm$ now only being locally defined. Let now L be the complex line bundle $\det(S^+)$ over M ($c_1(L) = w_2(M) \pmod 2$). We put

$$W^\pm = S^\pm \otimes L^{1/2}$$

($L^{1/2}$ is in general only locally defined, but $W^\pm$ is globally defined.) We have an induced Clifford multiplication

$$TM \otimes W^+ \to W^- \ .$$

If A is connection on L, we get a "twisted" Dirac operator

$$\mathcal{D}_A : \Gamma(W^+) \to \Gamma(W^-) \ ,$$

(42)
$$\mathcal{D}_A = \sum_{i=1}^{4} e_i \cdot \nabla_A \ ,$$

where ∇_A is induced by the Levi–Civita connection ∇^S on S and the connection A on L. The curvature of A is again written as $-iF_A$. We let F_A^+ again be the selfdual part of F_A.

The Seiberg–Witten functional is defined for connections A on L and sections φ of W^+,

(43)
$$SW(\varphi, A) := \int_M \left(|\nabla_A \varphi|^2 + \left|F_A^+\right|^2 + \frac{S}{4} |\varphi|^2 + \frac{1}{8} |\varphi|^4 \right) d\,\mathrm{vol}(g) \ ,$$

where S is the scalar curvature of M. The corresponding Euler–Lagrange equations are

(44)
$$\Delta_A \varphi = \frac{S}{4}\varphi + \frac{1}{4} |\varphi|^2 \varphi$$

(45)
$$d^* F_A^+ = -\frac{1}{2} Im \langle \nabla_{A, e^k} \varphi, \varphi \rangle e^k$$

(a summation convention is used, here and in the sequel). Note that (45) is linear in A, because A is a connection on a line bundle with abelian structure group. Again, the functional SW may be rewritten in such a way as to exhibit selfduality, namely using the Weitzenböck formula

(46)
$$\mathcal{D}_A^* \mathcal{D}_A \varphi = -\Delta_A \varphi + \frac{S}{4}\varphi + \frac{i}{2} F_A^+ \cdot \varphi$$

(here, "$\cdot$" denotes again a Clifford multiplication; more precisely, if $F_A = i\omega_{jk} e^j \wedge e^k$, then $F_A \cdot \varphi = i\omega_{jk} e_j \cdot e_k \cdot \varphi$), one obtains

(47)
$$SW(\varphi, A) = \int_M \left(|\mathcal{D}_A \varphi|^2 + \left| F_A^+ - \frac{i}{4} \langle e_j \cdot e_k \cdot \varphi, \varphi \rangle e^j \wedge e^k \right|^2 \right) d\,\mathrm{vol}(g)$$

($(e^j)_{j=1,\dots,4}$ is the basis of T^*M dual to $(e_j)_{j=1,\dots,4}$). Absolute minima now are given by the solutions of the Seiberg–Witten-equations

$$\mathcal{D}_A\varphi = 0 \tag{48}$$

$$F_A^+ = \frac{i}{4}\langle e_j \cdot e_k \cdot \varphi, \varphi\rangle e^j \wedge e^k \,. \tag{49}$$

In order to verify (47), we write

$$\begin{aligned} \left|F_A^+ - \frac{i}{4}\langle e_j \cdot e_k \cdot \varphi, \varphi\rangle e^j \wedge e^k\right|^2 &= \left|F_A^+\right|^2 + \frac{1}{16}\left|\langle e_j \cdot e_k \cdot \varphi, \varphi\rangle e^j \wedge e^k\right|^2 \\ &\quad + \frac{i}{2} F_A^+ \langle e_j, e_k\rangle \langle e_j \cdot e_k \cdot \varphi, \varphi\rangle \\ &= \left|F_A^+\right|^2 + \frac{1}{8}\left|\varphi\right|^4 + \frac{i}{2}\langle F_A^+ \cdot \varphi, \varphi\rangle \,, \end{aligned} \tag{50}$$

while (46) yields

$$\int \left|\mathcal{D}_A\varphi\right|^2 = \int \left(\left|\nabla_A\varphi\right|^2 + \frac{S}{4}\left|\varphi\right|^2 - \frac{i}{2}\langle F_A^+ \cdot \varphi, \varphi\rangle\right) , \tag{51}$$

and (47) follows from (43), (50), (51).

Sometimes it is useful to consider, instead of (48), (49), the perturbed equations

$$\mathcal{D}_A\varphi = 0 \tag{48}$$

$$F_A^+ = \frac{i}{4}\langle e_j \cdot e_k \cdot \varphi, \varphi\rangle e^j \wedge e^k - \mu \tag{49μ}$$

for some fixed, real valued, selfdual 2-form μ on M. The reason is similar to the one for prefering $\lambda > 0$ to $\lambda = 0$ in the modified Ginzburg–Laudau equations above, namely in order to avoid $\varphi \equiv 0$ as a solution.

3. Scaling invariance

Let us return to the Ginzburg–Landau functional (33). This functional originated in the theory of superconductivity, and there, φ represents a complex order parameter. $|\varphi(x)| = 1$ corresponds to a superconducting phase, and therefore, one wishes to constrain φ to have absolute value 1. However, there are topological obstructions for that: if d in (35) or $\deg L$ in (39) is $\neq 0$, then any section φ of L will have at least $|d|$ zeroes (counted with multiplicities), the so-called vortices. Therefore, one tried to penalize $|\varphi| \neq 1$ by giving a stronger weight to the term $(1-|\varphi|^2)^2$ in $\mathcal{L}(\varphi, A)$ and considered the family of functionals

$$\tilde{\mathcal{L}}_\varphi(\varphi, A) := \int_\Sigma \Big(|dA|^2 + |D_A\varphi|^2 + \frac{1}{4\varepsilon^2}(1-|\varphi|^2)^2\Big) d\,\mathrm{vol}(g)$$

for $\varepsilon > 0$ and investigated the limiting behavior for $\varepsilon \to 0$. The first mathematical treatment is due to Berger–Chen [BC] who obtained very explicit results in the class of rotationally symmetric solutions. More recently, the general case was studied in detail by Bethuel–Rivière [BR], Qing [Q] and Orlandi [O]. A disadvantage of $\tilde{\mathcal{L}}_\varepsilon$, however, is that for $\varepsilon \neq 1$, selfduality is lost. Also, in general

$$\lim_{\varepsilon\to 0} \inf_{(\varphi,A)} \tilde{\mathcal{L}}_\varepsilon(\varphi, A) = \infty \quad \text{in case} \quad d \neq 0(\deg L \neq 0)\ .$$

Therefore, in work of Hong–Jost–Struwe [HJS], the functional

$$\mathcal{L}_\varepsilon(\varphi, A) := \int_\Sigma \big(\varepsilon^2 |dA|^2 + |D_A\varphi|^2 + \frac{1}{4\varepsilon^2}(1-|\varphi|^2)^2\big) d\,\mathrm{vol}(g) \tag{52}$$

was studied instead which preserves selfduality. The Euler–Lagrange equations for (52) are

$$\Delta_A\varphi = -\frac{1}{2\varepsilon^2}(1-|\varphi|^2)\varphi \tag{53}$$

$$\varepsilon^2 d^* F = -Im\langle D_A\varphi, \varphi\rangle\ , \tag{54}$$

while the corresponding 1st order equations are

$$\bar\partial_A\varphi = 0 \tag{55}$$

$$-i\varepsilon^2 * F = \frac{1}{2}(1-|\varphi|^2) \tag{56}$$

Theorem (M.C. Hong–J. Jost–M. Struwe): *Let $(\varphi_\varepsilon, A_\varepsilon)$ be solutions of (55), (56) on some compact Riemann-surface Σ, with fixed $d = \deg L \geq 0$. Then, for some sequence $\varepsilon_n \to 0$, there exist points x_j, $j = 1, \ldots, d$ (not necessarily distinct) with*

$$|\varphi_{\varepsilon_n}| \to 1\ , \quad \mathcal{D}_{A_{\varepsilon_n}}\varphi_{\varepsilon_n} \to 0\ , \quad dA_{\varepsilon_n} \to 0 \tag{57}$$

uniformly on compact subsets of $\Sigma\backslash\{x_1, \ldots, x_d\}$, and

$$*dA_{\varepsilon_n} \to 2\pi \sum_{j=1}^d \delta(x_j) \qquad \textit{(sum of delta distributions).} \tag{58}$$

An analogous result holds on $\mathbb{R}^2$; however, one has to deal with dichotomy, i.e. vortices moving off to infinity.

The Theorem yields a method for degenerating a line bundle L on Σ of degree d into a flat line bundle with $|d|$ singular points (counted with multiplicity) and a covariantly constant section.

One may also obtain similar (but somewhat less precise) results for minimizers of $\mathcal{L}_\varepsilon$ that are not necessarily selfdual, for example on a domain $\Omega \subset \mathbb{R}^2$ with prescribed boundary values. In order to get some feeling for the asymptotic behavior of $\mathcal{L}_\varepsilon$ and its minimizers as $\varepsilon \to 0$, we shall verify that $\inf \mathcal{L}_\varepsilon$ is bounded independently of ε as $\varepsilon \to 0$. The issue arises only in case $d \neq 0$ because then any section φ of L must have zeroes. It suffices to study the case $d = 1$, $\Sigma = B(0,1) =: B$ (the unit disc) with prescribed boundary values

$$(59) \qquad \varphi(1,\vartheta) = e^{i\vartheta}\,, \quad A(1,\vartheta) = d\vartheta \quad \text{in polar coordinates} \quad (r,\vartheta)\,.$$

We consider rotationally symmetric objects

$$\varphi(r,\vartheta) = R(r)e^{i\vartheta}$$

$$A(r,\vartheta) = S(r)d\vartheta\,.$$

R and S then have to satisfy the side conditions

$$R(0) = 0 = S(0)\,, \quad R(1) = 1 = S(1)\,.$$

We have

$$dA = S'(r)dr\,d\vartheta$$

$$D_A\varphi = \big(d - iS(r)d\vartheta\big)R(r)e^{i\vartheta} = \Big(R'(r)dr + i\big(1 - S(r)\big)R(r)d\vartheta\Big)e^{i\vartheta}\,.$$

Therefore

$$|dA|^2 = \frac{1}{r^2}S'(r)^2$$

$$|D_A\varphi|^2 = R'(r)^2 + \frac{1}{r^2}\big(1 - S(r)\big)^2 R(r)^2$$

and

$$(61) \qquad \mathcal{L}_\varepsilon(\varphi, A) = \int_0^{2\pi}\int_0^1 \Big\{R'^2 + \frac{1}{r^2}(1-S)^2R^2 + \frac{1}{4\varepsilon^2}(1-R^2)^2 + \varepsilon^2\frac{1}{r^2}S'^2\Big\} r\,dr\,d\vartheta\,.$$

If we put

$$S(r) := \begin{cases} 1 & \text{for } \varepsilon \le r \le 1 \\ \left(\frac{1}{\varepsilon}r\right)^\alpha & \text{for } 0 \le r \le \varepsilon \end{cases} \quad \text{for some } \alpha > 1$$

and

$$R(r) := \begin{cases} 1 & \text{for } \varepsilon \le r \le 1 \\ \frac{1}{\varepsilon}r & \text{for } 0 \le r \le \varepsilon\,, \end{cases}$$

we obtain

$$\mathcal{L}_\varepsilon(\varphi, A) \le \text{const.} \qquad (\text{independent of } \varepsilon)\ .$$

Of course, (φ, A) as just constructed is not a minimizer for $\mathcal{L}_\varepsilon$. Nevertheless, it exhibits some of the qualitative behavior of a minimizer near a vortex.

A similar scaling invariance holds for the Seiberg–Witten equations: We recall the perturbed equations, i.e.

$$\mathcal{D}_A\varphi = 0$$

$$F^+ = \frac{i}{4}\langle e_j \cdot e_k \cdot \varphi, \varphi\rangle e^j \wedge e^k - \mu$$

coming from the perturbed functional

$$\begin{aligned} SW_\mu(\varphi, A) :&= \int_M \Big(|\mathcal{D}_A\varphi|^2 + \Big| F_A^+ - \frac{i}{4}\langle e_j \cdot e_k \cdot \varphi, \varphi\rangle e^j \wedge e^k + \mu\Big|^2 \Big) \\ &= \int_M \Big(|\nabla_A\varphi|^2 + \big|F_A^+\big|^2 + \frac{S}{4}\,|\varphi|^2 + \frac{1}{8}\Big|\mu - \frac{i}{4}\langle e_j \cdot e_k \cdot \varphi, \varphi\rangle e^j \wedge e^k\Big|^2 \Big)\ , \end{aligned}$$

using (50) again and noting that

$$\langle F_A^+, \mu\rangle = 0$$

since F_A^+ is antiselfdual (Because $-iF_A^+$ is selfdual and $*$ is antilinear over $\mathbb{C}$), μ selfdual.

We may again introduce a scaling factor ε^2 and consider the functionals

$$\begin{aligned} SW_{\mu,\varepsilon}(\varphi, A) :&= \int_M \Big(|\nabla_A\varphi|^2 + \varepsilon^2\,\big|F_A^+\big|^2 + \frac{S}{4}\,|\varphi|^2 + \frac{1}{8\varepsilon^2}\Big|\mu - \frac{i}{4}\langle e_j \cdot e_k \cdot \varphi, \varphi\rangle e^j \wedge e^k\Big|^2 \Big) \\ &= \int_M \Big(|\mathcal{D}_A\varphi|^2 + \Big|\varepsilon F_A^+ - \frac{1}{\varepsilon}\big(\frac{i}{4}\langle e_j \cdot e_k \cdot \varphi, \varphi\rangle e^j \wedge e^k - \mu\big)\Big|^2 \Big)\ . \end{aligned}$$

In the case where M is symplectic, Taubes [T2], [T3], has (independently) performed the analogous (but more difficult) limit analysis as $\varepsilon \to 0$, and he has shown that the limiting zero set of the solutions φ_ε is a pseudoholomorphic curve. It turns out that the analogy between the Ginzburg–Landau and the Seiberg–Witten equations can be pursued further in order to gain additional insights into his construction. Let ω be a symplectic form on M, i.e.

$$d\omega = 0\ , \quad \omega \wedge \omega \ne 0 \quad \text{everywhere.}$$

An almost complex structure on M, compatible with ω, defines a Riemannian metric g on M for which ω is selfdual. Normalize s.t. $|\omega|^2 = 2$. There then exists a canonical spin structure on M; it satisfies

$$W^+ = I \oplus K^{-1}$$

where I is the trivial line bundle, K the canonical bundle [1]. The splitting is induced by ω: ω acts (via Clifford multiplication) on the I-summand with eigenvalue $-2i$, on the K^{-1} summand with eigenvalue $2i$.

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[1] defined by the almost complex structure. We have $\Lambda^{2,+}_{\mathbb{C}} = \mathbb{C}\omega \oplus K \oplus K^{-1}$.

For any connection A on K^{-1} $(= \det W^+)$, together with the Levi–Civita connection on T^*M, we get an induced covariant derivative $\bar{\nabla}_A$ on W^+. Using the above splitting, we then also get an induced covariant derivative ∇_A on the I-summand of W^+ by applying $\bar{\nabla}_A$ and projecting back to I:

$$\nabla_A = \frac{1}{2}\big(1 + i\frac{\omega}{2}\big)\bar{\nabla}_A$$

(using that ω acts with eigenvalue $2i$ on the K^{-1} summand). We may now determine a connection A_o on K^{-1} uniquely by the requirement that

$$\nabla_{A_0} = d \ .$$

We may therefore find a nontrivial section ψ_0 of I, of norm $\equiv 1$, with

$$\nabla_{A_0}\psi = 0 \ .$$

One may check, by using $d\omega = 0$, that also

$$\mathcal{D}_{A_0}\psi_0 = 0 \ . \tag{63}$$

The set of spin$^{\mathbb{C}}$ structures on M is identified with the set of complex line bundles E over M, with the correspondence

$$W^+ = E \oplus (K^{-1} \otimes E) \ .$$

A section φ of W^+ may be decomposed as

$$\varphi = \alpha\psi_0 + \beta \ , \tag{64}$$

with α a section of E, β a section of $K^{-1} \otimes E$. With these notations, the SW equations for a connection A on $\det W^+ = K^{-1} \otimes E^2$ and a section φ of W^+ becomes

$$(\mathcal{D}_A\varphi =)\sigma(\psi_0 \otimes \nabla_a\alpha) + \mathcal{D}_A\beta = 0 \tag{65}$$

$$F_A^+ = -\frac{1}{4}(|\alpha|^2 - |\beta|^2)\omega - \frac{1}{2}(\alpha\beta^* + \alpha^*\beta) \ , \tag{66}$$

where $\sigma : W^+ \otimes T^*M \to W^-$ is induced by Clifford multiplication and ∇_a is the covariant derivative on E induced by the connection A_0 on K^{-1} and the connection A on $K^{-1} \otimes E^2$. We have

$$F_a = \frac{1}{2}(F_A - F_{A_0}) \ .$$

Furthermore, $\alpha\beta^*$ is a section of K, $\alpha^*\beta$ one of K^{-1}, and since

$$\Lambda^{2,+}_{\mathbb{C}} = \mathbb{C}\omega \oplus K \oplus K^{-1} \ ,$$

they can be considered as sections of $\Lambda^{2,+}_{\mathbb{C}}$. Also, the derivation of the first part of (65) uses (63).

We now perturb (66) by adding

$$\mu = \frac{1}{4}\omega + F^+_{A_0}$$

on the right hand side. The new equation is

$$F^+_a = \frac{1}{8}(1 - |\alpha|^2 + |\beta|^2)\omega - \frac{1}{4}(\alpha\beta^* + \alpha^*\beta) \,. \tag{67}$$

We finally scale the left hand side of the equation with a parameter ε^2 to get

$$\varepsilon^2 F^+_a = \frac{1}{8}(1 - |\alpha|^2 + |\beta|^2)\omega - \frac{1}{4}(\alpha\beta^* + \alpha^*\beta) \,. \tag{68}$$

(65) and (68) are equations for absolute minimizes of the functional

$$\begin{aligned} SW_{\mu,\varepsilon}(\varphi, a) &:= \int_M \Big(|\nabla_A \varphi|^2 + 4\varepsilon^2 \left|F^+_a\right|^2 + \frac{S}{4}|\varphi|^2 + \frac{1}{\varepsilon^2}\left|\frac{1}{4}\big(\omega - \langle e_j \cdot e_k \cdot \varphi, \varphi\rangle e^j \wedge e^k\big)\right|^2 \Big) \\ &= \int_M \Big(|\mathcal{D}_A \varphi|^2 + \left| 2\varepsilon F^+_a - \frac{i}{4\varepsilon}\big(\langle e_j \cdot e_k \cdot \varphi, \varphi\rangle e^j \wedge e^k - \omega\big)\right|^2 \Big) \,. \end{aligned}$$

Using the decomposition (64), $SW_{\mu,\varepsilon}$ can also be written as

$$\begin{aligned} SW_{\mu,\varepsilon}(\varphi, a) = \int_M \Big(&|\nabla_a \alpha|^2 + |\nabla_{a'}\beta|^2 + 4\varepsilon^2 \left|F^+_a\right|^2 + \frac{S}{4}(|\alpha|^2 + |\beta|^2) \\ &+ \frac{1}{8\varepsilon^2}(1 - |\alpha|^2 + |\beta|^2)^2 + \frac{1}{2\varepsilon^2}|\alpha|^2|\beta|^2 \Big) \,, \end{aligned}$$

where $\nabla_{a'}$ is the covariant derivative on $K^{-1} \otimes E$ induced by A_0 and A,

$$\begin{aligned} = \int_M \Big(&|\nabla_a \alpha|^2 + |\nabla_{a'}\beta|^2 + 4\varepsilon^2 \left|F^+_a\right|^2 + \frac{S}{4}(|\alpha|^2 + |\beta|^2) \\ &+ \frac{1}{8\varepsilon^2}(1 - |\alpha|^2)^2 + \frac{1}{8\varepsilon^2}|\beta|^4 + \frac{1}{4\varepsilon^2}|\beta|^2(1 + |\alpha|^2) \Big) \,. \end{aligned} \tag{69}$$

Special solutions of (65), (67) can be obtained as follows: Consider $M = \mathbb{C} \times \Sigma$ where Σ is some Riemann surface. On $\mathbb{C}$, take a solution of the abelian vortex equations, (α_1, A_1), i.e.

$$\bar\partial_{A_1}\alpha_1 = 0 \tag{70}$$

$$*F_{A_1} = \frac{1}{2}(1 - |\alpha_1|^2) \,, \tag{71}$$

and take the solution to be constant in the Σ-direction. We then obtain a solution of (65), (67) on $\mathbb{C} \times \Sigma$ with $\beta = 0$. Of course, in order to obtain a solution of (68), one has to scale the second equation:

$$\varepsilon^2 * F_{A_1} = \frac{1}{2}(1 - |\alpha_1|^2)$$

Now, by locally patching together solutions of this type, one may show

$$\lim_{\varepsilon \to 0} \inf_{(\varphi, a)} SW_{\mu,\varepsilon}(\varphi, a) < \infty \, . \tag{72}$$

Theorem (Taubes): *Let (M, ω) be a compact symplectic 4-manifold, equipped with a compatible almost complex structure that induces a Riemannian metric on M such that ω is (selfdual and) of length $\sqrt{2}$. Let $E \to M$ be a complex lie bundle and consider the associated* spin$^{\mathbb{C}}$*-structure with $W^+ = E \oplus (K^{-1} \otimes E)$. Suppose there exists a sequence $\varepsilon_n \to 0$ for which (7), (11) has a solution (A_n, φ_n) (with $\varepsilon = \varepsilon_n$). Then, after selection of a subsequence, $\alpha_n^{-1}(0)$ (the zero set of α_n) converges to a pseudoholomorphic curve $\varphi(\Sigma)$ (Σ a smooth compact Riemann surface, $\varphi : \Sigma \to M$ a pseudoholomorphic map) for which $\varphi_*[\Sigma]$ is Poincaré dual to $c_1(E)$. Moreover, the sequence $\mathcal{F}_n$ of currents defined by*

$$\mathcal{F}_n(\mu) = \frac{1}{2\pi} \int F_{A_n} \wedge \mu \quad \textit{for} \quad \mu \in \Lambda^2 T^* M$$

converges to a current with support $\varphi_*[\Sigma]$ (and in the a homology class defined by $\varphi_*[\Sigma]$). Also, for a generic choice of almost complex structure, φ is an embedding, except possibly on some genus 0 components, corresponding to spheres with selfintersection number -1, and on some genus 1 components where φ may be a higher multiplicity covering of an embedded curve.

Outline of a proof: Let $(\varphi_\varepsilon, a_\varepsilon)$, with $\varphi_\varepsilon : \alpha_\varepsilon \psi_0 + \beta_\varepsilon$, be solutions of (65), (68), for $\varepsilon = \varepsilon_n$ as in the statement.

1) From (69), (72)

$$\beta_\varepsilon \to 0 \quad \text{in} \quad L^4 \quad \text{as} \quad \varepsilon \to 0 \, .$$

2) Projecting (65) onto $K^{-1} \otimes E$ gives an equation for β. Use this equation and 1) to show that $\beta_\varepsilon \to 0$ also in higher norms. In particular

$$\nabla_{a'_\varepsilon} \beta_\varepsilon \to 0 \quad \text{in} \quad L^2 \, ,$$

and all terms involving β_ε in $SW_\varepsilon(\varphi_\varepsilon, a_\varepsilon)$ can be neglected.

3) From (69), (72)

$$1 - |\alpha_\varepsilon|^2 \to 0 \quad \text{in} \quad L^2 \quad \text{as} \quad \varepsilon \to 0 \, .$$

This convergence, however, cannot be so easily improved, because the topology of E will force α_ε to have zeroes. Nevertheless, a maximum principle shows

$$|\alpha_\varepsilon| \leq 1 + c_1 \varepsilon^2 \quad \text{for some constant} \quad c_1$$

(see Lemmas 1, 2 below).

4) Get estimates like

$$|\nabla_{a_\varepsilon}\alpha_\varepsilon|^2 \le \frac{c_2}{\varepsilon^2}(1-|\alpha|^2)+c_3$$

from the equations, using elliptic regularity theory.

5) Similarly first

$$\left|F^+_{a_\varepsilon}\right| \le \frac{c_3}{\varepsilon^2}(1-|\alpha|^2)+c_4$$

and then also, using among other things Bianchi's identity

$$\left|F^-_{a_\varepsilon}\right| \le \frac{c_5}{\varepsilon^2}(1-|\alpha|^2)+c_6\ .$$

6) Use compactness results for solutions to show that after a suitable blow-up, the rescalings of $(a_\varepsilon, \alpha_\varepsilon)$ approach a solution of the Abelian vortex equations (70), (71) on $\mathbb{C}\times\mathbb{C}$. This shows in particular that α_ε is, in a certain sense, close to being holomorphic.

7) Let

$$e_\varepsilon(x_0,s) := \int_{B(x_0,s)} \left(\frac{1}{\varepsilon^2}(1-|\alpha_\varepsilon|^2)^2 + |\nabla_{a_\varepsilon}\alpha_\varepsilon|^2\right)$$

($B(x_0,s)$ = ball of radius s with center $x_0\in M$).

Assume $\alpha_\varepsilon(x_0)=0$. Then

$$e_\varepsilon(x_0,s)\sim cs^2$$

with a constant c independent of x_0,ε.
("Monotonicity formula"). See also §5 below in this direction.

8) The monotonicity formula controls the size of the zero set of α_ε:

$$\exists c_0<\infty, \varepsilon_0>0 \quad \forall \varepsilon\le\varepsilon_0, \delta>0\ \exists N=N(\delta)\le c_0\frac{1}{\delta^2}:$$

$$\{\alpha_\varepsilon=0\}\subset \cup_{i=1}^N B(x_i,\delta) \quad \text{for suitable} \quad x_i\in M$$

9) Use the local control from 6) and the estimates from 5), 8) to accomplish the proof.

Taubes' result in particular implies an existence result for pseudoholomorphic curves, because one may verify the hypothesis on the existence of solutions for the perturbed Seiberg–Witten equations on symplectic manifolds in his theorem. Conversely, he also showed how to obtain a solution of the Seiberg–Witten equations from a pseudoholomorphic curve by "grafting" a solution of the Ginzburg–Landau equations (70), (71) on the normal bundle of a pseudoholomorphic curve Σ in order to obtain an approximate solution of the Seiberg–Witten equations on M that then can be deformed into an actual solution with the help of an implicit function theorem argument. He used these results to relate invariants obtained from the solution space of the (perturbed) Seiberg–Witten equations to invariants for M defined by Gromov [Gr] with the help of pseudoholomorphic curves.

Finally, one may attempt a similar perturbation analysis with a suitable 2-form μ on a general, not necessarily symplectic 4-manifold. In general, however, such a 2-form μ will have zeroes, and these will cause additional difficulties in the asymptotic analysis that are not yet properly understood.

4. Regularity aspects

First of all, weobserve the following maximum principle for solutions of the Ginzburg–Landau equations:

Lemma 1. *For any solution (φ, A) of (40), (41)*

$$|\varphi| \leq 1 \quad \text{on} \quad \Sigma \tag{73}$$

In order to have regularity results with estimates, one needs to eliminate the gauge invariance (38) by fixing the gauge, for example by choosing a Coulomb gauge, i.e. requiring

$$d^*A = 0\ , \tag{74}$$

(74) and (41) imply

$$\Delta A = -(d^*d + dd^*)A = Im\langle D_A\varphi, \varphi\rangle\ , \tag{75}$$

(40) and (75) constitute an elliptic system. From the lemma and elliptic regularity theory, one obtains smoothness of solutions. The same holds for the equations (53), (54) with parameter $\varepsilon > 0$.

Let us now consider the Euler–Lagrange equations for the Seiberg–Witten functional, (44), (45), recalled here

$$\Delta_A\varphi = \frac{S}{4}\varphi + \frac{1}{4}|\varphi|^2\varphi \tag{76}$$

$$d^*F_A^+ = -\frac{1}{2}Im\langle\nabla_{A,e^k}\varphi, \varphi\rangle e^k\ . \tag{77}$$

The maximum principle implies

Lemma 2: *For any solution φ of (76),*

$$\sup_M |\varphi|^2 \leq \max\bigl(-S(x), 0\bigr)\ . \tag{78}$$

The same result also holds for weak solutions.

Corollary: *If the scalar curvature S of M is nonnegative, then for any solution of (76), (77), we have $\varphi \equiv 0$.*

This result applies in particular to solutions of the Seiberg–Witten equations (48), (49), and it is the starting point for several applications to 4-dimensional differential topology. For example, one has the following result of Taubes [T4].

Let M be a compact, oriented, 4-dimensional symplectic manifold with $b_2^+(M) \geq 2$ (dimension of the space of selfdual cohomology classes). Then, for a suitable $\text{spin}^{\mathbb{C}}$ *structure on M and a suitable Riemannian metric on M, the number of solutions of (48), (49) up to gauge equivalence is odd.*

In particular, M cannot carry a metric of positive scalar curvature.

Let us return now to the regularity question for solutions of (76), (77) as developed in [JPW]. As always, one has to deal with the issue of gauge invariance. If (φ, A) is a solution, and if

$$g : M \to S^1$$

is a (sufficiently regular) map, then $\big(g^{-1}\varphi, g(A)\big)$ with $g(A) = A + g^{-1}dg$ is a solution as well. (There are some technical points here in the case of weak solutions (of Sobolev class $H^{1,2}$) where one needs to admit gauge transformations that are of class $H^{2,2}$ and thus not necessarily continuous.)

We now describe the regularity results of J. Jost–X.W. Peng–G.F. Wang. We first state without proof

Lemma 3: *The Seiberg–Witten functional is coercive:*

$$\exists \lambda > 0 \ \forall (\varphi, A) \in H^{1,2} \ \exists \text{ gauge transformation } g$$

$$SW(\varphi, A) \geq \lambda\big(\left\|g^{-1}\varphi\right\|_{H^{1,2}} + \|g(A)\|_{H^{1,2}}\big) - \lambda^{-1} \ .$$

With the help of Lemmas 2, 3, one obtains

Theorem: *Let (φ, A) be a critical point of $SW(\varphi, A)$. Then there exists a gauge transformation g for which*

$$\big(g^{-1}\varphi, g(A)\big)$$

is smooth.

Proof: Since $SW(\varphi, A)$ is bounded, Lemma 3 yields bounds for $\left\|g^{-1}\varphi\right\|_{H^{1,2}}$ and $\|g(A)\|_{H^{1,2}}$. Henceforth, we write for simplicity (φ, A) instead of $\big(g^{-1}\varphi, g(A)\big)$. We have from (79)

$$d^* F_A = 2d^* F_A^+ = -Im\langle \nabla_{A,e^k}\varphi, \varphi\rangle e^k \ .$$

Since $\|\varphi\|_{L^\infty}$ is bounded by Lemma 2, we get

$$\|d^* F_A\|_{L^2} \leq c\big(\|\nabla\varphi\|_{L^2} + \|A\|_{L^2}\big) \ . \tag{80}$$

The Bianchi identity is

$$dF_A = 0 \, . \tag{81}$$

Since $d + d^*$ is elliptic, we obtain from (80), (81)

$$\|F_A\|_{H^{1,2}} = \|d^* F_A\|_{L^2} + \|F_A\|_{L^2} \le c\big(\|\nabla\varphi\|_{L^2} + \|A\|_{H^{1,2}}\big) \, . \tag{82}$$

Since we obtain from Lemma 3 that the right hand side is bounded, we get altogether

$$\|A\|_{H^{2,2}} \le c \tag{83}$$

and by the Sobolev embedding theorem then

$$\|A\|_{L^p} \le c_p \quad \text{for all} \quad p < \infty \text{ and } \|\nabla A\|_{L^r} \le c_r \quad \text{for} \quad 1 < r < 4 \, . \tag{84}$$

Next, for $1 < q < 4$, from the boundedness of $\|\varphi\|_{L^\infty}$, (83), (84)

$$\begin{aligned} \|\Delta\varphi\|_{L^q} &\le c\big(\|\Delta_A\varphi\|_{L^q} + \|\nabla A\|_{L^q} + \||A|\,|\nabla\varphi|\|_{L^q} + \left\||A|^2\right\|_{L^q}\big) \\ &\le c\big(1 + \||A|\,|\nabla\varphi|\|_{L^q}\big) \end{aligned} \, . \tag{85}$$

¿From the Hölder inequality, for $1 < q < 2$, with (84) and the $H^{1,2}$-bound for φ

$$\||A|\,|\nabla\varphi|\|_{L^q} \le c\,\|A\|_{L^{\frac{2q}{2-q}}} \|\nabla\varphi\|_{L^2} \le c \, . \tag{86}$$

Using this in (85), we get by Sobolev's embedding theorem

$$\|\nabla\varphi\|_{L^{\frac{4q}{4-q}}} \le c \quad \text{for} \quad 1 < q < 2 \tag{87}$$

We may then choose $q = 2$ in (85) and get

$$\|\varphi\|_{H^{2,2}} \le c \, . \tag{88}$$

If we fix a gauge s.t.

$$d^* A \quad \text{is smooth} \tag{89}$$

(this is possible), we obtain smoothness of (φ, A) by a standard bootstrap argument. □

Further, one has the following result of Jost–Peng–Wang [JPW].

Theorem: *SW satisfies the following Palais–Smale condition: For any sequence* $(\varphi_n, A_n) \in H^{1,2}$ *satisfying*
(i) $dSW(\varphi_n, A_n) \to 0$ *in* $H^{-1,2}$
(ii) $SW(\varphi_n, A_n) \le \text{const}$,

after selection of a subsequence, there exist gauge transformations g_n s.t. $\left(g_n^{-1}\varphi_n, g_n(A_n)\right)$ converges in $H^{1,2}$ to a critical point (φ, A) of SW with

$$SW(\varphi, A) = \lim_{n\to\infty} SW(\varphi_n, A_n)\ .$$

5. Bochner type inequalities

If one has a variational problem of the type considered,

$$\mathcal{L}(\varphi, A) = \int_M e(\varphi, A) dvol(g) \tag{90}$$

on a Riemannian manifold (M, g), one often derives a useful formula from computing

$$\Delta e(\varphi, A)$$

for a solution (φ, A) of the equations.

Let us present this strategy in thesame of our Ginzburg–Landau functional, i.e.

$$e(\varphi, A) = |D_A\varphi|^2 + |F|^2 + \frac{1}{4}\left(1 - |\varphi|^2\right)^2 , \tag{91}$$

with $F = dA$ as usual. One obtains the following formula of Hong–Jost–Struwe [HJS] ($\Delta = \partial_j\partial_j = -d^*d$ on functions, R the curvature of Σ).

$$\begin{aligned}
\Delta e(\varphi, A) &= 2\,|D_AD_A\varphi|^2 + 2\langle D_A\varphi, \varphi\rangle^2 + 2\,|DF|^2 \\
&\quad - 6\langle F, Im\langle D_A\varphi, D_A\varphi\rangle\rangle + 2\,|F|^2\,|\varphi|^2 \\
&\quad - 2(1-|\varphi|^2)\,|D_A\varphi|^2 + 2\,Re\langle R(D_A\varphi), D_A\varphi\rangle \\
&\quad + \frac{1}{2}\left|D\,|\varphi|^2\right|^2 + \frac{1}{2}\,|\varphi|^2\,(1-|\varphi|^2)^2 \\
&\geq 2\,|\varphi|^2\left(|F|^2 + \frac{1}{4}(1-|\varphi|^2)^2\right) \\
&\quad - 2\,|D_A\varphi|^2\left(6\,|F| + (1-|\varphi|^2) + 2\,|R|\right) \\
&= 2\,|\varphi|^2\,e(\varphi, A) - 2\,|D_A\varphi|^2\,(6\,|F| + 2\,|R| + 1) \\
&\geq -ce(\varphi, A)^{3/2} - 4(|R|+1)e(\varphi, A)\ .
\end{aligned} \tag{92}$$

¿From such an inequality, one derives a so-called ε-regularity estimate in a standard manner as we now wish to explain.

Lemma 4: *Let (φ, A) be a solution of the Ginzburg–Landau equations, i.e.*

$$\Delta_A\varphi = -\frac{1}{2}(1-|\varphi|^2)\varphi \tag{93}$$

(94) $$d^*F = -Im\langle D_A\varphi, \varphi\rangle$$

on some disk $B(z_0, 2R)$ on a Riemann surface Σ with bounded curvature. There exist $\varepsilon > 0$ and $c_0 < \infty$ with: If

(95) $$\mathcal{L}\big(\varphi, A; B(z_0, 2R)\big)\Big(= \int_{B(z_0,2R)} e(\varphi, A)\Big) < \varepsilon \ ,$$

then

(96) $$\sup_{B(z_0,R/2)} e(\varphi, A) \le c_0 \frac{1}{R^2} E\big(\varphi, A; B(z_0, R)\big) \ .$$

In other words, if the value of the functional on some disk is sufficiently small, we obtain a pointwise estimate for a solution.

Proof: We choose $r_0 < R$ with

$$(R - r_0)^2 \sup_{B(z_0,r_0)} e(\varphi, A) = \max_{0\le r\le R}\Big((R - r)^2 \sup_{B(z_0,r)} e(\varphi, A)\Big)$$

and $x_0 \in \overline{B(z_0, r_0)}$ with

(98) $$e_0 := e(\varphi, A)(x_0) = \sup_{B(z_0,r_0)} e(\varphi, A) \ .$$

We claim

(100) $$e_0 \le \frac{4}{(R - r_0)^2} \ .$$

Assume on the contrary that

(101) $$\rho_0 := e_0^{-1/2} \le \frac{R - r_0}{2} \ .$$

We rescale:

(102) $$v(x) := \varphi(x_0 + \rho_0 x)$$

(103) $$B(x) := \rho_0 A(x_0 + \rho_0 x)$$

(104) $$D_B v = \rho_0 D_A u, \quad dB = \rho_0^2 dA \ .$$

With

(105) $$e_{\rho_0}(v, B) := |D_B v|^2 + \frac{1}{\rho_0^2}|dB|^2 + \frac{\rho_0^2}{4}(1 - |v|^2)^2 = \rho_0^2 e(\varphi, A) \ ,$$

we have

$$1 = e_{\rho_0}(v, B)(0) \ , \tag{106}$$

whereas

$$\begin{aligned}
\sup_{B(0,1)} e_{\rho_0}(v, B) &= \rho_0^2 \sup_{B(x_0,\rho_0)} e(\varphi, A) \\
&\le \rho_0^2 \sup_{B\left(z_0, \frac{R+r_0}{2}\right)} e(\varphi, A) && \text{by (101)} \\
&\le 4\rho_0^2 \sup_{B(z_0,r_0)} e(\varphi, A) && \text{by (97)} \\
&\le 4 \ . && \text{by (98), (101)}
\end{aligned}$$

¿From (92), we therefore get for some constant c

$$\Delta e_{\rho_0} \ge -c e_{\rho_0} \ . \tag{107}$$

¿From Moser's Harnack inequality, we get

$$\begin{aligned}
1 = e_{\rho_0}(v, B)(0) &\le c_1 \int_{B(0,1)} e_{\rho_0}(v, B) \\
&= c_1 \int_{B(x_0,\rho_0)} e(\varphi, A) \le c_1 \int_{B(z_0,R)} e(\varphi, A) \ .
\end{aligned} \tag{108}$$

If we take $\varepsilon = \frac{1}{c_1}$, (95) implies that the last quantity is smaller than 1, and a contradiction results.

Hence

$$e_0 \Big(\frac{R - r_0}{2}\Big)^2 \le 1 \ ,$$

and

$$\Big(\frac{R}{2}\Big)^2 \sup_{B(z_0, \frac{R}{2})} e(\varphi, A) \le (R - r_0)^2 e_0 \le 4 \ ,$$

i.e.

$$\sup_{B(z_0,\frac{R}{2})} e(\varphi, A) \le \frac{16}{R^2} \ . \tag{109}$$

The desired conclusion follows from (108) if one scales with R instead of ρ_0. References for the preceding method in a somewhat different context are [Hz] and [Sc]. □

As a consequence, one obtains the following proof of Hong–Jost–Struwe of a gap theorem that is a consequence of Taubes' work.

Corollary: *Let (φ, A) be a solution of (93), (94) on $\mathbb{R}^2$. Then there exists $\delta_0 > 0$ with the property that if*

$$\mathcal{L}(\varphi, A; \mathbb{R}^2) < \delta_0 \ , \tag{110}$$

then

$$|\varphi|^2 \equiv 1 \ , \quad D_A\varphi \equiv 0 \ , \quad A \equiv 0 \quad \text{on} \quad \mathbb{R}^2 \ . \tag{111}$$

Proof: Take $\delta_0 = \varepsilon$ and let $R \to \infty$ in Lemma 4. □

The same strategy works for other variational problems of the type considered here. Let us discuss the case of harmonic maps

$$f : (M, \gamma) \to (N, g) \ .$$

In that case, the relevant Bochner type identity (discovered by Eells–Sampson, Al'ber) is

$$\begin{aligned} \Delta \|df\|^2 = 2 \|Ddf\|^2 &+ 2\langle df\big(Ric^M(e_\alpha)\big), df(e_\alpha)\rangle \\ &- 2\langle R^N\big(df(e_\alpha), df(e_\beta)\big)df(e_\beta), df(e_\alpha)\rangle \ , \end{aligned} \tag{112}$$

where $(e_\alpha)_{\alpha=1,\ldots,\dim M}$ is an orthonormal frame on (M, γ) Ric^M is the Ricci tensor of (M, γ) and R^N is the curvature tensor of (N, g).

¿From (112), one obtains a similar inequality as above, namely with

$$e(f) := \|df\|^2 \ ,$$

$$\frac{1}{2}\Delta e(f) \geq -c_1 e(f) - c_2 e(f)^2 \ , \tag{113}$$

where $-c_1$ is a lower bound for the Ricci curvature of (M, γ), c_2 an upper bound for the sectional curvature of (N, g). As in Lemma 4, one may deduce an ε-regularity result for harmonic maps. In general, however, one does not have a complete regularity theory for harmonic maps, and as the quadratic term in (113) indicates, the difficulties result from the sectional curvature of (N, g). If one assumes, however, that N has nonpositive sectional curvature, then (113) simplifies to the linear inequality

$$\frac{1}{2}\Delta e(f) \geq -c_1 e(f) \tag{114}$$

from which one may conclude estimates for $e(f)$ in terms of $E(f)$ via Moser's Harnack inequality. On the other hand, the natural setting for harmonic maps is more general than the class of Riemannian manifolds, and in the last §, the author wishes to describe some recent considerations of his towards an axiomatic treatment of generalized harmonic maps.

6. Generalized Dirichlet forms and harmonic maps

The theory of Dirichlet forms was created by Beurling and Deny as an axiomatic setting for the theory of harmonic functions. Here, we shall present such a setting for generalized harmonic maps with values in metric spaces, following [J4]. A reference for the theory of Dirichlet forms is [FOT].

Let (X, μ) be a measure space. For two maps $f, g : X \to (Y, d)$ with values in some metric space (Y, d), we may define an L^2-distance

$$d^2(f,g) := \int_X d^2\big(f(x), g(x)\big)\mu(dx) \in \mathbb{R}^+ \cup \{\infty\} \, . \tag{115}$$

Selecting some base map

$$f_0 : X \to (Y, d) \, ,$$

we may put

$$L^2(X, \mu; Y, d; f_0) := \big\{ f : X \to (Y, d) : d^2(f_0, f) < \infty \big\} \, .$$

We consider functionals

$$E : \big\{ f \in L^2(X, \mu, Y, d, f_0) \text{ for some metric space } (Y, d) \text{ and some base map } f_0 \big\} \to \mathbb{R}^+ \cup \{\infty\}$$

with

$$D(E) := \big\{ f : E(f) < \infty \big\} \, .$$

Definition: Such a functional E is called generalized Dirichlet form or energy form if

(i) Quadratic contraction property:
If $f : X \to (Y, d)$ is in $D(E)$ and $\varphi : \big(f(X), d\big) \subset (Y, d) \to (Z, d')$ for some metric space
(Z, d') is Lipschitz with Lipschitz constant L, i.e.

$$d'\big(\varphi(p), \varphi(q)\big) \le L d(p, q) \ \forall p, q \in f(X)$$

then

$$E(\varphi \circ f) \le L^2 E(f) \, . \tag{117}$$

(ii) Closedness:
If $u_n : X \to Y$, $u_n \in D(E)$, $\lim_{n\to\infty} u_n = u$ in L^2 and if for every Lipschitz

$$\varphi : \bigcup_{n \in \mathbb{N}} u_n(X)(\subset Y) \to \mathbb{R}$$

with finite Lipschitz constant,

$$E\big(\varphi(u_n) - \varphi(u_m)\big) \to 0 \text{ for } n, m \to \infty \, , \tag{118}$$

then $n \in D(E)$ and

$$E\big(\varphi(u_n) - \varphi(n)\big) \to 0 \text{ for } n \to \infty \tag{119}$$

for all such φ.

(iii) Density:
$D(E) \cap L^2(X, \mu, \mathbb{R})$ is dense in $L^2(X, \mu, \mathbb{R})$. If for every Lipschitz $\varphi : (Y, d) \to \mathbb{R}$ with bounded Lipschitz constant, $\varphi \circ f \in D(E)$, then also $f \in D(E)$.

Some easy properties of energy forms:
- $E(g_0) = 0$ for every constant map
- $E(\varphi \circ f) = E(f)$ if $\varphi : (Y, d) \to (Y, d)$ is a bijective isometry.
If $u : X \to \mathbb{R}$, $\lambda \in \mathbb{R}$, $a \in \mathbb{R}$, then

$$E(\lambda u) = \lambda^2 E(u) \ ,$$

hence

$$E(-u) = E(u) \ ,$$

$$E(u + a) = E(u) \ .$$

By polarization, one obtains a symmetric bilinear form

$$E(u, v) := \frac{1}{4}\big(E(u + v) - E(u - v)\big)$$

which then is a Dirichlet form in the usual sense of Beurling–Deny. The representation theorem of Beurling–Deny says that under some technical conditions, a Dirichlet form can be represented for $u, v \in D(E) \cap C_0^0(X, \mathbb{R})$ as

$$\begin{aligned} E(u, v) = E^c(u, v) &+ \int_{X \times X \setminus \text{diagonal}} \big(u(x) - u(y)\big)\big(v(x) - v(y)\big) J(dx, dy) \\ &+ \int_X u(x) v(x) k(dx) \end{aligned} \tag{120}$$

where E^c is strongly local in the sense that

$$E^c(u, v) = 0 \text{ whenever } v \equiv \textit{const} \text{ on some neighborhood of } \operatorname{supp} u \ .$$

J is a symmetric nonnegative Radon measure on $X \times X \setminus$diagonal, k a nonnegative Radon measure on X, the jumping and the killing measure, resp.

For a Dirichlet form that comes from an energy form in our sense, the contraction property implies in fact that the killing measure vanishes.

For a Dirichlet form with vanishing killing measure, one has

$$E(u,u) = \lim_{t\searrow 0} \frac{1}{2t} \int_{X\times X} \big(u(x) - u(y)\big)^2 p_t(x,dy)\mu(dx) \tag{121}$$

where $(p_t)_{t\geq 0}$ is the family of transition functions for the Markov process associated with E (for details and references, see e.g. Fukushima's book). In this case, one may define an energy form (for $f : X \to (Y,d)$) by

$$E(f) := \lim_{t\searrow 0} \frac{1}{2t} \int d^2\big(f(x), f(y)\big) p_t(x,dy)\mu(dx) \tag{122}$$

Also, a Dirichlet form defines a negative semidefinite selfadjoint operator on $L^2(X,\mu;\mathbb{R})$ via

$$E(u,v) = -(Lu,v)_{L^2}\ . \tag{123}$$

Finally, any such Dirichlet form admits a representation

$$E(u,v) = \int \eta(u,v)(dx) \tag{124}$$

with a Radon measure valued, nonnegative, bilinear form η on $D(E)$.

As in the theory of harmonic maps between Riemannian manifolds, it is necessary to impose geometric restrictions on the target in order to have a successful theory. In fact, a generalization of nonpositive sectional curvature for metric spaces that was found by Alexandrov is perfect for our purposes. In order to state this property, let (Y,d) be a complete metric space with the property that any $p,q \in Y$ can be connected by a length minimizing geodesic, i.e. a curve

$$\gamma : [0, d(p,q)] \to Y$$

with $\gamma(0) = p, \gamma\big(d(p,q)\big) = q,$

$$d\big(\gamma(t_1), \gamma(t_2)\big) = |t_1 - t_2| \quad \text{for all} \quad t_1, t_2 \in [0, d(p,q)] \tag{125}$$

Y is said to have nonpositive curvature if for any such geodesic γ and any $p_0 \in Y$, we have

$$d\big(\gamma(t), p_0\big) \leq |\bar{\gamma}(t) - \bar{p}_0| \quad \text{for all} \quad t \in [0, d(p,q)], \tag{126}$$

where $\bar{\gamma}(t)$ is a straight line in the Euclidean plane $\mathbb{R}^2$ parametrized by arclength and $\bar{p}_0 \in \mathbb{R}^2$ satisfies $|\bar{p}_0 - \bar{\gamma}(0)| = d(p_0, p)$, $\big|\bar{p}_0 - \bar{\gamma}\big(d(p,q)\big)\big| = d(p_0, q)$.

A general reference is [BN].

If (Y,d) has nonpositive curvature in Alexandrov's sense, then an energy form E defines a convex functional on $L^2(X,\mu;Y,d)$, essentially since $d^2(p_0,\cdot)$ is strictly convex on

Y for any $p_0 \in Y$, and a general result of Jost [J2], [J3] shows the existence of minimizers for E among maps with values in (Y, d). (There are some topological points like lifting to universal covers that are suppressed here.)

Lemma: *If (Y, d) has nonpositive curvature, and if u is a minimizer for E, then*

$$L\big(d^2(u(\cdot), p)\big) \geq 2\eta(u, u) \quad \text{weakly} . \tag{127}$$

Thus, if u is a minimizer, we may use the convex functions $d^2(\cdot, p)$ for $p \in Y$, in order to produce scalar valued subsolutions for L.

In order to derive regularity properties one needs additional assumptions on (X, μ), however. It turns out that the proper set of assumptions was already found by Biroli–Mosco, and we shall now describe those, leaving aside some technical specifications, however:

(A1) Suppose E induces a metric on X via

$$d_E(x, y) := \sup\big\{\varphi(x) - \varphi(y) : \varphi \in D(E) \cap C_0^0 \ , \ \eta(\varphi, \varphi) \leq \mu \text{ on } X\big\}$$

that induces on X the same topology that we started with and makes X into a locally compact, separable metric space, and satisfies the ball doubling property:
$\exists c_0 < \infty, r_0 > 0 \ \forall x \in X, 0 < r \leq r_0$:

$$\mu\big(B(x, 2r)\big) \leq c_0 \mu\big(B(x, r)\big)$$

for $B(x, r) := \big\{y \in X : d_E(x, y) < r\big\}$.

(A2) Poincaré inequality $\forall X_0 \subset\subset X \ \exists 0 < \lambda \leq 1, \ c_1 < \infty \ \forall x_0 \in X_0, \ B(x_0, r) \subset X_0$, $f : X \to (Y, d)$ with $f \in D(E)$:

$$\int_{B(x_0, \lambda r)} d^2\big(f(x), \bar{f}_{B(x_0, \lambda r)}\big)\mu(dx) \leq c_1 r^2 \int_{B(x_0, r)} \eta(f, f)(dx) \ ,$$

where $\bar{f}_U$ denotes the mean value of f on a set U, i.e. $\bar{f}_U \in Y$ minimizes $\varphi(q) = \int_U d^2\big(f(x), q\big)\mu(dx)$.

(This inequality is implied by the following one

$$\int_{B(x_0, \lambda r)} \int_{B(x_0, \lambda r)} d^2\big(f(x), f(y)\big)\mu(dx)\mu(dy) \leq c_2 r^2 \int_{B(x_0, r)} \eta(f, f)(dx) \ .)$$

It was shown by Biroli–Mosco [BM] that the ball doubling property and the Poincaré inequality imply a Sobolev inequality and Harnack type inequalities in the sense of Moser for sub- and supersolutions of L. Important extensions and generalizations of these results have e.g. been found by Hajlasz–Koskelo [HK] and Sturm [St].

Using the Harnack inequalities of Biroli–Mosco and Sturm, the author showed [J4]:

Theorem: *Under the preceding assumptions on X and E, and if (Y,d) has nonpositive curvature in the sense of Alexandrov, then any minimizer for E is Hölder continuous.*

The proof makes use of a telescoping lemma of Giaquinta–Giusti and a method of Meier for deriving Hölder continuity of harmonic maps from Harnack inequalities for the Laplace–Beltrami operator.

In their study of harmonic maps from singular algebraic varieties with values in spaces of nonpositive curvature, P. Li and G. Tian also found a Hölder estimate based on a ball doubling property and a Poincaré inequality.

Under the much stronger assumption, that the domain X is a Riemannian manifold, Korevaar–Schoen [KS] had earlier shown Lipschitz continuity of generalized harmonic maps with values in spaces of nonpositive curvature. It is an open problem to find the proper set of assumptions on the domain that is necessary for Lipschitz continuity.

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