

Correlation dimension of intermittent signals

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Abstract

We investigate the breaking of proper self similarity of attractors in the presence of intermittency. We show that this can lead to dramatically too small values of the numerically estimated correlation dimension D_2 , which we relate, in the case of type I intermittency, to universal scaling properties in the vicinity of the critical value of the control parameter. For spatially extended systems we study the influences of space time intermittency on the correlation dimension.

The estimation of invariant quantities like dimensions, entropies or Lyapunov exponents has become a quite ubiquitous task in the analysis of chaotic time series. It is impossible to perform numerically the limits given in the mathematical definition of these quantities, since the attractors formed by time series are usually represented by a finite number of points and experimental observables are additionally corrupted by noise. In the case of 'well behaving' low dimensional systems this is not a severe problem at all. There one can estimate e.g. the correlation dimension even on relatively large length scales with good accuracy. With 'well behaving' we mean that the structure of the attractor is more or less independent of the length scales we look at (or in other words, the self similarity is visible on large scales, already). But this may not always be the case. The properties might differ quite strongly with respect to the length scales. A trivial example for this length scale dependency is a deterministic system, which is corrupted by noise. In this case the true dimension is infinity, but on large scales one might see (if the noise is not too large) the dimension of the noise free attractor.

Another example is the one we want to investigate in this article in more detail. The system we are looking at is supposed to show intermittent behaviour. That means, the time evolution of the system looks periodic for a certain time, but occasionally shows chaotic bursts.

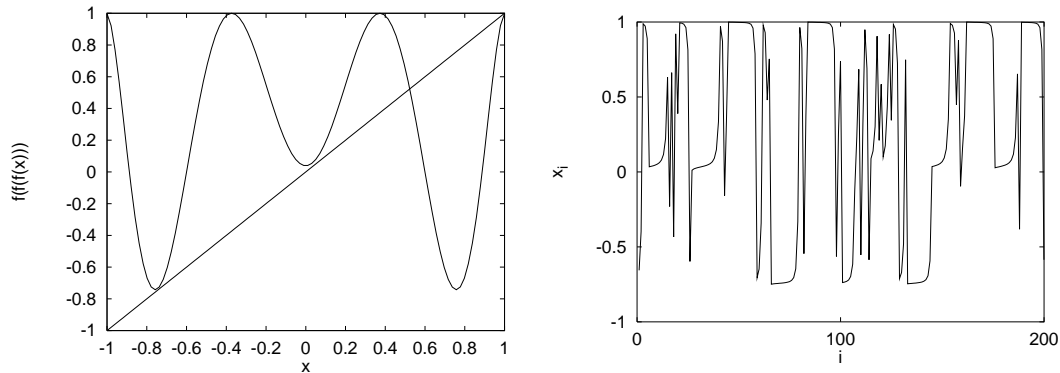


Figure 1: Third iterate of the logistic map with a parameter a close to the bifurcation to the period 3 orbit (left pannel) and the resulting time series (right pannel).

One possible mechanism for this behaviour (the so called type I intermittency [1]) is illustrated in fig. 1. There we show the graph of the third iterate of the logistic map ($f(x) = 1 - ax^2$) for $a = 1.7425$ (left plot). For $a = a_c = 7/4$ the map undergoes a bifurcation from chaotic behaviour $a < a_c$ to a stable period three orbit $a > a_c$. Thus, for the parameter value of a shown in the figure, the map is chaotic. But one can see that close to the values of x where the fixed points arise for $a = a_c$ the map is already near the $y = x$ line. That means that once the trajectory of $f \circ f \circ f$ comes close to these points, it stays there for a certain time (right plot in fig. 1), and a time series of f looks roughly periodic. It is known that the average length $\langle l \rangle$ of the almost periodic segments scales

like

$$\langle l \rangle \sim (a - a_c)^{1/2} \quad (1)$$

for type I intermittency [2, 3]. As we will see in the following, this behaviour has dramatic consequences for the correlation integral.

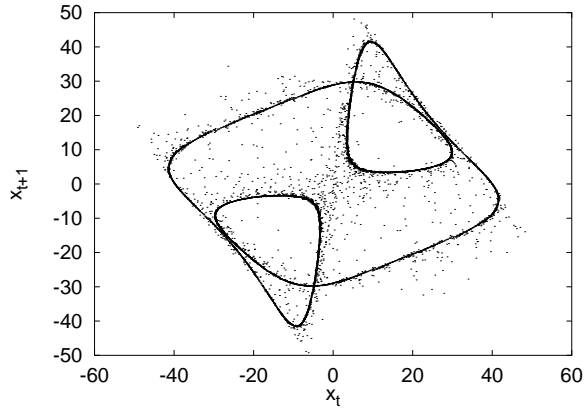


Figure 2: Delay plot of the x coordinate of the Lorenz system with strong intermittency. The parameters were: $b = 8/3$, $\sigma = 10$, $r = 166.07$.

As an other example for a low dimensional system we treat the Lorenz system [4]. It has a periodic solution for the parameter values $b = 8/3$, $\sigma = 10$ and $r \approx 166$ [2]. For these parameters we will see a one dimensional, non chaotic limit cycle. Increasing r a bit will lead to intermittent behaviour [1]. A delay plot of the x -coordinate for $r = 166.07$ is shown in fig. 2. Most of the points are close to the periodic orbit, whereas the remainder of the attractor is poorly sampled by the short chaotic bursts.

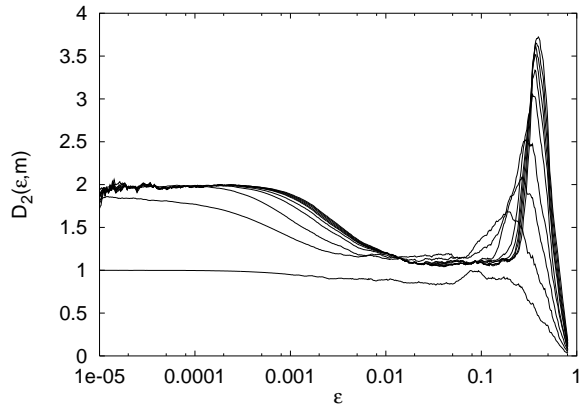


Figure 3: Results for the estimate of the correlation dimension for the data shown in fig. 2. The length of the time series was 10^6 .

Fig. 3 shows the results for the correlation dimension estimate of these data. On large length scales the dimension of the system is close to 1. Only on much smaller length scales we find the crossover to the correct dimension (> 2).

This behaviour can be explained by studying the properties of the correlation dimension. The correlation integral is defined by [5]

$$C_2(\epsilon, m) = \frac{1}{N(N-1)} \sum_{c=1}^N \sum_{n \neq c} \Theta(\epsilon - \|\vec{x}_c - \vec{x}_n\|) , \quad (2)$$

where m is the embedding dimension, ϵ the length scale and N the number of vectors $\vec{x}_i = (x_i, x_{i-1}, \dots, x_{i-m+1})$. Now we can divide the first sum on the rhs into two parts: one containing the centres inside the laminar phase, the other one containing centres inside the chaotic bursts. This gives

$$C_2(\epsilon, m) = p_{\text{in}} C_2^{\text{in}}(\epsilon, m) + p_{\text{out}} C_2^{\text{out}}(\epsilon, m) \quad (3)$$

with $p_{\text{in(out)}} = N_{\text{in(out)}}/N$ being the fraction of centres lying in the laminar and the chaotic phase, respectively. From this correlation integral one gets the correlation dimension by

$$D_2(\epsilon, m) = \frac{\partial \log C_2(\epsilon, m)}{\partial \log \epsilon} . \quad (4)$$

Using eq. (3) one easily gets

$$D_2(\epsilon, m) = p'_{\text{in}} D_2^{\text{in}}(\epsilon, m) + p'_{\text{out}} D_2^{\text{out}}(\epsilon, m) , \quad (5)$$

with

$$p'_{\text{in(out)}} = \frac{\sum_{\text{in(out)}} \sum_n \Theta(\epsilon - \|\vec{x}_{\text{in(out)}} - \vec{x}_n\|)}{\sum_c \sum_n \Theta(\epsilon - \|\vec{x}_c - \vec{x}_n\|)} ,$$

which is the fraction of pairs found in the laminar phase and the chaotic bursts, respectively.

On large length scales the 'partial' dimension $D_2^{\text{in}}(\epsilon, m)$ is close to one, while $D_2^{\text{out}}(\epsilon, m)$ is close to two¹. Furthermore, most of the points (see fig. 2) are located in the laminar phase, which means that p'_{in} is large. This explains the behaviour of the correlation dimension on large length scales ϵ . If we decrease ϵ the 'fine structure' of the laminar phase is resolved. That means, it becomes visible that it is not a limit cycle, but only close to one and $D_2^{\text{in}}(\epsilon, m)$ increases, so that we eventually see the global dimension of the attractor.

What can we say about the behaviour of $D_2(\epsilon)$ for values of ϵ where the laminar parts still look like a periodic solution? Let $\langle l \rangle$ be the average length

¹Of course, on length scales comparable to the attractor size, the correlation integral is mainly dominated by folding effects and $D_2(\epsilon, m)$ may be much larger than 2.

of the laminar parts and $\langle t \rangle$ the average length of the chaotic bursts. Then, the probability of a point lying in the laminar phase is given by

$$p_{\text{in}} = \frac{\langle l \rangle}{\langle l \rangle + \langle t \rangle}. \quad (6)$$

From eq. (1) we know that $\langle l \rangle$ scales like $1/\sqrt{a_c - a}$. On the other hand $\langle t \rangle$ does not depend on $a_c - a$ (at least not strongly), since it is determined by the global properties of the map, which do not change (strongly) under small changes of a . Thus, for small variations of a , we can expect $\langle t \rangle$ being a constant. With this we can conclude that the large scale dimension behaves ² like

$$D_2(\epsilon, m) = \frac{\langle l \rangle}{\langle l \rangle + \langle t \rangle} D_2^{\text{in}}(\epsilon, m) + \frac{\langle t \rangle}{\langle l \rangle + \langle t \rangle} D_2^{\text{out}}(\epsilon, m). \quad (7)$$

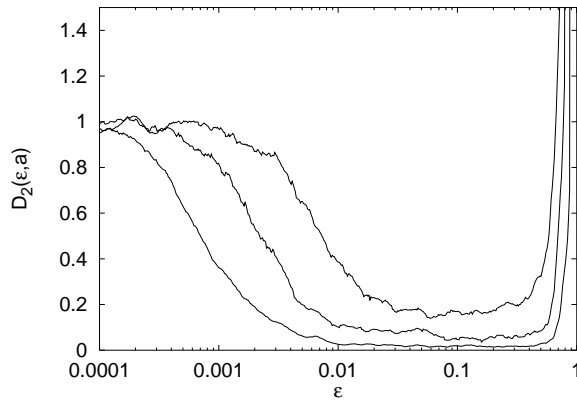


Figure 4: Shown are the dimension estimates of the logistic map for three values of a ($a = 1.7499, 1.74999, 1.749999$ from above to below) close to a_c . The embedding dimension was set to $m = 5$ for all a .

We want to check the scaling behaviour eq. (7) on the logistic map $f(x) = 1 - ax^2$. There we find the intermittent behaviour close to $a_c = 7/4$ [3], where the period three window occurs. For the case of the logistic map we have $D_2^{\text{in}}(\epsilon, m) = 0$ and $D_2^{\text{out}}(\epsilon, m) = 1$ and eq. (7) reduces to

$$D_2(\epsilon, m) \approx \sqrt{a_c - a} \Psi(a), \quad (8)$$

where

$$\Psi(a) = \frac{1}{1 + \langle t \rangle / \langle l \rangle} = \frac{1}{1 + \text{const} \sqrt{a_c - a}}$$

²For the dimension we have to use p'_{in} instead of p_{in} . But the scaling law is for both quantities the same, if the neighbourhoods of the points in the laminar and the chaotic parts of the phase space do not overlap too strongly.

gives the corrections to the square-root behaviour and can easily be calculated from eq. (7).

Fig. 4 shows the estimate of the dimensions on large length scales for different values of a . The closer a is to a_c , the smaller is the minimum of the estimated dimension.

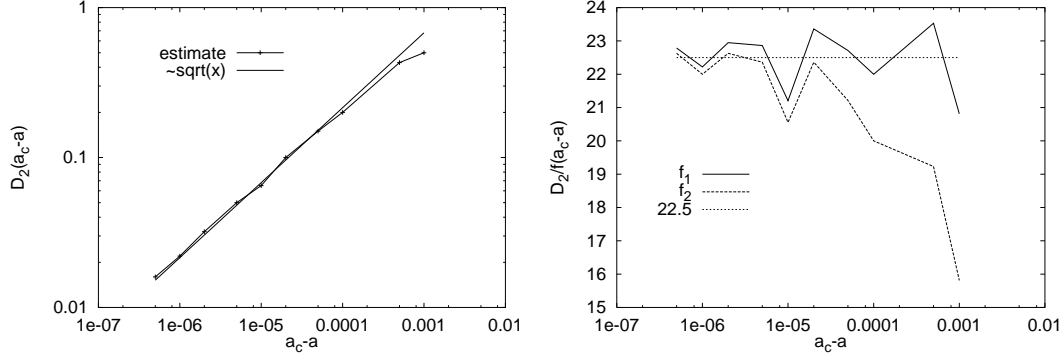


Figure 5: Estimated large scale dimension as a function of $a_c - a$. The left plot shows the scaling of D_2 . The right plot shows the behaviour of $D_2/f(a_c - a)$, where f is $f_1 = \sqrt{a_c - a}$ and $f_2 = \sqrt{a_c - a}\Psi(a)$, respectively.

In fig 5 (left plot) we show the estimated dimension on large scales as a function of $a_c - a$. The plot shows that the dimensions scale pretty well like the expected square root. The right plot shows $D_2(\epsilon, m)/f(a_c - a)$, with f being $f_1 = \sqrt{a_c - a}$ and $f_2 = \sqrt{a_c - a}\Psi(a)$, respectively, which is more sensitive to using the correct scaling law, and we see that eq. (8) describes the results correctly. The constant in Ψ was fitted to be 7.

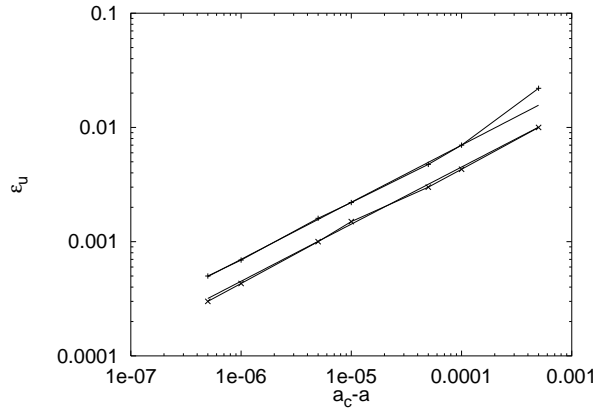


Figure 6: ϵ_u as a function of $a - a_c$ for two different values of t ($t = 0.5$ (upper curve) and $t = 0.7$ (lower curve)). Additionally the square roots are plotted.

Also for the length scales, where the crossover of the correlation dimension takes place, a scaling law should hold. We can define a characteristic length scale

of the laminar phases. The average 'time' the laminar phases last is given by $\langle l \rangle$ and the velocity with which the trajectory 'travels' through the laminar part is in first approximation proportional to $a - a_c$. This defines a length scale

$$\epsilon_c \propto (a - a_c) \langle l \rangle \propto \sqrt{a - a_c}. \quad (9)$$

It is very difficult to get this ϵ_c directly from data of the dimension estimate. Instead, we show in fig. 6 the values of a length scale ϵ_u , which is defined as the length scale, where the correlation dimension crosses a given threshold u . In fig. 6 we show ϵ_u as a function of $a - a_c$ for two different values of u , namely $u = 0.5$ and $u = 0.7$, and one can see that the scaling law (9) holds pretty well.

In spatially extended systems one often observes another kind of intermittency, which is called space–time intermittency [6, 7]. Space–time intermittency means that we have intermittent behaviour as well in time, as in space. This leads to a very complex and inhomogeneous structure of the system.

As for low dimensional systems we must expect that the estimates of the correlation dimension of this kind of intermittent systems might show similar 'misleading' results as they do for low dimensional systems.

As an example for a spatially extended system we study a coupled map lattice (CML). This is a system of N maps spatially arranged as a linear chain (in the one dimensional case), which are usually coupled by a nearest neighbour mechanism:

$$x_i^{t+1} = (1 - 2\sigma)f(x_i^t) + \sigma (f(x_{i+1}^t) + f(x_{i-1}^t)) , \quad (10)$$

where $i = 1, \dots, N$ is the spatial index of the maps, t the (discrete) time and f a chaotic map. Here f is the logistic map $f(x) = 1 - ax^2$ in the regime of fully developed chaos ($a = 2$). $\sigma \in [0, 1/2]$ is the coupling constant. Compared to the more physical partial differential equations, the advantage of this system is that it is discrete in space and time and thus very easy to iterate. Additionally, the Jacobian of this system is easy to calculate and one can estimate the Lyapunov exponents to whatever precision (in a numerical sense) one wants. Using the Pesin identity and the Kaplan–Yorke formula one easily obtains the entropy and the dimension of the system, so that we can check the results from the dimension estimate based on time series analysis directly with the results obtained by the Lyapunov exponents. Of course, time series analysis is not able to yield the full dimension of the system. The reason is that for a typical system size $N = 100$ and a coupling constant $\sigma = 1/3$ (which is sometimes called the democratic coupling) the Kaplan–Yorke formula yields a dimension of $D \approx 55$. Even if we assume that the system is multifractal, so that the correlation dimension is smaller than the information dimension, it is impossible to reach such values using time series analysis [8, 9].

Fig. 7 shows a typical plot of the correlation dimension estimate based on a scalar time series $\{x_i^t\}_{i=1, \dots, T}$ for the above mentioned parameters. For our treatments the interesting part of the plot is the breakdown of the estimate for

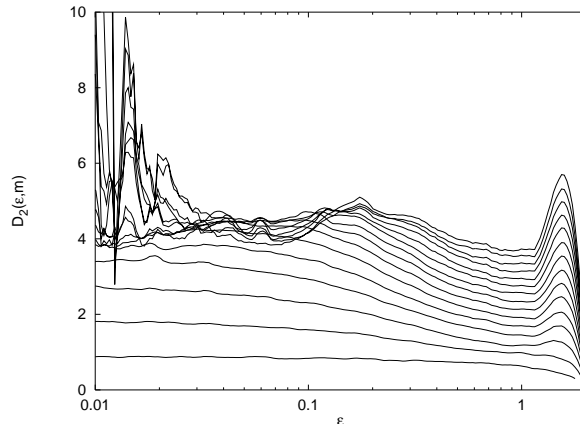


Figure 7: Dimension estimate of a system of 100 coupled Ulam maps based on a scalar time series. The coupling constant was set to $\sigma = 1/3$. The embedding dimensions range from $m = 1$ to $m = 15$.

$\epsilon \leq 0.2$. This breakdown shows a similar signature as observed in the dimension estimate for systems with strong linear correlations [10]. We will see that the effects shown in fig. 7 are of completely different origin.

The behaviour in our system is determined by a kind of space time intermittency. If one observes the time evolution of the system one sees that the spatial degrees of freedom behave most of the time irregularly. But occasionally, this irregularity is replaced by laminar windows occurring temporarily in parts of the system. To study the structure of these laminar parts we reduced the system size. If the system size is small enough, the whole system eventually 'relaxes' to the laminar solution and one can investigate its properties in detail. For our special parameter ($\sigma = 1/3$) this is very difficult. Depending on the system size the laminar parts are periodic in time with either period 2 or period 4 or even quasiperiodic. Also the spatial structure of the (quasi-) periodic solutions is ambiguous. We found period 7 in space but also a structure which shows no clear periodicity. This complicated behaviour also appears in the large system. Depending on the size of the laminar phases in the large system, one can find any of the mentioned solutions. Thus one cannot relate the properties of the correlation dimension to a single periodic solution. To study its behaviour in more detail, let us treat another value of σ , namely $\sigma = 0.16$. For this coupling the breakdown is even more pronounced (see fig. 8).

For this value of σ there exists a unique periodic solution [11], which has period 2 in time and period 4 in space with the spatial structure: $\dots xy y x x y y \dots$, where x and y are

$$y(x) = \frac{1}{4(1-2\sigma)} \left(1 \pm \sqrt{9 - \frac{4(1-\sigma)^2}{(1-2\sigma)^2}} \right). \quad (11)$$

Linear stability analysis shows, that this solution is stable in the interval $\sigma \in$

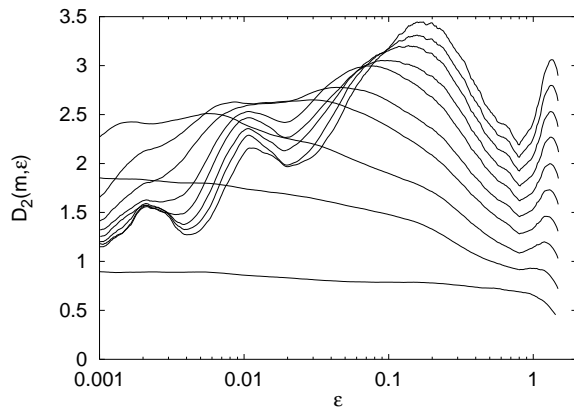


Figure 8: Similar to fig. 7, but for $\sigma = 0.16$ and $m = 1, \dots, 10$

[0.14 : 0.19] for all $N = 4n$ with $n \in \mathbf{N}$ and periodic boundary conditions. Thus, the chaotic behaviour we observe in this case is only transient and represents a chaotic repeller.

Independent of N , parts of the system are temporarily in the laminar (periodic) phase. Of course, these periodic subsystems are disturbed by the rest of the systems and therefore have finite life times only. This is one mechanism for space time intermittency and we can expect some behaviour for the correlation integral similar to what we found for the low dimensional intermittent systems.

The difference in the dimension estimate of a low dimensional and of the above system is the following: For low dimensional systems, we saw the laminar behaviour on large length scales, while for the CML the laminar part appears on small length scales. This is due to the frequency the periodic parts appear. They are relatively seldom, which means that $p'_{\text{out}} \gg p'_{\text{in}}$, so that the coarse grained invariant measure of the system on large scales is dominated by the chaotic parts and the correlation dimension is roughly identical to $D_2^{\text{out}}(\epsilon, m)$. The singularities (and thus $D_2^{\text{in}}(\epsilon, m)$) only appear if we reduce the length scale further and further. From fig. 8 we see that the higher the embedding dimension, the more abrupt is the breakdown. The reason for this behaviour is that due to the high dimensionality of the attractor, all centre points in the chaotic parts lose their neighbours on large length scales, so that finally the only pairs contributing to the correlation sum stem from the laminar phase. This holds for arbitrarily high embedding dimension, since the temporal length of the laminar phases $\langle l \rangle$ is not bounded. Of course, this loss of neighbours is faster the higher the embedding dimension is (at least as long as the embedding dimension is smaller the attractor's dimension, which is the case here).

To further illustrate that the breakdown is indeed caused by the laminar parts we checked it directly the following way: When computing the correlation integral we excluded all points as centre points which lay in the laminar parts. That is,

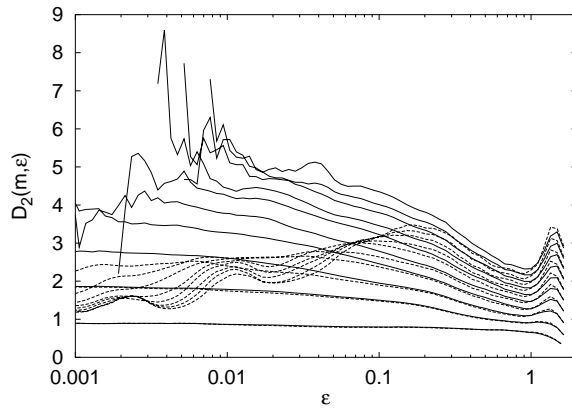


Figure 9: Shown are the data from fig. 8 and the estimate of $D_2^{\text{out}}(\epsilon, m)$ ($m = 1, \dots, 10$) only, for the same system.

we only computed $D_2^{\text{out}}(\epsilon, m)$. The result is shown in fig. 9. There one clearly sees that the breakdown does no longer appear and that the dimension grows more or less linearly with $\log(\epsilon)$.

We showed that the existence of laminar parts in a system can produce spurious dimension estimates for low as well as for high dimensional (spatially extended) systems. While this effect may not be too severe for low dimensional systems, since we may have enough data to resolve the intermittent structure, for high dimensional systems these effects could pretend a dimensionality of the system, which cannot be resolved by going to smaller length scales. This becomes clear if we look at fig. 7 where the breakdown is not as pronounced as in fig. 8. The 'breakdown' in the former figure could also be interpreted as a plateau, which would lead to a dimension estimate of about 4.5. This means, one has to be very careful in interpreting the results of the correlation dimension estimate, especially in spatially extended system, where space time intermittency seems to be a ubiquitous phenomenon.

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