Maximal Regularity for a Degenerate Operator of Fourth Order

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Maximal regularity for a degenerate operator of fourth order*

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Abstract

This paper is a first step towards a short–time regularity theory for the thin–film equation. The focus is on the free boundary — the moving contact line. We apply a hodograph transformation and linearize around a typical shape. To leading order near the contact line, this leads to the degenerate fourth–order operator

$$L_0 f \;=\; f_t + \frac{1}{x}(x^3 f_{xx})_{xx} \quad \text{for } x > 0.$$

In this paper, we establish maximal regularity of L_0 in Hölder spaces. The Hölder semi–norms are defined with respect to the intrinsic distance $|t_1 - t_2|^{1/4} + |\sqrt{x_1} - \sqrt{x_2}|$. Following Safonov, we reduce the Hölder estimates to high order local estimates for the homogeneous equation and low order global estimates for the inhomogeneous equation. These estimates are derived from L^2–type estimates which rely on the gradient structure of L_0.

Contents

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1 Introduction

1.1 Motivation

This paper is the first step towards a short–time regularity theory for the thin–film equation. The thin–film equation describes the spreading of a liquid film on a planar solid driven by capillarity. The film is described by its continuous height $h(t,y) \geq 0$, hence $\{h > 0\}$ denotes the wetted region and $\partial\{h > 0\}$ the moving contact set. We focus on the simplest of all thin–film equations: one–dimensional, zero–contact angle and Darcy–dynamics. We write it in the following conservative form

$$h_t + j_y \;=\; 0 \quad \text{with} \quad \left\{ \begin{array}{rcll} j &=& hh_{yyy} & \text{in} \quad \{h > 0\} \\ h_y &=& 0 & \text{on} \quad \partial\{h > 0\} \end{array} \right\}. \tag{1}$$

Notice that this is a fourth order parabolic equation for h in $\{h > 0\}$ coupled to a free boundary $\partial\{h > 0\}$. The fourth order parabolic equation is endowed with two boundary conditions: At $\partial\{h > 0\}$, both h and h_y vanish. Since the equation can be written as $h_t + (hu)_y = 0$ with velocity $u = h_{yyy}$ in $\{h.0\}$, we see in particular that the velocity of moving contact points $\partial\{h(t,\cdot) > 0\}$ is given by $h_{yyy}(t,\cdot)$. Hence one expects (1) to be well–posed.

Notice however that the equation is of the form $h_t + (mh_{yyy})_y = 0$ with a mobility m depending on h itself: $m = h$. On one hand, this degeneracy in the mobility (m goes to zero as h goes to zero) is necessary to keep h non–negative (recall that there is no maximum principle for a fourth order parabolic equation). On the other hand, it means that the parabolicity degenerates near the free boundary $\partial\{h > 0\}$. Over the last decade, a rather comprehensive theory for weak solutions of (1) has been worked out — from existence to the finite speed of propagation property and the occurrence of waiting–time phenomena, we refer to [4, 6] for an overview. Interestingly, the issue of short–time existence of regular solutions has not been addressed.

The prototype of such an analysis does exist for second–order degenerate parabolic equations. The second–order analogue of (1) is given by

$$h_t + j_y \;=\; 0 \quad \text{with} \quad j \;=\; -hh_y \;\text{ in } \{h > 0\}. \tag{2}$$

Here, the second order equation on $\{h > 0\}$ is endowed with the boundary condition $h = 0$ and the velocity of $\partial\{h > 0\}$ is given by $-h_y$. Angenent [1]–[3], Daskalopoulos & Hamilton [5] and Koch [8] have worked out a short–time regularity theory for (2) and its generalizations. However, there is a qualitative difference between (2) and (1). Near a contact point $y_0 \in \partial\{h > 0\}$ (say at the left end of $\{h > 0\}$) with velocity V, we generically expect the behavior

$$h \;\approx\; \left\{ \begin{array}{ll} V\,(y-y_0) & \text{for (2)} \\ C\,\frac{1}{2}(y-y_0)^2 + V\,\frac{1}{6}(y-y_0)^3 & \text{for (1)} \end{array} \right\}. \tag{3}$$

(This for instance is the behavior of the self–similar solutions.) In particular, $V \geq 0$ for (2) whereas (1) allows for $V < 0$. Moreover, the constant $C \geq 0$ for (1) is left undetermined by a purely local expansion.

The general strategy for short–time existence for nonlinear evolution problems is to linearize around the initial data. The general strategy for short–time existence of a free boundary problem like (2) or (1) is to first transform onto a fixed domain and then to linearize around the initial data. For (2), there are at least two options: A time–dependent transformation of the spatial variable y ([1]-[3]) or a hodograph transformation ([8] and [5]), which means to interchange dependent and independent variables. It is a local transformation near a contact point, where h is expected to be strictly monotone in y, cf. (3). The first strategy does not seem to be an option for (1). Its failure seems to be related to the fact that the velocity V is not the leading order term in the local expansion (3). We therefore opt for a hodograph transform.

In view of (3), we choose the following hodograph transformation:

$$h(t, Y(t,x)) \;=\; \frac{1}{2}x^2.$$

In the new coordinates, the free boundary is fixed at $x = 0$. At the same time Y has linear growth in x (comparable to the "pressure formulation" of variants of (2)), i. e.

$$Y_x \;\approx\; \frac{1}{\sqrt{2C}} \quad \text{near } x = 0. \tag{4}$$

In Y, the thin–film equation reads as

$$Y_t + \frac{1}{2x}\left(Y_x^{-1}(x^3 Y_x^{-3} Y_{xx})_x\right)_x \;=\; 0. \tag{5}$$

As it should, (5) keeps the structure of (1): It has the conserved quantity $\int h\,dy = -\int x\,Y\,dx$ (volume) and the Liapunov functional $\int \frac{1}{2}h_y^2\,dy = \int \frac{1}{2}x^2\,(Y_x)^{-1}\,dx$ (capillary energy).

We now linearize (5) around the initial data Y_0, i. e. we write $Y \;=\; Y_0 + f$ and keep the first order terms in f. This leads to the following linear fourth order operator with degenerate coefficients:

$$\begin{aligned} Lf \;&=\; f_t + \frac{1}{2x}\left(Y_{0x}^{-1}(x^3 Y_{0x}^{-3} f_{xx})_x\right)_x \\ &\quad -\frac{3}{2x}\left(Y_{0x}^{-1}(x^3 Y_{0x}^{-4} f_x Y_{0xx})_x\right)_x \\ &\quad -\frac{1}{2x}\left(Y_{0x}^{-2} f_x (x^3 Y_{0x}^{-3} Y_{0xx})_x\right)_x . \end{aligned}$$

In view of (4), L is a small perturbation of

$$L_0 f \;=\; f_t + \frac{2C_0}{x}(x^3 f_{xx})_{xx}$$

near $x = 0$. The constant C_0 can be scaled out:

$$L_0 f \;=\; f_t + \frac{1}{x}(x^3 f_{xx})_{xx} \;=\; f_t + x^2 f_{xxxx} + 6x f_{xxx} + 6 f_{xx}. \tag{6}$$

Notice that also L_0 preserves the basic structure of the original problem: $\int x\,f\,dx$ is a conserved quantity and $\int \frac{1}{2}x^3\,f_{xx}^2\,dx$ a Liapunov functional. In fact, it is the gradient flow of $\int \frac{1}{2}x^3\,f_{xx}^2\,dx$ w. r. t. the inner product $\int x\,f^2\,dx$. This symmetric structure is important for our subsequent analysis.

The above motivates the study maximal regularity of L_0, which is the subject of this paper. Also the short–time existence theory for (2) passes via maximal regularity for a linear degenerate parabolic operator (of second order). In [1]–[3], this maximal regularity is reduced to ODE considerations by a semi–group approach. In [5] and [8], maximal regularity is directly treated in time–space. We opt for this second approach since it generalizes to higher space dimensions. However, the maximum principle in [5], and estimates for the kernel in [8], are the key element in the analysis of the linearized operator. Instead, we base our analysis on energy methods which rely on the inherited gradient flow structure of (6).

1.2 The intrinsic geometry of L_0

Which regularity can we expect from equation (6)? Which geometry does it induce? On one hand, (6) is invariant under the scaling of the standard second order parabolic equation, i. e.

$$(t, x) \ \mapsto \ (r^2 t, rx). \tag{7}$$

On the other hand, near $x = 1$ the equation is to leading order similar to the standard fourth order parabolic equation

$$f_t + f_{xxxx} \ = \ 0. \tag{8}$$

The appropriate metric for (8) is given by

$$|t_1 - t_2|^{\frac{1}{4}} + |x_1 - x_2|. \tag{9}$$

Up to equivalence classes, the only metric which is homogeneous under (7) and looks like (9) for $x_1 \sim x_2 \sim 1$ is given by

$$s(z_1, z_2) \ := \ |t_1 - t_2|^{\frac{1}{4}} + |\sqrt{x_1} - \sqrt{x_2}| \tag{10}$$

(hereafter, $z = (t, x)$ denotes a point in $\mathbb{R} \times \mathbb{R}_+$).

There is another way to motivate this metric: Under the scaling $\hat{x} = 2\sqrt{x}$, (6) transforms into the standard parabolic fourth order equations plus terms which have the same scaling as $f_{\hat{x}\hat{x}\hat{x}\hat{x}}$:

$$L_0 f \ = \ f_t + f_{\hat{x}\hat{x}\hat{x}\hat{x}} + \cdots .$$

The standard metric (9) for this operator gives (10) once scaled back.

This motivates measuring the regularity of a function in the intrinsic Hölder semi–norm

$$\|g\|_{H_s^\beta(A)} \;:=\; \sup_{z_1,z_2\in A} \frac{|g(z_1)-g(z_2)|}{s(z_1,z_2)^\beta}.$$

Maximal regularity means that each of the single terms, f_t, $x^2 f_{xxxx}$, $x f_{xxx}$, f_{xx}, has the same regularity as the sum $L_0 f$. This motivates the Hölder norm

$$\|g\|_{H_s^{2+\beta}(A)} \;:=\; \|g_t\|_{H_s^\beta(A)} + \|x^2 g_{xxxx}\|_{H_s^\beta(A)} + \|x g_{xxx}\|_{H_s^\beta(A)} + \|g_{xx}\|_{H_s^\beta(A)}.$$

2 Statement of the result

2.1 The main result

By a smooth function f in D, we mean a infinitely differentiable function f in $\overline{D}$. Our main result is the following:

Theorem 2.1 (Schauder estimates). *For any $\beta \in (0,1)$ there exists a constant $C < \infty$ such that*

$$\|f\|_{H_s^{2+\beta}(\mathbb{R}\times\mathbb{R}_+)} \;\le\; C\,\|L_0 f\|_{H_s^\beta(\mathbb{R}\times\mathbb{R}_+)}$$

for any smooth and bounded f in $\mathbb{R}\times\mathbb{R}_+$.

To prove Theorem 2.1, we follow the method of Safonov [7]. It does not rely on the fundamental solution. Instead, it is based on the following two ingredients:

- local L^∞–estimates on derivatives of a local solution for the homogeneous equation ("inner estimates"),
- and a global L^∞–bound on a bounded solution of the inhomogeneous equation ("L^∞–bound").

We denote (square-shaped) neighbourhoods of $(0,0)$ in the metric space $(\mathbb{R}_-\times\mathbb{R}_+, s)$ by

$$P_r \;:=\; \{z : 0 \le x \le r, -r^2 \le t \le 0\} \;=\; I_r \times B_r, \tag{11}$$

where

$$I_r \;=\; [-r^2, 0], \quad B_r \;=\; [0, r].$$

Proposition 2.2 (Inner estimates). *For any $k, l \in N_0$ and $\gamma < 1$, a positive constant C exists such that*

$$\|\partial_x^k \partial_t^l f\|_{C^0(P_{\gamma R})} \;\le\; C\,R^{-k-2l}\,\|f\|_{C^0(P_R)}$$

for any $R > 0$ and any smooth function f in P_R which satisfies $L_0 f = 0$.

A smooth function f in $I \times \mathbb{R}_+$ is said to have exponential decay if

$$\lim_{x \to \infty} x^{-m} \partial^k f(t, x) = 0$$

for all $m, k \in \mathbb{N}_0$ and all $t \in I$. With this understanding, we have:

Proposition 2.3 (L^∞–bound). *A universal constant C exists such that*

$$\|f\|_{C^0(\mathbb{R}\times\mathbb{R}_+)} \;\leq\; C\, s^2 \|L_0 f\|_{C^0(P_s)}$$

for any $s > 0$ and any smooth function f in $\mathbb{R} \times \mathbb{R}_+$ with exponential decay which satisfies

$$\begin{array}{rcl} supp(L_0 f) & \subseteq & P_s \\ f(-s^2, \cdot) & = & 0. \end{array} \tag{12}$$

In Section 3 we will outline the proof of Propositions 2.2 and 2.3, and the main steps in going from these to Theorem 2.1. The proof of Theorem 2.1 itself can be found in Section 7. Proofs of the various steps involved are given in Sections 4 to 7. Before all that, let us summarize the subsequent notations.

2.2 Notations

In what follows, $\alpha \in \left(0, \frac{1}{2}\right)$ and $\beta = 2\alpha$ denote Hölder exponents. Positive constants $\gamma < 1$, $\mu < \lambda < 1$, $\delta < 1$ will serve as scaling factors between, say, an inner square $P_{\gamma r}$ and an outer square P_r. The integer constants k, l, m will be related to the order of differentiation and to powers of x. We will write

$$A \lesssim B$$

whenever a positive constant C, depending (at most) on the aforementioned parameters α, β, γ, δ, λ, μ, k, l, m, exists such that $A \leq C\, B$.
Differentiation of a function f with respect to space, resp. time, will be denoted either by f_x and by ∂f, resp. by f_t and by $\partial_t f$.
Hölder spaces in the metric s [say, H^α_s] are defined in paragraph 1.2. Their Euclidean counterparts are denoted by removing the subindex s [say, H^α].

3 Outline of the proof

3.1 The inner estimates

Because of the lack of maximum principle, we base our analysis on energy methods. Observe that the spatial part of equation (6) is symmetric and positive definite with respect to the inner product

$$(f, f) = \int_{\mathbb{R}_+} x\, f^2\, dx. \tag{13}$$

This guarantees us a countable number of energy estimates — the time derivatives of (13):

Lemma 3.1 (Global energy estimates). *Let f be a smooth, exponentially decaying solution of $L_0 f = 0$ in $\mathbb{R} \times \mathbb{R}_+$. Then for any $k \in \mathbb{N}_0$*

$$\frac{d}{dt} \int_{\mathbb{R}_+} \frac{1}{2} x^{2k+1} \left(\partial^{2k} f\right)^2 \;=\; - \int_{\mathbb{R}_+} x^{2k+3} \left(\partial^{2k+2} f\right)^2. \tag{14}$$

By a localization argument, (14) can be transformed into the following local estimates at the boundary points $x = 0$:

Lemma 3.2 (Local energy estimates). *Under the assumptions of Proposition 2.2,*

$$\iint_{P_{\gamma R}} x^{2k+3} \left(\partial^{2k+2} f\right)^2 \;\lesssim\; R^{-2} \iint_{P_R} x^{2k+1} \left(\partial^{2k} f\right)^2$$

for any $R > 0$.

To convert these weighted integrals into norms, we need the following localized form of Hardy inequality:

Lemma 3.3 (Local Hardy inequality). *For all $m \in \mathbb{N}_0$ and $k \in \mathbb{N}$,*

$$\int_{B_{\gamma R}} x^m \left(\partial^k \varphi\right)^2 \;\lesssim\; \int_{B_R} x^m \left(\partial^{k-1} \varphi\right)^2 + \int_{B_R} x^{m+2} \left(\partial^{k+1} \varphi\right)^2$$

for any $R > 0$ and any smooth φ in B_R.

The combination of Lemmas 3.2 and 3.3 gives as intermediate step:

Lemma 3.4. *Under the assumptions of Proposition 2.2,*

$$\iint_{P_\gamma R} x \left(\partial_t^l \partial^k f\right)^2 \;\lesssim\; R^{-4l-2k} \iint_{P_R} x f^2$$

for every $R > 0$.

A standard interpolation — $\sup_t |\varphi| \lesssim \int (|\varphi| + |\varphi_t|)$ — converts it into a pointwise bound in time:

Lemma 3.5. *Under the assumptions of Proposition 2.2,*

$$\sup_{t\in I_{\gamma R}} \int_{B_{\gamma R}} \left(\partial_t^l \partial_x^k f\right)^2 \lesssim R^{-4l-2k} \iint_{P_R} x f^2$$

for every $R > 0$.

Another interpolation, in space — $\sup \varphi^2 \lesssim \int x(\varphi^2 + \varphi_{xx}^2)$ — allows us to remove the remaining weight:

Lemma 3.6. *Under the assumptions of Proposition 2.2,*

$$\sup_{(t,x)\in P_{\gamma R}} \left|\partial_t^l \partial_x^k f\right|^2 \lesssim R^{-4l-2k} \iint_{P_R} x f^2$$

for every $R > 0$.

This obviously implies Proposition 2.2.

3.2 The L^∞–bound

We now consider the inhomogeneous equation with zero "initial" data and compactly supported right-hand side, i.e.

$$\begin{aligned} \operatorname{supp}(L_0 f) &\subseteq P_s, \\ f(-s^2,\cdot) &= 0. \end{aligned}$$

Generically speaking, an L^∞-bound follows from one integral estimate for the function and one integral estimates for the derivatives. The Liapunov functional (13) provides the first ingredient:

Lemma 3.7. *Under the assumptions of Proposition 2.3,*

$$\sup_t \int_{\mathbb{R}_+} x f^2 \lesssim s^2 \, \|L_0 f\|^2_{C^0(P_s)}$$

for every $s > 0$.

The second ingredient follows from the existence of a second Liapunov functional:

$$\int_{R_+} x^2 f_{xx}^2.$$

It is noteworthy that this functional does not seem to have a counterpart in the original (y, h) space for the nonlinear problem. It yields the following:

Lemma 3.8. *Under the assumptions of Proposition 2.3,*

$$\sup_t \int_{R_+} x^2 f_{xx}^2 \;\lesssim\; \|L_0 f\|^2_{C^0(P_s)}$$

for every $s > 0$.

A simple interpolation between Lemmas 3.7 and 3.8 will yield Proposition 2.3.

3.3 The remainder estimates

We will derive Schauder estimates along the lines of the polynomial approximation method introduced by Safonov [7]. This approach was used by Daskalopoulos and Hamilton in [5] to analyze the second order equation (2) and its variants in two space dimensions. It is based on estimating the remainder between a function and its Taylor polynomial. The Taylor polynomials we shall need are of degree at $n = 2$ resp. $n = 4$ in space, and 1 in time. We use the following notation

$$T_n^{z_0} f := \sum_{i=0}^{n} \frac{1}{i!} \partial^i f(z_0)(x - x_0)^i + \partial_t f(z_0)(t - t_0), \quad z_0 = (t_0, x_0).$$

The remainder is denoted by

$$R_n^{z_0} f \;:=\; f - T_n^{z_0} f.$$

The remainder estimates are divided into three parts. First we obtain remainder estimates at the boundary:

Lemma 3.9 (Remainder estimate at the boundary). *Let f be smooth in P_1. Then*

$$\|R_2^0 f\|_{C^0(P_r)} \;\lesssim\; r^{2+\alpha} \left(\|f\|_{C^0(P_1)} + \|L_0 f\|_{H_s^\beta(P_1)} \right)$$

for all $0 < r \leq 1$.

We then appeal to standard Schauder estimates for uniformly parabolic equations. Rewritten in terms of our metric s, these hold on parabolic squares away from the boundary

$$Q_r \;:=\; \left\{ z \,:\; r/4 \leq x \leq r, -r^2 \leq t \leq 0 \right\}.$$

Lemma 3.10 (Scaled remainder estimate in the interior). *Let f be smooth in P_1. Then*

$$\|R_4^{(0,r/2)} f\|_{C^0(Q_r)} \lesssim \|f\|_{C^0(Q_r)} + r^{2+\alpha}\|L_0 f\|_{H_s^\beta(Q_r)}$$

for all $0 < r \leq 1$.

The combination of Lemmas 3.9 and 3.10 yields

Lemma 3.11 (Remainder estimate in the interior). *Let f be smooth in P_1. Then*

$$\|R_4^{(0,r/2)} f\|_{C^0(Q_r)} \lesssim r^{2+\alpha}\left(\|f\|_{C^0(P_1)} + \|L_0 f\|_{H_s^\beta(P_1)}\right)$$

for all $0 < r \leq 1$.

3.4 The Schauder estimate

The first step in the Schauder estimate is to combine the remainder estimates from Lemma 3.9 and 3.10 to obtain Hölder continuity in space at the boundary in the following sense:

Lemma 3.12 (Hölder continuity in space at the boundary). *Let f be smooth in P_2. Then*

$$\begin{aligned}
&|x^2 f_{xxxx}(t,x)| + |x f_{xxx}(t,x)| \\
&\quad + |f_{xx}(t,x) - f_{xx}(t,0)| + |f_t(t,x) - f_t(t,0)| \\
&\quad \lesssim s((t,x),(t,0))^\beta \left(\|f\|_{C^0(P_2)} + \|L_0 f\|_{H_s^\beta(P_2)}\right)
\end{aligned}$$

for all $(t,x) \in P_{1/2}$.

We combine standard Schauder estimates for fourth order parabolic operator with Lemma 3.11 to obtain

Lemma 3.13 (Hölder continuity in the interior). *Let f be smooth in P_1. Then*

$$\|f\|_{H_s^{2+\beta}(Q_{1/2})} \lesssim \|f\|_{C^0(P_1)} + \|L_0 f\|_{H_s^\beta(P_1)}.$$

We combine Lemmas 3.12 with 3.13 to obtain Hölder continuity in time and space at the boundary:

Lemma 3.14 (Hölder continuity at the boundary). *Let f be smooth in P_4. Then*

$$\begin{aligned} &|x_1^2 f_{xxxx}(t_1,x_1)| + |x_1 f_{xxx}(t_1,x_1)| \\ &\quad + |f_{xx}(t_1,x_1) - f_{xx}(t_2,0)| + |f_t(t_1,x_1) - f_t(t_2,0)| \\ &\quad \lesssim\ s((t_1,x_1),(t_2,0))^\beta \left(\|f\|_{C^0(P_4)} + \|L_0 f\|_{H_s^\beta(P_4)}\right) \end{aligned}$$

for all $(t_1,x_1),(t_2,0) \in P_{1/4}$.

Lemmas 3.13 and 3.14 combine to

Lemma 3.15 (Local Hölder continuity). *Let f be smooth in P_8. Then*

$$\|f\|_{H_s^{2+\beta}(P_{1/4})} \ \lesssim\ \|f\|_{C^0(P_8)} + \|L_0 f\|_{H_s^\beta(P_8)}.$$

Theorem 2.1 is an easy corollary of Lemma 3.15.

4 Proof of the inner estimates

The assumptions of Proposition 2.2 are in force throughout the section.

4.1 Local energy estimates

Proof of Lemma 3.1 We need to show that

$$\frac{d}{dt}\int_{\mathbb{R}_+} \frac{1}{2} x^{2k+1}\left(\partial^{2k} f\right)^2 \ =\ -\int_{\mathbb{R}_+} x^{2k+3}\left(\partial^{2k+2} f\right)^2. \tag{15}$$

The case $k = 0$ is straightforward. For $k \geq 1$, integrations by parts give

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}_+} \frac{1}{2}\, x^{2k+1}\ \left(\partial^{2k} f\right)^2 \\ &= \ -\int_{\mathbb{R}_+} x^{2k+1}\,\partial^{2k} f\,\partial^{2k}\big(\frac{1}{x}(x^3 f_{xx})_{xx}\big) \\ &= \ -\int_{\mathbb{R}_+} \left(x^{2k+1}\partial^{2k} f\right)_{xx} \partial^{2k-2}(6f_{xx} + 6x f_{xxx} + x^2 f_{xxxx}). \end{aligned}$$

We compute (using appropriate binomial coefficients)

$$\begin{aligned}
&\left(x^{2k+1}\partial^{2k} f\right)_{xx} \\
&\quad = (2k)(2k+1)x^{2k-1}\partial^{2k} f + 2(2k+1)x^{2k}\partial^{2k+1} f + x^{2k+1}\partial^{2k+2} f \\
&\quad = x^{2k-1}\left((2k)(2k+1)\partial^{2k} f + 2(2k+1)x\partial^{2k+1} f + x^2\partial^{2k+2} f\right),
\end{aligned}$$

$$\begin{aligned}
&\partial^{2k-2}(6f_{xx} + 6xf_{xxx} + x^2 f_{xxxx}) \\
&\quad = 6\partial^{2k} f \\
&\qquad + 6(2k-2)\partial^{2k} f + 6x\partial^{2k+1} f \\
&\qquad + (2k-2)(2k-3)\partial^{2k} f + 2(2k-2)x\partial^{2k+1} + x^2\partial^{2k+2} f \\
&\quad = (2k)(2k+1)\partial^{2k} f + 2(2k+1)x\partial^{2k+1} f + x^2\partial^{2k+2} f.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}_+} \frac{1}{2}\, x^{2k+1}\left(\partial^{2k} f\right)^2 \\
&= -\int_{\mathbb{R}} x^{2k-1}\left((2k)(2k+1)\partial^{2k} f + 2(2k+1)x\partial^{2k+1} f + x^2\partial^{2k+2} f\right)^2 \\
&= -\int_{\mathbb{R}} (2k)^2(2k+1)^2 x^{2k-1}(\partial^{2k} f)^2 + 4(2k)(2k+1)^2 x^{2k}(\partial^{2k} f)(\partial^{2k+1} f) \\
&\qquad +2(2k)(2k+1)x^{2k+1}(\partial^{2k} f)(\partial^{2k+2} f) + 4(2k+1)^2 x^{2k+1}(\partial^{2k+1} f)^2 \\
&\qquad +4(2k+1)x^{2k+2}(\partial^{2k+1} f)(\partial^{2k+2} f) + x^{2k+3}(\partial^{2k+2} f)^2.
\end{aligned}$$

All summands have the same scaling in x. By partial integration we can convert all of them into a “diagonal” form, that means into a form where they quadratically depend on some derivative of f. This is already the case for the first, fourth and sixth term; the second and fifth ones are straightforward in view of $(\partial^{2k} f)(\partial^{2k+1} f) = \frac{1}{2}\partial(\partial^{2k} f)^2$ and $(\partial^{2k+1} f)(\partial^{2k+2} f) = \frac{1}{2}\partial(\partial^{2k+1} f)^2$. For the third, we write

$$\begin{aligned}
&\int_{\mathbb{R}_+} 2x^{2k+1}(\partial^{2k} f)(\partial^{2k+2} f) \\
&\quad = -\int_{\mathbb{R}_+} 2(2k+1)x^{2k}(\partial^{2k} f)(\partial^{2k+1} f) + 2x^{2k+1}(\partial^{2k+1} f)^2 \\
&\quad = \int_{\mathbb{R}_+} (2k)(2k+1)x^{2k-1}(\partial^{2k} f)^2 - 2x^{2k+1}(\partial^{2k+1} f)^2.
\end{aligned}$$

Therefore

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\frac{1}{2}x^{2k+1}\left(\partial^{2k}f\right)^2\\
&= -\int_{\mathbb{R}}(2k)^2(2k+1)^2x^{2k-1}(\partial^{2k}f)^2-2(2k)^2(2k+1)^2x^{2k-1}(\partial^{2k}f)^2\\
&\qquad +(2k)^2(2k+1)^2x^{2k-1}(\partial^{2k}f)^2-2(2k)(2k+1)x^{2k+1}(\partial^{2k+1}f)^2\\
&\qquad +4(2k+1)^2x^{2k+1}(\partial^{2k+1}f)^2-2(2k+1)(2k+2)x^{2k+1}(\partial^{2k+1}f)^2\\
&\qquad +x^{2k+3}(\partial^{2k+2}f)^2\\
&= -\int_{\mathbb{R}}x^{2k+3}(\partial^{2k+2}f)^2.
\end{aligned}
$$

□

Proof of Lemma 3.2 By the scale invariance (7) and the definition (11) of P_R, it suffices to prove the statement for $R=1$, that is,

$$\iint_{P_\gamma}x^{2k+3}\left(\partial^{2k+2}f\right)^2\lesssim\iint_{P_1}x^{2k+1}\left(\partial^{2k}f\right)^2.$$

We look at $k\geq 1$ (the case $k=0$ is simpler). Choose a smooth non-negative cut-off function ζ such that $\zeta=1$ on P_γ, $\zeta=0$ outside of P_1. Using a product ansatz, we can assume that in addition

$$\zeta_x \;=\; 0 \quad \text{for } x\in B_\gamma. \tag{16}$$

Let q be a sufficiently large integer. Similarly to the proof of the global statement, we compute

$$
\begin{aligned}
&\frac{d}{dt}\int_{R_+}\zeta^q x^{2k+1}(\partial^{2k}f)^2\\
&= \int_{R_+}(\zeta^q)_t x^{2k+1}(\partial^{2k}f)^2+2\int_{R_+}\zeta^q x^{2k+1}(\partial^{2k}f)(\partial^{2k}f_t)\\
&= \int_{R_+}(\zeta^q)_t x^{2k+1}(\partial^{2k}f)^2-2\int_{R_+}\zeta^q x^{2k+1}(\partial^{2k}f)\,\partial^{2k}(\frac{1}{x}(x^3f_{xx})_{xx}).
\end{aligned}
$$

Integrating in time from $-\infty$ to 0 yields

$$0\;\lesssim\;\iint_{P_1}(\zeta^q)_t x^{2k+1}(\partial^{2k}f)^2-2\iint_{P_1}\zeta^q x^{2k+1}(\partial^{2k}f)\,\partial^{2k}(\frac{1}{x}(x^3f_{xx})_{xx}).$$

Therefore

$$\iint_{P_1}\zeta^q x^{2k+1}(\partial^{2k}f)\,\partial^{2k}(\frac{1}{x}(x^3f_{xx})_{xx})\;\lesssim\;\iint_{P_1}x^{2k+1}(\partial^{2k}f)^2.$$

The terms containing no derivatives on ζ can be treated as in the proof of Lemma 3.1, yielding

$$\iint_{P_\gamma} \zeta^q x^{2k+3}(\partial^{2k+2} f)^2 \;\lesssim\; \iint_{P_1} x^{2k+1}(\partial^{2k} f)^2 + R,$$

where R contains only terms which contain at least one derivation on ζ:

$$R \;=\; \iint_{P_1} \sum_{\substack{i\in\{0,1\},\ell\in\{0,1\},m\in\{0,1,2\}\\ i+\ell+m=j}} a_{i,j,\ell,m}\,(\partial^{1+i}\zeta^q)\,x^{2k+j}\,(\partial^{2k+\ell} f)\,(\partial^{2k+m} f)$$

for suitable constants $a_{i,j,\ell,m}$. Having a better scaling in space w.r.t. (x, f), all the terms can be absorbed into the two other terms, i.e.

$$R \;\le\; \varepsilon \iint_{P_1} \zeta^q x^{2k+3}(\partial^{2k+2} f)^2 + C_\varepsilon \iint_{P_1} x^{2k+1}(\partial^{2k} f)^2$$

for a sufficiently small $\varepsilon > 0$, which completes the proof. Let us work this out in two typical cases. The first example is:

$$\begin{aligned}
&\left|\iint_{P_1} (\zeta^q)_{xx} x^{2k+3}(\partial^{2k} f)(\partial^{2k+2} f)\right| \\
&\quad= \left|\iint_{P_1} \left(q(q-1)\zeta^{q-2}\zeta_x^2 + q\zeta^{q-1}\zeta_{xx}\right) x^{2k+3}(\partial^{2k} f)(\partial^{2k+2} f)\right| \\
&\quad\lesssim \varepsilon \iint_{P_1} \zeta^q x^{2k+3}(\partial^{2k+2} f)^2 + C_\varepsilon \iint_{P_1} (\zeta^{q-4}\zeta_x^4 + \zeta^{q-2}\zeta_{xx}^2) x^{2k+3}(\partial^{2k} f)^2 \\
&\quad\le \varepsilon \iint_{P_1} \zeta^q x^{2k+3}(\partial^{2k+2} f)^2 + C_\varepsilon \iint_{P_1} x^{2k+1}(\partial^{2k} f)^2, \qquad (17)
\end{aligned}$$

where we have used that $x^2 \lesssim 1$ in P_1. The second example is:

$$\begin{aligned}
&\left|\iint_{P_1} (\zeta^q)_x x^{2k+3} \partial^{2k+1} f \partial^{2k+2} f\right| \\
&\quad= \frac{1}{2}\left|\iint_{P_1} ((\zeta^q)_x x^{2k+3})_x (\partial^{2k+1} f)^2\right| \\
&\quad\lesssim \left|\iint_{P_1} ((\zeta^q)_x x^{2k+3})_x (\partial^{2k} f)(\partial^{2k+2} f)\right| \qquad (18)\\
&\qquad+ \left|\iint_{P_1} ((\zeta^q)_x x^{2k+3})_{xxx} (\partial^{2k} f)^2\right|, \qquad (19)
\end{aligned}$$

where we have used the identity

$$(\partial^{2k+1} f)^2 \;=\; -(\partial^{2k} f)(\partial^{2k+2} f) + (\frac{1}{2}(\partial^{2k} f)^2)_{xx}.$$

The integral in line (18) is estimated as in (17). For the integral in line (19) we recall (16) which ensures that $|((\zeta^q)_x x^{2k+3})_{xxx}| \lesssim x^{2k+1}$ in P_1. □

4.2 Local Hardy inequalities

We recall the Hardy inequality:

Lemma 4.1 (Global Hardy inequality). *Let φ be smooth in $\mathbb{R}_+$ with exponential decay. Then*

$$\int_{\mathbb{R}_+} x^m \left(\partial^k \varphi\right)^2 \leq 2(m+1)^{-1} \int_{\mathbb{R}_+} x^{m+2} \left(\partial^{k+1} \varphi\right)^2$$

for all $k, m \in \mathbb{N}_0$.

Proof By integration by parts:

$$\begin{aligned}\int_{\mathbb{R}_+} x^m \left(\partial^k \varphi\right)^2 &= 2(m+1)^{-1} \int_{\mathbb{R}_+} x^{m+1} (\partial^k \varphi)(\partial^{k+1} \varphi) \\ &\leq 4(m+1)^{-2} \left(\int_{\mathbb{R}_+} x^m (\partial^k \varphi)\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}_+} x^{m+2} (\partial^{k+1} \varphi)^2\right)^{\frac{1}{2}}.\end{aligned}$$

□

The proof of its localized version is based on a similar reasoning.

Proof of Lemma 3.3 . By scaling, it suffices to prove that for all $m \in \mathbb{N}_0$ and $k \in \mathbb{N}$,

$$\int_{B_\gamma} x^m \left(\partial^k \varphi\right)^2 \lesssim \int_{B_1} x^m \left(\partial^{k-1} \varphi\right)^2 + \int_{B_1} x^{m+2} \left(\partial^{k+1} \varphi\right)^2 \qquad (20)$$

for all $\varphi \in C^\infty(\mathbb{R}_+)$. Choose a non-negative cut-off $\zeta \in C^\infty(\mathbb{R}_+)$ with $\zeta = 1$ in B_γ and $\zeta = 0$ outside of B_1. Then

$$\int_{\mathbb{R}_+} \zeta x^m \left(\partial^k \varphi\right)^2 \sim -\int_{\mathbb{R}_+} \zeta x^{m+1} (\partial^k \varphi)(\partial^{k+1} \varphi) - \int_{\mathbb{R}_+} \zeta_x x^{m+1} (\partial^k \varphi)^2$$

The first term can be estimated as in (19), yielding

$$\left|\int_{\mathbb{R}_+} \zeta x^{m+1} (\partial^k \varphi)(\partial^{k+1} \varphi)\right| \lesssim \varepsilon \int_{\mathbb{R}_+} \zeta x^m (\partial^k \varphi)^2 + C_\varepsilon \int_{R_+} \zeta x^{m+2} (\partial^{k+1} \varphi)^2.$$

For the second one we use the identity

$$(\partial^k \varphi)^2 = -(\partial^{k-1} \varphi)(\partial^{k+1} \varphi) + (\frac{1}{2} (\partial^{k-1} \varphi)^2)_{xx}$$

to infer that

$$\left|\int_{\mathbb{R}_+} x^{m+1}\zeta_x(\partial^k\varphi)^2\right| \lesssim \left|\int_{\mathbb{R}_+} x^{m+1}\zeta_x(\partial^{k-1}\varphi)(\partial^{k+1}\varphi)\right| + \left|\int_{\mathbb{R}_+} (x^{m+1}\zeta_x)_{xx}(\partial^{k-1}\varphi)^2\right|.$$

The first integral is estimated by the right-hand side of (20) by Hölder inequality, whereas for the second one we just need to observe that

$$(x^{m+1}\zeta_x)_{xx} \lesssim x^m,$$

since ζ_x is supported in $B_2 - B_1$. Collecting the estimates yields (20). □

4.3 Removing weights

Here we prove Lemmas 3.4 - 3.6, thus completing the proof of Proposition 2.2.

Proof of Lemma 3.4 By the scale invariance (7) and the definition (11) of P_R, it suffices to prove the statement for $R = 1$, that is,

$$\iint_{P_\gamma} x\left(\partial_t^l\partial^k f\right)^2 \lesssim \iint_{P_1} xf^2. \tag{21}$$

Let us first consider the case $l = 0$. We shall start by arguing that for any $k \in \mathbb{N}_0$

$$\iint_{P_\gamma} x^{2k+1}(\partial^{2k+1}f)^2 \lesssim \iint_{P_1} xf^2. \tag{22}$$

Indeed, fix k and select a sequence $\{\gamma_j\}$ strictly increasing from γ to 1. The local energy estimates yield

$$\iint_{P_{\gamma_1}} x^{2k+3}(\partial^{2k+2}f)^2 \lesssim \iint_{P_{\gamma_2}} x^{2k+1}(\partial^{2k}f)^2 \lesssim \cdots \lesssim \iint_{P_1} xf^2. \tag{23}$$

Then we appeal to the localized Hardy estimate

$$\iint_{P_\gamma} x^{2k+1}(\partial^{2k+1}f)^2 \overset{(20)}{\lesssim} \iint_{P_{\gamma_2}} x^{2k+1}(\partial^{2k}f)^2 + \iint_{P_{\gamma_1}} x^{2k+3}(\partial^{2k+2}f)^2. \tag{24}$$

The combination of (23) with (24) yields (22). We once more appeal to the localized Hardy estimate in form of

$$\begin{aligned}\iint_{P_\gamma} x^{2k+1}(\partial^{2k+2}f)^2 &\overset{(20)}{\lesssim} \iint_{P_{\gamma_1}} x^{2k+1}(\partial^{2k+1}f)^2 + \iint_{P_{\gamma_1}} x^{2k+3}(\partial^{2k+3}f)^2 \\ &\overset{(22)}{\lesssim} \iint_{P_1} xf^2.\end{aligned}$$

A finite number of steps yields (21) for $l = 0$.

For $l = 1$ we use the fact that $\partial_t f$ is again a solution. Hence we obtain by the above

$$\iint_{P_\gamma} x(\partial_t\partial^k f)^2 \lesssim \iint_{P_{\gamma_1}} x(\partial_t f)^2.$$

Using the equation $\partial_t f = -x^2\partial^4 f - 6x\partial^3 f - 6\partial^2 f$, this turns into

$$\begin{aligned}&\iint_{P_\gamma} x(\partial_t\partial^k f)^2 \\ &\quad\lesssim \iint_{P_{\gamma_1}} x^5(\partial^4 f)^2 + \iint_{P_{\gamma_1}} x^3(\partial^3 f)^2 + \iint_{P_{\gamma_1}} x(\partial^2 f)^2 \\ &\quad\lesssim \iint_{P_{\gamma_1}} x(\partial^4 f)^2 + \iint_{P_{\gamma_1}} x(\partial^3 f)^2 + \iint_{P_{\gamma_1}} x(\partial^2 f)^2 \\ &\quad\lesssim \int_{P_1} xf^2.\end{aligned}$$

The case of general l follows by iteration of the above argument.

□

Proof of Lemma 3.5 By scaling, it suffices to prove the statement for $R = 1$, that is,

$$\sup_{t\in I_\gamma} \int_{B_\gamma} x\left(\partial_t^l\partial^k f\right)^2 \lesssim \iint_{P_1} xf^2. \tag{25}$$

This follows at once from the straightforward inequality

$$\sup_{t\in I_\gamma} |\varphi| \lesssim \int_{I_\gamma} |\varphi| + \int_{I_\gamma} |\varphi_t|.$$

Indeed:

$$\begin{aligned}
\sup_{t\in I_\gamma}\int_{B_\gamma} x\left(\partial_t^l\partial^k f\right)^2 &\lesssim \int_{I_\gamma}\int_{B_\gamma} x\left(\partial_t^l\partial^k f\right)^2 + \int_{I_\gamma}\left|\frac{d}{dt}\int_{B_\gamma} x\left(\partial_t^l\partial^k f\right)^2\right| \\
&\lesssim \iint_{P_\gamma} x\left(\partial_t^l\partial^k f\right)^2 + \iint_{P_\gamma} x\left|\left(\partial_t^l\partial^k f\right)\left(\partial_t^{l+1}\partial_x^k f\right)\right| \\
&\leq \iint_{P_\gamma} x\left(\partial_t^l\partial^k f\right)^2 \\
&\quad + \left(\iint_{P_\gamma} x\left(\partial_t^l\partial^k f\right)^2\right)^{\frac{1}{2}}\left(\iint_{P_\gamma} x\left(\partial_t^{l+1}\partial_x^k f\right)^2\right)^{\frac{1}{2}} \\
&\overset{(21)}{\lesssim} \iint_{P_1} x f^2.
\end{aligned}$$

□

As a preparation to Lemma 3.6 we need:

Lemma 4.2. *For any $\varphi\in C^\infty((0,1))$,*

$$\sup_{x\in(0,1)}|\varphi| \lesssim \left(\int_0^1 x\varphi^2\right)^{\frac{1}{2}} + \left(\int_0^1 x\varphi_{xx}^2\right)^{\frac{1}{2}}. \tag{26}$$

Proof Let E denote the right-hand side of (26). First we note that

$$\left(\int_{1/4}^{1/2}\varphi_x^2\right)^{1/2} \lesssim E. \tag{27}$$

Indeed, select a non–negative cut–off function ζ with $\zeta = 1$ on $(1/4, 1/2)$ and $\zeta = 0$ outside of $(1/8, 1)$. We have

$$\begin{aligned}
\int\zeta^2\varphi_x^2 &= -2\int\zeta\zeta_x\varphi\varphi_x - \int\zeta^2\varphi_{xx}\varphi \\
&\lesssim \left(\int\zeta^2\varphi_x^2\int\zeta_x^2\varphi^2\right)^{1/2} + \left(\int\zeta^2\varphi_{xx}^2\int\zeta^2\varphi^2\right)^{1/2}
\end{aligned}$$

and thus

$$
\begin{aligned}
\int_{1/4}^{1/2} \varphi_x^2 &\le \int \zeta^2 \varphi_x^2 \\
&\lesssim \int (\zeta_x^2 + \zeta^2)\varphi^2 + \int \zeta^2 \varphi_{xx}^2 \\
&\le \int_{1/8}^{1} \varphi^2 + \int_{1/8}^{1} \varphi_{xx}^2 \\
&\lesssim \int_0^1 x\varphi^2 + \int_0^1 x\varphi_{xx}^2 \lesssim E^2.
\end{aligned}
$$

This establishes (27).

This implies that for all $y \in (0,1)$,

$$
\sup_{(y,1)} |\varphi_x| \lesssim \left(1 + \left(\ln \frac{1}{y}\right)^{\frac{1}{2}}\right) E. \tag{28}
$$

Indeed,

$$
\begin{aligned}
\sup_{(y,1)} |\varphi_x| &\lesssim \left(\int_{1/4}^{1/2} \varphi_x^2\right)^{1/2} + \int_y^1 |\varphi_{xx}| \\
&\overset{(27)}{\lesssim} E + \left(\int_y^1 x^{-1}\right)^{\frac{1}{2}} \left(\int_y^1 x\varphi_{xx}^2\right)^{\frac{1}{2}} \\
&\lesssim \left(1 + \left(\ln \frac{1}{y}\right)^{\frac{1}{2}}\right) E.
\end{aligned}
$$

Therefore, we conclude:

$$
\begin{aligned}
\sup_{x \in (0,1)} |\varphi| &\lesssim \int_{1/2}^{1} |\varphi| + \int_0^1 |\varphi_x| \\
&\lesssim E + E \int_0^1 \left(1 + \left(\ln \frac{1}{y}\right)^{\frac{1}{2}}\right) dy \\
&\lesssim E.
\end{aligned}
$$

□

Proof of Lemma 3.6 By scaling, it suffices to prove that

$$
\sup_{(t,x) \in P_\gamma} \left|\partial_t^l \partial^k f\right|^2 \lesssim \iint_{P_1} x f^2, \tag{29}
$$

which follows at once from (25) and (26):

$$\begin{aligned}\sup_{(t,x)\in P_\gamma} \left|\partial_t^l \partial^k f\right|^2 &= \sup_{t\in I_\gamma} \sup_{x\in B_\gamma} \left|\partial_t^l \partial^k f\right|^2 \\ &\overset{(26)}{\lesssim} \sup_{t\in I_\gamma} \left\{ \int_{B_\gamma} x\left(\partial_t^l \partial^k f\right)^2 + \int_{B_\gamma} x\left(\partial_t^l \partial^{k+2} f\right)^2 \right\} \\ &\overset{(25)}{\lesssim} \iint_{P_1} x f^2 .\end{aligned}$$

□

5 Proof of the L^∞–bound

In this section we consider the inhomogeneous equation:

$$f_t + \frac{1}{x}(x^3 f_{xx})_{xx} = g.$$

The assumptions of Proposition 2.3 are in force throughout, and without loss of generality $s = 1$, that is, g is supported in P_1.

Proof of Lemma 3.7 We need to show that

$$\sup_t \int_{\mathbb{R}_+} x f^2 \;\lesssim\; \|g\|^2_{C^0(\mathbb{R}\times\mathbb{R}_+)}. \tag{30}$$

We have

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}_+} \frac{1}{2} x f^2 &= -\int_{\mathbb{R}_+} f(x^3 f_{xx})_{xx} + \int_{\mathbb{R}_+} x f g \\ &= -\int_{R_+} x^3 f_{xx}^2 + \int_{\mathbb{R}_+} x f g \\ &\lesssim \left\{ \begin{array}{ll} \int_{\mathbb{R}_+} x f^2 + \int_{\mathbb{R}_+} x g^2 & \text{for } -1 \le t \le 0 \\ 0 & \text{else} \end{array} \right\}.\end{aligned}$$

Since $f(t = -1) = 0$, integration in time gives

$$\sup_t \int_{\mathbb{R}_+} x f^2 \;\lesssim\; \int_0^1 \int_0^1 x g^2 \lesssim \|g\|^2_{C^0(\mathbb{R}\times\mathbb{R}_+)}.$$

□

Proof of Lemma 3.8 We claim that

$$\sup_t \int_{R_+} x^2 f_{xx}^2 \lesssim \|g\|^2_{C^0(\mathbb{R}\times\mathbb{R}_+)}. \tag{31}$$

We have

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}_+} x^2 f_{xx}^2 &= \int_{\mathbb{R}_+} (x^2 f_{xx})_{xx} f_t \\ &= -\int_{\mathbb{R}_+} (x^2 f_{xx})_{xx} \frac{1}{x}(x^3 f_{xx})_{xx} + \int_{\mathbb{R}_+} (x^2 f_{xx})_{xx} g. \end{aligned}$$

A few integrations by part show that the homogeneous part has a sign:

$$\begin{aligned} &-\int_{\mathbb{R}_+} (x^2 f_{xx})_{xx} \frac{1}{x}(x^3 f_{xx})_{xx} \\ &= -\int_{\mathbb{R}_+} (2f_{xx} + 4x f_{xxx} + x^2 f_{xxxx})(6 f_{xx} + 6x f_{xxx} + x^2 f_{xxxx}) \\ &= -\int_{\mathbb{R}_+} (12 f_{xx}^2 + 36 x f_{xx} f_{xxx} + 8x^2 f_{xx} f_{xxxx} \\ &\qquad + 10 x^3 f_{xxx} f_{xxxx} + 24 x^2 f_{xxx}^2 + x^4 f_{xxxx}^2) \\ &= -\int_{\mathbb{R}_+} (x^4 f_{xxxx}^2 + x^2 f_{xxx}^2 + 2 f_{xx}^2). \end{aligned}$$

This means that the inhomogeneous part

$$\int_{\mathbb{R}_+} (x^2 f_{xx})_{xx} g = \int_{\mathbb{R}_+} (2 f_{xx} + 4x f_{xxx} + x^2 f_{xxxx}) g$$

can be absorbed into the homogeneous part. Arguing as in the proof of Lemma 3.7 yields (31). □

Proof of Proposition 2.3 It follows from (31) and Hardy inequality (cf. Lemma 4.1) that

$$\sup_t \left(\int_{\mathbb{R}_+} f_x^2\right)^{\frac{1}{2}} \lesssim \|g\|_{C^0(\mathbb{R}\times\mathbb{R}_+)}. \tag{32}$$

Therefore

$$\begin{aligned} \sup_{x\in(1,\infty)} |f| &\lesssim \left(\int_1^\infty f^2 \int_1^\infty f_x^2\right)^{1/4} \\ &\lesssim \left(\int_1^\infty x f^2 \int_1^\infty f_x^2\right)^{1/4} \\ &\overset{(30),(32)}{\lesssim} \|g\|_{C^0(\mathbb{R}\times\mathbb{R}_+)}. \end{aligned} \tag{33}$$

In turn, this gives

$$\sup_{x\in[0,1)} |f| \quad \lesssim \quad |f(x=1)| + \left(\int_{\mathbb{R}_+} f_x^2\right)^{1/2}$$
$$\overset{(32),(33)}{\lesssim} \quad \|g\|_{C^0(\mathbb{R}\times\mathbb{R}_+)},$$

and the proof is complete. □

6 Proof of the remainder estimates

6.1 Remainder estimates at the boundary

The first lemma is at the core of Safonov's approach to Hölder estimates, see also [9]. Indeed, here we see that inner estimates and L^∞-bound are the two ingredients of a polynomial approximation argument.

Lemma 6.1. *Assume that f is smooth in P_s. Then there exists a polynomial p of degree 2 in x and degree 1 in t such that*

$$\|f-p\|_{C^0(P_r)} \;\lesssim\; \frac{r^3}{s^3}\|f\|_{C^0(P_s)} + s^2\|L_0 f\|_{C^0(P_s)}$$

for all $0 < r \leq s$.

Proof By the scale invariance (7), we may let $s = 1$. The idea is to split f into a near–field part h and a far–field part k:

$$f \;=\; h + k. \tag{34}$$

To this purpose, we choose a cut-off function ζ such that $\zeta = 1$ in $P_{\frac{1}{2}}$ and $\zeta = 0$ outside of P_1. The near–field part is defined as the solution of

$$L_0 h \;=\; \zeta\, L_0 f,$$

with vanishing "initial" data $h(-1,\cdot) \equiv 0$. The L^∞-bound in Proposition 2.3 implies that

$$\|h\|_{C^0(P_1)} \;\lesssim\; \|L_0 f\|_{C^0(P_1)}. \tag{35}$$

The far field part is defined via $k := f - h$ so that (34) holds.

We choose p to be the Taylor polynomial in zero of the far–field part, i. e.

$$p(t,x) \;=\; (T_2^0 k)(t,x) \;=\; k(0,0) + k_x(0,0)x + \frac{1}{2}\,k_{xx}(0,0)x^2 + k_t(0,0)t.$$

Since by construction

$$L_0 k\ x = \ L_0 f - \zeta L_0 f \ = \ 0 \quad \text{in } P_{1/2},$$

the inner estimate in Proposition 2.2 yields

$$\|k\|_{C^3(P_{1/2})} \ \lesssim \ \|k\|_{C^0(P_1)}. \tag{36}$$

We thus obtain for $r \le 1/2$

$$\|k - p\|_{C^0(P_r)} \ \lesssim \ \|k\|_{C^3(P_{1/2})} \sup_{P_r}(x^3, |t|x, t^2) \ \overset{(36)}{\lesssim} \ r^3 \, \|k\|_{C^0(P_1)}.$$

In case of $r \ge 1/2$ we have

$$\begin{aligned} \|k - p\|_{C^0(P_r)} \quad &\le \quad \|k\|_{C^0(P_1)} + \|p\|_{C^0(P_1)} \\ &\lesssim \quad \|k\|_{C^0(P_1)} + \max(|k(0)|, |k_x(0)|, |k_{xx}(0)|, |k_t(0)|) \\ &\overset{(36)}{\lesssim} \quad \|k\|_{C^0(P_1)} \ \sim \ r^3 \|k\|_{C^0(P_1)}. \end{aligned}$$

We thus have for any $r \le 1$

$$\|k - p\|_{C^0(P_r)} \ \lesssim \ r^3 \|k\|_{C^0(P_1)}. \tag{37}$$

Now we are ready to complete the proof:

$$\begin{aligned} \|f - p\|_{C^0(P_r)} \ &\overset{(34)}{\le} \ \ \|k - p\|_{C^0(P_r)} + \|h\|_{C^0(P_r)} \\ &\overset{(37)}{\lesssim} \ \ r^3 \|k\|_{C^0(P_1)} + \|h\|_{C^0(P_1)} \\ &\overset{(34)}{\le} \ \ r^3 \|f\|_{C^0(P_1)} + 2\|h\|_{C^0(P_1)} \\ &\overset{(35)}{\lesssim} \ \ r^3 \|f\|_{C^0(P_1)} + \|L_0 f\|_{C^0(P_1)}. \end{aligned}$$

□

The next next lemma connects polynomial approximation to Hölder norms.

Lemma 6.2. *For every smooth function f in P_1 with $T_2^0 f = 0$ we have*

$$\sup_{0 \le \rho \le 1} \frac{\|f\|_{C^0(P_\rho)}}{\rho^{2+\alpha}} \ \lesssim \ \|f\|_{C^0(P_1)} + \sup_{0 \le \rho \le 1} \frac{\|L_0 f\|_{C^0(P_\rho)}}{\rho^\alpha}. \tag{38}$$

Proof Let the numbers q, r, s satisfy

$$0 \leq q \leq r \leq s \leq 1.$$

Note that for any polynomial p of degree 2 in x and degree 1 in t we have the inverse estimate

$$\|p\|_{C^0(P_r)} \lesssim \frac{r^2}{q^2}\|p\|_{C^0(P_q)}. \tag{39}$$

We use this to conclude

$$\begin{aligned}
\|f\|_{C^0(P_r)} &\leq \|f-p\|_{C^0(P_r)} + \|p\|_{C^0(P_r)} \\
&\overset{(39)}{\lesssim} \|f-p\|_{C^0(P_r)} + \frac{r^2}{q^2}\|p\|_{C^0(P_q)} \\
&\leq \|f-p\|_{C^0(P_r)} + \frac{r^2}{q^2}\|f-p\|_{C^0(P_q)} + \frac{r^2}{q^2}\|f\|_{C^0(P_q)} \\
&\leq 2\frac{r^2}{q^2}\|f-p\|_{C^0(P_r)} + \frac{r^2}{q^2}\|f\|_{C^0(P_q)}.
\end{aligned}$$

According to Lemma 6.1 we thus obtain

$$\begin{aligned}
\|f\|_{C^0(P_r)} &\lesssim \frac{r^2}{q^2}\left(\frac{r^3}{s^3}\|f\|_{C^0(P_s)} + s^2\|L_0 f\|_{C^0(P_s)}\right) + \frac{r^2}{q^2}\|f\|_{C^0(P_q)} \\
&= \frac{r^5}{q^2 s^3}\|f\|_{C^0(P_s)} + \frac{r^2}{q^2}\|f\|_{C^0(P_q)} + \frac{r^2 s^2}{q^2}\|L_0 f\|_{C^0(P_s)}.
\end{aligned}$$

We rewrite this as

$$\begin{aligned}
&\frac{\|f\|_{C^0(P_r)}}{r^{2+\alpha}} \\
&\lesssim \frac{r^{3-\alpha}}{q^2 s^{1-\alpha}}\frac{\|f\|_{C^0(P_s)}}{s^{2+\alpha}} + \frac{q^\alpha}{r^\alpha}\frac{\|f\|_{C^0(P_q)}}{q^{2+\alpha}} + \frac{s^{2+\alpha}}{q^2 r^\alpha}\frac{\|L_0 f\|_{C^0(P_s)}}{s^\alpha} \\
&\lesssim \left(\frac{r^{3-\alpha}}{q^2 s^{1-\alpha}} + \frac{q^\alpha}{r^\alpha}\right)\sup_{0\leq\rho\leq 1}\frac{\|f\|_{C^0(P_\rho)}}{\rho^{2+\alpha}} + \frac{s^{2+\alpha}}{q^2 r^\alpha}\sup_{0\leq\rho\leq 1}\frac{\|L_0 f\|_{C^0(P_\rho)}}{\rho^\alpha}.
\end{aligned}$$

Since $\alpha \in (0,1)$ we may first choose $q \leq r$ sufficiently small and then $s \geq r$ sufficiently large so that the term in the brackets becomes smaller than, say, $\frac{1}{2}$. Hence there exists an $C < \infty$ such that

$$\sup_{0\leq r\leq\frac{1}{C}}\frac{\|f\|_{C^0(P_r)}}{r^{2+\alpha}} \leq \frac{1}{2}\sup_{0\leq\rho\leq 1}\frac{\|f\|_{C^0(P_\rho)}}{\rho^{2+\alpha}} + C\sup_{0\leq\rho\leq 1}\frac{\|L_0 f\|_{C^0(P_\rho)}}{\rho^\alpha}. \tag{40}$$

On the other hand, we trivially have

$$\sup_{\frac{1}{C}\le r\le 1}\frac{\|f\|_{C^0(P_r)}}{r^{2+\alpha}} \;\lesssim\; \|f\|_{C^0(P_1)}. \tag{41}$$

The lemma follows from combining (40) with (41) since $\sup_{0\le\rho\le 1}\frac{\|f\|_{C^0(P_\rho)}}{\rho^{2+\alpha}} < \infty$ by our assumption that $T_2^0 f = 0$. $\square$

Now we are in position to prove the remainder estimates at the boundary.

Proof of Lemma 3.9 Recall that we have to show

$$\sup_{0\le\rho\le 1}\frac{\|R_2^0 f\|_{C^0(P_\rho)}}{\rho^{2+\alpha}} \;\lesssim\; \|f\|_{C^0(P_1)} + \|L_0 f\|_{H_s^\beta(P_1)}. \tag{42}$$

We apply (38) to $R_2^0 f$ and obtain

$$\sup_{0\le\rho\le 1}\frac{\|R_2^0 f\|_{C^0(P_\rho)}}{\rho^{2+\alpha}} \;\lesssim\; \|R_2^0 f\|_{C^0(P_1)} + \sup_{0\le\rho\le 1}\frac{\|L_0 R_2^0 f\|_{C^0(P_\rho)}}{\rho^{\alpha}}. \tag{43}$$

We start by remarking that $L_0 T_2^0 f = (L_0 f)(0)$ so that

$$L_0 R_2^0 f \;=\; L_0 f - (L_0 f)(0).$$

Therefore

$$\begin{aligned}
\|L_0 R_2^0 f\|_{C^0(P_\rho)} &= \|L_0 f - (L_0 f)(0)\|_{C^0(P_\rho)} \\
&\le \left(\sup_{z\in P_\rho} c(z,0)\right)^{\beta} \|L_0 f\|_{H_s^\beta(P_1)} \\
&\lesssim \rho^{\alpha}\|L_0 f\|_{H_s^\beta(P_1)}
\end{aligned}$$

with $\beta = 2\alpha$. Hence (43) turns into

$$\sup_{0\le\rho\le 1}\frac{\|R_2^0 f\|_{C^0(P_\rho)}}{\rho^{2+\alpha}} \;\lesssim\; \|R_2^0 f\|_{C^0(P_1)} + \|L_0 f\|_{H_s^\beta(P1)}. \tag{44}$$

It remains to show

$$\|R_2^0 f\|_{C^0(P_1)} \;\le\; \|f\|_{C^0(P_1)} + \|L_0 f\|_{H_s^\beta(P_1)}. \tag{45}$$

By polynomial scaling, i. e. (39), we have for every $\rho \leq 1$:

$$
\begin{aligned}
\|T_2^0 f\|_{C^0(P_1)} &\lesssim \frac{1}{\rho^2}\|T_2^0 f\|_{C^0(P_\rho)} \\
&\leq \frac{1}{\rho^2}(\|R_2^0 f\|_{C^0(P_\rho)} + \|f\|_{C^0(P_\rho)}) \\
&\overset{(44)}{\lesssim} \frac{1}{\rho^2}\Big(\rho^{2+\alpha}\big(\|R_2^0 f\|_{C^0(P_1)} + \|L_0 f\|_{H_s^\beta(P_1)}\big) + \|f\|_{C^0(P_1)}\Big) \\
&\leq \rho^\alpha\left(\|f\|_{C^0(P_1)} + \|T_2^0 f\|_{C^0(P_1)}\right) \\
&\quad + \rho^\alpha \|L_0 f\|_{H_s^\beta(P_1)} + \frac{1}{\rho^2}\|f\|_{C^0(P_1)}.
\end{aligned}
$$

By choosing a sufficiently small $\rho > 0$, the above turns into

$$
\|T_2^0 f\|_{C^0(P_1)} \lesssim \|f\|_{C^0(P_1)} + \|L_0 f\|_{H_s^\beta(P_1)}.
$$

This implies (45):

$$
\begin{aligned}
\|R_2^0 f\|_{C^0(P_1)} &\leq \|f\|_{C^0(P_1)} + \|T_2^0 f\|_{C^0(P_1)} \\
&\lesssim \|f\|_{C^0(P_1)} + \|L_0 f\|_{H_s^\beta(P_1)}.
\end{aligned}
$$

□

6.2 Scaled interior estimates

Proof of Lemma 3.10 Since L_0 is uniformly fourth–order parabolic in the cube $(-1,0) \times (1/4, 1)$, we have by standard local Schauder estimates

$$
\begin{aligned}
&\|R_4^{(0,1/2)} f\|_{C^0((-1/2,0)\times(1/3,2/3))} \\
&\quad \lesssim \|f\|_{C^0((-1,0)\times(1/4,1))} + \|L_0 f\|_{H^\beta((-1,0)\times(1/4,1))},
\end{aligned}
$$

where H^β is the Hölder semi–norm w. r. t. the standard metric for fourth–order parabolic operators, i. e. $|t_1 - t_2|^{1/4} + |x_1 - x_2|$. Since this standard metric is equivalent to our metric $|t_1 - t_2|^{1/4} + |\sqrt{x_1} - \sqrt{x_2}|$ on $(-1,0)\times(1/4,1)$, the above estimate turns into

$$
\begin{aligned}
&\|R_4^{(0,1/2)} f\|_{C^0((-1/2,0)\times(1/3,2/3))} \\
&\quad \lesssim \|f\|_{C^0((-1,0)\times(1/4,1))} + \|L_0 f\|_{H_s^\beta((-1,0)\times(1/4,1))}.
\end{aligned}
$$

Since the operator is invariant under the rescaling $(t,x) \rightsquigarrow (r^2 t, rx)$, the last estimate entails

$$
\|R_4^{(0,r/2)} f\|_{C^0((-r^2/2,0)\times(r/3,2r/3))} \lesssim \|f\|_{C^0(Q_r)} + r^{2+\alpha}\|L_0 f\|_{H_s^\beta(Q_r)}. \tag{46}
$$

In the last step, we appeal to an inverse estimate for polynomials of order at most four:

$$\begin{aligned}\|R_4^{(0,r/2)}f\|_{C^0(Q_r)} &\le \|T_4^{(0,r)}f\|_{C^0(Q_r)} + \|f\|_{C^0(Q_r)} \\ &\lesssim \|T_4^{(0,r)}f\|_{C^0((-r^2/2,0)\times(r/3,2r/3))} + \|f\|_{C^0(Q_r)} \\ &\le \|R_4^{(0,r)}f\|_{C^0((-r^2/2,0)\times(r/3,2r/3))} + 2\|f\|_{C^0(Q_r)}. \quad (47)\end{aligned}$$

The combination of (46) and (47) yields as desired

$$\|R_4^{(0,r/2)}f\|_{C^0(Q_r)} \lesssim \|f\|_{C^0(Q_r)} + r^{2+\alpha}\|L_0 f\|_{H_s^\beta(Q_r)}. \quad (48)$$

□

6.3 Remainder estimates in the interior

Proof of Lemma 3.11 Recall that we want to prove for $r \le 1$:

$$\|R_4^{(0,r/2)}f\|_{C^0(Q_r)} \lesssim r^{2+\alpha}\left(\|f\|_{C^0(P_1)} + \|L_0 f\|_{H_s^\beta(P_1)}\right). \quad (49)$$

We notice that $R_4^{(0,r/2)}T_2^0 f = 0$ which yields

$$R_4^{(0,r/2)}f \;=\; R_4^{(0,r/2)}R_2^0 f.$$

This allows us to make use Lemma 3.10 (applied to $R_0^2 f$) and then of Lemma 3.9 (applied to f):

$$\begin{aligned}&\|R_4^{(0,r/2)}f\|_{C^0(Q_r)} \\ &= \|R_4^{(0,r/2)}R_2^0 f\|_{C^0(Q_r)} \\ &\overset{(48)}{\lesssim} \|R_2^0 f\|_{C^0(Q_r)} + r^{2+\alpha}\|L_0 R_2^0 f\|_{H_s^\beta(Q_r)} \\ &\le \|R_2^0 f\|_{C^0(P_r)} + r^{2+\alpha}\|L_0 R_2^0 f\|_{H_s^\beta(P_1)} \\ &\overset{(42)}{\lesssim} r^{2+\alpha}\|f\|_{C^0(P_1)} + r^{2+\alpha}\|L_0 f\|_{H_s^\beta(P_1)} + r^{2+\alpha}\|L_0 R_2^0 f\|_{H_s^\beta(P_1)}. \quad (50)\end{aligned}$$

We now appeal to

$$L_0 R_2^0 f - L_0 f \;=\; L_0 T_2^0 f \;=\; const,$$

so that

$$\|L_0 R_2^0 f\|_{H_s^\beta(P_1)} \;=\; \|L_0 f\|_{H_s^\beta(P_1)}. \quad (51)$$

Now (49) follows from (50) and (51). □

7 Proof of the Schauder estimate

Proof of Lemma 3.12 Recall that we want to show for $(t,x) \in P_{1/2}$:

$$\begin{aligned}
&|x^2 f_{xxxx}(t,x)| + |x f_{xxx}(t,x)| \\
&\quad +|f_{xx}(t,x) - f_{xx}(t,0)| + |f_t(t,x) - f_t(t,0)| \\
&\quad \lesssim\ s((t,x),(t,0))^\beta \left(\|f\|_{C^0(P_2)} + \|L_0 f\|_{H_s^\beta(P_2)}\right).
\end{aligned}$$

By the translation in time invariance, it suffices to show that for $0 \le x \le \frac{1}{2}$:

$$\begin{aligned}
&|x^2 f_{xxxx}(0,x)| + |x f_{xxx}(0,x)| \\
&\quad +|f_{xx}(0,x) - f_{xx}(0,0)| + |f_t(0,x) - f_t(0,0)| \\
&\quad \lesssim\ x^\alpha (\|f\|_{C^0(P_1)} + \|L_0 f\|_{H_s^\beta(P_1)}).
\end{aligned} \tag{52}$$

In order to establish this, we will argue that

$$\begin{aligned}
&|x^2 f_{xxxx}(0,x)| + |x f_{xxx}(0,x)| \\
&\quad +|f_{xx}(0,x) - f_{xx}(0,0)| + |f_t(0,x) - f_t(0,0)| \\
&\quad \lesssim\ \frac{1}{x^2}\|R_4^{(0,x)} f\|_{C^0(Q_{2x})} + \frac{1}{x^2}\|R_2^0 f\|_{C^0(P_{2x})}.
\end{aligned} \tag{53}$$

Indeed, the remainder estimates (42) and (49) allow us to conclude (52) from (53).

We now address (53). We start with $x^2 f_{xxxx}$. By definition of $T_4^{(0,x)}$, we have

$$f_{xxxx}(0,x) \ = \ (T_4^{(0,x)} f)_{xxxx}(0,x) \ = \ const.$$

Since $(T_2^0 f)_{xxxx} = 0$, this can be reformulated as

$$f_{xxxx}(0,x) \ = \ (T_4^{(0,x)} f - T_2^0 f)_{xxxx}(0,x).$$

We now appeal to the following inverse estimate for polynomials p of order at most 4:

$$\begin{aligned}
x^4 \, |p_{xxxx}(0,x)| + x^3 \, |p_{xxx}(0,x)| + x^2 \, |p_{xx}(0,x)| + x^2 \, |p_t(0,x)| \\
\lesssim \ \|p\|_{C^0(Q_{2x})}.
\end{aligned} \tag{54}$$

Applied to $T_4^{(0,x)} f - T_2^0 f$, (54) yields

$$\begin{aligned}
|x^2 f_{xxxx}(0,x)| \ &\lesssim\ \frac{1}{x^2}\|T_4^{(0,x)} f - T_2^0 f\|_{C^0(Q_{2x})} \\
&=\ \frac{1}{x^2}\|R_4^{(0,x)} f - R_2^0 f\|_{C^0(Q_{2x})} \\
&\le\ \frac{1}{x^2}\|R_4^{(0,x)} f\|_{C^0(Q_{2x})} + \frac{1}{x^2}\|R_2^0 f\|_{C^0(P_{2x})}.
\end{aligned}$$

Also the next term in (53) can be written in a similar way

$$f_{xxx}(0,x) \;=\; (T_4^{(0,x)}f - T_2^0 f)_{xxx}(0,x),$$

so that likewise

$$\begin{aligned} |xf_{xxx}(0,x)| &\overset{(54)}{\lesssim} \frac{1}{x^2}\|T_4^{(0,x)}f - T_2^0 f\|_{C^0(Q_{2x})} \\ &\le \frac{1}{x^2}\|R_4^{(0,x)}f\|_{C^0(Q_{2x})} + \frac{1}{x^2}\|R_2^0 f\|_{C^0(P_{2x})}. \end{aligned}$$

For the third term in (53) we notice that

$$f_{xx}(0,x) - f_{xx}(0,0) \;=\; (T_4^{(0,x)}f)_{xx}(0,x) - (T_2^0 f)_{xx}(0,0).$$

Since $(T_2^0 f)_{xx} = \mathit{const}$, this can be reformulated as

$$f_{xx}(0,x) - f_{xx}(0,0) \;=\; (T_4^{(0,x)}f - T_2^0 f)_{xx}(0,x),$$

so that as above

$$\begin{aligned} |f_{xx}(0,x) - f_{xx}(0,0)| &\overset{(54)}{\lesssim} \frac{1}{x^2}\|T_4^{(0,x)}f - T_2^0 f\|_{C^0(Q_{2x})} \\ &\le \frac{1}{x^2}\|R_4^{(0,x)}f\|_{C^0(Q_{2x})} + \frac{1}{x^2}\|R_2^0 f\|_{C^0(P_{2x})}. \end{aligned}$$

Finally, the same argument applies to the fourth term in (53): Since $(T_2^0 f)_t = \mathit{const}$, we have

$$f_{xx}(0,x) - f_{xx}(0,0) \;=\; (T_4^{(0,x)}f - T_2^0 f)_t(0,x),$$

so that also here, using (54),

$$|f_t(0,x) - f_t(0,0)| \;\lesssim\; \frac{1}{x^2}\|R_4^{(0,x)}f\|_{C^0(Q_{2x})} + \frac{1}{x^2}\|R_2^0 f\|_{C^0(P_{2x})}.$$

This establishes (53). □

Proof of Lemma 3.13 Since L_0 is uniformly fourth–order parabolic in the cube $(-2,0)\times(1/8,2)$, we have by standard local Schauder estimates

$$\begin{aligned} &\|x^2 f_{xxxx}, xf_{xxx}, f_{xx}, f_t\|_{H^\beta((-1,0)\times(1/4,1))} \\ &\quad\lesssim \|f\|_{C^0((-2,0)\times(1/8,2))} + \|L_0 f\|_{H^\beta((-2,0)\times(1/8,2))}. \end{aligned}$$

Since the standard metric for fourth–order parabolic operators is equivalent to our metric on $(-2,0)\times(1/8,2)$, the above estimate turns into

$$\begin{aligned}&\|x^2 f_{xxxx}, x f_{xxx}, f_{xx}, f_t\|_{H_s^\beta((-1,0)\times(1/4,1))}\\&\quad\lesssim \|f\|_{C^0((-2,0)\times(1/8,2))} + \|L_0 f\|_{H_s^\beta((-2,0)\times(1/8,2))}.\end{aligned}$$

Since the operator L_0 and its components, i. e. $x^2 f_{xxxx}$, $x f_{xxx}$, f_{xx}, f_t, scale the same under $(t,x)\rightsquigarrow(r^2 t, rx)$, the last estimate entails

$$\begin{aligned}\|f\|_{H_s^{2+\beta}(Q_r)} &= \|x^2 f_{xxxx}, x f_{xxx}, f_{xx}, f_t\|_{H_s^\beta(Q_r)}\\&\lesssim \frac{1}{r^{2+\alpha}}\|f\|_{C^0((-2r^2,0)\times(r/8,2r))} + \|L_0 f\|_{H_s^\beta((-2r^2,0)\times(r/8,2r))}.\end{aligned} \tag{55}$$

We now apply (55) to $R_2^0 f$. Since $(R_2^0 f)_{xxxx} = f_{xxxx}$, $(R_2^0 f)_{xxx} = f_{xxx}$, $(R_2^0 f)_{xx} - f_{xx} = const$, $(R_2^0 f)_t - f_t = const$ and thus $L_0 R_2^0 f - L_0 f = const$, (55) turns into

$$\|f\|_{H_s^{2+\beta}(Q_r)} \lesssim \frac{1}{r^{2+\alpha}}\|R_2^0 f\|_{C^0((-2r^2,0)\times(r/8,2r))} + \|L_0 f\|_{H_s^\beta((-2r^2,0)\times(r/8,2r))}. \tag{56}$$

We now evoke Lemma 3.9: Since $(-2r^2,0)\times(r/8,2r)\subset P_1$ for $0\le r\le\frac{1}{2}$ we have

$$\frac{1}{r^{2+\alpha}}\|R_2^0 f\|_{C^0((-2r^2,0)\times(r/8,2r))} \lesssim \|f\|_{C^0(P_1)} + \|L_0 f\|_{H_s^\beta(P_1)}. \tag{57}$$

The lemma follows from the combination of estimates (56) & (57). □

Proof of Lemma 3.14 In view of Lemma 3.12, it remains to show for $(t_1,0),(t_2,0)\in P_{1/4}$:

$$\begin{aligned}&|f_{xx}(t_1,0)-f_{xx}(t_2,0)| + |f_t(t_1,0)-f_t(t_2,0)|\\&\quad\lesssim s((t_1,0),(t_2,0))^\beta\left(\|f\|_{C^0(P_4)} + \|L_0 f\|_{H_s^\beta(P_4)}\right).\end{aligned}$$

By translation in time invariance, it suffices to show for $-1/4\le t\le 0$

$$\begin{aligned}&|f_{xx}(t,0)-f_{xx}(0,0)| + |f_t(t,0)-f_t(0,0)|\\&\quad\lesssim (-t)^{\beta/4}\left(\|f\|_{C^0(P_2)} + \|L_0 f\|_{H_s^\beta(P_2)}\right).\end{aligned} \tag{58}$$

We introduce $x := (-t)^{1/2}\le 1/2$ and write

$$\begin{aligned}&|f_{xx}(t,0)-f_{xx}(0,0)| + |f_t(t,0)-f_t(0,0)|\\&\quad\le |f_{xx}(t,0)-f_{xx}(t,x)| + |f_t(t,0)-f_t(t,x)|\\&\quad\quad+|f_{xx}(t,x)-f_{xx}(0,x)| + |f_t(t,x)-f_t(0,x)|\\&\quad\quad+|f_{xx}(0,x)-f_{xx}(0,0)| + |f_t(0,x)-f_t(0,0)|.\end{aligned} \tag{59}$$

Since $(t,x),(0,x) \in P_{1/2}$, Lemma 3.12 yields

$$\begin{aligned} & |f_{xx}(t,0) - f_{xx}(t,x)| + |f_t(t,0) - f_t(t,x)| \\ & \quad + |f_{xx}(0,x) - f_{xx}(0,0)| + |f_t(0,x) - f_t(0,0)| \\ & \quad \lesssim \sqrt{x}^{\beta} \left(\|f\|_{C^0(P_2)} + \|L_0 f\|_{H_s^\beta(P_2)} \right) \\ & \quad = (-t)^{\beta/4} \left(\|f\|_{C^0(P_2)} + \|L_0 f\|_{H_s^\beta(P_2)} \right). \end{aligned} \tag{60}$$

Since $(t,x),(0,x) \in Q_x$, Lemma 3.13 yields

$$\begin{aligned} & |f_{xx}(t,x) - f_{xx}(0,x)| + |f_t(t,x) - f_t(0,x)| \\ & \quad \lesssim (-t)^{\beta/4} \left(\|f\|_{C^0(P_1)} + \|L_0 f\|_{H_s^\beta(P_1)} \right) \end{aligned} \tag{61}$$

Inserting (60) and (61) into (59) yields (58).

□

Proof of Lemma 3.15 Let g denote any of the four functions xxf_{xxxx}, xf_{xxx}, f_{xx}, f_t. Recall that we want to show for any $z_1, z_2 \in P_{1/4}$

$$|g(z_1) - g(z_2)| \lesssim s(z_1,z_2)^{\beta} \left(\|f\|_{C^0(P_8)} + \|L_0 f\|_{H_s^\beta(P_8)} \right).$$

W. l. o. g. we may assume $x_2 \leq x_1$. We distinguish two cases:

$$\left\{ \begin{array}{ll} \text{Case I:} & \sqrt{x_1} \geq 2s(z_1,z_2) \\ \text{Case II:} & \sqrt{x_1} \leq 2s(z_1,z_2) \end{array} \right\}. \tag{62}$$

In case I we have

$$x_2 \geq \frac{x_1}{4} \quad \text{and} \quad |t_1 - t_2| \leq \frac{x_1^2}{8}.$$

This implies that there exists a $t \in (-1/4, 0)$ such that

$$z_1,\ z_2 \in (t,0) + Q_{x_1}.$$

By translation invariance in time, we may appeal to Lemma 3.13 which yields as desired

$$|g(z_1) - g(z_2)| \lesssim s(z_1,z_2)^{\beta} \left(\|f\|_{C^0((t,0)+P_4)} + \|L_0 f\|_{H_s^\beta((t,0)+P_4)} \right).$$

In case II we write

$$|g(z_1) - g(z_2)| \leq |g(z_1) - g(t_1,0)| + |g(t_1,0) - g(t_2,0)| + |g(t_2,0) - g(z_2)|$$

and evoke Lemma 3.14 which yields

$$
\begin{aligned}
|g(z_1) - g(z_2)| &\lesssim (s(z_1,(t_1,0)) + s((t_1,0),(t_2,0)) + s((t_2,0),z_2))^\beta \\
&\quad \times \left(\|f\|_{C^0(P_4)} + \|L_0 f\|_{H_s^\beta(P_4)}\right) \\
&= \left(\sqrt{x_1} + |t_1 - t_2|^{1/4} + \sqrt{x_2}\right)^\beta \left(\|f\|_{C^0(P_4)} + \|L_0 f\|_{H_s^\beta(P_4)}\right) \\
&\overset{(62)}{\le} (5s(z_1,z_2))^\beta \left(\|f\|_{C^0(P_4)} + \|L_0 f\|_{H_s^\beta(P_4)}\right).
\end{aligned}
$$

□

Proof of Theorem 2.1 According to Lemma 3.15 we have

$$
\|f\|_{H_s^{2+\beta}(P_{1/4})} \lesssim \|f\|_{C^0(P_8)} + \|L_0 f\|_{C_s^\beta(P_8)} \tag{63}
$$

for all f smooth in P_8. Let f be smooth and bounded in $\mathbb{R} \times \mathbb{R}_+$. Consider

$$
f_R(t,x) := f(R^2(t - 1/8), Rx).
$$

Then

$$
\begin{aligned}
\|f_R\|_{H_s^{2+\beta}(P_{1/4})} &= R^{2+\frac{\beta}{2}} \|f\|_{H_s^{2+\beta}((-R/8,R/8)\times(0,R/4))}, \\
\|f_R\|_{C^0(P_8)} &\le \|f\|_{C^0(\mathbb{R}\times\mathbb{R}_+)}, \\
\|L_0 f_R\|_{H_s^\beta(P_8)} &\le R^{2+\frac{\beta}{2}} \|L_0 f\|_{H_s^\beta(\mathbb{R}\times\mathbb{R}_+)}.
\end{aligned}
$$

Therefore (63) turns into

$$
\|f\|_{H_s^{2+\beta}((-R/8,R/8)\times(0,R/4))} \lesssim R^{-2-\frac{\beta}{2}} \|f\|_{C^0(\mathbb{R}\times\mathbb{R}_+)} + \|L_0 f\|_{H_s^\beta(\mathbb{R}\times\mathbb{R}_+)}.
$$

Since $\|f\|_{C^0(\mathbb{R}\times\mathbb{R}_+)} < \infty$, the theorem follows as $R \to \infty$. □

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141. Albeverio, Sergio; Liang, Song: A Remark on Different Lattice Approximations and Continuum Limits for ϕ_2^4-Fields

143. Albeverio, Sergio; Nizhnik, Leonid: A Schrödinger Operator with Point Interactions on the Sobolev Spaces; eingereicht bei: Letters in Mathematical Physics

144. Labesse, Jean-Pierre; Müller, Werner: Weak Weyl's Law for Congruence Subgroups

145. Arndt, Marcel; Griebel, Michael: Higher Order Gradient Continuum Description of Atomistic Models for Crystalline Solids; erscheint in: Proceedings der Konferenz ECCOMAS 2004, Jyväskylä, Finnland

146. Albeverio, Sergio; Herzberg, Frederik S.: On an Internal Random Walk Representation of Measurable Lévy Processes and their Stochastic Integrals

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152. Arndt, Marcel; Griebel, Michael: Derivation of Higher Order Gradient Continuum Models from Atomistic Models for Crystalline Solids; eingereicht bei: Multiscale Modeling and Simulation

153. Griebel, Michael; Preusser, Tobias; Rumpf, Martin; Schweitzer, Marc Alexander; Telea, Alexandru: Flow Field Clustering via Algebraic Multigrid; eingereicht bei: Visualization 2004

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155. Griebel, Michael; Jager, Lukas; Voigt, Axel: Computing Diffusion Coefficients of Intrinsic Point Defects by Atomistic Simulations

156. Engel, Martin; Griebel, Michael: Flow Simulation on Moving Boundary-Fitted Grids and Application to Fluid-Structure Interaction Problems

157. Croce, Roberto; Griebel, Michael; Schweitzer, Marc Alexander: A Parallel Level-Set Approach for Two-Phase Flow Problems with Surface Tension in Three Space Dimensions; eingereicht bei: Journal of Computational Physics

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