Slow Motion of Gradient Flows

Felix Otto, Maria G. Reznikoff

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Felix Otto*and Maria G. Reznikoff†

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Abstract

We present sufficient conditions on an energy landscape in order for the associated gradient flow to exhibit slow motion or "dynamic metastability." The first condition is a weak form of convexity transverse to the so–called slow manifold, $\mathcal{N}$. The second condition is that the energy restricted to $\mathcal{N}$ is Lipschitz with a constant $\delta \ll 1$. We apply the abstract result to give a new proof of the exponentially slow motion of transition layers in the one–dimensional Allen–Cahn equation. The analysis is more nonlinear than previous work: It relies on the nonlinear convexity condition or "energy–energy–dissipation inequality." Our result demonstrates that a broad class of initial data relaxes with an exponential rate into a δ–neighborhood of the slow manifold, where it is then trapped for an exponentially long time. One feature of the abstract result that makes it of broader interest is that it does not rely on maximum principles; we use the maximum principle in the application to Allen–Cahn, but this is only for convenience.

Keywords. Energy methods, nonlinear partial differential equation, dynamic metastability, slow coarsening.

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1 Introduction

Although local energy minimizers are the only stable states of a gradient flow system, so–called dynamic metastability is characterized by evolution that is

*Inst. für Angewandte Math., Universität Bonn, Germany, otto@iam.uni-bonn.de

†Math. Dept., Princeton University, Princeton, USA, mrezniko@math.princeton.edu

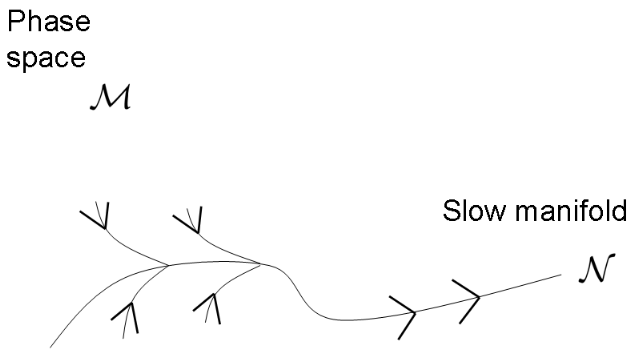

Figure 1: Fast relaxation to the slow manifold, slow motion along it.

so slow that solutions *appear* to be stable. Typically, this metastable behavior is misleading: The solution will eventually undergo drastic change.

Our goal is to convert information about the energy landscape into information about the dynamics. The main result is a pair of sufficient conditions for metastable behavior (see Theorem 1.1 below for a precise statement). We have in mind a pde with gradient–flow structure. Viewing state space as an abstract manifold $\mathcal{M}$, metastability means that generic initial data is drawn quickly to a *slow manifold* $\mathcal{N} \subset \mathcal{M}$, where it remains trapped for a long time (Figure 1).

Main result: sufficient conditions for metastability. Consider the gradient flow

$$\begin{aligned} \partial_t u &= -\nabla E(u), & t > 0, \\ u &= u(0), & t = 0. \end{aligned} \tag{1.1}$$

Our assumptions about the energy landscape are:

(i) For every $u \in \mathcal{M}$, there exists a $v \in \mathcal{N}$ such that

$$\frac{1}{2}|u-v|^2 \;\le\; E(u) - E(v) \;\le\; \frac{1}{2}|\nabla E(u)|^2. \tag{1.2}$$

(ii) There exists a finite constant δ such that for every v_1, $v_2 \in \mathcal{N}$,

$$|E(v_1) - E(v_2)| \ \le\ \delta |v_1 - v_2|. \tag{1.3}$$

Assumption (i) is a weak form of strict convexity of E transverse to $\mathcal{N}$. Assumption (ii) is a Lipschitz condition on E restricted to $\mathcal{N}$. Our result is:

Theorem 1.1. *Suppose that Assumptions (i) and (ii) hold, and let v be such that $v(t)$ and $u(t)$ satisfy (1.2). Then the solution of (1.1) is drawn into a δ-neighborhood of $\mathcal{N}$ with an exponential rate close to 1:*

$$\begin{aligned}|u(t) - v(t)| &+ \big(E(u(t)) - E(v(t))\big)^{1/2} \\ &\lesssim\ \exp(-(1-\varepsilon)t)\big(E(u(0)) - E(v(0))\big)^{1/2} + C_\varepsilon \delta,\end{aligned} \tag{1.4}$$

where ε is any fixed number in $(0,1)$. Moreover, we have that:

$$|u(t) - u(s)| \ \lesssim\ \big(E(u(s)) - E(v(s))\big)^{1/2} + \delta\,(t - s + 1), \tag{1.5}$$

for any $0 < s < t$.

Remark 1. *In the case $\delta \ll 1$, Theorem 1.1 reflects dynamic metastability. According to (1.4), u is trapped within a small neighborhood of $\mathcal{N}$ after an initial layer of scale*

$$t_1 \sim \log\Big(\big(E(u(0)) - E(v(0))\big)/\delta^2\Big),$$

and (1.5) identifies δ^{-1} as the slow motion timescale. To be precise, for $t \gtrsim t_1$, the change in u is of order δ until $(t - t_1) \sim \delta^{-1}$.

The proof of Theorem 1.1 is elementary and relies on the general inequality

$$\begin{aligned}|u(t) - u(t_0)|^2 \le\ & \left(\int_{t_0}^{t} (-\dot{\widehat{E(u(t))}})^{1/2}\, ds\right)^2 \\ =\ & \left(\int_{t_0}^{t} \frac{1}{w(s)^{1/2}} w(s)^{1/2} (-\dot{\widehat{E(u(t))}})^{1/2}\, ds\right)^2 \\ \le\ & \int_{t_0}^{t} \frac{1}{w(s)}\, ds \Big(w(t_0) E(u(t_0)) - w(t) E(u(t)) \\ & \qquad + \int_{t_0}^{t} \dot{w}(s) E(u(s))\, ds \Big),\end{aligned} \tag{1.6}$$

for any positive weight w and any $t \ge t_0$. For (1.4), we use $w = 1$ and develop a differential inequality for the energy gap between u and v. For (1.5), we use (1.4) and equation (1.6) with two different weights: an exponential weight for the "initial layer" of rapid energy relaxation, and a constant weight for the stagnant phase.

Application to Allen–Cahn: background. A classic example of metastable behavior is the exponentially slow motion of transition layers in the one–dimensional Allen–Cahn equation,

$$u_t \;=\; u_{xx} - G'(u), \qquad (x,t) \in (0,L) \times (0,\infty). \tag{1.7}$$

We will show how the abstract result of Theorem 1.1 may be applied to give a new proof of the exponentially slow coarsening time–scale. Moreover, the result (Theorem 1.2, below) shows that closeness to the slow manifold is not only *propagated*, but also *generated*: A broad class of initial data is quickly drawn into a δ–neighborhood of the slow manifold, where it is then trapped for a time of order δ^{-1}. We now give some brief background. For a more thorough introduction, see for instance [CP, FH, BK, W, C].

The Allen–Cahn equation (1.7) is the L^2–gradient flow for the scalar Ginzburg–Landau energy,

$$E(u) \;=\; \int_0^L \left(\frac{1}{2} u_x^2 + G(u) \right) \, dx. \tag{1.8}$$

The potential, G, has nondegenerate minima at two preferred phases, which for simplicity we normalize as $u = \pm 1$. To be precise, we assume that G is a smooth, even potential satisfying

- $G(v) \geq 0$ and $= 0$ only at ± 1,
- $G'(v) \geq 0$ on $[-1, 0]$, $G'(v) \leq 0$ on $[0, 1]$,
- $G''(\pm 1) > 0$.

A standard choice of potential is

$$G(u) \;:=\; \frac{(1-u^2)^2}{4}. \tag{1.9}$$

The two characteristic length–scales of equation (1.7) are the minimal distance ℓ between zeros of u and the width of an optimal transition layer between the preferred phases, which is order one in our scaling of the equation. In the case in which there is a clear separation between the two scales — i.e. $\ell \gg 1$ — states with bounded energy are characterized by large regions of $u \approx \pm 1$, separated by order one interfaces on which the energy concentrates.

It is the degeneracy of the energy for large ℓ that makes the motion slow. By degeneracy, we mean that for a finite–sized system with well–separated transition layers, the change in energy from translating an interface is exponentially small with respect to the distance between layers. Thus, until two layers come close, not much energy is dissipated, and if there is not much energy dissipated, then the interfaces hardly move. Driven only by the exponentially small correction terms to the energy, the motion is exponentially slow.

Thus, the heuristics suggest three distinct stages for the evolution problem: A fast, initial stage of energy relaxation, an exponentially slow stage of layer motion, and a collision stage in which the two closest layers come together and annihilate. Then the process repeats. A detailed analysis of the exponentially slow motion of transition layers was carried out by Carr and Pego [CP] and Fusco and Hale [FH]. Subsequently, Ward [W] studied all three stages using a combination of numerical and asymptotic methods. Eckmann and Rougemont [ER] and Rougemont [R] studied the coarsening problem on $\mathbb{R}$, analyzing also the collision stage. Most recently, Chen [C] analyzed the initial relaxation stage: He proved that initial data that is order one away from the slow manifold is drawn into a small neighborhood of it and then trapped in the slow motion phase. The proof uses an idea of de Mottoni and Schatzman [MS] and a result of Fife and McLeod [FM] on the stability of the travelling wave solution on $\mathbb{R}$.

Bronsard and Kohn [BK] introduced an alternate, energy–based analysis: Via an elementary method requiring weaker hypotheses than [CP] but returning weaker results, they prove that initial data that is algebraically close in energy to the slow manifold stays close for an algebraically long time. In an extension, Grant [G] proved that initial data that is exponentially close to the slow manifold stays close for an exponentially long time.

Here, we use an energy–based method — natural for a gradient flow — to derive stronger information: Namely, we start with initial data whose energy is order one away from the slow manifold and capture the fast, initial relaxation, followed by the exponentially long stage of layer motion. Thus, the main result is similar to Chen's, but the method is different. Perhaps the most salient feature of our method is that it exploits the nonlinearity: By passing from the linearized estimates of energy and energy dissipation (which appear already in [CP]) to their nonlinear counterparts, we gain a strong advantage; see the discussion just after Theorem 1.2 below for a heuristic illustration.

Remark 2. *For convenience, we rely on the maximum principle in two ways. First, a maximum of the initial data that is greater than* 1 *(resp. minimum less than* -1*) is driven exponentially quickly to* 1 *(resp.* -1*); for simplicity, we assume throughout the paper that* $u \in [-1, 1]$. *Second, the zeros of* (1.7) *move continuously and can only decrease in number [A]; we use this fact when proving energy–energy–dissipation. We emphasize that the abstract result is independent of the maximum principle, so that with some work it should be possible to prove energy–energy–dissipation and use Theorem 1.1 for the higher*

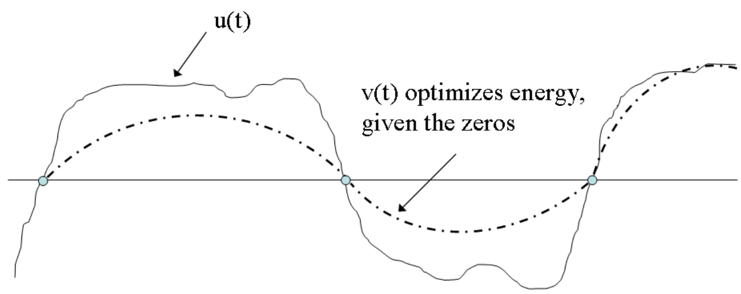

Figure 2: We associate to u the function v that has the same zeros, the same sign in–between zeros, and minimal energy.

order equations (e.g. Cahn Hilliard) or systems, where the maximum principle does not hold.

Application to Allen–Cahn: main result. We state our main result for the Allen–Cahn equation for the case of initial data with two zeros and periodic boundary conditions. This is for simplicity; one can generalize to N zeros, in which case the timescale for motion is controlled by the distance between the two nearest zeros. The "energy–optimal profiles" play the role of the slow manifold: Associated to u is the function v that has the same simple zeros as u, the same sign as u, and minimal energy given the location of the zeros (Figure 2). We assume that the energy of the initial data is bounded by $4c_0$, where c_0 is the energy of an optimal transition layer on $\mathbb{R}$,

$$c_0 := \sqrt{2} \int_{-1}^{1} G(u)^{1/2} \, du.$$

This makes sense since we have in mind the successive collision and annihilation of neighboring layers, and just after a collision event has reduced the number of layers from four to two, the energy is close to $4c_0$. This assumption means that our constants are universal, depending only on G. (One can instead allow for any order one energy E_0 of the initial condition, and then the constants depend also on E_0.) We formulate our result as:

Theorem 1.2. *Let $u : [0, L] \times (0, \infty) \to \mathbb{R}$ be a smooth, periodic solution of*

$$\begin{aligned} u_t &= u_{xx} - G'(u), && (x,t) \in (0, L] \times (0, \infty), \\ u &= u_0, && (x,t) \in (0, L] \times \{0\}. \end{aligned} \tag{1.10}$$

Suppose that $u(0)$ has exactly two simple zeros, $x(0)$, $y(0)$, and that

$$E(u(0)) \leq 4c_0, \quad \ell(0) \gg 1, \tag{1.11}$$

where $\ell(t)$ denotes the minimal distance between the zeros of $u(t)$ (bearing in mind the periodicity). Then for all

$$t \ll \exp(\sqrt{G''(1)}\ell(0)),$$

$u(t)$ has exactly two simple zeros, $x(t)$, $y(t)$, and

$$|x(t) - x(0)| + |y(t) - y(0)| \lesssim 1. \tag{1.12}$$

Moreover, not only are the zeros of u trapped within an order one neighborhood of their starting location, but after an initial layer, u is trapped exponentially close in H^1 to the corresponding energy–optimal profile: For all t with

$$\ell(0) \ll t \ll \exp(\sqrt{G''(1)}\ell(0)),$$

we have

$$||u(t) - v(t)||_{H^1([0,L])} \lesssim \exp(-\sqrt{G''(1)}\ell(0)). \tag{1.13}$$

The main ingredient in Theorem 1.2 is energy–energy–dissipation. Roughly, the idea is: Suppose u is a solution of (1.10) where the initial data $u(0)$ has N zeros. Suppose that the optimal energy of a function with N zeros on $(0, L)$ is well–approximated by $c_0 N$. By the gradient flow structure, we have

$$\frac{d}{dt}(E(u(t)) - c_0\, N) = -\int_0^L (u_{xx} - G'(u))^2\, dx. \tag{1.14}$$

Now suppose that we have the energy–energy–dissipation (e–e–d) inequality:

$$\int_0^L (u_{xx} - G'(u))^2\, dx \geq \frac{1}{C}(E(u(t)) - N\, c_0).$$

Then (1.14) becomes

$$\frac{d}{dt}(E(u(t)) - c_0\, N) \;\leq\; -\frac{1}{C}(E(u(t)) - N\, c_0).$$

This estimate implies the exponential relaxation of the energy. After this relaxation, u is very close in energy to the optimal N–layer configuration. Subsequently, because small translations of the zeros hardly change the energy, one expects u to be trapped for a long time, its zeros barely moving.

To make these ideas precise, we will apply the abstract result of Theorem 1.1. Therefore, the first step is to establish Assumptions (i) and (ii). Energy–energy–dissipation takes the form:

Proposition 1.1. *There exist constants $\ell_1, C_1 < \infty$ with the following properties. Suppose $u : [0, L] \to \mathbb{R}$ is a periodic function that satisfies the conditions:*

$$\text{The minimal distance between simple zeros of } u \text{ is at least } \ell_1. \tag{1.15}$$

$$\text{The energy between adjacent zeros is less than } (G(0)/2)\ell_1. \tag{1.16}$$

Then u and the corresponding energy–optimal profile v satisfy

$$\frac{1}{C_1}||u - v||^2_{H^1([0,L])} \;\leq\; E(u) - E(v) \;\leq\; C_1 \int_0^L (u_{xx} - G'(u))^2\, dx.$$

Proposition 1.1 says that u and the energy-optimal profile satisfy a relationship of the form (1.2). The hypothesis (1.15) of well–separated zeros allows us to compare v to its infinite–system limit (cf. Subsection 3.2). The hypothesis (1.16) allows us to bound u away from $u \equiv 0$ (cf. Lemma 3.8).

Verification of the Lipschitz condition (1.3) follows from direct calculations on the energy. The Lipschitz property, stated as Lemma 3.1, was proved already in [CP], Section 7, but for completeness, we include a proof in Subsection 3.1.

Organization. The proof of Theorem 1.1 is given in Section 2. In Section 3 we apply the abstract theorem to the Allen–Cahn equation. The proof of Theorem 1.2 is given in Subsection 3.1, assuming energy–energy–dissipation (Proposition 1.1) and the Lipschitz condition on the slow manifold (Lemma 3.1). Then to prove the e–e–d relationship, we begin by proving the linearized estimates in Subsection 3.2. In Subsection 3.3, we show how to extend from the linear to the nonlinear estimates, proving Proposition 1.1. Finally, in Subsection 3.4 we prove the Lipschitz condition and an auxiliary lemma.

Notation 1. *We write* $A(u) \lesssim B(u)$ *if and only if there exists a finite, positive constant* C *depending only on* G *such that*

$$A(u) \leq C\, B(u),$$

and analogously for $\gtrsim$ *and* $\sim$. *We write* $A(u) \ll B(u)$ *if and only if for a given* $c \in (0,1)$, *we have that*

$$A(u) \leq cB(u)$$

and analogously for $\gg$.

2 Abstract Result: Proof of Theorem 1.1

We break the proof of Theorem 1.1 into two parts, stated as Lemmas 2.1 and 2.2:

Lemma 2.1. *Let* u *satisfy* (1.1). *Under assumptions* (1.2) *and* (1.3), u *and the associated function* v *satisfy:*

$$E(u) - E(v) \lesssim \exp(-(2-\varepsilon)t)\big(E(u(0)) - E(v(0))\big) + C_\varepsilon\, \delta^2,$$

where $\varepsilon \in (0,1)$ *is arbitrary and* $C_\varepsilon < \infty$ *depends only on* ε.

Lemma 2.2. *Let* u *satisfy* (1.1). *Under assumptions* (1.2) *and* (1.3), *we have*

$$|u(t) - u(0)| \lesssim \big(E(u(0)) - E(v(0))\big)^{1/2} + \delta + \delta\, t. \qquad (2.1)$$

In particular, for $t_1 := \frac{2}{3}\log\Big(\big(E(u(0)) - E(v(0))\big)/\delta^2\Big)$, *we have*

- *For* $t \in (0, t_1)$, $|u(t) - u(0)| \lesssim \big(E(u(0)) - E(v(0))\big)^{1/2} + \delta$.
- *For* $t \geq t_1$, $|u(t) - u(t_1)| \lesssim \delta + \delta(t - t_1)$.

The combination of Lemma 2.1 and (1.2) implies (1.4). Statement (2.1) of Lemma 2.2 is (1.5). Thus, Theorem 1.1 is established as soon as we prove the lemmas.

Proof of Lemma 2.1. Let $e(t) := E(u(t)) - E(v(t))$. Recall that by the gradient flow dynamics and (1.2), we have

$$\frac{d}{dt}E(u(t)) = -|\nabla E|^2 \leq -2\, e(t).$$

Integrating from s to t, we deduce

$$e(t) - e(s) + 2\int_s^t e(\tau)\,d\tau \;\le\; E(v(s)) - E(v(t)). \tag{2.2}$$

Now we would like to take advantage of (1.3). To begin, we use the triangle inequality to estimate $|v(s) - v(t)|$:

$$\begin{aligned} |v(s) - v(t)| &\le |v(s) - u(s)| + |v(t) - u(t)| + |u(t) - u(s)| \\ &\overset{(1.2)}{\le} \sqrt{2\,e(s)} + \sqrt{2\,e(t)} + |u(t) - u(s)|. \end{aligned} \tag{2.3}$$

On the other hand, taking the weight $w \equiv 1$ in (1.6),

$$\begin{aligned} &|u(t) - u(s)| \\ &\le \Big((t-s)\big(E(u(s)) - E(u(t))\big)\Big)^{1/2} \\ &= \Big((t-s)\big(E(u(s)) - E(v(s)) + E(v(s)) - E(v(t)) + E(v(t)) - E(u(t))\big)\Big)^{1/2} \\ &\le \Big((t-s)\big(e(s) + |E(v(s)) - E(v(t))|\big)\Big)^{1/2}, \end{aligned} \tag{2.4}$$

where we have dropped the nonpositive term, $E(v(t)) - E(u(t))$. The combination of (1.3), (2.3), and (2.4) yields

$$\begin{aligned} &|E(v(t)) - E(v(s))| \\ &\le \delta |v(s) - v(t)| \\ &\le \delta \left(\sqrt{2\,e(s)} + \sqrt{2\,e(t)} + \Big((t-s)\big(e(s) + |E(v(s)) - E(v(t))|\big)\Big)^{1/2} \right) \\ &\le \delta \left(\sqrt{2\,e(s)} + \sqrt{2\,e(t)} + \sqrt{(t-s)e(s)} + \sqrt{(t-s)|E(v(s)) - E(v(t))|} \right). \end{aligned}$$

By Young's inequality, this becomes

$$\begin{aligned} &|E(v(t)) - E(v(s))| \\ &\le 2\delta\Big((\sqrt{2} + \sqrt{t-s})\sqrt{e(s)} + \sqrt{2e(t)}\Big) + \delta^2(t-s). \end{aligned} \tag{2.5}$$

Substituting (2.5) into (2.2),

$$\begin{aligned} &e(t) - e(s) + 2\int_s^t e(\tau)\,d\tau \\ &\le 2\delta\Big((\sqrt{2} + \sqrt{t-s})\sqrt{e(s)} + \sqrt{2e(t)}\Big) + \delta^2(t-s), \end{aligned}$$

which by Young's inequality may be expressed

$$(1-\varepsilon)e(t)-(1+\varepsilon)e(s)+2\int_s^t e(\tau)\,d\tau \;\le\; C_\varepsilon\delta^2(1+t-s)$$

for any fixed $\varepsilon \in (0,1)$. Here C_ε is a constant depending only on ε that may change from line to line. Dividing through by $(1+\varepsilon)$ and rearranging terms,

$$\frac{1-\varepsilon}{1+\varepsilon}\,e(t) \;\le\; -\frac{2}{1+\varepsilon}\int_s^t e(\tau)\,d\tau + e(s) + C_\varepsilon\delta^2(1+t-s). \tag{2.6}$$

We will use (2.6) to derive a differential inequality.

Let $s \in (t-1,t)$ and use $(1+\varepsilon)^{-1} \ge 1-\varepsilon$ for $\varepsilon > 0$ to reexpress (2.6) as:

$$(1-\varepsilon)^2 e(t) \;\le\; -2(1-\varepsilon)\int_s^t e(\tau)d\tau + e(s) + C_\varepsilon\delta^2. \tag{2.7}$$

As initial data for our inequality, we need the following: Notice that for $s=0$ and $t \le 1$, (2.7) implies

$$(1-\varepsilon)^2 e(t) \;\le\; e(0) + C_\varepsilon\delta^2,$$

and as a trivial consequence, we also have

$$\begin{aligned}\int_0^1 \exp(-\tilde{\varepsilon}\tau)e(\tau)\,d\tau \;&\le\; \frac{1}{(1-\varepsilon)^2}(e(0)+C_\varepsilon\delta^2)\\ &\lesssim\; e(0)+C_\varepsilon\delta^2 \end{aligned} \tag{2.8}$$

for any fixed $\tilde{\varepsilon} > 0$. Now let $\tilde{\varepsilon} := -\ln(1-\varepsilon)^2$ and rewrite (2.7), choosing $s = t-1$:

$$\exp(-\tilde{\varepsilon})e(t) \;\le\; -2(1-\varepsilon)\int_{t-1}^t e(\tau)\,d\tau + e(t-1) + C_\varepsilon\delta^2. \tag{2.9}$$

Multiplying (2.9) by $\exp(-\tilde{\varepsilon}(t-1))$:

$$\begin{aligned}\exp(-\tilde{\varepsilon}t)e(t) \;&\le\; -2(1-\varepsilon)\int_{t-1}^t \exp(-\tilde{\varepsilon}(t-1))e(\tau)\,d\tau\\ &\qquad + \exp(-\tilde{\varepsilon}(t-1))(e(t-1)+C_\varepsilon\delta^2)\\ &\le\; -2(1-\varepsilon)\int_{t-1}^t \exp(-\tilde{\varepsilon}\tau)e(\tau)d\tau\\ &\qquad + \exp(-\tilde{\varepsilon}(t-1))(e(t-1)+C_\varepsilon\delta^2).\end{aligned} \tag{2.10}$$

Letting

$$F(t) := \int_{t-1}^{t} \exp(-\tilde{\varepsilon}\tau)e(\tau)d\tau, \tag{2.11}$$

(2.10) reads

$$F'(t) \;\leq\; -2(1-\varepsilon)F(t) + \exp(-\tilde{\varepsilon}(t-1))C_\varepsilon\delta^2,$$

so that

$$\begin{aligned} F(t) \;&\lesssim\; \exp(-2(1-\varepsilon)t)F(1) + \exp(-\tilde{\varepsilon}(t-1))C_\varepsilon\delta^2 \\ &\overset{(2.8)}{\lesssim}\; \exp(-2(1-\varepsilon)t)e(0) + \exp(-\tilde{\varepsilon}(t-1))C_\varepsilon\delta^2. \end{aligned} \tag{2.12}$$

From (2.11) and (2.12) it follows that

$$\begin{aligned} \exp(-\tilde{\varepsilon}(t-1))\int_{t-1}^{t} e(\tau)\,d\tau \;&\lesssim\; \int_{t-1}^{t} \exp(-\tilde{\varepsilon}\tau)e(\tau)\,d\tau \\ &\lesssim\; \exp(-2(1-\varepsilon)t)e(0) + \exp(-\tilde{\varepsilon}(t-1))C_\varepsilon\delta^2, \end{aligned}$$

which implies

$$\int_{t-1}^{t} e(\tau)\,d\tau \;\lesssim\; \exp(-(2(1-\varepsilon)-\tilde{\varepsilon})t)e(0) + C_\varepsilon\delta^2. \tag{2.13}$$

Thus,

$$\inf_{(t-1,t)} e(\tau) \;\lesssim\; \exp(-(2-2\varepsilon-\tilde{\varepsilon})t))e(0) + C_\varepsilon\delta^2. \tag{2.14}$$

This is enough to bound the supremum: Suppose the supremum on $[t-1,t]$ is achieved at t_{max} and the infimum at t_{min}. From (2.6) with $t = t_{max}$ and $s = t_{min}$,

$$\begin{aligned} e(t_{max}) \quad&\lesssim\quad -\int_{t_{min}}^{t_{max}} e(\tau)d\tau + e(t_{min}) + C_\varepsilon\delta^2 \\ &\leq\quad \int_{t-1}^{t} e(\tau)d\tau + e(t_{min}) + C_\varepsilon\delta^2 \\ &\overset{(2.13),(2.14)}{\lesssim}\quad \exp(-(2-2\varepsilon-\tilde{\varepsilon})t)e(0) + C_\varepsilon\delta^2. \end{aligned}$$

□

Proof of Lemma 2.2. For simplicity, set $\varepsilon = 1/2$ in Lemma 2.1. We begin by considering $t \geq t_1$, where by (1.4),

$$\frac{1}{2}|u(t) - v(t)|^2 + E(u(t)) - E(v(t)) \;\lesssim\; \delta^2. \tag{2.15}$$

Therefore, recalling the definition $e(t) := E(u(t)) - E(v(t))$ and combining equations (2.3) and (2.4) from the proof of Lemma 2.1, we have:

$$\begin{aligned}
&|v(t) - v(t_1)| \\
&\quad \leq \quad \sqrt{2e(t)} + \sqrt{2e(t_1)} + \Big((t-t_1)\big(e(t_1) + |E(v(t_1)) - E(v(t))|\big)\Big)^{1/2} \\
&\quad \overset{(1.3)}{\leq} \quad \sqrt{e(t_1)}(1 + \sqrt{t-t_1}) + \sqrt{e(t)} + \delta^{1/2}\sqrt{t-t_1}|v(t) - v(t_1)|^{1/2} \\
&\quad \overset{(2.15)}{\lesssim} \quad \delta(1 + \sqrt{t-t_1}) + \delta^{1/2}\sqrt{t-t_1}|v(t) - v(t_1)|^{1/2}.
\end{aligned}$$

By Young's inequality, this implies

$$|v(t) - v(t_1)| \;\lesssim\; \delta(1 + t - t_1).$$

Together with the triangle inequality and (2.15), this yields

$$|u(t) - u(t_1)| \;\lesssim\; \delta(1 + t - t_1) \quad \text{for } t \geq t_1.$$

We now turn to $t \leq t_1$. (The "initial layer.") Here we use a different weight in the gradient flow inequality (1.6):

$$w(s) := \exp(s/4).$$

This yields the inequality

$$\begin{aligned}
&|u(t) - u(0)|^2 \\
&\leq\; E(u(0)) - \exp(t/4)E(u(t)) + \frac{1}{4}\int_0^t \exp(s/4)E(u(s))ds \\
&=\; E(u(0)) - E(v(0)) + E(v(0)) - \exp(t/4)E(v(t)) \\
&- \exp(t/4)\big(E(u(t)) - E(v(t))\big) + \frac{1}{4}\int_0^t \exp(s/4)\big(E(u(s)) - E(v(s))\big)\,ds \\
&+ \int_0^t \exp(s/4)E(v(s))\,ds.
\end{aligned} \tag{2.16}$$

Without loss, we may assume that $E(v(0)) = 0$. Observe then that by (1.3),

$$|E(v(t))| \;\leq\; \delta|v(t) - v(0)|. \tag{2.17}$$

The combination of (2.16), (2.17), and Lemma 2.1 gives for $t \le t_1$:

$$
\begin{aligned}
&|u(t) - u(0)|^2 \\
&\lesssim\ e(0) + \delta \exp(t/4)|v(t) - v(0)| + \int_0^t \exp(s/4) e(0) \exp(-(2-\varepsilon)s)\, ds \\
&\qquad + \delta \int_0^t \exp(s/4)|v(s) - v(0)|\, ds \\
&\lesssim\ e(0) + \delta \exp(t_1/4) \sup_{t \le t_1} |v(t) - v(0)|.
\end{aligned} \tag{2.18}
$$

From the triangle inequality,

$$
|v(t) - v(0)|^2 \ \lesssim\ |v(t) - u(t)|^2 + |v(0) - u(0)|^2 + |u(t) - u(0)|^2,
$$

we deduce, after another application of (1.4), that

$$
\sup_{t \le t_1} |v(t) - v(0)| \ \lesssim\ \sqrt{e(0)} + \sup_{t \le t_1} |u(t) - u(0)|. \tag{2.19}
$$

Taking the supremum on the left–hand side of (2.18), substituting (2.19), and applying Young's inequality one more time, we arrive at:

$$
\begin{aligned}
\sup_{t \le t_1} |u(t) - u(0)|^2 &\lesssim\ e(0) + \delta^2 \exp(t_1/2) \\
&=\ e(0) + \delta^2 \left(\frac{e(0)}{\delta^2} \right)^{1/2} \\
&\lesssim\ e(0) + \delta^2,
\end{aligned}
$$

as desired.

□

3 Application to coarsening in Allen–Cahn

In Subsection 3.1, we apply the abstract result to prove Theorem 1.2. In order to invoke Theorem 1.1, we need to show that Assumptions (i) and (ii) are satisfied. Most of the work lies in proving the energy–energy–dissipation relationship. We prove a linearized version in Subsection 3.2. Then in Subsection 3.3, we improve from the linearized estimates to the nonlinear estimates, proving Proposition 1.1. The ingredients for Assumption (ii) (Lemmas 3.1 and 3.2) are proved in Subsection 3.4.

Throughout this section, we use the fact that for the Allen–Cahn equation, the number of zeros can only decrease in time [A], so that the sign of u in–between two adjacent zeros is well–defined and constant for as long as the zeros exist.

3.1 Proof of Theorem 1.2

Proof of Theorem 1.2. As discussed in the introduction, we associate to u the function v that has the same zeros as u, minimal energy in–between zeros, and the same sign as u (well–defined, by Remark 2). The main task is to verify Assumptions (i) and (ii) so that we can invoke Theorem 1.1. Proposition 1.1 is, up to the constant C_1, a verification of the first assumption for u and the associated v. For Assumption (ii), we will need two lemmas. First, it is convenient to introduce:

Notation 2. *Given a periodic function $v : [0, L] \to \mathbb{R}$ with two simple zeros, let x_v denote the zero at which v changes from negative to positive, and y_v the one at which it changes back to negative. Moreover, let ℓ_v denote the minimal distance between zeros of v, taking into account the periodicity.*

The basic Lipschitz property is formulated:

Lemma 3.1. *Let v and w be periodic, energy–optimal profiles with two simple zeros. There exist constants $\ell_2, C_2 < \infty$ such that if $\min\{\ell_v, \ell_w\} \geq \ell_2$, then*

$$|E(v) - E(w)| \;\leq\; \delta\,(|x_w - x_v| + |y_w - y_v|), \tag{3.1}$$

with $\delta := C_2 \exp(-\sqrt{G''(1)}\,\min\{\ell_v, \ell_w\})$.

The estimate in (3.1) is formulated in terms of the positions of the zeros, the natural distance on the submanifold $\mathcal{N}$. We will need to be able to go back and forth between this distance and the L^2–distance associated to the ambient space. Lemma 3.2 serves this purpose:

Lemma 3.2. *Let v and w be periodic, energy–optimal profiles with two simple zeros. If $\min\{\ell_v, \ell_w\} \gg 1$ and*

$$|x_v - x_w| + |y_v - y_w| \ll \min\{\ell_v, \ell_w\},$$

then:

$$\begin{aligned}
&|x_v - x_w| + |y_v - y_w| \\
&\lesssim\; ||v - w||^2_{L^2([0,L])} + ||v - w||_{L^2([0,L])} \qquad (3.2)\\
&\lesssim\; ||v - w||^2_{L^2([0,L])} + 1, \qquad (3.3)
\end{aligned}$$

and

$$||v - w||_{L^2([0,L])} \;\lesssim\; |x_v - x_w| + |y_v - y_w|. \tag{3.4}$$

Now consider the evolution on some time interval $(0,T]$. Recall from the statement of the theorem that $\ell(t)$, and $x(t), y(t)$ denote the minimal distance and the location of the zeros at time t, respectively. By Lemma 3.1 and (3.2), we have that Assumption (ii) with the norm given by $H^1([0,L])$ is verified with

$$\begin{aligned}\delta \quad &\sim \quad \left(\sup_{t\le T}\left(\int_0^L (v(t)-v(0))^2\,dx\right)^{1/2}+1\right)\exp\Big(-\sqrt{G''(1)}\inf_{t\le T}\ell(t)\Big)\\ &\overset{(3.4)}{\lesssim} \quad \big(\Delta x(T)+1\big)\exp\Big(-\sqrt{G''(1)}\inf_{t\le T}\ell(t)\Big), \qquad (3.5)\end{aligned}$$

where we have set for abbreviation

$$\Delta x(T) := \sup_{t\le T}(|x(t)-x(0)|+|y(t)-y(0)|).$$

Let T be such that $\Delta x(T) \ll \ell(0)$ and $\delta \ll 1$. (We will see that it is possible to choose $T \sim \exp(\sqrt{G''(1)}\ell(0))$.) Notice that our choice of T implies in particular that

$$\inf_{t\le T}\ell(t) \ge \min\{\ell_1,\ell_2\},$$

so we may apply Proposition 1.1 and Lemma 3.1. Now let $\varepsilon \ll 1$ and define

$$T^* := \min\left\{\varepsilon\exp\left(\sqrt{G''(1)}\ell(0)\right), T\right\}. \qquad (3.6)$$

From (1.5) of Theorem 1.1, we have that

$$\begin{aligned}&\left(\int_0^L (u(t)-u(0))^2\,dx\right)^{1/2}\\ &\quad\lesssim \quad 1+\delta+\delta T^*\\ &\overset{(3.5),(3.6)}{\lesssim} \quad 1+\varepsilon\big(\Delta x(T^*)+1\big)\exp\Big(\sqrt{G''(1)}\sup_{t\le T^*}|\ell(0)-\ell(t)|\Big). \qquad (3.7)\end{aligned}$$

On the one hand, we claim:

$$\sup_{t\le T^*}|\ell(0)-\ell(t)| \;\le\; \Delta x(T^*). \qquad (3.8)$$

To see this, we observe that

$$\begin{aligned}\ell(t)-\ell(0) \;&=\; \min\{|(x(t),y(t))|,|(y(t),x(t))|\}-\min\{|(x(0),y(0))|,|(y(0),x(0))|\}\\ &\le\; \max\{|(x(t),y(t))|-|(x(0),y(0))|,|(y(t),x(t))|-|(y(0),x(0))|\}\\ &\le\; \max\{|(x(t),y(t))\cap(y(0),x(0))|,|(y(t),x(t))\cap(x(0),y(0))|\}\\ &\le\; |x(t)-x(0)|+|y(t)-y(0)|.\end{aligned}$$

Using symmetry and taking the supremum, (3.8) follows. On the other hand, we claim that

$$(\Delta x(T^*))^{1/2} \lesssim \sup_{t\leq T^*}\left(\int_0^L (u(t)-u(0))^2 dx\right)^{1/2} + 1. \tag{3.9}$$

This is a simple consequence of (3.3) together with

$$\begin{aligned}
&\int_0^L (v(t)-v(0))^2\,dx \\
&\lesssim \int_0^L (u(t)-u(0))^2\,dx + \int_0^L (u(0)-v(0))^2\,dx + \int_0^L (u(t)-v(t))^2\,dx \\
&\overset{(3.35)}{\lesssim} \int_0^L (u(t)-u(0))^2\,dx + E(u(0)) - E(v(0)) + E(u(t)) - E(v(t)) \\
&\leq \int_0^L (u(t)-u(0))^2\,dx + 2E(u(0)) - E(v(0)) - E(v(t)) \\
&\overset{(1.11)}{\lesssim} \int_0^L (u(t)-u(0))^2\,dx + 1.
\end{aligned}$$

The combination of (3.7), (3.8), and (3.9) gives:

$$(\Delta x(T^*))^{1/2} \lesssim 1 + \varepsilon(\Delta x(T^*)+1)\exp\left(\sqrt{G''(1)}\Delta x(T^*)\right).$$

By continuity of $\Delta x(t)$, $\Delta x(0)=0$, and $\varepsilon \ll 1$, this implies

$$\Delta x(T^*) \lesssim 1. \tag{3.10}$$

Moreover, since $\ell(0) \gg 1$, we have that

$$\ell(t) \gg 1, \text{ for all } 0 \leq t \leq T^*, \tag{3.11}$$

and in particular, (3.10), (3.11), and (3.5) imply

$$\delta \lesssim \exp(-\sqrt{G''(1)}\ell(0)) \ll 1. \tag{3.12}$$

According to (3.10) and (3.12), we may choose $T \geq T^*$, so that

$$T^* = \varepsilon \exp(\sqrt{G''(1)}\ell(0)).$$

Thus,

$$|x(t)-x(0)| + |y(t)-y(0)| \lesssim 1$$

provided $t \leq \varepsilon \exp(\sqrt{G''(1)}\ell(0))$, that is, for all $t \ll \exp(\sqrt{G''(1)}\ell(0))$.

In addition, for $t \gg \ell(0)$, we have by statement (1.4) of Theorem 1.1 precisely

$$||u(t)-v(t)||_{H^1([0,L])} \lesssim \delta \overset{(3.12)}{\lesssim} \exp(-\sqrt{G''(1)}\ell(0)).$$

□

3.2 Linear energy-energy-dissipation estimates

The goal of this subsection is to develop the estimate:

Proposition 3.1 (Finite system). *There exist constants $\ell_3, C_3 < \infty$ such that for all $\ell \geq \ell_3$, for all smooth f with $f(0) = f(\ell) = 0$,*

$$||f||^2_{H^1([0,\ell])} \ \leq \ C_3 \int_0^\ell \left(f_x^2 + G''(v) f^2 \right) \, dx, \tag{3.13}$$

$$||f||^2_{H^2([0,\ell])} \ \leq \ C_3 \int_0^\ell \left(-f_{xx} + G''(v) f \right)^2 \, dx, \tag{3.14}$$

where v solves (3.34).

To prove the proposition, we first state and prove a related result for the large–system limit.

Proposition 3.2 (Infinite system). *There exists a constant $C_3' > 0$ such that if the smooth function v is the solution of*

$$\begin{aligned} -v_{xx} + G'(v) \ &= \ 0, \qquad && x \in \mathbb{R}, \\ v \ &\to \ \pm 1, \qquad && x \to \pm\infty, \end{aligned} \tag{3.15}$$

and if $f \in H^1(\mathbb{R})$ satisfies:

$$(i)\ f(0) \ = \ 0 \qquad or \qquad (ii) \int_{\mathbb{R}} f\, v_x \, dx \ = \ 0,$$

then

$$||f||^2_{L^2(\mathbb{R})} \ \leq \ C_3' \int_{\mathbb{R}} \left(f_x^2 + G''(v) f^2 \right) \, dx.$$

Proof of Proposition 3.2. We begin by introducing two lemmas.

Lemma 3.3. *Let v be as in Proposition 3.2. For all $f \in H^1(\mathbb{R})$,*

$$\int_{\mathbb{R}} \left(f_x^2 + G''(v) f^2 \right) \, dx \ \geq \ 0.$$

Proof. This is a consequence of the well–known fact that v minimizes the energy over functions satisfying the boundary conditions. To see that v is the minimizer, one observes that (3.15) implies:

$$\partial_x \left(\frac{v_x^2}{2} - G(v) \right) \ = \ 0,$$

and from the boundary conditions, one may conclude equipartition of energy, meaning that

$$\frac{v_x^2}{2} - G(v) \;=\; 0. \tag{3.16}$$

This implies

$$E(v) \;=\; \int_{\mathbb{R}} \left| v_x \sqrt{2G(v)} \right| \, dx \;=\; \int_{-1}^{1} \sqrt{2G(v)} \, dv.$$

At the same time, the "Modica–Mortola calculation" shows that the energy of any admissible function w is at least this big:

$$\frac{1}{2} \int_{\mathbb{R}} \left(w_x^2 + 2G(w) \right) \, dx \;\geq\; \int_{\mathbb{R}} |w_x \sqrt{G(w)}| \, dx \;\geq\; \int_{-1}^{1} \sqrt{2G(w)} \, dw.$$

□

Lemma 3.4. *For all $f \in C^2(\mathbb{R}) \cap H^1(\mathbb{R})$,*

$$-f_{xx} + G''(v) f \;=\; 0 \quad \Rightarrow \quad f \;=\; \alpha \, v_x.$$

Proof. Let $\phi := v_x$. We know by the maximum principle that $v \in (-1, 1)$ on $\mathbb{R}$, so $G(v) \in (0, 1]$ and by (3.16), ϕ is of one sign. Furthermore, ϕ satisfies $-\phi_{xx} + G''(v)\phi = 0$. Suppose f is another solution. From the o.d.e.,

$$\frac{d}{dx}(f'\phi - \phi' f) \;=\; 0.$$

Since f and $\phi \in H^1(\mathbb{R})$, $f'\phi - \phi' f = 0$. Since ϕ is nonzero, we can consider the ratio f/ϕ, and we see that

$$\frac{d}{dx}\left(\frac{f}{\phi}\right) \;=\; \frac{f'\phi - f\phi'}{\phi^2} \;=\; 0.$$

□

The proof of the proposition now follows by contradiction. Assume that there exists an H^1–sequence $\{f_n\}_{n=1}^{\infty}$ with $f_n(0) = 0$, $||f_n||_{L^2(\mathbb{R})} = 1$, and

$$\int_{\mathbb{R}} \left(f_{n,x}^2 + G''(v) f_n^2 \right) \, dx \;<\; \frac{1}{n} \int_{\mathbb{R}} f_n^2 \, dx. \tag{3.17}$$

This implies that f_n is uniformly bounded in $H^1(\mathbb{R})$, so there is a subsequence converging weakly in H^1 and strongly in L^2 to a limit f, and by lower semi–continuity,

$$\int_{\mathbb{R}} (f_x^2 + G''(v) f^2)\, dx \ \leq \ 0.$$

Therefore, by Lemma 3.3, f minimizes this functional and thus satisfies the Euler–Lagrange equation

$$-f_{xx} + G''(v) f \ = \ 0.$$

From here we can conclude that $f \in C^2(\mathbb{R})$, so Lemma 3.4 implies $f = \alpha\, v_x$. But $f(0) = 0$ and $v_x(0) \neq 0$ implies $\alpha = 0$, so $f \equiv 0$. We now show that this leads to a contradiction.

First of all, there exists an $X < \infty$ such that $G''(v(x)) \geq \frac{1}{2} G''(1) > 0$ on $\mathbb{R} \setminus (-X, X)$. Because an H^1 bound gives a Hölder bound in one dimension, the Arzela–Ascoli theorem implies that f_n converges locally uniformly (up to a subsequence). Thus, we may assume that $f_n \to 0$ uniformly on $(-X, X)$, and by the L^2–convergence, $\int_{\mathbb{R}\setminus(-X,X)} f_n^2\, dx \to 1$. But then

$$\begin{aligned}
&\lim_{n\to\infty} \int_{\mathbb{R}} G''(v) f_n^2\, dx \\
&\quad \geq \ -\sup_{\mathbb{R}} |G''(v(x))| \lim_{n\to\infty} \int_{-X}^{X} f_n^2\, dx + \frac{1}{2} G''(1) \lim_{n\to\infty} \int_{\mathbb{R}\setminus(-X,X)} f_n^2\, dx \\
&\quad = \ 0 + \frac{1}{2} G''(1) \\
&\quad > \ 0.
\end{aligned}$$

This contradicts (3.17) and proves the proposition under condition (i). The proof under condition (ii) is similar; assume the existence of an H^1–sequence $\{f_n\}_{n=1}^{\infty}$ with $\int_{\mathbb{R}} f_n v_x\, dx \ = \ 0$, $||f_n||_{L^2(\mathbb{R})} = 1$, and (3.17). One may conclude, as above, convergence to a limit f and, by Lemma 3.4, that $f = \alpha v_x$. But then

$$0 \ = \ \lim_{n\to\infty} \int_{\mathbb{R}} f_n v_x\, dx \ = \ \int_{\mathbb{R}} f v_x\, dx \ = \ \alpha \int_{\mathbb{R}} v_x^2\, dx \ \Rightarrow \ \alpha = 0.$$

This implies $f \equiv 0$, which leads to a contradiction, as in the first case. □

Proof of Proposition 3.1. We turn now to the proof of the main proposition. We begin by showing

$$\int_0^{\ell} f^2\, dx \ \leq \ C_3 \int_0^{\ell} \left(f_x^2 + G''(v) f^2 \right)\, dx, \tag{3.18}$$

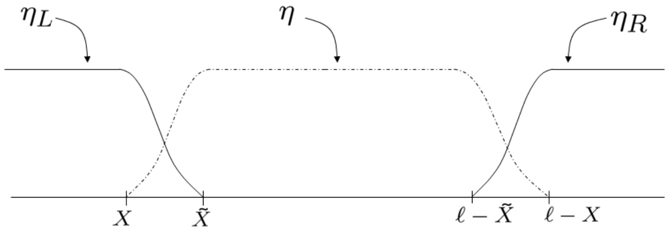

Figure 3: The partition of unity.

and then complete the proof by bootstrapping.

Let v_ℓ denote the solution of (3.34) on $[0, \ell]$. The idea is to split up the domain, using on one part that $G''(v_\ell)$ is bounded away from zero, and using on the other part that $v_\ell \to v_\infty$ uniformly as $\ell \to \infty$, allowing us to take advantage of Proposition 3.2. (Here, v_∞ represents the solution of (3.15) with $v_\infty(0) = 0$.) To implement the idea, we use a partition of unity. We split the domain in the following way. (See Figure 3.) Fix X such that

$$G''(v_\infty(x)) \; > \; \frac{G''(1)}{2} \; =: \; c_* \;\; \text{for } x > X.$$

The functions v_ℓ converge in the sense:

$$v_\infty(x) \; = \; \lim_{\ell\to\infty} \begin{cases} v_\ell(x), & x \in [0, \ell/2), \\ v_\ell(\ell - x), & x \in (\ell/2, \ell], \end{cases} \tag{3.19}$$

with the convergence uniform on bounded sets. Also, v_ℓ is symmetric about $\ell/2$, and monotone on $(0, \ell/2)$. Thus, there exists an $\ell_3(X)$ such that for all $\ell \geq \ell_3$, we have that

$$G''(v_\ell(x)) \; > \; \frac{c_*}{2} \;\; \text{for } x \in (X, \ell - X). \tag{3.20}$$

Define $c_{**} := \min\{c_*, 1/C_3'\}$ (where C_3' is from Proposition 3.2) and

$$\tilde{X} \;:=\; X + \left(\frac{16}{c_{**}}\right)^{1/2}, \tag{3.21}$$

a choice that will become clear later. Finally, in view of (3.19), we may assume that ℓ_3 is so large that for all $\ell \geq \ell_3$ we have:

$$G''(v_\ell(x)) \;\geq\; G''(v_\infty(x)) - \frac{1}{2C_3'} \qquad \text{for} \qquad x \in [0, \tilde{X}]. \tag{3.22}$$

Choose a cut–off function $\eta \in C^\infty([0,\ell])$ with $|\eta| \leq 1$,

$$\eta \;=\; \begin{cases} 1 & x \in [\tilde{X}, \ell - \tilde{X}] \\ 0 & x \in [0, X] \cap [\ell - X, \ell], \end{cases}$$

and

$$|\eta_{xx}| \leq \frac{2}{(\tilde{X} - X)^2}. \tag{3.23}$$

Define also the "left" and "right" functions:

$$\eta_L := \begin{cases} 1 - \eta & x \in [0, \tilde{X}] \\ 0 & x \in (\tilde{X}, \ell], \end{cases} \qquad \eta_R := \begin{cases} 0 & x \in [0, \ell - \tilde{X}), \\ 1 - \eta & x \in [\ell - \tilde{X}, \ell]. \end{cases}$$

Then $\{\eta_L, \eta, \eta_R\}$ is a smooth partition of unity for $[0,\ell]$, which we will use to prove (3.18). By (3.20),

$$\int_0^\ell \Big(((\eta f)_x)^2 + G''(v_\ell)(\eta f)^2 \Big)\; dx \;\geq\; \frac{c_*}{2} \int_0^\ell (\eta f)^2\; dx. \tag{3.24}$$

By (3.22) and Proposition 3.2,

$$\begin{aligned} &\int_0^\ell \Big(((\eta_L f)_x)^2 + G''(v_\ell)(\eta_L f)^2 \Big)\; dx \\ &\qquad \geq\; \int_0^\ell \left(((\eta_L f)_x)^2 + G''(v_\infty)(\eta_L f)^2 - \frac{1}{2C_3'}(\eta_L f)^2 \right)\; dx \\ &\qquad \geq\; \frac{1}{2C_3'} \int_0^\ell (\eta_L f)^2\; dx. \end{aligned} \tag{3.25}$$

Similarly,

$$\int_0^\ell \Big(((\eta_R f)_x)^2 + G''(v_\ell)(\eta_R f)^2 \Big)\; dx \;\geq\; \frac{1}{2C_3'} \int_0^\ell (\eta_R f)^2\; dx. \tag{3.26}$$

We add (3.24), (3.25), and (3.26). For the right–hand side, we claim that

$$\frac{1}{2C_3'}\int_0^\ell ((\eta_L f)^2 + (\eta_R f)^2)\, dx + \frac{c_*}{2}\int_0^\ell (\eta f)^2\, dx \geq \frac{c_{**}}{2}\int_0^\ell f^2(\eta_L^2 + \eta^2 + \eta_R^2)\, dx \geq \frac{c_{**}}{4}\int_0^\ell f^2\, dx. \tag{3.27}$$

Clearly, (3.27) is true wherever $\eta = 1$ or $\eta = 0$. We check what happens on $[X, \tilde{X}]$. By the definition of η_L and the fact that $\eta(x) \in [0, 1]$,

$$\eta^2 + \eta_L^2 \;=\; \eta^2 + (1-\eta)^2 \;=\; 1 - 2\eta(1-\eta) \;\geq\; \frac{1}{2},$$

and similarly on $[\ell - \tilde{X}, \ell - X]$, so (3.27) is true. Next, we claim that the left–hand side of the inequality satisfies

$$\int_0^\ell \Big(((\eta_L f)_x)^2 + ((\eta f)_x)^2 + ((\eta_R f)_x)^2 + (\eta_L^2 + \eta^2 + \eta_R^2) G''(v_\ell) f^2 \Big)\, dx \leq \int_0^\ell \left(f_x^2 + G''(v_\ell) f^2 \right)\, dx + \frac{c_{**}}{8}\int_0^\ell f^2\, dx, \tag{3.28}$$

which, together with (3.27), proves (3.18) (with $C_3 := 8/c_{**}$). Again, the equality is obvious when $\eta = 1$ or $\eta = 0$, and we check what happens on $[X, \tilde{X}]$ and $[L - \tilde{X}, \ell - X]$. There, we have $G''(v_\ell) \geq 0$, so that $\eta_L^2 + \eta^2 + \eta_R^2 \leq 1$ implies for the zeroth order terms:

$$\int_{[X,\tilde{X}]\cup[\ell-\tilde{X},\ell-X]} G''(v_\ell) f^2 (\eta_L^2 + \eta^2 + \eta_R^2)\, dx \;\leq\; \int_{[X,\tilde{X}]\cup[\ell-\tilde{X},\ell-X]} G''(v_\ell) f^2\, dx.$$

For the derivative terms, we use the following identity, which comes from expanding the square and integrating by parts:

$$\int_0^\ell ((wf)_x)^2\, dx \;=\; \int_0^\ell (w^2\, f_x^2 - w\, w_{xx} f^2)\, dx.$$

Adding the three contributions and recalling the definition of η_L and η_R,

$$\begin{aligned}\int_0^\ell ((\eta_L f)_x)^2 + ((\eta f)_x)^2 + ((\eta_R f)_x)^2 \, dx \\ &= \int_0^\ell (\eta_L^2 + \eta^2 + \eta_R^2) f_x^2 - \eta \eta_{xx} f^2 - \eta_L \, \eta_{\ell,xx} f^2 - \eta_R \eta_{r,xx} f^2 \, dx \\ &= \int_0^\ell (\eta_L^2 + \eta^2 + \eta_R^2) f_x^2 - \eta \eta_{xx} f^2 - (1-\eta)(-\eta_{xx}) f^2 \, dx \\ &= \int_0^\ell (\eta_L^2 + \eta^2 + \eta_R^2) f_x^2 + (1-2\eta)\eta_{xx} f^2 \, dx \\ &\leq \int_0^\ell f_x^2 + |\eta_{xx}| f^2 \, dx.\end{aligned}$$

Recall that by (3.21) and (3.23), we have chosen η and $\tilde{X}$ such that

$$|\eta_{xx}| \leq \frac{2}{(\tilde{X} - X)^2} = \frac{c_{**}}{8}.$$

Therefore, (3.28) holds and the proof of (3.18) is complete.

It is not hard to complete the proof of (3.13) by bootstrapping from (3.18) to the H^1–norm. Letting $\delta > 0$ be such that

$$\delta \; \leq \; \frac{1}{2C_3 \sup_{[0,1]} \; |G''(y)|}, \tag{3.29}$$

we have:

$$\begin{aligned}(1+\delta) \int_0^\ell (f_x^2 + G''(v_\ell) f^2) \, dx \\ &= \int_0^\ell \left(f_x^2 + G''(v_\ell) f^2\right) \, dx + \delta \int_0^\ell f_x^2 \, dx + \delta \int_0^\ell G''(v_\ell) f^2 \, dx \\ &\geq \frac{1}{C_3} \int_0^\ell f^2 \, dx + \delta \int_0^\ell f_x^2 \, dx - \frac{1}{2C_3} \int_0^\ell f^2 \, dx \\ &\geq \min\left\{\frac{1}{2C_3}, \delta\right\} \int_0^\ell (f^2 + f_x^2) \, dx,\end{aligned}$$

which proves (3.13) with C_3 appropriately redefined.

Turning to (3.14), it is convenient to introduce the linear operator $\mathcal{L}$ defined by $\mathcal{L} f := f_{xx} - G''(v_\ell) f$, whereby (3.14) reads:

$$||f||^2_{H^2([0,\ell])} \; \leq \; C_3 ||\mathcal{L} f||^2_{L^2([0,\ell])}.$$

Let $(\cdot,\cdot)$ denote the L^2 inner product on $[0,\ell]$. First, by (3.13) and Hölder's inequality, we have

$$||f||^2_{L^2([0,\ell])} \;\le\; C_3|(f,\mathcal{L}f)| \;\le\; C_3||f||_{L^2([0,\ell])}||\mathcal{L}f||_{L^2([0,\ell])}, \tag{3.30}$$

which implies

$$||f||_{L^2([0,\ell])} \;\le\; C_3||\mathcal{L}f||_{L^2([0,\ell])}.$$

Next, substituting for $||f||_{L^2([0,\ell])}$ in the right–hand side of (3.30), we observe

$$|(f,\mathcal{L}f)| \;\le\; C_3||\mathcal{L}f||^2_{L^2([0,\ell])}.$$

Invoking (3.13) again, we conclude:

$$||f||^2_{H^1([0,\ell])} \;\le\; C_3^2||\mathcal{L}f||^2_{L^2([0,\ell])}. \tag{3.31}$$

Finally, the second derivatives are also estimated, in the following way:

$$\begin{aligned}\int_0^\ell (\mathcal{L}f)^2\,dx \;&=\; \int_0^\ell f_{xx}^2 + G''(v_\ell)^2 f^2 - 2f_{xx}G''(v_\ell)f\,dx\\ &\ge\; \int_0^\ell f_{xx}^2 - \frac{1}{2}f_{xx}^2 - 4G''(v_\ell)^2f^2\,dx\\ &\ge\; \int_0^\ell \frac{1}{2}f_{xx}^2 - 4\Big(\sup_{[0,1]}|G''(y)|\Big)^2 f^2\,dx.\end{aligned}$$

Rearranging terms and applying (3.31),

$$\begin{aligned}\int_0^\ell f_{xx}^2\,dx \;&\le\; 2\int_0^\ell(\mathcal{L}f)^2\,dx + 8\Big(\sup_{[0,1]}|G''(y)|\Big)^2\int_0^\ell f^2\,dx\\ &\le\; \left(2+8\Big(\sup_{[0,1]}|G''(y)|\Big)^2C_3^2\right)\int_0^\ell(\mathcal{L}f)^2\,dx.\end{aligned} \tag{3.32}$$

Combining and (3.31) and (3.32) and redefining the constant C_3 appropriately, we arrive at (3.14). □

3.3 Nonlinear energy-energy-dissipation estimate

3.3.1 Proof of the main proposition

In this subsection, we extend from the linear estimates of Proposition 3.1 to the nonlinear energy–energy–dissipation estimates. We will prove:

Proposition 3.3. *There exist constants ℓ_1, $C_1 < \infty$ with the following properties. Let $\ell \geq \ell_1$ and suppose that u satisfies*

$$\begin{aligned} u &\geq 0, \qquad x \in (0, \ell), \\ u &= 0, \qquad x \in \{0, \ell\}. \end{aligned} \tag{3.33}$$

Let v be the smooth "energy–optimal profile" that satisfies

$$\begin{aligned} -v_{xx} + G'(v) &= 0, \qquad x \in (0, \ell), \\ v &> 0, \qquad x \in (0, \ell), \\ v &= 0, \qquad x \in \{0, \ell\}. \end{aligned} \tag{3.34}$$

Then we have the energy–gap inequality

$$||u - v||^2_{H^1([0,\ell])} \leq C_1(E(u) - E(v)). \tag{3.35}$$

Moreover, if u is "bounded away from $u \equiv 0$" in the sense that

$$E(u) \leq \frac{G(0)}{2}\,\ell_1,$$

then we have the energy–dissipation inequality

$$||u - v||^2_{H^2([0,\ell])} \leq C_1 \int_0^\ell (u_{xx} - G'(u))^2 \, dx. \tag{3.36}$$

Remark 3. *Since it is easy to bound the energy–gap above by the H^1–norm of $u - v$, the combination of* (3.35) *and* (3.36) *implies*

$$\frac{1}{C_1}||u - v||^2_{H^1([0,\ell])} \leq E(u) - E(v) \leq C_1 \int_0^\ell (u_{xx} - G'(u))^2 \, dx, \tag{3.37}$$

where C_1 has been appropriately redefined. Proposition 1.1 follows by applying (3.37) *on each subinterval in–between zeros.*

Proof of Proposition 3.3. Let ℓ_3 be as in Proposition 3.1. We use six lemmas, proved in Subsections 3.3.2 and 3.3.3. For the energy–gap inequality, we need the following three lemmas.

Lemma 3.5. *Suppose (3.33) holds and v is defined as usual. For every $\varepsilon > 0$, there exists $\gamma > 0$ such that for all $\ell \geq \ell_3$,*

$$E(u) - E(v) \leq \gamma \quad \Rightarrow \quad \sup_{[0,\ell]} |u - v| \leq \varepsilon.$$

Lemma 3.5 is used to prove the energy gap inequality in the case of small energy gap:

Lemma 3.6. *There exists $\gamma > 0$ such that for all $\ell \geq \ell_3$,*

$$E(u) - E(v) \;\leq\; \gamma \quad \Rightarrow \quad ||u - v||^2_{H^1([0,\ell])} \;\leq\; 4C_3(E(u) - E(v)).$$

In the case of order one energy gap, we use the rough estimate:

Lemma 3.7. *For every $\gamma > 0$, there exists $C_\gamma < \infty$ such that for $u, v \geq 0$,*

$$E(u) - E(v) \;\geq\; \gamma \quad \Rightarrow \quad ||u - v||^2_{H^1([0,\ell])} \;\leq\; C_\gamma(E(u) - E(v)).$$

The combination of Lemmas 3.6 and 3.7 proves (3.35) with

$$C_1 := \max\{4C_3, C_\gamma\}.$$

Similarly, (3.36) is proved by introducing three lemmas. We abbreviate:

$$g := -u_{xx} + G'(u).$$

Lemma 3.8. *Suppose (3.33) holds and v is defined as usual. For every $\varepsilon > 0$, there exists $\gamma > 0$ such that for all $\ell \geq \ell_3$, as long as*

$$E(u) \;\leq\; \frac{G(0)}{2}\,\ell_3, \tag{3.38}$$

then

$$||g||^2_{L^2([0,\ell])} \;\leq\; \gamma \quad \Rightarrow \quad \sup_{[0,\ell]} |u - v| \;\leq\; \varepsilon. \tag{3.39}$$

Lemma 3.8 is used to prove the energy dissipation inequality in the case of small dissipation:

Lemma 3.9. *There exists $\gamma > 0$ such that for all $\ell \geq \ell_3$, as long as* (3.38) *holds, then*

$$||g||^2_{L^2([0,\ell])} \;\leq\; \gamma \quad \Rightarrow \quad ||u - v||^2_{H^2([0,\ell])} \;\leq\; 4C_3||g||^2_{L^2([0,\ell])}.$$

In the case of order one dissipation, we use the rough estimate:

Lemma 3.10. *For every $\gamma > 0$, there exists $C_\gamma < \infty$ such that for $u, v \geq 0$ and $E(u) \lesssim 1$,*

$$||g||^2_{L^2([0,\ell])} \;\geq\; \gamma \quad \Rightarrow \quad ||u - v||^2_{H^2([0,\ell])} \;\leq\; C_\gamma||g||^2_{L^2([0,\ell])}.$$

The combination of Lemmas 3.9 and 3.10 yields (3.36) with

$$C_1 := \max\{4C_3, C_\gamma\}.$$

□

3.3.2 Proofs of the energy gap lemmas

The main idea in the proof of Lemma 3.5 is to use the function $F : [0,1] \to \mathbb{R}$,

$$F(w) \;:=\; 2\int_0^w \sqrt{2(G(v)-G(w))}\,dv + \ell\, G(w),$$

to prove that for ℓ large,

$$\begin{aligned} E(u)-E(v) \;\geq\; & \frac{1}{2}\int_0^{x_m} \Big(u_x - \sqrt{2(G(u)-G(u_m))}\Big)^2\,dx \\ & + \frac{1}{2}\int_{x_m}^{\ell} \Big(u_x + \sqrt{2(G(u)-G(u_m))}\Big)^2\,dx. \end{aligned}$$

Roughly, one then concludes that when the energy gap is small, u cannot be far from the function v that solves

$$v_x \;=\; \begin{cases} \sqrt{2(G(v)-G(v_m))} & x \in (0,\ell/2), \\ -\sqrt{2(G(v)-G(v_m))} & x \in (\ell/2,\ell). \end{cases} \tag{3.40}$$

Notation 3. *As usual, let $f := u - v$.*

Proof of Lemma 3.5. We prove the lemma by establishing:

(1) For every $\varepsilon > 0$, there exist $\ell_* < \infty$ and $\gamma_* > 0$ such that if $\ell \geq \ell_*$ and (3.33) holds, then

$$E(u)-E(v) \;\leq\; \gamma_* \quad\Rightarrow\quad \sup_{[0,\ell]}|f| \;\leq\; \varepsilon.$$

(2) For every $\varepsilon > 0$ and $\ell_* < \infty$, there exists $\gamma_{**} > 0$ such that if $\ell \in [\ell_3, \ell_*]$ and (3.33) holds, then

$$E(u)-E(v) \;\leq\; \gamma_{**} \quad\Rightarrow\quad \sup_{[0,\ell]}|f| \;\leq\; \varepsilon.$$

ad (2): We give an indirect argument. Suppose for a contradiction that there exists a sequence of interval lengths ℓ_n and functions u_n satisfying (3.33) and

$$E_{\ell_n}(u_n) - E_{\ell_n}(v_n) \underset{n\to\infty}{\longrightarrow} 0, \tag{3.41}$$

but

$$\lim_{n\to\infty}\left(\sup_{[0,\ell_n]}|f_n|\right) \;>\; 0. \tag{3.42}$$

Without loss of generality, we may assume that $\ell_n \to \ell \in [\ell_3, \ell_*]$, and that

$$\begin{aligned} u_n &\overset{H^1}{\rightharpoonup} u, \\ u_n &\to u \quad \text{uniformly}, \end{aligned}$$

where the limit $u \in H^1([0,\ell]) \cap C([0,\ell])$ satisfies (3.33). By (3.41), continuity of $E_\ell(u)$ in ℓ, and lower semi–continuity of $E_\ell(u)$ in u,

$$E_\ell(u) \;\le\; E_\ell(v_\ell).$$

Hence, by uniqueness of the minimizer among nonnegative functions, $u = v_\ell$. That is, $f = 0$, and by the uniform convergence,

$$\lim_{n\to\infty} \left(\sup_{[0,\ell_n]} |f_n| \right) \;=\; 0,$$

contradicting (3.42).
ad (1): Let $u(x_m) = u_m$ be the maximum of u and $v_m = v(\ell/2)$ the maximum of v on $[0,\ell]$. We will prove below that for $\ell \gg 1$,

$$\begin{aligned} E(u) - E(v) \;\ge\; & \frac{1}{2}\int_0^{x_m} \left(u_x - \sqrt{2(G(u) - G(u_m))} \right)^2 dx \\ & + \frac{1}{2}\int_{x_m}^{\ell} \left(u_x + \sqrt{2(G(u) - G(u_m))} \right)^2 dx. \end{aligned} \tag{3.43}$$

In view of (3.43), it remains to show that $\sup_{[0,\ell]} |f| \ll 1$ if

$$\begin{aligned} & \int_0^{x_m} \left(u_x - \sqrt{2(G(u) - G(u_m))} \right)^2 dx \\ & \qquad + \int_{x_m}^{\ell} \left(u_x + \sqrt{2(G(u) - G(u_m))} \right)^2 dx \;\ll\; 1, \end{aligned} \tag{3.44}$$

$$E(u) \;\lesssim\; 1, \tag{3.45}$$

$$\ell \;\gg\; 1. \tag{3.46}$$

We notice that because $u \ge 0$, (3.45) and (3.46) imply

$$u_m \;\approx\; 1, \quad \text{and} \quad \min\{x_m, \ell - x_m\} \;\gtrsim\; 1. \tag{3.47}$$

Furthermore, (3.46) implies $v_m \approx 1$, so that in particular,

$$G(u_m) \;\approx\; G(v_m). \tag{3.48}$$

From (3.44) and (3.48), we deduce that

$$u_x \;=\; \sqrt{2(G(u)-G(u_m))}+r, \quad x\in(0,x_m),$$

with a small error term r:

$$\int_0^{x_m} r^2\,dx \;\ll\; 1.$$

Hence, we have by standard ODE theory:

$$\sup_{[0,\min\{x_m,X\}]} |f| \;\ll\; 1 \quad \text{for some } X \;\gg\; 1. \tag{3.49}$$

We claim that in fact $x_m \gg 1$, so that (3.49) holds on a large neighborhood of zero. To see this, observe that if $x_m \gg 1$ were false, we would have

$$u_m \;=\; u(x_m) \;\overset{(3.49)}{\approx}\; v(x_m) \;\not\approx\; 1,$$

contradicting the first part of (3.47). Hence, (3.49) improves to

$$\sup_{[0,X]} |f| \;\ll\; 1 \quad \text{for some } X \;\gg\; 1.$$

Likewise,

$$\sup_{[\ell-X,\ell]} |f| \;\ll\; 1 \quad \text{for some } X \;\gg\; 1.$$

Since $v \approx 1$ on $(X,\ell-X)$ and $u \le 1$, it remains to show that $u \;\gtrapprox 1$ on $(X,\ell-X)$. Suppose that there exists an $x_* \in (X,\ell-X)$ with

$$u(x_*) \;\not\gtrapprox\; 1. \tag{3.50}$$

Without loss of generality, $x_* \le x_m$. We compare u to the solution w of

$$\begin{aligned} w_x &= \sqrt{2(G(w)-G(u_m))} & & x<x_*,\\ w &= u(x_*) & & x=x_*. \end{aligned}$$

Again using (3.44) and standard ODE theory, we deduce

$$\sup_{(x_*-X',x_*)} |u-w| \;\ll\; 1 \quad \text{for some } X' \;\gg\; 1. \tag{3.51}$$

(Without loss, we may assume that $X'=X$.) On the other hand, because of (3.50), $G(u_m)\ll 1$, and the properties of G,

$$w \;\approx\; -1 \quad \text{on} \quad (-\infty, x_*-X/2). \tag{3.52}$$

From (3.51) and (3.52), we deduce

$$u \;<\; 0 \quad \text{on} \quad (x_* - X, x_* - X/2),$$

a contradiction.

Finally, to prove (3.43), we introduce the function $F : [0,1] \to \mathbb{R}$:

$$F(w) \;:=\; 2\int_0^w \sqrt{2(G(v) - G(w))}\, dv + \ell\, G(w).$$

We will establish the following:

(a) We have the identity:

$$\begin{aligned} E(u) \;=\; F(u_m) &+ \frac{1}{2}\int_0^{x_m} \left(u_x - \sqrt{2(G(u) - G(u_m))}\right)^2 \, dx \\ &+ \frac{1}{2}\int_{x_m}^{\ell} \left(u_x + \sqrt{2(G(u) - G(u_m))}\right)^2 \, dx. \end{aligned}$$

(b) $E(v) \;=\; F(v_m) \;\le\; F(w)$ for $w \;\le\; v_m$.

(c)

$$F'(w) \;=\; G'(w)\left(\ell - \int_0^w \frac{2}{\sqrt{2(G(v) - G(w)}}\, dv\right).$$

(d) $F'(v_m) \;=\; 0$ and $v_m \approx 1$ for $\ell \;\gg\; 1$.

(e) $F(v_m) \;\le\; F(w)$ for all $w \in [0,1]$, provided $\ell \;\gg\; 1$.

This is enough to conclude (3.43):

$$\begin{aligned} E(u) \;&-\; \int_0^{x_m} \left(u_x - \sqrt{2(G(u) - G(u_m))}\right)^2 \, dx \\ &-\; \int_{x_m}^{\ell} \left(u_x + \sqrt{2(G(u) - G(u_m))}\right)^2 \, dx \\ &\overset{(a)}{=}\; F(u_m) \\ &\overset{(e)}{\ge}\; F(v_m) \\ &\overset{(b)}{=}\; E(v). \end{aligned}$$

ad (a): Observe that $G(u) \geq G(u_m)$, so we can express the energy as:

$$\begin{aligned}
E(u) &= \ell G(u_m) + \frac{1}{2}\int_0^{x_m} \left(u_x^2 + \left(\sqrt{2(G(u)-G(u_m))}\right)^2 \right) dx \\
&\quad + \frac{1}{2}\int_{x_m}^{\ell} \left(u_x^2 + \left(\sqrt{2(G(u)-G(u_m))}\right)^2 \right) dx \\
&= \ell G(u_m) + \frac{1}{2}\int_0^{x_m} \left(u_x - \sqrt{2(G(u)-G(u_m))}\right)^2 dx \\
&\quad + \int_0^{x_m} u_x \sqrt{2(G(u)-G(u_m))}\, dx \\
&\quad + \frac{1}{2}\int_{x_m}^{\ell} \left(u_x + \sqrt{2(G(u)-G(u_m))}\right)^2 dx \\
&\quad - \int_{x_m}^{\ell} u_x \sqrt{2(G(u)-G(u_m))}\, dx \\
&= F(u_m) + \frac{1}{2}\int_0^{x_m} \left(u_x - \sqrt{2(G(u)-G(u_m))}\right)^2 dx \\
&\quad + \frac{1}{2}\int_{x_m}^{\ell} \left(u_x + \sqrt{2(G(u)-G(u_m))}\right)^2 dx.
\end{aligned}$$

ad (b): The first equality is clear, from (3.40). For the inequality, observe that for $w \leq v_m$ there exists an interval length $\ell_w \leq \ell$ such that the maximum of the solution of (3.34) on ℓ_w is equal to w. Consider the comparison function u that is equal to this solution on $[0, \ell_w/2]$, takes the constant value w on $[\ell_w/2, \ell/2]$, and is even about $\ell/2$. Then

$$\begin{cases}
u_x - \sqrt{2(G(u)-G(w))} = 0 & x \in (0, \ell_w/2), \\
u_x + \sqrt{2(G(u)-G(w))} = 0 & x \in (\ell - \ell_w/2, \ell), \\
u_x = \sqrt{2(G(u)-G(w))} = 0 & x \in (\ell_w/2, \ell - \ell_w/2).
\end{cases}$$

Consequently,

$$F(w) = E(u) \geq E(v) = F(v),$$

where we have used the minimality of the energy of v.

ad (c): This is a direction calculation.

ad (d): For $x \in [0, \ell/2]$, we have:

$$x = \int_0^{v(x)} \frac{1}{\sqrt{2(G(v)-G(v_m))}}\, dv,$$

and in particular,

$$\frac{\ell}{2} = \int_0^{v_m} \frac{1}{\sqrt{2(G(v)-G(v_m))}}\, dv.$$

Thus, (c) implies that $F'(v_m) = 0$. The fact that $v_m \approx 1$ for large ℓ follows, for instance, from the equality in (b) and the definition of F.

ad (e): In view of (b), it suffices to show that $F(v_m) \le F(w)$ for $w \in [v_m, 1]$ when $\ell \gg 1$. This will follow from $F'(w) \ge 0$ (and the continuity of F) on this interval.

Since $G'(w) \le 0$ for all $w \in [v_m, 1]$, this means (from (c)) that we would like to show

$$\frac{\ell}{2} \le \int_0^{w} \frac{dv}{\sqrt{2(G(v)-G(w))}}, \quad w \in [v_m, 1].$$

We have equality at v_m (by (d)), therefore it suffices to show

$$\frac{d}{dw}\int_0^{w} \frac{dv}{\sqrt{G(v)-G(w)}} \ge 0, \quad w \in [v_m, 1],$$

(where we have dropped an irrelevant factor of $\sqrt{2}$). For $\ell \gg 1$, $v_m \approx 1$ (by (d)), therefore we would like to show

$$\frac{d}{dw}\int_0^{w} \frac{dv}{\sqrt{G(v)-G(w)}} \ge 0, \quad \forall\, w\,;\ 1-w \ll 1. \tag{3.53}$$

In order to calculate the derivative, we change variables. Let $s := v/w$. Then

$$\begin{aligned}
&\frac{d}{dw}\int_0^{w} \frac{dv}{\sqrt{G(v)-G(w)}} \\
&= \frac{d}{dw}\int_0^{1} \frac{w\, ds}{\sqrt{G(sw)-G(w)}} \\
&= \int_0^{1} \frac{ds}{\sqrt{G(sw)-G(w)}} - w\int_0^{1} \frac{sG'(sw)-G'(w)}{2(G(sw)-G(w))^{3/2}}\, ds \\
&= \frac{1}{w}\left(\int_0^{w} \frac{dv}{\sqrt{G(v)-G(w)}} - \int_0^{w} \frac{vG'(v)-wG'(w)}{2(G(v)-G(w))^{3/2}}\, dv\right) \\
&= \frac{1}{w}\left(\int_0^{w} \frac{G(v)-G(w)+G(v)-vG'(v)-(G(w)-wG'(w))}{2(G(v)-G(w))^{3/2}}\, dv\right). \qquad (3.54)
\end{aligned}$$

By hypothesis, $G(v) \geq G(w)$ for $0 \leq v \leq w \leq 1$. We claim that in addition, $w \approx 1$ implies

$$G(v) - v\,G'(v) - (G(w) - w\,G'(w)) \;\geq\; 0, \tag{3.55}$$

which shows (3.53) and completes the proof of (e). To see this, define

$$h(v) \;:=\; G(v) - v\,G'(v).$$

The inequality (3.55) follows from two observations:

- On a neighborhood of $v = 1$, h is monotone decreasing, since $G''(1) > 0$ and

$$h'(v) \;=\; -v\,G''(v).$$

- Away from $v = 1$, h is bounded away from zero, while on the other hand $h(w) \ll 1$ (since $w \approx 1$).

□

Proof of Lemma 3.6. We reexpress the energy difference in a form such that the first term can be bounded using Proposition 3.1, and the second by a Taylor expansion.

$$\begin{aligned}
E(u) - E(v) \;&=\; \int_0^\ell \left(\frac{1}{2}(u_x^2 - v_x^2) + G(u) - G(v)\right) dx \\
&=\; \int_0^\ell \left(\frac{1}{2} f_x^2 + v_x\, f_x + G(u) - G(v)\right) dx \\
&=\; \int_0^\ell \left(\frac{1}{2} f_x^2 - v_{xx} f + G(u) - G(v)\right) dx \\
&=\; \int_0^\ell \left(\frac{1}{2} f_x^2 - G'(v) f + G(u) - G(v)\right) dx \\
&=\; \frac{1}{2}\int_0^\ell \left(f_x^2 + G''(v) f^2\right)\, dx \\
&\quad\; + \int_0^\ell \left(G(u) - G(v) - G'(v) f - \frac{G''(v)}{2} f^2\right) dx \\
&\geq\; \frac{1}{2C_3}\int_0^\ell (f^2 + f_x^2)\, dx - C_* \sup_{[0,\ell]} |f| \int_0^\ell f^2\, dx
\end{aligned}$$

with $C_* := \sup_{[0,1]} |G'''(\cdot)|$. By Lemma 3.5, we may choose γ such that

$$\sup_{[0,\ell]} |f| \le \frac{1}{4\,C_*\,C_3},$$

so that

$$E(u) - E(v) \;\ge\; \frac{1}{4\,C_3} \int_0^\ell (f^2 + f_x^2)\,dx.$$

□

Proof of Lemma 3.7. From the properties of G, there exists a c_* such that for $u \ge 0$,

$$G(u) \ge c_*(u-1)^2. \tag{3.56}$$

We use the rough estimate:

$$\begin{aligned}
\int_0^\ell (f^2 + f_x^2)\,dx \quad &\le \quad 2\int_0^\ell \left((u-1)^2 + (v-1)^2\right)\,dx + 2\int_0^\ell (u_x^2 + v_x^2)\,dx \\
&\overset{(3.56)}{\le} \quad \frac{2}{c_*}\int_0^\ell (G(u) + G(v))\,dx + 2\int_0^\ell (u_x^2 + v_x^2)\,dx \\
&\le \quad C_{**}\,(E(u) + E(v)),
\end{aligned} \tag{3.57}$$

where $C_{**} := \max\{2/c_*, 4\}$. Letting C_{***} denote the bound on the energy of v for $\ell \in (\ell_3, \infty)$ and adding and subtracting the energy of v in (3.57),

$$\begin{aligned}
\int_0^\ell (f^2 + f_x^2)\,dx \quad &\le \quad C_{**}(E(u) - E(v)) + 2C_{**}E(v) \\
&\le \quad C_{**}(E(u) - E(v)) + 2C_{**}C_{***} \\
&\le \quad C_{**}\left(1 + \frac{2C_{***}}{\gamma}\right)(E(u) - E(v)),
\end{aligned}$$

since the energy gap is bounded below by γ. □

3.3.3 Proof of the energy dissipation lemmas

Notation 4. *As usual, let $f := u - v$.*

Proof of Lemma 3.8. We prove the lemma by establishing:

(1) For every $\varepsilon > 0$, there exist $\ell_* < \infty$ and $\gamma_* > 0$ such that if $\ell \ge \ell_*$ and u satisfies (3.33) as well as condition (3.38), then

$$\int_0^\ell g^2\,dx \;\le\; \gamma_* \quad \Rightarrow \quad \sup_{[0,\ell]} |f| \le \varepsilon.$$

(2) For every $\varepsilon > 0$ and $\ell_* < \infty$, there exists $\gamma_{**} > 0$ such that if $\ell \in [\ell_3, \ell_*]$ and u satisfies (3.33) as well as condition (3.38), then

$$\int_0^\ell g^2\, dx \ \le\ \gamma_{**} \quad \Rightarrow \quad \sup_{[0,\ell]} |f| \le \varepsilon.$$

ad (2): As with the analogous statement in the case of the energy gap, we give an indirect argument. Suppose for a contradiction that there exists a sequence of interval lengths ℓ_n and functions u_n satisfying (3.33), (3.38), and

$$\int_0^{\ell_n} g_n^2\, dx \underset{n\to\infty}{\longrightarrow} 0, \tag{3.58}$$

but

$$\lim_{n\to\infty} \left(\sup_{[0,\ell_n]} |f_n| \right) \ >\ 0. \tag{3.59}$$

Without loss of generality, we may assume that $\ell_n \to \ell \in [\ell_3, \ell_*]$, and that

$$\begin{aligned} u_n &\overset{H^2}{\rightharpoonup} u, \\ u_n &\to u \quad \text{uniformly}, \\ g_n &\to 0 \quad \text{pointwise } a.e. \end{aligned} \tag{3.60}$$

Consequently, we have:

$$u_{xx} - G'(u) = 0 \quad a.e.,$$

and by the uniform convergence, u satisfies (3.33). This implies the dichotomy:

$$\text{either} \quad u \equiv 0 \quad \text{or} \quad u = v.$$

Condition (3.38) survives in the limit $n \to \infty$ under the convergence (3.60), which rules out the first alternative. Thus, $u = v$, and by the uniform convergence,

$$\lim_{n\to\infty} \left(\sup_{[0,\ell_n]} |f_n| \right) \ =\ 0,$$

contradicting (3.59).

ad (1): We introduce the discrepancy function:

$$\xi := \frac{1}{2}u_x^2 - G(u),$$

so–called because it measures the amount by which u deviates from the equipartition of energy expressed in (3.16). Because

$$\int_0^\ell |\xi|\, dx \;\le\; E(u) \;\lesssim\; 1,$$

we conclude

$$\inf_{[0,\ell]} |\xi| \;\lesssim\; \frac{1}{\ell}. \tag{3.61}$$

Moreover, the total variation of ξ is controlled by the energy dissipation:

$$\int_0^\ell |\xi_x|\, dx \;=\; \int_0^\ell |g\, u_x|\, dx \;\le\; \left(\int_0^\ell g^2\, dx \int_0^\ell u_x^2\, dx\right)^{1/2} \;\lesssim\; \gamma_*^{1/2}. \tag{3.62}$$

Together, (3.61) and (3.62) imply

$$\sup_{[0,\ell]} |\xi| \;\lesssim\; \frac{1}{\ell} + \gamma_*^{1/2}.$$

Hence it is enough to show

$$\sup_{[0,\ell]} |\xi| \;\ll\; 1, \quad \ell \;\gg\; 1 \quad\Rightarrow\quad \sup_{[0,\ell]} |f| \;\ll\; 1.$$

To this end, we use ξ to write u as a perturbed solution of the ODE solved by v:

$$u_x \;=\; \pm\sqrt{2(G(u)+\xi)}.$$

Consider $x = 0$. The condition (3.33) implies $u_x(0) \;\ge\; 0$, and we have

$$u_x(0) \;=\; \sqrt{2(G(u(0))+\xi(0))} \;=\; \sqrt{2(G(0)+\xi(0))} \;\approx\; \sqrt{2G(0)},$$

since $G(0)$ is order one and $|\xi| \ll 1$. Therefore, u_x takes the positive square root on a neighborhood of zero:

$$u_x = \sqrt{2(G(u)+\xi)}, \quad x \in [0, x_m],$$

where x_m denotes the first maximum of u. Notice that $|\xi| \ll 1$ and $u \geq 0$ imply

$$u_m = u(x_m) \approx 1. \tag{3.63}$$

Recall now that on $[0, \ell/2]$,

$$v_x \;=\; \sqrt{2(G(v) - G(v_m))}\,,$$

and also, $\ell \gg 1$ implies $G(v_m) \ll 1$. Therefore, by ODE theory,

$$\sup_{[0,\min\{x_m,X\}]} |f| \;\ll\; 1 \quad \text{for some } X \;\gg\; 1. \tag{3.64}$$

Combining (3.63) and (3.64), we see that $x_m \gg 1$ and (3.64) improves to

$$\sup_{[0,X]} |f| \;\ll\; 1 \quad \text{for some } X \;\gg\; 1.$$

The parallel argument implies

$$\sup_{[\ell-X,\ell]} |f| \;\ll\; 1 \quad \text{for some } X \;\gg\; 1.$$

Finally, observe that $v \approx 1$ on $(X, \ell - X)$ and that $|\xi| \ll 1$ implies $u_{min} \approx 1$ for any minimum point of u. Thus, we also have $|f| \ll 1$ on $(X, \ell - X)$. □

Proof of Lemma 3.9. Because v satisfies the stationary equation (3.34), we can reexpress the dissipation as:

$$\begin{aligned}
\int_0^\ell g^2\,dx &= \int_0^\ell (-u_{xx} + G'(u))^2\,dx \\
&= \int_0^\ell (-f_{xx} + G'(u) - G'(v))^2\,dx \\
&= \int_0^\ell (-f_{xx} + G''(v)f + G'(u) - G'(v) - G''(v)f)^2\,dx \\
&\geq \frac{1}{2}\int_0^\ell (-f_{xx} + G''(v)f)^2\,dx \\
&\qquad - \int_0^\ell (G'(u) - G'(v) - G''(v)f)^2\,dx.
\end{aligned} \tag{3.65}$$

On the one hand, from Proposition 3.1,

$$\int_0^\ell (-f_{xx} + G''(v)f)^2\,dx \;\geq\; \frac{1}{C_3}\int_0^\ell (f^2 + f_x^2 + f_{xx}^2)\,dx.$$

On the other hand,

$$\int_0^\ell (G'(u) - G'(v) - G''(v)f)^2\, dx \;\le\; C_*^2 \sup_{[0,\ell]} |f|^2 \int_0^\ell f^2\, dx,$$

with $C_* := \sup_{[0,1]} |G'''(\cdot)|$. Thus, (3.65) turns into

$$\int_0^\ell g^2\, dx \;\ge\; \frac{1}{2C_3} \int (f^2 + f_x^2 + f_{xx}^2)\, dx - C_*^2 \sup_{[0,\ell]} |f|^2 \int_0^\ell f^2\, dx.$$

By Lemma 3.8, we can choose γ so small that

$$\sup_{[0,\ell]} |f| \;\le\; \left(\frac{1}{4C_3 C_*^2}\right)^{1/2},$$

which yields

$$\int_0^\ell g^2\, dx \;\ge\; \frac{1}{4C_3} \int_0^\ell (f^2 + f_x^2 + f_{xx}^2)\, dx.$$

□

Proof of Lemma 3.10. It is enough to show

$$\int_0^\ell (f^2 + f_x^2 + f_{xx}^2)\, dx \;\lesssim\; \int_0^\ell g^2\, dx + E(u) + 1.$$

Because

$$\int_0^\ell f_x^2\, dx \;=\; -\int_0^\ell f\, f_{xx}\, dx \;\le\; \frac{1}{2} \int_0^\ell (f^2 + f_{xx}^2)\, dx,$$

it is enough to show

$$\int_0^\ell (f^2 + f_{xx}^2)\, dx \;\lesssim\; \int_0^\ell g^2\, dx + E(u) + 1.$$

Because $g = -f_{xx} + G'(u) - G'(v)$,

$$\int_0^\ell f_{xx}^2\, dx \;\lesssim\; \int_0^\ell g^2\, dx + \int_0^\ell f^2\, dx,$$

and it is enough to show

$$\int_0^\ell f^2\, dx \;\lesssim\; E(u) + 1.$$

Finally, since $f^2 \le 2((u-1)^2 + (v-1)^2)$ and $u, v \;\ge 0$, we have

$$\int_0^\ell f^2\, dx \;\overset{(3.56)}{\lesssim}\; E(u) + E(v) \;\lesssim E(u) + 1.$$

□

3.4 Lipschitz constant for the energy

Proof of Lemma 3.1. It suffices to consider the energy in–between a pair of zeros and show that there exists a $C_* < \infty$ such that

$$\left|\frac{d}{d\ell}E(\ell)\right| \;\le\; C_* \exp\left(-\sqrt{G''(1)}\ell\right). \tag{3.66}$$

Let $v : [0,\ell] \to \mathbb{R}$ be an energy–optimal profile satisfying (3.34). The energy is

$$\begin{aligned} E(\ell) &= \int_0^\ell \left(\frac{1}{2}v_x^2 + G(v)\right) dx \\ &= 2\int_0^{\ell/2} \left(\frac{1}{2}v_x^2 + G(v)\right) dx \\ &= 2\int_0^{v_m(\ell)} \frac{\left(\frac{1}{2}v_x^2 + G(v)\right)}{v_x}\, dv, \end{aligned} \tag{3.67}$$

where $v_m(\ell)$ denotes the maximum of v on $(0,\ell)$. Recalling that

$$\frac{1}{2}v_x^2 \;=\; G(v) - G(v_m),$$

we can rewrite (3.67) as

$$\begin{aligned} E(\ell) &= 2\int_0^{v_m(\ell)} \frac{2G(v) - G(v_m(\ell))}{\sqrt{2(G(v) - G(v_m(\ell)))}}\, dv \\ &= 2\int_0^{v_m(\ell)} \sqrt{2(G(v) - G(v_m(\ell)))}\, dv \\ &\qquad + 2G(v_m(\ell))\int_0^{v_m(\ell)} \frac{1}{\sqrt{2(G(v) - G(v_m(\ell)))}}\, dv \\ &= 2\int_0^{v_m(\ell)} \sqrt{2(G(v) - G(v_m(\ell)))}\, dv + G(v_m(\ell))\ell, \end{aligned} \tag{3.68}$$

where in the last step we have used the identity

$$\begin{aligned} \frac{\ell}{2} = \int_0^{\ell/2} dx &= \int_0^{\ell/2} \frac{v_x}{\sqrt{2(G(v) - G(v_m(\ell)))}} dx \\ &= \int_0^{v_m(\ell)} \frac{1}{\sqrt{2(G(v) - G(v_m(\ell)))}}\, dv. \end{aligned} \tag{3.69}$$

Differentiation of (3.68) leads to

$$E'(\ell) \;=\; G(v_m(\ell)) \;=\; \frac{G''(1)}{2}(1-v_m(\ell))^2 + O((1-v_m(\ell))^3). \tag{3.70}$$

Thus, it remains to estimate $1 - v_m(\ell)$ in terms of ℓ. To this end, we multiply (3.69) by $\sqrt{G''(v_m(\ell))}$ and then add and subtract the dominant term

$$\int_0^{v_m(\ell)} \frac{dv}{\sqrt{2\lambda(v-v_m(\ell)) + (v-v_m(\ell))^2}} \;=\; \int_0^{v_m(\ell)} \frac{dv}{\sqrt{-2\lambda\, v + v^2}}$$

with

$$\lambda := G'(v_m)/G''(v_m) \;=\; -(1-v_m) + O((1-v_m)^2). \tag{3.71}$$

This leads to

$$\begin{aligned}
&\frac{\sqrt{G''(v_m(\ell))}\,\ell}{2} \\
&= \int_0^{v_m(\ell)} \left(\frac{\sqrt{G''(v_m(\ell))}}{\sqrt{2(G(v)-G(v_m(\ell)))}} - \frac{1}{\sqrt{-2\lambda\, v + v^2}} \right) dv \\
&\qquad + \int_0^{v_m(\ell)} \frac{dv}{\sqrt{-2\lambda\, v + v^2}}.
\end{aligned} \tag{3.72}$$

The second integral may be calculated analytically:

$$\int_0^{v_m(\ell)} \frac{dv}{\sqrt{-2\lambda\, v + v^2}} \;=\; -\log(-\lambda) + \log\left(v_m(\ell) - \lambda + \sqrt{v_m(\ell)^2 - 2\lambda\, v_m(\ell)}\right),$$

so that

$$\int_0^{v_m(\ell)} \frac{dv}{\sqrt{-2\lambda\, v + v^2}} + \log(-\lambda) \;\overset{(3.71)}{=}\; \log 2 + O(1-v_m(\ell)). \tag{3.73}$$

Combining (3.71), (3.72), and (3.73), we conclude

$$\begin{aligned}
\frac{\sqrt{G''(v_m(\ell))}\,\ell}{2} + \log(1-v_m(\ell)) \;=\; &\int_0^1 \left(\frac{\sqrt{G''(1)}}{\sqrt{2G(v)}} - \frac{1}{v} \right) dv + \log 2 \\
&+ O(1-v_m(\ell)).
\end{aligned}$$

We deduce that to leading order,

$$1 - v_m(\ell) \;=\; 2\exp(-\sqrt{G''(1)}\ell/2)\exp(c),$$

with

$$c := \int_0^1 \left(\frac{\sqrt{G''(1)}}{\sqrt{2G(v)}} - \frac{1}{v} \right) dv.$$

Insertion into (3.70) gives

$$E'(\ell) \;=\; 2G''(1)\exp(2c)\exp(-\sqrt{G''(1)}\,\ell),$$

which verifies (3.66) with $C_* = 2G''(1)\exp(2c)$. □

Proof of Lemma 3.2. First consider (3.2). For $\ell_v, \ell_w \gg 1$ and

$$|x_v - x_w| + |y_v - y_w| \ll \min\{\ell_v, \ell_w\},$$

the estimate is a perturbation of the calculation for a *single* transition layer on $\mathbb{R}$, so consider this simpler case. For small translations, the mean value formula gives:

$$\int_{\mathbb{R}} (v(x+h) - v(x))^2\,dx \;\sim\; h^2 ||v_x||^2_{L^2(\mathbb{R})} \;\sim\; h^2. \tag{3.74}$$

On the other hand, for translations of at least order one,

$$h \;\sim\; \int_{\mathbb{R}} |v(x+h) - v(x)|\,dx \;\sim\; \int_{\mathbb{R}} (v(x+h) - v(x))^2\,dx. \tag{3.75}$$

The combination of (3.74) and (3.75) implies on the one hand:

$$|x_v - x_w| \lesssim \left(\int_{\mathbb{R}} (v-w)^2\,dx \right)^{1/2} + \int_{\mathbb{R}} (v-w)^2\,dx,$$

and on the other hand

$$\left(\int_{\mathbb{R}} (v-w)^2\,dx \right)^{1/2} \;\lesssim\; |x_v - x_w|.$$

□

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Bestellungen nimmt entgegen:

Institut für Angewandte Mathematik
der Universität Bonn
Sonderforschungsbereich 611
Wegelerstr. 6
D - 53115 Bonn

Telefon: 0228/73 3411
Telefax: 0228/73 7864
E-mail: anke@iam.uni-bonn.de Homepage: http://www.iam.uni-bonn.de/sfb611/

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