Domain Branching in Uniaxial Ferromagnets – Asymptotic Behavior of the Energy

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Abstract

In this article we analyze the ground state energy of a ferromagnetic bulk sample with strong uniaxial anisotropy in a regime featuring domain branching. We derive the Γ-convergence of the micromagnetic energy towards a sharp interface energy. Then we use this convergence to rigorously justify the notion of a minimal energy per cross section area. Compared to known results, the scaling bounds for the minimal energy are improved to an asymptotic equality.

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1 Introduction

The scaling of the micromagnetic energy in various material regimes has been intensely studied, particularly in connection to the explanation of branching phenomena. In this article we are interested in a bulk regime for uniaxial ferromagnets. Our work thus continues the push for a good mathematical understanding started with [CK98] and [CKO99]. A concise overview of the physical observations, the heuristic explanation for the branching behaviour and the energy scaling as well as a short elementary rigorous proof of the lower bound for the scaling can be found in [DKMO05, Chapter 6.8], but let us give a brief description of the nature of the physical structures:

The energy appears to favor the formation of two phases of almost uniform magnetization called *domains*, see Figure 1. These are separated by fairly sharp *walls*, almost lower-dimensional transition regions of a certain *wall width*. In the regime that we are going to study, the domains themselves exhibit a pattern featuring a typical *domain width* when we take a slice parallel to the x_1x_2-plane. This domain width is a function of the third coordinate and decreases as the translated planes approach the boundary of the sample and the domains refine by branching. Of particular interest are the limiting domain widths, the *bulk domain width* in the interior and the *surface domain width* at the boundary.

Our goal reaches beyond upper and lower energy bounds which just match in *scaling* in the two non-dimensional parameters Q, the *quality factor* and t/d, the quotient of the thickness of the sample by the *exchange length*. We show that the appropriately normalized minimal energy per area in (x_1, x_2) converges to a finite universal limit in the parameter regime $t \gg Q^{1/2}d$. This analysis essentially consists of two parts:

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- The first part establishes a Γ-convergence result on a domain whose lateral size l is large but fixed in terms of the (expected) intrinsic lengthscale of the microstructure, the domain width in the bulk. It is based on an anisotropic rescaling of variables (x_1 and x_2 are rescaled by the domain width, x_3 is rescaled by the geometry t). Interestingly, the Γ-limit turns out to be the 3-d generalization of the original functional proposed by Kohn & Müller for twin-branching [KM92, KM94]. In the rescaled variables, it is given by

$$E_{KM}(m_3) = \int |\nabla' m_3|\, dx + \int ||\nabla'|^{-1}\partial_3 m_3|^2 dx \quad \text{where } m_3 \in \{-1, 1\}. \tag{1}$$

  The subtle part is the construction of a "recovery sequence" in the full parameter regime. It requires a version of the Modica-Mortola construction that is quantitative in the parameter $\frac{\text{wall width}}{\text{domain width}} \ll 1$, since only this quantification allows us to "paste" this construction into a domain construction which relies on the independently small parameter $\frac{\text{domain width}}{\text{sample thickness}} \ll 1$.

- The second part shows that a notion of "minimal energy per area in the (x_1, x_2)-plane" is well-defined in the sense that $l^{-2} \min E$, where the minimum is taken over m's which are l-periodic in (x_1, x_2), converges to a finite constant if the artificial "system size" l tends to infinity with respect to the domain width, an intrinsic lengthscale of the microstructure. This means that we establish an "extensive" behavior reminiscent of the hydrodynamic limits of Ising-type models.

We consider the micromagnetic model for a sample $\Omega \subset \mathbb{R}^3$ with magnetization $m : \mathbb{R}^3 \to \mathbb{R}^3$ satisfying

$$|m|^2 = \begin{cases} 1 & \text{in } \Omega, \\ 0 & \text{elsewhere,} \end{cases}$$

and energy with exchange, anisotropy and stray field contribution

$$E(m) = d^2 \int_\Omega |\nabla m|^2 dx + Q \int_\Omega |m'|^2 dx + \int_{\mathbb{R}^3} |h|^2 dx.$$

Here $m' = \binom{m_1}{m_2}$ and the stray field h is given by $\nabla \cdot (h + m) = 0$ and $\nabla \times h = 0$, both understood in the sense of distributions on $\mathbb{R}^3$. These equations stem from Maxwell's equations in the case of magnetostatics, for notes on the derivation, see e.g. [DKMO05]. We remark that h can be computed as $h = -\nabla u$ where

$$\begin{aligned} \Delta u &= \nabla \cdot m && \text{in } \Omega, \\ \left[\tfrac{\partial u}{\partial \nu}\right] &= m \cdot \nu && \text{on } \partial\Omega, \\ \Delta u &= 0 && \text{outside } \Omega. \end{aligned}$$

Let us briefly recall the dimensions of the quantities involved. While m, h, and Q are dimensionless, x and d have units of length. Thus the energy E has units of (length)^3.

We analyze the magnetic ground state, i.e. the minimizer of E among all $m : \mathbb{R}^3 \to \mathbb{R}^3$ such that m has unit length inside Ω and vanishes elsewhere.

Micromagnetics and related phenomena have been extensively studied. In [DKMO05], experimental observations are compared to the state of mathematical analysis for a broad range of physical regimes. The paramount resources for the specific regime under consideration are [CK98] and [CKO99] mentioned above.

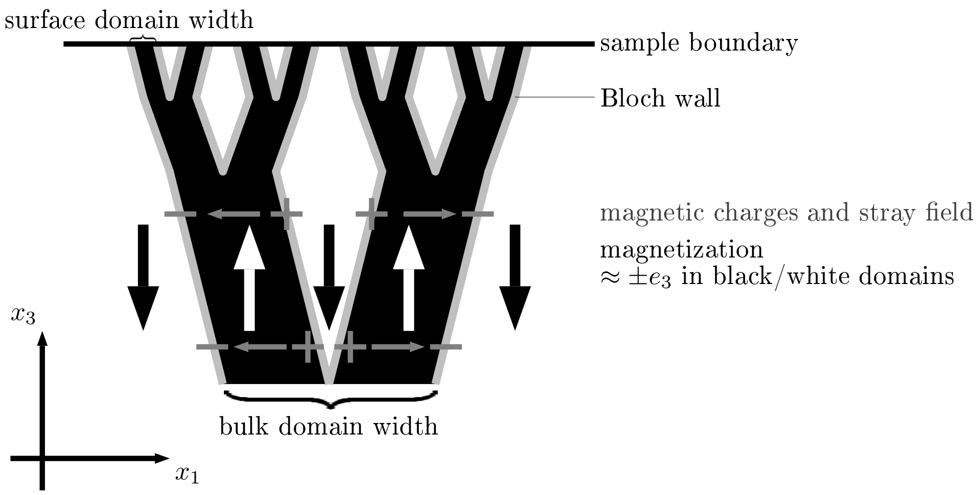

Figure 1: Microstructure in more than 1000 words

2 Model and scaling behavior of the energy

We approach the model with the desire to predict qualitative properties of ground states such as the formation of domains and the structural refinement of domains. We also want to quantitatively predict the width of domains in the bulk. However, our analysis does not contribute to justifying the (observed and heuristically explained) domain width at the surface.

As an idealized sample geometry we take $\Omega = \mathbb{R}^2 \times (-t, t)$. In order to deal with this unbounded sample we need to replace the absolute energy by the energy per area of cross-section. To overcome this conceptual problem, we introduce some artificial periodicity l with respect to the first two coordinates and then consider the limit $l \uparrow \infty$.

Hence we consider the energy per cross-section area

$$e_{Q,d,t,l}(m) := \frac{1}{4l^2}\left(d^2 \int_{(-l,l)^2\times(-t,t)} |\nabla m|^2 dx + Q \int_{(-l,l)^2\times(-t,t)} |m'|^2 dx + \int_{(-l,l)^2\times\mathbb{R}} |h|^2 dx \right)$$

for $(-l,l)^2$-periodic (in x') vector fields $m, h : \mathbb{R}^3 \to \mathbb{R}^3$ satisfying

$$|m|^2 = \begin{cases} 1 & \text{for } x_3 \in (-t,t), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\nabla \cdot (m + h) = 0 \qquad \text{and} \qquad \nabla \times h = 0$$

distributionally in $\mathbb{R}^3$.

We wish to investigate the behavior of the energy functional in the regime of

$$Q \gg 1 \qquad \text{(large anisotropy)}$$

and

$$t \gg dQ^{1/2} \qquad \text{(bulk sample).}$$

We denote the minimal energy (per area) among such functions by $e(Q, d, t, l)$.

In [Hub67], see [HS00, Chapter 3.7.1] for a recent treatment, the type (called Bloch walls) and width of the domain walls, the scaling of domain widths and energy is heuristically determined to be

$$\begin{aligned} \frac{\text{Bloch wall width}}{d} &\sim Q^{-1/2}, \\ \frac{\text{surface domain width}}{d} &\sim Q^{1/2}, \\ \frac{\text{bulk domain width}}{d} &\sim (Q^{1/2})^{1/3} \left(\frac{t}{d}\right)^{2/3}, \text{ and} \\ e(Q, d, t, l) &\sim (dQ^{1/2})^{2/3} t^{1/3}. \end{aligned}$$

Note that the scaling of the domain widths implies surface domain width $\ll$ bulk domain width.

The heuristics for bulk domain width and energy scaling are echoed by the rigorous results of [CKO99].

3 Γ-limit of the energy and a more precise scaling result

In this section we present our two main results. We start with the energy scaling and prove the following theorem.

Theorem 1. *In the regime of bulk sample and strong anisotropy the energy per surface area $e(Q, d, t, l)$ is asymptotically proportional to $(dQ^{1/2})^{2/3} t^{1/3}$. More precisely, the limit*

$$\lim_{Q, \frac{t}{dQ^{1/2}}, \frac{l}{(dQ^{1/2})^{1/3} t^{2/3}} \uparrow \infty} \frac{e(Q, d, t, l)}{(dQ^{1/2})^{2/3} t^{1/3}} \in (0, \infty)$$

exists.

In other words, the theorem states that there is a universal constant $e^* \in (0, \infty)$ such that for any sequence $\{(d^\nu, Q^\nu, t^\nu)\}_{\nu \in \mathbb{N}} \subset \mathbb{R}^3_+$ satisfying

$$Q^\nu \to \infty, \qquad d^\nu Q^\nu / t^\nu \to 0, \text{ and } (d^\nu (Q^\nu)^{1/2})^{1/3} (t^\nu)^{2/3} / l \to 0$$

the energy per cross section area behaves as

$$\frac{e(Q^\nu, d^\nu, t^\nu, l)}{(d^\nu (Q^\nu)^{1/2})^{2/3} (t^\nu)^{1/3}} \to e^*.$$

Proof. We make two preparatory steps, reformulation and rescaling, before proceeding to the core of the argument.

Localizing the stray field. First, we reformulate the magnetostatic energy to include h in the minimization in order to make the problem more local. Observe that the L^2-norm of $(-l, l)^2$-periodic $h : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$\nabla \cdot (h + m) = 0 \text{ and } \qquad \nabla \times h = 0$$

distributionally in $\mathbb{R}^3$ can be rewritten, see the appendix for a bit more context, in terms of the minimization problem

$$\int_{(-l,l)^2\times\mathbb{R}} |h|^2 dx = \min\Big\{ \int_{(-l,l)^2\times\mathbb{R}} |\tilde h|^2 dx \Big| \tilde h : \mathbb{R}^3 \to \mathbb{R}^3 \text{ is } (-l,l)^2\text{-periodic in } x', \\ \nabla\cdot(\tilde h + m) = 0 \text{ distributionally in } \mathbb{R}^3 \Big\}.$$

Hence, setting

$$e_{Q,d,t,l}(m,h) := \frac{1}{4l^2}\Big(d^2 \int_\Omega |\nabla m|^2 dx + Q\int_\Omega |m'|^2 dx + \int_{\mathbb{R}^3} |h|^2 dx\Big)$$

we have

$$e(Q,d,t,l) = \min\Big\{ e_{Q,d,t,l}(m,h) \Big| m,h : \mathbb{R}^3 \to \mathbb{R}^3 \text{ are } (-l,l)^2\text{-periodic in } x', \\ |m|^2 = \begin{cases} 1 & \text{for } x_3 \in (-t,t), \\ 0 & \text{otherwise,} \end{cases} \\ \nabla\cdot(h+m) = 0 \text{ distributionally in } \mathbb{R}^3 \Big\}.$$

Rescaling. As a second step in preparation, we rescale the lengths, fields and energy to more convenient units.

The horizontal lengths are scaled so that the width of bulk domains is of order 1, i.e.

$$x' = (dQ^{1/2})^{1/3} t^{2/3} \hat x' \quad\text{and}\quad l = (dQ^{1/2})^{1/3} t^{2/3} \hat l,$$

the vertical length is normalized so that the sample is on the interval $(-1,1)$ in $\hat x_3$-direction, i.e.

$$x_3 = t\hat x_3.$$

In order to retain the structure of the magnetic field, we have to rescale the horizontal field according to

$$h' = \frac{(dQ^{1/2})^{1/3} t^{2/3}}{t} \hat h' = \Big(\frac{dQ^{1/2}}{t}\Big)^{1/3} \hat h'$$

and keep the vertical component $h_3 = \hat h_3$ to ensure $\nabla\cdot h = \frac{1}{t}\hat\nabla\cdot\hat h$. For consistency we write $m = \hat m$.

With these rescalings in the coordinates, it is convenient to also rescale the energy density as

$$e = (dQ^{1/2})^{2/3} t^{1/3} \hat e$$

in order to non-dimensionalize the weights in the energy.

We write the energy in the new coordinates and quantities

$$e = (dQ^{1/2})^{2/3} t^{1/3} \hat e = \frac{t}{4\hat l^2}\Big(d^2 \int_{(-\hat l,\hat l)\times(-1,1)} \left| \begin{pmatrix} \frac{1}{(dQ^{1/2})^{1/3} t^{2/3}} \hat\nabla' \\ \frac{1}{t}\frac{\partial}{\partial \hat x_3} \end{pmatrix} \hat m \right|^2 d\hat x \\ + Q\int_{(-\hat l,\hat l)\times(-1,1)} |\hat m'|^2 d\hat x + \int_{(-\hat l,\hat l)^2\times\mathbb{R}} \left| \begin{pmatrix} \frac{(dQ^{1/2})^{1/3} t^{2/3}}{t} \hat h' \\ \hat h_3 \end{pmatrix} \right|^2 d\hat x \Big),$$

and thus

$$
\begin{aligned}
\hat{e} &= \frac{1}{4\hat{l}^2}\Bigg(\left(\frac{d}{tQ}\right)^{2/3}\int_{(-\hat{l},\hat{l})^2\times(-1,1)}\left|\begin{pmatrix}\hat{\nabla}' \\ (\frac{dQ^{1/2}}{t})^{1/3}\frac{\partial}{\partial\hat{x}_3}\end{pmatrix}\hat{m}\right|^2 d\hat{x} \\
&\qquad + \left(\frac{tQ}{d}\right)^{2/3}\int_{(-\hat{l},\hat{l})\times(-1,1)}|\hat{m}'|^2 d\hat{x} + \int_{(-\hat{l},\hat{l})^2\times\mathbb{R}}\left|\begin{pmatrix}\hat{h}' \\ (\frac{t}{dQ^{1/2}})^{1/3}\hat{h}_3\end{pmatrix}\right|^2 d\hat{x}\Bigg) \\
&= \frac{1}{4\hat{l}^2}\left(\delta\int_{(-\hat{l},\hat{l})^2\times(-1,1)}\left|\begin{pmatrix}\hat{\nabla}' \\ \varepsilon\frac{\partial}{\partial\hat{x}_3}\end{pmatrix}\hat{m}\right|^2 d\hat{x} + \frac{1}{\delta}\int_{(-\hat{l},\hat{l})\times(-1,1)}|\hat{m}'|^2 d\hat{x} + \int_{(-\hat{l},\hat{l})^2\times\mathbb{R}}\left|\begin{pmatrix}\hat{h}' \\ \frac{1}{\varepsilon}\hat{h}_3\end{pmatrix}\right|^2 d\hat{x}\right) \\
&=: \hat{e}_{\delta,\varepsilon,\hat{l}}(\hat{m},\hat{h})
\end{aligned}
$$

when we set

$$
\delta := \left(\frac{d}{tQ}\right)^{2/3} = \frac{d/Q^{1/2}}{(dQ^{1/2})^{1/3}t^{2/3}} = \frac{\text{Bloch wall width}}{\text{bulk domain width}}
$$

and

$$
\varepsilon := \left(\frac{dQ^{1/2}}{t}\right)^{1/3} = \frac{(dQ^{1/2})^{1/3}t^{2/3}}{t} = \frac{\text{bulk domain width}}{\text{sample thickness}}.
$$

The rescaling of h' now reads $h' = \varepsilon\hat{h}'$ and the constraints turn into

$$
|\hat{m}|^2 = \begin{cases} 1 & \text{for } \hat{x}_3 \in (-1,1), \\ 0 & \text{otherwise} \end{cases}
$$

and

$$
\hat{\nabla}' \cdot (\hat{h} + \frac{1}{\varepsilon}\hat{m}') + \frac{\partial}{\partial\hat{x}_3}(\hat{h}_3 + \hat{m}_3) = 0.
$$

Finally, observe that we can now conveniently characterize the parameter regime of interest because $Q \gg 1$, $dQ^{1/2} \ll t$, and $(dQ^{1/2})^{1/3}t^{2/3} \ll l$ are equivalent to $\delta \ll \varepsilon^2$, $\varepsilon^2 \ll 1$, and $1 \ll \hat{l}$, respectively. Combining, we are interested in

$$
\delta \ll \varepsilon^2 \ll 1 \ll \hat{l}.
$$

Hence, defining

$$
\begin{aligned}
\hat{e}(\delta,\varepsilon,\hat{l}) := \min\Big\{\hat{e}_{\delta,\varepsilon,\hat{l}}(\hat{m},\hat{h})\Big|\hat{m},\hat{h}:\mathbb{R}^3\to\mathbb{R}^3,\quad (-\hat{l},\hat{l})^2\text{-periodic},\quad |\hat{m}|^2 = \begin{cases} 1 & \text{if } \hat{x}_3\in(-1,1), \\ 0 & \text{otherwise,} \end{cases} \\
\hat{\nabla}'\cdot(\hat{h}' + \frac{1}{\varepsilon}\hat{m}') + \frac{\partial}{\partial\hat{x}_3}(\hat{h}_3+\hat{m}_3) = 0 \text{ distributionally in } \mathbb{R}^3\Big\}
\end{aligned}
$$

we have to show

$$
\lim_{\delta\ll\varepsilon^2\ll1\ll\hat{l}} \hat{e}(\delta,\varepsilon,\hat{l}) \in (0,\infty).
$$

Having adequately reformulated the problem, we proceed using the rescaled quantities exclusively and drop all »ˆ«.

Comparing energies. With the following inequalities we can nail the asymptotic behavior of the energy. We shall consider the minimization problem with periodic and free boundary values and introduce sharp interface versions in the next section, the corresponding minimal energies are denoted by $e^p(\varepsilon,\delta,l)$, $e^f(\varepsilon,\delta,l)$, $e^p(l)$, and $e^f(l)$. Then we proceed to establish the following relations between them:

$$\begin{aligned}
\limsup_{\delta\ll\varepsilon^2\ll 1\ll l} e^p(\delta,\varepsilon,l) &\le \liminf_{1\ll l}\limsup_{\delta\ll\varepsilon^2\ll 1} e^p(\delta,\varepsilon,l) && \text{(Lemma 5)},\\
\limsup_{\delta\ll\varepsilon^2\ll 1} e^p(\delta,\varepsilon,l) &\le e^p(l) && \text{(Theorem 2, part 1)},\\
\limsup_{1\ll l} e^p(l) &\le \liminf_{1\ll l} e^f(l) && \text{(Lemma 7)},\\
e^f(l) &\le \liminf_{\delta\ll\varepsilon^2\ll 1} e^f(\delta,\varepsilon,l) && \text{(Theorem 2, part 2)},\\
\limsup_{1\ll l}\liminf_{\delta\ll\varepsilon^2\ll 1} e^f(\delta,\varepsilon,l) &\le \liminf_{\delta\ll\varepsilon^2\ll 1\ll l} e^f(\delta,\varepsilon,l) && \text{(Lemma 4)}.
\end{aligned}$$

These combined with the trivial

$$e^f(\delta,\varepsilon,l)\le e^p(\delta,\varepsilon,l) \qquad \text{and} \qquad e^f(l)\le e^p(l)$$

imply that the limits under consideration exist and coincide for all energies. Then

$$\liminf e^p(l) > 0 \qquad \text{(Lemma 3) and} \tag{2}$$

$$\limsup e^f(l) < \infty \qquad \text{(Lemma 6)} \tag{3}$$

show that the limit indeed is a finite positive number. Thus the theorem is reduced to above inequalities to be proven in the next sections. □

Two of the above inequalities are provided by our second main theorem, the Γ-limit of the energy functional.

Theorem 2. *For fixed length l, the reduced energy is an upper and lower Γ-limit of the full energy for*

$$\delta/\varepsilon^2 \to 0 \qquad \text{and} \qquad \varepsilon^2\to 0. \tag{4}$$

More precisely

1. *The energy of any pair (m_3, h') admissible in $e^p(l)$ can in the regime (4) be approximated by that of pairs $(m^{(\varepsilon,\delta)}, h^{(\varepsilon,\delta)})$ admissible for $e^p(\varepsilon,\delta,l)$ such that*
$$\lim_{\delta\ll\varepsilon^2\ll 1} e_{\delta,\varepsilon,l}(m^{(\varepsilon,\delta)}, h^{(\varepsilon,\delta)}) \le e_l(m_3,h').$$
2. *If $\delta^{(\nu)},\varepsilon^{(\nu)}$ converge as in (4) and $(m^{(\nu)},h^{(\nu)})$ is admissible for $e^f(\delta^{(\nu)},\varepsilon^{(\nu)},l)$ with*
$$m_3^{(\nu)} \xrightarrow{w*} m_3 \text{ in } L^\infty((-l,l)^2\times\mathbb{R}) \qquad \text{and} \qquad h^{(\nu)\prime} \xrightarrow{w} h' \text{ in } L^2((-l,l)^2\times\mathbb{R})$$
then (m_3,h') is admissible for $e^f(l)$ and
$$e_l(m_3,h') \le \liminf_{\nu\uparrow\infty} e_{\delta^{(\nu)},\varepsilon^{(\nu)},l}(m^{(\nu)},h^{(\nu)}).$$

Note that the theorem does not provide a Γ-limit result because the lower bound and the approximation are done in regimes with different boundary conditions and we do not actually verify the approximation property of our prospective recovery sequence. This could be fixed, but as our main interest is Theorem 1, we omit stating and proving a theorem concerning a proper Γ-limit.

4 Energy functionals and configuration spaces

We first discuss the various energy functionals of concern.

For a configuration (m,h) consider the energy

$$e_{\delta,\varepsilon,l}(m,h)=\frac{1}{4l^2}\left(\delta\int_{(-l,l)^2\times(-1,1)}\left|\left(\begin{smallmatrix}\nabla'\\ \varepsilon\partial_3\end{smallmatrix}\right)m\right|^2dx+\frac{1}{\delta}\int_{(-l,l)^2\times(-1,1)}|m'|^2\,dx+\int_{(-l,l)^2\times\mathbb{R}}\left|\left(\begin{smallmatrix}h'\\ \frac{1}{\varepsilon}h_3\end{smallmatrix}\right)\right|^2dx\right),$$

and its minimum on the periodic functions

$$e^p(\delta,\varepsilon,l)=\min\Big\{e_{\delta,\varepsilon,l}(m,h)\,\Big|\,m,h:\mathbb{R}^3\to\mathbb{R}^3,\quad(-l,l)^2\text{-periodic},\quad|m|^2=\begin{cases}1&\text{if }x_3\in(-1,1),\\0&\text{otherwise},\end{cases}$$
$$\nabla'\cdot(h'+\frac{1}{\varepsilon}m')+\partial_3(h_3+m_3)=0\Big\}.$$

Here and in the following the differential equations as in the last condition are understood in the sense of distributions.

We also introduce the renormalized energy where m' and h_3 have vanished, the exchange and anisotropy terms have been replaced by a BV-norm, and $\partial_3 m_3$ has ceased to play a role as

$$e_l(m_3,h')=\frac{1}{4l^2}\left(2\int_{[-l,l)^2\times(-1,1)}|\nabla'm_3|\,dx+\int_{(-l,l)^2\times\mathbb{R}}|h'|^2\,dx\right)$$

and the seek the corresponding minimization amongst periodic m_3,h', i.e.

$$e^p(l)=\min\Big\{e_l(m_3,h')\Big|\,m_3:\mathbb{R}^3\to\mathbb{R},h':\mathbb{R}^3\to\mathbb{R}^2,\quad(-l,l)^2\text{-periodic},\quad m_3^2=\begin{cases}1&\text{if }x_3\in(-1,1),\\0&\text{otherwise},\end{cases}$$
$$\nabla'\cdot h'+\partial_3m_3=0\Big\}.$$

The first term in the energy is understood in the sense of BV-functions periodic in x', i.e.

$$\begin{aligned}&\int_{[-l,l)^2\times(-1,1)}|\nabla'm_3|dx\\&\quad=\sup\Big\{\int_{(-l,l)^2\times(-1,1)}m_3\,\nabla'\cdot\xi'dx\,\Big|\,\xi'\in C^\infty(\mathbb{R}^2\times\mathbb{R}),\ (-l,l)^2\text{-periodic in }x',\\&\qquad\qquad|\xi'|\le1\text{ in }(-l,l)^2\times(-1,1)\Big\}.\end{aligned}$$

Although the half-open fundamental cell isn't used on the right hand side, we use it to be consistent with the interpretation the integral as the measure $|\nabla'm_3|$ of the domain of integration.

Note that as $m_3\in\{\pm1\}$ *a.e.* in $(-l,l)^2\times(-1,1)$, we have an interpretation of the reduced BV-gradient a "slicewise measure" of the interface that corresponds to the usual geometric interpretation of the gradient of a characteristic function of a set as perimeter. To be precise,

$$\int_{[-l,l)^2\times(-1,1)}|\nabla'm_3|dx=2\int_{-1}^{1}\mathscr{H}^1(\partial\{m_3(\,\cdot\,,x_3)=1\})dx_3.$$

In analogy of the analysis of infinite Gibbs states, we broaden the class of admissible functions by not imposing periodicity and consider minima with free boundary conditions, namely

$$e^f(\delta,\varepsilon,l) = \min\Big\{ e_{\delta,\varepsilon,l}(m,h) \,\Big|\, m,h : (-l,l)^2\times\mathbb{R}\to\mathbb{R}^3, \quad |m|^2 = \begin{cases} 1 & \text{if } x_3\in(-1,1), \\ 0 & \text{otherwise,}\end{cases}$$
$$\nabla'\cdot(h' + \frac{1}{\varepsilon}m') + \partial_3(h_3+m_3) = 0 \Big\},$$

and

$$e^f(l) = \min\Big\{ e_l(m_3,h') \,\Big|\, m_3 : (-l,l)^2\times\mathbb{R}\to\mathbb{R}, h' : (-l,l)^2\times\mathbb{R}\to\mathbb{R}^2, \quad m_3^2 = \begin{cases} 1 & \text{if } x_3\in(-1,1), \\ 0 & \text{otherwise,}\end{cases}$$
$$\nabla'\cdot h' + \partial_3 m_3 = 0 \Big\}.$$

As in the proof of Theorem 1, the role of h or h' is to allow a local formulation of the stray field energy. Equivalently, we could write the corresponding terms as negative norms, characterized either by a Fourier-multiplier or via solving an auxiliary problem, and the energies as only depending on m or m_3. Inspired by these interpretations of h and h' we introduce the formal notation

$$\int_{(-l,l)^2\times\mathbb{R}} \big||\nabla'|^{-1}\partial_3 m_3\big|^2\,dx = \min\Big\{ \int_{(-l,l)^2\times\mathbb{R}} |h'|^2 dx \Big| h' : \mathbb{R}^2\times\mathbb{R}\to\mathbb{R}^2 \text{ is } (-l,l)^2\text{-periodic in } x',$$
$$\nabla'\cdot h' + \partial_3 m_3 = 0 \text{ distributionally in } \mathbb{R}^2\times\mathbb{R}\Big\},$$

and

$$\int_{(-l,l)^2\times\mathbb{R}} \Big|(|\nabla'|^2+\varepsilon^2|\partial_3|^2)^{-1/2}\partial_3 m_3\Big|^2 dx$$
$$= \min\Big\{ \int_{(-l,l)^2\times\mathbb{R}} \Big|\begin{pmatrix} h' \\ \frac{1}{\varepsilon}h_3\end{pmatrix}\Big|^2 dx \,\Big|\, h : \mathbb{R}^2\times\mathbb{R}\to\mathbb{R}^3 \text{ is } (-l,l)^2\text{-periodic in } x',$$
$$\nabla\cdot h + \partial_3 m_3 = 0 \text{ distributionally in } \mathbb{R}^2\times\mathbb{R}\Big\}. \qquad (5)$$

5 Proof of Theorem 2

This section is devoted to the proof of the almost-Γ-limit Theorem 2. We start with the more straightforward lower bound, which is demonstrated by a compensated compactness argument that takes into account the anisotropy. Then we address the upper bound which requires a more involved proof.

We thus begin with the proof of Theorem 2, part 2, used to show

$$\liminf_{\delta\ll\varepsilon^2\ll 1} e^f(\delta,\varepsilon,l) \geq e^f(l).$$

in the proof of Theorem 1.

Proof of Theorem 2, part 2. We fix l and recall that we aim to show the following: Given any sequences $\{\delta^{(\nu)}, \varepsilon^{(\nu)}\} \subset (0,\infty)$ and $(m^{(\nu)}, h^{(\nu)})$ admissible for $e^f(\delta^{(\nu)}, \varepsilon^{(\nu)}, l)$ with

$$\begin{aligned} &\delta^{(\nu)} \to 0, \quad \varepsilon^{(\nu)} \to 0, \quad \frac{\delta^{(\nu)}}{(\varepsilon^{(\nu)})^2} \to 0, \\ &m_3^{(\nu)} \overset{w*}{\rightharpoonup} m_3 \text{ in } L^\infty((-l,l)^2 \times \mathbb{R}), \end{aligned} \tag{6}$$

and

$$h^{(\nu)\prime} \overset{w}{\rightharpoonup} h' \text{ in } L^2((-l,l)^2 \times \mathbb{R})$$

as $\nu \to \infty$,

$$\begin{aligned} &(m_3, h') \text{ is admissible for } e^f(l) \text{ and} \\ &e_l(m_3, h') \le \liminf_{\nu\uparrow\infty} e_{\delta^{(\nu)},\varepsilon^{(\nu)},l}(m^{(\nu)}, h^{(\nu)}). \end{aligned}$$

Without compromising generality, we may assume that the energy $e_{\delta^{(\nu)},\varepsilon^{(\nu)},l}(m^{(\nu)}, h^{(\nu)})$ remains bounded.

In particular, this implies that

$$\frac{1}{\delta^{(\nu)}} \int_{(-l,l)^2\times(-1,1)} |m^{(\nu)\prime}|^2 dx \qquad \text{and} \qquad \int_{(-l,l)^2\times\mathbb{R}} \left|\frac{1}{\varepsilon^{(\nu)}} h_3^{(\nu)}\right|^2 dx$$

are bounded and with $\frac{\delta^{(\nu)}}{(\varepsilon^{(\nu)})^2} \to 0$ we see

$$\int_{(-l,l)^2\times(-1,1)} \left|\frac{1}{\varepsilon^{(\nu)}} m^{(\nu)\prime}\right|^2 dx \to 0 \qquad \text{as well as} \qquad \int_{(-l,l)^2\times\mathbb{R}} |h_3^{(\nu)}|^2 dx \to 0. \tag{7}$$

Thus, the differential equation

$$\nabla' \cdot \big(h^{(\nu)\prime} + \frac{1}{\varepsilon^{(\nu)}} m^{(\nu)\prime}\big) + \partial_3(h_3^{(\nu)} + m_3^{(\nu)}) = 0 \qquad \text{distributionally}$$

yields

$$\nabla' \cdot h' + \partial_3 m_3 = 0 \qquad \text{distributionally}$$

in the limit as desired.

We first bound $4l^2 e_{\delta^{(\nu)},\varepsilon^{(\nu)},l}(m^{(\nu)}, h^{(\nu)})$ from below. In the sample the magnetization satisfies the pointwise estimate

$$\begin{aligned} \delta^{(\nu)} \left|\begin{pmatrix} \nabla' \\ \varepsilon^{(\nu)}\partial_3 \end{pmatrix} m^{(\nu)}\right|^2 + \frac{1}{\delta^{(\nu)}} \left|m^{(\nu)\prime}\right|^2 &\ge \delta^{(\nu)} \left|\nabla' m^{(\nu)}\right|^2 + \frac{1}{\delta^{(\nu)}} \left|m^{(\nu)\prime}\right|^2 \\ &\ge \delta^{(\nu)} \left|\nabla' |m^{(\nu)}|\right|^2 + \frac{1}{\delta^{(\nu)}} \left|m^{(\nu)\prime}\right|^2 \\ &= \delta^{(\nu)} \frac{(m_3^{(\nu)})^2}{1-(m_3^{(\nu)})^2} \left|\nabla' m_3^{(\nu)}\right|^2 + \frac{1}{\delta^{(\nu)}} \Big(1-(m_3^{(\nu)})^2\Big) \\ &\ge 2\left|\nabla' m_3^{(\nu)}\right|. \end{aligned}$$

Dropping the third component in h we see that the renormalized energy provides a lower bound

$$\begin{aligned}
&4l^2 e_{\delta^{(\nu)},\varepsilon^{(\nu)},l}(m^{(\nu)},h^{(\nu)})\\
&= \delta^{(\nu)} \int_{(-l,l)^2\times(-1,1)} \left| \begin{pmatrix} \nabla' \\ \varepsilon^{(\nu)}\partial_3 \end{pmatrix} m^{(\nu)} \right|^2 dx + \frac{1}{\delta^{(\nu)}} \int_{(-l,l)^2\times(-1,1)} \left| m^{(\nu)\prime} \right|^2 dx \\
&\quad + \int_{(-l,l)^2\times\mathbb{R}} \left| \begin{pmatrix} h^{(\nu)\prime} \\ \frac{1}{\varepsilon} h_3^{(\nu)} \end{pmatrix} \right|^2 dx \\
&\geq 2 \int_{(-l,l)^2\times(-1,1)} \left| \nabla' m_3^{(\nu)} \right| dx + \int_{(-l,l)^2\times\mathbb{R}} \left| h^{(\nu)\prime} \right|^2 dx.
\end{aligned}$$

By the lower semicontinuity of convex functionals under weak convergence this implies

$$\begin{aligned}
\liminf_{\nu\uparrow\infty} 4l^2 e_{\delta^{(\nu)},\varepsilon^{(\nu)},l}(m^{(\nu)},h^{(\nu)}) &\geq 2\int_{(-l,l)^2\times(-1,1)} |\nabla' m_3|\, dx + \int_{(-l,l)^2\times\mathbb{R}} |h'|^2\, dx \\
&= 4l^2 e_l(m_3,h').
\end{aligned}$$

It remains to show

$$m_3^2 = 1 \text{ a.e. in } (-l,l)^2\times(-1,1).$$

Since

$$|m^{(\nu)}|^2 = 1 \text{ if } x_3 \in (-1,1),$$

this follows upon establishing

$$\begin{aligned}
m^{(\nu)\prime} &\longrightarrow 0 \text{ in } L^2((-l,l)^2\times(-1,1)),\\
m_3^{(\nu)} &\longrightarrow m_3 \text{ pointwise a.e. in } (-l,l)^2\times(-1,1).
\end{aligned}$$

In other words, we need compactness (only) for the nonconvex part. The first convergence follows readily from (7). The second is a consequence of (6) if we can show compactness of $\{m_3^{(\nu)}\}_{\nu\uparrow\infty}$ in form of

$$\int_{(-\tilde{l},\tilde{l})^2\times\mathbb{R}} |m_3^{(\nu)}(x'+h,x_3) - m_3^{(\nu)}(x',x_3)|dx \leq C|h| \text{ for } |h|\leq l-\tilde{l}, \tag{8}$$

$$\left(\int_{(-\tilde{l},\tilde{l})^2\times\mathbb{R}} |m_3^{(\nu)}(x',x_3+\tau) - m_3^{(\nu)}(x',x_3)|^2 dx\right)^{1/2} \leq C|\tau|^{1/3} + o(1) \quad \text{for } |\tau| \leq (l-\tilde{l})^{3/2}, \tag{9}$$

for $\tilde{l}\leq l$, that is, the modulus of continuity w.r.t. x' and x_3 must decrease uniformly in L^1 and L^2, respectively, as $\nu\uparrow\infty$. Then, by the usual L^p-compactness criterion of M. Riesz (see e.g. [Ada75, Theorem 2.21]), boundedness and vanishing of the norm for thin boundary layers is evident because $|m_3^{(\nu)}|\leq 1$ and l is fixed, the sequence is precompact in L^1. As we already know the limit of converging subsequences, $m_3^{(\nu)}\to m_3$ in L^1 and thus pointwise a.e., so we are done upon establishing (8) and (9).

Inequality (8) is an immediate consequence of our bound on $e_l(m_3^{(\nu)}, h^{(\nu)\prime})$ because for $|h| \leq l - \tilde{l}$

$$\begin{aligned}
&\int_{(-\tilde{l},\tilde{l})^2\times\mathbb{R}} |m_3^{(\nu)}(x'+h,x_3) - m_3^{(\nu)}(x',x_3)|dx \\
&= \int_{(-\tilde{l},\tilde{l})^2\times(-1,1)} |m_3^{(\nu)}(x'+h,x_3) - m_3^{(\nu)}(x',x_3)|dx \\
&\leq |h| \int_{(-l,l)^2\times(-1,1)} |\nabla' m_3^{(\nu)}|dx \leq C|h|.
\end{aligned}$$

For the second inequality, (9), we use a compensated compactness argument in the sense that we can combine the uniform modulus of continuity in x' in a strong norm (cf. (8)) with the uniform modulus of continuity in x_3 in a weak (negative) norm provided by the field energy to obtain a uniform modulus of continuity in x_3 also in a strong norm (cf. (9)). We fix a smooth convolution kernel $\varphi : \mathbb{R}^2 \to R$ such that $\varphi \geq 0$, $\varphi(x') = 0$ for $|x'| \geq 1$, $\varphi(-x') = \varphi(x')$, and $\int_{\mathbb{R}^2} \varphi\, dx' = 1$ and denote by a subscript $_\alpha$ the convolution with $\frac{1}{\alpha^2}\varphi(\frac{\cdot}{\alpha})$. We observe that the distributional equation

$$\partial_3(h_3^{(\nu)} + m_3^{(\nu)}) = -\nabla' \cdot (h^{(\nu)\prime} + \frac{1}{\varepsilon^{(\nu)}} m^{(\nu)\prime})$$

in $(-l,l)^2 \times \mathbb{R}$ implies for $\alpha \leq l - \tilde{l}$

$$\partial_3(h_3^{(\nu)} + m_3^{(\nu)})_\alpha = -\nabla' \cdot (h^{(\nu)\prime} + \frac{1}{\varepsilon^{(\nu)}} m^{(\nu)\prime})_\alpha$$

in $(-\tilde{l},\tilde{l})^2 \times \mathbb{R}$.

Hence

$$\begin{aligned}
&\left(\int_{(-\tilde{l},\tilde{l})^2\times\mathbb{R}} \left|\partial_3(h_3^{(\nu)} + m_3^{(\nu)})_\alpha\right|^2 dx\right)^{1/2} \\
&= \left(\int_{(-\tilde{l},\tilde{l})^2\times\mathbb{R}} \left|\nabla' \cdot (h^{(\nu)\prime} + \frac{1}{\varepsilon^{(\nu)}} m^{(\nu)\prime})_\alpha\right|^2 dx\right)^{1/2} \\
&\leq \int_{\mathbb{R}^2} \left|\frac{1}{\alpha^3}\nabla'\varphi\left(\frac{y'}{\alpha}\right)\right| dy' \left(\int_{(-l,l)^2\times\mathbb{R}} \left|h^{(\nu)\prime} + \frac{1}{\varepsilon^{(\nu)}} m^{(\nu)\prime}\right|^2 dx\right)^{1/2} \\
&\leq C\frac{1}{\alpha}\left(\left(\int_{(-l,l)^2\times\mathbb{R}} \left|h^{(\nu)\prime}\right|^2 dx\right)^{1/2} + \frac{\delta^{(\nu)}}{(\varepsilon^{(\nu)})^2}\frac{1}{\delta^{(\nu)}}\left(\int_{(-l,l)^2\times\mathbb{R}} \left|m^{(\nu)\prime}\right|^2 dx\right)^{1/2}\right) \\
&\leq C\frac{1}{\alpha} \cdot (C + o(1)\,C) \quad \leq \quad C\frac{1}{\alpha}.
\end{aligned}$$

As a consequence,

$$\left(\int_{(-\tilde{l},\tilde{l})^2\times\mathbb{R}} |(m_3^{(\nu)} + h_3^{(\nu)})_\alpha(x', x_3+\tau) - (m_3^{(\nu)} + h_3^{(\nu)})(x',x_3)|^2 dx\right)^{1/2} = C\frac{|\tau|}{\alpha}.$$

Finally, we observe that

$$\int_{(-l,l)^2\times\mathbb{R}} |h_3^{(\nu)}|^2 dx \leq (\varepsilon^{(\nu)})^2 \int_{(-l,l)^2\times\mathbb{R}} \left|\frac{1}{\varepsilon^{(\nu)}} h_3^{(\nu)}\right|^2 dx = o(1).$$

Combining these two estimates with inequality (8), we obtain for $\alpha \leq l - \tilde{l}$

$$\begin{aligned}
&\left(\int_{(-\tilde{l},\tilde{l})^2 \times \mathbb{R}} |m_3^{(\nu)}(x', x_3 + \tau) - m_3^{(\nu)}(x', x_3)|^2 dx \right)^{1/2} \\
&\leq \left(\int_{(-\tilde{l},\tilde{l})^2 \times \mathbb{R}} |(m_3^{(\nu)} + h_3^{(\nu)})_\alpha(x', x_3 + \tau) - (m_3^{(\nu)} + h_3^{(\nu)})_\alpha(x', x_3)|^2 dx \right)^{1/2} \\
&\quad + 2 \left(\int_{(-\tilde{l},\tilde{l})^2 \times \mathbb{R}} |(h_3^{(\nu)})_\alpha|^2 dx \right)^{1/2} \\
&\quad + 2 \left(\int_{(-\tilde{l},\tilde{l})^2 \times \mathbb{R}} |(m_3^{(\nu)})_\alpha - m_3^{(\nu)}|^2 dx \right)^{1/2} \\
&\leq C \frac{|\tau|}{\alpha} + 2 \left(\int_{(-l,l)^2 \times \mathbb{R}} |h_3^{(\nu)}|^2 dx \right)^{1/2} + 2^{3/2} \left(\int_{(-\tilde{l},\tilde{l})^2 \times \mathbb{R}} |(m_3^{(\nu)})_\alpha - m_3^{(\nu)}| dx \right)^{1/2} \\
&\leq C \frac{|\tau|}{\alpha} + o(1) + 2^{3/2} \left(\sup_{|k'| \leq \alpha} \int_{(-\tilde{l},\tilde{l})^2 \times \mathbb{R}} |m_3^{(\nu)}(x' + k', x_3) - m_3^{(\nu)}(x', x_3)| dx \right)^{1/2} \\
&\leq C \frac{|\tau|}{\alpha} + o(1) + C\alpha^{1/2}.
\end{aligned}$$

With the choice of $\alpha = |\tau|^{2/3}$ this is (9), and the above reasoning yields the desired $m_3^2 = 1$ for a.e. $x \in (-l, l)^2 \times (-1, 1)$.

Thus (m_3, h') is admissible and the proof of our claim is complete. □

We now wish to prove Theorem 2, part 1, needed to obtain

$$\limsup_{\delta \ll \varepsilon^2 \ll 1} e^p(\delta, \varepsilon, l) \leq e^p(l)$$

in the proof of Theorem 1.

The approximation is done in two steps. In the first, given by Lemma 1, we energetically approximate (m_3, h') admissible for $e^p(l)$ by a pair (m_3, h) for which we allow a small third field component but require some additional regularity of m_3 in the third direction. This can be seen as a counterpart to taking the limit of extreme anisotropy. In the second step, Proposition 1, we will "revert" the Modica-Mortola type passage from a diffuse to a sharp interface energy. We will defer the proof of the latter lemma to Section 7.

More precisely we introduce an intermediate energy

$$e_{\varepsilon,l}(m_3, h) = (2l)^{-2} \left(2 \int_{[-l,l)^2 \times (-1,1)} \left| \begin{pmatrix} \nabla' \\ \varepsilon \partial_3 \end{pmatrix} m_3 \right| dx + \int_{(-l,l)^2 \times \mathbb{R}} \left| \begin{pmatrix} h' \\ \frac{1}{\varepsilon} h_3 \end{pmatrix} \right|^2 dx \right)$$

for pairs (m_3, h) satisfying

$$\begin{aligned}
&m_3 : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}, h : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^3 \text{ are } (-l, l)^2\text{-periodic in } x', \\
&m_3^2 = \begin{cases} 1 & \text{if } x_3 \in (-1, 1), \\ 0 & \text{otherwise,} \end{cases} \quad \text{and } \nabla \cdot h + \partial_3 m_3 = 0 \text{ distributionally in } \mathbb{R}^2 \times \mathbb{R}
\end{aligned} \tag{10}$$

for use in the following approximation lemma. As discussed for the other energies, we can also split the minimization in h and m_3, replace the second term by the expression

$$\int_{(-l,l)^2\times\mathbb{R}} \Big|(|\nabla'|^2+\varepsilon^2|\partial_3|^2)^{-1/2}\partial_3 m_3\Big|^2 dx$$
$$= \min\Big\{ \int_{(-l,l)^2\times\mathbb{R}} \Big| \begin{pmatrix} h' \\ \frac{1}{\varepsilon}h_3 \end{pmatrix} \Big|^2 dx \,\Big|\, h:\mathbb{R}^2\times\mathbb{R}\to\mathbb{R}^3 \text{ is } (-l,l)^2\text{-periodic in } x',$$
$$\nabla\cdot h+\partial_3 m_3=0 \text{ distributionally in } \mathbb{R}^2\times\mathbb{R}\Big\}$$

and speak of the energy $e_{l,\varepsilon}(m_3)$ as only depending on m_3.

Lemma 1. *Fix m_3 and the associated h' in the admissible class in the minimization problem in for $e^p(l)$, i.e.*

$$m_3:\mathbb{R}^3\to\mathbb{R},\ h':\mathbb{R}^3\to\mathbb{R}^2, \quad (-l,l)^2\text{-periodic},$$
$$m_3^2=\begin{cases}1 & \text{if } x_3\in(-1,1),\\ 0 & \text{otherwise,}\end{cases} \qquad \nabla'\cdot h'+\partial_3 m_3=0.$$

Then there exists a sequence $\{(m_3^{(\varepsilon)},h^{(\varepsilon)})\}_{\varepsilon\downarrow 0}$ satisfying (10) *such that*

$$\limsup_{\varepsilon\ll 1} e_{\varepsilon,l}(m_3^{(\varepsilon)},h^{(\varepsilon)}) \le e_l(m_3,h'). \tag{11}$$

We postpone the proof of this lemma and first present the path from this approximation to the desired result of Theorem 2, part 1.

Proof of Theorem 2, part 1. Fix an $\alpha\ll 1$. Using Proposition 1 (see Section 7) with the slight generalization of Remark 1 we obtain from $m_3^{(\varepsilon)}$ functions $m_3^{(\varepsilon,\delta)}$ such that

$$m_3^{(\varepsilon,\delta)} \text{ is } (-l,l)^2 \text{ periodic in } x' \text{ and } (m_3^{(\varepsilon,\delta)})^2 \begin{cases}\le 1 & \text{for } x_3\in(-1,1),\\ =0 & \text{otherwise}\end{cases}$$

such that

$$\delta\int_{(-l,l)^2\times(-1,1)} \frac{1}{1-(m_3^{(\varepsilon,\delta)})^2}\Big|\begin{pmatrix}\nabla'\\ \varepsilon\partial_3\end{pmatrix} m_3^{(\varepsilon,\delta)}\Big|^2 dx + \frac{1}{\delta}\int_{(-l,l)^2\times(-1,1)} 1-(m_3^{(\varepsilon,\delta)})^2\,dx$$
$$\le (1+\alpha)2\int_{[-l,l)^2\times(-1,1)} \Big|\begin{pmatrix}\nabla'\\ \varepsilon\partial_3\end{pmatrix} m_3^{(\varepsilon)}\Big|\,dx \tag{12}$$

and

$$\int_{(-l,l)^2\times(-1,1)} (m_3^{(\varepsilon,\delta)}-m_3^{(\varepsilon)})^2 dx \le C(\alpha)\delta\int_{[-l,l)^2\times(-1,1)} \Big|\begin{pmatrix}\nabla'\\ \varepsilon\partial_3\end{pmatrix} m_3^{(\varepsilon)}\Big|\,dx. \tag{13}$$

We then set

$$m_1^{(\varepsilon,\delta)}=\begin{cases}\sqrt{1-(m_3^{(\varepsilon,\delta)})^2} & \text{for } x_3\in(-1,1),\\ 0 & \text{otherwise,}\end{cases}$$

and

$$m_2^{(\varepsilon,\delta)}\equiv 0$$

so that

$$\left|m^{(\varepsilon,\delta)}\right|^2 = \begin{cases} 1 & \text{for } x_3 \in (-1,1), \\ 0 & \text{otherwise.} \end{cases}$$

We also set

$$h^{(\varepsilon,\delta)\prime} = h^{(\varepsilon)\prime} - \frac{1}{\varepsilon} m^{(\varepsilon,\delta)\prime} \qquad \text{and} \qquad h_3^{(\varepsilon,\delta)} = h_3^{(\varepsilon)} + m_3^{(\varepsilon)} - m_3^{(\varepsilon,\delta)},$$

so that (10) turns into

$$\nabla' \cdot (h^{(\varepsilon,\delta)\prime} + \frac{1}{\varepsilon} m^{(\varepsilon,\delta)\prime}) + \partial_3 (h_3^{(\varepsilon,\delta)} + m_3^{(\varepsilon,\delta)}) = 0$$

distributionally in $\mathbb{R}^2 \times \mathbb{R}$ and see that $(m^{(\varepsilon,\delta)}, h^{(\varepsilon,\delta)})$ is admissible for $e^p(\delta,\varepsilon,l)$. We rewrite the first two terms of the energy using $|m|^2 = 1$ and $m_2 = 0$

$$\begin{aligned}
&\delta \int_{(-l,l)^2\times(-1,1)} \left| \left(\begin{smallmatrix} \nabla' \\ \varepsilon\partial_3 \end{smallmatrix}\right) m^{(\varepsilon,\delta)} \right|^2 dx + \frac{1}{\delta} \int_{(-l,l)^2\times(-1,1)} |m^{(\varepsilon,\delta)\prime}|^2 dx + \int_{(-l,l)^2\times\mathbb{R}} \left| \left(\begin{smallmatrix} h^{(\varepsilon,\delta)\prime} \\ \frac{1}{\varepsilon} h_3^{(\varepsilon,\delta)} \end{smallmatrix}\right) \right|^2 dx \\
&= \delta \int_{(-l,l)^2\times(-1,1)} \frac{1}{1-(m_3^{(\varepsilon,\delta)})^2} \left| \left(\begin{smallmatrix} \nabla' \\ \varepsilon\partial_3 \end{smallmatrix}\right) m_3^{(\varepsilon,\delta)} \right|^2 dx + \frac{1}{\delta} \int_{(-l,l)^2\times(-1,1)} 1-(m_3^{(\varepsilon,\delta)})^2 \, dx \\
&\quad + \int_{(-l,l)^2\times\mathbb{R}} \left| \left(\begin{smallmatrix} h^{(\varepsilon,\delta)\prime} \\ \frac{1}{\varepsilon} h_3^{(\varepsilon,\delta)} \end{smallmatrix}\right) \right|^2 dx
\end{aligned}$$

and apply Young's inequality in the form $(a+b)^2 \le (1+\alpha)a^2 + (1+\alpha^{-1})b^2$ to split the field term and obtain

$$\begin{aligned}
&\le \delta \int_{(-l,l)^2\times(-1,1)} \frac{1}{1-(m_3^{(\varepsilon,\delta)})^2} \left| \left(\begin{smallmatrix} \nabla' \\ \varepsilon\partial_3 \end{smallmatrix}\right) m_3^{(\varepsilon,\delta)} \right|^2 dx + \frac{1}{\delta} \int_{(-l,l)^2\times(-1,1)} 1-(m_3^{(\varepsilon,\delta)})^2 \, dx \\
&\quad + (1+\alpha) \int_{(-l,l)^2\times\mathbb{R}} \left| \left(\begin{smallmatrix} h^{(\varepsilon)\prime} \\ \frac{1}{\varepsilon} h_3^{(\varepsilon)} \end{smallmatrix}\right) \right|^2 dx + \frac{C}{\alpha} \int_{(-l,l)^2\times(-1,1)} \left| \left(\begin{smallmatrix} \frac{1}{\varepsilon} m^{(\varepsilon,\delta)\prime} \\ \frac{1}{\varepsilon}(m_3^{(\varepsilon)} - m_3^{(\varepsilon,\delta)}) \end{smallmatrix}\right) \right|^2 dx,
\end{aligned}$$

expanding the last integrand and applying (12) and (13) we estimate

$$\begin{aligned}
&= \delta \int_{(-l,l)^2\times(-1,1)} \frac{1}{1-(m_3^{(\varepsilon,\delta)})^2} \left| \left(\begin{smallmatrix} \nabla' \\ \varepsilon\partial_3 \end{smallmatrix}\right) m_3^{(\varepsilon,\delta)} \right|^2 dx + \frac{1}{\delta} \int_{(-l,l)^2\times(-1,1)} 1-(m_3^{(\varepsilon,\delta)})^2 \, dx \\
&\quad + (1+\alpha) \int_{(-l,l)^2\times\mathbb{R}} \left| \left(\begin{smallmatrix} h^{(\varepsilon)\prime} \\ \frac{1}{\varepsilon} h_3^{(\varepsilon)} \end{smallmatrix}\right) \right|^2 dx \\
&\quad + \frac{C(\alpha)}{\varepsilon^2} \left(\int_{(-l,l)^2\times(-1,1)} 1-(m_3^{(\varepsilon,\delta)})^2 dx + \int_{(-l,l)^2\times(-1,1)} (m_3^{(\varepsilon,\delta)} - m_3^{(\varepsilon)})^2 dx \right) \\
&\overset{(12),(13)}{\le} \left(1+\alpha+C(\alpha)\frac{\delta}{\varepsilon^2}\right) \left(2\int_{[-l,l)^2\times(-1,1)} \left| \left(\begin{smallmatrix} \nabla' \\ \varepsilon\partial_3 \end{smallmatrix}\right) m_3^{(\varepsilon)} \right| dx + \int_{(-l,l)^2\times\mathbb{R}} \left| \left(\begin{smallmatrix} h^{(\varepsilon)\prime} \\ \frac{1}{\varepsilon} h_3^{(\varepsilon)} \end{smallmatrix}\right) \right|^2 dx \right) \\
&= \left(1+\alpha+C(\alpha)\frac{\delta}{\varepsilon^2}\right) 4l^2 e_{l,\varepsilon}(m_3^{(\varepsilon)}, h^{(\varepsilon)}).
\end{aligned}$$

Combining this estimate in the limit $\delta \ll \varepsilon^2 \ll 1$ with (11) from Lemma 1 we conclude

$$\limsup_{\delta\ll\varepsilon^2\ll1} e^p(\delta,\varepsilon,l) \le \limsup_{\varepsilon^2\ll1} (1+\alpha)\, e_{l,\varepsilon}(m_3^{(\varepsilon)}, h^{(\varepsilon)}) \overset{(11)}{\le} (1+\alpha) e_l(m_3,h).$$

As $0 < \alpha \ll 1$ was arbitrary, we obtain the desired conclusion by using a diagonal sequence for the above limit relation and $\alpha \downarrow 0$. $\square$

Proof of Lemma 1. Before we begin with the proof in full technical detail, let us point out the key ideas. In order to approximate m_3 with functions having some regularity in x_3-direction, the first thing that comes to mind is taking a convolution. This, however, does not play well with the requirement that the magnetization only having a third component of unit length. So instead we approximate m_3 by piecewise (in x_3-direction) constant functions and obtain some BV-regularity also in this direction in the following way: For a third component in the BV-norm, we need to control the L^1-norm (with respect to the Hausdorff measure) of the jumps. The field term only yields a bound in the H^{-1}-norm, so we have to resort to interpolation (using Lemma 2) with the x'-perimeter over which we have control. The jump norm is small in L^2 (and thus also L^1) if the weight ε is small compared to the discretization lengthscale τ.

We incur, however, the problem that now the field energy measured as $\int ||\nabla'|^{-1}\partial_3 m_3|^2 dx$ is infinite in the presence of jumps. We thus need to allow an ε-small third component in the field term, i.e. introduce a ∂_3-term the inverted operator (cf. (5)). As the jumps are essentially a surface phenomenon, we want, roughly speaking, to control the $H^{-1/2}$-norm. To that end we need to interpolate again between the H^{-1}-norm on a slowly changing component and the domain perimeter or, more precisely, the L^2-norm on the oscillations of short wave length (this happens on the level of Fourier series in (21)).

We begin with a few preparations in order to be able to define $m_3^{(\varepsilon)}$. Let us denote by $\mathcal{P}_\lambda$ the projection on the Fourier modes n' with $\pi\lambda|\frac{n'}{l}| \geq 1$, i.e.

$$\mathcal{F}'(\mathcal{P}_\lambda\zeta)(n') = \begin{cases} (\mathcal{F}'\zeta)(n') & \text{if } \pi\lambda|\frac{n'}{l}| \geq 1, \\ 0 & \text{otherwise,} \end{cases} \tag{14}$$

where $\mathcal{F}'(\zeta)(n')$ is the Fourier coefficient

$$\mathcal{F}'(\zeta)(n') = \frac{1}{2l}\int_{(-l,l)^2} \exp(-\pi i n' \cdot \frac{x'}{l})\zeta(x')dx'.$$

Hence $\mathcal{P}_\lambda\zeta$ only sees the (horizonal) wavelengths smaller than λ.

The Fourier space representation of negative Sobolev norm appearing in the sharp interface field energy is

$$\int_{(-l,l)^2\times\mathbb{R}} |h'|^2 dx = \int_{(-l,l)^2\times\mathbb{R}} \left||\nabla'|^{-1}\partial_3 m_3\right|^2 dx = \int_{\mathbb{R}} \sum_{n'\in\mathbb{Z}^2} \frac{l^2}{\pi^2|n'|^2}|(\mathcal{F}'\partial_3 m_3)(n')|^2 dx_3,$$

see also the appendix. We need to make more precise the notion that small lengthscale oscillations in the magnetization, for our purposes $\mathcal{P}_\lambda\partial_3 m_3$ with λ small, contribute little to the field energy. Note that an admissible m_3 cannot be constant in x_3-direction. As such we have

$$\int_{(-l,l)^2\times\mathbb{R}} \left||\nabla'|^{-1}\partial_3 m_3\right|^2 dx > 0,$$

and thus

$$\int_{(-l,l)^2\times\mathbb{R}} \left||\nabla'|^{-1}\mathcal{P}_\lambda\partial_3 m_3\right|^2 dx \leq (\omega(\lambda))^2 \int_{(-l,l)^2\times\mathbb{R}} \left||\nabla'|^{-1}\partial_3 m_3\right|^2 dx \tag{15}$$

with some modulus function $\omega > 0$ satisfying $\lim_{\lambda \downarrow 0} \omega(\lambda) = 0$ and depending only on m_3. We rewrite (15) as

$$\frac{1}{\omega(\lambda)} \int_{(-l,l)^2 \times \mathbb{R}} \left| |\nabla'|^{-1} \mathcal{P}_\lambda \partial_3 m_3 \right|^2 dx \leq \omega(\lambda) l^2 e_l(m_3).$$

We wish to find good layers to introduce the discontinuities in our envisioned x_3-piecewise constant approximation. This means that we want to limit the "slice" energy in these layers. Still integrating over all the domain we note that replacing x_3-derivatives by difference quotients does not enlarge the norms involved, thus

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \left(2 \int_{(-l,l)^2} |\nabla' m_3| \, dx' + \int_{(-l,l)^2} \left| |\nabla'|^{-1} \frac{1}{\tau} (m_3(x', x_3 + \tau) - m_3(x', x_3)) \right|^2 dx' \right. \\
&\quad \left. + \frac{1}{\omega(\lambda)} \int_{(-l,l)^2} \left| |\nabla'|^{-1} \frac{1}{\tau} \mathcal{P}_\lambda (m_3(x', x_3 + \tau) - m_3(x', x_3)) \right|^2 dx' \right) dx_3 \\
&\leq (1 + \omega(\lambda)) 4 l^2 e_l(m_3, h').
\end{aligned}$$

We are now able to select a set of slices that is good for the energy on the left hand side of this inequality. For any fixed $N \in \mathbb{N}$ we set $\tau = \frac{1}{N}$. By Fubini's theorem not all slices can be above average and so there exists a $x_3^0 \in (0, \tau)$ depending only on m_3, λ and N such that

$$\begin{aligned}
&\sum_{k \in \mathbb{Z}} \tau \left(2 \int_{(-l,l)^2} \left| \nabla' m_3^{(k)} \right| dx' + \int_{(-l,l)^2} \left| |\nabla'|^{-1} \frac{1}{\tau} [m_3]^{(k)} \right|^2 dx' \right. \\
&\qquad \left. + \frac{1}{\omega(\lambda)} \int_{(-l,l)^2} \left| |\nabla'|^{-1} \frac{1}{\tau} \mathcal{P}_\lambda [m_3]^{(k)} \right|^2 dx' \right) \\
&\leq \sum_{k \in \mathbb{Z}} \tau \fint_{(0,\tau)} \left(2 \int_{(-l,l)^2} \left| \nabla' m_3^{(k)} \right| dx' + \int_{(-l,l)^2} \left| |\nabla'|^{-1} \frac{1}{\tau} [m_3]^{(k)} \right|^2 dx' \right. \\
&\qquad \left. + \frac{1}{\omega(\lambda)} \int_{(-l,l)^2} \left| |\nabla'|^{-1} \frac{1}{\tau} \mathcal{P}_\lambda [m_3]^{(k)} \right|^2 dx' \right) dx_3 \\
&\leq (1 + \omega(\lambda)) 4 l^2 e_l(m_3).
\end{aligned} \tag{16}$$

Here we use the abbreviations $m_3^{(k)}(x') = m_3(x', k\tau + x_3^0)$ and $[m_3]^{(k)} = m_3^{(k+1)} - m_3^{(k)}$. To finally obtain a candidate for $m_3^{(\varepsilon)}$ we take the piecewise constant (w.r.t. x_3) interpolant

$$\tilde{m}_3(x', x_3) = m_3^{(k)}(x') \qquad \text{for } x_3 \in [k\tau, (k+1)\tau).$$

Note that $\tilde{m}_3$ is admissible for the ε-energy $e_{\varepsilon,l}$. We want to estimate $e_{\varepsilon,l}(\tilde{m}_3)$ with $\tilde{h}$ minimized out. More precisely we are going to use an equivalent representation of the field energy. To prepare well, we use the interpolation estimate of Lemma 2 for $(-l,l)^2$-periodic φ

$$\left(\int_{(-l,l)^2} |\mathcal{P}_\lambda \varphi|^2 dx' \right)^{1/2} \leq C \left(\int_{[-l,l)^2} |\nabla' \varphi| dx' \right)^{1/3} \left(\sup_{(-l,l)^2} |\varphi| \right)^{1/3} \left(\int_{(-l,l)^2} \left| |\nabla'|^{-1} \mathcal{P}_\lambda \varphi \right|^2 dx' \right)^{1/6}$$

and conclude that for $\lambda \ll 1$ and small enough such that $\omega(\lambda) \le C$

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} \int_{(-l,l)^2} |\mathcal{P}_\lambda[m_3]^{(k)}|^2 dx' \\
\le\ & C \sum_{k\in\mathbb{Z}} \left(\int_{[-l,l)^2} |\nabla' m_3^{(k)}| dx' + \int_{[-l,l)^2} |\nabla' m_3^{(k+1)}| dx' \right)^{2/3} \left(\int_{(-l,l)^2} \left| |\nabla'|^{-1} \mathcal{P}_\lambda [m_3]^{(k)} \right|^2 dx' \right)^{1/3} \\
\le\ & C \left(\sum_{k\in\mathbb{Z}} \int_{[-l,l)^2} |\nabla' m_3^{(k)}| dx' \right)^{2/3} \left(\sum_{k\in\mathbb{Z}} \int_{(-l,l)^2} \left| |\nabla'|^{-1} \mathcal{P}_\lambda [m_3]^{(k)} \right|^2 dx' \right)^{1/3} \\
\overset{(16)}{\le}\ & C \left(\frac{1}{\tau} 4l^2 e_l(m_3) \right)^{2/3} \left(\omega(\lambda) \tau 4l^2 e_l(m_3) \right)^{1/3} = C\omega(\lambda)^{1/3} \frac{1}{\tau^{1/3}} 4l^2 e_l(m_3). \qquad (17)
\end{aligned}$$

Taking into account that $|[m_3]^{(k)}|$ is either 0 or at least 1 (in fact 0 or 2 except at the boundary), we likewise have

$$\sum_{k\in\mathbb{Z}} \int_{[-l,l)^2} |[m_3]^{(k)}|\, dx' \le \sum_{k\in\mathbb{Z}} \int_{(-l,l)^2} |[m_3]^{(k)}|^2 dx' \le C \frac{1}{\tau^{1/3}} 4l^2 e_l(m_3). \qquad (18)$$

With these preparations we turn to estimate $e_l(\tilde{m}_3)$. We first bound the surface term

$$\begin{aligned}
2 \int_{[-l,l)^2\times\mathbb{R}} \left| \begin{pmatrix} \nabla' \\ \varepsilon\partial_3 \end{pmatrix} \tilde{m}_3 \right| dx \ \le\ & 2 \int_{[-l,l)^2\times\mathbb{R}} |\nabla' \tilde{m}_3| dx + 2\varepsilon \int_{[-l,l)^2\times\mathbb{R}} |\partial_3 \tilde{m}_3|\, dx \\
=\ & 2 \sum_{k\in\mathbb{Z}} \tau \int_{[-l,l)^2} |\nabla' m_3^{(k)}| dx' + 2\varepsilon \sum_{k\in\mathbb{Z}} \int_{(-l,l)^2} |[m_3]^{(k)}| dx' \\
\overset{(18)}{\le}\ & \sum_{k\in\mathbb{Z}} \tau 2 \int_{[-l,l)^2} |\nabla' m_3^{(k)}| dx' + C \frac{\varepsilon}{\tau^{1/3}} 4l^2 e_l(m_3). \qquad (19)
\end{aligned}$$

To tackle the field term we take the Fourier series in x' and the Fourier transform in x_3. Writing

$G_\alpha(z_3) = \frac{1}{\alpha}G\left(\frac{z_3}{\alpha}\right)$ and $G(\hat{z}_3) = \frac{1}{2}\exp(-|\hat{z}_3|)$, the field term is

$$
\begin{aligned}
&\int_{(-l,l)^2\times\mathbb{R}} \left|(|\nabla'|^2+\varepsilon^2|\partial_3|^2)^{-1/2}\partial_3\tilde{m}_3\right|^2 dx \\
&= \int_{\mathbb{R}} \sum_{n'\in\mathbb{Z}^2} \frac{1}{\pi^2(|n'|^2/l^2+4\varepsilon^2\xi^2)} \left|\int_{\mathbb{R}} \exp(-2\pi i\xi x_3)\partial_3\mathcal{F}'(\tilde{m}_3)(n',x_3)dx_3\right|^2 d\xi \\
&= \int_{\mathbb{R}} \sum_{n'\in\mathbb{Z}^2} \frac{1}{\pi^2(|n'|^2/l^2+4\varepsilon^2\xi^2)} \int_{\mathbb{R}}\int_{\mathbb{R}} \exp(-2\pi i\xi x_3)\partial_3\mathcal{F}'(\tilde{m}_3)(n',x_3) \\
&\qquad\qquad \overline{\exp(-2\pi i\xi y_3)\partial_3\mathcal{F}'(\tilde{m}_3)(n',y_3)}dx_3dy_3d\xi \\
&= \sum_{n'\in\mathbb{Z}^2} \frac{l^2}{\pi^2|n'|^2} \int_{\mathbb{R}}\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{1}{1+4\varepsilon^2l^2\xi^2/|n'|^2}\exp(-2\pi i\xi(x_3-y_3))d\xi\right) \\
&\qquad\qquad \partial_3\mathcal{F}'(\tilde{m}_3)(n',x_3)\overline{\partial_3\mathcal{F}'(\tilde{m}_3)(n',y_3)}dx_3dy_3 \\
&= \sum_{n'\in\mathbb{Z}^2} \frac{l^2}{\pi^2|n'|^2} \int_{\mathbb{R}}\int_{\mathbb{R}} \left(\frac{\pi}{2\varepsilon l/|n'|}\exp(-2\pi|x_3-y_3||n'|/2\varepsilon l)\right) \\
&\qquad\qquad \partial_3\mathcal{F}'(\tilde{m}_3)(n',x_3)\overline{\partial_3\mathcal{F}'(\tilde{m}_3)(n',y_3)}dx_3dy_3 \\
&= \sum_{n'\in\mathbb{Z}^2} \frac{l^2}{\pi^2|n'|^2} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G_{\frac{\varepsilon l}{\pi|n'|}}(x_3-y_3)\partial_3(\mathcal{F}'(\tilde{m}_3))(n',x_3)\overline{\partial_3(\mathcal{F}'(\tilde{m}_3))(n',y_3)}dx_3dy_3.
\end{aligned}
$$

The above calculation is certainly valid for smooth $\tilde{m}_3$ and by approximation also for our piecewise constant $\tilde{m}_3(x',\cdot)$. In this case the two integrals on the right hand side are in fact sums, thus

$$
\begin{aligned}
&\int_{(-l,l)^2\times\mathbb{R}} \left|(|\nabla'|^2+\varepsilon^2|\partial_3|^2)^{-1/2}\partial_3\tilde{m}_3\right|^2 dx \\
&= \sum_{n'\in\mathbb{Z}^2} \frac{l^2}{\pi^2|n'|^2} \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} G_{\frac{\varepsilon l}{\pi|n'|}}((j-k)\tau)\mathcal{F}'([m_3]^{(j)})(n')\overline{\mathcal{F}'([m_3]^{(k)})(n')} \\
&= \sum_{n'\in\mathbb{Z}^2} \frac{l^2}{\pi^2|n'|^2}\frac{1}{\tau} \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} G_{\frac{\varepsilon l}{\pi\tau|n'|}}(j-k)\mathcal{F}'([m_3]^{(j)})(n')\overline{\mathcal{F}'([m_3]^{(k)})(n')} \\
&\le \sum_{n'\in\mathbb{Z}^2} \frac{l^2}{\pi^2|n'|^2}\frac{1}{\tau} \left(\sum_{j\in\mathbb{Z}} G_{\frac{\varepsilon l}{\pi\tau|n'|}}(j)\right)\left(\sum_{k\in\mathbb{Z}}|\mathcal{F}'([m_3]^{(k)})(n')|^2\right).
\end{aligned}
$$

Using the inequality $\exp(1/\alpha) \ge 1 + 1/\alpha$ we observe that

$$
\sum_{j\in\mathbb{Z}} G_\alpha(j) = \frac{1}{2\alpha}\left(-1+2\sum_{j=0}^{\infty}\exp\left(-\frac{1}{\alpha}\right)^j\right) = \frac{1}{2\alpha}\frac{1+\exp(-1/\alpha)}{1-\exp(-1/\alpha)} \le 1+\frac{1}{2\alpha},
$$

and so

$$
\begin{aligned}
&\int_{(-l,l)^2\times\mathbb{R}} \left|(|\nabla'|^2+\varepsilon^2|\partial_3|^2)^{-1/2}\partial_3\tilde{m}_3\right|^2 dx \\
&\le \frac{1}{\tau}\sum_{n'\in\mathbb{Z}^2} \frac{l^2}{\pi^2|n'|^2}\sum_{k\in\mathbb{Z}}|\mathcal{F}'([m_3]^{(k)})(n')|^2 + \frac{1}{2\varepsilon}\sum_{n'\in\mathbb{Z}^2}\frac{l}{\pi|n'|}\sum_{k\in\mathbb{Z}}|\mathcal{F}'([m_3]^{(k)})(n')|^2. \qquad (20)
\end{aligned}
$$

The second sum is an $H^{-1/2}$-norm which we estimate by interpolating between the H^{-1}-norm and the L^2-norm for high wave numbers. More precisely, we estimate splitting the second sum

$$
\begin{aligned}
&\sum_{n'\in\mathbb{Z}^2} \frac{l}{\pi|n'|}\sum_{k\in\mathbb{Z}}|\mathcal{F}'([m_3]^{(k)})(n')|^2 \\
&= \sum_{\substack{n'\in\mathbb{Z}^2\\ \pi\lambda|n'|/l<1}} \frac{l}{\pi|n'|}\sum_{k\in\mathbb{Z}}|\mathcal{F}'([m_3]^{(k)})(n')|^2 + \sum_{\substack{n'\in\mathbb{Z}^2\\ \pi\lambda|n'|/l\geq 1}} \frac{l}{\pi|n'|}\sum_{k\in\mathbb{Z}}|\mathcal{F}'([m_3]^{(k)})(n')|^2 \\
&\leq \frac{1}{\lambda}\sum_{n'\in\mathbb{Z}^2}\frac{l^2}{\pi^2|n'|^2}\sum_{k\in\mathbb{Z}}|\mathcal{F}'([m_3]^{(k)})(n')|^2 + \lambda\sum_{\substack{n'\in\mathbb{Z}^2\\ \pi\lambda|n'|/l\geq 1}}\sum_{k\in\mathbb{Z}}|\mathcal{F}'([m_3]^{(k)})(n')|^2. \qquad (21)
\end{aligned}
$$

Thus using (16) and (17) we obtain for λ small enough such that $\omega(\lambda)\leq C$

$$
\begin{aligned}
&\int_{(-l,l)^2\times\mathbb{R}} \left|(|\nabla'|^2+\varepsilon^2|\partial_3|^2)^{-1/2}\partial_3\tilde{m}_3\right|^2 dx \\
&\overset{(20),(21)}{\leq} \left(\frac{1}{\tau}+\frac{1}{2\varepsilon\lambda}\right)\sum_{n'\in\mathbb{Z}^2}\frac{l^2}{\pi^2|n'|^2}\sum_{k\in\mathbb{Z}}|\mathcal{F}'([m_3]^{(k)})(n')|^2 + \frac{\lambda}{2\varepsilon}\sum_{\substack{n'\in\mathbb{Z}^2\\ \lambda\pi|n'|/l\geq 1}}\sum_{k\in\mathbb{Z}}|\mathcal{F}'([m_3]^{(k)})(n')|^2 \\
&= \left(1+\frac{\tau}{2\varepsilon\lambda}\right)\sum_{k\in\mathbb{Z}}\tau\int_{(-l,l)^2}\left|\frac{1}{\tau}|\nabla'|^{-1}[m_3]^{(k)}\right|^2 dx' + \frac{\lambda}{2\varepsilon}\sum_{k\in\mathbb{Z}}\int_{(-l,l)^2}\left|\mathcal{P}_\lambda[m_3]^{(k)}\right|^2 dx' \\
&\overset{(16),(17)}{\leq} \sum_{k\in\mathbb{Z}}\tau\int_{(-l,l)^2}\left|\frac{1}{\tau}|\nabla'|^{-1}[m_3]^{(k)}\right|^2 dx' + C\frac{\tau}{\varepsilon\lambda}4l^2e_l(m_3) + C\frac{\lambda\omega(\lambda)^{1/3}}{\varepsilon\tau^{1/3}}4l^2e_l(m_3). \qquad (22)
\end{aligned}
$$

Combining (19) with (22) and employing (16) we see that

$$
\begin{aligned}
&2\int_{[-l,l)^2\times\mathbb{R}}\left|\left(\begin{smallmatrix}\nabla'\\ \varepsilon\partial_3\end{smallmatrix}\right)\tilde{m}_3\right| dx + \int_{(-l,l)^2\times\mathbb{R}}\left|(|\nabla'|^2+\varepsilon^2|\partial_3|^2)^{-1/2}\partial_3\tilde{m}_3\right|^2 dx \\
&\overset{(19),(22)}{\leq} \sum_{k\in\mathbb{Z}}\tau\left(2\int_{(-l,l)^2}\left|\nabla' m_3^{(k)}\right|^2 dx' + \int_{(-l,l)^2}\left|\frac{1}{\tau}|\nabla'|^{-1}[m_3]^{(k)}\right|^2 dx'\right) \\
&\qquad + C\left(\frac{\varepsilon}{\tau^{1/3}}+\frac{\tau}{\varepsilon\lambda}+\frac{\lambda\omega(\lambda)^{1/3}}{\varepsilon\tau^{1/3}}\right)4l^2e_l(m_3) \\
&\overset{(16)}{\leq} 4l^2e_l(m_3) + C\left(\omega(\lambda)+\frac{\varepsilon}{\tau^{1/3}}+\frac{\tau}{\varepsilon\lambda}+\frac{\lambda\omega(\lambda)^{1/3}}{\varepsilon\tau^{1/3}}\right)4l^2e_l(m_3). \qquad (23)
\end{aligned}
$$

In the last inequality we have used our choice of "good slices" again. To finish the proof, we need to arrange for the second term to vanish in the limit, so choosing $\tau = M^3\varepsilon^3$ and $\lambda = M^4\varepsilon^2$ we compute

$$
\omega(\lambda)+\frac{\varepsilon}{\tau^{1/3}}+\frac{\tau}{\varepsilon\lambda}+\frac{\lambda\omega(\lambda)^{1/3}}{\varepsilon\tau^{1/3}} = \omega(M^4\varepsilon^2)+\frac{1}{M}+\frac{1}{M}+M^3\omega(M^4\varepsilon^2)^{1/3}.
$$

Since

$$
\lim_{M\uparrow\infty}\lim_{\varepsilon\downarrow 0}\left(\omega(M^4\varepsilon^2)+\frac{1}{M}+M^3\omega(M^4\varepsilon^2)^{1/3}\right)=0
$$

we can select sequences $\{N^{(\varepsilon)} = \frac{1}{\tau^{(\varepsilon)}}\}_{\varepsilon\downarrow 0}$ and $\{\lambda^{(\varepsilon)}\}_{\varepsilon\downarrow 0}$ such that

$$\lim_{\varepsilon\downarrow 0}\left(\omega(\lambda^{(\varepsilon)}) + \frac{\varepsilon^{(\varepsilon)}}{(\tau^{(\varepsilon)})^{1/3}} + \frac{\tau^{(\varepsilon)}}{\varepsilon\lambda^{(\varepsilon)}} + \frac{\lambda^{(\varepsilon)}\omega(\lambda^{(\varepsilon)})^{1/3}}{\varepsilon(\tau^{(\varepsilon)})^{1/3}}\right) = 0.$$

Application of (23) for the corresponding sequence of $\tilde{m}_3^{(\varepsilon)}$ yields the assertion of the lemma. □

We now provide the interpolation inequality used in the proof of Lemma 1. It originally appeared in [CKO99, Lemma 2.3], but we wish to present a simplified argument here.

Lemma 2. *There exists a universal constant C such that*

$$\left(\int_{(-l,l)^2}|P\zeta|^2dx'\right)^{1/2} \leq C\left(\sup_{(-l,l)^2}|\zeta|\right)^{1/3}\left(\int_{[-l,l)^2}|\nabla'\zeta|dx'\right)^{1/3}\left(\int_{(-l,l)^2}\left||\nabla'|^{-1}P\zeta\right|^2dx'\right)^{1/2}$$

for all $(-l,l)^2$-periodic $\zeta:\mathbb{R}^2\to\mathbb{R}$ and P either be the identity or the projection $\mathcal{P}_\lambda$ on the Fourier modes as defined in (14).

Proof. We fix a smooth convolution kernel $\varphi:\mathbb{R}^2\to\mathbb{R}$ with

$$\varphi\geq 0,\quad \varphi = 0 \text{ for } |x'|\geq 1,\quad \varphi(-x') = \varphi(x'),\ \text{and}\quad \int_{\mathbb{R}^2}\varphi dx' = 1$$

and denote by subscript α the convolution with $\frac{1}{\alpha^2}\varphi(\frac{\cdot}{\alpha})$. Note that P commutes with convolution and is indeed a projection in L^2.

We observe that for any $(-l,l)^2$-periodic $\zeta:\mathbb{R}^2\to\mathbb{R}$ we have

$$\begin{aligned}\int_{(-l,l)^2}|\zeta(x'+h')-\zeta(x')|^2dx' &\leq \sup_{x'\in(-l,l)^2}|\zeta(x'+h')-\zeta(x')|\int_{(-l,l)^2}|\zeta(x'+h')-\zeta(x')|dx'\\ &\leq 2\left(\sup_{(-l,l)^2}|\zeta|\right)\cdot|h'|\int_{[-l,l)^2}|\nabla'\zeta|dx'. \quad (24)\end{aligned}$$

A standard convolution argument using Jensen's inequality shows

$$\begin{aligned}\int_{(-l,l)^2}|\zeta-\zeta_\alpha|^2dx' &\leq \int_{\mathbb{R}^2}\frac{1}{\alpha^2}\varphi\left(\frac{y'}{\alpha}\right)\int_{(-l,l)^2}|\zeta(x')-\zeta(x'-y')|^2dx'dy'\\ &\leq \int_{\mathbb{R}^2}\frac{1}{\alpha^2}\varphi\left(\frac{y'}{\alpha}\right)dy'\alpha\sup_{h'}\left(\frac{1}{|h'|}\int_{(-l,l)^2}|\zeta(x')-\zeta(x'+h')|^2dx'\right)\\ &= \alpha\sup_{h'}\left(\frac{1}{|h'|}\int_{(-l,l)^2}|\zeta(x')-\zeta(x'+h')|^2dx'\right)\\ &\overset{(24)}{\leq} 2\alpha\left(\sup_{(-l,l)^2}|\zeta|\right)\int_{[-l,l)^2}|\nabla'\zeta|dx'. \quad (25)\end{aligned}$$

By duality (see (73) in the appendix) the standard estimate

$$\begin{aligned}\int_{(-l,l)^2}|\nabla'\psi_\alpha|^2dx' &\leq \int_{\mathbb{R}^2}\frac{1}{\alpha^3}\left|\nabla'\varphi\left(\frac{x'}{\alpha}\right)\right|dx'\int_{(-l,l)^2}|\psi|^2dx'\\ &\leq C\frac{1}{\alpha^2}\int_{(-l,l)^2}|\psi|^2dx', \quad (26)\end{aligned}$$

valid for all $(-l,l)^2$-periodic $\psi:\mathbb{R}^2\to\mathbb{R}$, entails

$$\begin{aligned}\int_{(-l,l)^2}|(P\zeta)_\alpha|^2dx' &= \int_{(-l,l)^2}(P\zeta)_{\alpha\alpha}P\zeta dx' \\ &\overset{(73)}{\leq} \left(\int_{(-l,l)^2}|\nabla'(P\zeta)_{\alpha\alpha}|^2dx'\right)^{1/2}\left(\int_{(-l,l)^2}\left||\nabla'|^{-1}(P\zeta)\right|^2dx'\right)^{1/2} \\ &\overset{(26)}{\leq} C\frac{1}{\alpha}\left(\int_{(-l,l)^2}|(P\zeta)_\alpha|^2dx'\right)^{1/2}\left(\int_{(-l,l)^2}\left||\nabla'|^{-1}(P\zeta)\right|^2dx'\right)^{1/2}\end{aligned}$$

and thus

$$\int_{(-l,l)^2}|(P\zeta)_\alpha|^2dx' \leq C\frac{1}{\alpha^2}\int_{(-l,l)^2}\left||\nabla'|^{-1}(P\zeta)\right|^2dx'. \tag{27}$$

Note that convolution and the projection P on Fourier modes are (pointwise) multiplications in Fourier space and thus commute, in particular $P\zeta-(P\zeta)_\alpha = P\zeta - P(\zeta_\alpha) = P(\zeta-\zeta_\alpha)$. Combining (25) and (27) after using the triangle inequality and the projection property of P we obtain the assertion of the lemma from the estimate

$$\begin{aligned}&\left(\int_{(-l,l)^2}|P\zeta|^2dx'\right)^{1/2} \\ &\leq \left(\int_{(-l,l)^2}|(P\zeta)_\alpha|^2dx'\right)^{1/2}+\left(\int_{(-l,l)^2}|P\zeta-(P\zeta)_\alpha|^2dx'\right)^{1/2} \\ &\leq \left(\int_{(-l,l)^2}|(P\zeta)_\alpha|^2dx'\right)^{1/2}+\left(\int_{(-l,l)^2}|\zeta-\zeta_\alpha|^2dx'\right)^{1/2} \\ &\overset{(25),(27)}{\leq} C\left(\frac{1}{\alpha}\left(\int_{(-l,l)^2}\left||\nabla'|^{-1}P\zeta\right|^2dx'\right)^{1/2}+\alpha^{1/2}\left(\sup_{(-l,l)^2}|\zeta|\int_{[-l,l)^2}|\nabla'\zeta|dx'\right)^{1/2}\right)\end{aligned}$$

by choosing the optimal

$$\alpha = \left(\frac{\int_{(-l,l)^2}\left||\nabla'|^{-1}P\zeta\right|^2dx'}{\sup_{(-l,l)^2}|\zeta|\int_{[-l,l)^2}|\nabla'\zeta|dx'}\right)^{1/3}. \qquad \square$$

6 Inequalities used in the proof of Theorem 1

In this section we wrap up the proof of Theorem 1 by proving the remaining inequalities. Our main tools are the energy scaling, some elementary reflection and extension arguments. For the lower bound, we also need the interpolation inequality of Lemma 2, the upper bound is provided by an explicit construction. As we shall need to fix a few constants for later reference, we recall that C denotes an arbitrary constant (universal or depending only on the parameters indicated in parantheses) that can change between any two occurrences, while the numbered constants C_1, C_2, etc. are fixed within this section.

Lemma 3. *The sharp interface energy per cross-section area on configurations with periodic boundary conditions is bounded from below, i.e.*

$$\liminf_{1\ll l} e^p(l) > 0.$$

Proof. At core of the proof is is the interpolation estimate of Lemma 2. Fix an arbitrary l and let (m_3, h') be admissible for $e^p(l)$. Recall that

$$4l^2 e_l^p(m_3, h') = 2\int_{[-l,l)\times(-1,1)} |\nabla' m_3| dx + \int_{(-l,l)^2\times\mathbb{R}} |h'|^2 dx. \tag{28}$$

We estimate the second term from below by

$$\begin{aligned}\int_{(-l,l)^2\times\mathbb{R}} |h'|^2 dx &= \int_{-\infty}^{+\infty}\int_{(-l,l)^2} |h'|^2 dx' dx_3\\ &\geq \int_{-\infty}^{+\infty}\int_{(-l,l)^2} \left||\nabla'|^{-1}\partial_3 m_3\right|^2 dx' dx_3\\ &\geq \left(\frac{\pi}{2}\right)^2 \int_{-1}^{1}\int_{(-l,l)^2} \left||\nabla'|^{-1} m_3\right|^2 dx' dx_3,\end{aligned}$$

where we use the Poincaré estimate on $(-1, 1)$. For the first term in (28) we observe

$$\begin{aligned}2\int_{[-l,l)^2\times(-1,1)} |\nabla' m_3| dx &= 2\int_{-1}^{1}\int_{[-l,l)^2} |\nabla' m_3| dx' dx_3\\ &\geq 2\int_{-1}^{1} \sup_{(-l,l)^2} |m_3| \int_{[-l,l)^2} |\nabla' m_3| dx' dx_3,\end{aligned}$$

because $|m_3| \leq 1$, and hence using Young's inequality and Lemma 2 we can estimate the energy from below as

$$\begin{aligned}4l^2 e_l^p(m_3, h') &\geq \frac{1}{C}\int_{-1}^{1}\left(\sup_{(-l,l)^2}|m_3| \int_{[-l,l)^2} |\nabla' m_3| dx' + \int_{(-l,l)^2} \left||\nabla'|^{-1} m_3\right|^2 dx'\right) dx_3\\ &\geq \frac{1}{C}\int_{-1}^{1}\left(\sup_{(-l,l)^2}|m_3| \int_{[-l,l)^2} |\nabla' m_3| dx'\right)^{2/3} \left(\int_{(-l,l)^2} \left||\nabla'|^{-1} m_3\right|^2 dx'\right)^{1/3} dx_3\\ &\geq \frac{1}{C}\int_{-1}^{1}\int_{(-l,l)^2} m_3^2 dx' dx_3 = \frac{1}{C} 4l^2,\end{aligned}$$

that is

$$e_l^p(m_3, h') \geq \frac{1}{C}.$$

Since (m_3, h') was an arbitrary admissible pair, $e^p(l) \geq \frac{1}{C}$, as claimed. □

Lemma 4. *In the setting of free boundary conditions, decoupling the passage to the limit $l \to \infty$ from the limits in δ and ε does not increase the limiting energy of the minimizers. More precisely*

$$\liminf_{\delta\ll\varepsilon^2\ll 1\ll l} e^f(\varepsilon, \delta, l) \geq \limsup_{1\ll l}\, \liminf_{\delta\ll\varepsilon^2\ll 1} e^f(\varepsilon, \delta, l).$$

Proof. A key ingredient to establish the claim is

$$e^f(\varepsilon,\delta,Nl_0) \geq e^f(\varepsilon,\delta,l_0) \quad \text{for } N \in \mathbb{N}. \tag{29}$$

To establish this inequality, consider a minimizer (m,h) for $e^f(\varepsilon,\delta,Nl_0)$ and decompose the domain $(-Nl_0,Nl_0)^2$ into N^2 squares $\{Q_n\}_{=\{0..N-1\}^2}$ of edge length $2l_0$. Denote by (m_n,h_n) the restriction of (m,h) onto $Q_n \times \mathbb{R}$ translated back to $(-l_0,l_0)^2 \times \mathbb{R}$. As these are admissible in the minimization problem for $e^f(\varepsilon,\delta,l_0)$ and the energy functional is translation invariant we conclude with

$$e_{\varepsilon,\delta,Nl_0}(m,h) = \sum_{n\in\{0,\ldots,N-1\}^2} e_{\varepsilon,\delta,Nl_0}(m_n,h_n)$$

that

$$(Nl_0)^2 e^f(\varepsilon,\delta,Nl_0) \geq \sum_{n\in\{0,\ldots,N-1\}^2} l_0^2 e^f(\varepsilon,\delta,l_0).$$

As a second item we need that

$$e^f(\varepsilon,\delta,l) \geq \left(\frac{\tilde{l}}{l}\right)^2 e^f(\varepsilon,\delta,\tilde{l}) \quad \text{for } l \geq \tilde{l}, \tag{30}$$

which is evident when considering that the restriction of a minimizer (m,h) for $e^f(\varepsilon,\delta,l)$ to $(-\tilde{l},\tilde{l})$ is admissible for $e^f(\varepsilon,\delta,\tilde{l})$.

Now fix l_0 and let $l \geq l_0$ be arbitrary. Write $l = Nl_0 + r$ with $N \in \mathbb{N}$ and $r \in [0,l_0)$. By above estimates (29) and (30) we have

$$e^f(\varepsilon,\delta,l) \geq \left(\frac{Nl_0}{l}\right)^2 e^f(\varepsilon,\delta,Nl_0) \geq \left(\frac{l-l_0}{l}\right)^2 e^f(\varepsilon,\delta,l_0) = \left(1-\frac{l_0}{l}\right)^2 e^f(\varepsilon,\delta,l_0),$$

thus

$$\liminf_{\delta\ll\varepsilon^2\ll 1\ll l} e^f(\varepsilon,\delta,l) \geq \liminf_{\delta\ll\varepsilon^2\ll 1} e^f(\varepsilon,\delta,l_0),$$

and so as l_0 was arbitrary

$$\liminf_{\delta\ll\varepsilon^2\ll 1\ll l} e^f(\varepsilon,\delta,l) \geq \limsup_{1\ll l} \liminf_{\delta\ll\varepsilon^2\ll 1} e^f(\varepsilon,\delta,l).$$ □

Lemma 5. *In the setting of periodic boundary conditions, coupling the passage to the limit $l \to \infty$ with the limits in δ and ε does not increase the limiting energy of the minimizers. More precisely*

$$\limsup_{\delta\ll\varepsilon^2\ll 1\ll l} e^p(\varepsilon,\delta,l) \leq \liminf_{1\ll l} \limsup_{\delta\ll\varepsilon^2\ll 1} e^p(\varepsilon,\delta,l).$$

Proof. Because $(-l,l)^2$-periodicity implies $(-Nl,Nl)^2$-periodicity and the resulting inclusion of the admissible classes we have

$$e^p(\varepsilon,\delta,Nl) \leq e^p(\varepsilon,\delta,l) \quad \text{for } N \in \mathbb{N}. \tag{31}$$

We claim that

$$e^p(\varepsilon,\delta,l) \leq \left(2\frac{l}{\tilde{l}}-1\right)^2 e^p(\varepsilon,\delta,\tilde{l}) \quad \text{for } l \geq \tilde{l}, \varepsilon^2 \geq \delta. \tag{32}$$

Indeed, let $(\tilde m, \tilde h)$ be a minimizer for $e^p(\varepsilon,\delta,\tilde l)$. We define an admissible (m,h) for $e^p(\varepsilon,\delta,l)$ as follows: Let

$$\begin{aligned} m(x',x_3) &= \tilde m\left(\frac{\tilde l}{l}x',x_3\right),\\ h'(x',x_3) &= \left(\frac{l}{\tilde l}\tilde h' + \left(\frac{l}{\tilde l}-1\right)\frac{1}{\varepsilon}\tilde m'\right)\left(\frac{\tilde l}{l}x',x_3\right), \text{ and}\\ h_3(x',x_3) &= \tilde h_3\left(\frac{\tilde l}{l}x',x_3\right). \end{aligned}$$

Thus h' is defined such that

$$\left(h' + \frac{1}{\varepsilon}m'\right)(x',x_3) = \frac{l}{\tilde l}\left(\tilde h' + \frac{1}{\varepsilon}\tilde m'\right)\left(\frac{\tilde l}{l}x',x_3\right),$$

ensuring

$$\left(\nabla'\cdot\left(h' + \frac{1}{\varepsilon}m'\right)\right)(x',x_3) = \left(\nabla'\cdot\left(\tilde h' + \frac{1}{\varepsilon}\tilde m'\right)\right)\left(\frac{\tilde l}{l}x',x_3\right),$$

i.e. the admissibility of (m,h).

Furthermore using $\tilde l \le l$ and $\delta \le \varepsilon^2$ we have

$$\begin{aligned} \frac{1}{4l^2}\delta\int_{(-l,l)^2\times(-1,1)}\left|\left(\begin{smallmatrix}\nabla'\\ \varepsilon\partial_3\end{smallmatrix}\right)m\right|^2 dx &= \frac{1}{4\tilde l^2}\delta\int_{(-\tilde l,\tilde l)^2\times(-1,1)}\left|\left(\begin{smallmatrix}\frac{\tilde l}{l}\nabla'\\ \varepsilon\partial_3\end{smallmatrix}\right)\tilde m\right|^2 dx\\ &\le \frac{1}{4\tilde l^2}\delta\int_{(-\tilde l,\tilde l)^2\times(-1,1)}\left|\left(\begin{smallmatrix}\nabla'\\ \varepsilon\partial_3\end{smallmatrix}\right)\tilde m\right|^2 dx, \end{aligned}$$

as well as

$$\frac{1}{4l^2}\frac{1}{\delta}\int_{(-l,l)^2\times(-1,1)}|m'|^2\,dx = \frac{1}{4\tilde l^2}\frac{1}{\delta}\int_{(-\tilde l,\tilde l)^2\times(-1,1)}|\tilde m'|^2\,dx,$$

and

$$\begin{aligned} &\frac{1}{4l^2}\int_{(-l,l)^2\times\mathbb{R}}\left|\left(\begin{smallmatrix}h'\\ \frac{1}{\varepsilon}h_3\end{smallmatrix}\right)\right|^2 dx\\ &= \frac{1}{4\tilde l^2}\int_{(-\tilde l,\tilde l)^2\times\mathbb{R}}\left|\left(\begin{smallmatrix}\frac{l}{\tilde l}\tilde h' + (\frac{l}{\tilde l}-1)\frac{1}{\varepsilon}\tilde m'\\ \frac{1}{\varepsilon}\tilde h_3\end{smallmatrix}\right)\right|^2 dx\\ &\le \left(\frac{l}{\tilde l}\right)^2\frac{1}{4\tilde l^2}\int_{(-\tilde l,\tilde l)^2\times\mathbb{R}}\left|\left(\begin{smallmatrix}\tilde h'\\ \frac{1}{\varepsilon}\tilde h_3\end{smallmatrix}\right)\right|^2 dx\\ &\quad + 2\frac{l}{\tilde l}\left(\frac{l}{\tilde l}-1\right)\left(\frac{1}{4\tilde l^2}\int_{(-\tilde l,\tilde l)^2\times\mathbb{R}}|\tilde h'|^2 dx\right)^{1/2}\left(\frac{1}{4\tilde l^2}\frac{1}{\varepsilon^2}\int_{(-\tilde l,\tilde l)^2\times(-1,1)}|\tilde m'|^2 dx\right)^{1/2}\\ &\quad + \left(\frac{l}{\tilde l}-1\right)^2\frac{1}{4\tilde l^2}\frac{1}{\varepsilon^2}\int_{(-\tilde l,\tilde l)^2\times(-1,1)}|\tilde m'|^2 dx\\ &\le \left(\frac{l}{\tilde l}\right)^2\frac{1}{4\tilde l^2}\int_{(-\tilde l,\tilde l)^2\times\mathbb{R}}\left|\left(\begin{smallmatrix}\tilde h'\\ \frac{1}{\varepsilon}\tilde h_3\end{smallmatrix}\right)\right|^2 dx + \left(2\frac{l}{\tilde l}\left(\frac{l}{\tilde l}-1\right) + \left(\frac{l}{\tilde l}-1\right)^2\right)e^p(\varepsilon,\delta,\tilde l). \end{aligned}$$

Hence

$$e^p(\varepsilon,\delta,l) \le \left(\left(\frac{l}{\tilde{l}}\right)^2 + 2\frac{l}{\tilde{l}}\left(\frac{l}{\tilde{l}}-1\right) + \left(\frac{l}{\tilde{l}}-1\right)^2\right) e^p(\varepsilon,\delta,\tilde{l}) = \left(2\frac{l}{\tilde{l}}-1\right)^2 e^p(\varepsilon,\delta,\tilde{l}),$$

as claimed.

With (31) and (32) at our disposal, we are able to proceed as in Lemma 4. Fix l_0 and let $l \ge l_0$. We write

$$l = Nl_0 + r \text{ with } N \in \mathbb{N} \text{ and } r \in [0, l_0).$$

Then for $\varepsilon^2 \ge \delta$ we can estimate

$$e^p(\varepsilon,\delta,l) \le \left(2\frac{l}{Nl_0}-1\right)^2 e^p(\varepsilon,\delta,Nl_0) \le \left(2\frac{l}{l-l_0}-1\right)^2 e^p(\varepsilon,\delta,l_0) = \left(\frac{l+l_0}{l-l_0}\right)^2 e^p(\varepsilon,\delta,l_0).$$

Hence

$$\limsup_{\delta \ll \varepsilon^2 \ll 1 \ll l} e^p(\varepsilon,\delta,l) \le \liminf_{\delta \ll \varepsilon^2 \ll 1} e^p(\varepsilon,\delta,l_0),$$

and again the assertion of the lemma follows because l_0 was arbitrary. □

Lemma 6. *The minimal sharp-interface energy per cross-section area amongst admissible configurations with free boundary conditions is bounded, i.e.*

$$\limsup_{1 \ll l} e^f(l) < \infty.$$

Proof. We use four main estimates for the proof. First, completely analogous to (30) we have

$$e^f(l) \le \left(\frac{\tilde{l}}{l}\right)^2 e^f(\tilde{l}) \quad \text{for } l \le \tilde{l}, \tag{33}$$

then because the inclusion of the corresponding admissible classes we obviously have

$$e^f(l) \le e^p(l). \tag{34}$$

Thirdly, as in (31) we have

$$e^p(N) \le e^p(1) \quad \text{for } N \in \mathbb{N}. \tag{35}$$

And as a fourth ingredient we need the estimate

$$e^p(1) < \infty. \tag{36}$$

To establish the latter, we have to construct $m_3 : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ such that

$$m_3 \text{ is } (-1,1)^2\text{-periodic in } x', \quad m_3^2 = \begin{cases} 1 & \text{if } x_3 \in (-1,1), \\ 0 & \text{otherwise,} \end{cases}$$

with

$$2\int_{[-1,1)^2 \times (-1,1)} |\nabla' m_3| dx + \int_{(-1,1)^2 \times \mathbb{R}} \left||\nabla'|^{-1} \partial_3 m_3\right|^2 dx < \infty.$$

By symmetry and translation invariance, it suffices to construct $m_3 : \mathbb{R}^2 \times (0,1) \to \mathbb{R}$ such that

$$m_3 \text{ is } (-1,1)^2\text{-periodic in } x', \quad m_3^2 = 1, \text{ and } \quad m_3(\,\cdot\,,) \xrightarrow{x_3 \uparrow 1} 0 \text{ (weakly) in } L^\infty((-1,1)^2)$$

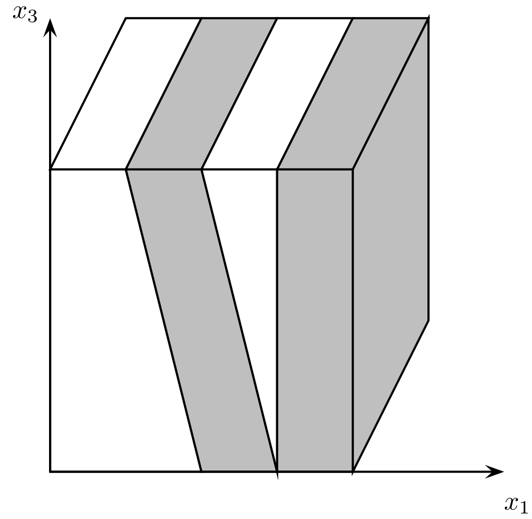

Figure 2: The refinement (constant in x_2)

with

$$2\int_{[-1,1)^2\times(0,1)} |\nabla' m_3| dx + \int_{(-1,1)^2\times(0,1)} \left||\nabla'|^{-1}\partial_3 m_3\right|^2 dx < \infty.$$

Denote by $m_3^0 : \mathbb{R}^2 \to \mathbb{R}$ the $(-1,1)^2$-periodic function given by

$$m_3^0(x_1, x_2) = \operatorname{sign} x_1.$$

Obviously (see e.g. Figure 2), one can construct $m_3^{01} : \mathbb{R}^2 \times (0,1) \to \mathbb{R}$ such that

$$\begin{aligned} &m_3^{01} \text{ is } (-1,1)\text{-periodic in } x', \quad (m_3^{01})^2 = 1, \\ &m_3^{01}(x', 0) = m_3^0(x'), \text{ and } \quad m_3^{01}(x', 1) = m_3^0(2x'), \end{aligned}$$

and

$$2\int_{[-1,1)^2\times(0,1)} |\nabla' m_3^{01}| dx + \int_{(-1,1)^2\times(0,1)} \left||\nabla'|^{-1}\partial_3 m_3^{01}\right|^2 dx < \infty.$$

We now glue rescaled versions of m_3^{01}. Let $\Delta_k = \theta^k(1-\theta)$ with $\theta \in (0,1)$ to be chosen later. Note that $\sum_{k=0}^{\infty} \Delta_k = 1$. Let $x_3^{(0)} = 0$ and $x_3^{(k+1)} = x_3^{(k)} + \Delta_k$ and define

$$m_3(x', x_3) = m_3^{01}\left(2^k x', \frac{x_3 - x_3^{(k)}}{\Delta_k}\right) \quad \text{for } x_3 \in (x_3^{(k)}, x_3^{(k+1)}).$$

By construction we have

$$m_3(\,\cdot\,, x_3^{(k)}-) \equiv m_3(\,\cdot\,, x_3^{(k)}+) \text{ and } \quad m_3(\,\cdot\,, x_3) \xrightarrow{x_3\uparrow 1} 0,$$

thus

$$\int_{[-1,1)^2\times(0,1)} |\nabla' m_3| dx = \sum_{k=0}^{\infty} \int_{[-1,1)^2\times(x_3^{(k)},x_3^{(k+1)})} |\nabla' m_3| dx$$
$$= \sum_{k=0}^{\infty} \Delta_k 2^k \int_{[-1,1)^2\times(0,1)} |\nabla' m_3^{01}| dx$$

and

$$\int_{(-1,1)^2\times(0,1)} \left||\nabla'|^{-1}\partial_3 m_3\right|^2 dx = \sum_{k=0}^{\infty} \int_{(-1,1)^2\times(x_3^{(k)},x_3^{(k+1)})} \left||\nabla'|^{-1}\partial_3 m_3\right|^2 dx$$
$$= \sum_{k=0}^{\infty} \frac{1}{\Delta_k}\left(\frac{1}{2^k}\right)^2 \int_{(-1,1)^2\times(0,1)} \left||\nabla'|^{-1}\partial_3 m_3^{01}\right|^2 dx.$$

Combining these two, we see that

$$2\int_{[-1,1)^2\times(0,1)} |\nabla' m_3| dx + \int_{(-1,1)^2\times(0,1)} \left||\nabla'|^{-1}\partial_3 m_3\right|^2 dx$$
$$\le \max\left\{\sum_{k=0}^{\infty} \Delta_k 2^k, \sum_{k=0}^{\infty} \frac{1}{\Delta_k}\left(\frac{1}{2^k}\right)^2\right\}\left(2\int_{[-1,1)^2\times(0,1)} |\nabla' m_3^{01}| dx + \int_{(-1,1)^2\times(0,1)} \left||\nabla'|^{-1}\partial_3 m_3^{01}\right|^2 dx\right)$$
$$= \max\left\{(1-\theta)\sum_{k=0}^{\infty}(2\theta)^k, \frac{1}{1-\theta}\sum_{k=0}^{\infty}\left(\frac{1}{4\theta}\right)^k\right\}$$
$$\cdot\left(2\int_{[-1,1)^2\times(0,1)} |\nabla' m_3^{01}| dx + \int_{(-1,1)^2\times(0,1)} \left||\nabla'|^{-1}\partial_3 m_3^{01}\right|^2 dx\right),$$

to ensure that the bound is finite, we need to choose $\frac{1}{4} < \theta < \frac{1}{2}$. The natural choice based on the energy scaling is $\theta = \left(\frac{1}{2}\right)^{3/2}$, but this is not of further interest here. This construction entails (36).

To finish the proof of the lemma, fix $l \ge 1$ and write

$$l = N + r \text{ with } N \in \mathbb{N} \text{ and } r \in [0,1).$$

By consecutively applying (33), (34), (35), and (36) we see

$$e^f(l) \le \left(\frac{N+1}{l}\right)^2 e^f(N+1) \le \left(\frac{N+1}{N}\right)^2 e^p(N+1)$$
$$\le \left(\frac{N+1}{N}\right)^2 e^p(1) = \left(1+\frac{1}{N}\right)^2 e^p(1) < \infty$$

and because l was arbitrary, this entails the assertion of the lemma. □

Lemma 7. *In the limit $l \to \infty$ of the sharp interface model, the minimal energy among periodic configuration is no larger than that among admissible configurations with free boundary conditions, more precisely*

$$\limsup_{1\ll l} e^p(l) \le \liminf_{1\ll l} e^f(l).$$

Proof. The main ingredients for the proof are

$$e^f(Nl_0) \geq e^f(l_0) \quad \text{for } N \in \mathbb{N} \tag{37}$$

resembling (29),

$$e^f(l) \geq \left(\frac{\tilde{l}}{l}\right)^2 e^f(\tilde{l}) \quad \text{for } \tilde{l} \leq l \tag{38}$$

analogous to (30), and

$$e^p(2l) \leq e^f(l) + \frac{4}{l}. \tag{39}$$

To verify (39), we consider an admissible pair (m_3, h') for $e^f(l)$. We translate (m_3, h') such that the domain is $(0, 2l)^2 \times \mathbb{R}$. We aim at the construction of an admissible pair $(\tilde{m}_3, \tilde{h}')$ for $e^p(2l)$. Unique extensions $(\tilde{m}_3, \tilde{h}')$ of (m_3, h') to $\mathbb{R}^2 \times \mathbb{R}$ exist with the following properties

$$\begin{aligned}
&(\tilde{m}_3, \tilde{h}') \text{ are } (-2l, 2l)^2\text{-periodic in } x',\\
&(\tilde{h}_1, \tilde{h}_2, \tilde{m}_3)(-x_1, x_2, x_3) = (\tilde{h}_1, -\tilde{h}_2, -\tilde{m}_3)(x_1, x_2, x_3), \text{ and}\\
&(\tilde{h}_1, \tilde{h}_2, \tilde{m}_3)(x_1, -x_2, x_3) = (-\tilde{h}_1, \tilde{h}_2, -\tilde{m}_3)(x_1, x_2, x_3).
\end{aligned}$$

We observe

$$\begin{aligned}
\left(\partial_3\tilde{m}_3 + \partial_1\tilde{h}_1' + \partial_2\tilde{h}_2'\right)(-x_1, x_2, x_3) &= -\left(\partial_3\tilde{m}_3 + \partial_1\tilde{h}_1' + \partial_2\tilde{h}_2'\right)(x_1, x_2, x_3),\\
\left(\partial_3\tilde{m}_3 + \partial_1\tilde{h}_1' + \partial_2\tilde{h}_2'\right)(x_1, -x_2, x_3) &= -\left(\partial_3\tilde{m}_3 + \partial_1\tilde{h}_1' + \partial_2\tilde{h}_2'\right)(x_1, x_2, x_3),\\
\tilde{h}_1'(0-, x_2, x_3) &= \tilde{h}_1'(0+, x_2, x_3),\\
\tilde{h}_2'(x_1, 0-, x_3) &= \tilde{h}_2'(x_1, 0+, x_3).
\end{aligned}$$

Hence

$$\nabla' \cdot h' + \partial_3 m_3 = 0 \text{ distributionally in } (0, 2l)^2 \times \mathbb{R}$$

turns into

$$\nabla' \cdot \tilde{h}' + \partial_3 \tilde{m}_3 = 0 \text{ distributionally in } \mathbb{R}^2 \times \mathbb{R}.$$

Additionally

$$\tilde{m}_3^2 = \begin{cases} 1 & \text{for } x_3 \in (-1, 1),\\ 0 & \text{otherwise,} \end{cases}$$

so that $(\tilde{m}_3^2, \tilde{h}')$ is admissible for $e^p(2l)$. We estimate the energy as

$$\begin{aligned}
(4l)^2 e_{2l}(\tilde{m}_3, \tilde{h}') &= 2\int_{[-2l,2l)^2\times(-1,1)} |\nabla'\tilde{m}_3| dx + \int_{(-2l,2l)^2\times\mathbb{R}} |\tilde{h}'|^2 dx\\
&= 4\left(2\int_{(0,2l)^2\times(-1,1)} |\nabla'\tilde{m}_3| dx + \int_{(0,2l)^2\times\mathbb{R}} |\tilde{h}'|^2 dx\right) + 64l\\
&= 16l^2 e_l(m_3, h') + 64l,
\end{aligned}$$

we incur a new term because $\tilde{m}_3$ has a jump of height 2 at the reflection lines. Dividing by $16l^2$ yields (39).

To wrap up the proof, fix $l_0 \geq 0$ and let $l \geq l_0$ be arbitrary. Write

$$l = Nl_0 + r \text{ with } N \in \mathbb{N} \text{ and } r \in [0, l_0).$$

Having prepared our three ingredients we combine them to compute

$$\begin{aligned} e^p(2l) &\overset{(37)}{\leq} e^f(l) + \frac{4}{l} \\ &\overset{(38)}{\leq} \left(\frac{(N+1)l_0}{l}\right)^2 e^f((N+1)l_0) + \frac{4}{l} \\ &\overset{(39)}{\leq} \left(\frac{l+l_0}{l}\right)^2 e^f(l_0) + \frac{4}{l} \\ &= \left(1 + \frac{l_0}{l}\right)^2 e^f(l_0) + \frac{4}{l} \end{aligned}$$

and thus find

$$\limsup_{1 \ll l} e^p(l) = \limsup_{1 \ll l} e^p(2l) \leq e^f(l_0).$$

Since $l_0 > 0$ was arbitrary we have

$$\limsup_{1 \ll l} e^p(l) \leq \liminf_{1 \ll l} e^f(l),$$

completing the proof. □

7 Quantification of the construction in a Modica-Mortola problem

In this largely self-contained section we provide a quantification of the construction used to show the Γ-convergence result of Modica and Mortola, [MM77] that we use in the proof of part 1 of Theorem 2. Throughout this section, we work with the half-open cubes $Q_l(x) = x + (\frac{-l}{2}, \frac{l}{2}]^n$. Our goal is to prove the following proposition:

Proposition 1. *For all $\alpha > 0$ a constant $C_5(\alpha, n) < \infty$ exists such that for any domain size $L > 0$, all functions $\chi : Q_L \to \{-1, 1\}$ and all $\delta > 0$ there is an approximation $u : Q_L \to [-1, 1]$ such that*

$$\int_{Q_L} \frac{\delta}{2} \frac{1}{1-u^2} |\nabla u|^2 + \frac{1}{2\delta}(1-u^2) dx \leq (1+\alpha) \int_{Q_L} |\nabla \chi| dx$$

and

$$\int_{Q_L} |\chi - u| \leq C_5(\alpha, n) \delta \int_{Q_L} |\nabla \chi| dx.$$

Remark 1. We apply the lemma in a rescaled version with $n = 3$ and $\hat{x}_3 = \frac{1}{\varepsilon} x_3$ and a size in x_3-direction that differs from that in the other two directions. This does not affect the viability of the proposition.

First, we approximate characteristic functions by those from a set of finite cardinality.

Lemma 8. *For all $L \in \mathbb{N}$ there is a finite set $F \subset BV(Q_L, \{-1,1\})$ with cardinality*

$$\#F \leq 2^{L^n}$$

such that all $\chi : Q_L \to \{-1,1\}$ can be approximated by a $\tilde{\chi} \in F$ in the sense of

$$\int_{Q_L} |\nabla \tilde{\chi}| dx \leq \int_{Q_L} |\nabla \chi| dx \text{ and} \tag{40}$$

$$\int_{Q_L} |\chi - \tilde{\chi}| dx \leq C_0(n) \int_{Q_L} |\nabla \chi| dx. \tag{41}$$

The exponent in the cardinality estimate is, of course, the volume of Q_L.

Proof. We proceed in two steps. First we approximate χ in L^1 by functions constant on unit cubes with an error bound proportional to the total variation, i.e. with a bound resembling (41). We then replace these initial approximation functions by minimizers of the total variation within appropriately sized closed L^1-neighborhoods. This ensures that the approximation statisfies (40) without making the L^1-error larger than twice that of the first step.

Let us decompose Q_L into L^n translated unit cubes $\{Q_1^k\}_{k \in \{1,\ldots,L\}^n}$ and denote by F_0 the set of all functions $\chi_0 : Q_L \to \{-1,1\}$ that are piecewise constant on each Q_1^k. Clearly $\#F_0 = 2^{L^n}$. We claim that any $\chi : Q_L \to \{-1,1\}$ can be approximated by a function $\chi_0 \in F_0$ in the sense that

$$\int_{Q_L} |\chi - \chi_0| dx \leq C_1(n) \int_{Q_L} |\nabla \chi| dx. \tag{42}$$

Indeed, let χ_0 be the piecewise constant function given by

$$\chi_0\big|_{Q_1^k} = \begin{cases} 1 & \text{if } \int_{Q_1^k} \chi dx \geq 0, \\ -1 & \text{if } \int_{Q_1^k} \chi dx < 0 \end{cases}$$

for $k \in \{1, \ldots, L\}^n$. Since Q_1^k has unit size, we can use the Poincaré inequality to estimate the deviation from the mean as

$$\int_{Q_1^k} \left|\chi - \int_{Q_1^k} \chi\right| \leq C_2(n) \int_{Q_1^k} |\nabla \chi| dx. \tag{43}$$

If $\int_{Q_1^k} \chi dx \geq 0$, this implies

$$\begin{aligned} \int_{Q_1^k} |\chi_0 - \chi| dx = \int_{Q_1^k} |1 - \chi| dx = 2\mathscr{L}^n(\{x \in Q_1^k | \chi(x) = -1\}) \\ \leq 2 \int \left|\chi - \int_{Q_1^k} \chi\right| dx \leq 2C_2(n) \int_{Q_1^k} |\nabla \chi| dx. \end{aligned} \tag{44}$$

The same calculation also works if $\int_{Q_1^k} \chi dx \leq 0$. Summing over $k \in \{1, \ldots, L\}^n$ we establish our claim (43) with $C_1(n) = 2C_2(n)$.

We now want to improve our choice of the approximation functions to have small total variation. To this end, consider for any $\chi_0 \in F_0$ the smallest L^1-neighborhood containing a good approximation. More specifically we define

$$S_P(\chi_0) := \left\{\chi : Q_L \to \{-1,1\} \Big| \int_{Q_L} |\chi - \chi_0| dx \leq C_1(n) P\right\}$$

and then find the minimal radius

$$P^* := P^*(\chi_0) := \inf\Big\{P\Big|\inf_{\chi\in S_P(\chi_0)}\int_{Q_L}|\nabla\chi|dx \le P\Big\}$$

that is of relevance to our approximation needs. By the standard compactness and lower semicontinuity properties of BV-functions both infima are, in fact, minima. We thus find $\tilde\chi \in S_{P^*}(\chi_0)$ that minimizes the total variation in the L^1-closed set $S_{P^*}(\chi_0)$, i.e.

$$\int_{Q_L}|\nabla\tilde\chi|dx = P^*.$$

We claim that the set

$$F = \{\tilde\chi|\chi_0 \in F_0\}$$

has the desired approximation properties. Indeed, given any $\chi : Q_L \to \{-1,1\}$ with total variation $P = \int_{Q_L}|\nabla\chi|dx$, we find by the first step a $\chi_0 \in F_0$ satisfying (42). In particular, $\chi \in S_P(\chi_0)$ and thus $P \ge P^*(\chi_0)$, which is (40). By the triangle inequality

$$S_{2P}(\tilde\chi) \supseteq S_P(\chi_0) \ni \chi,$$

in other words (41) is satisfied with $C_0(n) = 2C_1(n)$, completing the proof. □

We now use this approximation by functions from a finite set to improve the Modica-Mortola result to a uniform version, first for bounded and later for arbitary system sizes.

Lemma 9. *For any system size bound $L_0 \in \mathbb{N}$ and approximation parameter $R \in \mathbb{N}$ there is a scaling coefficient $0 < \delta_0 \le 1$ such that for all $0 < \delta \le \delta_0$, all $L \le L_0$, and all functions $\chi : Q_L \to \{-1,1\}$ there is an approximating $u : Q_L \to [-1,1]$ such that the diffuse interface energy is bounded by*

$$\int_{Q_L}\frac{\delta}{2}\frac{1}{1-u^2}|\nabla u|^2 + \frac{1}{2\delta}(1-u^2)dx \le \Big(1+\frac{1}{R}\Big)\int_{Q_L}|\nabla\chi|dx \tag{45}$$

and u is close to χ in the sense that

$$\int_{Q_L}|\chi-u|dx \le C(n)\int_{Q_L}|\nabla\chi|dx. \tag{46}$$

Proof. Recall from [MM77, Theorema 2] that for given fixed $L \in \mathbb{N}$ and $\delta \to 0$

$$E_\delta(u) := E_\delta(u,Q_L) := \int_{Q_L}\frac{\delta}{2}\frac{1}{1-u^2}|\nabla u|^2 + \frac{1}{2\delta}(1-u^2)dx$$

Γ-converges with respect to the $L^1(Q_L)$-topology to

$$E_0(u) := \begin{cases}\int_{Q_L}|\nabla u|dx & \text{if } u\in\{-1,+1\} \text{ a.e.,}\\ +\infty & \text{otherwise.}\end{cases}$$

With this in mind, we begin the proof. Fix an arbitrary $L_0 \in \mathbb{N}$ and $R \in \mathbb{N}$. Let us assume for the moment that $L = L_0$.

Since the set F of Lemma 8 is finite, there exists a $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$ and all $\tilde{\chi} \in F$, there is a $u_{\tilde{\chi}} : Q_L \to [-1,1]$ satisfying

$$E_\delta(u_{\tilde{\chi}}) \leq \left(1 + \frac{1}{R}\right) \int_{Q_L} |\nabla \tilde{\chi}| dx \tag{47}$$

and

$$\int_{Q_L} |\tilde{\chi} - u_{\tilde{\chi}}| \leq C_0(n) \int_{Q_L} |\nabla \tilde{\chi}| dx. \tag{48}$$

We do have the choice of using the constant in the approximation property of Lemma 8 as $C_0(n)$.

Now let $\chi : Q_L \to \{-1,1\}$ be given. According to Lemma 8, there exists $\tilde{\chi} \in F$ with (40) and (41). This allows us to estimate

$$E_\delta(u_{\tilde{\chi}}) \overset{(47)}{\leq} \left(1 + \frac{1}{R}\right) \int_{Q_L} |\nabla \tilde{\chi}| dx \overset{(40)}{\leq} \left(1 + \frac{1}{R}\right) \int_{Q_L} |\nabla \chi| dx,$$

establishing (45), and

$$\begin{aligned} \int_{Q_L} |\chi - u_{\tilde{\chi}}| dx &\leq \int_{Q_L} |\chi - \tilde{\chi}| dx + \int_{Q_L} |\tilde{\chi} - u_{\tilde{\chi}}| dx \\ &\overset{(41),(48)}{\leq} C_0(n) \int_{Q_L} |\nabla \chi| dx + C_0(n) \int_{Q_L} |\nabla \tilde{\chi}| dx \\ &\overset{(40)}{\leq} 2C_0(n) \int_{Q_L} |\nabla \chi| dx, \end{aligned}$$

i.e. (46), so that $u_{\tilde{\chi}}$ has properties claimed in the lemma for δ and $C(n) = 2C_0(n)$, completing its proof if $L = L_0$.

It remains to consider the case $L < L_0$. If $L \geq 1$ we rescale lengths according to

$$\hat{x} = \frac{L_0}{L} x, \qquad \hat{\delta} = \frac{L_0}{L} \delta, \qquad \hat{L} = \frac{L_0}{L} L = L_0$$

and, as this puts us in the case already dealt with, obtain for $\hat{\chi} : Q_{\hat{L}} \to \{-1,1\}$ an approximation $\hat{u} : Q_{\hat{L}} \to [-1,1]$ with (45) and (46) in the new coordinates, i.e.

$$\int_{Q_{\hat{L}}} \frac{\hat{\delta}}{2} \frac{1}{1 - \hat{u}^2} |\hat{\nabla} \hat{u}|^2 + \frac{1}{2\hat{\delta}} (1 - \hat{u}^2) d\hat{x} \leq \left(1 + \frac{1}{R}\right) \int_{Q_{\hat{L}}} |\hat{\nabla} \hat{\chi}| d\hat{x}$$

and

$$\int_{Q_{\hat{L}}} |\hat{\chi} - \hat{u}| d\hat{x} \leq C(n) \int_{Q_{\hat{L}}} |\hat{\nabla} \hat{\chi}| d\hat{x},$$

provided $\hat{\delta} \leq \delta_0$. Rescaling back, we notice that the constant for (46) only improves (by a factor $\frac{L}{L_0} < 1$ on the right hand side which we may drop) and (45) remains valid with $\delta = \frac{L}{L_0} \hat{\delta} \geq \frac{1}{L_0} \hat{\delta}$. As δ_0 may depend on L_0, this is not a problem and so the claim of the lemma is established for $1 \leq L \leq L_0$ when we replace the original δ_0 by $\frac{1}{L_0} \delta_0$.

Finally, we need to address the case $0 < L < 1$. Without loss of generality, we assume $\int_{Q_L} \chi dx \geq 0$. Obviously $u \equiv 1$ satisfies (45). We claim that it is also a good approximation in the sense of (46). Note that by our assumption of χ having non-negative average

$$\int_{Q_L} |\chi - u| dx = \int_{Q_L} |\chi - 1| dx = 2\mathscr{L}^n(\{x \in Q_L | x = -1\}) \leq 1,$$

so we are done if $\int_{Q_L} |\nabla\chi| dx \geq 1$. Otherwise, we can estimate similarly to (44) but this time using the Poincaré-Sobolev inequality

$$\begin{aligned}
\int_{Q_L} |\chi - u| dx &\leq \int_{Q_L} |\chi - L^{-n} \int_{Q_L} \chi| dx + \int_{Q_L} |1 - L^{-n} \int_{Q_L} \chi| dx \\
&= \int_{Q_L} |\chi - L^{-n} \int_{Q_L} \chi| dx + 2\mathscr{L}^n(\{x \in Q_L | x = -1\}) \\
&\leq 3 \int_{Q_L} |\chi - L^{-n} \int_{Q_L} \chi| dx \\
&\leq 3C_3(n) \Big(\int_{Q_L} |\nabla\chi| dx \Big)^{\frac{n}{n-1}} \\
&\leq C(n) \int_{Q_L} |\nabla\chi| dx,
\end{aligned}$$

and so (46) is verified, concluding the proof of the lemma. □

We now proceed to the core argument, where we decompose very large cubic domains into such of moderate size in order to improve above convergence result by eliminating the dependence of the approximation length scale δ on the system size L.

Lemma 10. *For all $R > 0$ there is a $\delta > 0$ such that for any system size $\tilde{L} > 0$ and for all functions $\chi : Q_{\tilde{L}} \to \{-1, 1\}$ periodically extended to $\mathbb{R}^n$ there exists a periodic approximation $u : Q_{\tilde{L}} \to [-1, 1]$ such that*

$$\int_{Q_{\tilde{L}}} \Big(\frac{\delta}{2} \frac{1}{1 - u^2} |\nabla u|^2 + \frac{1}{2\delta}(1 - u^2) \Big) dx \leq \Big(1 + \frac{1}{R}\Big) \int_{Q_{\tilde{L}}} |\nabla\chi| dx, \tag{49}$$

and

$$\int_{Q_{\tilde{L}}} |\chi - u| dx \leq C_4(n) \int_{Q_{\tilde{L}}} |\nabla\chi| dx. \tag{50}$$

Proof. It is well to develop a plan before delving into the minutiae. Our basic idea is to split $Q_{\tilde{L}}$, which we think of as being very large, into cubes of a suitably chosen intermediate size L. Then we apply Lemma 9 to these subcubes in order to obtain approximating functions on each piece and glue together one on $Q_{\tilde{L}}$. In proceeding we need to apply some care to appropriately choose the width Λ and position of the overlap during the cutting in order to keep a lid on $|\nabla\chi|$. We also need to make a considerate choice of the region of glueing to not lose the approximation property.

Let us now fix an arbitrary $\tilde{L}$-periodic $\chi : \mathbb{R}^n \to \{-1, 1\}$. Given δ small enough, smallness depending only on R, our goal is to construct some u satisfying

$$E_\delta(u, Q_{\tilde{L}}) \leq \Big(1 + C(n)\Big(\frac{1}{R} + \frac{\Lambda}{L} + \frac{R}{\Lambda}\Big)\Big) \int_{Q_{\tilde{L}}} |\nabla\chi| dx. \tag{51}$$

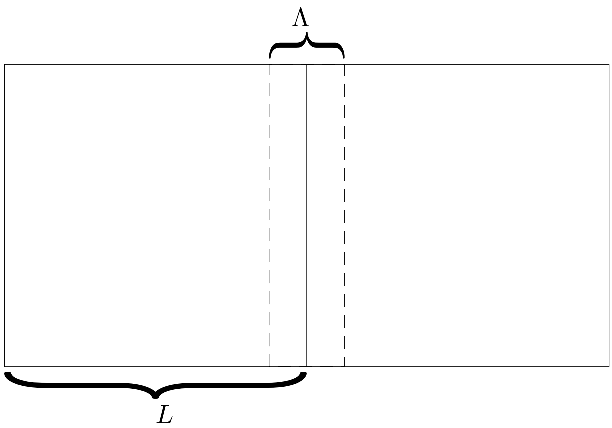

Figure 3: decomposing $Q_{\tilde{L}}$ into overlapping regions

Then we optimize the coefficient to determine Λ and L such that

$$\frac{1}{R} = \frac{\Lambda}{L} = \frac{R}{\Lambda},$$

i.e. we set

$$\Lambda = R^2 \qquad \text{and } L = R^3,$$

establishing (49) with a renaming of R to compensate the constant $3C(n)$. Thus we need to achieve (51) and (50) to prove the lemma.

To begin in earnest we decompose $\mathbb{R}^n$ into cubes $\{Q^k_{L-\Lambda}\}_{k\in\mathbb{Z}^n}$ of size $L-\Lambda > 0$ and denote their centers by $x_k = (L-\Lambda)k$. For given $k \in \mathbb{Z}^n$, let $Q^k_L = Q_L(x_k)$ be the cube of size L with the same center as $Q^k_{L-\Lambda}$. As alluded to above the cubes $\{Q^k_L\}_{k\in\mathbb{Z}^n}$ overlap with width Λ, see also Figure 3. Without loss of generality, we assume $R \geq 2$. This entails that the overlap width is not too large compared to the size of the Q^k_L, more precisely, it will be of use that

$$\Lambda \leq \frac{L}{2}. \tag{52}$$

For convenience we also assume that $\tilde{L} = M(L-\Lambda)$ for some $M \in \mathbb{N}$ so that M^n cubes $Q^k_{L-\Lambda}$ cover exactly one fundamental cell in the domain of the $\tilde{L}$-periodic functions. A variation of the rescaling used in the proof of Lemma 9 can be used to deal with nonintegral ratios greater than 1 and in the remaining case of small $\tilde{L}$ the present lemma does not claim any improvement over the previous. We remark that δ, which we want to depend only on (the dimension n and) the approximation quality R, may by above considerations also depend on the quantities L and Λ determined by R, a fact that shall be of use to us.

With these preparations, let us determine good areas of overlap, i.e. a good offset for the x_k. Using the $\tilde{L}$-periodicity we claim that there exists a translation vector $h \in \mathbb{R}^n$ such that

$$\sum_{k\in\{1,\ldots,M\}^n} \int_{Q_L(x_k+h)} |\nabla\chi| dx \leq \left(\frac{L}{L-\Lambda}\right)^n \int_{Q_{\tilde{L}}} |\nabla\chi| dx. \tag{53}$$

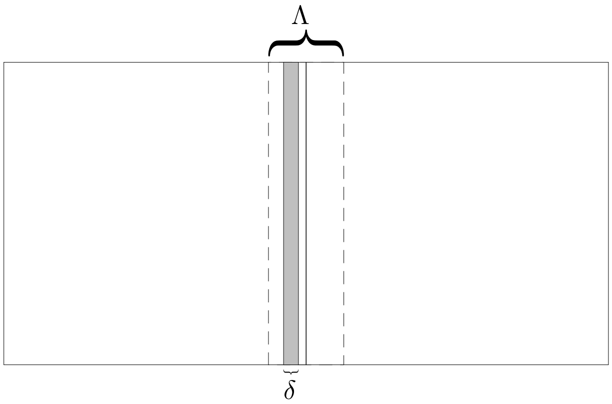

Figure 4: choosing a good set of stripes of width δ in the overlap

Indeed, we have for the average over $h' \in Q_{\tilde{L}}$

$$\frac{1}{\mathscr{L}^n(Q_{\tilde{L}})} \int_{Q_{\tilde{L}}} \sum_{k \in \{1,\ldots,M\}^n} \int_{Q_L(x_k+h)} |\nabla \chi| dx dh'$$
$$= \sum_{k \in \{1,\ldots,M\}^n} \frac{\mathscr{L}^n(Q_L^k)}{\mathscr{L}^n(Q_{\tilde{L}})} \int_{Q_{\tilde{L}}} |\nabla \chi| dx$$
$$= M^n \frac{L^n}{M^n (L-\Lambda)^n} \int_{Q_{\tilde{L}}} |\nabla \chi| dx,$$

and there must be an h for which the integrant is bounded by the average. Without loss of generality, we assume $h = 0$.

According to Lemma 9 there exists a $0 < \delta = \delta(L, R) \leq \frac{L}{2}$ with the property that for any $k \in \mathbb{Z}^n$ a function $u_k : Q_L^k \to [-1, 1]$ exists such that

$$\int_{Q_L^k} \frac{\delta}{2} \frac{1}{1-u_k^2} |\nabla u_k|^2 + \frac{1}{2\delta}(1-u_k^2) dx \leq \Big(1 + \frac{1}{R}\Big) \int_{Q_L^k} |\nabla \chi| dx \tag{54}$$

and

$$\int_{Q_L^k} |\chi - u_k| dx \leq C(n) \int_{Q_L^k} |\nabla \chi| dx. \tag{55}$$

For given $k \in \mathbb{Z}^n$, consider $Q_{L-\Lambda+\delta/2}(x_k + h) \subset Q_L^k$ for translation vectors $h \in Q_{\Lambda-\delta/2}(0)$. In order to be able to glue functions together we are interested in the approximation quality in the boundary layer of thickness δ, i.e. the set $Q_{L-\Lambda+\delta/2}(x_k+h) \setminus Q_{L-\Lambda-\delta/2}(x_k+h)$. We claim that there exists $h \in Q_{\Lambda-\delta/2}(0)$ such that

$$\sum_{k \in \{1,\ldots,M\}^n} \int_{h+(Q^k_{L-\Lambda+\delta/2} \setminus Q^k_{L-\Lambda-\delta/2})} |u_k - \chi| dx \leq \frac{4n\delta}{\Lambda} \sum_{k \in \{1,\ldots,M\}^n} \int_{Q_L^k} |u_k - \chi| dx. \tag{56}$$

This is shown similarly to (53), this time with a one-dimensional optimization (see Figure 4):

Considering stripes

$$S^k(h^1) := Q_L^k \cap \{x | x^1 \in x_k^1 + h^1 + \big([-(L-\Lambda)/2 - \delta/2, -(L-\Lambda)/2 + \delta/2) \\ \cup [(L-\Lambda)/2 - \delta/2, (L-\Lambda)/2 + \delta/2)\big)\}$$

there is a h^{1*} with

$$\begin{aligned}
&\sum_{k\in\{1,\dots,M\}^n} \int_{S^k(h^{1*})} |u_k - \chi| dx \\
&\le \frac{1}{\Lambda - \delta/2} \int_{-\Lambda/2+\delta/4}^{\Lambda/2-\delta/4} \sum_{k\in\{1,\dots,M\}^n} \int_{S^k(h_1)} |u_k - \chi| dx\, dh^1 \\
&\le \frac{2\delta}{\Lambda - \delta/2} \sum_{k\in\{1,\dots,M\}^n} \int_{Q_L^k} |u_k - \chi| dx.
\end{aligned}$$

As noted in the beginning of the proof Λ depends only on R, so we may assume $\delta \le \Lambda/2$. Optimization for and summation over all coordinate directions yields the desired estimate (56).

Let $\{\eta_k : \mathbb{R}^n \to [0,1]\}_{k\in\mathbb{Z}^n}$ be a partition of unity subordinate to $Q_{L-\Lambda+\delta/2}(x_k + h)$. More precisely we ask that

$$\sum_{k\in\mathbb{Z}^n} \eta_k = 1 \text{ in } \mathbb{R}^n, \tag{57}$$

$$\eta_k \equiv 1 \text{ on } Q_{L-\Lambda-\delta/2}(x_k + h),$$

$$\eta_k = 0 \text{ on } \mathbb{R}^n \setminus Q_{L-\Lambda+\delta/2}(x_k + h) \supset \mathbb{R}^n \setminus Q_L^k. \tag{58}$$

In addition we choose η_k such that

$$|\nabla \eta_k|^2 \le \frac{C}{\delta^2} \eta_k (1 - \eta_k). \tag{59}$$

Let us emphasize that this partition of unity is uniformly locally finite in the sense that for any k the number of cutoff functions with support overlapping that of η_k is bounded by a constant depending only on n, i.e. for all $k \in \mathbb{Z}^n$

$$\#\{k' \in \mathbb{Z}^n |\, \operatorname{supp} \eta_{k'} \cap \operatorname{supp} \eta_k \ne \emptyset\} \le C(n). \tag{60}$$

We can now define the $\tilde{L}$-periodic function $u : \mathbb{R}^n \to [-1,1]$ as $u = \sum_{k\in\mathbb{Z}^n} \eta_k u_k$ and set out to verify (51) and later (50). We begin by noticing that

$$\begin{aligned}
1-u^2 &= 1-\sum_{k\in\mathbb{Z}^n}\sum_{k'\in\mathbb{Z}^n}\eta_k\eta_{k'}u_ku_{k'}\\
&= 1-\sum_{k\in\mathbb{Z}^n}\eta_k^2u_k^2-\sum_{k\in\mathbb{Z}^n}\sum_{k'\neq k}\eta_k\eta_{k'}u_ku_{k'}\\
&= 1-\sum_{k\in\mathbb{Z}^n}\eta_ku_k^2+\sum_{k\in\mathbb{Z}^n}\Big((\eta_k-\eta_k^2)u_k^2-\sum_{k'\neq k}\eta_k\eta_{k'}u_ku_{k'}\Big)\\
&\overset{(57)}{=} \sum_{k\in\mathbb{Z}^n}\eta_k(1-u_k^2)+\sum_{k\in\mathbb{Z}^n}\Big(\eta_k(1-\eta_k)u_k^2-\sum_{k'\neq k}\eta_k\eta_{k'}u_ku_{k'}\Big)\\
&\overset{(57)}{=} \sum_{k\in\mathbb{Z}^n}\eta_k(1-u_k^2)+\sum_{k\in\mathbb{Z}^n}\sum_{k'\neq k}\Big(\eta_k\eta_{k'}u_k^2-\eta_k\eta_{k'}u_ku_{k'}\Big)\\
&= \sum_{k\in\mathbb{Z}^n}\eta_k(1-u_k^2)+\sum_{k\in\mathbb{Z}^n}\sum_{k'\neq k}\eta_k\eta_{k'}u_k(u_k-u_{k'})\\
&= \sum_{k\in\mathbb{Z}^n}\eta_k(1-u_k^2)+\frac{1}{2}\sum_{k\in\mathbb{Z}^n}\sum_{k'\in\mathbb{Z}^n}\eta_k\eta_{k'}(u_k-u_{k'})^2 \qquad (61)\\
&\overset{(57)}{=} \frac{1}{2}\sum_{k\in\mathbb{Z}^n}\sum_{k'\in\mathbb{Z}^n}\eta_k\eta_{k'}\left((1-u_k^2)+(1-u_{k'}^2)+(u_k-u_{k'})^2\right). \qquad (62)
\end{aligned}$$

Using $\sum_{k\in\mathbb{Z}^n}\nabla\eta_k\overset{(57)}{=}0$ we see that

$$\begin{aligned}
\nabla u &= \sum_{k\in\mathbb{Z}^n}\eta_k\nabla u_k+\sum_{k\in\mathbb{Z}^n}u_k\nabla\eta_k\\
&= \sum_{k\in\mathbb{Z}^n}\eta_k\nabla u_k+\sum_{k\in\mathbb{Z}^n}\Big(u_k-\sum_{k'\in\mathbb{Z}}\eta_{k'}u_{k'}\Big)\nabla\eta_k\\
&\overset{(57)}{=} \sum_{k\in\mathbb{Z}^n}\eta_k\nabla u_k+\sum_{k\in\mathbb{Z}}\sum_{k'\in\mathbb{Z}}(u_k-u_{k'})\eta_{k'}\nabla\eta_k. \qquad (63)
\end{aligned}$$

We can thus estimate with Young's inequality

$$|\nabla u|^2\le\Big(1+\frac{1}{R}\Big)\Big|\sum_{k\in\mathbb{Z}^n}\eta_k\nabla u_k\Big|^2+(1+R)\Big|\sum_{k\in\mathbb{Z}}\sum_{k'\in\mathbb{Z}}(u_k-u_{k'})\eta_{k'}\nabla\eta_k\Big|^2.$$

Combining this with (61) and (62) we get

$$\begin{aligned}
\frac{1}{1-u^2}|\nabla u|^2 &\le \Big(1+\frac{1}{R}\Big)\frac{\big|\sum_{k\in\mathbb{Z}^n}\eta_k\nabla u_k\big|^2}{\sum_{k\in\mathbb{Z}^n}\eta_k(1-u_k^2)}\\
&\quad+(1+R)\frac{\big|\sum_{k\in\mathbb{Z}}\sum_{k'\in\mathbb{Z}}(u_k-u_{k'})\eta_{k'}\nabla\eta_k\big|^2}{\frac{1}{2}\sum_{k\in\mathbb{Z}^n}\sum_{k'\in\mathbb{Z}^n}\eta_k\eta_{k'}\left((1-u_k^2)+(1-u_{k'}^2)+(u_k-u_{k'})^2\right)}.
\end{aligned}$$

We use the convexity of $(v,g)\mapsto\frac{1}{v}|g|^2$ on $(0,\infty)\times\mathbb{R}^n$ to estimate by pulling the (locally finite) summation in the first term out of the fraction and obtain

$$\begin{aligned}
\frac{1}{1-u^2}|\nabla u|^2 &\le \Big(1+\frac{1}{R}\Big)\sum_{k\in\mathbb{Z}^n}\eta_k\frac{|\nabla u_k|^2}{1-u_k^2}\\
&\quad+(1+R)\frac{\big|\sum_{k\in\mathbb{Z}}\sum_{k'\in\mathbb{Z}}(u_k-u_{k'})\eta_{k'}\nabla\eta_k\big|^2}{\frac{1}{2}\sum_{k\in\mathbb{Z}^n}\sum_{k'\in\mathbb{Z}^n}\eta_k\eta_{k'}\left((1-u_k^2)+(1-u_{k'}^2)+(u_k-u_{k'})^2\right)}.
\end{aligned}$$

In combination with (61), this entails

$$
\begin{aligned}
&\frac{\delta}{2}\frac{1}{1-u^2}|\nabla u|^2+\frac{1}{2\delta}(1-u^2)\\
&\le\Big(1+\frac{1}{R}\Big)\sum_{k\in\mathbb{Z}^n}\eta_k\Big(\frac{\delta}{2}\frac{|\nabla u_k|^2}{1-u_k^2}+\frac{1}{2\delta}(1-u_k^2)\Big)\\
&\quad+(1+R)\frac{\delta\big|\sum_{k\in\mathbb{Z}}\sum_{k'\in\mathbb{Z}}(u_k-u_{k'})\eta_{k'}\nabla\eta_k\big|^2}{\sum_{k\in\mathbb{Z}^n}\sum_{k'\in\mathbb{Z}^n}\eta_k\eta_{k'}\left((1-u_k^2)+(1-u_{k'}^2)+(u_k-u_{k'})^2\right)}\\
&\quad+\frac{1}{4\delta}\sum_{k\in\mathbb{Z}^n}\sum_{k'\in\mathbb{Z}^n}\eta_k\eta_{k'}(u_k-u_{k'})^2\\
&=: S_1+S_2+S_3. \qquad (64)
\end{aligned}
$$

We address the terms on the right hand side separately. Starting with S_1 we write

$$
\begin{aligned}
\int_{Q_{\tilde L}}S_1dx &= \Big(1+\frac{1}{R}\Big)\int_{Q_{\tilde L}}\sum_{k\in\mathbb{Z}^n}\eta_k\Big(\frac{\delta}{2}\frac{|\nabla u_k|^2}{1-u_k^2}+\frac{1}{2\delta}(1-u_k^2)\Big)dx\\
&\overset{(58)}{\le}\Big(1+\frac{1}{R}\Big)\sum_{k\in\{1,\dots,M\}^n}E_\delta\big(u_k,Q_{L-\Lambda+\delta/2}(x_k+h)\big)\\
&\le\Big(1+\frac{1}{R}\Big)\sum_{k\in\{1,\dots,M\}^n}E_\delta\big(u_k,Q_L^k\big)\\
&\overset{(54)}{\le}\Big(1+\frac{1}{R}\Big)^2\sum_{k\in\{1,\dots,M\}^n}\int_{Q_L^k}|\nabla\chi|dx\\
&\overset{(53)}{\le}\Big(1+\frac{1}{R}\Big)^2\Big(\frac{L}{L-\Lambda}\Big)^n\int_{Q_{\tilde L}}|\nabla\chi|dx. \qquad (65)
\end{aligned}
$$

We proceed to estimate S_2+S_3 at any point $x\in\mathbb{R}^n$. To this end, assume without loss of generality $\chi(x)=1$ and let $J=J(x)=\{k\in\mathbb{Z}^n|x\in\operatorname{supp}\eta_k\}$. Using the local finiteness (60) and $R \geq 1$ we see

$$
\begin{aligned}
& S_2 + S_3 \\
= \quad & (1+R)\frac{\delta\big|\sum_{k\in\mathbb{Z}}\sum_{k'\in\mathbb{Z}}(u_k-u_{k'})\eta_{k'}\nabla\eta_k\big|^2}{\sum_{k\in\mathbb{Z}^n}\sum_{k'\in\mathbb{Z}^n}\eta_k\eta_{k'}\left((1-u_k^2)+(1-u_{k'}^2)+(u_k-u_{k'})^2\right)} \\
& +\frac{1}{4\delta}\sum_{k\in\mathbb{Z}^n}\sum_{k'\in\mathbb{Z}^n}\eta_k\eta_{k'}(u_k-u_{k'})^2 \\
\leq \quad & C(n)R\frac{\delta\sum_{k\in\mathbb{Z}}\sum_{k'\in\mathbb{Z}}(u_k-u_{k'})^2\eta_{k'}^2|\nabla\eta_k|^2}{\sum_{k\in\mathbb{Z}^n}\sum_{k'\in\mathbb{Z}^n}\eta_k\eta_{k'}\left((1-u_k^2)+(1-u_{k'}^2)+(u_k-u_{k'})^2\right)} \\
& +\frac{1}{4\delta}\sum_{k\in\mathbb{Z}^n}\sum_{k'\in\mathbb{Z}^n}\eta_k\eta_{k'}(u_k-u_{k'})^2 \\
\overset{(59)}{\leq} \quad & \frac{1}{\delta}\sum_{k\in\mathbb{Z}^n}\sum_{k'\in\mathbb{Z}^n}\eta_k\eta_{k'}(u_k-u_{k'})^2 \\
& \quad \Big(\frac{1}{4}+\frac{C(n)R\eta_{k'}(1-\eta_k)}{\sum_{k\in\mathbb{Z}^n}\sum_{k'\in\mathbb{Z}^n}\eta_k\eta_{k'}\left((1-u_k^2)+(1-u_{k'}^2)+(u_k-u_{k'})^2\right)}\Big) \\
\overset{(\eta_k\leq 1)}{\leq} \quad & \frac{1}{\delta}\sum_{k\in\mathbb{Z}^n}\sum_{k'\in\mathbb{Z}^n}\eta_k\eta_{k'}(u_k-u_{k'})^2\Big(\frac{1}{4}+\frac{C(n)R}{(1-u_k^2)+(1-u_{k'}^2)+(u_k-u_{k'})^2}\Big). \qquad (66)
\end{aligned}
$$

We now claim

$$
\frac{(u_k-u_{k'})^2}{(1-u_k^2)+(1-u_{k'}^2)+(u_k-u_{k'})^2} \leq |u_k-u_{k'}|+(1-u_k)+(1-u_{k'}). \qquad (67)
$$

If $u_k \leq 0$ or $u_{k'} \leq 0$, the left hand side smaller than 1 while the right hand side is larger, so that the inequality is trivial in this case. For $u_k \geq 0$ and $u_{k'} \geq 0$ we start with the elementary observation

$$
(a-b)^2 \leq |a-b||a+b|,
$$

which, for $a, b \geq 0$ is equivalent to

$$
\frac{(a-b)^2}{a+b} \leq |a-b|.
$$

Plugging in $a = 1-u_k$ and $b = 1-u_{k'}$, this becomes

$$
\frac{(u_k-u_{k'})^2}{(1-u_k)+(1-u_{k'})} \leq |u_k-u_{k'}|,
$$

which, by $1-u^2 = (1-u)(1+u) \geq 1-u$ for $u \geq 0$ and adding non-negative terms to the denominator and right hand side implies

$$
\begin{aligned}
& \frac{(u_k-u_{k'})^2}{(1-u_k^2)+(1-u_{k'}^2)+(u_k-u_{k'})^2} \\
& \qquad \leq \frac{(u_k-u_{k'})^2}{(1-u_k)+(1-u_{k'})} \leq |u_k-u_{k'}| \leq |u_k-u_{k'}|+(1-u_k)+(1-u_{k'}).
\end{aligned}
$$

Thus (67) is established.

We can now continue with our estimation (66), we start with using $(u_1-u_2)^2 \le 2|u_1-u_2|$

$$
\begin{aligned}
&S_2+S_3 \\
&\overset{(66)}{\le} \quad \frac{1}{\delta}\sum_{k\in\mathbb{Z}^n}\sum_{k'\in\mathbb{Z}^n}\eta_k\eta_{k'}(u_k-u_{k'})^2\Big(\frac{1}{4}+\frac{C(n)R}{(1-u_k^2)+(1-u_{k'}^2)+(u_k-u_{k'})^2}\Big) \\
&\le \quad \frac{1}{\delta}\sum_{k\in\mathbb{Z}^n}\sum_{k'\in\mathbb{Z}^n}\eta_k\eta_{k'}\Big(\frac{1}{2}|u_k-u_{k'}|+\frac{C(n)R(u_k-u_{k'})^2}{(1-u_k^2)+(1-u_{k'}^2)+(u_k-u_{k'})^2}\Big) \\
&\overset{(67)}{\le} \quad \frac{C(n)R}{\delta}\sum_{k\in\mathbb{Z}^n}\sum_{k'\ne k}\eta_k\eta_{k'}(|u_k-u_{k'}|+(1-u_k)+(1-u_{k'})) \\
&\le \quad \frac{C(n)R}{\delta}\sum_{k\in\mathbb{Z}^n}\sum_{k'\ne k}\eta_k\eta_{k'}(|\chi-u_k|+|\chi-u_{k'}|) \\
&\overset{(57)}{\le} \quad \frac{C(n)R}{\delta}\sum_{k\in\mathbb{Z}^n}\eta_k(1-\eta_k)|\chi-u_k|,
\end{aligned}
\tag{68}
$$

in the last two estimates we use the triangle inequality and our assumption $\chi(x)=1$.

Using our choice of the boundary layer we estimate the integral over S_2+S_3 (which are supported only on the boundary layers) as

$$
\begin{aligned}
&\int_{Q_{\tilde{L}}} S_2+S_3dx \\
&\overset{(68)}{\le} \quad \frac{C(n)R}{\delta}\sum_{k\in\{1,\dots,M\}^n}\int_{Q_{\tilde{L}}}\eta_k(1-\eta_k)|\chi-u_k|dx \\
&\overset{(58)}{\le} \quad \frac{C(n)R}{\delta}\sum_{k\in\{1,\dots,M\}^n}\int_{Q_L^k}\eta_k(1-\eta_k)|\chi-u_k|dx \\
&\overset{(56),(58)}{\le} \quad \frac{C(n)R}{\Lambda}\sum_{k\in\{1,\dots,M\}^n}\int_{Q_L^k}|\chi-u_k|dx \\
&\overset{(55)}{\le} \quad \frac{C(n)R}{\Lambda}\sum_{k\in\{1,\dots,M\}^n}\int_{Q_L^k}|\nabla\chi|dx \\
&\overset{(53),(52)}{\le} \quad \frac{C(n)R}{\Lambda}\int_{Q_{\tilde{L}}}|\nabla\chi|dx.
\end{aligned}
\tag{69}
$$

Combining (64), (65), and (69), we see that

$$
\begin{aligned}
&\int_{Q_{\tilde{L}}}\frac{\delta}{2}\frac{1}{1-u^2}|\nabla u|^2+\frac{1}{2\delta}(1-u^2)dx \\
&\overset{(64)}{\le} \quad \int_{Q_{\tilde{L}}} S_1+S_2+S_3dx \\
&\overset{(65),(69)}{\le} \quad \Big(\Big(1+\frac{1}{R}\Big)^2\Big(\frac{L}{L-\Lambda}\Big)^n+\frac{C(n)R}{\Lambda}\Big)\int_{Q_{\tilde{L}}}|\nabla\chi|dx \\
&\overset{(52),(R\ge1)}{\le} \quad \Big(1+C(n)\Big(\frac{1}{R}+\frac{\Lambda}{L}+\frac{R}{\Lambda}\Big)\Big)\int_{Q_{\tilde{L}}}|\nabla\chi|dx.
\end{aligned}
$$

But this is (51), which we know from above to imply (49). To complete the proof of the lemma we need to verify the approximation property (50). By definition of u

$$\begin{aligned}\int_{Q_{\tilde L}} |\chi - u| dx &= \int_{Q_{\tilde L}} \Big|\chi - \sum_{k\in\mathbb{Z}^n} \eta_k u_k\Big| dx \\ &\overset{(57)}{=} \int_{Q_{\tilde L}} \Big|\sum_{k\in\mathbb{Z}^n} \eta_k(\chi - u_k)\Big| dx \\ &\overset{(58)}{\leq} \sum_{k\in\{1,\ldots,M\}^n} \int_{Q_L^k} \Big|\chi - u_k\Big| dx \\ &\overset{(55)}{\leq} C(n) \sum_{k\in\{1,\ldots,M\}^n} \int_{Q_L^k} |\nabla\chi| dx \\ &\overset{(53)}{\leq} C(n)\Big(\frac{L}{L-\Lambda}\Big)^n \int_{Q_{\tilde L}} |\nabla\chi| dx \\ &\overset{(52)}{\leq} C(n) \int_{Q_{\tilde L}} |\nabla\chi| dx,\end{aligned}$$

which is (50), the missing piece in the proof of our lemma. □

Finally, we prove Proposition 1.

Proof of Proposition 1. Let $\alpha > 0$ be given and set $R = \frac{1}{\alpha}$. Denote by $\hat\delta = \hat\delta(R)$ the parameter of Lemma 10 and let $C_5(n,R) = \frac{1}{\hat\delta} C_4(n)$.

We rescale the lengths according to

$$x = \frac{\delta}{\hat\delta}\hat x, \qquad L = \frac{\delta}{\hat\delta}\hat L.$$

According to Lemma 10, there exists $\hat u : Q_{\hat L} \to [-1,1]$ such that

$$\int_{Q_{\hat L}} \left(\frac{\hat\delta}{2}\frac{1}{1-\hat u^2}|\hat\nabla \hat u|^2 + \frac{1}{2\hat\delta}(1-\hat u^2)\right) d\hat x \leq \Big(1+\frac{1}{R}\Big) \int_{Q_{\hat L}} |\hat\nabla\hat\chi| d\hat x,$$

and

$$\int_{Q_{\hat L}} |\hat\chi - \hat u| d\hat x \leq C_4(n) \int_{Q_{\hat L}} |\hat\nabla\hat\chi| d\hat x.$$

Rescaling back this gives $u : Q_L \to [-1,1]$ such that

$$\int_{Q_L} \left(\frac{\hat\delta}{2}\frac{1}{1-u^2}\Big(\frac{\delta}{\hat\delta}\Big)^2|\nabla u|^2 + \frac{1}{2\hat\delta}(1-u^2)\right) dx \leq \Big(1+\frac{1}{R}\Big)\frac{\delta}{\hat\delta} \int_{Q_L} |\nabla\chi| dx,$$

and

$$\int_{Q_L} |\chi - u| dx \leq C_4(n)\frac{\delta}{\hat\delta} \int_{Q_L} |\nabla\chi| dx$$

as desired. □

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9 Appendix

For the reader's convenience we collect some facts related to the treatment of the stray field and our notation involving the inverted divergence in this appendix.

As the magnetization induces a stray field h, the conceptually simplest way to include its contribution to the energy is to explicitly include the squared L^2-norm

$$\int_{\mathbb{R}^3} |h|^2 dx$$

in the energy. The stray field h satisfies Maxwell's equations (greatly reduced in the magnetostatic case to)

$$\nabla \cdot (h + m) = 0 \text{ and } \quad \nabla \times h = 0, \tag{70}$$

both understood in the sense of distributions on $\mathbb{R}^3$. For notes on the derivation, see e.g. [DKMO05]. Being curl-free, h is a gradient field and, in fact, the Helmholtz projection of $-m$ onto the space of gradient fields. One way to compute h is setting $h = -\nabla u$ where

$$\begin{aligned} \Delta u &= \nabla \cdot m && \text{in } \Omega, \\ \left[\tfrac{\partial u}{\partial \nu}\right] &= m \cdot \nu && \text{on } \partial\Omega, \\ \Delta u &= 0 && \text{outside } \Omega. \end{aligned} \tag{71}$$

We can similarly define h for periodic domains, then (71) reduces to the first equation $\Delta u = \nabla \cdot m$.

An alternative approach to the stray-field energy is to include h in the minimization in order to make the problem more local. Observe that the L^2-norm of $(-l, l)^2$-periodic $h : \mathbb{R}^3 \to \mathbb{R}^3$ defined by (70) can be rewritten in terms of the minimization problem

$$\int_{(-l,l)^2 \times \mathbb{R}} |h|^2 dx = \min \Big\{ \int_{(-l,l)^2 \times \mathbb{R}} |\tilde{h}|^2 dx \Big| \, \tilde{h} : \mathbb{R}^3 \to \mathbb{R}^3 \text{ is } (-l,l)^2\text{-periodic in } x', \\ \nabla \cdot (\tilde{h} + m) = 0 \text{ distributionally in } \mathbb{R}^3 \Big\}, \tag{72}$$

and the second equation in (70) is just the Euler-Lagrange-equation for the minimization. Hence, setting

$$e_{Q,d,t,l}(m, h) := \frac{1}{4l^2} \left(d^2 \int_{\Omega} |\nabla m|^2 dx + Q \int_{\Omega} |m'|^2 dx + \int_{\mathbb{R}^3} |h|^2 dx \right)$$

we have

$$e(Q, d, t, l) = \min \Big\{ e_{Q,d,t,l}(m, h) \Big| \, m, h : \mathbb{R}^3 \to \mathbb{R}^3 \text{ are } (-l,l)^2\text{-periodic in } x', \\ |m|^2 = \begin{cases} 1 & \text{for } x_3 \in (-t, t), \\ 0 & \text{otherwise}, \end{cases} \\ \nabla \cdot (h + m) = 0 \text{ distributionally in } \mathbb{R}^3 \Big\}.$$

There is a third way to think about h that we want to illustrate with the stray-field term in the reduced energy concerning $m_3 : \mathbb{R}^3 \to \{-1,1\}$ and $h' : \mathbb{R}^3 \to \mathbb{R}^2$, both $(-l,l)^2$-periodic in x_3 and satisfying $m_3^2 = 1$ if $x_3 \in (-1,1)$ and $m_3^2 = 0$ otherwise and

$$\nabla' h + \partial_3 m_3 = 0.$$

We are tempted to invert the operator ∇' in the above equation and indeed define for any distribution f (with zero slicewise average)

$$\int_{(-l,l)^2\times\mathbb{R}} \left||\nabla'|^{-1} f\right|^2 dx = \min\Big\{ \int_{(-l,l)^2\times\mathbb{R}} |\tilde{h}'|^2 dx \Big| \, \tilde{h} : \mathbb{R}^3 \to \mathbb{R}^3 \text{ is } (-l,l)^2\text{-periodic in } x', \\ \nabla' \cdot \tilde{h}' = f \text{ distributionally in } \mathbb{R}^3 \Big\}$$

and can thus write in the spirit of (72)

$$\int_{(-l,l)^2\times\mathbb{R}} |h'|^2 dx = \int_{(-l,l)^2\times\mathbb{R}} \left||\nabla'|^{-1} \partial_3 m\right|^2 dx.$$

Another way to look at $|\nabla'|^{-1}$ is by taking Fourier series in x'-direction. With

$$\mathcal{F}'(\zeta)(n') = \frac{1}{2l} \int_{(-l,l)^2} \exp\left(-\pi i n' \cdot \frac{x'}{l}\right) \zeta(x') dx'$$

and

$$\int_{(-l,l)^2\times\mathbb{R}} \left||\nabla'|^{-1} f\right|^2 dx = \int_{\mathbb{R}} \sum_{n'\in\mathbb{Z}^2} \frac{l^2}{\pi^2 |n'|^2} |(\mathcal{F}' f)(n')|^2 dx_3$$

we can rewrite the energy as

$$\int_{(-l,l)^2\times\mathbb{R}} |h'|^2 dx = \int_{(-l,l)^2\times\mathbb{R}} \left||\nabla'|^{-1} \partial_3 m_3\right|^2 dx = \int_{\mathbb{R}} \sum_{n'\in\mathbb{Z}^2} \frac{l^2}{\pi^2 |n'|^2} |(\mathcal{F}'(\partial_3 m_3))(n')|^2 dx_3.$$

This also aligns well to the method of defining the energy via (71), when we plug in the usual Fourier-series solution formula for Poisson's equation on periodic domains.

Let us briefly look at the rôle of this inverse norm as a dual of the H^1-seminorm making a brief appearance in the proof of the interpolation inequality Lemma 2. Fix two $(-l,l)^2$-periodic functions f, g with average 0, thought of as smooth, and let u be a solution to $\Delta u = g$. Then by the divergence theorem and the Cauchy-Schwarz inequality, the duality estimate is but a simple calculation

$$\begin{aligned} \int_{(-l,l)^2} f g \, dx' &= -\int_{(-l,l)^2} f \Delta u \, dx' \\ &= \int_{(-l,l)^2} \nabla f \cdot \nabla u \, dx' \\ &\le \left(\int_{(-l,l)^2} |\nabla f|^2 dx' \right)^{1/2} \left(\int_{(-l,l)^2} |\nabla u|^2 dx' \right)^{1/2} \\ &= \left(\int_{(-l,l)^2} |\nabla f|^2 dx' \right)^{1/2} \left(\int_{(-l,l)^2} \left||\nabla|^{-1} g\right|^2 dx' \right)^{1/2}. \end{aligned} \tag{73}$$

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