A Rigidity Result for a Perturbation of the Geometrically Linear Three-Well Problem

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Abstract

We study a 3-d model for alloys which undergo a cubic-to-tetragonal phase transition, in the martensitic phase. Any pair of the three martensitic variants can form a stress-free laminate. However, this laminate is only compatible on average with the remaining variant. The resulting local stresses favor a microstructure if all three variants are present (for instance because of an externally imposed average strain).

Next to the linearized elastic energy, the variational model features an interfacial energy between the three variants. This introduces a material length scale, which together with the sample size (which we mimic by periodic boundary conditions) gives rise to a non-dimensional parameter η.

We rigorously establish the scaling of the minimal energy e per volume in case of an externally imposed volume fractions of the martensitic variants in η. More precisely, we show $e \sim \eta^{2/3}$. The upper bound construction is achieved by few patches of branched laminates, the lower bound relies on suitable interpolation inequalities. This is in the spirit of a celebrated work by Kohn & Müller and relies on techniques developed for domain branching in micromagnetics.

We also prove a rigidity result in the sense that if the energy per volume e of a configuration is much smaller than $\eta^{2/3}$, the configuration is approximately a simple unbranched laminate. In particular, one of the three volume fractions has to be small. This is related to similar rigidity results by Dolzmann & Müller and by Kirchheim.

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1 Introduction

In the study of solid-solid phase transitions, and in particular of shape memory alloys, one is concerned with variational models of the form

$$\int_\Omega W(\nabla u)\,dx, \tag{1}$$

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where u represents the displacement. The energy density $W : \mathbb{R}^{3\times 3} \to \mathbb{R}$ is minimized in sets of the form

$$K = SO(3)U_1 \cup \cdots \cup SO(3)U_m;$$

$U_i \in \mathbb{R}^{3\times 3}$ are the stress-free strains for the different phases, and $SO(3)$ is the set of proper rotations. The existence of gradient fields ∇u taking values in K is referred to as the m-well problem. The simplest nontrivial solutions to this problem are the so called simple laminates. That is, functions u whose gradient takes only two values $\nabla u = U_a$ and $\nabla u = RU_b$, $R \in SO(3)$, and are constant along hyperplanes perpendicular to some vector n. It is easy to see that this is only possible if U_a and U_b are rank-one connected, that is $U_a - RU_b = a \otimes n$ for some vector $a \in \mathbb{R}^3$. Depending on the form of U_a and U_b there can be zero, one or two rank-one connections. Since in the case of two rank-one connections one can argue that two different simple laminates cannot cross, one may be tempted to conjecture that all solutions are of this type. In general this *rigidity* result is false [13].

The first proof of a rigidity result is due to Dolzmann and Müller. For the nonlinear two-well problem (i.e. $m = 2$) they show in [8] that, if the set where the gradient lies in one of the wells has finite perimeter, the zero energy states consist of simple laminates. This result was later generalized by Kirchheim [10] to the case of three wells. In [9], Dolzmann and Müller also prove that for the geometrically linearized three-well problem the rigidity holds. In the later case no assumption on the perimeter of the transition is required.

The first quantitative version of Dolzmann and Müller's result, for the nonlinear two-well problem, was proved by Lorent [12] for bi-Lipshitz maps and two-wells with equal determinant. More precisely, he proved that if the perimeter of the transition in a ball, as controlled by $\|D^2u\|(B_1)$, is sufficiently small, there exist a certain power $\gamma < 1$ of the L^1-norm of the distance between ∇u and the set K that controls the L^1-norm of the distance of ∇u to one of the wells in some smaller ball. Lorent's result was later improved by Conti and Schweizer [7] to $\gamma = 1$, in the case of two wells with positive determinant, and maps $u \in W^{1,1}$ such that $\|D^2u\|(\Omega)$ is small compared with a quantity that depends on the geometry of Ω. Chermisi and Conti [3] generalized the later result for any number of *well-separated wells* and $u \in W^{1,1}$ with $\nabla u \in BV$, satisfying as before a smallness condition on the perimeter of the transitions.

In this paper we focus on the effect of surface energy in low-energy states of the geometrically linearized version of (1) of the three-well problem. In particular we study cubic-to-tetragonal phase transitions (that sets the form of the wells) and include a penalization of the interfacial energy between variants (see (2) below). The particular form of the interfacial energy term is debated in the physical literature. However, there is a consensus that such an energy contribution has to be included in any realistic model.

We consider an energy of the form

$$\begin{aligned} E(\chi) &= A\int_\Omega |\nabla(\chi_1-\chi_2)| + |\nabla(\chi_2-\chi_3)| + |\nabla(\chi_3-\chi_1)|dx \qquad (2) \\ &+ \inf_{\substack{u\,:\,\mathbb{R}^3\to\mathbb{R}^3 \\ \nabla u\ \Omega-\text{periodic}}} \left\{ \mathbb{C}\int_\Omega \left|\frac{1}{2}(\nabla u + (\nabla u)^T) - \chi_1 U_1 - \chi_2 U_2 - \chi_3 U_3\right|^2 dx \right\}, \end{aligned}$$

where $\Omega = (0, \ell)^3 \subset \mathbb{R}^3$ is a cubic domain of lateral size ℓ, and $\chi = (\chi_1, \chi_2, \chi_3)$ is an order parameter that indicates the support of the three different phases (in this case the martensitic variants). More precisely we have that $\chi_i : \mathbb{R}^3 \to \{0, 1\}$ for $i = 1, 2, 3$ are Ω–periodic and such that

$$\chi_1 + \chi_2 + \chi_3 = 1 \qquad \text{in } \Omega. \tag{3}$$

Using the undistorted austenitic phase as a reference configuration, the preferred values of the strain tensor (that correspond to the three wells of the martensitic phase) are given by

$$U_i = \varepsilon_0(Id - 3e_i \otimes e_i), \qquad i = 1, 2, 3 \tag{4}$$

where ε_0 represents the deformation parameter and $\{e_i\}$ is the canonical basis of $\mathbb{R}^3$.

The first term in (2) models the interfacial energy of the transition between variants, and the second one models the geometrically linear bulk-energy response of the crystal. The material parameters A and $\mathbb{C}$ represent the interfacial energy per area and the elastic modulus, respectively. We note that the assumption of periodic boundary conditions is a rather convenient one. It serves as a model of the sample size, and let us to define negative Sobolev norms in terms of Fourier series simplifying the analysis carried out in this paper.

In Theorem 1 we show that, in the regime

$$\frac{A}{\ell} \frac{1}{\mathbb{C}\, \varepsilon_0^2} \ll 1, \tag{5}$$

if we have

$$E(\chi) \ll \left(\frac{A}{\ell}\right)^{2/3} (\mathbb{C}\varepsilon_0^2)^{1/3} |\Omega|, \tag{6}$$

then one of the variants has negligible volume fraction and χ is close to a function that depends only on one Cartesian coordinate. In particular, χ is close to a simple laminate.

Note that in contrast with [8], [9], and [10], we do not assume zero elastic energy states. Theorem 1 includes the geometrically linear three-well problem of Dolzmann and Müller [8], and in a certain sense can be regarded as a perturbation of it. With respect to the quantitative results of [12], [7] and [3], we stress that, in despite of being also rigidity results for non-zero energy states, they are essentially different from the ones of this paper, not only in spirit, because they rely on estimates involving the distances between ∇u and K, and ∇u and a particular energy well (at which we don't aim), but also on the mathematical tools used in their proofs.

The second results of this paper (Theorem 2) establishes the sharpness of Theorem 1. More precisely, we show that in the regime (5) for any given volume fractions, there exists a state of the system χ with

$$E(\chi) \lesssim \left(\frac{A}{\ell}\right)^{2/3} (\mathbb{C}\varepsilon_0^2)^{1/3} |\Omega|.$$

On the one hand, since all the three variants are present in χ, χ can not be a simple laminate formed only of two variants. This shows the sharpness of Theorem 1. On the other hand, the combination of Theorem 1 and 2 shows that in the regime (5), for prescribed non-vanishing volume fractions, the minimal energy scales as

$$\min E \sim \left(\frac{A}{\ell}\right)^{2/3} (\mathbb{C}\varepsilon_0^2)^{1/3}|\Omega|.$$

We now describe the state of the system χ constructed in Theorem 2. Within the periodic cell Ω, χ consist of two patches. Each patch consists of a laminate of two variants. The width of the domains which form the laminate decreases when approaching the interface between the patches. This change in domain width is mediated by a branching of the domains. This "twin branching" is actually observed in experiments [14]. Branching is energetically advantageous, since on the one hand, the small domain width near the interface between the patches reduces the local stresses due to the incompatibility, whereas on the other hand, the larger domain width in the bulk of the patches saves interfacial energy. The amount of branching itself is limited by the local stresses generated.

This energetic advantage of branching has been mathematically capture in the seminal paper by Kohn and Müller [11]. In [11], branching is studied "in vitro" in a 2-d scalar model at an externally imposed interface between a patch consisting of a laminate of 2 martensitic variants and a patch formed by austenite. In our language, [11] proves that in the regime (5), unbranched laminates are not optimal. Considering the same model, Conti [6] proves that any minimizers must display branching. Our paper shows that the simplified model from [11] indeed captures the essence of (2). Surprisingly, it seams to have not been noticed before that it is also branching which limits the range of rigidity results.

Rescaled energy

From (6) it is clear that the considered energy regime depends only on one non-dimensional parameter. To stress this fact and simplify notation, we reformulate the energy functional (2) in a non-dimensional form.

We rescale length by $x \to x/\ell$, the displacement by $u \to \varepsilon_0\, u$, and define

$$\mathcal{E}_\eta(\chi) := \frac{E(\chi)}{\ell^3\, \mathbb{C}\, \varepsilon_0^2}.$$

Setting

$$\eta = \frac{A}{\ell}\,\frac{1}{\mathbb{C}\,\varepsilon_0^2} \quad \text{and} \quad Q = \ell^{-3}\Omega = (0,1)^3$$

we obtain

$$\begin{aligned} \mathcal{E}_\eta(\chi) \;&=\; \eta \int_Q |\nabla(\chi_1-\chi_2)| + |\nabla(\chi_2-\chi_3)| + |\nabla(\chi_3-\chi_1)|dx \qquad (7)\\ &+ \inf_{\substack{u\,:\,\mathbb{R}^3\to\mathbb{R}^3\\ \nabla u\;Q-\text{periodic}}} \left\{ \int_Q \left| \frac{1}{2}(\nabla u + (\nabla u)^T) - \chi_1 U_1 - \chi_2 U_2 - \chi_3 U_3 \right|^2 dx \right\}. \end{aligned}$$

The results of this paper are stated in terms of this energy functional.

Energy wells

We consider a crystal with a cubic to tetragonal solid-solid phase transition. This choice fixes the form of the energy wells of the linearized problem cf. (4), and the geometry of the simple laminates that are permitted. For simplicity we will choose $\varepsilon_0 = 1$. By cubic symmetry, the variants are related by

$$U_2 = R_2^T U_1 R_2, \quad U_3 = R_3^T U_1 R_3,$$

where R_2 and R_3 are two 180^o rotation matrices. Following Bhattacharya [1] we conclude that if the three energy wells are pairwise rank-one connected, there exist three unit vectors $n_{i,j}$, and three vectors $a_{i,j}$ with $i, j = 1, 2, 3$, $i < j$ such that

$$U_i - U_j = \frac{1}{2}(a_{i,j} \otimes n_{i,j} + n_{i,j} \otimes a_{i,j}),$$

where the vectors $n_{i,j}$ point in the direction perpendicular to the transition between U_i and U_j. In this case, we have

$$\begin{aligned} n_{1,2} &= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), & a_{1,2} &= \frac{3}{2}\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right), \\ n_{1,3} &= \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), & a_{1,3} &= \frac{3}{2}\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right), \\ n_{2,3} &= \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), & a_{2,3} &= \frac{3}{2}\left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right). \end{aligned}$$

By exchanging the roles of $n_{i,j}$ and $a_{i,j}$ it is clear that two types of planar interfaces are possible between each pair of different variants. It is also straight forward to check that the simple laminates formed by combinations of $\{U_1, U_2\}$ and $\{U_1, U_3\}$ are compatible only on average. These type of laminates are used in the proof of Theorem 2.

The rest of the paper is organized as follows: in Section 2 we present our main results and a proposition on which the proof of Theorem 1 is based. In Section 3 we show an auxiliary lemma that introduces a reformulation of the elastic energy. The proof of the proposition and the main theorems are presented in Sections 4 and 5.

2 Rigidity result: dichotomy on the volume fractions

First we introduce some notation:

- We write $A \gtrsim B$ whenever there exist a positive constant c independent of A, B or η such that
$$A \geq c B.$$
- Let $f : Q \to \mathbb{R}$, then
$$\langle f \rangle = \int_Q f(x)\, dx.$$

- Let θ_i represent the volume fraction of the variant χ_i, i.e.

$$\theta_1 = \langle \chi_1 \rangle, \quad \theta_2 = \langle \chi_2 \rangle, \quad \theta_3 = \langle \chi_3 \rangle \qquad \text{with} \quad \theta_1 + \theta_2 + \theta_3 = 1. \tag{8}$$

- We denote the Fourier transform of a Q-periodic function $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$\hat{f}(k) = \frac{1}{(2\pi)^{3/2}} \int_Q e^{ik\cdot x} f(x)\, dx \quad \text{for } k \in 2\pi\mathbb{Z}^3.$$

- Let $f : \mathbb{R}^3 \to \mathbb{R}$ and $\mathbf{h} \in \mathbb{R}^3$ then

$$\Delta_{\mathbf{h}} f(x) = f(x + \mathbf{h}) - f(x).$$

Main results

The first theorem establishes a rigidity result: In regime (9), one of the variants is very small and the remaining two variants form a simple laminate.

Theorem 1. *Let $\eta \leq 1$, and $\chi_1, \chi_2, \chi_3 : \mathbb{R}^3 \to \{0,1\}$ be $(0,1)^3$-periodic functions with*

$$\chi_1 + \chi_2 + \chi_3 = 1.$$

Assume

$$\mathcal{E}_\eta(\chi) \ll \eta^{2/3}. \tag{9}$$

Then:

i) We have

$$\theta_1\theta_2\theta_3 \ll 1. \tag{10}$$

ii) Without loss of generality we may assume $\theta_3 \ll 1$. Then:

– either there exist $(0,1)^3$-periodic functions $\widetilde{\chi}_1, \widetilde{\chi}_2 : \mathbb{R}^3 \to \{0,1\}$ that only depend on $x_1 + x_2$ with

$$\int_Q |\chi_1 - \widetilde{\chi}_1|^2\, dx \; + \; \int_Q |\chi_2 - \widetilde{\chi}_2|^2\, dx \ll 1,$$

– or there exist $(0,1)^3$-periodic functions $\widetilde{\chi}_1, \widetilde{\chi}_2 : \mathbb{R}^3 \to \{0,1\}$ that only depend on $x_1 - x_2$ with

$$\int_Q |\chi_1 - \widetilde{\chi}_1|^2\, dx \; + \; \int_Q |\chi_2 - \widetilde{\chi}_2|^2\, dx \ll 1.$$

The precise meaning of the symbol "$\ll$" used in the statement of Theorem 1 is the following: We say that (9) holds whenever

$$\eta^{-2/3}\mathcal{E}_\eta(\chi) \to 0 \qquad \text{as} \quad \eta \to 0,$$

and by (10) we simply mean that $\theta_1\theta_2\theta_3 \to 0$ as $\eta \to 0$. Note that (10) in the above sense is equivalent to $\theta_1\theta_2\theta_3 = o(1)$ as $\eta \to 0$.

In our next result we show the optimality of Theorem 1 in the sense that if (9) does not hold, then there exists a construction χ of bounded energy for any given volume fractions.

Theorem 2. *For any $\theta_1, \theta_2, \theta_3 \geq 0$ with $\theta_1 + \theta_2 + \theta_3 = 1$ and $\eta \ll 1$ there exist $(0,1)^3$-periodic functions*

$$\chi_1, \chi_2, \chi_3 : \mathbb{R}^3 \to \{0, 1\},$$

such that

$$\chi_1 + \chi_2 + \chi_3 = 1,$$

with

$$\theta_i = \int_{(0,1)^3} \chi_i dx \quad \textit{for} \quad i = 1, 2, 3,$$

and

$$\mathcal{E}_\eta(\chi) \lesssim \eta^{2/3}.$$

Theorems 1 and 2 taken together imply the following corollary:

Corollary 1. *Let $\eta \ll 1$ and χ_1, χ_2, χ_2 as in Theorem 1. Assume further that the three volume fractions $\theta_1, \theta_2, \theta_3$ are strictly positive, then*

$$\frac{1}{c}\eta^{2/3} \leq \mathcal{E}_\eta(\chi) \leq c\eta^{2/3}.$$

for some constant c independent on η, but that in principle may depend on θ_1, θ_2 and θ_3.

The functional dependence of c with respect to θ_1, θ_2 and θ_3 was out of the scope of the present paper.

In the proof of Theorem 1 we decompose the characteristic functions $\chi = (\chi_1, \chi_2, \chi_3)$ into functions that are "close" to functions depending only on one coordinate (that is χ is "close" to a simple laminate). This decomposition is controlled in terms of the energy regime. The next proposition contains this result.

Proposition 1. *Under the same assumptions as in Theorem 1, for every $i = 1, 2, 3$ there exist $(0,1)^3$-periodic functions $f^+_{i,j}$, $f^-_{i,j}$, $f^+_{i,l}$, $f^-_{i,l}$ for $j, l \in \{1, 2, 3\}$ and i, j, l pairwise different (i.e. $i \neq j$, $j \neq l$ and $l \neq i$) satisfying a universal L^6-bound such that*

$$\chi_i - \theta_i = f^+_{i,j} + f^-_{i,j} + f^+_{i,l} + f^-_{i,l} \tag{11}$$

and we have

$$\langle |\Delta_{\mathbf{h}} f^\pm_{i,j}|^2 \rangle \lesssim \eta^{-2/3}\mathcal{E}_\eta(\chi), \tag{12}$$

for $\mathbf{h} \in \mathbb{R}^3$ provided $\mathbf{h}$ is of the form

$$\mathbf{h} \in \mathbb{R}(e_i \pm e_j) \quad \textit{or} \quad \mathbf{h} \in \mathbb{R}e_l. \tag{13}$$

3 Reformulation of the elastic energy

In the following lemma, we re-express the elastic energy in terms of (χ_1, χ_2, χ_3), using Fourier series. We obtain a highly anisotropic, zero order Fourier multiplier.

Lemma 1. *Let* $Q = (0,1)^3$ *and* $\chi_1, \chi_2, \chi_3 : \mathbb{R}^3 \to \{0,1\}$ *be Q-periodic such that* (3) *holds. Then*

$$\inf_{\substack{u\,:\,\mathbb{R}^3 \to \mathbb{R}^3 \\ \nabla u\; Q-periodic}} \left\{ \int_Q \left| \frac{1}{2}(\nabla u + (\nabla u)^T) - \chi_1 U_1 - \chi_2 U_2 - \chi_3 U_3 \right|^2 dx \right\}$$

$$= 9 \sum_{\substack{k \in 2\pi\mathbb{Z}^3 \\ k \neq 0}} |k|^{-4} \Big[|k_3^2 \hat{\chi}_2 + k_2^2 \hat{\chi}_3|^2 + |k_3^2 \hat{\chi}_1 + k_1^2 \hat{\chi}_3|^2 + |k_1^2 \hat{\chi}_2 + k_2^2 \hat{\chi}_1|^2$$

$$+ 2k_1^2 k_2^2 |\hat{\chi}_3|^2 + 2k_2^2 k_3^2 |\hat{\chi}_1|^2 + 2k_3^2 k_1^2 |\hat{\chi}_2|^2 \Big]. \tag{14}$$

Proof. We set for abbreviation

$$U_0 := \chi_1 U_1 + \chi_2 U_2 + \chi_3 U_3. \tag{15}$$

We divide the proof into two steps:

Step 1 *In this first step, we argue that*

$$\inf_{\substack{u\,:\,\mathbb{R}^3 \to \mathbb{R}^3 \\ \nabla u\; Q-periodic}} \left\{ \int_Q \left| \frac{1}{2}(\nabla u + (\nabla u)^T) - U_0 \right|^2 dx \right\}$$

$$= \sum_{\substack{k \in 2\pi\mathbb{Z}^3 \\ k \neq 0}} |k|^{-4} \Big[|k|^4 |\widehat{U}_0|^2 - 2|k|^2 |\widehat{U}_0 k|^2 + |k \cdot \widehat{U}_0 k|^2 \Big] \tag{16}$$

We note that

$$\inf_{\substack{u\,:\,\mathbb{R}^3 \to \mathbb{R}^3 \\ \nabla u\; Q-\text{periodic}}} \left\{ \int_Q \left| \frac{1}{2}(\nabla u + (\nabla u)^T) - U_0 \right|^2 dx \right\}$$

$$= \inf_{\substack{u\,:\,\mathbb{R}^3 \to \mathbb{R}^3 \\ \nabla u\; Q-\text{periodic}}} \left\{ \int_Q \left| \frac{1}{2}\Big(\langle \nabla u \rangle + \langle \nabla u \rangle^T\Big) - \langle U_0 \rangle \right|^2 dx \right.$$

$$\left. + \int_Q \left| \frac{1}{2}\Big((\nabla u - \langle \nabla u \rangle) + (\nabla u - \langle \nabla u \rangle)^T\Big) - \Big(U_0 - \langle U_0 \rangle\Big) \right|^2 dx \right\}$$

$$= \inf_{\substack{v\,:\,\mathbb{R}^3 \to \mathbb{R}^3 \\ Q-\text{periodic}}} \left\{ \int_Q \left| \frac{1}{2}\Big(\nabla v + \nabla v^T\Big) - \Big(U_0 - \langle U_0 \rangle\Big) \right|^2 dx \right\}.$$

In the last identity we used the fact that, since $\nabla u - \langle \nabla u \rangle$ is a symmetric, Q-periodic gradient field of mean zero, it can be written as the gradient of a Q-periodic function v. Thus, since we are now on the level of periodic functions, we may use Fourier series to re-express the elastic energy as

$$\int_Q \left| \frac{1}{2}\Big(\nabla v + \nabla v^T\Big) - \Big(U_0 - \langle U_0 \rangle\Big) \right|^2 dx \quad = \sum_{\substack{k \in 2\pi\mathbb{Z}^3 \\ k \neq 0}} \left| \frac{1}{2}\big(k \otimes \hat{v} + \hat{v} \otimes k\big) - i\,\widehat{U}_0 \right|^2 .$$

Hence we obtain the Euler-Lagrange equation

$$\sum_{\substack{k \in 2\pi\mathbb{Z}^3 \\ k \neq 0}} \mathrm{Re}\left[\overline{\big(k \otimes \hat{\varphi} + \hat{\varphi} \otimes k\big)} : \left(\frac{1}{2}\big(k \otimes \hat{v} + \hat{v} \otimes k\big) - i\,\widehat{U}_0 \right) \right] = 0, \qquad \begin{array}{c} \forall\, \varphi : \mathbb{R}^3 \to \mathbb{R}^3 \\ Q - \text{periodic} \end{array}$$

where $\mathrm{Re}[z]$ denotes the real part of a complex number z and : stands for a double contraction. Since $1/2\big(k \otimes \hat{v} + \hat{v} \otimes k\big) - i\,\widehat{U}_0$ is symmetric, this can be rewritten as

$$\sum_{\substack{k \in 2\pi\mathbb{Z}^3 \\ k \neq 0}} \mathrm{Re}\left[\overline{\hat{\varphi}} \cdot \left(\frac{1}{2}\big(k \otimes \hat{v} + \hat{v} \otimes k\big)k - i\,\widehat{U}_0 k \right) \right] = 0. \qquad \begin{array}{c} \forall\, \varphi : \mathbb{R}^3 \to \mathbb{R}^3 \\ Q - \text{periodic} \end{array}$$

Since $\hat{\varphi}$ and $1/2\big(k \otimes \hat{v} + \hat{v} \otimes k\big) - i\,\widehat{U}_0$ satisfy the symmetry $\hat{\varphi}(-k) = \overline{\hat{\varphi}}(k)$, we deduce

$$\frac{1}{2}\big(k \otimes \hat{v} + \hat{v} \otimes k\big)k - i\,\widehat{U}_0 k = 0, \qquad \begin{array}{c} \forall\, k \in 2\pi\mathbb{Z}^3 \\ k \neq 0. \end{array}$$

In turn the last equation can be rewritten as

$$|k|^2 \hat{v} + (k \cdot \hat{v})k = 2i\,\widehat{U}_0 k. \tag{17}$$

Next, we take the scalar product of (17) with k and obtain

$$k \cdot \hat{v} = i\,\frac{k \cdot \widehat{U}_0 k}{|k|^2}. \tag{18}$$

Inserting (18) into (17) we find

$$\hat{v} = \frac{i}{|k|^2}\left(2\,\hat{U}_0 - \frac{k \cdot \hat{U}_0 k}{|k|^2} \right) k. \tag{19}$$

Let us recall that the energy density in Fourier space is given by

$$\left| \frac{1}{2}(\hat{v} \otimes k + k \otimes \hat{v}) - i\,\widehat{U}_0 \right|^2 = \left| \frac{1}{2}(\hat{v} \otimes k + k \otimes \hat{v}) \right|^2 - \mathrm{Re}\left[(\hat{v} \otimes k + k \otimes \hat{v}) : \overline{i\,\widehat{U}_0} \right] + |\widehat{U}_0|^2. \tag{20}$$

Using (17) and (19) in the first term of the right hand side of (20) yields

$$\begin{aligned}\Big|\frac{1}{2}(\hat v\otimes k+k\otimes\hat v)\Big|^2 &= \frac{1}{2}|k|^2|\hat v|^2+\frac{1}{2}|\hat v\cdot k|^2\\ &= \frac{1}{|k|^2}\Big(2|\widehat{U}_0k|^2-\frac{|k\cdot\widehat{U}_0k|^2}{|k|^2}\Big).\end{aligned}\tag{21}$$

In the same manner, for the second term of the right hand side of (20) we obtain

$$\mathrm{Re}\left[(\hat v\otimes k+k\otimes\hat v):\overline{i\,\widehat{U}_0}\right]=\frac{2}{|k|^2}\left(2|\widehat{U}_0k|^2-\frac{\left|k\cdot\widehat{U}_0k\right|^2}{|k|^2}\right).\tag{22}$$

By inserting (21) and (22) back into (20) we find for the energy density

$$\left|\frac{1}{2}\big(k\otimes\hat v+\hat v\otimes k\big)-\widehat{U}_0\right|^2=\frac{1}{|k|^4}\Big[|k|^4|\widehat{U}_0|^2-2|k|^2|\widehat{U}_0k|^2+|k\cdot\widehat{U}_0k|^2\Big].\tag{23}$$

Step 2 *In this step we argue that for* U_0 *of the form* (15) *and* U_1,U_2,U_3 *as in* (4) *we have*

$$\begin{aligned}|k|^4|\widehat{U}_0|^2-2|k|^2|\widehat{U}_0k|^2+|k\cdot\widehat{U}_0k|^2 &= |k_3^2\hat\chi_2+k_2^2\hat\chi_3|^2+|k_3^2\hat\chi_1+k_1^2\hat\chi_3|^2\\ &\quad+|k_1^2\hat\chi_2+k_2^2\hat\chi_1|^2+2k_1^2k_2^2|\hat\chi_3|^2+2k_2^2k_3^2|\hat\chi_1|^2+2k_3^2k_1^2|\hat\chi_2|^2.\end{aligned}$$

Recalling (3) and the specific form (4) of U_1,U_2 and U_3, we get for $k\neq 0$

$$\widehat{U}_0=\begin{pmatrix}-3\hat\chi_1 & 0 & 0\\ 0 & -3\hat\chi_2 & 0\\ 0 & 0 & -3\hat\chi_3\end{pmatrix}.$$

Using this explicit expression for $\widehat{U}_0$ we find

$$|k|^4|\widehat{U}_0|^2 = 9(k_1^4+k_2^4+k_3^4+2k_1^2k_2^2+2k_2^2k_3^2+2k_3^2k_1^2)(|\hat\chi_1|^2+|\hat\chi_2|^2+|\hat\chi_3|^2),\tag{24}$$

$$\begin{aligned}|k|^2|\widehat{U}_0k|^2 &= 9\Big((k_1^4+k_1^2k_2^2+k_3^2k_1^2)|\hat\chi_1|^2+\\ &\quad (k_1^2k_2^2+k_2^4+k_2^2k_3^2)|\hat\chi_2|^2+(k_3^2k_1^2+k_2^2k_3^2+k_3^4)|\hat\chi_1|^2\Big),\end{aligned}\tag{25}$$

$$\begin{aligned}|k\cdot\widehat{U}_0k|^2 &= 9\Big(k_1^4|\hat\chi_1|^2+k_2^4|\hat\chi_2|^2+k_3^4|\hat\chi_3|^2\\ &\quad+2\,\mathrm{Re}\left[k_1^2\overline{\hat\chi}_1k_2^2\hat\chi_2\right]+2\,\mathrm{Re}\left[k_2^2\overline{\hat\chi}_2k_3^2\hat\chi_3\right]+2\,\mathrm{Re}\left[k_3^2\overline{\hat\chi}_3k_1^2\hat\chi_1\right]\Big).\end{aligned}\tag{26}$$

Combining (24), (25) and (26) and rearranging the factors within real-part function, we obtain

$$\begin{aligned}|k|^4|\widehat{U}_0|^2-2|k|^2|\widehat{U}_0k|^2+|k\cdot\hat U_0k|^2 &= 9\Big((k_2^4+k_3^4+2k_2^2\,k_3^2)|\hat\chi_1|^2\\ &\quad+(k_1^4+k_3^4+2k_3^2\,k_1^2)|\hat\chi_2|^2+(k_1^4+k_2^4+2k_1^2\,k_2^2)|\hat\chi_3|^2\\ &\quad+2\,\mathrm{Re}\left[k_1^2\overline{\hat\chi}_2k_2^2\hat\chi_1\right]+2\,\mathrm{Re}\left[k_2^2\overline{\hat\chi}_3k_3^2\hat\chi_2\right]+2\,\mathrm{Re}\left[k_3^2\overline{\hat\chi}_1k_1^2\hat\chi_3\right]\Big).\end{aligned}$$

Finally, rearranging terms to complete the complex modulus square terms as in (14), we get the result. □

4 Proof of the rigidity result

4.1 Proof of Proposition 1

We divide the proof into several steps:

Step 1 *In the first step, we replace the actual Fourier multiplier F (cf (14)) defining the elastic energy by an equivalent expression $\tilde{F}$ that separates χ_1, χ_2 and χ_3. It is equivalent in the sense that $\tilde{F}$ scales as the original Fourier multiplier F.*

Consider the functions of $k \in 2\pi\mathbb{Z}^3$ with $k \neq 0$, and $\hat{\chi} \in \mathbb{C}^3$ with $\hat{\chi}_1 + \hat{\chi}_2 + \hat{\chi}_3 = 0$ given by

$$F(k, \hat{\chi}) = |k|^{-4}\Big(|k_3^2\hat{\chi}_2 + k_2^2\hat{\chi}_3|^2 + |k_3^2\hat{\chi}_1 + k_1^2\hat{\chi}_3|^2 + |k_1^2\hat{\chi}_2 + k_2^2\hat{\chi}_1|^2$$

$$+2k_1^2k_2^2|\hat{\chi}_3|^2 + 2k_2^2k_3^2|\hat{\chi}_1|^2 + 2k_3^2k_1^2|\hat{\chi}_2|^2\Big),$$

and

$$\tilde{F}(k, \hat{\chi}) := |k|^{-8}\Big(\tag{27}$$

$$(k_2^2 + (k_3 - k_1)^2)(k_2^2 + (k_3 + k_1)^2)(k_3^2 + (k_1 - k_2)^2)(k_3^2 + (k_1 + k_2)^2)|\hat{\chi}_1|^2$$

$$+(k_3^2 + (k_2 - k_1)^2)(k_3^2 + (k_2 + k_1)^2)(k_1^2 + (k_3 - k_2)^2)(k_1^2 + (k_3 + k_2)^2)|\hat{\chi}_2|^2$$

$$+(k_1^2 + (k_2 - k_3)^2)(k_1^2 + (k_2 + k_3)^2)(k_2^2 + (k_3 - k_1)^2)(k_2^2 + (k_3 + k_1)^2)|\hat{\chi}_3|^2\Big).$$

Then,

$$\tilde{F}(k, \hat{\chi}) \lesssim F(k, \hat{\chi}) \lesssim \tilde{F}(k, \hat{\chi}). \tag{28}$$

By homogeneity of F and $\tilde{F}$ in k and $\hat{\chi}$, we need to show (28) only on

$$\mathbb{S}^2 \times \big(\mathbb{S}^5 \cap \{\hat{\chi}_1 + \hat{\chi}_2 + \hat{\chi}_3 = 0\}\big) \tag{29}$$

with the understanding that

$$\mathbb{S}^5 = \big\{(\hat{\chi}_1, \hat{\chi}_2, \hat{\chi}_3) \in \mathbb{C}^3 \text{ such that } |\hat{\chi}_1|^2 + |\hat{\chi}_2|^2 + |\hat{\chi}_3|^2 = 1\big\}.$$

Since F and $\tilde{F}$ are continuous on the compact set (29) it suffices to show that

$$\big\{F = 0\big\} = \big\{\tilde{F} = 0\big\} \qquad \text{on} \quad (29),$$

where $\{F = 0\}$ stands for $\{(k, \hat{\chi}) \in 2\pi\mathbb{Z}^3 \times \mathbb{C}^3 \text{ such that } F(k, \hat{\chi}) = 0\}$. In fact, we shall argue that

$$\begin{aligned} \{\tilde{F} = 0\} &= \mathcal{N} \quad \text{on} \quad (29), \qquad (30)\\ \{F = 0\} &= \mathcal{N} \quad \text{on} \quad (29), \qquad (31) \end{aligned}$$

where

$$\begin{aligned} \mathcal{N} \;:=\; & \Big(\{k_1 = 0 \quad \text{and} \quad k_2^2 = k_3^2\} \cap \{\hat{\chi}_1 = 0\}\Big) \\ & \cup\Big(\{k_2 = 0 \quad \text{and} \quad k_3^2 = k_1^2\} \cap \{\hat{\chi}_2 = 0\}\Big) \\ & \cup\Big(\{k_3 = 0 \quad \text{and} \quad k_1^2 = k_2^2\} \cap \{\hat{\chi}_3 = 0\}\Big). \end{aligned}$$

We start with (30). By definition of $\tilde{F}$ we have

$$\begin{aligned} \{\tilde{F} = 0\} \;=\; & \Big(\{k_2 = 0 \text{ and } k_3^2 = k_1^2\} \cup \{k_3 = 0 \text{ and } k_1^2 = k_2^2\} \cup \{\hat{\chi}_1 = 0\}\Big) \\ & \cap\Big(\{k_3 = 0 \text{ and } k_1^2 = k_2^2\} \cup \{k_1 = 0 \text{ and } k_2^2 = k_3^2\} \cup \{\hat{\chi}_2 = 0\}\Big) \\ & \cap\Big(\{k_1 = 0 \text{ and } k_2^2 = k_3^2\} \cup \{k_2 = 0 \text{ and } k_3^2 = k_1^2\} \cup \{\hat{\chi}_3 = 0\}\Big). \qquad (32) \end{aligned}$$

Notice that on (29), the three sets

$$\{k_1 = 0 \quad \text{and} \quad k_2^2 = k_3^2\}, \; \{k_2 = 0 \quad \text{and} \quad k_3^2 = k_1^2\} \text{ and } \{k_3 = 0 \quad \text{and} \quad k_1^2 = k_2^2\}$$

are pairwise disjoint. Likewise, the sets

$$\{\hat{\chi}_1 = 0\}, \quad \{\hat{\chi}_2 = 0\} \quad \text{and} \quad \{\hat{\chi}_3 = 0\}$$

are pairwise disjoint. Hence, using the distributional law on (32) we conclude (30).

Now we turn to (31), which we split into

$$\begin{aligned} \mathcal{N} \subset \{F = 0\} & \quad \text{on} \quad (29), \qquad (33)\\ \{F = 0\} \subset \mathcal{N} & \quad \text{on} \quad (29). \qquad (34) \end{aligned}$$

For (33) we notice that by cyclical symmetry, that it suffices to show that for $(k, \hat{\chi})$ in (29),

$$k_1 = 0 \text{ and } k_2^2 = k_3^2 \text{ and } \hat{\chi}_1 = 0 \qquad (35)$$

implies

$$F(k, \hat{\chi}) = 0.$$

Notice that because of (35), F reduces to $F(k, \hat{\chi}) = |k|^{-4}|k_3^2\hat{\chi}_2 + k_2^2\hat{\chi}_3|^2$. Also this last term vanishes since (35) also yields

$$k_2^3\hat{\chi}_2 + k_2^2\hat{\chi}_3 = \frac{1}{2}|k|^2(\hat{\chi}_1 + \hat{\chi}_2 + \hat{\chi}_3) = 0.$$

We finally address (34). Let $(k, \hat{\chi})$ be in (29). In particular, at least one component of (k_1, k_2, k_3) must be non-zero and two components of $(\hat{\chi}_1, \hat{\chi}_2, \hat{\chi}_3)$ must not vanish. Since F is invariant under permutation of the indices $(1, 2, 3)$, we may assume

$$k_2 \neq 0 \quad \text{and} \quad \hat{\chi}_3 \neq 0.$$

Since $F(k, \hat{\chi}) = 0$ implies $k_1^2 k_2^2 |\hat{\chi}_3|^2 = 0$ this yields

$$k_1 = 0, \tag{36}$$

so that $F(k, \hat{\chi})$ reduces to

$$\begin{aligned} F(k, \hat{\chi}) &= |k_3^2 \hat{\chi}_2 + k_2^2 \hat{\chi}_3|^2 + |k_3^2 \hat{\chi}_1|^2 + |k_2^2 \hat{\chi}_1|^2 + 2k_2^2 k_3^2 |\hat{\chi}_1|^2 \\ &= |k_3^2 \hat{\chi}_2 + k_2^2 \hat{\chi}_3|^2 + (k_2^2 + k_3^2)^2 |\hat{\chi}_1|^2 \\ &\overset{(36)}{=} |k_3^2 \hat{\chi}_2 + k_2^2 \hat{\chi}_3|^2 + |\hat{\chi}_1|^2. \end{aligned}$$

Hence $F(k, \hat{\chi}) = 0$ implies

$$\hat{\chi}_1 = 0. \tag{37}$$

However $\hat{\chi}_1 = 0$ yields $\hat{\chi}_2 + \hat{\chi}_3 = 0$. Together with $|\hat{\chi}_2|^2 + |\hat{\chi}_3|^2 = 1$ we see that $F(k, \hat{\chi})$ reduces to

$$F(k, \hat{\chi}) = \frac{1}{2}(k_2^2 - k_3^2)^2,$$

so that $F(k, \hat{\chi}) = 0$ yields

$$k_2^2 = k_3^2. \tag{38}$$

The fact that $F(k, \hat{\chi}) = 0$ yields (36), (37) and (38) establishes (34).

Step 2 *In this second step, we use the modified Fourier multiplier $\tilde{F}$ from Step 1 to split each χ_i into 4 components which are such that each of them essentially depend only on one variable if the elastic energy is small.*

For every $\chi_i : Q \to \{0, 1\}$ with $i = 1, 2, 3$ there exist functions $f_{i,j}^+$, $f_{i,j}^-$, $f_{i,l}^+$, $f_{i,l}^-$ for $j, l \in \{1, 2, 3\}$ and i, j, l pairwise different satisfying a universal L^6-bound, such that

$$\chi_i - \theta_i = f_{i,j}^+ + f_{i,j}^- + f_{i,l}^+ + f_{i,l}^-. \tag{39}$$

Moreover, we have

$$\begin{aligned} &\frac{1}{|k|^8}\Big((k_j^2 + (k_i + k_l)^2)(k_j^2 + (k_i - k_l)^2)(k_l^2 + (k_i + k_j)^2)(k_l^2 + (k_i - k_j)^2)\Big)|\hat{\chi}_i|^2 \\ &\quad \gtrsim \frac{1}{|k|^2}\Big((k_j^2 + (k_i + k_l)^2)|\hat{f}_{i,l}^+|^2 + (k_j^2 + (k_i - k_l)^2)|\hat{f}_{i,l}^-|^2 \\ &\qquad + (k_l^2 + (k_i + k_j)^2)|\hat{f}_{i,j}^+|^2 + (k_l^2 + (k_i - k_j)^2)|\hat{f}_{i,j}^-|^2\Big) \end{aligned} \tag{40}$$

and

$$\int_Q |\Delta_{\mathbf{h}} f_{i,j}^{\pm}|^2 \, dx \lesssim \int_Q |\Delta_{\mathbf{h}} \chi_i|^2 \, dx \quad \textit{for every} \quad \mathbf{h} \in \mathbb{R}^3. \tag{41}$$

Without loss of generality we may assume $i = 1$, $j = 2$ and $l = 3$, and set

$$\begin{aligned} \hat{f}_{1,2}^{\pm} &:= \frac{1}{|k|^6}(k_2^2 + (k_1 + k_3)^2)(k_2^2 + (k_1 - k_3)^2)(k_3^2 + (k_1 \mp k_2)^2)m(k)\hat{\chi}_1, \\ \hat{f}_{1,3}^{\pm} &:= \frac{1}{|k|^6}(k_2^2 + (k_1 \mp k_3)^2)(k_3^2 + (k_1 + k_2)^2)(k_3^2 + (k_1 - k_2)^2)m(k)\hat{\chi}_1, \end{aligned}$$

where $m(k)$ is defined in order to satisfy (39), namely $m(k) := |k|^6/M(k)$ with

$$\begin{aligned} M(k) := &\ (k_2^2 + (k_1 + k_3)^2)(k_2^2 + (k_1 - k_3)^2)(k_3^2 + (k_1 + k_2)^2) \\ &+(k_2^2 + (k_1 + k_3)^2)(k_2^2 + (k_1 - k_3)^2)(k_3^2 + (k_1 - k_2)^2) \\ &+(k_2^2 + (k_1 + k_3)^2)(k_3^2 + (k_1 + k_2)^2)(k_3^2 + (k_1 - k_2)^2) \\ &+(k_2^2 + (k_1 - k_3)^2)(k_3^2 + (k_1 + k_2)^2)(k_3^2 + (k_1 - k_2)^2). \end{aligned} \tag{42}$$

We claim that

$$m(k) = \frac{|k|^6}{M(k)} \lesssim 1 \quad \text{for all } k \in \mathbb{R}^3. \tag{43}$$

Therefore, $m(k)$ is bounded, homogeneous of degree 0 and continuous outside the origin. Thus, Hörmander-Mikhlin's multiplier theorem (see e.g. pp 135 in [2]) ensures the boundedness of $f_{1,2}^{\pm}$, $f_{1,3}^{\pm}$ in $L^6(Q)$.

Now we present the argument for (43). By homogeneity of $M(k)$ and $|k|^6$, we only need to show that

$$|k|^6 \lesssim M(k) \quad \text{on} \quad \mathbb{S}^2.$$

Since $M(k)$ and $|k|^6$ are continuous on the compact set $\mathbb{S}^2$, and $|k|^6 = 1$ for every $k \in \mathbb{S}^2$, it suffices to show that

$$M(k) \neq 0 \quad \text{on} \quad \mathbb{S}^2. \tag{44}$$

Indeed, by contradiction assume that $M(k) = 0$ for some $k \in \mathbb{S}^2$. Since the four lines in the definition (42) of $M(K)$ don't have a common factor, at least two of the following terms

$$k_2^2 + (k_1 + k_3)^2, \quad k_2^2 + (k_1 - k_3)^2, \quad k_3^2 + (k_1 + k_2)^2, \quad \text{and} \quad k_3^2 + (k_1 + k_2)^2,$$

have to vanish. In all six possible cases it is straight forward to find that $k = 0$, a contradiction. Hence, (44) follows.

Now, because of (43) it holds

$$\begin{aligned} |\hat{f}_{1,2}^{+}|^2 &\leq \frac{(k_2^2 + (k_1 + k_3)^2)(k_2^2 + (k_1 - k_3)^2)(k_3^2 + (k_1 - k_2)^2)}{|k|^6} \\ &\times \underbrace{\frac{(k_2^2 + (k_1 + k_3)^2)(k_2^2 + (k_1 - k_3)^2)(k_3^2 + (k_1 - k_2)^2)}{|k|^6}\, m(k)}_{\leq 1} \underbrace{m(k)}_{\overset{(43)}{\lesssim} 1} |\hat{\chi}_1|^2 \\ &\lesssim \frac{1}{2}\frac{(k_2^2 + (k_1 + k_3)^2)(k_2^2 + (k_1 - k_3)^2)(k_3^2 + (k_1 - k_2)^2)}{|k|^6} |\hat{\chi}_1|^2. \end{aligned} \tag{45}$$

Thus,

$$\frac{(k_3^2+(k_1+k_2)^2)}{|k|^2}|\hat{f}^+_{1,2}|^2 \lesssim$$
$$\frac{(k_2^2+(k_1+k_3)^2)(k_2^2+(k_1-k_3)^2)(k_3^2+(k_1+k_2)^2(k_3^2+(k_1-k_2)^2)}{|k|^8}\ |\hat{\chi}_1|^2.$$

Hence, by cyclical symmetry in the indices (40) follows.

It only remains to show (41). To this end we notice that from (45) we have

$$|\hat{f}^\pm_{1,2}|^2 \ \lesssim\ |\hat{\chi}_1|^2.$$

This yields

$$\sum_{k\in 2\pi\mathbb{Z}^3} 4\sin^2\left(\frac{k\cdot\mathbf{h}}{2}\right)|\hat{f}^\pm_{1,2}|^2 \ \lesssim\ \sum_{k\in 2\pi\mathbb{Z}^3} 4\sin^2\left(\frac{k\cdot\mathbf{h}}{2}\right)|\hat{\chi}_1|^2.$$

which is the Fourier version of (41).

Step 3 *This step provides the crucial estimate which interpolates between the interfacial energy and the elastic energy. It is in the spirit of the estimate introduced in [5, Lemma 2.3].*

Let f be a Q-periodic function. Then the following interpolation estimate holds

$$\sup_{|\mathbf{h}_0|>0}\frac{1}{|\mathbf{h}_0|^{\frac{2}{3}}}\int_Q|\Delta_{\mathbf{h}_0}f|^2\,dx \lesssim \left(\sup_{|\mathbf{h}|>0}\frac{1}{|\mathbf{h}|}\int_Q|\Delta_{\mathbf{h}}f|^2\,dx\right)^{\frac{2}{3}}\left(\int_Q\left||\nabla|^{-1}(\mathbf{e}_0\cdot\nabla f)\right|^2\,dx\right)^{\frac{1}{3}}, \tag{46}$$

where $\mathbf{h}\in\mathbb{R}^3$ and $\mathbf{h}_0$ is of the form $\mathbf{h}_0=\mathbb{R}(e_i\pm e_j)$ or $\mathbf{h}_0=\mathbb{R}e_l$ with $i,j,l\in\{1,2,3\}$ pairwise different, and $\mathbf{e}_0\cdot\nabla f$ stands for

$$(e_i\pm e_j)\cdot\nabla f \quad or \quad e_l\cdot\nabla f,$$

respectively.

Without loss of generality we may assume $\mathbf{h}_0=\mathbb{R}e_l$ and denote $h_0=|\mathbf{h}_0|$. If an estimate of the form

$$\sup_{h_0>0}\frac{1}{h_0^{2/3}}\int_Q|\Delta_{\mathbf{h}_0}f|^2\,dx \lesssim \beta\sup_{|\mathbf{h}|>0}\frac{1}{|\mathbf{h}|}\int_Q|\Delta_{\mathbf{h}}f|^2\,dx+\frac{1}{\beta^2}\int_Q\left||\nabla|^{-1}\partial_l f\right|^2\,dx \tag{47}$$

holds for any $\beta>0$, then (46) follows by optimizing in β.

Next, we prove (47). In Fourier space, estimate (47) reads

$$\sup_{h_0>0}\frac{1}{h_0^{2/3}}\sum_{\substack{k\,\in\,2\pi\mathbb{Z}^3\\ k\,\neq\,0}}4\sin^2\left(\frac{k_l h_0}{2}\right)|\hat{f}|^2 \tag{48}$$
$$\lesssim\beta\sup_{|\mathbf{h}|>0}\frac{1}{|\mathbf{h}|}\sum_{\substack{k\,\in\,2\pi\mathbb{Z}^3\\ k\,\neq\,0}}4\sin^2\left(\frac{k\cdot\mathbf{h}}{2}\right)|\hat{f}|^2+\beta^{-2}\sum_{\substack{k\,\in\,2\pi\mathbb{Z}^3\\ k\,\neq\,0}}|k|^{-2}k_l^2|\hat{f}|^2.$$

Given a fixed h_0, let us split the Fourier space according to

$$\begin{aligned} \beta h_0^{2/3}|k| &\leq 1 \qquad && \text{long wave-length,} \\ \beta h_0^{2/3}|k| &\geq 1 \qquad && \text{short wave-length.} \end{aligned}$$

For long wave-lengths we have

$$\begin{aligned} \frac{1}{h_0^{2/3}} \sum_{\substack{k \,\in\, 2\pi\mathbb{Z}^3 \setminus \{0\} \\ \beta h_0^{2/3}|k| \,\leq\, 1}} 4\sin^2\left(\frac{k_l h_0}{2}\right)|\hat{f}|^2 \ &\lesssim \sum_{\substack{k \,\in\, 2\pi\mathbb{Z}^3 \setminus \{0\} \\ \beta h_0^{2/3}|k| \,\leq\, 1}} \underbrace{h_0^{4/3}}_{\leq(\beta|k|)^{-2}} k_l^2 \quad |\hat{f}|^2 \\ &\lesssim \ \beta^{-2} \sum_{\substack{k \,\in\, 2\pi\mathbb{Z}^3 \\ k \,\neq\, 0}} |k|^{-2} k_l^2 |\hat{f}|^2. \end{aligned} \tag{49}$$

For short wave-length we have

$$\frac{1}{h_0^{2/3}} \sum_{\substack{k \,\in\, 2\pi\mathbb{Z}^3 \\ \beta h_0^{2/3}|k| \,\geq\, 1}} 4\sin^2\left(\frac{k_l h_0}{2}\right)|\hat{f}|^2 \ \leq \ \frac{4}{h_0^{2/3}} \sum_{\substack{k \,\in\, 2\pi\mathbb{Z}^3 \\ \beta h_0^{2/3}|k| \,\geq\, 1}} |\hat{f}|^2. \tag{50}$$

We now argue that for k with $\beta h_0^{2/3}|k| \geq 1$ we have

$$1 \ \lesssim \fint_{B_{\beta h_0^{2/3}}} \sin^2\left(\frac{k \cdot \mathbf{h}}{2}\right)\, d\mathbf{h}. \tag{51}$$

Indeed, by rescaling, (51) follows from the estimate

$$\fint_{B_1} \sin^2\left(\frac{k \cdot \mathbf{h}}{2}\right)\, d\mathbf{h} \ \gtrsim \ 1 \quad \text{for all} \quad |k| \geq 1,$$

which is a consequence of the continuity and the limiting behavior

$$\lim_{|k|\to\infty} \fint_{B_1} \sin^2\left(\frac{k \cdot \mathbf{h}}{2}\right)\, d\mathbf{h} \ = \ \frac{1}{2}.$$

Thus, from (50) and (51) we find

$$\begin{aligned} \frac{1}{h_0^{2/3}} \sum_{\beta h_0^{2/3}|k|\geq 1} 4\sin^2\left(\frac{k_l h_0}{2}\right)|\hat{f}|^2 \ &\lesssim \ \frac{4}{h_0^{2/3}} \sum_{\beta h_0^{2/3}|k|\geq 1} \left(\fint_{B_{\beta h_0^{2/3}}} \sin^2\left(\frac{k \cdot \mathbf{h}}{2}\right) d\mathbf{h}\right) |\hat{f}|^2 \\ &\lesssim \ \frac{1}{h_0^{2/3}} \fint_{B_{\beta h_0^{2/3}}} \left(\sum_{\beta h_0^{2/3}|k|\geq 1} 4\sin^2\left(\frac{k \cdot \mathbf{h}}{2}\right)|\hat{f}|^2 \right) d\mathbf{h} \\ &\lesssim \ \beta \sup_{\mathbf{h}\in\mathbb{R}^3} \left(\frac{1}{|\mathbf{h}|} \sum_{k\in 2\pi\mathbb{Z}^3} 4\sin^2\left(\frac{k \cdot \mathbf{h}}{2}\right)|\hat{f}|^2 \right). \end{aligned} \tag{52}$$

Estimate (48) now follows from (49) and (52).

Step 4 *Conclusion*

The decomposition (11) has been introduced in Step 1. It only remains to show (12). First, we notice

$$\sup_{|\mathbf{h}|>0} \frac{1}{|\mathbf{h}|} \int_Q |\Delta_{\mathbf{h}} \chi_i|^2 \, dx \ \lesssim \ \sup_Q |\chi_i| \int_Q |\nabla \chi_i| \, dx. \tag{53}$$

Indeed, assume that χ_i is a Q-periodic smooth function, we have

$$\begin{aligned} \int_Q |\Delta_{\mathbf{h}} \chi_i|^2 \, dx &\leq \sup_Q |\Delta_{\mathbf{h}} \chi_i| \int_Q |\Delta_{\mathbf{h}} \chi_i| dx \\ &\leq 2 \sup_Q |\chi_i| \int_Q |\mathbf{h} \cdot \nabla \chi_i| dx. \end{aligned} \tag{54}$$

Next, we divide (54) by $|\mathbf{h}|$ and take the supremum. By density of smooth functions in the space of functions of bounded variations, we obtain (53). In view of (53) and (41), we have via the control of the interfacial energy

$$\eta \sup_{|\mathbf{h}|>0} \frac{1}{|\mathbf{h}|} \int_Q |\Delta_{\mathbf{h}} f^{\pm}_{i,j}|^2 \, dx \ \lesssim \ \mathcal{E}_\eta(\chi) \tag{55}$$

for $i, j \in \{1, 2, 3\}$ with $i \neq j$. Combining Lemma 1, (28) and (40) we get via the control of the elastic energy

$$\int_Q \left\{ \left| |\nabla|^{-1} (\partial_i \pm \partial_j) f^{\pm}_{i,j} \right|^2 + \left| |\nabla|^{-1} \partial_l f^{\pm}_{i,j} \right|^2 \right\} dx \lesssim \mathcal{E}_\eta(\chi) \tag{56}$$

for every $i, j, l \in \{1, 2, 3\}$ pairwise different. Without loss of generality we may assume $i = 1$, $j = 2$ and $l = 3$. Taking $\mathbf{h}_0$ of the form $\mathbb{R}e_3$, estimate (46) yields

$$\begin{aligned} \sup_{|\mathbf{h}_0|>0} \frac{1}{|\mathbf{h}_0|^{\frac{2}{3}}} \int_Q |\Delta_{\mathbf{h}_0} f^{+}_{1,2}|^2 \, dx &\leq \Bigg(\underbrace{\sup_{|\mathbf{h}|>0} \frac{1}{|\mathbf{h}|} \int_Q |\Delta_{\mathbf{h}} f^{+}_{1,2}|^2 \, dx}_{\overset{(55)}{\lesssim} \eta^{-1} \mathcal{E}_\eta(\chi)} \Bigg)^{\frac{2}{3}} \Bigg(\underbrace{\int_Q \left| |\nabla|^{-1} \partial_3 f^{+}_{1,2} \right|^2 dx}_{\overset{(56)}{\lesssim} \mathcal{E}_\eta(\chi)} \Bigg)^{\frac{1}{3}} \\ &\lesssim \eta^{-2/3} \mathcal{E}_\eta(\chi). \end{aligned} \tag{57}$$

Analogously, taking $\mathbf{h}_0$ of the form $\mathbb{R}(e_1 + e_2)$, we conclude

$$\begin{aligned} &\sup_{|\mathbf{h}_0|>0} \frac{1}{|\mathbf{h}_0|^{\frac{2}{3}}} \int_Q |\Delta_{\mathbf{h}_0} f^{+}_{1,2}|^2 \, dx \\ &\quad \lesssim \Bigg(\underbrace{\sup_{|\mathbf{h}|>0} \frac{1}{|\mathbf{h}|} \int_Q |\Delta_{\mathbf{h}} f^{+}_{1,2}|^2 \, dx}_{\overset{(55)}{\lesssim} \eta^{-1} \mathcal{E}_\eta(\chi)} \Bigg)^{\frac{2}{3}} \Bigg(\underbrace{\int_Q \left| |\nabla|^{-1} (\partial_1 + \partial_2) f^{+}_{1,2} \right|^2 dx}_{\overset{(56)}{\lesssim} \mathcal{E}_\eta(\chi)} \Bigg)^{\frac{1}{3}} \\ &\quad \lesssim \eta^{-2/3} \mathcal{E}_\eta(\chi). \end{aligned} \tag{58}$$

In turn, by cyclical symmetry, (57) and (58) yield (12).

4.2 Proof of Theorem 1

First, let us consider the decomposition as given in Proposition 1:

$$\left.\begin{aligned} \chi_1 - \theta_1 &= f_{1,2}^+ + f_{1,2}^- + f_{1,3}^+ + f_{1,3}^-, \\ \chi_2 - \theta_2 &= f_{2,3}^+ + f_{2,3}^- + f_{2,1}^+ + f_{2,1}^-, \\ \chi_3 - \theta_3 &= f_{3,1}^+ + f_{3,1}^- + f_{3,2}^+ + f_{3,2}^-. \end{aligned}\right\} \tag{59}$$

Without loss of generality we may assume

$$\langle f_{i,j}^{\pm} \rangle = 0. \tag{60}$$

Now, we proceed in several steps:

Step 1 *In this step, we show that two functions enjoying particular symmetry properties depend on orthogonal variables, respectively.*

Let $\{b_1, b_2, b_3\}$ be a basis of $\mathbb{R}^3$, and f, g be Q-periodic square integrable functions such that in distributional sense

$$b_1 \cdot \nabla f = 0, \quad b_2 \cdot \nabla f = 0, \quad \textit{and} \quad b_3 \cdot \nabla g = 0. \tag{61}$$

Then we have

$$\langle fg \rangle = \langle f \rangle \langle g \rangle.$$

Without loss of generality we may assume that f and g are smooth. On the level of Fourier series, since $\widehat{\nabla h}(k) = -i\hat{h}(k)k$, (61) translates into

$$\begin{aligned} \hat{f}(k) &= 0 \quad \text{for} \quad k \in 2\pi\mathbb{Z}^3 \quad \text{with} \quad b_1 \cdot k \neq 0 \text{ or } b_2 \cdot k \neq 0, \\ \hat{g}(k) &= 0 \quad \text{for} \quad k \in 2\pi\mathbb{Z}^3 \quad \text{with} \quad b_3 \cdot k \neq 0. \end{aligned}$$

Since $\{b_1, b_2, b_3\}$ is a basis, this yields

$$\hat{f}(k)\overline{\hat{g}(k)} = 0 \quad \text{for} \quad k \in 2\pi\mathbb{Z}^3 \quad \text{with} \quad k \neq 0.$$

Therefore by Plancherel we obtain

$$\langle f\,\overline{g} \rangle \;=\; \sum_{k \in 2\pi\mathbb{Z}^3} \hat{f}(k)\overline{\hat{g}(k)} \;=\; \hat{f}(0)\overline{\hat{g}(0)} \;=\; \langle f \rangle \langle \overline{g} \rangle.$$

Step 2 *In this step, we argue that based on* (12) *and* (9)*, $f_{i,j}^{\pm}$ is close to a function $g_{i,j}^{\pm}$ which only depends on one variable.*

There exist functions $g_{i,j}^{\pm}$ such that

$$(\partial_i \pm \partial_j)\, g_{i,j}^{\pm} \;=\; 0 \quad \textit{and} \quad \partial_l\, g_{i,j}^{\pm} \;=\; 0, \tag{62}$$

in distributional sense, where $i, j, l \in \{1, 2, 3\}$ are pairwise different, and

$$\langle |f_{i,j}^{\pm} - g_{i,j}^{\pm}|^4 \rangle \;\ll 1. \tag{63}$$

We define $g^{\pm}_{i,j}$ to be the $L^2(Q)$-orthogonal projection of $f^{\pm}_{i,j}$ onto the space of Q-periodic functions with the symmetry requirement (62). We argue that instead of (63), it is sufficient to show

$$\langle |f^{\pm}_{i,j} - g^{\pm}_{i,j}|^2 \rangle \ll 1. \tag{64}$$

Indeed, (63) follows from (64),

$$\langle |f^{\pm}_{i,j}|^6 \rangle \lesssim 1 \quad \text{and} \quad \langle |g^{\pm}_{i,j}|^6 \rangle \lesssim 1, \tag{65}$$

via Hölder's inequality and the triangle inequality in $L^6(Q)$.

We now give the argument for (65). Since according to Proposition 1 we have $\langle |f^{\pm}_{i,j}|^6 \rangle \lesssim 1$, it suffices to show

$$\langle |g^{\pm}_{i,j}|^6 \rangle \;\le\; \langle |f^{\pm}_{i,j}|^6 \rangle.$$

This inequality follows from the fact that with $g^{\pm}_{i,j}$ also

$$h^{\pm}_{i,j} := (g^{\pm}_{i,j})^5$$

satisfies the symmetry requirement (62), thus

$$\begin{aligned}
\langle |g^{\pm}_{i,j}|^6 \rangle &= \langle g^{\pm}_{i,j}\, h^{\pm}_{i,j} \rangle && \text{by definition of } h^{\pm}_{i,j} \\
&= \langle f^{\pm}_{i,j}\, h^{\pm}_{i,j} \rangle && \text{by definition of } g^{\pm}_{i,j} \\
&\le \langle |f^{\pm}_{i,j}|^6 \rangle^{1/6} \langle |h^{\pm}_{i,j}|^{6/5} \rangle^{5/6} && \text{by Hölder's inequality} \\
&= \langle |f^{\pm}_{i,j}|^6 \rangle^{1/6} \langle |g^{\pm}_{i,j}|^6 \rangle^{5/6}.
\end{aligned}$$

Now, we present the argument in favor of (64). Notice that because of $\widehat{\Delta_{\mathbf{h}} f^{\pm}_{i,j}} = (\exp(ik\cdot\mathbf{h}) - 1)\hat{f}^{\pm}_{i,j}$ and Parseval, (12) translates into

$$\sum_{k\in 2\pi\mathbb{Z}^3} \sin^2(k\cdot\mathbf{h}) |\hat{f}^{\pm}_{i,j}|^2 \lesssim \eta^{-2/3}\mathcal{E}_\eta(\chi) \overset{(9)}{\ll} 1. \tag{66}$$

In view of (13), (66) holds uniformly for

$$\begin{aligned}
\mathbf{h} &= s\,(e_i \pm e_j), \quad s \in \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right] \quad \text{or} \\
\mathbf{h} &= s\,e_l, \qquad\qquad s \in [-1,1].
\end{aligned}$$

Since $k \in 2\pi\mathbb{Z}^3, k \neq 0$ implies

$$\begin{aligned}
|k\cdot(e_i \pm e_j)| &\gtrsim 1 \quad \text{for} \quad k\cdot(e_j \pm e_i) \neq 0 \quad \text{and} \\
|k\cdot e_l| &\gtrsim 1 \quad \text{for} \quad k\cdot e_l \neq 0.
\end{aligned}$$

Hence we have

$$\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \sin^2(s\, k\cdot(e_i \pm e_j))\, ds \gtrsim 1 \qquad \text{for} \quad k\cdot(e_j \pm e_i) \neq 0 \quad \text{and}$$
$$\int_{-1}^{1} \sin^2(s\, k\cdot e_l)\, ds \gtrsim 1 \qquad \text{for} \quad k\cdot e_l \neq 0.$$

Therefore, (66) yields

$$\sum_{\substack{k\in 2\pi\mathbb{Z}^3\\ k\cdot(e_i\pm e_j)\neq 0 \text{ or } k\cdot e_l\neq 0}} |\hat{f}^{\pm}_{i,j}|^2 \ll 1. \tag{67}$$

We notice that on the level of Fourier series, the definition of $g^{\pm}_{i,j}$ turns into

$$\hat{g}^{\pm}_{i,j} = \begin{cases} \hat{f}^{\pm}_{ij} & \text{for} \quad k\cdot(e_i \pm e_j) = 0 \quad \text{or} \quad k\cdot e_l = 0, \\ 0 & \text{otherwise,} \end{cases}$$

so that by Parseval

$$\langle |f^{\pm}_{i,j} - g^{\pm}_{i,j}|^2\rangle = \sum_{\substack{k\in 2\pi\mathbb{Z}^3\\ k\cdot(e_i\pm e_j)\neq 0 \text{ or } k\cdot e_l\neq 0}} |\hat{f}^{\pm}_{i,j}|^2.$$

Hence, (64) follows from (67).

Step 3 *In this step, we argue that only 8 instead of 16 functions of one variable each suffice to approximately describe the χ_i's.*

For $i \neq j$ we have that

$$\langle |f^{+}_{i,j} + f^{+}_{j,i}|^2\rangle \ll 1 \quad \textit{and} \quad \langle |f^{-}_{i,j} + f^{-}_{j,i}|^2\rangle \ll 1. \tag{68}$$

Hence, since the image of the 3×3 matrix

$$\begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

is given by the plane $(\chi_1-\theta_1)+(\chi_2-\theta_2)+(\chi_3-\theta_3)=0$, we may modify $f^{\pm}_{i,j}$ such that

$$\left.\begin{aligned} \chi_1-\theta_1 &= (f^{+}_{1,2}+f^{-}_{1,2}) - (f^{+}_{3,1}+f^{-}_{3,1}), \\ \chi_2-\theta_2 &= (f^{+}_{2,3}+f^{-}_{2,3}) - (f^{+}_{1,2}+f^{-}_{1,2}), \\ \chi_3-\theta_3 &= (f^{+}_{3,1}+f^{-}_{3,1}) - (f^{+}_{2,3}+f^{-}_{2,3}), \end{aligned}\right\} \tag{69}$$

and where the $f^{\pm}_{i,j}$ are such that $g^{\pm}_{i,j}$ exist as in Step 1.

Indeed, because (3) and (8), summation of (59) yields

$$F^{+}_{1,2} + F^{-}_{1,2} + F^{+}_{2,3} + F^{-}_{2,3} + F^{+}_{3,1} + F^{-}_{3,1} = 0, \tag{70}$$

where for $i \neq j$,

$$F_{i,j}^{+} := f_{i,j}^{+} + f_{j,i}^{+}, \quad F_{i,j}^{-} := f_{i,j}^{-} + f_{j,i}^{-}. \tag{71}$$

Next, take the square of (70) and integrate on Q to get

$$\begin{aligned}0 = \langle |F_{1,2}^{+}|^2\rangle + \langle |F_{1,2}^{-}|^2\rangle + \langle |F_{2,3}^{+}|^2\rangle + \langle |F_{2,3}^{-}|^2\rangle + \langle |F_{3,1}^{+}|^2\rangle + \langle |F_{3,1}^{-}|^2\rangle \\ +2\langle F_{1,2}^{+} F_{1,2}^{-}\rangle + 2\langle F_{1,2}^{+} F_{2,3}^{+}\rangle + 2\langle F_{1,2}^{+} F_{2,3}^{-}\rangle + 2\langle F_{1,2}^{+} F_{3,1}^{+}\rangle + 2\langle F_{1,2}^{+} F_{3,1}^{-}\rangle \\ +2\langle F_{1,2}^{-} F_{2,3}^{+}\rangle + 2\langle F_{1,2}^{-} F_{2,3}^{-}\rangle + 2\langle F_{1,2}^{-} F_{3,1}^{+}\rangle + 2\langle F_{1,2}^{-} F_{3,1}^{-}\rangle \\ +2\langle F_{2,3}^{+} F_{2,3}^{-}\rangle + 2\langle F_{2,3}^{+} F_{3,1}^{+}\rangle + 2\langle F_{2,3}^{+} F_{3,1}^{-}\rangle \\ +2\langle F_{2,3}^{-} F_{3,1}^{+}\rangle + 2\langle F_{2,3}^{-} F_{3,1}^{-}\rangle \\ +2\langle F_{3,1}^{+} F_{3,1}^{-}\rangle.\end{aligned}$$

This yields (68) provided we can show that the mixed terms are small. Up to symmetry (permutation and reflection of the Cartesian coordinates), there are two types of mixed terms, namely

$$\langle F_{1,2}^{+} F_{1,2}^{-}\rangle \quad \text{and} \quad \langle F_{1,2}^{+} F_{2,3}^{+}\rangle.$$

In view of (63), it suffices to show

$$\langle G_{1,2}^{+} G_{1,2}^{-}\rangle = 0 \quad \text{and} \quad \langle G_{1,2}^{+} G_{2,3}^{+}\rangle = 0 \tag{72}$$

where the $G_{i,j}^{\pm}$ are defined analogously to (71). In particular,

$$(\partial_i \pm \partial_j) G_{i,j}^{\pm} = 0 \quad \text{and} \quad \partial_l G_{i,j}^{\pm} = 0$$

distributionally for i, j, l pairwise different. Hence the first item in (72) follows from Step 1 since

$$b_1 = e_1 + e_2, \quad b_2 = e_3, \quad b_3 = e_1 - e_2$$

form a basis of $\mathbb{R}^3$. Likewise, the second item in (72) follows from Step 1 since

$$b_1 = e_1 + e_2, \quad b_2 = e_3, \quad b_3 = e_1$$

form a basis of $\mathbb{R}^3$.

Step 4 *In this step, we argue that at least one of the volume fractions should be small.*

We claim that

$$\theta_1 \, \theta_2 \, \theta_3 \ll 1. \tag{73}$$

On the one hand, since the χ_i's have disjoint support, we have

$$(\chi_1 - \theta_1)(\chi_2 - \theta_2)(\chi_3 - \theta_3) = \chi_1\theta_2\theta_3 + \theta_1\chi_2\theta_3 + \theta_1\theta_2\chi_3 - \theta_1\theta_2\theta_3.$$

Thus, because of (8),

$$\langle (\chi_1 - \theta_1)(\chi_2 - \theta_2)(\chi_3 - \theta_3)\rangle = 2\theta_1 \, \theta_2 \, \theta_3. \tag{74}$$

On the other hand, using the abbreviations

$$f_1 := f^+_{2,3} + f^-_{2,3}, \quad f_2 := f^+_{3,1} + f^-_{3,1}, \text{ and } f_3 := f^+_{1,2} + f^-_{1,2}, \tag{75}$$

we obtain from (69)

$$\begin{aligned}(\chi_1 - \theta_1)(\chi_2 - \theta_2)(\chi_3 - \theta_3) &= (f_3 - f_2)(f_1 - f_3)(f_2 - f_1) \\ &= -f_1 f_2^2 - f_1^2 f_3 + f_1^2 f_2 - f_2 f_3^2 + f_2^2 f_3 + f_1 f_3^2. \end{aligned} \tag{76}$$

We claim that for $i \neq j$

$$\langle f_i\, f_j^2 \rangle \ll 1. \tag{77}$$

Assuming the claim for a moment, (73) follows from (74), (76), and (77).

Now, we prove (77). It suffices to argue that for $i \neq j$

$$\langle g_i\, g_j^2 \rangle = 0,$$

where the g_j's are defined analogously to (75). This follows from

$$\langle g^+_{i+1,i+2}\, g_j^2 \rangle = 0 \quad \text{and} \quad \langle g^-_{i+1,i+2}\, g_j^2 \rangle = 0, \tag{78}$$

where the indices are cyclically defined modulo 3. As in the previous step, we apply Step 1. For the first item in (78) we note that

$$(\partial_{i+1} + \partial_{i+2}) g^+_{i+1,i+2} = 0, \quad \partial_i g^+_{i+1,i+2} = 0, \quad \text{and} \quad \partial_j g_j^2 = 0$$

distributionally and that for $i \neq j$

$$b_1 = e_{i+1} + e_{i+2}, \quad b_2 = e_i, \quad b_3 = e_j$$

form a basis of $\mathbb{R}^3$. The argument for the second item in (78) relies on the fact that

$$b_1 = e_{i+1} - e_{i+2}, \quad b_2 = e_i, \quad b_3 = e_j$$

also form a basis of $\mathbb{R}^3$.

Step 5 *In this step, we argue that the two remaining χ_i's can approximately be written as the sum of two functions of one variable each.*

In view of (73) *we may assume*

$$\theta_3 \ll 1. \tag{79}$$

In particular, this implies

$$\langle |f^+_{3,1}|^2 \rangle \ll 1, \ \ \langle |f^-_{3,1}|^2 \rangle \ll 1, \ \ \langle |f^+_{2,3}|^2 \rangle \ll 1, \ \ \textit{and} \ \ \langle |f^-_{2,3}|^2 \rangle \ll 1. \tag{80}$$

Hence, we may modify $f^\pm_{1,2}$ such that

$$\chi_1 - \theta_1 = f^+_{1,2} + f^-_{1,2}, \tag{81}$$

and where $f^{\pm}_{1,2}$ are such that $g^{\pm}_{1,2}$ exists as in Step 1.

Indeed, since

$$\langle|\chi_3 - \theta_3|^2\rangle \overset{(8)}{=} \theta_3(1-\theta_3)$$

because of (69) and (79) we have

$$\langle\big|f^+_{3,1} + f^-_{3,1} - f^+_{2,3} - f^-_{2,3}\big|^2\rangle \ll 1.$$

In turn, this implies (80) provided

$$\langle f^+_{3,1} f^-_{3,1}\rangle \ll 1, \quad \langle f^+_{3,1}\, f^+_{2,3}\rangle \ll 1, \quad \langle f^-_{3,1}\, f^-_{2,3}\rangle \ll 1, \quad \langle f^+_{2,3}\, f^-_{2,3}\rangle \ll 1. \tag{82}$$

Finally, to prove (82) we can argue as in the proof of (72).

Step 6 *In this step, we show that the two remaining χ_i's can be approximated by a single function of one variable.*

With the assumption of Step 5, we have the dichotomy

$$\langle|f^+_{1,2}|^2\rangle \ll 1 \quad \textit{or} \quad \langle|f^-_{1,2}|^2\rangle \ll 1. \tag{83}$$

First, in view of (63) (and Jensen's inequality) it suffices to show that

$$\langle|g^+_{1,2}|^2\rangle \ll 1 \quad \text{or} \quad \langle|g^-_{1,2}|^2\rangle \ll 1. \tag{84}$$

Second, in view of (63) and (81) we have

$$\langle|\chi_1 - \theta_1 - (g^+_{1,2} + g^-_{1,2})|^2\rangle \ll 1. \tag{85}$$

In addition, since $g^+_{1,2}$ and $g^-_{1,2}$ are uniformly bounded in $L^6(Q)$, by Hölder's inequality, (85) also implies

$$\langle|\chi_1 - \theta_1 - (g^+_{1,2} + g^-_{1,2})|^4\rangle \ll 1. \tag{86}$$

Next, we need the following formula:

$$\begin{aligned}
&\big|g^+_{1,2} + g^-_{1,2} + \theta_1\big|^2 \big|g^+_{1,2} + g^-_{1,2} + \theta_1 - 1\big|^2 \\
&\quad = \big|(g^+_{1,2} + \theta_1)(g^+_{1,2} + \theta_1 - 1) + (g^-_{1,2} + \theta_1)(g^-_{1,2} + \theta_1 - 1) - \theta_1(\theta_1 - 1)\big|^2 \\
&\qquad +4|g^+_{1,2}|^2|g^-_{1,2}|^2 + 4\big[(g^+_{1,2})^3 g^-_{1,2} + g^+_{1,2}(g^-_{1,2})^3\big] + 4\theta_1(\theta_1-1)g^+_{1,2}g^-_{1,2} \\
&\qquad\quad +4(2\theta_1 - 1)\Big[|g^+_{1,2}|^2 g^-_{1,2} + g^+_{1,2}|g^-_{1,2}|^2\Big].
\end{aligned} \tag{87}$$

The validity of this formula is best seen as follows:

$$\begin{aligned}
&\big|g^+ + g^- + \theta_1\big|^2 \big|g^+ + g^- + \theta_1 - 1\big|^2 \\
&\quad -\big|(g^+ + \theta_1)(g^+ + \theta_1 - 1) + (g^- + \theta_1)(g^- + \theta_1 - 1) - \theta_1(\theta_1 - 1)\big|^2 \\
&\qquad = \quad \big||g^+|^2 + 2g^+g^- + |g^-|^2 + (2\theta - 1)(g^+ + g^-) + \theta(1-\theta)\big|^2 \\
&\qquad\quad - \big||g^+|^2 \qquad\qquad + |g^-|^2 + (2\theta - 1)(g^+ + g^-) + \theta(1-\theta)\big|^2 \\
&\qquad = 4g^+g^-\left(|g^+|^2 + g^+g^- + |g^-|^2 + (2\theta-1)(g^+ + g^-) + \theta(1-\theta)\right) \\
&\qquad = 4\left((g^+)^3 g^- + |g^+|^2|g^-|^2 + g^+(g^-)^3\right) \\
&\qquad\quad +4(2\theta_1 - 1)\left(|g^+|^2 g^- + g^+|g^-|^2\right) + 4\theta_1(\theta_1 - 1)g^+g^-.
\end{aligned}$$

On the one hand, since the first term in the right hand side of (87) is positive, we find

$$\begin{aligned}\int_Q \left|g_{1,2}^+ + g_{1,2}^- + \theta_1\right|^2 \left|g_{1,2}^+ + g_{1,2}^- + \theta_1 - 1\right|^2 dx \quad &\geq \quad 4\int_Q |g_{1,2}^+|^2 |g_{1,2}^-|^2 dx \\ &+4\int_Q \left[(g_{1,2}^+)^3 g_{1,2}^- + g_{1,2}^+ (g_{1,2}^-)^3\right] dx + 4\theta_1(\theta_1 - 1)\int_Q g_{1,2}^+ g_{1,2}^- dx \\ &+4(2\theta_1 - 1)\theta_1 \int_Q \left[|g_{1,2}^+|^2 g_{1,2}^- + g_{1,2}^+ |g_{1,2}^-|^2\right] dx. \qquad (88)\end{aligned}$$

We note that, for any $p, q \in \mathbb{N}$

$$(\partial_1 + \partial_2)(g_{1,2}^+)^p = 0, \qquad \partial_3 (g_{1,2}^+)^p = 0 \quad \text{and} \quad (\partial_1 - \partial_2)(g_{1,2}^-)^q = 0$$

distributionally, and that

$$b_1 = e_1 + e_2, \qquad b_2 = e_3 \quad \text{and} \quad b_3 = e_1 - e_2$$

form a basis in $\mathbb{R}^3$. Then, because of Step 1 and $\langle g_{1,2}^{\pm}\rangle = 0$, (88) yields

$$\int_Q \left|g_{1,2}^+ + g_{1,2}^- + \theta_1\right|^2 \left|g_{1,2}^+ + g_{1,2}^- + \theta_1 - 1\right|^2 dx \gtrsim \left(\int_Q |g_{1,2}^+|^2 dx\right)\left(\int_Q |g_{1,2}^-|^2 dx\right). \qquad (89)$$

On the other hand, we notice that because of (86) and $\chi_1 - \theta_1 \in \{-\theta_1, 1 - \theta_1\}$

$$\int_Q \left|g_{1,2}^+ + g_{1,2}^- + \theta_1\right|^2 \left|g_{1,2}^+ + g_{1,2}^- + \theta_1 - 1\right|^2 dx \ll 1,$$

holds. Thus, from (89) we obtain

$$\left(\int_Q |g_{1,2}^+|^2 dx\right)\left(\int_Q |g_{1,2}^-|^2 dx\right) \ll 1,$$

which in turn yields (84).

5 Optimality and branching

In the proof of Theorem 2 we use, as basic building block, the 2-d construction in the following lemma. It is the construction of a branched laminate which fills part of the 2-d periodic cell. The laminate refines by branching towards the boundary of the part it fills. This construction is a minor adaptation of [4, Proposition 3.1]. The only difference is the periodicity in y_2 and the arbitrary volume fraction λ.

Lemma 2. *For any $\lambda, \theta \in [0,1]$ there exists an $(0,1)^2$- periodic function $\sigma_\theta : \mathbb{R}^2 \to \{-1, 2\lambda - 1, 1\}$, with*

$$\sigma_\theta(y_1, y_2) \begin{cases} \in \{-1, 1\} & \text{for } y_2 \in (0,\theta), \\ = 2\lambda - 1 & \text{for } y_2 \in (\theta, 1), \end{cases}$$

and

$$\int_{(0,1)^2} \sigma_\theta \, dy = 2\lambda - 1,$$

such, that for sufficiently small η,

$$\mathcal{E}_\eta^2(\sigma_\theta) := \eta \int_{(0,1)^2} |\nabla \sigma_\theta| dy + \sum_{\substack{k \in 2\pi\mathbb{Z}^2 \\ k \neq 0}} |k|^{-2} k_2^2 |\hat{\sigma}_\theta|^2 \lesssim \eta^{2/3}.$$

Proof. We prove this lemma by constructing a function that enjoys the prescribed properties in $[0,1]^2$ and extend it to all of $\mathbb{R}^2$ by periodicity. The basic structure of σ_θ in $[0,1]^2$ is summarized in Figure 1. For $y_2 \in (\theta, 1)$, σ_θ takes the constant value $2\lambda - 1$ and for $y_2 \in (0,\theta)$ the function oscillates between the values -1 and 1 with volume fractions $1-\lambda$ and λ, respectively. With this structure we have as desired

$$\begin{aligned} \int_{(0,1)^2} \sigma_\theta \, dy &= \int_{(0,1)\times(0,\theta)} \sigma_\theta \, dy + \int_{(0,1)\times(\theta,1)} (2\lambda - 1) \, dy \\ &= \lambda\theta - (1-\lambda)\theta + (1-\theta)(2\lambda - 1) \\ &= 2\lambda - 1. \end{aligned}$$

For the region $y_2 \in (0,\theta)$ the structure of σ_θ is defined as follows: Let N and w_1^{-1} be two natural numbers, and for $n \in \{1, 2, \dots, N\}$ define

$$w_n = 2^{-(n-1)} w_1 \qquad \text{and} \qquad \ell_n = 2^{-\beta(n-1)} \ell_1, \tag{90}$$

where $\beta > 0$ is a number to be chosen later and ℓ_1 is such that

$$\sum_{n=1}^{N} \ell_n = \theta/2. \tag{91}$$

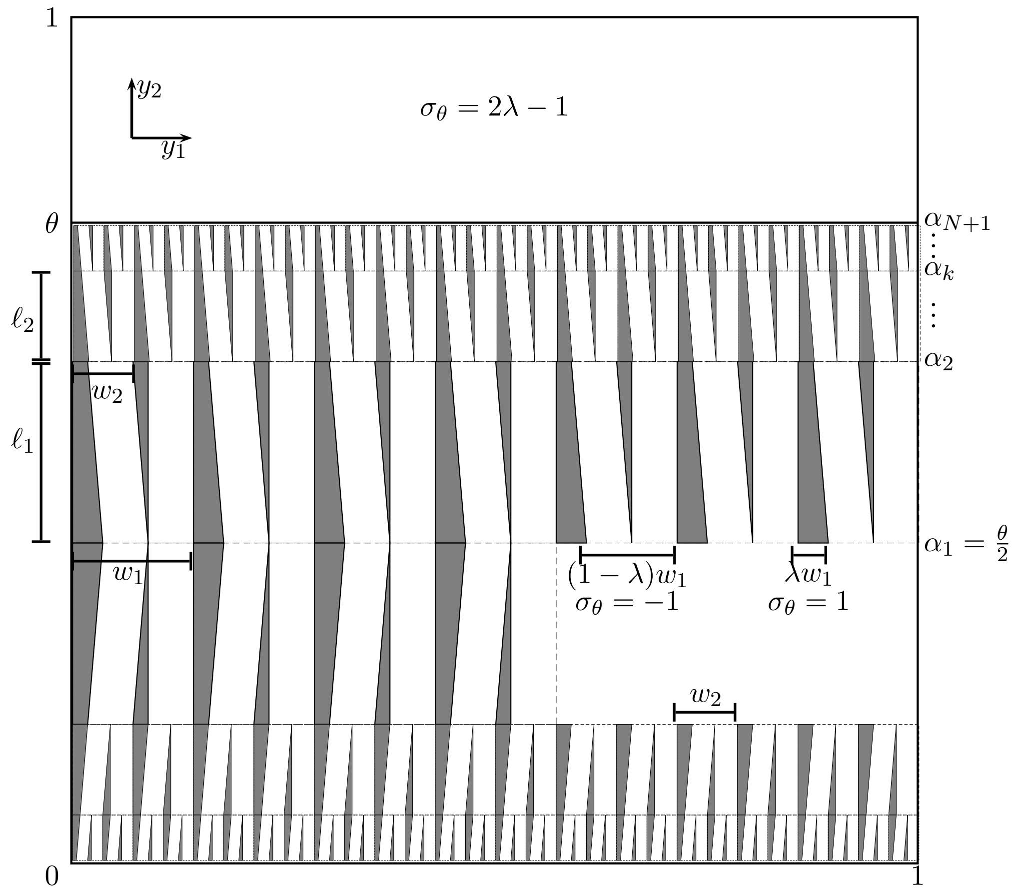

Figure 1: Construction of σ_θ

For the y_2-values $\alpha_n = \theta/2 + \sum_{k=2}^{n} \ell_{k-1}$, $n \in \{1, \dots, N+1\}$, we let for each n the function $\sigma_\theta(\cdot, \alpha_n)$ to oscillate between the values -1 and 1 on horizontal segments of length $(1-\lambda)w_n$ and λw_n, respectively. For the regions in-between, namely $\alpha_n < y_2 < \alpha_{n+1}$, we define σ_θ as w_n^{-1} copies of a basic cell of width w_n and high ℓ_n as given in Figure 2. Finally, for $y_2 \in (0, \theta/2)$ we extend σ_θ by an even reflection along the line $y_2 = \theta/2$.

In order to prove the lemma it only remains to show that, for sufficiently small η, we may chose β, w_1 and N such that

$$\mathcal{E}_\eta^2(\sigma_\theta) \lesssim \eta^{2/3}. \tag{92}$$

First, we consider $(0,1)^2$-periodic vector fields h satisfying

$$\int_{(0,1)^2} h \cdot \nabla\varphi \, dy \;=\; \int_{(0,1)^2} \sigma_\theta \, \partial_2 \varphi \, dy \qquad \begin{array}{c} \forall \varphi : \mathbb{R}^3 \to \mathbb{R}^3 \\ (0,1)^2 - \text{periodic} \end{array} \tag{93}$$

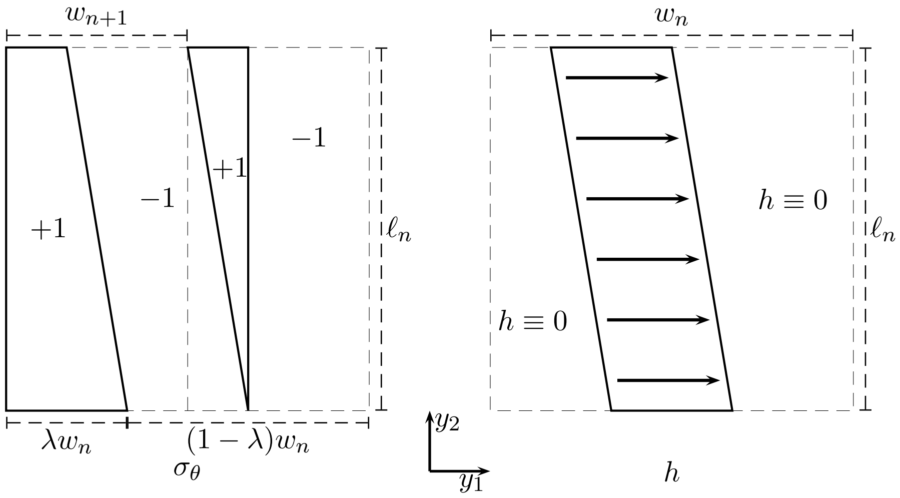

Figure 2: Basic splitting cell

and notice that the bulk energy may be characterized as follows

$$\sum_{\substack{k \,\in\, 2\pi\mathbb{Z}^2 \\ k \neq 0}} |k|^{-2} k_2^2 |\hat{\sigma_\theta}|^2 = \int_{(0,1)^2} ||\nabla|^{-1} \partial_2 \sigma_\theta|^2 \, dy = \inf_{h \ \text{ with (93)}} \left\{ \int_{(0,1)^2} |h|^2 \, dy \right\}. \tag{94}$$

Hence, we may estimate the bulk energy of σ_θ by a test vector field h that is admissible in the sense that (93) holds. In Figure 2 we describe an admissible h corresponding to σ_θ for each splitting cell. For this specific test vector field we have $h = (h_1, 0)$, so that (93) turns into

$$\int_{(0,1)^2} h_1 \partial_1 \varphi \, dy \ = \int_{(0,1)^2} \sigma_\theta \, \partial_2 \varphi \, dy \qquad \begin{array}{c} \forall \varphi : \mathbb{R}^3 \to \mathbb{R}^3 \\ (0,1)^2 - \text{periodic.} \end{array} \tag{95}$$

The test vector field h is not continuous through the lines $y_2 = \alpha_n$. Nevertheless, since $h_2 \equiv 0$ in the branching region $(0,1) \times (0,\theta)$ these jumps do not contribute to the distributional divergence $\nabla \cdot h$ and therefore do not affect the estimate on the energy.

Second, we estimate the energy by a scaling argument. Because of (95), it is clear that for each cell the test vector field $h = (h_1, 0)$ scales as

$$|h| = |h_1| \lesssim w_n \, \ell_n^{-1}.$$

Therefore, on each cell the energy is bounded by

$$\eta \int |\nabla \sigma| \, dy + \int |h|^2 \, dy \ \lesssim \ \eta \ \max\{w_n, \ell_n\} + \left(w_n \, \ell_n^{-1}\right)^2 (w_n \, \ell_n). \tag{96}$$

Assuming

$$w_n \leq \ell_n, \tag{97}$$

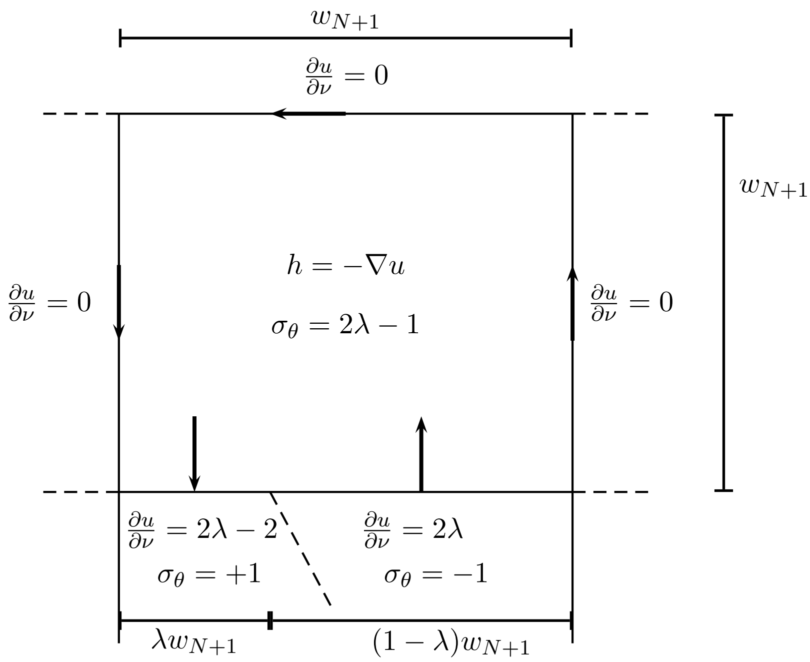

Figure 3: Closing domains

we have

$$\eta \int |\nabla \sigma| \, dy + \int |h|^2 \, dy \;\lesssim\; \eta \, \ell_n + w_n^3 \, \ell_n^{-1}. \tag{98}$$

Thus, since the number of basic cells in the n^{th} row is w_n^{-1}, the energy in the branching region $\mathcal{B} = (0, \theta) \times (0, 1)$ is estimated by

$$\begin{aligned} \eta \int_{\mathcal{B}} |\nabla \sigma| \, dy + \int_{\mathcal{B}} |h|^2 \, dy \;&\lesssim\; \sum_{n=1}^{N} w_n^{-1} \left(\eta \, \ell_n + w_n^3 \ell_n^{-1} \right) \\ &\lesssim\; \sum_{n=1}^{N} \eta \left(\ell_n \, w_n^{-1} + w_n^2 \, \ell_n^{-1} \right). \end{aligned} \tag{99}$$

Next, we have to estimate the energy due to the transition between the branching region $\mathcal{B}$ and the region $\mathcal{C}$ where σ_θ takes the constant value $2\lambda - 1$. The surface energy on these transitions is simply estimated by η.

Now we estimate the bulk energy. First, notice that σ_θ jumps between ± 1 and $2\lambda - 1$ along the y_2-direction along the transition between $\mathcal{B}$ and $\mathcal{C}$. Therefore, we need to define a test vector field h on $\mathcal{C}$ such that (95) holds not only in $\mathcal{C}$ alone, but in all of $(0,1)^2$. With this aim, we consider a region $(0,1) \times (\theta, \theta + w_{N+1}) \cup (0,1) \times (1 - w_{N+1}, 1)$ of closure domains of h as represented as in Figure 3. These closure domains have to be small enough to fit inside the region $y_2 \in (\theta, 1)$, thus we have the constraint

$$w_{N+1} \le \frac{1-\theta}{2}. \tag{100}$$

We define h on each closure domain cell as follows: let u be the solution (unique up to an additive constant) of the Neumann problem

$$\begin{cases} -\Delta u &=& 0 \quad \text{in} \quad (0, w_{N+1})^2, \\ \dfrac{\partial u}{\partial \nu} &=& \begin{cases} 2(1-\lambda) & \text{for } y_1 \in (0,\lambda) \text{ and } y_2 = 0, \\ -2\lambda & \text{for } y_1 \in (\lambda, 1) \text{ and } y_2 = 0, \\ 0 & \text{otherwise.} \end{cases} \end{cases}$$

Inside each closure domain cell we let $h = -\nabla u$. Thus,

$$\int_{(0,w_{N+1})^2} |h|^2 dy \;=\; \int_{(0,w_{N+1})^2} |\nabla u|^2 dy \;\lesssim\; w_N^2 \tag{101}$$

for each of the $2\, w_{N+1}^{-1}$ closure domain cells.

Hence, by combining estimate (99) in the branching region $\mathcal{B}$ with the estimate (101) in the closure region, we find

$$\begin{aligned} \eta \int_{(0,1)^2} |\nabla \sigma|\, dy + \int_{(0,1)^2} |h|^2\, dy \;&\lesssim\; \sum_{n=1}^{N} \left(\eta\, \ell_n\, w_n^{-1} + w_n^2\, \ell_n^{-1}\right) + \eta + w_N. \\ &\overset{(90)}{=} \sum_{n=1}^{N} \left(\eta\, 2^{(n-1)(1-\beta)} \frac{\ell_1}{w_1} + 2^{(n-1)(\beta-2)}\, \frac{w_1^2}{\ell_1}\right) + \eta + 2^{-N} w_1. \end{aligned}$$

The right hand side suggests to select $\beta \in (1,2)$, say $\beta = 3/2$. We obtain

$$\eta \int_{(0,1)^2} |\nabla \sigma|\, dy + \int_{(0,1)^2} |h|^2\, dy \;\lesssim\; \eta\, \frac{\ell_1}{w_1} + \frac{w_1^2}{\ell_1} + \eta + 2^{-N} w_1.$$

Since

$$\frac{\theta}{2} \;\overset{(91)}{=}\; \sum_{n=1}^{N} \ell_n \;\overset{(90)}{=}\; \sum_{n=1}^{N} 2^{-\beta(n-1)} \ell_1 \;\sim\; \ell_1, \tag{102}$$

this turns into

$$\eta \int_{(0,1)^2} |\nabla \sigma|\, dy + \int_{(0,1)^2} |h|^2\, dy \;\lesssim\; \eta\, \frac{\theta}{w_1} + \frac{w_1^2}{\theta} + \eta + 2^{-N} w_1.$$

In order to balance the first two terms on the right hand side in terms of η, we choose

$$w_1 = \theta\, \eta^{1/3}, \tag{103}$$

so that we obtain

$$\eta \int_{(0,1)^2} |\nabla \sigma|\, dy + \int_{(0,1)^2} |h|^2\, dy \;\lesssim\; \eta^{2/3} + \eta + 2^{-N} \eta^{1/3}. \tag{104}$$

In order to motivate the choice of N, we turn to the constraints, i.e. (97) and (100). In view of (90), (103), (102) and $\beta = 3/2$, (97) turns into

$$2^{-(n-1)}\,\theta\,\eta^{1/3} \;\lesssim\; 2^{-3/2(n-1)}\,\theta \quad \text{for} \quad n \in \{1,\dots,N+1\},$$

i.e.

$$\eta^{2/3} \;\lesssim\; 2^{-N}.$$

This motivates to choose the integer $N \in \mathbb{N}$ such that

$$\eta^{2/3} \sim 2^{-N}, \tag{105}$$

which we can since $\eta \ll 1$.

We now turn to the constraint (100), which in view of (90), (103) and (105) reads as

$$\theta\,\eta \ll 1 - \theta$$

and this is always satisfied provided $\eta \ll 1 - \theta$. In view of (105), (104) turns as desired into

$$\begin{aligned} \eta \int_{(0,1)^2} |\nabla \sigma|\,dy + \int_{(0,1)^2} |h|^2\,dy \quad &\lesssim \quad \eta^{2/3} + \eta \\ &\overset{\eta \ll 1}{\lesssim} \quad \eta^{2/3}. \end{aligned}$$

□

5.1 Proof of Theorem 2

Within a periodic cell, our construction will consist of two patches: one patch which is a branched laminate of variants 1 and 2, and a second patch which is a branched laminate of variants 1 and 3, see Figure 4. It is the variable $x_2 + x_3$ which distinguishes between the patches. The laminates refine by branching towards the interface of the patches. The main direction of the laminate is different for the two patches.

Let H_θ be a $(0,1)^2$-periodic function such that

$$H_\theta(y_1,y_2) = \begin{cases} 0 & \text{for } y_2 \in (0,\theta), \\ 2 - 2\lambda & \text{for } y_2 \in (\theta,1). \end{cases}$$

Let χ_1, χ_2, χ_3 be Q-periodic functions that we define (in Q) as follows:

$$\left.\begin{aligned} \chi_1(x_1,x_2,x_3) &= \tfrac{1}{2}\Big[\sigma_\theta(x_1 - x_2, x_2 + x_3) + \sigma_{1-\theta}(x_1 - x_3, -(x_2 + x_3)) + 2 - 2\lambda\Big], \\ \chi_2(x_1,x_2,x_3) &= \tfrac{1}{2}\Big[-(\sigma_\theta + H_\theta)(x_1 - x_2, x_2 + x_3) + 1\Big], \\ \chi_3(x_1,x_2,x_3) &= \tfrac{1}{2}\Big[-(\sigma_{1-\theta} + H_{1-\theta})(x_1 - x_3, -(x_2 + x_3)) + 1\Big], \end{aligned}\right\} \tag{106}$$

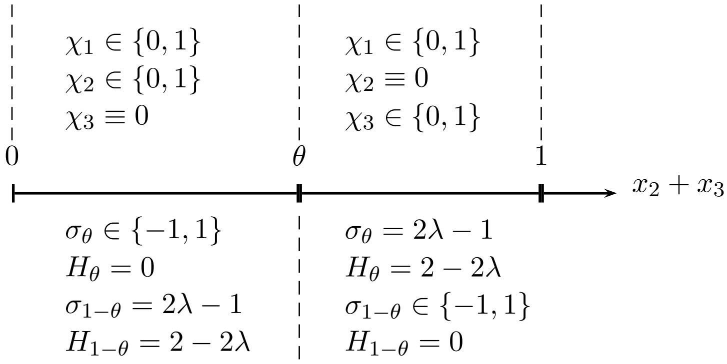

Figure 4: Support of χ_1, χ_2, χ_3

where σ_θ is as in Lemma 2.

We claim that

$$\chi_i \in \{0,1\}, \quad i = 1,2,3 \quad \text{with} \quad \chi_1 + \chi_2 + \chi_3 = 1, \tag{107}$$

and

$$\int_Q \chi_1 \, dx = \lambda, \quad \int_Q \chi_2 \, dx = \theta(1-\lambda) \quad \text{and} \quad \int_Q \chi_3 \, dx = (1-\theta)(1-\lambda). \tag{108}$$

Therefore, by choosing θ and λ we may achieve any prescribed combination of volume fractions. Now we present the argument for (107). The values that $\sigma_\theta, H_\theta, \sigma_{1-\theta}$ and $H_{1-\theta}$ as functions of the argument $x_2 + x_3$ are displayed in Figure 4. Thus, from (106) we get

$$\begin{aligned}
\chi_1(x) &= \frac{1}{2}\Big[\underbrace{\sigma_\theta(\cdot, x_2+x_3)}_{\in\{-1,1\}} + 2\lambda - 1 + 2 - 2\lambda\Big] \in \{0,1\} \quad \text{for} \quad x_2 + x_3 \in (0,\theta), \\
\chi_1(x) &= \frac{1}{2}\Big[2\lambda - 1 + \underbrace{\sigma_{1-\theta}(\cdot, -(x_2+x_3))}_{\in\{-1,1\}} + 2 - 2\lambda\Big] \in \{0,1\} \quad \text{for} \quad x_2 + x_3 \in (\theta,1).
\end{aligned}$$

Analogously,

$$\begin{aligned}
\chi_2(x) &= \frac{1}{2}\Big[\underbrace{-\sigma_\theta(\cdot, x_2+x_3)}_{\in\{-1,1\}} + 0 + 1\Big] \in \{0,1\}, \quad \text{for} \quad x_2 + x_3 \in (0,\theta), \\
\chi_2(x) &= \frac{1}{2}\Big[-(2\lambda - 1 + 2 - 2\lambda) + 1\Big] \equiv 0, \quad \text{for} \quad x_2 + x_3 \in (\theta,1),
\end{aligned}$$

and

$$\begin{aligned}
\chi_3(x) &= \frac{1}{2}\Big[-(2\lambda - 1 + 2 - 2\lambda) + 1\Big] \equiv 0, \quad \text{for} \quad x_2 + x_3 \in (0,\theta), \\
\chi_3(x) &= \frac{1}{2}\Big[\underbrace{-\sigma_{1-\theta}(\cdot, -(x_2+x_3))}_{\in\{-1,1\}} + 0 + 1\Big] \in \{0,1\}, \quad \text{for} \quad x_2 + x_3 \in (\theta,1).
\end{aligned}$$

The second item of (107) follows directly from (106) and the definition of H_θ. Finally, (108) holds because of (106),

$$\int_{(0,1)^2} \sigma_\theta(y)dy = 2\lambda - 1, \qquad \int_{(0,1)^2} H_\theta(y)dy = 2(\lambda-1)(1-\theta)$$

and the properties of the change of variables (113) defined below.

In order to prove the theorem it only remains to show that

$$\mathcal{E}_\eta(\chi) \lesssim \eta^{2/3}. \tag{109}$$

The surface energy contribution may be estimated in a straight forward manner:

$$\begin{aligned}
&\eta \int_Q |\nabla(\chi_1-\chi_2)| + |\nabla(\chi_2-\chi_3)| + |\nabla(\chi_3-\chi_1)|\, dx \\
&\quad\lesssim \eta \int_Q |\nabla\chi_1| dx + \eta \int_Q |\nabla\chi_2| dx + \eta \int_Q |\nabla\chi_3|\, dx \\
&\quad\lesssim \eta \int_{(0,1)^2} |\nabla\sigma_\theta| dy + \eta \int_{(0,1)^2} |\nabla\sigma_{1-\theta}| dy + \eta \int_{(0,1)^2} |\nabla H_\theta| dy + \eta \int_{(0,1)^2} |\nabla H_{1-\theta}| dy \\
&\quad\lesssim \eta^{2/3} + \eta|2-2\lambda|,
\end{aligned} \tag{110}$$

where we have use Lemma 2 in the last line. Next we estimate the bulk energy. We claim that if χ_0 and σ are two Q-periodic functions such that

$$\chi_0(x_1,x_2,x_3) = \sigma(x_1-x_3, x_2+x_3), \tag{111}$$

then their Fourier coefficients relate as follows

$$\hat{\chi}_0(k_1,k_2,k_3) = \begin{cases} \hat{\sigma}(k_1,k_2) & \text{if} \quad k_1-k_2+k_3=0, \\ 0 & \text{else.} \end{cases} \tag{112}$$

We prove (112) by considering the change of variables

$$y = Ax \quad \text{where} \quad A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}. \tag{113}$$

We note that

$$A^{-T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}. \tag{114}$$

Because of the relation (111), this change of variables yields

$$\begin{aligned}
\hat{\chi}_0(k) &= \frac{1}{(2\pi)^{3/2}} \int_Q \exp(ik\cdot x)\chi_0(x)\, dx \\
&\overset{(111)}{=} \frac{1}{(2\pi)^{3/2}} \int_{AQ} \exp(i\, A^{-T}k \cdot y)\sigma(y_1,y_2)\frac{1}{\det A}\, dy.
\end{aligned} \tag{115}$$

Since A^{-T} has integer coefficients, cf. (114), the function

$$y \ \to \ \exp\big(i\,A^{-T}k\cdot y\big)\ \sigma(y_1,y_2) =: h(y)$$

is Q-periodic. Since also A has integer coefficients, this yields

$$\int_{AQ} h(y)\,\frac{1}{\det A}\,dy \ = \ \int_Q h(y)\,dy,$$

so that (115) turns as desired into (112):

$$\begin{aligned}
\hat\chi_0(k) &= \frac{1}{(2\pi)^{3/2}}\int_{AQ}\exp(i\,A^{-T}k\cdot y)\sigma(y_1,y_2)\,dy\\
&\overset{(114)}{=} \frac{1}{(2\pi)^{3/2}}\int_{Q}\exp(i\{[k_1-k_2+k_3]y_3+k_1y_1+k_2y_2\})\sigma(y_1,y_2)\,dy\\
&= \begin{cases} \dfrac{1}{(2\pi)^{3/2}}\displaystyle\int_{(0,1)^2}\exp(i\,(k_1y_1+k_2y_2))\sigma(y_1,y_2)dy & \text{if} \quad k_1-k_2+k_3=0,\\[2ex] 0 & \text{else.}\end{cases}
\end{aligned}$$

Analogously, for

$$\chi_0(x_1,x_2,x_3)=\sigma(x_1-x_2,x_2+x_3),$$

we have

$$\hat\chi_0(k_1,k_2,k_3)=\begin{cases}\hat\sigma(k_1,k_3) & \text{if} \quad k_1+k_2-k_3=0,\\[1ex] 0 & \text{else.}\end{cases} \tag{116}$$

Now, because of (28), we have

$$\sum_{\substack{k\,\in\,2\pi\mathbb{Z}^3\\ k\neq 0}} F(k,\hat\chi) \quad\lesssim\quad \sum_{\substack{k\,\in\,2\pi\mathbb{Z}^3\\ k\neq 0}} \tilde F(k,\hat\chi). \tag{117}$$

By linearity of $\tilde F(k,\hat\chi)$ in $|\hat\chi_1|^2,|\hat\chi_2|^2,|\hat\chi_3|^2$ we may argue for each component of χ separately. In order to estimate the bulk energy of χ_1, note that under the condition $k_1-k_2+k_3=0$ the first line of (27) can be estimated as

$$|k|^{-8}\underbrace{(k_2^2+(k_3+k_1)^2)}_{=\,2k_2^2}\underbrace{(k_2^2+(k_3-k_1)^2)}_{\lesssim\,|k|^2}\underbrace{(k_3^2+(k_2+k_1)^2)}_{\lesssim\,|k|^2}\underbrace{(k_3^2+(k_2-k_1)^2)}_{=\,2k_3^2}\lesssim |k|^{-4}k_2^2k_3^2,$$

and in the same manner for $k_1+k_2-k_3=0$, we get

$$|k|^{-8}\underbrace{(k_2^2+(k_3+k_1)^2)}_{\lesssim\,|k|^2}\underbrace{(k_2^2+(k_3-k_1)^2)}_{=\,2k_2^2}\underbrace{(k_3^2+(k_2+k_1)^2)}_{=\,2k_3^2}\underbrace{(k_3^2+(k_2-k_1)^2)}_{\lesssim\,|k|^2}\lesssim |k|^{-4}k_2^2k_3^2.$$

Hence,

$$\sum_{\substack{k \in 2\pi\mathbb{Z}^3 \\ k \neq 0}} \tilde{F}(k, (\hat{\chi}_1, 0, 0)) \overset{(106),(112),(116)}{\lesssim} \sum_{\substack{k \in 2\pi\mathbb{Z}^3 \\ k \neq 0 \\ k_1 + k_2 - k_3 = 0}} \frac{k_2^2 k_3^2}{|k|^4} |\hat{\sigma}_\theta(k_1, k_3)|^2 + \sum_{\substack{k \in 2\pi\mathbb{Z}^3 \\ k \neq 0 \\ k_1 - k_2 + k_3 = 0}} \frac{k_2^2 k_3^2}{|k|^4} |\hat{\sigma}_{1-\theta}(k_1, k_2)|^2$$

$$\lesssim \sum_{\substack{(k_1, k_3) \in 2\pi\mathbb{Z}^2 \\ (k_1, k_3) \neq 0}} \frac{k_3^2}{k_1^2 + k_3^2} |\hat{\sigma}_\theta(k_1, k_3)|^2 + \sum_{\substack{(k_1, k_2) \in 2\pi\mathbb{Z}^2 \\ (k_1, k_2) \neq 0}} \frac{k_2^2}{k_1^2 + k_2^2} |\hat{\sigma}_{1-\theta}(k_1, k_2)|^2. \qquad (118)$$

Analogously, we obtain for χ_2

$$\sum_{\substack{k \in 2\pi\mathbb{Z}^3 \\ k \neq 0}} \tilde{F}(k, (0, \hat{\chi}_2, 0)) \overset{(106),(116)}{\lesssim} \sum_{\substack{(k_1, k_3) \in 2\pi\mathbb{Z}^2 \\ (k_1, k_3) \neq 0}} \frac{k_3^2 k_1^2}{|k|^4} |(\hat{\sigma}_\theta + \hat{H}_\theta)(k_1, k_3)|^2$$

$$\lesssim \sum_{\substack{(k_1, k_3) \in 2\pi\mathbb{Z}^2 \\ (k_1, k_3) \neq 0}} \left(\frac{k_3^2}{k_1^2 + k_3^2} |\hat{\sigma}_\theta(k_1, k_3)|^2 + \frac{k_3^2}{|k|^4} \underbrace{|k_1 \hat{H}_\theta(k_1, k_3)|^2}_{=0} \right), \qquad (119)$$

where in the last line we have use that $\hat{H}_\theta(k_1, k_3) = 0$ for $k_1 \neq 0$, since by definition, $H_\theta(y_1, y_2)$ does not depend in y_1. By a similar argument we find

$$\sum_{\substack{k \in 2\pi\mathbb{Z}^3 \\ k \neq 0}} \tilde{F}(k, (0, 0, \hat{\chi}_3)) \lesssim \sum_{\substack{(k_1, k_2) \in 2\pi\mathbb{Z}^2 \\ (k_1, k_2) \neq 0}} \frac{k_2^2}{k_1^2 + k_2^2} |\hat{\sigma}_{1-\theta}(k_1, k_2)|^2. \qquad (120)$$

Hence, by combining (117), (118), (119), (120), and Lemma 2, we find

$$\sum_{\substack{k \in 2\pi\mathbb{Z}^3 \\ k \neq 0}} F(k, \hat{\chi}) \lesssim \eta^{2/3}.$$

Finally, assuming η small enough so Lemma 2 holds, this last estimate and (110) shows (109).

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