The Concertina Pattern –
From Micromagnetics to Domain Theory

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This is a continuation of a series of papers on the concertina pattern.
The concertina pattern is a ubiquitous metastable, nearly periodic magnetization pattern in elongated thin film elements. In previous papers, a reduced variational model for this pattern was rigorously derived from 3-d micromagnetics. Numerical simulations of the reduced model reproduce the concertina pattern and show that its optimal period w is an increasing function of the applied external field h_{ext}. The latter is an explanation of the experimentally observed coarsening. Domain theory, which can be heuristically derived from the reduced model, predicts and quantifies this dependence of w on h_{ext}. In this paper, we rigorously extract these heuristic observations of domain theory directly from the reduced model. The main ingredient of the analysis is a new type of estimate on solutions of a perturbed Burgers equation.

Mathematics Subject Classification: 78A99, 49K20, 74G60

1 Introduction

The concertina pattern is a domain pattern frequently appearing in ferromagnetic thin-film elements of low crystalline anisotropy (like Permalloy) with an elongated cross-section. The pattern-forming quantity is the magnetization. The concertina pattern consists of an almost periodic array of folds separated by low-angle Néel walls. It is experimentally generated as follows: With the help of a strong homogeneous external field H_{ext}, the magnetization is saturated along the long axis, then the external field is slowly reduced and eventually reversed. At some critical strength of the external field, the uniform magnetization buckles into the concertina pattern.

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Figure 1 shows such a pattern seen from above, observed in an elongated Permalloy thin-film. The magnetization is in-plane and constant in the direction of the film thickness. The grey scales in the experimental picture allow to reconstruct the magnetization angle as shown in the schematical picture by the small arrows.

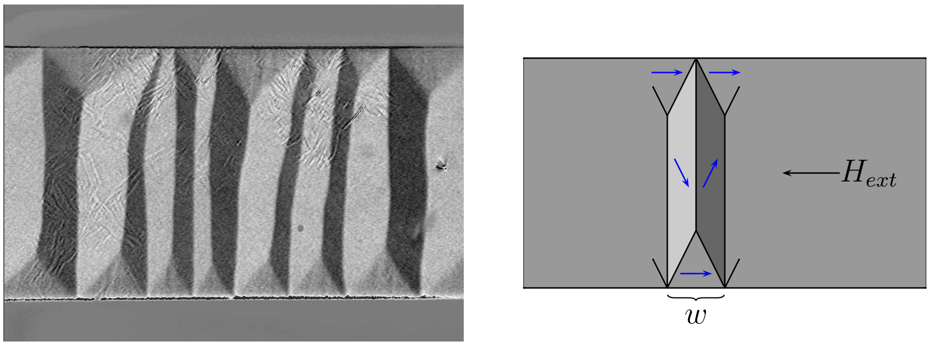

Figure 1: Experimentally observed concertina pattern

As the strength of the reversed external field is slowly increased after the concertina pattern has already formed, coarsening events are observed: A fold collapses, increasing the average period of the concertina pattern. Several generations of these coarsening events by which the average period increases can be observed before the pattern disappears; cf. Figure 2, which shows the same sample as Figure 1 after two coarsening events.

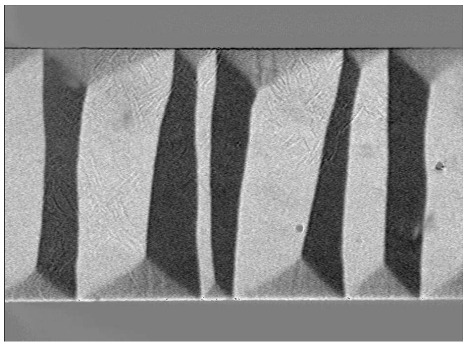

Figure 2: Coarsened concertina pattern

In this paper, we study the formation and coarsening of the concertina domain-wall pattern. We investigate the dependence of the optimal period w of the pattern and of the average inclination of the magnetization on the external field and prove in particular that the optimal period increases if the strength of the reversed external field is increased. This a first step towards the understanding of the coarsening phenomena.

1.1 Micromagnetic energy

The experiments described above, which lead to the formation of the concertina pattern and the subsequent coarsening, are quasi-static processes. The states observed in a quasi-static experiment are related to stationary points of some suitable energy. The analysis of the concertina pattern, which was started in [CÁO06a] and was pursued in [CÁOS07], is based on the 3-d micromagnetic energy which we introduce now. For a sample $\Omega \subset \mathbb{R}^3$ and a magnetization $m : \Omega \to \mathbb{R}^3$ the energy is given by

$$E(m) \quad = \quad d^2 \int_\Omega |\nabla m|^2 \, \mathrm{d}x + \int_{\mathbb{R}^3} |\nabla u_m|^2 \, \mathrm{d}x - 2 \int_\Omega H_{ext} \cdot m \, \mathrm{d}x. \tag{1.1}$$

The magnetization m is constrained to $|m|^2 = 1$. We note that the energy is non-dimensional except for length and comment on the energy terms:

- The first contribution is the *exchange energy*. The exchange length d, a material parameter, is of the order of a few nanometers (for Permalloy).
- By the static Maxwell equations in a medium, the *stray field* $-\nabla u_m$ is determined by:

  $$\int_{\mathbb{R}^3} \nabla u_m \cdot \nabla \varphi \;=\; \int_\Omega m \cdot \nabla \varphi \quad \text{for all test functions } \varphi.$$

  It is thus generated by two types of "magnetic charges",

  $$\begin{array}{lll} \text{volume charges :} & -\nabla \cdot m & \text{in } \Omega, \text{ and} \\ \text{surface charges :} & \nu \cdot m & \text{on } \partial\Omega, \end{array} \tag{1.2}$$

  where ν is the outward normal. The stray-field energy can equivalently be written as

  $$\int_{\mathbb{R}^3} |\nabla u_m|^2 \, \mathrm{d}x \;=\; \int_{\mathbb{R}^3} ||\nabla|^{-1} \nabla \cdot m|^2 \, \mathrm{d}x,$$

  where $|\nabla|^{-1}$ denotes the pseudo-differential operator with symbol $|k|^{-1}$.
- If a magnetic field $H_{ext} : \mathbb{R}^3 \to \mathbb{R}^3$ is applied, there is a contribution due to the *Zeeman effect*.

1.2 Previous results

In [CÁO06a] an idealized sample geometry of width ℓ and of thickness t with $t \ll \ell$ was considered. This sample geometry is idealized in the sense that instead of a finite extent in x_1-direction, we impose an (artificial) periodicity L, i.e.

$$\Omega \;=\; \mathbb{R} \times (0, \ell) \times (0, t) \quad L\text{-periodic w.r.t. } x_1.$$

The idea is that L is chosen so large that it does not interfere with the intrinsic period of the pattern. This setting has the advantage that $m \equiv (1, 0, 0)$ is a stationary point of E for all values of the external field $H_{ext} = (-h_{ext}, 0, 0)$.

The experimental observations suggest that a bifurcation at a critical value $h_{ext} = h_{crit}$ of the external field $H_{ext} = (-h_{ext}, 0, 0)$ is at the origin of the concertina pattern. Hence in [CÁO06a] a linear stability analysis of the saturated state $m \equiv (1, 0, 0)$ was carried out. Since the Hessian of E is not explicitly diagonalizable, the analysis is not obvious and several Ansätze for unstable modes, i.e. infinitesimal perturbations δm, have been proposed in the physics literature. In [CÁO06a], it was proven that there are exactly four qualitatively different regimes depending on the non-dimensional parameters $\frac{t}{d}$ and $\frac{\ell}{d}$. In the regime $\frac{d^2}{\ell} \ll t \ll \sqrt{d\ell}$ the unstable mode is in-plane and does not depend on the thickness direction, i.e. x_3, and displays an oscillatory behavior named "oscillatory buckling":

$$\delta m_2 = \cos(2\pi \tfrac{x_1}{w}) \sin(\pi \tfrac{x_2}{l}).$$

Obviously, this regime is the potentially relevant regime for the concertina pattern. The optimal period $w = w^*$ of the oscillation is determined by the balance of exchange and stray-field energy and asymptotically behaves as

$$w^* \approx \left(32\,\pi\,\tfrac{\ell^2 d^2}{t}\right)^{\frac{1}{3}}, \tag{1.3}$$

cf. [CÁO06b]. The experimentally observed period of the concertina pattern agrees with this theoretically predicted period of the unstable mode up to a factor of about 2 over a wide range of sample sizes t and ℓ. However, the experimentally observed period is systematically larger than the theoretically predicted one. It is natural to conjecture that this is due to coarsening: The experiments discern the low-amplitude concertina pattern only after the external field has been increased beyond its critical value - and thus after a certain number of coarsening events has occurred.

In [CÁOS07] the type of bifurcation in the oscillatory buckling regime was investigated. The goal was to find local minimizers near $m \equiv (1,0,0)$ for $h_{ext} > h_{crit}$ which are of concertina type. This required to take a parameter limit and to zoom in near $m \equiv (1,0,0)$ simultaneously. We used Γ-convergence to carry out this double limit. We performed an anisotropic rescaling of space variables and a rescaling of magnetization, field and energy:

- x_1 and in particular L and w^* are measured in units of $\left(\frac{\ell^2 d^2}{t}\right)^{\frac{1}{3}}$, cf. (1.3),
- x_2 is measured in units of the width ℓ,
- x_3 is measured in units of the thickness t,
- m_2 and m_3 are measured in units of $\left(\frac{d^2}{\ell t}\right)^{1/3}$,
- h_{ext} is measured in units of $\left(\frac{d\,t}{\ell^2}\right)^{2/3}$, and
- E is measured in units of $\left(\frac{d^8 t^2}{\ell}\right)^{1/3}$.

In the limit $\frac{d^2}{\ell} \ll t \ll \sqrt{d\ell}$, the magnetization indeed becomes in-plane and x_3-independent, i.e.

$$m_3 = 0 \text{ and } m = m(x_1, x_2).$$

For rescaled L of order 1, i.e. $L \sim 1$, the Γ-limit w.r.t. the L^2-topology is given by minimizing

$$\begin{aligned} E_0(m_2) \; = & \int_0^L \int_0^1 (\partial_1 m_2)^2 \; \mathrm{d}x_2 \; \mathrm{d}x_1 \\ & + \tfrac{1}{2} \int_0^L \int_0^1 \left||\partial_1|^{-1/2}(-\partial_1(\tfrac{1}{2}m_2^2) + \partial_2 m_2)\right|^2 \; \mathrm{d}x_2 \; \mathrm{d}x_1 \\ & - h_{ext} \int_0^L \int_0^1 m_2^2 \; \mathrm{d}x_2 \; \mathrm{d}x_1 \end{aligned} \tag{1.4}$$

among all $m_2(x_1, x_2)$ with

$$\begin{aligned} m_2(x_1, x_2) \; &= \; 0 \text{ for } x_2 \in \{0, 1\}, \\ m_2(x_1 + L, x_2) \; &= \; m_2(x_1, x_2), \\ \textstyle\int_0^L m_2(x_1, x_2) \, \mathrm{d}x_1 \; &= \; 0, \end{aligned} \tag{1.5}$$

where $|\partial_1|^{-1/2}$ denotes the pseudo-differential operator with symbol $|k_1|^{-1/2}$, [CÁOS07, Theorem 3].

Clearly, (1.4) has much reduced complexity compared to the original (1.1): The reduced functional just contains a single non-dimensional parameter h_{ext}, the configuration space consists of functions of two variables, and the non-locality is just a non-locality in the x_1-variable.

Notice that the appearance of Burgers' operator $-\partial_1\left(\frac{1}{2}m_2^2\right) + \partial_2 m_2$ (if we think of x_2 as the time variable and $-x_1$ as the space variable) is not surprising: Because of

$$m_1 \; = \; \sqrt{1 - m_2^2 - m_3^2} \; \approx \; 1 - \frac{1}{2} m_2^2 \quad \text{for} \quad |m_3| \ll |m_2| \ll 1,$$

the volume charge density (1.2) turns into

$$\sigma \; := \; -\nabla \cdot m \approx -\left(-\partial_1 \left(\tfrac{1}{2}m_2^2\right) + \partial_2 m_2\right). \tag{1.6}$$

Notice that m_2 is rescaled in such a way that (1.6) is preserved. As in the 3-d model, the (volume) charge density σ is penalized in a non-local way by (1.4).

On the level of the reduced energy (1.4), the following two statements were proven in [CÁOS07]:

- On the one hand, the bifurcation is subcritical. A particular implication of subcriticality is that at slightly supercritical fields, there are *no* local minimizers close to $m \equiv (1, 0, 0)$.

- On the other hand, the reduced energy E_0 is coercive for all values of h_{ext} (at fixed L). Thus given L there always exists an absolute minimizer of E_0, also for supercritical fields. Hence by the properties of Γ-convergence, there exists a local minimizer of the 3-d energy E near $m = (1, 0, 0)$.

Hence there are non-trivial local minimizers, but in view of the subcritical bifurcation, it is not clear whether these minimizers inherit properties of the unstable mode, like for instance its period.

The numerical simulations presented in [CÁOS07], where the magnetization was restricted to be w^*-periodic, i.e. $L = w^*$, show:

- Although the bifurcation is subcritical, the bifurcating branch has a turning point, see Figure 3. Beyond the turning point, the configurations along the branch are stable, at least under w^*-periodic perturbations.

- As the field h_{ext} increases, the w^*-periodic configurations along the branch develop the concertina domain pattern with the (almost) piecewise constant magnetization in the triangular and quadrangular domains which are separated by thin transition layers, cf. Figure 3.

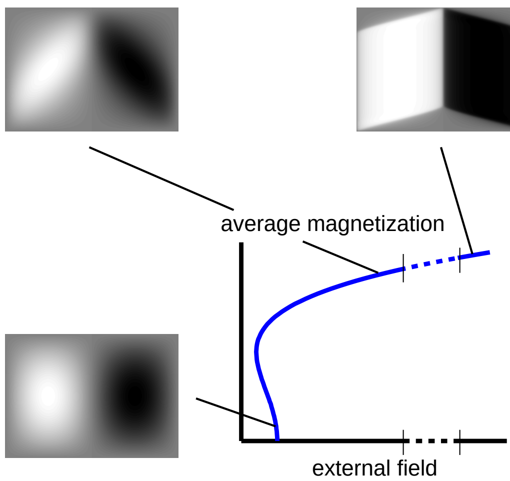

Figure 3: Bifurcation diagram and corresponding magnetization patterns

Moreover, the numerical simulations showed that the minimizers are indeed forced by the (reduced) stray-field energy to be close to weak solutions to the Burgers equation, i.e.

$$-\partial_1(\tfrac{1}{2}m_2^2) + \partial_2 m_2 = 0. \tag{1.7}$$

The shocks (in the language of the theory of conservation laws) of (1.7) correspond of the (mesoscopically) sharp walls of the concertina pattern.

In order to get a better understanding of the experimentally observed coarsening and to determine the dependence of the optimal period w of the concertina on the external field h_{ext}, let us now present some new numerical results based on the finite difference discretization of the energy introduced in [CÁOS07]. For details we refer to Appendix A. The numerical simulations showed that for given external field h_{ext}, there is an energetically optimal period of the concertina pattern, which increases with increasing field, see Figure 4. This is a first explanation why the experimentally observed period increases with increasing h_{ext}.

However, the experimentally observed period is rather expected to be the largest period that is still stable under long-wave length modulation of the pattern. This instability under long-wave length modulations can be related to the concavity of the energy of the concertina pattern as a function of the imposed period w. In a simplified context, this instability has been analyzed in [Sei08].

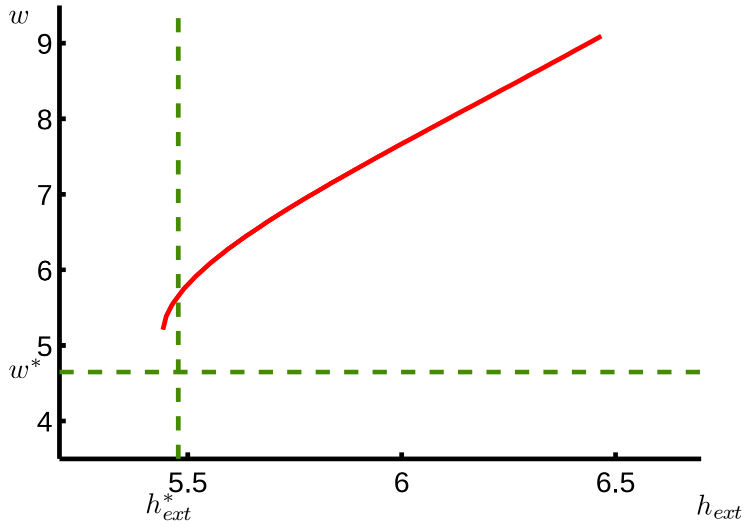

Figure 4: Optimal period $w(h_{ext})$ of the concertina pattern

1.3 Domain theory

To gain more insight into the properties of global minimizers of the reduced model (1.4), we apply domain theory. On the level of domain theory, which is a sharp-interface model, admissible magnetizations are given by weak solutions to the Burgers equation (1.7). In view of the boundary conditions (1.5), the method of characteristics shows that non-trivial weak solutions of (1.7) cannot be continuous. Typically, they will have line discontinuities, i.e. a one-dimensional jump set J. The energy which discriminates between these solutions is given by an appropriate line energy density e integrated over the jump set J, augmented by Zeeman energy:

$$E_{domain}(m_2) \;=\; \int_J e\Big(\frac{[m_2]}{2}\Big)\,\mathrm{d}\mathcal{H}^0\,\mathrm{d}x_2 - h_{ext}\int m_2^2\,\mathrm{d}x_1\,\mathrm{d}x_2.$$

Not surprisingly, the specific line energy e is a function of the jump $[m_2]$ of m_2 across J. Due to the “shear invariance”, cf. (2.8), of (1.4), the specific line – or wall – energy density e can be derived by restricting (1.4) to 1-d configurations with prescribed boundary data $\pm m_2^0$, i.e.

$$E_{\text{Néel}}(m_2) \;=\; \int (\partial_1 m_2)^2\;\mathrm{d}x_1 + \tfrac{1}{8}\int ||\partial_1|^{1/2} m_2^2|^2\;\mathrm{d}x_1. \tag{1.8}$$

The optimal transition layers are low-angle Néel walls whose line energy density is given by

$$e\Big(\frac{[m_2]}{2}\Big) \;=\; e(m_2^0) \;=\; \frac{\pi}{8}(m_2^0)^4 \ln^{-1}\frac{w_{tail}}{w_{core}}, \tag{1.9}$$

where w_{tail} and w_{core} are the two characteristic length scales of the Néel wall, namely the tail and core width, see [Mel03] and [DKMO05, Section 6]. For the scaling of these two parameters in case of the concertina pattern, see below.

We use an Ansatz which mimics the concertina pattern with its quadrangular and triangular domains and which is determined by just two parameters, namely the period w and the inclination $m_2 = \pm m_2^0$ in the quadrangular domains ($m_2 = 0$ in the triangular domains), cf. Figure 5. Indeed, the angles in the pattern are fixed by the

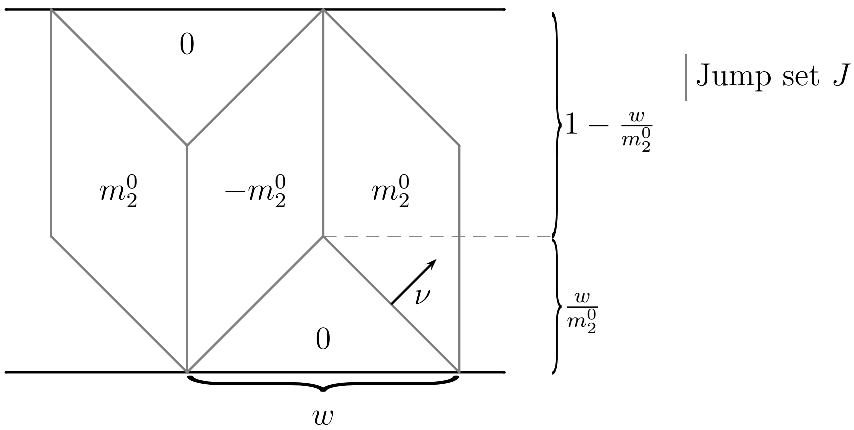

Figure 5: Domain theory: Ansatz function

constraint that m_2 is a weak solution to Burgers' equation: If ν is the normal of one of the diagonal jumps, then for a weak solution the jump of the normal component of the magnetization has to vanish:

$$0 \;=\; [\nu\cdot(-\tfrac{1}{2}m_2^2, m_2)] \;=\; \nu\cdot(-\tfrac{1}{2}(m_2^0)^2, m_2^0).$$

We note that it is always necessary to impose $2\,m_2^0 \;>\; w$ to avoid a degenerated pattern.

We claim that with our Ansatz, the energy per length in x_1 becomes a function of only two parameters, namely m_2^0 and w. For that purpose, we first turn to the two parameters w_{tail} and w_{core} in (1.9). The tails of the Néel wall spread as much as possible; in case of the concertina pattern, they are only limited by the neighboring walls – thus $w_{tail} \sim w$. A more careful inspection of (1.8) shows that the core width increases with increasing jump size, more precisely $w_{core} \sim \frac{1}{(m_2^0)^2}$, see [Ste06]. Hence (1.9) turns into

$$e(m_2^0) \;=\; \frac{\pi}{8}(m_2^0)^4 \ln^{-1} w(m_2^0)^2. \tag{1.10}$$

Notice that one period of the pattern in Figure 5 contains

- two vertical walls of height $1 - \frac{w}{m_2^0}$ and of jump size $2\,m_2^0$, leading to an energy contribution of $2\,(1 - \frac{w}{m_2^0})\, e(m_2^0)$,
- four diagonal walls of projected height $\frac{w}{m_2^0}$ and of jump size m_2^0, leading to an energy contribution of $4\,\frac{w}{m_2^0}\, e(\frac{m_2^0}{2})$,
- two quadrangular domains of total area $w - \frac{w^2}{m_2^0}$, leading to a Zeeman energy of $-h_{ext}(m_2^0)^2\,(w - \frac{w^2}{m_2^0})$.

Hence, the total energy per unit length in x_1 is given by

$$E_{domain}(m_2^0, w) \;=\; \frac{1}{w}\left(2\left(1 - \tfrac{w}{m_2^0}\right) e(m_2^0) + 4\,\frac{w}{m_2^0}\, e\left(\tfrac{m_2^0}{2}\right) - h_{ext}(m_2^0)^2\left(w - \tfrac{w^2}{m_2^0}\right)\right). \tag{1.11}$$

In Appendix B we argue that for $h_{ext} \gg 1$

a) the minimal energy per length in x_1-direction scales as $-h_{ext}^3 \ln^2 h_{ext}$,

b) the optimal inclination of the magnetization scales as $m_2^0 \;\sim\; h_{ext} \ln h_{ext}$, and

c) the optimal period scales as $w \;\sim\; h_{ext} \ln h_{ext}$.

We point out that c) is the domain theoretic explanation why a larger period is preferred for a stronger external field, as observed in the experiments and the numerical simulations.

Until now there is no rigorous derivation of domain theory. Instead in this paper, we rigorously prove the above predictions by domain theory starting from the reduced energy E_0 and show in addition that global minimizers are close to weak solutions of Burgers' equation.

1.4 Main results

Theorem 1 below addresses: a) the scaling behavior of the minimal energy, b) the scaling behavior of the average inclination of minimizing magnetizations, and c) the scaling behavior of the period of global minimizers as predicted by domain theory – all statements including the logarithm.

Theorem 1. *Let $h_{ext} \gg 1$ and $L \;\geq\; h_{ext} \ln h_{ext}$. Then for E_0 and m_2 as in (1.5):*

a) $\quad \min\limits_{m_2} L^{-1} E_0 \;\sim\; -h_{ext}^3 \,\ln^2 h_{ext}.$

For any m_2 with $L^{-1}E_0(m_2) \;\sim\; \min_{m_2} L^{-1} E_0$ we have

b) $\quad A^2 \;:=\; L^{-1} \displaystyle\int_0^L \int_0^1 m_2^2 \,\mathrm{d}x_2\,\mathrm{d}x_1 \;\sim\; h_{ext}^2 \ln^2 h_{ext},$

c1) $$L^{-1}\int_0^L\int_0^1 (m_2(x_1+w,x_2)-m_2(x_1,x_2))^2\,\mathrm{d}x_2\,\mathrm{d}x_1 \lesssim \left(\tfrac{w}{h_{ext}\ln h_{ext}}\right)^\alpha A^2,$$

for $\alpha\in[0,\frac{2}{5})$, *and*

c2) $$L^{-1}\int_0^L\int_0^1 |(m_2)_w(x_1,x_2)|\,\mathrm{d}x_2\,\mathrm{d}x_1 \lesssim \left(\tfrac{h_{ext}\ln h_{ext}}{w}\right)^{1/2} A.$$

Here $(m_2)_w$ *denotes the mean of* m_2 *over cells of size* w, *i.e. for any* L*-periodic* $f(x_1,x_2)$

$$f_w(x_1,x_2) := w^{-1}\int_{-\frac{w}{2}}^{\frac{w}{2}} f(x_1+x_1',x_2)\,\mathrm{d}x_1'. \tag{1.12}$$

Remark 1. *Instead of a separate definition for the asymptotic relations* $\sim$, $\lesssim$ *and so forth, we explain their meaning in the context of Theorem 1:*

There exist universal constants $1\le C$, $C_a, C_{b1}, C_{b2} < +\infty$ *such that for all* $h_{ext}\ge C$ *all* $L\ge h_{ext}\ln h_{ext}$,

a) $$-C_a\, h_{ext}^3\ln^2 h_{ext} \le \min_{m_2} L^{-1}E_0 \le -\frac{1}{C_a}\, h_{ext}^3\ln^2 h_{ext}.$$

For any m_2 *with* $L^{-1}E_0(m_2)\le \frac{1}{C_{b1}}\min_{m_2} L^{-1}E_0$ *we have*

b) $$\frac{1}{C_{b2}} h_{ext}^2\ln^2 h_{ext} \le A^2 := L^{-1}\int_0^L\int_0^1 m_2^2\,\mathrm{d}x_2\,\mathrm{d}x_1 \le C_{b2}\, h_{ext}^2\ln^2 h_{ext}.$$

For any $\alpha\in[0,\frac{2}{5})$ *there exists* $C_\alpha>0$ *such that for all* $h_{ext}\ge C$ *and all* $L\ge h_{ext}\ln h_{ext}$,

c1) $$L^{-1}\int_0^L\int_0^1 (m_2(x_1+w,x_2)-m_2(x_1,x_2))^2\,\mathrm{d}x_2\,\mathrm{d}x_1 \le C_\alpha \left(\tfrac{w}{h_{ext}\ln h_{ext}}\right)^\alpha A^2.$$

There exists $C_{c2}>0$ *such that for all* $h_{ext}\ge C$ *and for all* $L\ge h_{ext}\ln h_{ext}$,

c2) $$L^{-1}\int_0^L\int_0^1 |(m_2)_w(x_1,x_2)|\,\mathrm{d}x_2\,\mathrm{d}x_1 \le C_{c2}\left(\tfrac{h_{ext}\ln h_{ext}}{w}\right)^{1/2} A.$$

The upper bound on the minimal energy for large external fields h_{ext} in a) is proven on the basis of the Ansätze from domain theory, where the discontinuities are replaced by the optimal 1-d transitions layers (low-angle Néel walls). The proof of the lower bound and b) and c1) is based on a non-linear interpolation estimate, cf. Lemma 4. As opposed to the result in [CÁOS07, Theorem 4], the new interpolation estimate provides also L-independent coercivity of the reduced energy E_0. The proof of c2) is based on standard convolution estimates combined with the coercivity of the energy, treated in Lemma 5, which is derived from Lemma 4.

In Theorem 2 we use again Lemma 4 to prove that global minimizers are close to weak solutions of Burgers' equation for $h_{ext}\gg 1$.

Theorem 2. *Let $h_{ext} \gg 1$ and $L \sim h_{ext} \ln h_{ext}$. Then for E_0 and any m_2 as in* (1.5) *with $L^{-1}E_0(m_2) \sim -h_{ext}^3 \ln^2 h_{ext}$ there exists m_2^* with*

$$-\partial_1 \tfrac{1}{2}(m_2^*)^2 + \partial_2 m_2^* = 0 \ \textit{distributionally}$$

and

$$L^{-1}\int_0^L \int_0^1 (m_2 - m_2^*)^2 \, \mathrm{d}x_2 \, \mathrm{d}x_1 \ll L^{-1}\int_0^L \int_0^1 m_2^2 \, \mathrm{d}x_2 \, \mathrm{d}x_1.$$

Although the lower and the upper bound on the energy in Theorem 1 agree in terms of scaling with the simple Ansatz from domain theory (see above), it cannot be excluded that additional substructures in the concertina Ansatz, such as branched structures sometimes observed in the experiments, further reduce the energy.

To our knowledge, Theorem 2 is the first example of a rigorous connection between minimizers of the 3-d micromagnetic energy functional and solutions to a (linearized) eikonal equation – Burgers' equation – via the Γ-convergence in [CÁOS07, Theorem 3] and Theorem 2 in this paper.

Rescaling. In view of Theorem 1, it is convenient to rescale length, magnetization and energy according to

$$\begin{aligned} x_1 &= h_{ext}\,(\ln h_{ext})\,\widehat{x}_1, \\ x_2 &= \widehat{x}_2, \\ m_2 &= h_{ext}\,(\ln h_{ext})\,\widehat{m}_2, \\ L^{-1}E_0 &= h_{ext}^3\,(\ln^2 h_{ext})\,\widehat{L}^{-1}\widehat{E}. \end{aligned}$$

In these new variables we obtain

$$\begin{aligned} \widehat{L}^{-1}\widehat{E}(\widehat{m}_2) = & \, h_{ext}^{-3}\,(\ln^{-2} h_{ext})\,\widehat{L}^{-1}\int_0^{\widehat{L}}\int_0^1 (\widehat{\partial}_1 \widehat{m}_2)^2 \, \mathrm{d}\widehat{x}_2 \, \mathrm{d}\widehat{x}_1 \\ & + (\ln h_{ext})\,\tfrac{1}{2}\widehat{L}^{-1}\int_0^{\widehat{L}}\int_0^1 ||\widehat{\partial}_1|^{-1/2}(-\widehat{\partial}_1(\tfrac{1}{2}\widehat{m}_2^2) + \partial_2 \widehat{m}_2)|^2 \, \mathrm{d}\widehat{x}_2 \, \mathrm{d}\widehat{x}_1 \\ & - \widehat{L}^{-1}\int_0^{\widehat{L}}\int_0^1 \widehat{m}_2^2 \, \mathrm{d}\widehat{x}_2 \, \mathrm{d}\widehat{x}_1. \end{aligned}$$

It is convenient to introduce

$$\varepsilon := h_{ext}^{-3} \ln^{-2} h_{ext}$$

such that for $h_{ext} \gg 1$

$$\ln \tfrac{1}{\varepsilon} = 3 \ln h_{ext} + 2 \ln\ln h_{ext} \approx 3 \ln h_{ext}.$$

Hence, to leading order

$$\begin{aligned} \widehat{E}(\widehat{m}_2) = & \, \varepsilon \int_0^{\widehat{L}}\int_0^1 (\widehat{\partial}_1 \widehat{m}_2)^2 \, \mathrm{d}\widehat{x}_2 \, \mathrm{d}\widehat{x}_1 \\ & + (\ln \tfrac{1}{\varepsilon})\,\tfrac{1}{6}\int_0^{\widehat{L}}\int_0^1 ||\widehat{\partial}_1|^{-1/2}(-\widehat{\partial}_1(\tfrac{1}{2}\widehat{m}_2^2) + \partial_2 \widehat{m}_2)|^2 \, \mathrm{d}\widehat{x}_2 \, \mathrm{d}\widehat{x}_1 \\ & - \int_0^{\widehat{L}}\int_0^1 \widehat{m}_2^2 \, \mathrm{d}\widehat{x}_2 \, \mathrm{d}\widehat{x}_1. \end{aligned}$$

With this rescaling, Theorem 1 assumes the form:

Theorem 3. *Let $0 < \varepsilon \ll 1$ and $\widehat{L} \geq 1$. Then*

a) $\displaystyle \min_{\widehat{m}_2} \widehat{L}^{-1}\widehat{E} \sim -1.$

For any $\widehat{m}_2$ with $\widehat{L}^{-1}\widehat{E}(\widehat{m}_2) \sim -1$ we have

b) $\displaystyle \widehat{L}^{-1}\int_0^{\widehat{L}}\int_0^1 \widehat{m}_2^2 \,\mathrm{d}\widehat{x}_2\,\mathrm{d}\widehat{x}_1 \sim 1,$

c1) $\displaystyle \widehat{L}^{-1}\int_0^{\widehat{L}}\int_0^1 (\widehat{m}_2(\widehat{x}_1+\widehat{w},\widehat{x}_2)-\widehat{m}_2(\widehat{x}_1,\widehat{x}_2))^2 \,\mathrm{d}\widehat{x}_2\,\mathrm{d}\widehat{x}_1 \lesssim \widehat{w}^{\alpha}$

for $\alpha \in [0,\frac{2}{5})$, and

c2) $\displaystyle \widehat{L}^{-1}\int_0^{\widehat{L}}\int_0^1 |(\widehat{m}_2)_{\widehat{w}}(\widehat{x}_1,\widehat{x}_2)| \,\mathrm{d}\widehat{x}_2\,\mathrm{d}\widehat{x}_1 \lesssim \left(\tfrac{1}{\widehat{w}}\right)^{1/2},$

where $(\widehat{m}_2)_{\widehat{w}}$ is defined as in (1.12).

With the rescaling above, Theorem 2 assumes the form:

Theorem 4. *Let $0 < \varepsilon \ll 1$ and $\widehat{L} \sim 1$. Then for any $\widehat{m}_2$ with $\widehat{L}^{-1}\widehat{E}(\widehat{m}_2) \sim -1$ there exists $\widehat{m}_2^*$ with*

$$-\widehat{\partial}_1\tfrac{1}{2}(\widehat{m}_2^*)^2 + \widehat{\partial}_2\widehat{m}_2^* = 0 \ \textit{distributionally}$$

and

$$\widehat{L}^{-1}\int_0^{\widehat{L}}\int_0^1 (\widehat{m}_2-\widehat{m}_2^*)^2 \,\mathrm{d}\widehat{x}_2\,\mathrm{d}\widehat{x}_1 \ll 1.$$

2 Proofs

For notational convenience, we drop the hats $\widehat{\cdot}$. In the following we will write u instead of m_2, x instead of x_1, and t instead of x_2.

2.1 Upper bound

Proposition 1. *For any $0 < \varepsilon \ll 1$ and any $L \geq 1$*

$$\min_u L^{-1}E \lesssim -1. \tag{2.1}$$

Proof of Proposition 1. Let us explain the main features of our construction.

Symmetry. Our Ansatz will have the following symmetry properties, cf. Figure 6:

- It will be periodic in x with period $w \sim 1$, i.e.

$$u(x+w,t) = u(x,t).$$

  The parameter $w \sim 1$ will be chosen later such that L is an integer multiple of w. (By $w \sim 1$ we mean that $w \in (\frac{1}{C}, C]$ for some universal constant $1 < C < \infty$.)

- It will be odd w.r.t. reflection at $x = 0$ (one of the vertical walls), i.e.

$$u(-x,t) = -u(x,t).$$

- It will be even w.r.t. rotation in $(\frac{w}{4}, \frac{1}{2})$ (the center of mass of one of the quadrangular domains), i.e.

$$u(\tfrac{w}{4}+x, \tfrac{1}{2}+t) \;=\; u(\tfrac{w}{4}-x, \tfrac{1}{2}-t).$$

Hence, u will be determined by its values

$$u(x,t) \text{ on the fundamental domain } (x,t) \in (0, \tfrac{w}{2}) \times (0, \tfrac{1}{2}).$$

Mesoscopic pattern. On a mesoscopic level, our u will be of the form

$$u_{meso}(x,t) \;=\; \begin{cases} 0 & \text{for } t \;\le\; \frac{x}{s}, \\ -2\,s & \text{for } t \;\ge\; \frac{x}{s}, \end{cases}$$

where the parameter $s \sim 1$ will be chosen later. Notice that $s > \frac{w}{2}$ is necessary to avoid a degenerated pattern. In favor of a clear presentation we only show the construction for the case $s \ge w$ in detail. This will be enough to obtain the desired upper bound on the minimal energy. We will comment on the differences for the case $\frac{w}{2} < s < w$ at the end of the proof.

The mesoscopic Ansatz u_{meso} satisfies

$$-\partial_x(\tfrac{1}{2}u_{meso}^2) + \partial_t u_{meso} \;=\; 0$$

distributionally. Notice that u_{meso} has the following discontinuity lines within $(0, \frac{w}{2}) \times (0, \frac{1}{2})$:

- a jump between $2\,s$ and $-2\,s$ across $x = 0$ for $0 \;\le\; t \;\le\; \frac{1}{2}$,
- a jump between $-2\,s$ and $2\,s$ across $x = \frac{w}{2}$ for $t \ge \frac{w}{2\,s}$,
- a jump between $-2\,s$ and 0 across $t = \frac{x}{s}$ for $0 \;\le\; x \;\le\; \frac{w}{2}$.

The first two discontinuity lines carry a weight of $\frac{1}{2}$, since they also belong to the neighboring cell, cf. Section 1.3.

Néel walls. We must choose appropriate transition layers, i.e. walls, in order to construct a microscopic u starting from u_{meso}. The construction will additionally depend on two parameters α and β, with $\varepsilon \ll \alpha \ll \beta \ll w$, which will be chosen later in function of ε. In fact, we distinguish 3 regions, cf. Figure 6:

- Bulk: Here we set $u = u_{meso}$.
- Walls: Here we use a one-dimensional construction. Within the fundamental domain $(0, \frac{w}{2}) \times (0, \frac{1}{2})$ the wall region is given by

$$\begin{aligned} &\{(x,t) \,|\, 0 \le x \le \beta,\, \tfrac{2\beta}{s} \le t \le \tfrac{1}{2}\} \\ \cup &\{(x,t) \,|\, \tfrac{w}{2} - \beta \le x \le \tfrac{w}{2},\, \tfrac{w}{2s} \le t \le \tfrac{1}{2}\} \\ \cup &\{(x,t) \,|\, s\,t - \beta \le x \le s\,t + \beta,\, \tfrac{2\beta}{s} \le t \le \tfrac{w}{2s} - \tfrac{\beta}{s}\}. \end{aligned}$$

  Notice that $\beta \le \frac{w}{4}$ is necessary.

- Corners: Here, we interpolate the x-dependent boundary data linearly in t. Within the fundamental domain $(0, \frac{w}{2}) \times (0, \frac{1}{2})$ the corner region is described by

$$\big((0, 3\,\beta) \times (0, \tfrac{2\,\beta}{s})\big) \cup \big((\tfrac{w}{2} - 2\,\beta, \tfrac{w}{2}) \times (\tfrac{w}{2s} - \tfrac{\beta}{s}, \tfrac{w}{2\,s})\big)\,.$$

  Notice that $3\,\beta \le \frac{w}{2}$ is necessary.

The function u will be constructed to be continuous across the regions. These regions contribute differently to the three parts of the energy:

- Exchange energy: This local energy contribution behaves in an additive way; only the walls and the corners contribute.
- Magnetostatic energy: Only walls and corners contribute to the charge density σ, i.e. the support of the charge density is a subset of the wall and corner region. Since the magnetostatic energy is non-local in the charge density σ, it behaves in a non-additive way. However, if $\sigma = \sigma_1 + \sigma_2 + \sigma_3$ is a decomposition, we have an upper bound by the triangle inequality

$$\begin{aligned} &\int \big||\partial_x|^{-1/2}\sigma\big|^2 \,\mathrm{d}x \\ &\quad \le 3\int \big||\partial_x|^{-1/2}\sigma_1\big|^2 \,\mathrm{d}x + 3\int \big||\partial_x|^{-1/2}\sigma_2\big|^2 \,\mathrm{d}x + 3\int \big||\partial_x|^{-1/2}\sigma_3\big|^2 \,\mathrm{d}x, \end{aligned} \tag{2.2}$$

  where we have to ascertain $\int_{-\frac{w}{2}}^{\frac{w}{2}} \sigma_1 \,\mathrm{d}x = \int_{-\frac{w}{2}}^{\frac{w}{2}} \sigma_2 \,\mathrm{d}x = \int_{-\frac{w}{2}}^{\frac{w}{2}} \sigma_3 \,\mathrm{d}x = 0$, so that the r.h.s. is finite. Since modulo w-periodicity in x, there are at most 3 walls or corners at a given t-value, (2.2) suffices.

- Zeeman energy: Here, we seek a *lower* bound for $\iint u^2 \,\mathrm{d}t \,\mathrm{d}x$. The main contribution will come from the bulk.

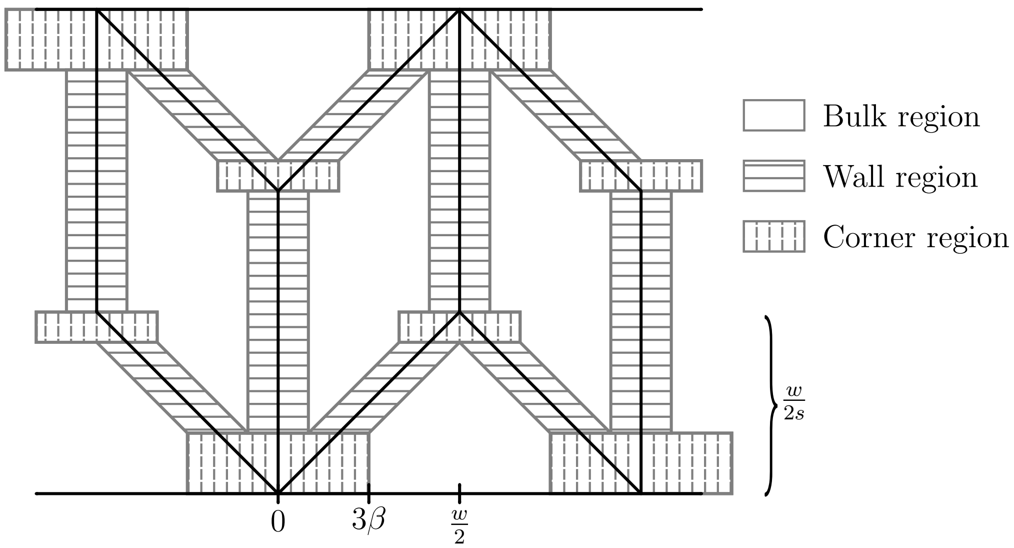

Figure 6: The Ansatz u

Vertical Néel walls. In this section, we construct the vertical Néel walls. W.l.o.g. we focus on the construction in the region

$$\big\{(x,t)\,\big|\, -\beta \,\le\, x \,\le\, \beta,\, \tfrac{2\,\beta}{s} \,\le\, t \,\le\, \tfrac{1}{2}\big\}. \tag{2.3}$$

We consider the exchange and magnetostatic energy E_{ex+ma}. Within (2.3), u coincides with an odd function v of the form

$$u \,=\, -2\,s\,v(x), \quad v(\pm\beta) \,=\, \pm 1,$$

which we think of as being w-periodic and $v(x+\frac{w}{2}) = -v(x)$. In terms of v, we have the estimate

$$\begin{aligned} E_{ex+ma} \,&=\, (\tfrac{1}{2} - \tfrac{2\beta}{s})\left(4\,s^2\,\varepsilon\int_{-\frac{w}{2}}^{\frac{w}{2}}(\partial_x v)^2\,\mathrm{d}x + \tfrac{8}{3}\,s^4\,(\ln\tfrac{1}{\varepsilon})\int_{-\frac{w}{2}}^{\frac{w}{2}}\big||\partial_x|^{-1/2}\partial_x(-\tfrac{1}{2}v^2)\big|^2\,\mathrm{d}x\right) \\ &\lesssim\, s^2\,\varepsilon\int_{-\frac{w}{2}}^{\frac{w}{2}}(\partial_x v)^2\,\mathrm{d}x + s^4\,(\ln\tfrac{1}{\varepsilon})\int_{-\frac{w}{2}}^{\frac{w}{2}}\big||\partial_x|^{1/2}\,v^2\big|^2\,\mathrm{d}x \\ &\lesssim\, s^2\,\varepsilon\int_{-\frac{w}{2}}^{\frac{w}{2}}\tfrac{1}{v^2}(\partial_x v^2)^2\,\mathrm{d}x + s^4\,(\ln\tfrac{1}{\varepsilon})\int_{-\frac{w}{2}}^{\frac{w}{2}}\big||\partial_x|^{1/2}\,v^2\big|^2\,\mathrm{d}x. \end{aligned}$$

It is convenient to think in terms of $\varrho = v^2$ which satisfies

$$\begin{aligned} &\varrho \,=\, 1 \text{ for } \beta \,\le\, |x| \,\le\, \tfrac{w}{2} - \beta, \\ &\varrho \,=\, 0 \text{ for } x \,=\, 0, \\ &\varrho \text{ is } \tfrac{w}{2}\text{-periodic,} \end{aligned}$$

so that

$$E_{ex+ma} \,\lesssim\, s^2\,\varepsilon\int_{-\frac{w}{4}}^{\frac{w}{4}}\frac{1}{\varrho}(\partial_x\varrho)^2\,\mathrm{d}x + s^4\,\ln\frac{1}{\varepsilon}\int_{-\frac{w}{4}}^{\frac{w}{4}}\big||\partial_x|^{1/2}\varrho\big|^2\,\mathrm{d}x.$$

We make the Néel-wall Ansatz, cf. [Mel03] and [DKMO05, Section 6],

$$\varrho(x) \;=\; \begin{cases} \frac{\ln \frac{\alpha^2+x^2}{\alpha^2}}{\ln \frac{\alpha^2+\beta^2}{\alpha^2}} & \text{for } |x| \;\leq\; \beta, \\ 1 & \text{for } \beta \;\leq\; |x| \;\leq\; \frac{w}{4}, \end{cases} \tag{2.4}$$

where ε and α with $\varepsilon \ll \alpha \ll \beta \ll w$ will be chosen later. We first turn to the magnetostatic part and use the trace characterization of the homogeneous $H^{1/2}$-norm, i.e.

$$\int_{-\frac{w}{4}}^{\frac{w}{4}} \left||\partial_x|^{1/2}\varrho\right|^2 \mathrm{d}x \;=\; \inf\{\int_{-\frac{w}{4}}^{\frac{w}{4}}\int_0^\infty (\partial_x\bar\varrho)^2 + (\partial_z\bar\varrho)^2 \,\mathrm{d}z\,\mathrm{d}x| \quad \bar\varrho(x,z) \text{ is } \tfrac{w}{2}\text{-periodic in } x \text{ and } \bar\varrho(x,0) \;=\; \varrho(x)\}, \tag{2.5}$$

which yields by extending ϱ in a radially symmetric way in the (x,z)-plane:

$$\begin{aligned}
\int_{-\frac{w}{4}}^{\frac{w}{4}} \left||\partial_x|^{1/2}\varrho\right|^2 \mathrm{d}x &\lesssim \frac{1}{\ln^2 \frac{\alpha^2+\beta^2}{\alpha^2}} \iint\limits_{x^2+z^2\,\leq\,\beta^2} \left(\partial_x\left(\ln\tfrac{\alpha^2+x^2+z^2}{\alpha^2}\right)\right)^2 + \left(\partial_z\left(\ln\tfrac{\alpha^2+x^2+z^2}{\alpha^2}\right)\right)^2 \mathrm{d}x\,\mathrm{d}z \\
&\overset{\alpha\ll\beta}{\lesssim} \frac{1}{\ln^2\frac{\beta}{\alpha}} \int_0^\beta (\partial_r(\ln\tfrac{\alpha^2+r^2}{\alpha^2}))^2 r\,\mathrm{d}r \\
&= \frac{1}{\ln^2\frac{\beta}{\alpha}} \int_0^{\frac{\beta}{\alpha}} (\partial_{\hat r}\ln(1+\hat r^2))^2 \hat r\,\mathrm{d}\hat r \\
&\lesssim \frac{1}{\ln^2\frac{\beta}{\alpha}} \int_0^{\frac{\beta}{\alpha}} \frac{\hat r^3}{(1+\hat r^2)^2}\,\mathrm{d}\hat r \\
&\sim \frac{1}{\ln\frac{\beta}{\alpha}}.
\end{aligned}$$

We now turn to the exchange energy. Since

$$\frac{\mathrm{d}\varrho}{\mathrm{d}x} \;=\; \frac{1}{\ln\frac{\alpha^2+\beta^2}{\alpha^2}} \begin{cases} \frac{2x}{\alpha^2+x^2} & \text{for } |x| < \beta, \\ 0 & \text{for } \beta < |x| < \frac{w}{4}, \end{cases}$$

and

$$\frac{1}{\varrho}\left(\frac{\mathrm{d}\varrho}{\mathrm{d}x}\right)^2 \;=\; \frac{1}{\ln\frac{\alpha^2+\beta^2}{\alpha^2}} \begin{cases} \frac{1}{\ln\frac{\alpha^2+x^2}{\alpha^2}}\,\frac{4x^2}{(\alpha^2+x^2)^2} & \text{for } |x| < \beta, \\ 0 & \text{for } \beta < |x| < \frac{w}{4}, \end{cases}$$

we have

$$\begin{aligned}
\int_{-\frac{w}{4}}^{\frac{w}{4}} \frac{1}{\varrho}\left(\frac{\mathrm{d}\varrho}{\mathrm{d}x}\right)^2 \mathrm{d}x &\overset{\alpha\ll\beta}{\lesssim} \frac{1}{\ln\frac{\beta}{\alpha}} \int_{-\beta}^{\beta} \frac{1}{\ln\frac{\alpha^2+x^2}{\alpha^2}}\,\frac{x^2}{(\alpha^2+x^2)^2}\,\mathrm{d}x \\
&= \frac{2}{\alpha\,\ln\frac{\beta}{\alpha}} \int_0^{\frac{\beta}{\alpha}} \frac{1}{\ln(1+\hat x^2)}\,\frac{\hat x^2}{(1+\hat x^2)^2}\,\mathrm{d}\hat x \\
&\sim \frac{1}{\alpha\,\ln\frac{\beta}{\alpha}}.
\end{aligned} \tag{2.6}$$

Hence we obtain

$$E_{ex+ma} \lesssim s^2\,\varepsilon \frac{1}{\alpha\,\ln\frac{\beta}{\alpha}} + s^4\,\left(\ln\tfrac{1}{\varepsilon}\right)\frac{1}{\ln\frac{\beta}{\alpha}} \;\approx\; s^4\,\left(\ln\tfrac{1}{\varepsilon}\right)\frac{1}{\ln\frac{\beta}{\alpha}}, \tag{2.7}$$

where the last asymptotic identity follows from $\varepsilon \ll \alpha \ll w \sim 1$ and $s \sim 1$.

Diagonal Néel walls. We now address the construction in the region

$$\{(x,t)|s\,t-\beta \le x \le s\,t+\beta,\ \tfrac{2\beta}{s} \le t \le \tfrac{w}{2s}-\tfrac{\beta}{s}\}.$$

Since exchange and magnetostatic energy E_{ex+ma} are invariant under the shear transform

$$x = s\,t+\tilde{x}, \quad t = \tilde{t}, \quad u = \tilde{u}-s, \tag{2.8}$$

we can reduce this construction to a construction of a vertical Néel wall in

$$\{(\tilde{x},\tilde{t})|-\beta \le \tilde{x} \le \beta,\ \tfrac{2\beta}{s} \le \tilde{t} \le \tfrac{w}{2s}-\tfrac{\beta}{s}\}.$$

The only difference to the vertical Néel wall before is that the construction connects $-s$ to s instead of $-2s$ to $2s$. Hence we obtain as there

$$E_{ex+ma} \lesssim s^4\left(\ln\tfrac{1}{\varepsilon}\right)\frac{1}{\ln\frac{\beta}{\alpha}}. \tag{2.9}$$

Corners. W.l.o.g. we consider the corner $(-3\,\beta,3\,\beta)\times(0,\frac{2\beta}{s})$. In view of (2.4) (for $\varrho = v^2$) and (2.8), we have to interpolate

$$u(x,0) = 0$$

and

$$u(x,\tfrac{2\beta}{s}) = \begin{cases} s\,(v(x+2\,\beta)+1) & \text{for } -3\,\beta \le x \le -\beta,\\ -2\,s\,v(x) & \text{for } -\beta \le x \le \beta,\\ s\,(v(x-2\,\beta)-1) & \text{for } \beta \le x \le 3\,\beta \end{cases} \tag{2.10}$$

in t, where

$$v(x) = (\operatorname{sign} x)\left(\frac{\ln\frac{\alpha^2+x^2}{\alpha^2}}{\ln\frac{\alpha^2+\beta^2}{\alpha^2}}\right)^{1/2}.$$

We opt for a linear interpolation, i.e.

$$u(x,t) = \tfrac{s\,t}{2\,\beta}u(x,\tfrac{2\beta}{s}).$$

We consider the exchange E_{ex} and the magnetostatic energy E_{ma} separately.

We first turn to the exchange energy E_{ex}. Because of the linear interpolation, we have

$$\begin{aligned} E_{ex} &= \varepsilon\int_0^{\frac{2\beta}{s}}\left(\frac{s\,t}{2\,\beta}\right)^2\mathrm{d}t\int_{-3\,\beta}^{3\,\beta}(\partial_x u(x,\tfrac{2\beta}{s}))^2\,\mathrm{d}x\\ &\lesssim \varepsilon\frac{\beta}{s}\int_{-3\,\beta}^{3\,\beta}(\partial_x u(x,\tfrac{2\beta}{s}))^2\,\mathrm{d}x\\ &\overset{(2.10)}{\lesssim} \varepsilon\,\beta\,s\int_{-\beta}^{\beta}(\partial_x v(x))^2\,\mathrm{d}x. \end{aligned}$$

From (2.6) we infer

$$\int_{-\beta}^{\beta}(\partial_x v(x))^2\,\mathrm{d}x \;\lesssim\; \tfrac{1}{\alpha\ln\frac{\beta}{\alpha}}, \tag{2.11}$$

so that we obtain

$$E_{ex} \;\lesssim\; \frac{\beta\,\varepsilon\,s}{\alpha\ln\frac{\beta}{\alpha}}.$$

We now address the magnetostatic energy E_{ma}. Notice

$$\sigma(x,t) \;=\; (-\partial_x(\tfrac{1}{2}u^2)+\partial_t u)(x,t) \;=\; -(\tfrac{s\,t}{2\beta})^2\partial_x(\tfrac{1}{2}u^2)(x,\tfrac{2\beta}{s}) + \tfrac{s}{2\beta}\,u(x,\tfrac{2\beta}{s}). \tag{2.12}$$

Because of the symmetry property

$$u(-x,\tfrac{2\beta}{s}) \;=\; -u(x,\tfrac{2\beta}{s}),$$

which entails

$$\partial_x(\tfrac{1}{2}u^2)(-x,\tfrac{2\beta}{s}) \;=\; -\partial_x(\tfrac{1}{2}u^2)(x,\tfrac{2\beta}{s}),$$

we have in particular for all $t\in(0,\frac{2\beta}{s})$

$$\int_{-\frac{w}{2}}^{\frac{w}{2}}\sigma(x,t)\,\mathrm{d}x \;=\; 0. \tag{2.13}$$

Since $\operatorname{supp}\sigma(\cdot,t)\subset[-3\,\beta,3\,\beta]$, we claim that

$$\int_{-\frac{w}{2}}^{\frac{w}{2}}\left|(|\partial_x|^{-1/2}\sigma)(\cdot,t)\right|^2\mathrm{d}x \;\lesssim\; \beta\int_{-\frac{w}{2}}^{\frac{w}{2}}\sigma(x,t)^2\,\mathrm{d}x. \tag{2.14}$$

Let us give the argument for (2.14). By duality, this is equivalent to

$$\int_{-\frac{w}{2}}^{\frac{w}{2}}\zeta(x)\sigma(x,t)\,\mathrm{d}x \;\lesssim\; \left(\beta\int_{-3\,\beta}^{3\,\beta}\sigma(x,t)^2\,\mathrm{d}x\int_{-\frac{w}{2}}^{\frac{w}{2}}\left||\partial_x|^{1/2}\zeta(x)\right|^2\mathrm{d}x\right)^{1/2},$$

for all w-periodic functions $\zeta(x)$. By the trace characterization of the homogeneous $H^{1/2}$-norm (2.5), this estimate is equivalent to

$$\int_{-\frac{w}{2}}^{\frac{w}{2}}\zeta(x,0)\sigma(x,t)\,\mathrm{d}x \;\lesssim\; \left(\beta\int_{-3\,\beta}^{3\,\beta}\sigma(x,t)^2\,\mathrm{d}x\int_{-\frac{w}{2}}^{\frac{w}{2}}\int_0^\infty(\partial_x\zeta)^2+(\partial_z\zeta)^2\,\mathrm{d}x\,\mathrm{d}z\right)^{1/2}$$

for all functions $\zeta(x,z)$ which are w-periodic in x. Because of (2.13) and $\operatorname{supp}\sigma(\cdot,t)\subset[-3\,\beta,3\beta]$, this estimate in turn follows from

$$\int_{-3\,\beta}^{3\,\beta}(\zeta(x,0)-\tfrac{1}{6\,\beta}\int_{-3\,\beta}^{3\,\beta}\zeta(\tilde{x},0)\,\mathrm{d}\tilde{x})^2\,\mathrm{d}x \;\lesssim\; \beta\int_{-\frac{w}{2}}^{\frac{w}{2}}\int_0^\infty(\partial_x\zeta)^2+(\partial_z\zeta)^2\,\mathrm{d}x\,\mathrm{d}z,$$

for all functions $\zeta(x,z)$. By rescaling length with β, this is equivalent to

$$\int_{-3}^{3}(\zeta(\widehat{x},0)-\tfrac{1}{6}\int_{-3}^{3}\zeta(\tilde{x},0)\,\mathrm{d}\tilde{x})^2\,\mathrm{d}\widehat{x} \;\lesssim\; \int_{-\frac{w}{2\,\beta}}^{\frac{w}{2\,\beta}}\int_0^\infty(\partial_{\widehat{x}}\zeta)^2+(\partial_{\widehat{z}}\zeta)^2\,\mathrm{d}\widehat{x}\,\mathrm{d}\widehat{z},$$

which because of $\beta \ll w$, thus $\frac{w}{2\beta} \gg 1$, follows from a standard trace estimate. This establishes (2.14).

Inserting (2.12) into (2.14) yields

$$
\begin{aligned}
&\int_{-\frac{w}{2}}^{\frac{w}{2}} \big|(|\partial_x|^{-1/2}\sigma)(\cdot,t)\big|^2 \,\mathrm{d}x \\
&\overset{\frac{st}{2\beta}\le 1}{\lesssim} \beta \left(\int_{-3\beta}^{3\beta} (\partial_x(u^2)(x,\tfrac{2\beta}{s}))^2 \,\mathrm{d}x + (\tfrac{s}{\beta})^2 \int_{-3\beta}^{3\beta} (u(x,\tfrac{2\beta}{s}))^2 \,\mathrm{d}x\right) \\
&\lesssim \beta \sup_{x\in(-3\beta,3\beta)} \{u^2(x,\tfrac{2\beta}{s})\} \int_{-3\beta}^{3\beta} (\partial_x u(x,\tfrac{2\beta}{s}))^2 \,\mathrm{d}x + \tfrac{s^2}{\beta} \sup_{x\in(-3\beta,3\beta)} \{u^2(x,\tfrac{2\beta}{s})\} \\
&\overset{(2.10)}{\lesssim} \beta \left(s^4 \int_{-\beta}^{\beta} (\partial_x v(x))^2 \,\mathrm{d}x + \tfrac{s^4}{\beta}\right) \\
&\overset{(2.11)}{\lesssim} \beta \left(\frac{s^4}{\alpha \ln\frac{\beta}{\alpha}} + \frac{s^4}{\beta}\right) \\
&= \left(\frac{\frac{\beta}{\alpha}}{\ln\frac{\beta}{\alpha}} + 1\right) s^4 \\
&\overset{\alpha\ll\beta}{\sim} \frac{\frac{\beta}{\alpha}}{\ln\frac{\beta}{\alpha}} s^4.
\end{aligned}
$$

Therefore

$$E_{ma} = \tfrac{1}{6}(\ln\tfrac{1}{\varepsilon}) \int_0^{\frac{2\beta}{s}} \int_{-\frac{w}{2}}^{\frac{w}{2}} \big||\partial_x|^{-1/2}\sigma\big|^2 \,\mathrm{d}x\,\mathrm{d}t \lesssim \frac{\beta^2 s^3}{\alpha} \frac{\ln\frac{1}{\varepsilon}}{\ln\frac{\beta}{\alpha}}.$$

Hence, we obtain for the sum E_{ex+ma} of exchange and magnetostatic energies

$$E_{ex+ma} \lesssim \frac{\beta\,\varepsilon\, s}{\alpha \ln\frac{\beta}{\alpha}} + \frac{\beta^2 s^3}{\alpha}\frac{\ln\frac{1}{\varepsilon}}{\ln\frac{\beta}{\alpha}} = \frac{\beta\, s}{\alpha\ln\frac{\beta}{\alpha}}(\varepsilon + \beta\, s^2 \ln\tfrac{1}{\varepsilon}),$$

so that because of $\varepsilon \ll \beta \ll w \sim 1$ and $s \sim 1$, this estimate asymptotically turns into

$$E_{ex+ma} \lesssim \frac{\beta^2 s^3}{\alpha}\frac{\ln\frac{1}{\varepsilon}}{\ln\frac{\beta}{\alpha}}. \tag{2.15}$$

Optimizing in the parameters. We first consider the exchange and magnetostatic energy E_{ex+ma} in $(-\frac{w}{2},\frac{w}{2}) \times (0,1)$. Collecting (2.7), (2.9) and (2.15) we obtain

$$E_{ex+ma} \lesssim s^4 \frac{\ln\frac{1}{\varepsilon}}{\ln\frac{\beta}{\alpha}} + \frac{\beta^2 s^3}{\alpha}\frac{\ln\frac{1}{\varepsilon}}{\ln\frac{\beta}{\alpha}}.$$

Choosing for instance

$$\alpha = \varepsilon^{2/3}, \quad \beta = \varepsilon^{1/2},$$

which is compatible with $\varepsilon \ll \alpha \ll \beta \ll w \sim 1$, the estimate asymptotically turns into

$$E_{ex+ma} \lesssim s^4 + \varepsilon^{1/3} s^3 \overset{\varepsilon\ll 1}{\approx} s^4.$$

Since $\beta \ll w$, we have for the Zeeman contribution that

$$\int_0^1 \int_{-\frac{w}{2}}^{\frac{w}{2}} u^2 \,\mathrm{d}x \,\mathrm{d}t \approx \int_0^1 \int_{-\frac{w}{2}}^{\frac{w}{2}} u_{meso}^2 \,\mathrm{d}x \,\mathrm{d}t \;=\; (2\,s)^2\, w\,(1 - \tfrac{w}{2\,s}).$$

Choosing $s = w$, we obtain for the total energy

$$E \;\leq\; C_1\, w^4 - \tfrac{1}{2} C_2\, w^3.$$

Hence we obtain for the total energy per length x

$$e(w) \;\leq\; C_1\, w^3 - \tfrac{1}{2} C_2\, w^2.$$

Obviously, there is a $\tilde{w} \leq 1$ s.t. for all $w \in (\frac{\tilde{w}}{2}, \tilde{w}]$

$$e(w) \;\lesssim\; -1. \tag{2.16}$$

Hence we can always choose w such that L is an integer multiple of w and (2.16) holds. The corresponding Ansatz function provides the upper bound on the energy.

The case $w > s > \frac{w}{2}$. Notice that the three discontinuity lines of the mesoscopic pattern have a common triple point at $(\frac{1}{2}, \frac{w}{2s})$ in the fundamental domain, cf. Figure 6. If we allowed for $w > s > \frac{w}{2}$ this triple point would be at $(0, 1-\frac{w}{2s})$ in the fundamental domain. The construction of the microscopic pattern with smooth transition layers can be carried out in the same way as in the case $s \geq w$. For the upper bound on the magnetostatic energy, we have to take into account (at most) 4 walls or corners at a given t-value modulo w-periodicity. $\square$

2.2 Lower bounds

Remark 2. *We introduce the notation for the average of an L-periodic function $\zeta(x,t)$ in x*

$$\langle \zeta \rangle \;:=\; L^{-1} \int_0^L \zeta \,\mathrm{d}x,$$

and the average both in x and t

$$\langle\!\langle \zeta \rangle\!\rangle \;:=\; L^{-1} \int_0^1 \int_0^L \zeta \,\mathrm{d}x \,\mathrm{d}t.$$

Proposition 2. *Let $0 < \varepsilon \ll 1$ and $L > 0$. Then*

$$\min_u L^{-1} E \;\gtrsim\; -1.$$

The main ingredient for the lower bound is a new estimate on smooth solutions u of the inhomogeneous, inviscid Burgers equation, i.e.

$$\partial_t u - \partial_x(\tfrac{1}{2} u^2) \;=\; \sigma. \tag{2.17}$$

This type of estimate was introduced in [Ott08, Section 2.6]; it relies on a generalization of Oleinik's E-principle [Ole63]. That principle states that for smooth solutions of the homogeneous inviscid Burgers equation, i.e.

$$\partial_t u - \partial_x(\tfrac{1}{2}u^2) \;=\; 0, \tag{2.18}$$

a one-sided Lipschitz bound improves over time in the sense that for any $\tau > 0$

$$\partial_x u(t=0,\cdot) \geq -\tau^{-1} \quad \Rightarrow \quad \partial_x u(t,\cdot) \;\geq\; -(\tau+t)^{-1}. \tag{2.19}$$

In fact, the main insight of [Ott08] is that in addition, the L^2-distance to the set of functions ζ with a one-sided Lipschitz bound improves over time. To make this more precise, we need

Definition 1. *Let $u(x)$ be L-periodic in x. Define*

$$\begin{aligned}
\mathcal{D}^-(u,\tau) \;&:=\; \inf\,\{\,\langle(\zeta-u)^2\rangle \mid \zeta \textit{ smooth and } L\textit{-periodic}, \tau\partial_x\zeta \geq -1\,\},\\
\mathcal{D}^+(u,\tau) \;&:=\; \inf\,\{\,\langle(\zeta-u)^2\rangle \mid \zeta \textit{ smooth and } L\textit{-periodic}, \tau\partial_x\zeta \;\leq\; 1\,\}.
\end{aligned}$$

If $u(x,t)$ is L-periodic in x we use the abbreviation

$$\mathcal{D}^\pm(t,\tau) \;:=\; \mathcal{D}^\pm(u(\cdot,t),\tau).$$

For $\mathcal{D}^\pm$ we denote the average w.r.t. t by

$$\langle\mathcal{D}^\pm\rangle(\tau) \;:=\; \int_0^1 \mathcal{D}^\pm(t,\tau)\,\mathrm{d}t.$$

It was shown in [Ott08] that if u satisfies the homogeneous Burgers equation (2.18), $\mathcal{D}^-$ satisfies the *linear* homogeneous differential inequality

$$\partial_t\mathcal{D}^- + \partial_\tau\mathcal{D}^- + \tau^{-1}\,\mathcal{D}^- \;\leq\; 0. \tag{2.20}$$

Obviously, (2.20) contains (2.19), which follows from $\partial_t\mathcal{D}^- + \partial_\tau\mathcal{D}^- \;\leq\; 0$. The new and crucial feature is the $\tau^{-1}\mathcal{D}^-$- term in (2.20).

It was also shown in [Ott08] that (2.20) survives for the inhomogeneous Burgers equation (2.17) in the form

$$\partial_t\mathcal{D}^- + \partial_\tau\mathcal{D}^- + \tau^{-1}\,\mathcal{D}^- \;\leq\; 2\,\langle\big||\partial_x|^{-1/2}\sigma\big|^2\rangle^{1/2}\,\langle\big||\partial_x|^{1/2}u\big|^2\rangle^{1/2}. \tag{2.21}$$

However, (2.21) is not of use to us since we do not control $\langle\big||\partial_x|^{1/2}u\big|^2\rangle$ independently of ε. The idea is to replace u on the r.h.s. of (2.21) by the optimal ζ in the definition of $\mathcal{D}^-(u,\tau)$, since a ζ with a one-sided Lipschitz bound has (up to a logarithm) half of a derivative in L^2. This is the content of the next two lemmas.

Lemma 1. *Let $\zeta(x,t)$ be smooth, L-periodic in x and satisfy*

$$\tau\partial_x\zeta \;\geq\; -1$$

for some $\tau > 0$. Then for $0 < r \leq R$

$$\begin{aligned}
\langle\big||\partial_x|^{1/2}\zeta\big|^2\rangle &\lesssim r\,\tfrac{1}{2}\,\langle(\partial_x\zeta)^2\rangle + (\ln\tfrac{R}{r})\,\tau^{-1}\,\langle|\zeta|\rangle + R^{-1}\,\tfrac{1}{2}\,\langle\zeta^2\rangle\\
&\leq r\,\tfrac{1}{2}\,\langle(\partial_x\zeta)^2\rangle + (\ln\tfrac{R}{r})\,\tau^{-1}\,\langle|\zeta|^2\rangle^{1/2} + R^{-1}\,\tfrac{1}{2}\,\langle\zeta^2\rangle.
\end{aligned} \tag{2.22}$$

This interpolation inequality in turn relies on

Lemma 2. *Let $\zeta(x,t)$ be smooth, L-periodic in x and satisfy*

$$\tau\,\partial_x\zeta \;\geq\; -1$$

for some $\tau > 0$. Then

$$\sup_{\Delta>0}\tfrac{1}{\Delta}\langle|\zeta^\Delta-\zeta|^2\rangle \;\lesssim\; \tau^{-1}\,\langle|\zeta|\rangle. \tag{2.23}$$

Let us comment on both lemmas: The estimate $\sup_{\Delta>0}\frac{1}{\Delta}\langle|\zeta^\Delta-\zeta|^2\rangle \lesssim \sup|\partial_x\zeta|\,\langle|\zeta|\rangle$ is obvious. The insight of (2.23) is that the two-sided control $\sup|\partial_x\zeta|$ can be replaced by the one-sided control.

We now turn to Lemma 1: Although $\langle\big||\partial_x|^{1/2}\zeta\big|^2\rangle$ and $\sup_{\Delta>0}\frac{1}{\Delta}\langle|\zeta^\Delta-\zeta|^2\rangle$ have the same scaling, the estimate

$$\langle\big||\partial_x|^{1/2}\zeta\big|^2\rangle \;\lesssim\; \sup_{\Delta>0}\tfrac{1}{\Delta}\langle|\zeta^\Delta-\zeta|^2\rangle$$

fails. However, if very short wave lengths ($\leq\, r$) and very long wave lengths ($\geq\, R$) are treated separately, one obtains the logarithmic estimate (2.22).

Mimicking the proof of (2.21), using Lemma 1, we will derive

Lemma 3. *For any smooth L-periodic $u(x,t)$ and* $0 \,<\, \varepsilon \,\leq\, 1$

$$\begin{aligned}&\partial_t\tfrac{1}{2}(\mathcal{D}^- - \langle u^2\rangle) + \partial_\tau\tfrac{1}{2}\mathcal{D}^- + \tau^{-1}\tfrac{1}{2}\mathcal{D}^-\\ &\qquad\lesssim\; \langle\big||\partial_x|^{-1/2}\sigma\big|^2\rangle^{1/2}\,[\,\varepsilon^{1/2}\,\langle(\partial_xu)^2\rangle^{1/2}\,\langle u^2\rangle^{1/2} + (\ln\tfrac{1}{\varepsilon})\,\tau^{-1}\,\langle u^2\rangle^{1/2}\,]^{1/2}.\end{aligned} \tag{2.24}$$

Note that the second factor on the r.h.s of (2.24) is related to the r.h.s. of (2.22) by optimizing in $r \,\leq\, R$ while keeping $\varepsilon \,=\, \frac{r}{R}$ fixed.

We use Lemma 3 to derive the following interpolation inequality:

Corollary 1. *For any smooth L-periodic $u(x,t)$ with $u(\cdot,0) \,=\, u(\cdot,1) \,=\, 0$ and $0 < \varepsilon \,\leq\, 1$ it holds*

$$\begin{aligned}\int_0^1\langle u^2\rangle\,\mathrm{d}t \;\lesssim\; &\left((\ln\tfrac{1}{\varepsilon})\int_0^1\langle\big||\partial_x|^{-1/2}\sigma\big|^2\rangle\,\mathrm{d}t\right)^{2/3}\\ &+\left(\int_0^1\langle\big||\partial_x|^{-1/2}\sigma\big|^2\rangle\,\mathrm{d}t\right)^{2/3}\left(\varepsilon\int_0^1\langle(\partial_xu)^2\rangle\,\mathrm{d}t\right)^{1/3}.\end{aligned} \tag{2.25}$$

We also use Lemma 3 to derive a regularity estimate:

Corollary 2. *For any smooth L-periodic $u(x,t)$ with $u(\cdot,0) \,=\, u(\cdot,1) \,=\, 0$ and $0 < \varepsilon \,\leq\, 1$ it holds*

$$\begin{aligned}\sup_{\tau>0}\tau^{-1/2}\int_0^1\mathcal{D}^+\,\mathrm{d}t + \sup_{\tau>0}\tau^{-1/2}\int_0^1\mathcal{D}^-\,\mathrm{d}t \;\lesssim\; &\left((\ln\tfrac{1}{\varepsilon})\int_0^1\langle\big||\partial_x|^{-1/2}\sigma\big|^2\rangle\,\mathrm{d}t\right)^{2/3}\\ &+\left(\int_0^1\langle\big||\partial_x|^{-1/2}\sigma\big|^2\rangle\,\mathrm{d}t\right)^{2/3}\left(\varepsilon\int_0^1\langle(\partial_xu)^2\rangle\,\mathrm{d}t\right)^{1/3}.\end{aligned} \tag{2.26}$$

Not surprisingly, the control of the L^2-distance to the set of functions with a (one-sided) Lipschitz-bound gives control of some fractional derivative in some L^p-norm. More precisely, $\sup_{\tau>0} \tau^{-1/2} \int_0^1 \mathcal{D}^+ \,\mathrm{d}t + \sup_{\tau>0} \tau^{-1/2} \int_0^1 \mathcal{D}^- \,\mathrm{d}t$ has the same scaling as $\sup_{\Delta>0} \Delta^{-1/2} \int_0^1 \langle |u^\Delta - u|^{5/2}\rangle \,\mathrm{d}t$. Using ideas from [Ott08, Proposition 4] and interpolation with Corollary 1 we indeed obtain:

Lemma 4. *For any smooth L-periodic $u(x,t)$ with $u(\cdot,0) = u(\cdot,1) = 0$ and $0 < \varepsilon \leq 1$ it holds*

$$\sup_{\Delta>0} \Delta^{-(p-2)} \int_0^1 \langle |u^\Delta - u|^p\rangle \,\mathrm{d}t \lesssim \left((\ln \tfrac{1}{\varepsilon}) \int_0^1 \langle \left| |\partial_x|^{-1/2}\sigma \right|^2 \rangle \,\mathrm{d}t \right)^{2/3} + \left(\int_0^1 \langle \left| |\partial_x|^{-1/2}\sigma \right|^2 \rangle \,\mathrm{d}t \right)^{2/3} \left(\varepsilon \int_0^1 \langle (\partial_x u)^2\rangle \,\mathrm{d}t \right)^{1/3} \tag{2.27}$$

with $p \in [2, \frac{5}{2})$.

Remark 3. *In [CÁOS07, Section 3.3], it was shown that admissible functions u as in* (1.5) *of finite energy can always be approximated by a sequence of smooth admissible functions $\{u_\alpha\}_{\alpha\downarrow 0}$ in the energy topology. Therefore Corollary 1 and Corollary 2 and Lemma 4, which were established for a smooth u, extend to our finite-energy u.*

We will apply Corollary 1 to derive the coercivity of the energy. To facilitate the notation we introduce the abbreviations

$$\begin{aligned} \Sigma &:= \langle\!\langle \left| |\partial_x|^{-1/2}\sigma \right|^2 \rangle\!\rangle, \\ DU &:= \langle\!\langle\, (\partial_x u)^2 \rangle\!\rangle, \text{ and} \\ U &:= \langle\!\langle\, u^2 \rangle\!\rangle. \end{aligned} \tag{2.28}$$

Lemma 5. *Let $0 < \varepsilon \ll 1$. Then for any L-periodic $u(x_1,x_2)$ with $u(\cdot,0) = u(\cdot,1)$ which is of finite energy, i.e.*

$$L^{-1}E(u) = \varepsilon DU + (\ln \tfrac{1}{\varepsilon})\Sigma - U < +\infty,$$

we have

$$\varepsilon DU,\ (\ln \tfrac{1}{\varepsilon})\Sigma,\ U \lesssim \begin{cases} 1 & \text{for } L^{-1}E(u) \leq 1, \\ L^{-1}E(u) & \text{for } L^{-1}E(u) \geq 1. \end{cases} \tag{2.29}$$

Proof of Lemma 1. We fix t. The fractional Sobolev norm can be expressed as a suitable average of the L^2-modulus of continuity of ζ (this can easily be seen in Fourier space, cf. [LM68, p.59]):

$$\int_0^L \left| |\partial_x|^{1/2}\zeta \right|^2 \,\mathrm{d}x \sim \int_0^\infty \tfrac{1}{\Delta} \int_0^L (\zeta(x+\Delta) - \zeta(x))^2 \,\mathrm{d}x \,\tfrac{1}{\Delta} \,\mathrm{d}\Delta. \tag{2.30}$$

We split the r.h.s. into a small scale part, an intermediate scale part, and a large scale part:

$$\begin{aligned}\int_0^\infty \tfrac{1}{\Delta}\int_0^L (\zeta(x+\Delta)-\zeta(x))^2 \,\mathrm{d}x\,\tfrac{1}{\Delta}\,\mathrm{d}\Delta \;=\; & \int_0^r \tfrac{1}{\Delta}\int_0^L (\zeta(x+\Delta)-\zeta(x))^2 \,\mathrm{d}x\,\tfrac{1}{\Delta}\,\mathrm{d}\Delta \\ & + \int_r^R \tfrac{1}{\Delta}\int_0^L (\zeta(x+\Delta)-\zeta(x))^2 \,\mathrm{d}x\,\tfrac{1}{\Delta}\,\mathrm{d}\Delta \\ & + \int_R^\infty \tfrac{1}{\Delta}\int_0^L (\zeta(x+\Delta)-\zeta(x))^2 \,\mathrm{d}x\,\tfrac{1}{\Delta}\,\mathrm{d}\Delta,\end{aligned} \tag{2.31}$$

where $0 < r \le R < +\infty$.

The most interesting term is the intermediate one, which we estimate as follows:

$$\int_r^R \tfrac{1}{\Delta}\int_0^L (\zeta(x+\Delta)-\zeta(x))^2 \,\mathrm{d}x\,\tfrac{1}{\Delta}\,\mathrm{d}\Delta \;\le\; (\ln \tfrac{R}{r})\,\sup_{\Delta>0}\tfrac{1}{\Delta}\int_0^L (\zeta(x+\Delta)-\zeta(x))^2\,\mathrm{d}x.$$

The application of Lemma 2, i.e.

$$\sup_{\Delta>0}\tfrac{1}{\Delta}\int_0^L (\zeta^\Delta-\zeta)^2\,\mathrm{d}x \;\lesssim\; \tau^{-1}\int_0^L |\zeta|\,\mathrm{d}x,$$

yields

$$\int_r^R \tfrac{1}{\Delta}\int_0^L (\zeta(x+\Delta)-\zeta(x))^2 \,\mathrm{d}x\,\tfrac{1}{\Delta}\,\mathrm{d}\Delta \;\lesssim\; (\ln \tfrac{R}{r})\,\tau^{-1}\int_0^L |\zeta|\,\mathrm{d}x. \tag{2.32}$$

We now turn to the large scale part in (2.31). Just using the triangle inequality in form of

$$\int_0^L (\zeta(x+\Delta)-\zeta(x))^2\,\mathrm{d}x \;\le\; 4\int_0^L \zeta^2\,\mathrm{d}x$$

we obtain

$$\int_R^\infty \tfrac{1}{\Delta}\int_0^L (\zeta(x+\Delta)-\zeta(x))^2 \,\mathrm{d}x\,\tfrac{1}{\Delta}\,\mathrm{d}\Delta \;\lesssim\; R^{-1}\int_0^L \zeta^2\,\mathrm{d}x. \tag{2.33}$$

Finally, we consider the small scale part in (2.31). We have by Jensen's inequality

$$\begin{aligned}\int_0^L (\zeta(x+\Delta)-\zeta(x))^2\,\mathrm{d}x \;&=\; \int_0^L \left(\int_x^{x+\Delta}\partial_x\zeta(x')\,\mathrm{d}x'\right)^2\mathrm{d}x \\ &\le\; \int_0^L \Delta\int_x^{x+\Delta}(\partial_x\zeta(x'))^2\,\mathrm{d}x'\,\mathrm{d}x \\ &=\; \Delta^2\int_0^L (\partial_x\zeta)^2\,\mathrm{d}x.\end{aligned}$$

Hence we obtain

$$\int_0^r \tfrac{1}{\Delta}\int_0^L (\zeta(x+\Delta)-\zeta(x))^2 \,\mathrm{d}x\,\tfrac{1}{\Delta}\,\mathrm{d}\Delta \;\le\; r\int_0^L (\partial_x\zeta)^2\,\mathrm{d}x. \tag{2.34}$$

Collecting (2.32), (2.33) and (2.34) we obtain from (2.30) and (2.31)

$$\int_0^L \left||\partial_x|^{1/2}\zeta\right|^2 \,\mathrm{d}x \lesssim r\,\tfrac{1}{2}\int_0^L (\partial_x\zeta)^2 \,\mathrm{d}x + (\ln\tfrac{R}{r})\,\tau^{-1}\int_0^L |\zeta| \,\mathrm{d}x + R^{-1}\,\tfrac{1}{2}\int_0^L \zeta^2 \,\mathrm{d}x,$$

which entails

$$\langle\left||\partial_x|^{1/2}\zeta\right|^2\rangle \lesssim r\,\tfrac{1}{2}\langle(\partial_x\zeta)^2\rangle + (\ln\tfrac{R}{r})\,\tau^{-1}\,\langle|\zeta|\rangle + R^{-1}\,\tfrac{1}{2}\langle\zeta^2\rangle.$$

□

Proof of Lemma 2. We shall actually prove that for any L-periodic function $\zeta(x)$ with

$$\tau\partial\zeta(x) \le 1 \quad \text{for all} \quad x,$$

we have

$$\int_0^L |\zeta(x+\Delta)-\zeta(x)|^2 \,\mathrm{d}x \lesssim \Delta\,\tau^{-1}\int_0^L |\zeta(x)| \,\mathrm{d}x \quad \text{for all} \quad \Delta > 0. \tag{2.35}$$

The statement of Lemma 2 follows by the application of (2.35) to $\zeta(x) = \tilde{\zeta}(-x,t)$. Because of the rescaling

$$x = \Delta\,\hat{x}, \quad L = \Delta\,\hat{L}, \quad \zeta = \Delta\,\tau^{-1}\,\hat{\zeta},$$

it is enough to show (2.35) for $\Delta = 1$ and $\tau^{-1} = 1$, that is under the assumption

$$\partial\zeta(x) \le 1 \quad \text{for all} \quad x. \tag{2.36}$$

We split (2.35) into a statement for positive and for negative increments:

$$\int_0^L (\zeta(x+1)-\zeta(x))_+^2 \,\mathrm{d}x \le 2\int_0^L |\zeta(x)| \,\mathrm{d}x, \tag{2.37}$$

$$\int_0^L (\zeta(x+1)-\zeta(x))_-^2 \,\mathrm{d}x \le 4\int_0^L |\zeta(x)| \,\mathrm{d}x. \tag{2.38}$$

The statement (2.37) is easy. Indeed, because of (2.36), we have the pointwise bound $\zeta(x+1)-\zeta(x) \le 1$, so that we obtain for the integrand

$$(\zeta(x+1)-\zeta(x))_+^2 \le (\zeta(x+1)-\zeta(x))_+ \le |\zeta(x+1)|+|\zeta(x)|.$$

This implies (2.37) after integration.

We now turn to (2.38). Because of L-periodicity we have

$$\int_0^L |\zeta(x)|(1-\partial\zeta(x)) \,\mathrm{d}x = \int_0^L |\zeta(x)| - \partial(\tfrac{1}{2}\mathrm{sign}\zeta|\zeta|^2)(x) \,\mathrm{d}x = \int_0^L |\zeta(x)| \,\mathrm{d}x.$$

Hence inequality (2.38) will follow by integration from

$$(\zeta(x+1)-\zeta(x))_-^2 \le 4\int_x^{x+1} |\zeta(x')|(1-\partial\zeta(x')) \,\mathrm{d}x',$$

which by translation invariance can be reduced to

$$(\zeta(1)-\zeta(0))_-^2 \;\leq\; 4\int_0^1 |\zeta(x)|(1-\partial\zeta(x))\,\mathrm{d}x. \tag{2.39}$$

Since by (2.36) the r.h.s. is positive, it is enough to consider the case $\zeta(1)\;\leq\;\zeta(0)$. Now (2.39) follows from

$$\begin{aligned}(\zeta(1)-\zeta(0))_-^2 \;&\leq\; -4\int_0^1 |\zeta(x)|\partial\zeta(x)\,\mathrm{d}x && (2.40)\\ &=\; -4\int_0^1 \tfrac{1}{2}\partial(\mathrm{sign}\zeta\,|\zeta|^2)(x)\,\mathrm{d}x && (2.41)\\ &=\; 2\,\mathrm{sign}\zeta(0)\,\zeta(0)^2-2\,\mathrm{sign}\zeta(1)\,\zeta(1)^2.\end{aligned}$$

In fact, to prove that (2.40) holds, we distinguish three cases:

Case $0\;\leq\;\zeta(1)\;\leq\;\zeta(0)$: In this case

$$\begin{aligned}(\zeta(1)-\zeta(0))_-^2 \;&=\; (\zeta(0)-\zeta(1))^2\\ &=\; (\zeta(0)-\zeta(1))\,\underbrace{(\zeta(0)-\zeta(1))}_{\leq\zeta(0)+\zeta(1)}\\ &\leq\; \zeta(0)^2-\zeta(1)^2\\ &=\; \mathrm{sign}\zeta(0)\,\zeta(0)^2-\mathrm{sign}\zeta(1)\,\zeta(1)^2.\end{aligned}$$

Case $\zeta(1)\;\leq\;0\leq\zeta(0)$: In this case

$$\begin{aligned}(\zeta(1)-\zeta(0))_-^2 \;&=\; (\zeta(0)-\zeta(1))^2\\ &=\; \zeta(0)^2+\zeta(1)^2-2\,\zeta(0)\,\zeta(1)\\ &\leq\; 2\,(\zeta(0)^2+\zeta(1)^2)\\ &=\; 2\,(\mathrm{sign}\zeta(0)\,\zeta(0)^2-\mathrm{sign}\zeta(1)\,\zeta(1)^2).\end{aligned}$$

Case $\zeta(1)\;\leq\;\zeta(0)\;\leq\;0$: In this case

$$\begin{aligned}(\zeta(1)-\zeta(0))_-^2 \;&=\; (\zeta(0)-\zeta(1))^2\\ &=\; (\zeta(0)-\zeta(1))\,\underbrace{(\zeta(0)-\zeta(1))}_{\leq-(\zeta(0)+\zeta(1))}\\ &\leq\; -(\zeta(0)^2-\zeta(1)^2)\\ &=\; \mathrm{sign}\zeta(0)\,\zeta(0)^2-\mathrm{sign}\zeta(1)\,\zeta(1)^2.\end{aligned}$$

□

Before we start with the other proofs, let us note that $\mathcal{D}\;=\;\mathcal{D}^{\pm}$ is locally Lipschitz continuous in (t,τ). Indeed, by the triangle inequality we easily obtain for t_1,t_0 and for $\tau_1\geq\tau_0$:

$$\begin{aligned}\mathcal{D}^{1/2}(t_1,\tau)-\mathcal{D}^{1/2}(t_0,\tau)\;&\leq\;\langle|u(t_1,\cdot)-u(t_0,\cdot)|^2\rangle^{1/2},\\ \mathcal{D}^{1/2}(t,\tau_1)-\mathcal{D}^{1/2}(t,\tau_0)\;&\leq\;(1-\frac{\tau_0}{\tau_1})\langle|u(t,\cdot)|^2\rangle^{1/2}.\end{aligned}$$

Clearly, $\mathcal{D}$ is monotonically increasing in τ. Indeed, let $\tau_2 > \tau_1 > 0$, and ζ be smooth and L-periodic with $\pm\,\tau_2\,\partial_x\zeta \;\le\; 1$, then also $\pm\,\tau_1\,\partial_x\zeta \;\le\; 1$ and hence

$$\begin{aligned}\mathcal{D}^{\pm}(u,\tau_1) \;&=\; \inf\,\{\,\langle(\zeta-u)^2\rangle \mid \zeta \text{ smooth and } L\text{-periodic}, \tau_1\partial_x\zeta \ge \pm 1\,\}\\ &\le\; \inf\,\{\,\langle(\zeta-u)^2\rangle \mid \zeta \text{ smooth and } L\text{-periodic}, \tau_2\partial_x\zeta \ge \pm 1\,\}\\ &=\; \mathcal{D}^{\pm}(u,\tau_2).\end{aligned} \tag{2.42}$$

Proof of Lemma 3. Let ζ_0 be admissible in the definition of $\mathcal{D}^{-}(0,\tau)$, i.e.

$$\tau\partial_x\zeta_0 \;\ge\; -1. \tag{2.43}$$

For $\lambda > 0$ define ζ as the solution to the initial value problem

$$\begin{aligned}\partial_t\zeta - \partial_x(\tfrac{1}{2}\zeta^2) + \lambda\partial\mathcal{A}(\tau+t)\,\zeta \;&=\; \tfrac{1}{2}(\partial_x\zeta + (\tau+t)^{-1})\,(u-\zeta)\\ \zeta(\cdot,0) \;&=\; \zeta_0.\end{aligned} \tag{2.44}$$

Here, the functional $\mathcal{A}$ is defined by

$$\mathcal{A}(\tau,\zeta) \;=\; \tfrac{1}{2}\big\langle\, r(\partial_x\zeta)^2 + \eta + \tfrac{1}{\eta}(\ln^2\tfrac{R}{r})\,\tau^{-2}\,\zeta^2 + R^{-1}\,\zeta^2\big\rangle \tag{2.45}$$

for $\eta > 0$, $0 < r \le R$, and $\tau > 0$, and the operator $\partial\mathcal{A}$ is (up to the factor L^{-1}) the functional derivative of (2.45) and thus given by

$$\partial\mathcal{A}(\tau)\,\zeta \;=\; -r\,\partial_x^2\zeta + \tfrac{1}{\eta}(\ln^2\tfrac{R}{r})\,\tau^{-2}\,\zeta + R^{-1}\,\zeta. \tag{2.46}$$

As we shall see, the reason for this choice of $\mathcal{A}$ is that

$$\min_{\eta}\mathcal{A}(\tau,\zeta) \;=\; \tfrac{1}{2}r\,\langle\,(\partial_x\zeta)^2\rangle + (\ln\tfrac{R}{r})\,\tau^{-1}\,\langle\zeta^2\rangle^{1/2} + \tfrac{1}{2}R^{-1}\,\langle\zeta^2\rangle$$

appears on the r.h.s. of the estimate of Lemma 1.

Because u is smooth and $r > 0$, a unique smooth solution to (2.44) always exists. Note that the solution ζ depends, next to the initial data and u, also on the parameters λ, η, τ, r, and R.

Step 1. Maximum principle. Here we argue that for ζ defined by (2.44) we have

$$(\tau+t)\partial_x\zeta(\cdot,t) \;\ge\; -1 \quad \text{for } t \ge 0. \tag{2.47}$$

To show (2.47) let us introduce

$$\varrho(x,t) \;:=\; \partial_x\zeta + (\tau+t)^{-1}. \tag{2.48}$$

We shall argue that (2.44) can be rewritten as an advection-diffusion equation in terms of the "density" ϱ:

$$\partial_t\varrho - \partial_x(\tfrac{1}{2}\varrho\,(u+\zeta)) + (\tau+t)^{-1}\,\varrho + \lambda\,\partial\mathcal{A}(\tau+t)\,\varrho = \lambda\tfrac{1}{\eta}(\ln^2\tfrac{R}{r})\,(\tau+t)^{-3} + \lambda R^{-1}\,(\tau+t)^{-1}. \tag{2.49}$$

For a solution to (2.49) with non-negative initial data, non-negativity is preserved since the r.h.s. is positive. Due to (2.43) and (2.48) this is a reformulation of (2.47).

To see that (2.49) holds, we first rewrite the r.h.s. of (2.44):

$$\begin{aligned}\partial_t\zeta - \partial_x(\tfrac{1}{2}\zeta^2) + \lambda\partial\mathcal{A}(\tau+t)\,\zeta \;&=\; \tfrac{1}{2}(\partial_x\zeta + (\tau+t)^{-1})\,(u-\zeta)\\ &\overset{(2.48)}{=}\; \tfrac{1}{2}\varrho\,(u+\zeta) - (\partial_x\zeta + (t+\tau)^{-1})\zeta\\ &=\; \tfrac{1}{2}\varrho\,(u+\zeta) - \partial_x(\tfrac{1}{2}\zeta^2) - (\tau+t)^{-1}\zeta.\end{aligned}$$

Therefore we obtain

$$\partial_t\zeta + \lambda\partial\mathcal{A}(\tau+t)\,\zeta \;=\; \tfrac{1}{2}\varrho\,(u+\zeta) - (\tau+t)^{-1}\zeta.$$

Differentiating this equation w.r.t. x yields by linearity of $\partial\mathcal{A}$

$$\partial_t\partial_x\zeta + \lambda\partial\mathcal{A}(\tau+t)\,\partial_x\zeta \;=\; \partial_x(\tfrac{1}{2}\varrho(u+\zeta)) - (\tau+t)^{-1}\,\partial_x\zeta.$$

Hence, by definition (2.48) and linearity of $\partial\mathcal{A}$ we obtain

$$\begin{aligned}\partial_t\varrho + (\tau+t)^{-2} + \lambda\partial\mathcal{A}(\tau+t)\,\varrho - \lambda\partial\mathcal{A}(\tau+t)\,(\tau+t)^{-1}&\\ =\; \partial_x(\tfrac{1}{2}\varrho\,(u+\zeta)) + (\tau+t)^{-2} - (\tau+t)^{-1}\,\varrho&\end{aligned}$$

and therefore

$$\partial_t\varrho - \partial_x(\tfrac{1}{2}\varrho\,(u+\zeta)) + (\tau+t)^{-1}\,\varrho + \lambda\partial\mathcal{A}(\tau+t)\,\varrho \;=\; \lambda\partial\mathcal{A}(\tau+t)\,(\tau+t)^{-1}.$$

Appealing to the definition (2.46) of $\partial\mathcal{A}$ this yields (2.49).

Step 2. L^2-Contraction. In this step we show that there exists a constant $C > 0$ s.t.

$$\partial_t(\tfrac{1}{2}\langle(u-\zeta)^2\rangle - \tfrac{1}{2}\langle u^2\rangle) + (\tau+t)^{-1}\tfrac{1}{2}\langle(u-\zeta)^2\rangle \;\le\; \lambda\mathcal{A}(\tau+t,u) \;+\tfrac{C}{4\lambda}\langle\big||\partial_x|^{-1/2}\sigma\big|^2\rangle. \tag{2.50}$$

We first rewrite equation (2.44) as

$$-\partial_t\zeta + \tfrac{1}{2}(\tau+t)^{-1}\,(u-\zeta) + u\,\partial_x\zeta - \tfrac{1}{2}(\partial_x\zeta)\,(u-\zeta) \;=\; \lambda\partial\mathcal{A}(\tau+t)\,\zeta$$

and combine it with $\partial_t u - u\,\partial_x u \;=\; \sigma$ which gives

$$\partial_t(u-\zeta) + \tfrac{1}{2}(\tau+t)^{-1}\,(u-\zeta) - u\,\partial_x(u-\zeta) - \tfrac{1}{2}(\partial_x\zeta)\,(u-\zeta) = \sigma + \lambda\partial\mathcal{A}(\tau+t)\,\zeta.$$

We multiply this equation by $u-\zeta$ and apply Leibniz' rule to obtain

$$\begin{aligned}\partial_t\tfrac{1}{2}(u-\zeta)^2 + \tfrac{1}{2}(\tau+t)^{-1}(u-\zeta)^2 - u\tfrac{1}{2}\partial_x(u-\zeta)^2 - (\partial_x\zeta)\tfrac{1}{2}(u-\zeta)^2&\\ =\; \sigma(u-\zeta) + \lambda(\partial\mathcal{A}(\tau+t)\,\zeta)\,(u-\zeta).&\end{aligned}$$

Taking averages w.r.t. x and integration by parts yields

$$\begin{aligned}\tfrac{1}{2}\partial_t\langle(u-\zeta)^2\rangle + \tfrac{1}{2}(\tau+t)^{-1}\langle(u-\zeta)^2\rangle + \langle(\partial_x u - \partial_x\zeta)\tfrac{1}{2}(u-\zeta)^2\rangle&\\ =\; \langle\sigma(u-\zeta)\rangle + \langle\lambda(\partial\mathcal{A}(\tau+t)\,\zeta)\,(u-\zeta)\rangle.&\end{aligned} \tag{2.51}$$

On the other hand, multiplying $\partial_t u - u\,\partial_x u \;=\; \sigma$ with u and taking averages w.r.t. x we have

$$\partial_t \tfrac{1}{2}\langle u^2\rangle \;=\; \langle \sigma\, u\rangle. \tag{2.52}$$

Because of $\langle(\partial_x u - \partial_x\zeta)\frac{1}{2}(u-\zeta)^2\rangle = 0$, the combination of (2.51) and (2.52) yields

$$\begin{aligned}
&\tfrac{1}{2}\partial_t\langle(u-\zeta)^2\rangle - \tfrac{1}{2}\partial_t\langle u^2\rangle + (\tau+t)^{-1}\tfrac{1}{2}\langle(u-\zeta)^2\rangle \\
&\quad= \;\langle\lambda(\partial\mathcal{A}(\tau+t)\,\zeta)\,(u-\zeta)\rangle - \langle\sigma\,\zeta\rangle, \\
&\overset{\text{Cauchy-Schwarz}}{\leq}\;\; \langle\lambda(\partial\mathcal{A}(\tau+t)\,\zeta)\,(u-\zeta)\rangle + \langle\big||\partial_x|^{-1/2}\sigma\big|^2\rangle^{1/2}\langle(|\partial_x|^{1/2}\zeta)^2\rangle^{1/2} \\
&\overset{\text{convexity of }\mathcal{A}\text{, Young's inequality}}{\leq}\;\; \lambda\mathcal{A}(\tau+t,u) - \lambda\mathcal{A}(\tau+t,\zeta) \;+\; \tfrac{C}{4\lambda}\langle\big||\partial_x|^{-1/2}\sigma\big|^2\rangle + \tfrac{\lambda}{C}\langle(|\partial_x|^{1/2}\zeta)^2\rangle,
\end{aligned} \tag{2.53}$$

where we choose $C>0$ to be the constant in the estimate of Lemma 1. Since $\zeta(\cdot,t)$ fulfills the assumptions of Lemma 1 according to Step 1, more precisely $(\tau+t)\,\partial_x\zeta(x,t)\geq -1$ for $t\geq 1$, we have by Young's inequality (w.r.t. η)

$$\langle(|\partial_x|^{1/2}\zeta)^2\rangle^{1/2} \;\leq\; C\,\mathcal{A}(\tau+t,\zeta).$$

Hence (2.53) turns into

$$\tfrac{1}{2}\partial_t\langle(u-\zeta)^2\rangle - \tfrac{1}{2}\partial_t\langle u^2\rangle + (\tau+t)^{-1}\tfrac{1}{2}\langle(u-\zeta)^2\rangle \leq \lambda\mathcal{A}(\tau+t,u) + \tfrac{C}{4\lambda}\langle\big||\partial_x|^{-1/2}\sigma\big|^2\rangle. \tag{2.54}$$

Step 3. The integration of (2.54) in t gives

$$\begin{aligned}
&\tfrac{1}{2}\langle(u(\cdot,t)-\zeta(\cdot,t))^2\rangle - \tfrac{1}{2}\langle u^2(\cdot,t)\rangle + \int_0^t(\tau+t')^{-1}\tfrac{1}{2}\langle(u(\cdot,t')-\zeta(\cdot,t'))^2\rangle\,\mathrm{d}t' \\
&\leq\; \tfrac{1}{2}\langle(u(\cdot,0)-\zeta(\cdot,0))^2\rangle - \tfrac{1}{2}\langle u^2(\cdot,0)\rangle + \int_0^t \lambda\mathcal{A}(\tau+t',u(\cdot,t')) + \tfrac{C}{4\lambda}\langle\big||\partial_x|^{-1/2}\sigma(\cdot,t')\big|^2\rangle\,\mathrm{d}t'.
\end{aligned}$$

According to Step 1, $\zeta(x,t')$ is admissible in the definition of $\mathcal{D}^-(t',\tau+t')$, so that we obtain

$$\begin{aligned}
&\tfrac{1}{2}\mathcal{D}^-(t,\tau+t) - \tfrac{1}{2}\langle u^2(\cdot,t)\rangle + \int_0^t(\tau+t')^{-1}\tfrac{1}{2}\mathcal{D}^-(t',\tau+t')\,\mathrm{d}t' \\
&\leq\; \tfrac{1}{2}\langle(u(\cdot,0)-\zeta_0)^2\rangle - \tfrac{1}{2}\langle u^2(\cdot,0)\rangle + \int_0^t \lambda\mathcal{A}(\tau+t',u(\cdot,t')) \;+\; \tfrac{C}{4\lambda}\langle\big|\partial_x|^{-1/2}\sigma(\cdot,t')\big|^2\rangle\,\mathrm{d}t'.
\end{aligned}$$

Finally, since ζ_0 was an arbitrary admissible function in $\mathcal{D}^-(0,\tau)$, this turns into

$$\begin{aligned}
&\tfrac{1}{2}(\mathcal{D}^-(t,\tau+t) - \langle u^2(\cdot,t)\rangle) + \tfrac{1}{2}\int_0^t(\tau+t')^{-1}\mathcal{D}^-(t',\tau+t')\,\mathrm{d}t' \\
&\quad\leq\; \tfrac{1}{2}(\mathcal{D}^-(0,\tau) - \langle u^2(\cdot,0)\rangle) + \int_0^t \lambda\mathcal{A}(\tau+t',u(\cdot,t')) \;+\; \tfrac{C}{4\lambda}\langle\big||\partial_x|^{-1/2}\sigma(\cdot,t')\big|^2\rangle\,\mathrm{d}t'
\end{aligned} \tag{2.55}$$

for all $t \geq 0$ and $\tau > 0$. Since $\mathcal{D}^-$ is locally Lipschitz continuous in both variables and by translation invariance in t, (2.55) entails a differential version:

$$\partial_t \tfrac{1}{2}(\mathcal{D}^-(t,\tau) - \langle u^2 \rangle) + \partial_\tau \tfrac{1}{2}\mathcal{D}^-(t,\tau) + \tau^{-1}\tfrac{1}{2}\mathcal{D}^-(t,\tau) \;\leq\; \lambda \mathcal{A}(\tau, u) + \tfrac{C}{4\lambda}\langle \big| |\partial_x|^{-1/2}\sigma \big|^2 \rangle. \tag{2.56}$$

Indeed, a Lipschitz function is classically differentiable almost everywhere and its classical derivative agrees with its weak derivative.

Step 4. Optimization.

The l.h.s. of (2.56) does not depend on $\lambda > 0$ and holds for all $t \geq 0$ and $\tau > 0$. Therefore, we can now optimize on the r.h.s. of (2.56) in λ to derive:

$$\partial_t \tfrac{1}{2}(\mathcal{D}^-(t,\tau) - \langle u^2 \rangle) + \partial_\tau \tfrac{1}{2}\mathcal{D}^-(t,\tau) + \tau^{-1}\tfrac{1}{2}\mathcal{D}^-(t,\tau) \lesssim \mathcal{A}(\tau, u)^{1/2}\, \langle \big| |\partial_x|^{-1/2}\sigma \big|^2 \rangle^{1/2}. \tag{2.57}$$

Since (2.57) holds true for all $\eta > 0$ and $0 < r \leq R$, we optimize at fixed $\varepsilon = \frac{r}{R} \leq 1$ in η and R:

$$\begin{aligned}
\min_{\eta,R} \mathcal{A}(\tau, u) \;&=\; \min_{\eta,R}\; \tfrac{1}{2}\langle\, r(\partial_x u)^2 + \eta + \tfrac{1}{\eta}(\ln^2 \tfrac{R}{r})\, \tau^{-2}\, u^2 + R^{-1}\, u^2 \rangle \\
&\sim \min_{R}\; R\,\varepsilon \langle (\partial_x u)^2 \rangle + (\ln \tfrac{1}{\varepsilon})\, \tau^{-1} \langle u^2 \rangle^{1/2} + R^{-1}\, \langle u^2 \rangle \\
&\sim\; \varepsilon^{1/2}\, \langle (\partial_x u)^2 \rangle^{1/2}\, \langle u^2 \rangle^{1/2} + (\ln \tfrac{1}{\varepsilon})\, \tau^{-1} \langle u^2 \rangle^{1/2}.
\end{aligned}$$

□

Proof of Corollary 1. In the following proof, we repeatedly use that due to (1.5)

$$\begin{aligned}
\langle\, u(t=0)^2 \,\rangle \;=\; \langle\, u(t=1)^2 \,\rangle \;=\; 0, \quad &\text{and thus} \\
\mathcal{D}^-(u(t=0), \tau) \;=\; \mathcal{D}^-(u(t=1), \tau) \;=\; 0 \quad &\text{for all } \tau > 0.
\end{aligned} \tag{2.58}$$

Step 1. We drop the positive terms $\tau^{-1}\mathcal{D}^-$ and $\partial_\tau \mathcal{D}^-$, cf. (2.42), on the l.h.s. of (2.24) and integrate backwards in t and get due to (2.58)

$$\begin{aligned}
&\langle u^2(\cdot, t) \rangle - \mathcal{D}^-(t,\tau) \\
&\quad\lesssim \int_t^1 \langle \big| |\partial_x|^{-1/2}\sigma \big|^2 \rangle^{1/2} \,(\, \varepsilon^{1/2}\, \langle (\partial_x u)^2 \rangle^{1/2}\, \langle u^2 \rangle^{1/2} + (\ln \tfrac{1}{\varepsilon})\, \tau^{-1} \langle u^2 \rangle^{1/2} \,)^{1/2}\, \mathrm{d}t' \\
&\quad\leq \int_0^1 \langle \big| |\partial_x|^{-1/2}\sigma \big|^2 \rangle^{1/2} \,(\, \varepsilon^{1/2}\, \langle (\partial_x u)^2 \rangle^{1/2}\, \langle u^2 \rangle^{1/2} + (\ln \tfrac{1}{\varepsilon})\, \tau^{-1} \langle u^2 \rangle^{1/2} \,)^{1/2}\, \mathrm{d}t'.
\end{aligned}$$

Applying Jensen's and Cauchy-Schwarz' inequality in t gives

$$\begin{aligned}
&\langle u^2(\cdot, t) \rangle - \mathcal{D}^-(t,\tau) \\
&\quad\lesssim \langle\!\langle \big| |\partial_x|^{-1/2}\sigma \big|^2 \rangle\!\rangle^{1/2} \,(\, \varepsilon^{1/2}\, \langle\!\langle (\partial_x u)^2 \rangle\!\rangle^{1/2}\, \langle\!\langle u^2 \rangle\!\rangle^{1/2} + (\ln \tfrac{1}{\varepsilon})\, \tau^{-1} \langle\!\langle u^2 \rangle\!\rangle^{1/2} \,)^{1/2}.
\end{aligned} \tag{2.59}$$

Averaging (2.59) w.r.t. t yields

$$\langle\!\langle u^2\rangle\!\rangle \lesssim \langle\mathcal{D}^-\rangle(\tau)+\langle\!\langle\big||\partial_x|^{-1/2}\sigma\big|^2\rangle\!\rangle^{1/2}(\,\varepsilon^{1/2}\,\langle\!\langle(\partial_x u)^2\rangle\!\rangle^{1/2}\,\langle\!\langle u^2\rangle\!\rangle^{1/2}+(\ln\tfrac{1}{\varepsilon})\,\tau^{-1}\langle\!\langle u^2\rangle\!\rangle^{1/2}\,)^{1/2}. \tag{2.60}$$

Step 2. Consider again (2.24). We drop the positive term $\partial_\tau\mathcal{D}^-$, cf. (2.42). We then average over $t\in[0,1]$. Because of (2.58), the $\partial_t\frac{1}{2}(\mathcal{D}^- - \langle u^2\rangle)$-term vanishes. Using Cauchy-Schwarz' and Jensen's inequality as above we obtain

$$\tau^{-1}\langle\mathcal{D}^-\rangle(\tau) \lesssim \langle\!\langle\big||\partial_x|^{-1/2}\sigma\big|^2\rangle\!\rangle^{1/2}\,(\,\varepsilon^{1/2}\,\langle\!\langle(\partial_x u)^2\rangle\!\rangle^{1/2}\langle\!\langle u^2\rangle\!\rangle^{1/2}+(\ln\tfrac{1}{\varepsilon})\,\tau^{-1}\langle\!\langle u^2\rangle\!\rangle^{1/2}\,)^{1/2}. \tag{2.61}$$

Combining inequalities (2.60) and (2.61) gives in our short hand notation, cf. (2.28),

$$U \lesssim (1+\tau)\,\Sigma^{1/2}\,(\varepsilon^{1/2}DU^{1/2}U^{1/2}+(\ln\tfrac{1}{\varepsilon})\tau^{-1}\,U^{1/2})^{1/2}.$$

Choosing $\tau\sim 1$ yields

$$\begin{aligned}U &\lesssim \Sigma^{1/2}\,(\varepsilon^{1/2}DU^{1/2}U^{1/2}+(\ln\tfrac{1}{\varepsilon})\,U^{1/2})^{1/2}\\ &\lesssim \Sigma^{1/2}\,(\varepsilon DU)^{1/4}U^{1/4}+((\ln\tfrac{1}{\varepsilon})\Sigma)^{1/2}\,U^{1/4},\end{aligned}$$

and by Young's inequality we absorb U into the l.h.s. to obtain (2.25):

$$U \lesssim \Sigma^{2/3}\ (\varepsilon\,DU)^{1/3}\ +\ ((\ln\tfrac{1}{\varepsilon})\Sigma)^{2/3}.$$

□

Proof of Corollary 2. We start from (2.61) in the proof of Corollary 1, i.e.

$$\tau^{-1}\langle\mathcal{D}^-\rangle(\tau) \lesssim \langle\!\langle\big||\partial_x|^{-1/2}\sigma\big|^2\rangle\!\rangle^{1/2}\,(\,\varepsilon^{1/2}\,\langle\!\langle(\partial_x u)^2\rangle\!\rangle^{1/2}\,\langle\!\langle u^2\rangle\!\rangle^{1/2}+(\ln\tfrac{1}{\varepsilon})\,\tau^{-1}\langle\!\langle u^2\rangle\!\rangle^{1/2}\,)^{1/2},$$

which in our short hand notation turns into

$$\begin{aligned}\tau^{-1}\langle\mathcal{D}^-\rangle(\tau) &\lesssim \Sigma^{1/2}\,(\,\tau^{-1}(\ln\tfrac{1}{\varepsilon})U^{1/2}+(\varepsilon DU)^{1/2}\,U^{1/2}\,)^{1/2}\\ &\lesssim \tau^{-1/2}((\ln\tfrac{1}{\varepsilon})\,\Sigma)^{1/2}\,U^{1/4}\ +\ \tau^{1/8}\Sigma^{1/2}(\varepsilon DU)^{1/4}\,(\tau^{-1/2}U)^{1/4}\\ &\overset{\text{Young}}{\lesssim}\ \tau^{-1/2}((\ln\tfrac{1}{\varepsilon})\,\Sigma)^{1/2}U^{1/4}\ +\tau^{1/6}\Sigma^{2/3}(\varepsilon DU)^{1/3}\ +\ \tau^{-1/2}U\\ &\overset{(2.25)}{\lesssim}\ \tau^{-1/2}((\ln\tfrac{1}{\varepsilon})\,\Sigma)^{1/2}\Big(\Sigma^{2/3}\ (\varepsilon\,DU)^{1/3}\ +\ ((\ln\tfrac{1}{\varepsilon})\Sigma)^{2/3}\Big)^{1/4}\\ &\qquad +\tau^{1/6}\Sigma^{2/3}(\varepsilon DU)^{1/3}\ +\ \tau^{-1/2}\ \Big(\Sigma^{2/3}\ (\varepsilon\,DU)^{1/3}\ +\ ((\ln\tfrac{1}{\varepsilon})\Sigma)^{2/3}\Big)\\ &\lesssim \tau^{-3/8}((\ln\tfrac{1}{\varepsilon})\,\Sigma)^{1/2}\Big(\tau^{-1/2}\Sigma^{2/3}\ (\varepsilon\,DU)^{1/3}\Big)^{1/4}\\ &\qquad +\tau^{1/6}\Sigma^{2/3}(\varepsilon DU)^{1/3}\ +\ \tau^{-1/2}\ \Big(\Sigma^{2/3}\ (\varepsilon\,DU)^{1/3}\ +\ ((\ln\tfrac{1}{\varepsilon})\Sigma)^{2/3}\Big)\\ &\overset{\text{Young}}{\lesssim}\ \tau^{-1/2}((\ln\tfrac{1}{\varepsilon})\Sigma)^{2/3}+\ (\tau^{1/6}+\tau^{-1/2})\Sigma^{2/3}(\varepsilon DU)^{1/3}.\end{aligned}$$

Therefore we deduce for $\tau \le 1$

$$\tau^{-1}\langle \mathcal{D}^-\rangle(\tau) \,\lesssim\, \tau^{-1/2}\left(((\ln\tfrac{1}{\varepsilon})\Sigma)^{2/3} + \Sigma^{2/3}(\varepsilon DU)^{1/3}\right). \tag{2.62}$$

On the other hand, for $\tau \ge 1$ we have

$$\begin{aligned}\langle \mathcal{D}^-(\tau)\rangle &\overset{\zeta=0}{\le} U \\ &\lesssim \tau^{1/2}U \\ &\overset{(2.25)}{\lesssim} \tau^{1/2}\left(((\ln\tfrac{1}{\varepsilon})\Sigma)^{2/3} + \Sigma^{2/3}(\varepsilon DU)^{1/3}\right)\end{aligned} \tag{2.63}$$

Collecting estimates (2.62) and (2.63), we now obtain

$$\sup_{\tau>0}\tau^{-1/2}\langle \mathcal{D}^-\rangle(\tau) \,\lesssim\, ((\ln\tfrac{1}{\varepsilon})\Sigma)^{2/3} + \Sigma^{2/3}(\varepsilon DU)^{1/3}. \tag{2.64}$$

For $\mathcal{D}^+$ note that the change of variables $\widehat{t} = 1-t$, $\widehat{u} = -u$ leaves the r.h.s. of (2.64) invariant whereas the l.h.s. turns into

$$\mathcal{D}^-(\widehat{u}(\cdot,\widehat{t}\,),\tau) \,=\, \mathcal{D}^-(-u(\cdot,1-t\,),\tau) \,=\, \mathcal{D}^+(u(\cdot,1-t\,),\tau),$$

which gives

$$\langle \mathcal{D}^-(\widehat{u},\tau)\rangle \,=\, \langle \mathcal{D}^+(u,\tau)\rangle.$$

Therefore we obtain (2.26) in our short hand notation, i.e.

$$\sup_{\tau>0}\tau^{-1/2}\langle \mathcal{D}^+\rangle(\tau) + \sup_{\tau>0}\tau^{-1/2}\langle \mathcal{D}^-\rangle(\tau) \,\lesssim\, ((\ln\tfrac{1}{\varepsilon})\Sigma)^{2/3} + \Sigma^{2/3}(\varepsilon DU)^{1/3}.$$

$\square$

Proof of Lemma 4. The main ingredient is the following estimate of the modulus of continuity in the weak $L^{5/2}$-norm

$$\sup_{\Delta>0}\Delta^{-1/2}\sup_{M>0}M^{5/2}\langle\!\langle I(|u^\Delta - u| > M)\rangle\!\rangle \,\lesssim\, \sup_{\tau>0}\tau^{-1/2}\langle \mathcal{D}^+\rangle + \sup_{\tau>0}\tau^{-1/2}\langle \mathcal{D}^-\rangle, \tag{2.65}$$

where I denotes the indicator function. To see that (2.65) holds, fix Δ, $M > 0$ and let $\zeta^+(x,t)$ and $\zeta^-(x,t)$ be L-periodic in x with $\pm\tau\partial_x\zeta^\pm \,\le\, 1$ for some $\tau > \frac{\Delta}{M}$ given. Then we have

$$\begin{aligned}|\{|u^\Delta - u| > M\}| \,&=\, |\{u^\Delta - u > M\}| + |\{u^\Delta - u < -M\}| \\ &\le\, |\{(u-\zeta^+)^\Delta - (u-\zeta^+) > (M - \tfrac{\Delta}{\tau})\}| \\ &\qquad + |\{(u-\zeta^-)^\Delta - (u-\zeta^-) < -(M - \tfrac{\Delta}{\tau})\}| \\ &\le\, (M-\tfrac{\Delta}{\tau})^{-2}\Big(\int_0^L((u-\zeta^+)^\Delta - (u-\zeta^+))^2\,\mathrm{d}x \\ &\qquad + \int_0^L((u-\zeta^-)^\Delta - (u-\zeta^-))^2\,\mathrm{d}x\Big) \\ &\le\, 4\,(M-\tfrac{\Delta}{\tau})^{-2}\Big(\int_0^L(u-\zeta^+)^2\,\mathrm{d}x + \int_0^L(u-\zeta^-)^2\,\mathrm{d}x\Big).\end{aligned}$$

Since $\zeta^\pm$ was arbitrary in the definition of $\mathcal{D}^\pm(\tau)$, we obtain

$$\langle I(|u^\Delta - u| > M)\rangle \le 4\,(M - \tfrac{\Delta}{\tau})^{-2}(\mathcal{D}^+(\tau) + \mathcal{D}^-(\tau)).$$

Therefore we have

$$\langle I(|u^\Delta - u| > M)\rangle \le \tau^{1/2}\, 4\,(M - \tfrac{\Delta}{\tau})^{-2}\,(\sup_{\tilde\tau>0} \tilde\tau^{-1/2}\mathcal{D}^+(\tilde\tau) + \sup_{\tilde\tau>0} \tilde\tau^{-1/2}\mathcal{D}^-(\tilde\tau)).$$

Now optimizing in $\tau > \frac{\Delta}{M}$ gives

$$\langle I(|u^\Delta - u| > M)\rangle \lesssim (\tfrac{\Delta}{M})^{1/2} M^{-2}\,(\sup_{\tilde\tau>0} \mathcal{D}^+(\tilde\tau) + \sup_{\tilde\tau>0} \mathcal{D}^-(\tilde\tau)),$$

which entails (2.65).

Plugging in Corollary 2 we obtain from (2.65) for all $\Delta > 0$

$$\Delta^{-1/2} \sup_{M>0} M^{5/2} \langle\!\langle I(|u^\Delta - u| > M)\rangle\!\rangle \lesssim ((\ln \tfrac{1}{\varepsilon})\Sigma)^{2/3} + \Sigma^{2/3}(\varepsilon DU)^{1/3}. \tag{2.66}$$

We can now interpolate the strong estimate on the modulus of continuity that we obtain from Corollary 1, i.e.

$$\langle\!\langle |u^\Delta - u|^2\rangle\!\rangle \lesssim \langle\!\langle u^2\rangle\!\rangle \overset{(2.25)}{\lesssim} ((\ln \tfrac{1}{\varepsilon})\Sigma)^{2/3} + \Sigma^{2/3}(\varepsilon DU)^{1/3},$$

and the weak estimate (2.66). By Marcinkiewicz interpolation, cf. [BL76, Section 5.3], we obtain for $0 \le \beta < 1$

$$\begin{aligned}\langle\!\langle |u^\Delta - u|^{2+\frac{1}{2}\beta}\rangle\!\rangle &\lesssim \langle\!\langle |u^\Delta - u|^2\rangle\!\rangle^{1-\beta}\,(\sup_{M>0} M^{5/2}\langle\!\langle I\,(|u^\Delta - u| > M)\rangle\!\rangle)^\beta \\ &\lesssim \Delta^{\frac{\beta}{2}}\,\left(((\ln \tfrac{1}{\varepsilon})\Sigma)^{2/3} + \Sigma^{2/3}(\varepsilon DU)^{1/3}\right).\end{aligned}$$

With the identification $p = 2 + \frac{\beta}{2}$ we obtain (2.27), i.e.

$$\sup_{\Delta>0} \Delta^{-(p-2)} \langle\!\langle |u^\Delta - u|^p\rangle\!\rangle \lesssim ((\ln \tfrac{1}{\varepsilon})\Sigma)^{2/3} + \Sigma^{2/3}(\varepsilon DU)^{1/3}$$

for $p \in [2, \frac{5}{2})$, in our short hand notation. □

Proof of Lemma 5. Let $C > 0$ be a generic constant. Due to Remark 3, Corollary 1, which was established for smooth u, extends to our finite-energy u:

$$U \lesssim ((\ln \tfrac{1}{\varepsilon})\Sigma)^{2/3}\,(1 + (\ln \tfrac{1}{\varepsilon})^{-2/3}\,(\varepsilon DU)^{1/3}) \overset{\varepsilon\ll 1}{\lesssim} ((\ln \tfrac{1}{\varepsilon})\Sigma)^{2/3}\,(1 + (\varepsilon DU)^{1/3}). \tag{2.67}$$

Hence we obtain by Young's inequality

$$\begin{aligned}L^{-1}E(u) &= \varepsilon DU + (\ln \tfrac{1}{\varepsilon})\Sigma - U \\ &\overset{\varepsilon\ll 1}{\ge} \varepsilon DU + (\ln \tfrac{1}{\varepsilon})\Sigma - C\,((\ln \tfrac{1}{\varepsilon})\Sigma)^{2/3}\,(1 + (\varepsilon DU)^{1/3}) \\ &\overset{\text{Young}}{\gtrsim} \varepsilon DU + (\ln \tfrac{1}{\varepsilon})\Sigma - C,\end{aligned}$$

where C is the constant in estimate (2.67). This entails

$$\varepsilon DU + (\ln \tfrac{1}{\varepsilon})\Sigma \overset{\varepsilon \ll 1}{\lesssim} \begin{cases} 1 & \text{for } L^{-1}E(u) \leq 1, \\ L^{-1}E(u) & \text{for } L^{-1}E(u) \geq 1. \end{cases}$$

Therefore we obtain if we once again apply Young's inequality to (2.67):

$$U \overset{\varepsilon \ll 1}{\lesssim} ((\ln \tfrac{1}{\varepsilon})\Sigma)^{2/3}\,(1 + (\varepsilon DU)^{1/3}) \overset{\varepsilon \ll 1}{\lesssim} \begin{cases} 1 & \text{for } L^{-1}E(u) \leq 1, \\ L^{-1}E(u) & \text{for } L^{-1}E(u) \geq 1. \end{cases}$$

□

Proof of Proposition 2. Due to Lemma 5, we have that for any u with $L^{-1}E(u) \leq 0$

$$\varepsilon DU,\ (\ln \tfrac{1}{\varepsilon})\Sigma,\ U \lesssim 1.$$

In particular

$$L^{-1}E(u) \geq -U \gtrsim -1.$$

□

Proof of Theorem 3. Let $0 < \varepsilon \ll 1$ and $L \geq 1$.

ad a) The upper bound on the minimal energy is the statement of Proposition 1, the lower bound is the statement of Proposition 2.

ad b) The upper bound

$$L^{-1} \int_0^L \int_0^1 u^2 \, \mathrm{d}t \, \mathrm{d}x \lesssim 1$$

was treated in Proposition 2. The lower bound

$$L^{-1} \int_0^L \int_0^1 u^2 \, \mathrm{d}t \, \mathrm{d}x \gtrsim 1$$

follows directly from the assumption $L^{-1}E(u) \sim -1$.

ad c1) Note that by Jensen's inequality

$$\begin{aligned} & w^{-2(p-2)/p}\, L^{-1} \int_0^L \int_0^1 (u(x+w,t) - u(x,t))^2 \, \mathrm{d}t \, \mathrm{d}x \\ & \quad \lesssim \left(w^{-(p-2)}\, L^{-1} \int_0^L \int_0^1 (u(x+w,t) - u(x,t))^p \, \mathrm{d}t \, \mathrm{d}x \right)^{2/p} \end{aligned} \qquad (2.68)$$

for $p \in [2, \infty)$. Due to Lemma 5 we have for any u with $L^{-1}E(u) \sim -1$, that εDU, $(\ln \frac{1}{\varepsilon})\Sigma$, $U \lesssim 1$ (uniformly in ε). Hence the r.h.s. in (2.68) is bounded for

$p \in [2, \frac{5}{2})$ (uniformly in ε) due to Lemma 4, which due to Remark 3 extends to our finite-energy u. Therefore with the identification $\alpha = 2(p-2)/p$ we have

$$L^{-1} \int_0^L \int_0^1 (u(x+w,t) - u(x,t))^2 \, \mathrm{d}t \, \mathrm{d}x \lesssim w^{\alpha}$$

for $\alpha \in [0, \frac{2}{5})$.

ad c2) We split the proof into an estimate for $w \le 1$ and an estimate for $w \ge 1$. For $w \le 1$ we have by Jensen's inequality

$$\begin{aligned} L^{-1} \int_0^L \int_0^L |u_w| \, \mathrm{d}t \, \mathrm{d}x &\le \left(L^{-1} \int_0^L \int_0^L u^2 \, \mathrm{d}t \, \mathrm{d}x \right)^{1/2} \\ &\le w^{-1/2} \left(L^{-1} \int_0^L \int_0^L u^2 \, \mathrm{d}t \, \mathrm{d}x \right)^{1/2}. \end{aligned}$$

Due to Lemma 5, for u with $L^{-1}E(u) \sim -1$ the energy contributions are separately bounded. Hence we obtain

$$L^{-1} \int_0^L |u_w| \, \mathrm{d}x \le w^{-1/2}.$$

We now turn to the case $w \ge 1$. By linearity we have that

$$\partial_t u_w - (\partial_x (\tfrac{1}{2} u^2))_w = \sigma_w.$$

Therefore by the triangle inequality we have

$$L^{-1} \int_0^L |\partial_t u_w| \, \mathrm{d}x \lesssim L^{-1} \int_0^L |(\partial_x (\tfrac{1}{2} u^2))_w| \, \mathrm{d}x + L^{-1} \int_0^L |\sigma_w| \, \mathrm{d}x. \tag{2.69}$$

We now appeal to the estimates

$$L^{-1} \int_0^L |(\partial_x (\tfrac{1}{2} u^2))_w| \, \mathrm{d}x \lesssim w^{-1} L^{-1} \int_0^L u^2 \, \mathrm{d}x \tag{2.70}$$

and

$$L^{-1} \int_0^L |\sigma_w| \, \mathrm{d}x \lesssim w^{-1/2} \left(L^{-1} \int_0^L \left| |\partial_x|^{-1/2} \sigma \right|^2 \mathrm{d}x \right)^{1/2}. \tag{2.71}$$

We first turn to (2.70). By definition (1.12),

$$\begin{aligned} \int_0^L |(\partial_x u^2)_w| \, \mathrm{d}x &= \int_0^L \left| w^{-1} \int_{-\frac{w}{2}}^{\frac{w}{2}} \partial_x u^2 (x + x') \, \mathrm{d}x' \right| \mathrm{d}x \\ &= w^{-1} \int_0^L |u^2(x + \tfrac{w}{2}) - u^2(x - \tfrac{w}{2})| \, \mathrm{d}x \\ &\le w^{-1} \int_0^L u^2 \, \mathrm{d}x. \end{aligned}$$

We now turn to (2.71), which is a standard convolution estimate. We start with Jensen's inequality in the form of

$$L^{-1}\int_0^L |\sigma_w|\,\mathrm{d}x \;\le\; \left(L^{-1}\int_0^L |\sigma_w|^2\,\mathrm{d}x\right)^{1/2}. \tag{2.72}$$

By definition (1.12),

$$\sigma_w(x,t) \;=\; \int_{\mathbb{R}} \eta^w(y)\,\sigma(x-y,t)\,\mathrm{d}y,$$

where $\eta^w(x) \;:=\; w^{-1}\eta(\frac{x}{w})$ and $\eta(x) \;:=\; I([-\frac{1}{2},\frac{1}{2}])(x)$. We appeal to the Fourier series $\mathcal{F}(\sigma)(\xi)=\frac{1}{\sqrt{L}}\int_0^L \sigma(x)\,\mathrm{e}^{-i\,x\xi}\,\mathrm{d}x$, $\xi\in 2\pi L^{-1}\mathbb{Z}$, of σ and to the Fourier transform $\mathcal{F}(\eta^w)(\xi)=\int_{\mathbb{R}}\eta^w(x)\,\mathrm{e}^{-i\,x\xi}\,\mathrm{d}x$, $\xi\in\mathbb{R}$, of η^w:

$$\begin{aligned}
\int_0^L |\sigma_w|^2\,\mathrm{d}x \;&=\; \sum_{\xi\in 2\pi L^{-1}\mathbb{Z}} |\mathcal{F}(\sigma_w)|^2(\xi)\,\mathrm{d}x \\
&=\; \sum_{\xi\in 2\pi L^{-1}\mathbb{Z}} \frac{1}{\sqrt{L}}\int_0^L\int_{\mathbb{R}} \eta^w(x-y)\sigma(y)\,\mathrm{d}y\;\mathrm{e}^{-i\,x\xi}\,\mathrm{d}x && (2.73)\\
&=\; \sum_{\xi\in 2\pi L^{-1}\mathbb{Z}} |\mathcal{F}(\eta^w)(\xi)|^2|\mathcal{F}(\sigma)(\xi)|^2 \\
&=\; \sum_{\xi\in 2\pi L^{-1}\mathbb{Z}} |\mathcal{F}(\eta)(w\,\xi)|^2|\mathcal{F}(\sigma)(\xi)|^2. && (2.74)
\end{aligned}$$

We explicitly calculate the Fourier transform of η:

$$\mathcal{F}(\eta)(\xi) \;=\; \int_{\mathbb{R}} \eta(x)\,\mathrm{e}^{-i\,x\,\xi}\,\mathrm{d}x \;=\; \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{e}^{-i\,x\,\xi}\,\mathrm{d}x \;=\; 2\sin(\tfrac{\xi}{2}).$$

Hence we have

$$|\mathcal{F}(\eta)(\xi)| \;\lesssim\; \frac{1}{1+|\xi|} \;\lesssim\; \frac{1}{|\xi|^{1/2}}.$$

Thus (2.74) turns into

$$\int_0^L |\sigma_w|^2\,\mathrm{d}x \;\lesssim\; \frac{1}{w}\sum_{\xi\in 2\pi L^{-1}\mathbb{Z}} \frac{1}{|\xi|}|\mathcal{F}(\sigma)(\xi)|^2 \;=\; \frac{1}{w}\int_0^L \left||\partial_x|^{-1/2}\sigma\right|^2\mathrm{d}x.$$

Now (2.71) follows from the last estimate together with (2.72).

In order to control the r.h.s. of (2.69), we collect estimates (2.70) and (2.71) and use again that for u with $L^{-1}E(u)\;\sim\;-1$ the energy contributions are separately bounded by Lemma 5 to obtain

$$L^{-1}\int_0^L |\partial_t u_w|\,\mathrm{d}x \;\lesssim\; w^{-1}+w^{-1/2} \;\overset{w\ge 1}{\lesssim}\; w^{-1/2}. \tag{2.75}$$

Hence we have for $w \geq 1$

$$
\begin{aligned}
L^{-1} \int_0^1 \int_0^L |u_w(x,t)| \,\mathrm{d}x \,\mathrm{d}t &\overset{u(\cdot,0)=0}{=} L^{-1} \int_0^1 \int_0^L |\int_0^t \partial_t u_w(x,t') \,\mathrm{d}t'| \,\mathrm{d}x \,\mathrm{d}t \\
&\leq L^{-1} \int_0^1 \int_0^L \int_0^1 |\partial_t u_w(x,t')| \,\mathrm{d}t' \,\mathrm{d}x \,\mathrm{d}t \\
&= L^{-1} \int_0^1 \int_0^L |\partial_t u_w| \,\mathrm{d}x \,\mathrm{d}t \\
&\overset{(2.75)}{\lesssim} w^{-1/2}. \qquad (2.76)
\end{aligned}
$$

□

2.3 Compactness

Proposition 3. *Let $L \sim 1$ be fixed and $\{u^\varepsilon\}_{\varepsilon \downarrow 0}$ be a sequence such that $L^{-1} E_\varepsilon(u^\varepsilon) \sim -1$. Then $\{u^\varepsilon\}_{\varepsilon \downarrow 0}$ is compact in $L^2((0,L) \times (0,1))$.*

Proof of Proposition 3. The proof is a classical compensated compactness argument, in the sense that the strong equi-continuity properties in x compensate the weak equi-continuity in t. To start, let us first list some direct consequences of the results in the previous section.

Let $\{u^\varepsilon\}_{\varepsilon \downarrow 0}$ be a sequence such that

$$
L^{-1} E_\varepsilon(u^\varepsilon) \sim -1. \qquad (2.77)
$$

We have due to Lemma 5 that the sequence $\{u^\varepsilon\}_{\varepsilon \downarrow 0}$ is bounded in $L^2((0,L) \times (0,1))$. Therefore, after extracting a subsequence we may assume that there exists $u^0 \in L^2((0,L) \times (0,1))$ such that :

$$
u^\varepsilon \xrightharpoonup{\varepsilon \downarrow 0} u^0 \text{ weakly in } L^2. \qquad (2.78)
$$

Hence our goal is to show that this weak convergence is in fact a strong convergence.

Let $\mathcal{F}$ denote the Fourier series w.r.t. x and the Fourier transform w.r.t. t. More precisely, for any L-periodic $g(x,t)$ we define $\mathcal{F}(g)(\xi,\theta) := \frac{1}{\sqrt{L}} \int_0^L \int_{\mathbb{R}} g(x,t)\, \mathrm{e}^{-it\theta} \mathrm{e}^{-ix\xi} \,dt \,\mathrm{d}x$, where $\xi \in \frac{2\pi}{L}\mathbb{Z}$ and $\theta \in \mathbb{R}$ denote the dual variables to x and t, respectively. Since u^ε is L-periodic in x with $L \sim 1$ and supported in $t \in [0,1]$ we automatically have

$$
|\mathcal{F}((u^\varepsilon)^2)(\xi,\theta)| \lesssim \int_0^1 \int_0^L (u^\varepsilon)^2 \,\mathrm{d}x \,\mathrm{d}t \lesssim 1. \qquad (2.79)
$$

By (2.78), we have

$$
\mathcal{F}(u^\varepsilon) \xrightarrow{\varepsilon \downarrow 0} \mathcal{F}(u^0) \text{ pointwise.}
$$

Therefore, we have for all $R > 0$

$$
\int_{B_R(0)} |\mathcal{F}(u^\varepsilon) - \mathcal{F}(u^0)|^2 \,\mathrm{d}\xi \,\mathrm{d}\theta \xrightarrow{\varepsilon \downarrow 0} 0, \qquad (2.80)
$$

where $B_R(0) = \{(\xi,\theta) \in \frac{2\pi}{L}\mathbb{Z} \times \mathbb{R} \mid |\theta| < R \text{ and } |\xi| < R\}$ and $\int \cdot \,\mathrm{d}\xi\,\mathrm{d}\theta$ denotes the integration w.r.t. ξ and the discrete summation w.r.t. θ. Hence for strong convergence in L^2, it is enough to show that there is no concentration in the high frequencies, i.e.

$$\int_{\left(\frac{2\pi}{L}\mathbb{Z}\times\mathbb{R}\right)-B_R(0)} |\mathcal{F}(u_\varepsilon)|^2 \,\mathrm{d}\xi\,\mathrm{d}\theta \xrightarrow{R\uparrow\infty} 0 \tag{2.81}$$

uniformly in ε.

Before embarking on (2.81), we note that

$$\int_0^1 \int_0^L \left||\partial_x|^s u^\varepsilon\right|^2 \mathrm{d}x\,\mathrm{d}t \lesssim 1 \tag{2.82}$$

uniformly in ε. Indeed, since $L \sim 1$ we have by Theorem 3 c1) that

$$\Delta^{-\alpha} \int_0^1 \int_0^L |(u^\varepsilon)^\Delta - u^\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t \lesssim 1$$

for $\alpha \in [0, \frac{2}{5})$ uniformly in ε. Therefore for $0 < r < 1$

$$\int_0^1 \int_0^1 \frac{1}{\Delta^{2/5+r}} \int_0^L |(u^\varepsilon)^\Delta - u^\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}\Delta\,\mathrm{d}t \lesssim 1$$

uniformly in ε, as well as for $1 < r < \infty$

$$\int_0^1 \int_1^\infty \frac{1}{\Delta^r} \int_0^L |(u^\varepsilon)^\Delta - u^\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}\Delta\,\mathrm{d}t \lesssim 1$$

uniformly in ε. This entails

$$\int_0^1 \int_0^\infty \Delta^{-2s} \int_0^L |(u^\varepsilon)^\Delta - u^\varepsilon|^2 \,\mathrm{d}x \frac{1}{\Delta}\,\mathrm{d}\Delta\,\mathrm{d}t \lesssim 1$$

for $s \in (0, \frac{1}{5})$ uniformly in ε. We once again refer to the characterization of fractional Sobolev spaces in [LM68, p.59] to deduce (2.82).

We now turn to the proof of (2.81). We will use the identity $\sigma^\varepsilon = -\partial_x(\frac{1}{2}(u^\varepsilon)^2) + \partial_t u^\varepsilon$ to provide for control of oscillations in t via its Fourier transformed version, namely

$$-i\,\theta\,\mathcal{F}(u^\varepsilon) \;=\; \mathcal{F}(\sigma^\varepsilon) - \tfrac{1}{2}\,i\,\xi\mathcal{F}((u^\varepsilon)^2). \tag{2.83}$$

Moreover, we have by assumption due to Lemma 5 that

$$\int_0^1 \int_0^L \left||\partial_x|^{-1/2}\sigma^\varepsilon\right|^2 \mathrm{d}x\,\mathrm{d}t \xrightarrow{\varepsilon\downarrow 0} 0. \tag{2.84}$$

Therefore we have for $M_2 \gg M_1 \gg 1$

$$
\begin{aligned}
&\int_{\{|\xi|>M_1\}\cup\{|\theta|>M_2\}} |\mathcal{F}(u^\varepsilon)|^2 \,\mathrm{d}\xi\,\mathrm{d}\theta \\
&\leq \int_{\{|\xi|>M_1\}} |\mathcal{F}(u^\varepsilon)|^2 \,\mathrm{d}\xi\,\mathrm{d}\theta + \int_{\{|\xi|\leq M_1\}\cap\{|\theta|>M_2\}} |\mathcal{F}(u^\varepsilon)|^2 \,\mathrm{d}\xi\,\mathrm{d}\theta \\
&\overset{(2.83)}{\lesssim} \int_{\{|\xi|>M_1\}} |\mathcal{F}(u^\varepsilon)|^2 \,\mathrm{d}\xi\,\mathrm{d}\theta \\
&\quad + \int_{\{|\xi|\leq M_1\}\cap\{|\theta|>M_2\}} \frac{|\mathcal{F}(\sigma^\varepsilon)|^2}{|\theta|^2} \,\mathrm{d}\xi\,\mathrm{d}\theta + \int_{\{|\xi|\leq M_1\}\cap\{|\theta|>M_2\}} \frac{1}{|\theta|^2}|\xi|^2|\mathcal{F}((u^\varepsilon)^2)|^2 \,\mathrm{d}\xi\,\mathrm{d}\theta \\
&\leq \frac{1}{M_1^{2s}} \int_{\{|\xi|>M_1\}} |\xi|^{2s}|\mathcal{F}(u^\varepsilon)|^2 \,\mathrm{d}\xi\,\mathrm{d}\theta\,\mathrm{d}\xi\,\mathrm{d}\theta \;+\; \frac{M_1}{M_2^2} \int_{\{|\xi|\leq M_1\}\cap\{|\theta|>M_2\}} \frac{|\mathcal{F}(\sigma^\varepsilon)|^2}{|\xi|} \,\mathrm{d}\xi\,\mathrm{d}\theta \\
&\quad + \int_{\{|\xi|\leq M_1\}\cap\{|\theta|>M_2\}} \frac{1}{|\theta|^2}|\xi|^2 \,\mathrm{d}\xi\,\mathrm{d}\theta\ (\sup|\mathcal{F}((u^\varepsilon)^2)|)^2 \\
&\overset{(2.82),(2.84),(2.79)}{\lesssim} \frac{1}{M_1^{2s}} + \frac{M_1}{M_2^2} + \frac{M_1^3}{M_2}.
\end{aligned}
$$

With the choice $M_1 = M^{1/4}$, $M_2 = M$, this implies

$$
\int_{\{|\xi|>M_1\}\cup\{|\theta|>M_2\}} |\mathcal{F}(u^\varepsilon)|^2 \lesssim \frac{1}{M^{s/2}} + \frac{M^{1/4}}{M^2} + \frac{M^{3/4}}{M} \xrightarrow{M\uparrow\infty} 0
$$

uniformly in ε, which yields (2.81). $\square$

Proof of Theorem 4. We give a proof by contradiction. Let $0 < \varepsilon \ll 1$ and $L \sim 1$. Assume there exists u with $L^{-1}E(u) \sim -1$ such that for any u^* with

$$
-\partial_x \tfrac{1}{2}(u^*)^2 + \partial_t u^* = 0
$$

distributionally

$$
L^{-1} \int_0^L \int_0^1 (u-u^*)^2 \,\mathrm{d}t\,\mathrm{d}x \gtrsim 1. \tag{2.85}
$$

Hence, there exist sequences $\{L_\varepsilon\}_{\varepsilon\downarrow 0}$, $\{u^\varepsilon\}_{\varepsilon\downarrow 0}$ with L_ε bounded and $L_\varepsilon^{-1}\,E(u^\varepsilon) \sim -1$ such that u^ε is not close to a weak solution to Burgers' equation. Rescaling according to $x = \frac{L_\varepsilon}{L}\,\widehat{x}$ and $u = \frac{L_\varepsilon}{L}\,\widehat{u}$, we may w.l.o.g. assume that $L_\varepsilon = L$. On the other hand, by Proposition 3, $\{u^\varepsilon\}_{\varepsilon\downarrow 0}$ is compact in L^2 and we claim that after extracting a subsequence, $\{u^\varepsilon\}_{\varepsilon\downarrow 0}$ converges in L^2 to a weak solution of Burgers' equation which is in contradiction to the assumption. Indeed, if we denote the L^2-limit of $\{u^\varepsilon\}_{\varepsilon\downarrow 0}$ by u^0 then $\{(u^\varepsilon)^2\}_{\varepsilon\downarrow 0}$ converges to $(u^0)^2$ in L^1. Therefore, like in (2.84) we have

$$
\int_0^1 \int_0^L \left||\partial_x|^{-1/2}\sigma^\varepsilon\right|^2 \mathrm{d}x\,\mathrm{d}t \xrightarrow{\varepsilon\downarrow 0} 0,
$$

and we obtain as desired

$$
-\partial_x \tfrac{1}{2}(u^0)^2 + \partial_t u^0 = \lim_{\varepsilon\downarrow 0} \left(-\partial_x \tfrac{1}{2}(u^\varepsilon)^2 + \partial_t u^\varepsilon\right) = \lim_{\varepsilon\downarrow 0} \sigma^\varepsilon = 0 \text{ distributionally.}
$$

$\square$

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A Numerical computation of the period of global minimizers

We want to determine the period of global minimizers of E_0 depending on the external field h_{ext}. Obviously, if we minimize the energy w.r.t. the magnetization m_2, the period L of the (computational) domain has to be an integer multiple of the period of the global minimizer. We are interested in periods L being so large that they do not affect the period of the global minimizer. Therefore we focus on an appropriate discretization of the following minimization problem: For given external field h_{ext}, minimize

$$\frac{E_0(m_2)}{L} \quad \text{among all } L\text{-periodic } m_2 \text{ for } 0 < L < +\infty. \tag{A.1}$$

This in turn is equivalent to the minimization of

$$\widehat{E}(\widehat{m}_2, L) := \frac{E_0(\widehat{m}_2(\frac{\cdot}{L}, \cdot))}{L} \quad \text{among all 1-periodic } \widehat{m}_2(x_1, x_2),\ 0 < L < \infty. \tag{A.2}$$

We discretize (A.2) with the help of the finite difference discretization $E_0^h(m^h)$ of $E_0(m_2)$, which was introduced in [Ste06] and also presented in [CÁOS07]. Here, $\widehat{m}^h = (\widehat{m}_{j,k})_{0 \le j \le N_1 - 1,\, 0 \le k \le N_2 - 1} \in \mathbb{R}^{N_1 \times N_2}$, which is N_1-periodic w.r.t. j, is the discrete approximation to $\widehat{m}_2$ on an equidistant grid on $(0,1) \times (0,1)$.

On the discrete level, the minimization of (A.2) therefore amounts to a minimization of

$$\widehat{E}^h(\widehat{m}^h, L) := \frac{E_0^h(\widehat{m}^h)}{L} \quad \text{among all } N_1\text{-periodic } \widehat{m}^h \in \mathbb{R}^{N_1 \times N_2},\ 0 < L < \infty. \tag{A.3}$$

Although $\widehat{E}^h$ in (A.3) is an explicit function of $\widehat{m}^h$ and L in contrast to the energy per length (A.1), we do not simultaneously minimize in both $\widehat{m}^h$ and L. Within our present implementation, we first have to compute an approximation to the branch of minimizers $(\widehat{m}^h_{L_i}, L_i)_{i=0,\dots,N_{max}}$ defined via

$$\widehat{m}^h_L := \arg\min\{\widehat{E}^h(\widehat{m}^h, L) \mid \widehat{m}^h \in \mathbb{R}^{N_1 \times N_2}\ N_1\text{-periodic }\}. \tag{A.4}$$

Then in a second step we obtain (after a one-dimensional interpolation)

$$L^* := \arg\min e(L) := \arg\min \widehat{E}^h(\widehat{m}^h_L, L), \quad \text{where } \widehat{m}^h_L \text{ is given by (A.4),}$$

by a one-dimensional minimization. In fact, a straightforward calculus argument shows, that generically the two-step minimization is equivalent to the simultaneous minimization.

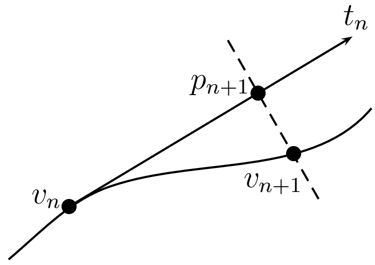

Figure 7: Tangent predictor-corrector continuation method

Computation of the branch For the computation of the approximation to the branch of solutions $(\widehat{m}_L^h, L)_L$ in (A.4) for given h_{ext}, we apply a tangent predictor-corrector path-following method. In order to apply this iterative method we need a good starting point $(\widehat{m}_0^h, L_0)$, i.e. a stationary point which indeed belongs to the minimal branch (A.4). Close to the critical field, we suggest to choose $L_0 = w^*$ and $\widehat{m}_0^h(h_{ext})$ from the w^*-periodic bifurcation branch. This is motivated by the following fact which was described in Section 1.2: The bifurcation at the critical field h_{crit} is just slightly subcritical, i.e. the critical field h_{crit} is just slightly smaller than the field at the turning point after which the branch becomes stable under w^*-periodic perturbations. Hence it is reasonable to suppose that for given h_{ext} close to the critical field h_{crit} the corresponding point on the branch after the turning point is in fact the global minimizer among all w^*-periodic m_2.

Tangent path-following algorithm. *Classical* iterative path-following takes into account the special role of the parameter L in the computation of the branch. Within the tangent continuation method the special role of the parameter L is dropped, cf. [Geo01]. The branch of solutions to the under-determined nonlinear system of equations

$$0 \;=\; F(v) \;:=\; \nabla_{m_2} \widehat{E}^h(\widehat{m}^h, L), \tag{A.5}$$
$$\text{where } F : \mathbb{R}^{N_1 \times N_2} \times (0, \infty) \to \mathbb{R}^{N_1 \times N_2} \text{ and } v \;:=\; (\widehat{m}^h, L).$$

is computed in the following way:

1. **Initialization.** Assume we have $v_0 = (\widehat{m}_{L_0}^h, L_0)$ with $F(v_0) = 0$. Approximate the tangent of the solution branch in $v = v_0$ by $t_0 \;=\; (0, \pm 1)$ (increasing or decreasing L).
2. **Iteration.** For $n \;=\; 0, 1, 2 \ldots, N_{max}$ we iterate the following two steps, cf. Figure 7:
   a) Compute the predictor $p_{n+1} \;=\; v_n + \Delta s\, t_n$, for some $\Delta s > 0$.
   b) Compute v_{n+1} as a solution to the augmented equation
   $$\begin{pmatrix} F(v_{n+1}) \\ (v_{n+1} - p_{n+1}) \cdot t_n \end{pmatrix} \;=\; 0. \tag{A.6}$$
   Compute the next tangent $t_{n+1} \;=\; (v_{n+1} - v_n)/|(v_{n+1} - v_n)|$.

Newton method for the augmented equation. In order to solve (A.6), we apply an inexact Newton method, i.e. the Newton directions are computed iteratively. This algorithm is well suited for our problem. In fact, the Jacobian of F has a block structure with a symmetric block of codimension 1, which consists of the Hessian of $\widehat{E}^h$ w.r.t. $\widehat{m}^h$. Therefore by a block gauss elimination, the computation of the Newton direction can be reduced to two applications of the conjugated gradient method to the Hessian. The contributions to the Hessian which come from the local energy terms, i.e. exchange and Zeeman, are sparse. However, the non-locality in the magnetostatic energy causes the Hessian of the discrete energy $\widehat{E}^h$ to be dense. Nevertheless, by the application of Fast Fourier Transform the matrix-vector multiplication, which is the essential and costly element in the conjugated gradient method, can be computed efficiently in Fourier space where the nonlocal operator becomes diagonal.

B Domain theory

Consider the total domain theoretic energy per unit length in x_1 given by

$$E_{domain}(m_2^0, w) = \frac{1}{w}\left(2\left(1 - \tfrac{w}{m_2^0}\right) e(m_2^0) + 4\,\frac{w}{m_2^0}\, e\left(\tfrac{m_2^0}{2}\right) - h_{ext}(m_2^0)^2\left(w - \tfrac{w^2}{m_2^0}\right)\right).$$

For $h_{ext} \gg 1$

a) the minimal energy per length in x_1-direction scales as $-h_{ext}^3 \ln^2 h_{ext}$,

b) the optimal inclination of the magnetization scales as $m_2^0 \sim h_{ext} \ln h_{ext}$, and

c) the optimal period scales as $w \sim h_{ext} \ln h_{ext}$.

This can easily be seen by the change of variables

$$\begin{aligned} w &= h_{ext}(\ln h_{ext})\,\widehat{w}, \text{ and thus } L = h_{ext}(\ln h_{ext})\,\widehat{L}, \\ m_2^0 &= h_{ext}(\ln h_{ext})\,\widehat{m}_2^0, \\ E &= h_{ext}^3(\ln^2 h_{ext})\,\widehat{E}, \text{ and thus } e = h_{ext}^4(\ln^3 h_{ext})\,\widehat{e}. \end{aligned} \tag{B.1}$$

Indeed, for $\widehat{m}_2^0$, $w \sim 1$ and $h_{ext} \gg 1$, we have that $\ln w(m_2^0)^2 \approx 3 \ln h_{ext}$ so that by (1.10)

$$\frac{1}{\widehat{w}}\,\widehat{e}(\widehat{m}_2^0) \approx \frac{1}{\widehat{w}}\frac{\pi}{24}(\widehat{m}_2^0)^4,$$

and

$$\widehat{E}_{domain}(\widehat{m}_2^0, \widehat{w}) = 2\left(1 - \tfrac{\widehat{w}}{\widehat{m}_2^0}\right)\frac{1}{\widehat{w}}\widehat{e}(\widehat{m}_2^0) + 4\,\frac{\widehat{w}}{\widehat{m}_2^0}\frac{1}{\widehat{w}}\,\widehat{e}\left(\tfrac{\widehat{m}_2^0}{2}\right) - (\widehat{m}_2^0)^2(1 - \tfrac{\widehat{w}}{\widehat{m}_2^0}).$$

Hence in the regime $h_{ext} \gg 1$, this change of variables leads to the parameter-free variational problem

$$\widehat{E}_{domain}(\widehat{m}_2^0, \widehat{w}) \approx \frac{\pi}{24}\left(2\,\tfrac{(\widehat{m}_2^0)^4}{\widehat{w}} - \tfrac{7}{4}(\widehat{m}_2^0)^3\right) - \left((\widehat{m}_2^0)^2 - \widehat{w}\,\widehat{m}_2^0\right).$$

Due to the constraint $\widehat{w} \leq \widehat{m}_2^0$, we have

$$\widehat{E}_{domain}(\widehat{m}_2^0, \widehat{w}) \gtrapprox \frac{\pi}{24}\frac{1}{4}(\widehat{m}_2^0)^3 - (\widehat{m}_2^0)^2 + \widehat{w}^2, \tag{B.2}$$

so that

$$\liminf_{|(\widehat{m}_2^0, \widehat{w})| \to \infty} = +\infty.$$

On the other hand, $\widehat{E}_{domain}$ assumes negative values for $0 < \widehat{w} \ll \widehat{m}_2^0 \lesssim 1$. Therefore we have $\min \widehat{E}_{domain} \sim -1$. Finally we note that from (B.2) we have

$$\liminf_{\widehat{m}_2^0 \to 0} \widehat{E}_{domain}(\widehat{m}_2^0, \widehat{w}) \geq 0 \quad \text{uniformly in } \widehat{w}^2,$$

and from

$$\widehat{E}_{domain}(\widehat{m}_2^0, \widehat{w}) \gtrapprox \frac{\pi}{24}\frac{1}{4}\frac{(\widehat{m}_2^0)^4}{\widehat{w}} - (\widehat{m}_2^0)^2$$

we gather

$$\liminf_{\widehat{w} \to 0} \widehat{E}_{domain}(\widehat{m}_2^0, \widehat{w}) \geq 0 \quad \text{for fixed } (\widehat{m}_2^0)^4 > 0.$$

Therefore, $\min \widehat{E}_{domain}$ is assumed for $\widehat{m}_2^0 \sim 1$ and $\widehat{w} \sim 1$.

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