Universal Bounds for the Littlewood-Paley First-Order Moments of the 3D Navier-Stokes Equations

Felix Otto, Fabio Ramos

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Felix Otto[1] Fabio Ramos[1,2]

[1] Institute of Applied Mathematics, University of Bonn
Bonn 53115, Germany
[2] Department of Computer Science and Applied Mathematics
Weizmann Institute of Science
Rehovot 76100, Israel

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Abstract. We derive upper bounds for the infinite-time and space average of the L^1-norm of the Littlewood-Paley decomposition of weak solutions of the $3D$ periodic Navier-Stokes equations. The result suggests that the Kolmogorov characteristic velocity scaling, $\mathbf{U}_\kappa \sim \epsilon^{1/3}\kappa^{-1/3}$, holds as an upper bound in a certain region before the dissipative cutoff.

1 Introduction

Consider the three-dimensional incompressible Navier-Stokes equations in the box $\Omega = [0, L]^3$:

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in} \quad \Omega \times [0, \infty), \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in} \quad \Omega \times [0, \infty), \tag{2}$$

$$\mathbf{u}(x, 0) = \mathbf{u}_0 \quad \text{in} \quad \Omega, \tag{3}$$

where $\mathbf{u}$, the velocity field, and p, the pressure field, are the unknowns. The constant $\nu > 0$ is the kinematic viscosity, $\mathbf{f}$, the body forcing, is a smooth time-independent, divergence-free vector field, and the initial velocity $\mathbf{u}_0$ is a divergence-free vector field belonging to $(L^2(\Omega))^3$. We impose Ω-periodic boundary conditions on $\mathbf{u}$, p, $\mathbf{f}$ and $\mathbf{u}_0$.

Clearly, because of the periodicity assumption, (1) preserves the spatial average of $\mathbf{u}$, i. e. $\frac{d}{dt}\langle \mathbf{u}\rangle = 0$, where for any Ω-periodic vector field $\mathbf{v}(x)$, we use the abbreviation $\langle \mathbf{v}\rangle := L^{-3}\int_\Omega \mathbf{v}\, dx$. Because (1) is invariant under the Galilean transformation

$$t = \hat{t}, \quad x = \hat{x} + U\,t, \quad u = \hat{u} + U, \tag{4}$$

one may restrict oneself to the study of solutions with vanishing spatial average, that is

$$\forall t \in [0,\infty), \quad \langle \mathbf{u}\rangle = 0. \tag{5}$$

We denote by $\dot{H}^1(\Omega)$ and $\dot{L}^2(\Omega)$ the space of functions belonging, respectively, to $H^1(\Omega)$ and $L^2(\Omega)$, with zero spatial average, and satisfying Ω-periodic boundary conditions.

In this work, we consider weak solutions of the Navier-Stokes equations, a notion first introduced by J. Leray in [7], and further developed by E. Hopf in [6].

Definition 1 *We say that $(\mathbf{u}, p)$ is a Leray-Hopf weak solution of the Navier-Stokes equations if*

(i) $(\mathbf{u}, p)$ *satisfies* (1) *and* (2) *in the sense of distributions, and*

$$\mathbf{u}(t) \to \mathbf{u}_0 \text{ weakly in } (L^2(\Omega))^3 \text{ as } t \to 0.$$

(ii)

$$\mathbf{u} \in L^2_{loc}([0,\infty), (\dot{H}^1(\Omega))^3) \cap L^\infty([0,\infty), (\dot{L}^2(\Omega))^3), \tag{6}$$

and if, for a.e. $T > 0$, $\mathbf{u}$ satisfies the following energy inequality

(iii)

$$\begin{aligned}\frac{1}{2}\|\mathbf{u}(\cdot,T)\|^2_{L^2} &+ \sum_{i,j=1}^{3}\int_0^T \left\|\frac{\partial u_i}{\partial x_j}(\cdot,s)\right\|^2_{L^2} ds \\ &\le \int_0^T (\mathbf{f}, \mathbf{u}(s))_{L^2} ds + \frac{1}{2}\|\mathbf{u}(\cdot,0)\|^2_{L^2},\end{aligned} \tag{7}$$

where $(\cdot,\cdot)_{L^2}$ stands for the inner product in $L^2(\Omega)$, and $\|\cdot\|_{L^2}$ is the norm in $L^2(\Omega)$.

The classical results concerning existence of Leray-Hopf weak solutions can be found in [4, 6, 7, 10, 11]. Notice that our definition of weak solutions is different from the usual one found, for example, in [4, 10, 11]. There, the authors consider only test functions φ satisfying $\nabla\cdot\varphi = 0$, which eliminates the pressure term in the weak formulation.

However, we remark that both formulations are equivalent since one can always recover the pressure term, see, e.g., [11, Theorem IV.2.2]. Moreover, one can also obtain from [11, Theorem IV.2.2] that Leray-Hopf weak solutions satisfy

$$\nabla p \in L^1_{loc}([0,T),(L^1(\Omega))^3). \tag{8}$$

The reason we chose to work with this definition of weak solutions comes from the fact that, in the course of this work, we may perform energy estimates with a non divergence-free test function. Furthermore, because we are interested in L^1 estimates, one may not circumvent this problem by using the Leray-Helmoltz orthogonal projector, since this is not a bounded operator in $(L^1(\Omega))^3$.

1.1 Main result

We begin with some definitions

Definition 2 *(i) For any suitable scalar Ω-periodic function $u(x)$, we define the spatial average by*

$$\langle u\rangle := \frac{1}{L^3}\int_\Omega u(x)dx.$$

(ii) For any suitable scalar periodic function $u(t,x)$, defined for $x\in\Omega$ and $t\geq 0$, which is Ω-periodic, and for any $p\in[1,\infty)$, we denote the finite time and space average of the p-th power by

$$\langle\langle |u|^p\rangle\rangle_T := \frac{1}{T}\int_0^T \langle |u(t,.)|^p\rangle dt.$$

(iii) *Similarly, for any vector field* $\mathbf{u}(t,x) = (u_1, u_2, u_3)$, *defined in* Ω *and for all* $t \geq 0$, *which is* Ω*-periodic, and for any* $p \in [1, \infty)$, *we define*

$$\langle |\mathbf{u}|_p^p \rangle := \langle \sum_{i=1}^{3} |u_i|^p \rangle, \qquad \langle\langle |\mathbf{u}|_p^p \rangle\rangle_T := \langle\langle \sum_{i=1}^{3} |u_i|^p \rangle\rangle_T,$$

and

$$\langle\langle \|\nabla \mathbf{u}\|^2 \rangle\rangle_T := \langle\langle \sum_{i,j=1}^{3} |\frac{\partial u_i}{\partial x_j}|^2 \rangle\rangle_T.$$

(iv) *For any vector field* $\mathbf{u}(t,x) = (u_1, u_2, u_3)$, *defined in* Ω *and for all* $t \geq 0$, *which is* Ω*-periodic, and for any* $p \in [1, \infty)$, *we define the infinite-time and space average by*

$$\langle\langle |\mathbf{u}|_p^p \rangle\rangle := \limsup_{T \to \infty} \langle\langle \sum_{i=1}^{3} |u_i|^p \rangle\rangle_T,$$

and

$$\langle\langle \|\nabla \mathbf{u}\|^2 \rangle\rangle := \limsup_{T \to \infty} \langle\langle \sum_{i,j=1}^{3} \left| \frac{\partial u_i}{\partial x_j} \right|^2 \rangle\rangle_T.$$

Definition 3 *(i)* *For any suitable* Ω*-periodic function* $u(x)$ *we define the Fourier series* $(\mathcal{F}u)(q)$ *via*

$$(\mathcal{F}u)(q) \; = \; L^{-3} \int_{[0,L]^3} \exp(iq \cdot x)\, u(x)\, dx, \quad \textit{for } q \in 2\pi L^{-1} \mathbb{Z}^3.$$

(ii) *For a Schwartz function* $\phi(x)$, $x \in \mathbb{R}^3$, *we define its Fourier transform* $(\mathcal{F}\phi)(q)$, $q \in \mathbb{R}^3$ *via*

$$(\mathcal{F}\phi)(q) \; = \; \int_{\mathbb{R}^3} \exp(iq \cdot x)\, \phi(x)\, dx. \tag{9}$$

(iii) *We select a family of Schwartz functions* $\{\phi_k(x)\}_{k\in\mathbb{Z}}$ *defined in* $\mathbb{R}^3$ *such that their Fourier transforms,* $\{(\mathcal{F}\phi_k)(q)\}_{k\in\mathbb{Z}}$*, satisfy*

$$(\mathcal{F}\phi_0)(q) \neq 0, \quad \text{only for } q \in \mathcal{A}_{2^{-1},2}, \tag{10}$$

$$(\mathcal{F}\phi_k)(q) = (\mathcal{F}\phi_0)(2^{-k}\,q), \quad \text{for all } k \text{ and } q, \tag{11}$$

$$\sum_{k\in\mathbb{Z}}(\mathcal{F}\phi_k)(q) = 1, \quad \text{for all } q, \tag{12}$$

where $\mathcal{A}_{2^{-1},2}$ *is the annulus defined as*

$$\mathcal{A}_{2^{-1},2} := \left\{q \in \mathbb{R}^3\,;\ |q| \in (2^{-1},2)\right\}. \tag{13}$$

(iv) *For any suitable* Ω*-periodic function* $u(x)$*, we define the Littlewood-Paley decomposition* $\{u_k\}_{k\in\mathbb{Z}}$ *by*

$$u_k := \phi_k * u,$$

where $*$ *denotes convolution in the* x *variable.*

Remark 1.1 *It is easy to prove, see [1, 9], that there exist universal constants* $c > 0$ *and* $C > 0$ *such that*

$$c\,2^{2k}\langle|u_k|^2\rangle \leq \langle|\nabla u_k|_2^2\rangle \leq C\,2^{2k}\langle|u_k|^2\rangle, \tag{14}$$

and so that

$$c\sum_{k=-\infty}^{\infty} 2^{2k}\langle|u_k|^2\rangle \leq \langle|\nabla u|_2^2\rangle \leq C\sum_{k=-\infty}^{\infty} 2^{2k}\langle|u_k|^2\rangle. \tag{15}$$

Remark 1.2 *Throughout this work, when a convolution product is taken between a scalar function and a vector or a tensor field, it is meant convolution coordinate by coordinate. In particular, the Littlewood-Paley decomposition,* $\{\mathbf{u}_k\}_{k\in\mathbb{Z}}$*, of a* Ω*-periodic vector field* $\mathbf{u}(x) = (u_1, u_2, u_3)$ *is defined by*

$$\mathbf{u}_k := \phi_k * \mathbf{u} := (\phi_k * u_1, \phi_k * u_2, \phi_k * u_3).$$

Definition 4 *Let* $\mathbf{u}$ *be a weak solution of the Navier-Stokes equations. We denote by* ϵ *the mean rate of dissipation of kinetic energy defined by*

$$\epsilon := \nu\langle\langle\|\nabla\mathbf{u}\|^2\rangle\rangle.$$

Remark 1.3 *We remark that Definition* 4 *makes sense because of energy inequality* (7)*, see [4] for details.*

The main result of this work states the following:

Theorem 1 *Let* $\mathbf{u}(x,t)$ *be a weak solution of the Navier-Stokes equations, and assume that the forcing term* $\mathbf{f}$ *in* (1) *satisfies*

$$\mathbf{f}_k = 0, \quad \textit{for every } k \geq k_0.$$

Then,

$$\langle\langle|\mathbf{u}_k|_1\rangle\rangle \leq C\epsilon\nu^{-2}(2^k)^{-3}, \tag{16}$$

for every $k \geq k_0$*, where* $C := C(\phi_0)$ *is a universal constant depending only on the choice of the partition function* ϕ_0 *in the definition of Littlewood-Paley decomposition.*

1.2 Physical interpretation

Let us now comment on the significance of Theorem 1. Our arguments are partially similar to the ones used by P. Constantin in [2, 3].

It is believed that the Navier-Stokes equations describe a wide range of incompressible Newtonian fluid flows, including the ones in turbulent regime. In 1941, in a celebrated series of papers, see [5] for details, Kolmogorov argued that there exists a range of wavenumbers, called the inertial range, such that the energy spectrum of a homogeneous turbulent flow, described by a velocity field $\mathbf{u}(x,t)$, satisfies

$$E(\kappa) \sim \epsilon^{2/3}\kappa^{-5/3}, \tag{17}$$

where the energy spectrum $E(\kappa)$ is a function with dimensions of energy per wavenumber defined by

$$E(\kappa) = \frac{1}{2}\int_{|\xi|=\kappa} \langle\langle|\hat{\mathbf{u}}(\xi,t)|_2^2\rangle\rangle\, dS(\xi), \tag{18}$$

where $\hat{\mathbf{u}}$ denotes the spatial Fourier transform. The asymptotic law (17) is expected to hold in the range of wavenumbers $[\kappa_0, \kappa_d]$, as the Reynolds number, Re, tends to ∞, where κ_0^{-1} is an energy input scale, κ_d is the Kolmogorov dissipation wavenumber, $\kappa_d := (\epsilon/\nu^3)^{1/4}$, and the Reynolds number, Re, can be defined by $Re := (\langle\langle|\mathbf{u}|_2^2\rangle\rangle^{1/2}\kappa_0^{-1})/\nu$.

Using (18), one may define a characteristic velocity at length scale $1/\kappa$ by

$$\mathbf{U}_\kappa := \left(\int_{\frac{\kappa}{2}}^{2\kappa} \int_{|\xi|=\kappa} \langle\langle |\hat{\mathbf{u}}(\xi,t)|^2 \rangle\rangle \, dS(\xi) \, d\kappa \right)^{1/2} = 2^{1/2} \left(\int_{\frac{\kappa}{2}}^{2\kappa} E(\kappa) \, d\kappa \right)^{1/2}. \tag{19}$$

Now, from (17), it is clear that $\mathbf{U}_\kappa$ satisfies

$$\mathbf{U}_\kappa \sim C\epsilon^{1/3}\kappa^{-1/3}. \tag{20}$$

Kolmogorov's heuristic laws are obtained by a universality hypothesis of the statistical properties of small scales in a homogeneous turbulent flow, i. e., that in the limit of infinite Reynolds number, all the small scale statistical properties are uniquely and universally determined by the scale κ^{-1}, and by the mean energy dissipation rate ϵ, see [5] for details.

The definition of the characteristic velocity at scale κ^{-1} given by (19), although natural, is arbitrary. Nonetheless, arguing by dimensional analysis, we have by the Kolmogorov's universality hypothesis that any working definition of characteristic velocity at scale κ^{-1}, $\tilde{\mathbf{U}}_\kappa$, must satisfy the scaling law (20), i. e.,

$$\tilde{\mathbf{U}}_\kappa \sim C\epsilon^{1/3}\kappa^{-1/3}. \tag{21}$$

We remark, however, that intermittent dynamics observed in many turbulent flows is inconsistent with the universality hypothesis in $K41$, and several modifications of this assumption have been proposed, see [5] for more details. The nature of the intermittent phenomena in turbulent flows is presently not well understood, and the question about the universality of the scaling law (21) for a given characteristic velocity at scale $1/\kappa$ remains elusive.

We refer the reader to [5, Ch. 7] for a phenomenological derivation of (21) using a second-order correlation function of the velocity field as the characteristic velocity, and with the universality hypothesis replaced by a self-similarity assumption.

Now, we interpret the result obtained in Theorem 1 by following the arguments used in [2, 3]. For any weak solution $\mathbf{u}(x,t)$ of the Navier-Stokes equations, let us define the Littlewood-Paley first-order characteristic velocity at length scale $1/\kappa$ by

$$\mathbf{U}_\kappa^{LP} := \langle\langle |\mathbf{u}_k|_1 \rangle\rangle, \tag{22}$$

for $\kappa \in [2^{k-1}, 2^k)$.

With this definition, Theorem 1 implies that for $\kappa \geq \kappa_0$,

$$\mathbf{U}_\kappa^{LP} \leq C\epsilon\nu^{-2}(2^k)^{-3} \leq C\epsilon\nu^{-2}\kappa^{-3}. \tag{23}$$

Arguing as in [2, 3], we can rewrite the inequality above as

$$\mathbf{U}_\kappa^{LP} \leq C\left(\frac{\kappa_d}{\kappa}\right)^{8/3} \epsilon^{1/3}\kappa^{-1/3}, \tag{24}$$

and, again, arguing as in [2], this implies that

$$\mathbf{U}_\kappa^{LP} \leq C\epsilon^{1/3}\kappa^{-1/3}, \tag{25}$$

holds for every $\kappa \in [\beta\kappa_d, \kappa_d]$, with $\frac{\kappa_0}{\kappa_d} \leq \beta \leq 1$, and $C := C_{\phi_0}\beta^{-8/3}$, where C_{ϕ_0} is a universal constant depending only on the choice of the partition function ϕ_0 in the definition of Littlewood-Paley decomposition.

Therefore, (25) states that the scaling $\epsilon^{1/3}\kappa^{-1/3}$ is obtained in the last decades before the dissipative cutoff, if we define the characteristic velocity as in (22).

The results discussed above are in the same vein as the ones obtained in [2] and [3]. There, similar bounds were obtained for the Littlewood-Paley second-order moments, which are somehow equivalent to the Fourier second-order moments by using the isometry of the Fourier transform in L^2, see [2] for details.

However, we remark an important difference in our work. The equivalent of our constant C displayed in the last inequality is not universal in [2] and [3]. Besides that, in [2], the author works with solutions of a regularized Navier-Stokes equations, and uses a different definition of ϵ, which involves a Besov norm.

Now, we proceed to the proof of Theorem 1.

2 Proof

Let $B_r(z)$ denote the ball in $\mathbb{R}^3$ of radius $r > 0$, and centered in z.

Lemma 1 *(Narrow-bandedness in Fourier space) Let $u : \Omega \to \mathbb{R}$ be a smooth Ω-periodic function, satisfying*

$$\mathcal{F}(u)(q) = 0, \quad \textit{for all } q \notin B_\delta(\eta), \tag{26}$$

where

$$0 < \delta < 1, \quad \textit{and} \quad \frac{1}{2} < |\eta| \le 2.$$

We claim that

$$\langle | - \Delta u - |\eta|^2 u|\rangle \le C\delta \langle |u|\rangle, \tag{27}$$

for some universal constant $C > 0$.

Proof. The proof is a direct modification of step 1 in the proof of Proposition 2 in [8]. Select a Schwartz function $\phi\colon \mathbb{R}^3 \to \mathbb{R}$, such that its Fourier transform $(\mathcal{F}\phi)(q) = \int_{\mathbb{R}^3} \exp(iq\cdot z)\,\phi(z)\,dz$, $q \in \mathbb{R}^3$, satisfies

$$(\mathcal{F}\phi)(q) \;=\; 1 \quad \text{for } |q| \le 1. \tag{28}$$

Consider the rescaled and modulated version ϕ_δ^η of ϕ:

$$\phi_\delta^\eta(z) := \;\exp\left(i\,\eta\cdot z\right)\,\delta^3\;\phi(\delta z). \tag{29}$$

An easy calculation shows that

$$(\mathcal{F}\phi)(q) = 1, \;\text{ for }\; q \in B_\delta(\eta). \tag{30}$$

Therefore, because $(\mathcal{F}(\phi_\delta^\eta * u))(q) = (\mathcal{F}\phi_\delta^\eta)(q)(\mathcal{F}u)(q)$, $q \in 2\pi L^{-1}\mathbb{Z}^3$, the narrow-bandedness assumption (26) implies that $(\mathcal{F}(\phi_\delta^\eta * u))(q) = (\mathcal{F}u)(q)$, which means that ϕ_δ^η leaves u invariant under convolution, i. e.,

$$u \;=\; \phi_\delta^\eta * u. \tag{31}$$

Now (27) follows easily because (31) implies the representation

$$-\Delta u - |\eta|^2 u \;=\; (-\Delta\phi_\delta^\eta - |\eta|^2\phi_\delta^\eta) * u.$$

Indeed, we obtain on the one hand

$$\langle | - \Delta u - |\eta|^2 u|\rangle \;\le\; \int_{\mathbb{R}^3} | - \Delta\phi_\delta^\eta - |\eta|^2\phi_\delta^\eta|dz\,\langle |u|\rangle. \tag{32}$$

On the other hand, because of

$$-\Delta \exp\left(i\,\eta\cdot z\right) = |\eta|^2 \exp\left(i\,\eta\cdot z\right),$$

we obtain

$$(-\Delta\phi_\delta^\eta - |\eta|^2\phi_\delta^\eta)(z) = -2i\exp\left(i\,\eta\cdot z\right)\delta^4\left(\nabla\phi\right)(\delta z)\cdot\eta + \exp\left(i\,\eta\cdot z\right)\delta^5\Delta\phi(\delta z),$$

so that, since ϕ is a Schwartz function, $|\eta| < 2$ and $\delta < 1$, we have

$$\int_{\mathbb{R}^3} |-\Delta\phi_\delta^\eta - |\eta|^2\phi_\delta^\eta| dz \ \leq \ C\,\delta. \tag{33}$$

Inserting (33) into (32) yields (27).

Lemma 2 *(Energy estimate) There exist universal $\delta > 0$ and $C < \infty$ with the following property: Assume that*

$$u \in L^\infty([0,\infty), L^2(\Omega)), \tag{34}$$

$$g \in L^1_{loc}([0,\infty), L^1(\Omega)), \tag{35}$$

and

$$\mathbf{v} \in L^2_{loc}([0,\infty), (H^1(\Omega))^3) \cap L^\infty_{loc}([0,\infty), (L^2(\Omega))^3). \tag{36}$$

Suppose that the following equation holds in the sense of distributions

$$\begin{cases} \partial_t u + \mathbf{v}\cdot\nabla u - \nu\Delta u \ = \ g, \\ \nabla\cdot\mathbf{v} = 0, \\ u, \mathbf{v} \text{ periodic, with periodic box } \Omega = [0,L]^3, \end{cases} \tag{37}$$

and

$$u(t) \to u_0 \text{ weakly in } (L^2(\Omega)) \text{ as } t \to 0.$$

Assume that u is narrow-banded in Fourier space, i. e.,

$$\mathcal{F}(u(\cdot,t))(q) = 0, \quad \text{for all } q \notin B_\delta(\eta), \text{ and } t \geq 0, \tag{38}$$

where

$$0 < \delta < 1, \quad \text{and } \frac{1}{2} \leq |\eta| \leq 2.$$

We claim that under these assumptions, for a.e. $T > 0$, we have

$$\frac{1}{T}\langle|u(.,T)|\rangle + \frac{\nu}{8}\langle\langle|u|\rangle\rangle_T \ \leq \ \langle\langle|g|\rangle\rangle_T + \frac{1}{T}\langle|u(.,0)|\rangle. \tag{39}$$

Proof. We first observe that because of (38), we have

$$\langle|\triangle u(t,\cdot)|_2^2\rangle + \langle\|\nabla u(t,\cdot)\|^2\rangle \ \lesssim \ \langle|u(t,\cdot)|^2\rangle,$$

for a. e. $t \in [0,\infty)$, so that (34) improves to

$$u \in L^\infty([0,\infty), H^2(\Omega)).$$

In particular,

$$-\mathbf{v}\cdot\nabla u+\nu\triangle u+g\in L^\infty(L^1)+L^\infty(L^2)+L^1_{loc}(L^1)\subset L^1_{loc}(L^1),$$

so that by (37) we obtain that

$$\frac{du}{dt}\in L^1_{loc}(L^1),$$

and that (37) also holds in the a. e. sense.

Now, let $A(z)$ be a smooth approximation of

$$A(z) \;=\; |z|. \tag{40}$$

By the chain rule for weak derivatives, we obtain from (37) in the a. e. sense that

$$\frac{d}{dt}A(u)+\mathbf{v}\cdot\nabla A(u)-\nu\triangle uA'(u)=gA'(u). \tag{41}$$

Since $\nabla\cdot\mathbf{v}=0$ in the weak sense, this yields

$$\frac{d}{dt}\langle A(u)\rangle-\nu\langle\triangle uA'(u)\rangle=\langle gA'(u)\rangle.$$

At this stage, we may carry out our approximation argument in A so that (41) holds for (40).

Now, we will show with help of Lemma 1 that the narrow-bandedness (26) implies

$$-\langle A'(u)\,\Delta u\rangle \;=\; -\langle \mathrm{sign} u\,\Delta u\rangle \;\geq\; \frac{1}{8}\,\langle|u|\rangle, \tag{42}$$

for δ small enough. Indeed, by Lemma 1 we have for δ small enough:

$$\langle|-\Delta u-|\eta|^2u|\rangle \;\leq\; \frac{1}{8}\,\langle|u|\rangle. \tag{43}$$

Therefore,

$$\begin{aligned}-\langle\mathrm{sign}\,u\,\Delta u\rangle \;&=\; \langle(\mathrm{sign}\,u)\,|\eta|^2u\rangle+\langle\mathrm{sign}\,u\,(-\Delta u-|\eta|^2u)\rangle\\ &\geq\; |\eta|^2\langle|u|\rangle-\frac{1}{8}\langle|u|\rangle\;\geq\;\frac{1}{8}\langle|u|\rangle,\end{aligned}$$

where we have used the fact that $|\eta|\geq\frac{1}{2}$ in the last inequality above. We now return to (41) with A given by (40), in which we insert (42), yielding

$$\partial_t\langle|u|\rangle+\frac{\nu}{8}\langle|u|\rangle\;\leq\;\langle|g|\rangle, \tag{44}$$

for a.e. $t > 0$. By (34) and (35), we may take the time average from 0 to T in the equation above, yielding (39).

Lemma 3 *(Commutator Estimates) Let $\phi : \mathbb{R}^3 \to \mathbb{R}$ be a Schwartz function. Let v be a smooth Ω-periodic function. Consider the commutator $[v, \phi *]$ of the operation "multiplication with v" and the operation "convolution with ϕ", that is,*

$$[v, \phi *]w \;:=\; v\,(\phi * w) - \phi * (v\,w), \tag{45}$$

for any smooth Ω-periodic function w. We claim that the two estimates

$$\langle |[v, \phi *]\partial_{x_i} w| \rangle \;\le\; C\,\langle |\nabla v|_2^2 \rangle^{1/2}\,\langle |w|^2 \rangle^{1/2}, \tag{46}$$

$$\langle |[v, \phi *]w| \rangle \;\le\; C\,\langle |\nabla v|_2^2 \rangle^{1/2}\,\langle |w|^2 \rangle^{1/2}, \tag{47}$$

hold for $i = 1, 2, 3$, with a constant $C(\phi) > 0$.

Proof. Both estimates rely on the elementary inequality, which proof we omit,

$$\frac{1}{L^3}\left(\int_\Omega |v(x-z) - v(x)|^2\,dx\right)^{1/2} \le |z|\,\langle |\nabla v|_2^2 \rangle^{1/2}, \tag{48}$$

for every $z \in \Omega$.

Indeed, in order to prove (47), we apply Fubini's Theorem, Cauchy-Schwarz' inequality and inequality (48), to obtain

$$\begin{aligned}
\langle |([v, \phi *]w)| \rangle &= \left\langle \left| v(\cdot)\int_{\mathbb{R}^3} \phi(z)\,w(\cdot - z)\,dz - \int_{\mathbb{R}^3} \phi(z)\,v(\cdot - z)\,w(\cdot - z)\,dz \right| \right\rangle \\
&\le \frac{1}{L^3}\int_{\mathbb{R}^3} |\phi(z)| \left(\int_\Omega |v(x) - v(x-z)|\,|w(x-z)|\,dx\right) dz \\
&\le C\int_{\mathbb{R}^3} |\phi(z)||z|\,dz \langle |\nabla v|_2^2 \rangle^{1/2} \langle |w|^2 \rangle^{1/2},
\end{aligned} \tag{49}$$

and (47) follows because ϕ is a Schwartz function.

For (46) we again use Fubini's Theorem, Cauchy-Schwarz' inequality and inequality (48) to write

$$
\begin{aligned}
\langle |([v, \phi *]\partial_{x_i} w)| \rangle &= \left\langle \left| \int_{\mathbb{R}^3} \phi(z)\,(v(\cdot) - v(\cdot - z))\,\partial_{x_i} w(\cdot - z)\,dz \right| \right\rangle \\
&\leq \frac{1}{L^3} \int_{\mathbb{R}^3} |\partial_{z_i}\phi(z)| \left(\int_\Omega |v(x) - v(x-z)||w(x-z)|\,dx \right) dz \\
&+ \frac{1}{L^3} \int_{\mathbb{R}^3} |\phi(z)| \left(\int_\Omega |\partial_{x_i} v(x-z)||w(x-z)|\,dx \right) dz \\
&\leq C \int_{\mathbb{R}^3} |\partial_{z_i}\phi(z)||z|\,dz\, \langle |\nabla v|_2^2 \rangle^{1/2} \langle |w|^2 \rangle^{1/2} \\
&+ C \int_{\mathbb{R}^3} |\phi(z)|\,dz \langle |\nabla v|_2^2 \rangle^{1/2} \langle |w|^2 \rangle^{1/2},
\end{aligned}
$$

and (46) follows because ϕ is a Schwartz function. □

Now, our aim is to estimate the component $\mathbf{u}_0$, and to extend this estimate for every $\mathbf{u}_k$ by using the scaling of the NSE as described in Lemma 6. In order to establish it, we first estimate a microlocal decomposition of $\mathbf{u}$ around a point $\eta \in \mathcal{A}_{2^{-1},2}$ by using Lemma 3, and, then, we recover the estimates for $\mathbf{u}_0$ by covering $\mathcal{A}_{2^{-1},2}$ with finite open balls centered at points belonging to $\mathcal{A}_{2^{-1},2}$. We split this strategy in two lemmas.

Let us first make a couple of definitions. Let $\delta > 0$ and $\eta \in \mathcal{A}_{2^{-1},2}$. For any $\mathbf{h} \in (L^2(\Omega))^3$, we define

$$
\mathbf{h}_\eta := ((h_1)_\eta, (h_2)_\eta, (h_3)_\eta) := (\phi_\delta^\eta * h_1, \phi_\delta^\eta * h_2, \phi_\delta^\eta * h_3), \tag{50}
$$

where ϕ_δ^η is a Schwartz function satisfying

$$
\mathcal{F}(\phi_\delta^\eta)(q) = 0, \quad \text{for all } q \notin B_\delta(\eta). \tag{51}
$$

We also define

$$
\mathbf{v}_0 \;:=\; \sum_{k \leq -1} \mathbf{u}_k, \quad \text{and} \quad \mathbf{w}_0 \;:=\; \sum_{k \geq 0} \mathbf{u}_k. \tag{52}
$$

Lemma 4 *(Micro-local estimates) There exist universal constants* $\delta > 0$ *and* $C = C(\phi_\delta^\eta) < \infty$ *with the following property: Let* $\mathbf{u}$ *be a weak solution of the*

Navier-Stokes equations (1)*. Let* $\mathbf{v}_0$*,* $\mathbf{w}_0$ *be as defined in* (52)*. Let* $\mathbf{u}_\eta$ *be as defined in* (50)*. Then, for a.e.* $T>0$*, we have*

$$\begin{aligned}\frac{1}{T}\langle|\mathbf{u}_\eta(\cdot,T)|_1\rangle+\frac{\nu}{8}\langle\langle|\mathbf{u}_\eta|_1\rangle\rangle_T &\le C(\langle\langle|\mathbf{u}_\eta|_2^2\rangle\rangle_T+\langle\langle|\mathbf{w}_0|_2^2\rangle\rangle_T+\langle\langle\|\nabla\mathbf{v}_0\|^2\rangle\rangle_T)\\ &\quad+\frac{1}{T}\langle|\mathbf{u}_\eta(\cdot,0)|_1\rangle+\frac{1}{T}\langle|\mathbf{f}_\eta|_1\rangle.\end{aligned} \tag{53}$$

Proof. We begin by treating the pressure gradient term in the Navier-Stokes equations. We remind the reader that the Leray-Helmholtz orthogonal projector

$$P_{LH}:L^2(\Omega)\to H=\left\{\mathbf{u}\in L^2(\Omega)\,;\,\mathbf{u}\text{ is }\Omega\text{-periodic , and }\nabla\cdot\mathbf{u}=0\right\} \tag{54}$$

can be explicitly described in Fourier domain by the tensor

$$\mathcal{F}(P_{LH}(\mathbf{u}))(k)=\left(Id-\frac{k\otimes k}{|k|^2}\right)\mathcal{F}(\mathbf{u})(k),\quad \forall k\in\mathbb{R}^3. \tag{55}$$

Now, let ψ_δ^η be the tensor defined componentwise by

$$\mathcal{F}((\psi_\delta^\eta)_{i\ell})(k)=\left(\delta_{i\ell}-\frac{k_ik_\ell}{|k|^2}\right)\mathcal{F}(\phi_\delta^\eta)(k), \tag{56}$$

for all $k\in\mathbb{R}^3$, and $i,\ell\in\{1,2,3\}$, where ϕ_δ^η is as described in (51), and $\delta_{i\ell}$ is the Kronecker delta tensor. It is easy to see that each component of the tensor ψ_δ^η is a Schwartz function. Moreover, because $\nabla\cdot\mathbf{u}=0$, and $\nabla\cdot\mathbf{f}=0$, the following identities hold

$$(u_i)_\eta=\phi_\delta^\eta * u_i=(\psi_\delta^\eta)_{i\ell}*u_\ell\,,\qquad (f_i)_\eta=\phi_\delta^\eta * f_i=(\psi_\delta^\eta)_{i\ell}*f_\ell\,, \tag{57}$$

where, throughout the work, repeated indices are summed over (Einstein summation). Moreover, because of (8), one easily obtains

$$\begin{aligned}\mathcal{F}((\partial_\ell\, p)*(\psi_\delta^\eta)_{i\ell})(k)&=\left(\delta_{i\ell}-\frac{k_ik_\ell}{|k|^2}\right)\mathcal{F}(\phi_\delta^\eta)(k)\mathcal{F}(\partial_\ell\, p)(k)\\ &=\left(\delta_{i\ell}-\frac{k_ik_\ell}{|k|^2}\right)\mathcal{F}(\partial_\ell\,(p*\phi_\delta^\eta))(k)=0.\end{aligned} \tag{58}$$

Now, we decompose the nonlinear term as

$$\mathbf{u}\cdot\nabla\mathbf{u}=\mathbf{v}_0\cdot\nabla\mathbf{u}+\mathbf{w}_0\cdot\nabla\mathbf{v}_0+\mathbf{w}_0\cdot\nabla\mathbf{w}_0,$$

and rewrite the Navier-Stokes equations (1) as

$$\partial_t \mathbf{u} + \mathbf{v}_0 \cdot \nabla \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f} - \mathbf{w}_0 \cdot \nabla \mathbf{v}_0 - \nabla \cdot (\mathbf{w}_0 \otimes \mathbf{w}_0). \tag{59}$$

Taking the convolution product of the equation above with the tensor ψ_δ^η, we obtain by (57) and (58), that the weak solution $\mathbf{u}$ satisfies the following equations in the distributional sense:

$$\partial_t (u_i)_\eta + (v_0)_j \, \partial_j (u_i)_\eta - \nu \Delta (u_i)_\eta \; = \; g_i^\eta, \quad \text{for} \quad i = 1, 2, 3, \tag{60}$$

where, for each $i = 1, 2, 3$ fixed, we define

$$\begin{aligned} g_i^\eta := &(f_i)_\eta - (\psi_\delta^\eta)_{i\ell} * ((w_0)_j \partial_j (v_0)_\ell) \\ &- (\psi_\delta^\eta)_{i\ell} * ((w_0)_j \, \partial_j (w_0)_\ell) + [(v_0)_j, (\psi_\delta^\eta)_{i\ell} *] \partial_j u_\ell \, . \end{aligned} \tag{61}$$

Now, we prove that there exists a universal $\delta > 0$, such that each coordinate of $\mathbf{u}_\eta$ satisfies the hypothesis of the function u in Lemma 2, with $\mathbf{v}$ replaced by $\mathbf{v}_0$, and g replaced by the correspondent coordinate of $\mathbf{g}^\eta$.

Indeed, by Definition 1 of weak solutions, $\mathbf{u}$ satisfies (6), and because ϕ_δ^η satisfies (51), it clearly implies that each coordinate of $\mathbf{u}_\eta$ satisfies (34) and (38). It is also clear that (6) implies that $\mathbf{v}_0$ satisfies (36). It remains only to prove that each coordinate $(g^\eta)_i$ satisfies (35). First, it is immediate to see that for each set of indices i, j, ℓ fixed, we have

$$\langle |(\psi_\delta^\eta)_{i\ell} * ((w_0)_j \partial_j (v_0)_\ell)| \rangle \le C \langle |((w_0)_j \partial_j (v_0)_\ell)| \rangle \le C \left(\langle |\mathbf{w}_0|_2^2 \rangle + \langle \|\nabla \mathbf{v}_0\|^2 \rangle \right). \tag{62}$$

The following inequality is also straightforward

$$\langle |(\psi_\delta^\eta)_{i\ell} * (\partial_j ((w_0)_j (w_0)_\ell))| \rangle = \langle |\partial_j (\psi_\delta^\eta)_{i\ell} * ((w_0)_j (w_0)_\ell)| \rangle \le C \langle |\mathbf{w}_0|_2^2 \rangle. \tag{63}$$

For the commutator term, by splitting $u_i = (v_i)_0 + (w_i)_0$, we have for each set of indices i, j, ℓ fixed

$$\langle |[(v_0)_j, (\psi_\delta^\eta)_{i\ell} *] \partial_j u_\ell| \rangle \le \left(\langle |[(v_0)_j, \psi_\delta^\eta *] \partial_j (v_0)_\ell| \rangle + \langle |[(v_0)_j, (\psi_\delta^\eta)_{i\ell} *] \partial_j (w_0)_\ell| \rangle \right).$$

Applying (47) to the first term on the right-hand side of the equation above, and, similarly, applying (46) to the second term on the right-hand side of the equation above yields

$$\langle |[(v_0)_j, (\psi_\delta^\eta)_{i\ell} *] \partial_j u_\ell| \rangle \le C(\phi_\delta^\eta) \left(\langle |\mathbf{w}_0|_2^2 \rangle + \langle \|\nabla \mathbf{v}_0\|^2 \rangle \right). \tag{64}$$

Thus, from (62), (63) and (64), we obtain that each coordinate $(g^\eta)_i$ satisfies (35). This concludes the proof that each coordinate $(u_i)_\eta$ of $\mathbf{u}_\eta$ satisfies the hypothesis of Lemma 2, with $\mathbf{v}$ replaced by $\mathbf{v}_0$, and g replaced by the correspondent coordinate of g^η. Therefore, by Lemma 2,

$$\frac{1}{T}\langle|(u_i)_\eta(.,T)|\rangle + \frac{\nu}{8}\langle\langle|(u_i)_\eta|\rangle\rangle_T \;\leq\; \langle\langle|g_i^\eta|\rangle\rangle_T + \frac{1}{T}\langle|(u_i)_\eta(.,0)|\rangle. \tag{65}$$

Taking the time-average of (62), (63), (64), summing in i, j, ℓ, and substituting into (65), yields (53). □

Now, we recover the result obtained in Lemma 4 to the annulus $\mathcal{A}_{2^{-1},2}$. From now on, throughout the paper, we fix $\delta > 0$ such that δ is small enough to satisfy the hypothesis of Lemma 2 and Lemma 4.

Lemma 5 *(Local dyadic estimates) There exist universal $\delta > 0$ and $C < \infty$ with the following property: Let $\mathbf{u}$ be a weak solution of the Navier-Stokes equations* (1)*. Let $\mathbf{v}_0$, $\mathbf{w}_0$ be defined as in* (52)*. Then, for every $T > 0$, we have*

$$\begin{aligned}\frac{1}{T}\langle|\mathbf{u}_0(\cdot,T)|_1\rangle + \frac{\nu}{8}\langle\langle|\mathbf{u}_0|_1\rangle\rangle_T \;\leq C(&\langle\langle|\mathbf{u}_0|_2^2\rangle\rangle_T + \langle\langle|\mathbf{w}_0|_2^2\rangle\rangle_T + \langle\langle\|\nabla\mathbf{v}_0\|^2\rangle\rangle_T \\ &+ \frac{1}{T}\langle|\mathbf{u}_0(\cdot,0)|_1\rangle) + \langle|\mathbf{f}_0|_1\rangle.\end{aligned} \tag{66}$$

Proof. Let $\mathcal{A}_{2^{-1},2}$ be as defined in (13). Consider a finite family of functions $\{\chi_j\}_j$ such that $\{\mathcal{F}(\chi_j)\}_j$ is a partition of unity of the annulus $\mathcal{A}_{2^{-1},2}$, subordinated to an open covering of $\mathcal{A}_{2^{-1},2}$ given by a finite family of open balls $\{B_\delta(\eta_j)\}_j$, with $|\eta_j| \in \mathcal{A}_{2^{-1},2}$. As already mentioned, we consider that $\delta > 0$ is small enough to satisfy the hypothesis of Lemma 2 and Lemma 4.

Now, because $\mathcal{F}(\mathbf{u}_0)$ is supported on the annulus $\mathcal{A}_{2^{-1},2}$, and because $\{\mathcal{F}(\chi_j)\}_j$ is a partition of unity of $\mathcal{A}_{2^{-1},2}$, we have that

$$\mathbf{u}_0 = \sum_j \chi_j * \mathbf{u}_0 = \sum_j (\chi_j * \phi_0) * \mathbf{u}. \tag{67}$$

Now, notice that because $\chi_j * \mathbf{u}_0 = (\chi_j * \phi_0) * \mathbf{u}$, and

$$\mathcal{F}(\chi_j * \phi_0)(q) = 0, \quad \text{for all } q \notin B_\delta(\eta_j), \tag{68}$$

we have that each function $(\chi_j * \phi_0) * \mathbf{u}$ satisfies the hypothesis of $\mathbf{u}_\eta$ in Lemma 4, with $\chi_j * \phi_0$ playing the role of ϕ_δ^η. Therefore, there are constants $C(\chi_j, \phi_0) > 0$, such that

$$\begin{aligned}
&\frac{1}{T}\langle|\mathbf{u}_0(\cdot,T)|_1\rangle + \frac{\nu}{8}\langle\langle|\mathbf{u}_0|_1\rangle\rangle_T \le \sum_j \left(\frac{1}{T}\langle|\chi_j * \mathbf{u}_0(\cdot,T)|_1\rangle + \frac{\nu}{8}\langle\langle|\chi_j * \mathbf{u}_0|_1\rangle\rangle_T\right) \\
&\le \sum_j C(\chi_j,\phi_0)\Big[\langle\langle|\chi_j * \mathbf{u}_0|_2^2\rangle\rangle_T + \langle\langle|\mathbf{w}_0|_2^2\rangle\rangle_T + \langle\langle\|\nabla \mathbf{v}_0\|^2\rangle\rangle_T \\
&\quad + \frac{1}{T}\langle|\chi_j * \mathbf{u}_0(\cdot,0)|_1\rangle + \langle|\chi_j * \mathbf{f}_0|_1\rangle\Big] \\
&\le C(\langle\langle|\mathbf{u}_0|_2^2\rangle\rangle_T + \langle\langle|\mathbf{w}_0|_2^2\rangle\rangle_T + \langle\langle\|\nabla \mathbf{v}_0\|^2\rangle\rangle_T + \frac{1}{T}\langle|\mathbf{u}_0(\cdot,0)|_1\rangle + \langle|\mathbf{f}_0|_1\rangle).
\end{aligned} \tag{69}$$

Thus, because the choice of the finite set of partition functions $\{\chi_j\}_j$, and of the function ϕ is independent of the particular setting of the Navier-Stokes equations, and its weak solutions, we conclude that (66) indeed holds with a universal $C > 0$.

Lemma 6 *(Scaling) For any $\ell \in \mathbb{Z}$,*

$$\begin{aligned}
\frac{2^{2\ell}}{T}\langle|\mathbf{u}_\ell(\cdot,T)|_1\rangle + \frac{\nu}{8}2^{2\ell}\langle\langle|\mathbf{u}_\ell|_1\rangle\rangle_T \;\le\; & C(2^\ell\langle\langle|\mathbf{u}_\ell|_2^2\rangle\rangle_T + 2^\ell\langle\langle|\mathbf{w}_\ell|_2^2\rangle\rangle_T \\
& + 2^{-\ell}\langle\langle\|\nabla \mathbf{v}_\ell\|^2\rangle\rangle_T) + \frac{2^{2\ell}}{T}\langle|\mathbf{u}_\ell(\cdot,0)|_1\rangle + \langle|\mathbf{f}_\ell|_1\rangle,
\end{aligned} \tag{70}$$

where $\mathbf{v}_\ell$ and $\mathbf{w}_\ell$ are defined analogously to $\mathbf{v}_0$ and $\mathbf{w}_0$ in (52), that is,

$$\mathbf{v}_\ell \;:=\; \sum_{k\le \ell-1} \mathbf{u}_k, \quad \mathbf{w}_\ell \;:=\; \sum_{k\ge\ell}\mathbf{u}_k. \tag{71}$$

Proof. Indeed, we notice that the change of variables

$$x \;=\; 2^{-\ell}\hat{x}, \quad t \;=\; 2^{-2\ell}\hat{t}, \quad \mathbf{u} \;=\; 2^\ell\hat{\mathbf{u}}, \quad p \;=\; 2^{2\ell}\hat{p}, \quad f \;=\; 2^{3\ell}\hat{f} \tag{72}$$

leaves (1) invariant. Notice that (11) translates into

$$\phi_k(z) \;=\; 2^{3k}\phi_0(2^k z), \tag{73}$$

so that

$$2^{-3\ell}\phi_k(2^{-\ell}\hat{z}) \;=\; \phi_{k-\ell}(\hat{z}).$$

Hence we deduce from (72) the following relation between the Littlewood-Paley decompositions

$$\mathbf{u}_k \;=\; 2^{\ell}\widehat{\mathbf{u}}_{k-\ell}, \quad f_k \;=\; 2^{3\ell}\widehat{f}_{k-\ell}.$$

In particular, we have

$$\mathbf{u}_\ell \;=\; 2^{\ell}\widehat{\mathbf{u}}_0, \quad \partial_x \mathbf{v}_\ell \;=\; 2^{2\ell}\partial_{\hat{x}_j}\widehat{\mathbf{v}}_0, \quad \mathbf{w}_\ell \;=\; 2^{\ell}\widehat{\mathbf{w}}_0.$$

Hence (66), applied to $(\hat{t}, \hat{x}, \hat{\mathbf{u}}, \hat{f})$ yields in terms of (t, x, u, f):

$$\begin{aligned}\frac{2^{-\ell}}{T}\langle|\mathbf{u}_\ell(\cdot,T)|_1\rangle + \frac{\nu}{8}2^{-\ell}\langle\langle|\mathbf{u}_\ell|_1\rangle\rangle_T \;\le\; & C(2^{-2\ell}\langle\langle|\mathbf{u}_\ell|_2^2\rangle\rangle_T + 2^{-2\ell}\langle\langle|\mathbf{w}_\ell|_2^2\rangle\rangle_T \\ & + 2^{-4\ell}\langle\langle\|\nabla\mathbf{v}_\ell\|^2\rangle\rangle_T) + \frac{2^{-\ell}}{T}\langle|\mathbf{u}_\ell(\cdot,0)|_1\rangle + 2^{-3\ell}\langle\langle|\mathbf{f}_\ell|_1\rangle\rangle_T.\end{aligned} \tag{74}$$

Multiplication with $2^{3\ell}$ yields (70). □

Proof of Theorem 1. We now proceed to the proof of our main result. From now on, we consider $k \ge k_0$, so that $\mathbf{f}_k = 0$. From Lemma 6 and Lemma 7, if $\mathbf{u}$ is a weak solution of (1), then

$$\begin{aligned}&\frac{2^{2k}}{T}\{\langle|\mathbf{u}_k(.,T)|_1\rangle - \langle|\mathbf{u}_k(.,0)|_1\rangle\} + \frac{\nu}{8}2^{2k}\langle\langle|\mathbf{u}_k|_1\rangle\rangle_T \\ &\le C(2^k\langle\langle|\mathbf{u}_k|_2^2\rangle\rangle_T + 2^k\langle\langle|\mathbf{w}_k|_2^2\rangle\rangle_T + 2^{-k}\langle\langle|\,\|\nabla\mathbf{v}_k\|^2\rangle\rangle_T).\end{aligned} \tag{75}$$

Now, taking the lim sup in T of both sides of the expression above, multiplying the result by 2^k, and using (14) and (15), we obtain

$$\begin{aligned}\nu 2^{3k}\langle\langle|\mathbf{u}_k|_1\rangle\rangle &\le C(2^{2k}\langle\langle|\mathbf{u}_k|_2^2\rangle\rangle + 2^{2k}\langle\langle|\mathbf{w}_k|_2^2\rangle\rangle + \langle\langle\|\nabla\mathbf{v}_k\|^2\rangle\rangle) \\ &\le C(2^{2k}\sum_{m=k}^{\infty}\langle\langle|\mathbf{u}_m|_2^2\rangle\rangle + \sum_{m=-\infty}^{k}\langle\langle\|\nabla\mathbf{u}_m\|^2\rangle\rangle) \\ &\le C(\sum_{m=k}^{\infty}2^{2m}\langle\langle|\mathbf{u}_m|_2^2\rangle\rangle + \sum_{m=-\infty}^{k}2^{2m}\langle\langle|\mathbf{u}_m|_2^2\rangle\rangle) \\ &\le C\epsilon\nu^{-1}.\end{aligned} \tag{76}$$

Therefore,

$$\langle\langle|\mathbf{u}_k|_1\rangle\rangle \le C\epsilon\nu^{-2}(2^k)^{-3}. \quad \square \tag{77}$$

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Sonderforschungsbereich 611
der Universität Bonn
Poppelsdorfer Allee 82
D - 53115 Bonn

Telefon: 0228/73 4882
Telefax: 0228/73 7864
E-Mail: astrid.link@ins.uni-bonn.de http://www.sfb611.iam.uni-bonn.de/

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