

**A Quantitative Rigidity Result for the Cubic to  
Tetragonal Phase Transition in the Geometrically  
Linear Theory with Interfacial Energy**

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A quantitative rigidity result  
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**ABSTRACT.** We are interested in the cubic to tetragonal phase transition in a shape memory alloy. We consider geometrically linear elasticity. In this framework, Dolzmann and Müller have shown the following rigidity result: The only stress-free configurations are (locally) twins (i. e. laminates of just two of the three Martensitic variants). However, configurations with arbitrarily small elastic energy are not necessarily close to these twins: The formation of microstructure allows to mix all three Martensitic variants at arbitrary volume fractions.

In this paper, we take an interfacial energy into account and establish a (local) lower bound on elastic + interfacial energy in terms of the Martensitic volume fractions. The introduction of an interfacial energy introduces a length scale and thus, together with the linear dimensions of the sample, a non-dimensional parameter. Our lower Ansatz-free bound has optimal scaling in this parameter. It is the scaling predicted by a reduced model introduced and analyzed by Kohn and Müller with the purpose to describe the microstructure near an interface between Austenite and twinned Martensite. The optimal construction features branching of the Martensitic twins when approaching this interface.

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# 1 Introduction

## 1.1 Twins

In the geometrically linear version of elasticity theory, the strain  $e$  (a field of symmetric  $3 \times 3$  tensors) generated by a displacement  $u$  (a vector field) is approximated by

$$e = \frac{1}{2}(\nabla u + \nabla^t u). \quad (1)$$

In the Martensitic phase of a shape memory material that undergoes a cubic to tetragonal phase transition, there are three different *stress-free* strains:

$$e^{(1)} := \epsilon \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad e^{(2)} := \epsilon \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad e^{(3)} := \epsilon \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad (2)$$

where  $\epsilon > 0$  is a (dimensionless) material parameter (typically of the order 0.1). The corresponding deformations  $u$  are taken with respect to the Austenite lattice, so that the stress-free strain in the Austenite phase is  $e = 0$ . The fact that  $e^{(1)}$ ,  $e^{(2)}$ , and  $e^{(3)}$  are trace-free means that we restricted ourselves to the case where the transformation is volume preserving. For further reading we refer for instance to [1, Chapters 4.1 and 11.1].

It is a classical result in the theory of elasticity that a displacement field  $u$  with a *constant* strain field  $e$  must be *affine*. It is well-known in the theory of shape memory alloys that there are *non-affine* (Lipschitz continuous) displacement fields  $u$  such that the corresponding strain fields  $e$  only assume the three values  $e^{(1)}$ ,  $e^{(2)}$ ,  $e^{(3)}$ . These are the so-called twins, which we shall describe presently. Dolzmann and Müller [5, Theorem 3.1] have rigorously shown that these twins are the *only* configurations with strain fields  $e$  that take only the three values  $e^{(1)}$ ,  $e^{(2)}$ ,  $e^{(3)}$  — a *rigidity* result. More precisely, they prove that

- $e$  (locally) only assumes at most *two* of the three values,
- $e$  (locally) depends only on a *single* variable (out of a fixed set of six possible single variables).

More precisely, if  $e$  locally assumes both values  $e^{(1)}$  and  $e^{(2)}$ , say, then  $e$  must be either locally constant along the planes  $x_1 + x_2 = \text{const}$  or along the planes  $x_1 - x_2 = \text{const}$ . The two other cases are similar by cubic symmetry. These configurations are called Martensitic twins (or “simple laminates” in the mathematical literature); there are exactly six twins in a cubic to tetragonal

phase transition. For further reading, see for instance [1, Chapters 5 and 11.3.1].

## 1.2 Microstructure

It is natural to ask how stable this rigidity result is under perturbations, i. e. when  $e$  is only *close* to the three values  $e^{(1)}$ ,  $e^{(2)}$ , and  $e^{(3)}$ , that is, if the configuration is not entirely stress-free. Since strain and stress are related via the elastic energy, we first address the modeling of the latter. The stress-free strains (2) are typically “embedded” into a piecewise quadratic elastic energy of  $e$  in the sample (or subset of the sample)  $B$ ,

$$E_{elast} = \int_B \min \{ eC^{(0)}e + \omega_0, (e - e^{(1)})C^{(1)}(e - e^{(1)}), \\ (e - e^{(2)})C^{(2)}(e - e^{(2)}), (e - e^{(3)})C^{(3)}(e - e^{(3)}) \} dx,$$

where  $C^{(0)}$ ,  $C^{(1)}$ ,  $C^{(2)}$ , and  $C^{(3)}$ , denote the elastic moduli (tensors of rank 4) of the Austenite and the three Martensitic phases. Moreover,  $\omega_0 > 0$  denotes the difference in Helmholtz free energy of the Austenite phase w. r. t. the Martensitic phases. In the mathematical literature, this model is known as the “three well problem”. In this paper, we will assume that the phases are elastically isotropic with identical shear modulus  $\mu$  and vanishing second Lamé constant  $\lambda$ . Moreover, we shall assume that  $\omega_0 = 0$ , since we are interested in the behavior of the energy at *given* volume fraction of the Austenite. Hence we obtain the simplest form possible for the elastic energy

$$E_{elast} = 2\mu \int_B \min \{ |e|^2, |e - e^{(1)}|^2, |e - e^{(2)}|^2, |e - e^{(3)}|^2 \} dx,$$

where  $|e|^2$  denotes the sum of the squares of all entries of the  $3 \times 3$ -tensor  $e$  (i. e. the Frobenius norm). For later reference, we will reformulate this elastic energy as

$$E_{elast} := 2\mu \int_B |e - \chi_1 e^{(1)} - \chi_2 e^{(2)} - \chi_3 e^{(3)}|^2 dx,$$

where the minimum is taken over all functions  $\chi_1, \chi_2, \chi_3$  with

$$\chi_1, \chi_2, \chi_3 \in \{0, 1\} \quad \text{and} \quad \chi_0 := 1 - \chi_1 - \chi_2 - \chi_3 \in \{0, 1\}. \quad (3)$$

These characteristic functions  $\chi_0, \chi_1, \chi_2$ , and  $\chi_3$  can be interpreted as the indicator functions of the Austenite phase and the three Martensitic phases, respectively. We refer to [1, Chapter 12.1] for further reading.

After these preliminaries, we address the question of stability of the rigidity result: It is well-known in the theory of shape memory alloys that the above-mentioned rigidity for strains  $e$  with exactly vanishing elastic energy  $E_{elast} = 0$  is destroyed on the level of sequences  $\{e_\nu\}_{\nu \uparrow \infty}$  of strains with energies  $E_{elast} \rightarrow 0$ . The best-studied rigidity result is related to a mixture of Austenite and Martensite: There is no displacement field  $u$  with a stress-free strain field  $e$  assuming the value zero and at least one of the three values  $e^{(1)}$ ,  $e^{(2)}$ , and  $e^{(3)}$  on a set of positive measure each! In particular, there is no exactly stress-free interface between a Martensitic twin (with twin plane  $x_1 + x_2 = const$ , say) on the one side and Austenite on the other side. However, the elastic energy of such a configuration can be made as small as desired

- by letting the twin width  $\ell$  tend to zero, and
- by choosing the twin volume fraction  $\lambda$  (i. e. the relative width of one layer w. r. t. the other layer) appropriately (i. e.  $\lambda = \frac{1}{3}$  or  $\lambda = \frac{2}{3}$ ), and
- by choosing the habit plane (i. e. the plane between the Martensitic twin and the Austenite) appropriately (i. e.  $x_1 + x_3 = const$ ,  $x_1 - x_3 = const$ ,  $x_2 + x_3 = const$ ,  $x_2 - x_3 = const$ ,  $x_1 - x_2 = const$ , or  $x_1 + x_2 = const$ ).

In fact, any volume fractions of the three Martensitic variants and of the Austenite can be reached at vanishing elastic energy. We refer to [1, 7.1 and 11.3.2] for more details.

### 1.3 Interfacial energy

Clearly, one way to remove this degeneracy of the energy functional is to take an interfacial energy between the variants into account. Again, since we are just interested in scaling properties of the energy, we assume that there is a single, isotropic energy per area of the interface  $\kappa$ . Typical values of  $\kappa$  are between 20 and 200  $mJ/m^2$  [10] and [12, Chapter 6, Table 5]. It is mathematically more convenient to attach this interfacial energy to the characteristic functions (3) instead of the strain  $e$ . The BV-norm  $\int_B |\nabla \chi| dx$  of the characteristic function  $\chi$  is a mathematically robust expression for the surface area of  $\partial\{\chi = 1\}$ . Hence we consider

$$E_{interf} := \kappa \int_B (|\nabla \chi_0| + |\nabla \chi_1| + |\nabla \chi_2| + |\nabla \chi_3|) dx.$$

Our result, Theorem 1, shows that there is a qualitative difference between the scaling of the minimal energy  $E_{interf} + E_{elast}$  for volume fractions that

allow for configurations with exactly vanishing elastic energy on the one hand and volume fractions for configurations that only allow for a sequence of configurations with vanishing elastic energy in the limit. More precisely,

- In the first case, just a single interface is needed so that the minimal energy scales as

$$\kappa L^2, \tag{4}$$

where  $L$  denotes the diameter of  $B$ .

- In the second case, a microstructure is needed and the minimal energy scales as

$$(\kappa L^2)^{2/3}(\epsilon^2 \mu L^3)^{1/3}. \tag{5}$$

In this sense, we may say that the interfacial energy *unfolds* the degenerate elastic energy. The pair of exponents  $\frac{2}{3}, \frac{1}{3} = 1 - \frac{2}{3}$  is familiar, as we shall point out next.

## 1.4 Twin branching

Kohn & Müller have studied the effect of the interfacial energy  $E_{interf}$  on the interface between a twinned Martensite and an Austenite phase in the seminal work [8, 9]. They work on the level of a simplified, in particular scalar and two-dimensional, model. As carefully explained in [8, Section 2.2], the simplifications of their model with respect to the above model  $E_{interf} + E_{elast}$  are the following:

- The (implicit) selection of one of the six possible twin planes, say  $x_1 + x_2 = const$ , by restricting to displacements that are of the form

$$u = \text{affine} + f \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix},$$

i. e. displacements that amount to a shear. In particular, the third Martensitic variant does not occur (i. e.  $\chi_3 \equiv 0$ ).

- The selection of one of the four possible habit planes (between the Austenite and a twin formed by the first two Martensitic variants), say  $x_2 + x_3 = const$ , by imposing that the half-space  $x_2 + x_3 \leq 0$  is the Austenitic phase (i. e.  $\chi_0$  is the characteristic function of the set  $\{x_2 + x_3 \leq 0\}$ ).
- The restriction that all quantities only depend on the two variables  $x_1 + x_2$  and  $x_2 + x_3$ .

Naively, one would expect that the minimal energy is found by optimizing the twin width  $\ell$ : On the one hand, the interfacial energy scales as  $\kappa L^3 \ell^{-1}$ , since the interfacial density scales as  $\ell^{-1}$ ; on the other hand, the elastic energy scales as  $\mu \epsilon^2 L^2 \ell$ , since the stresses are confined to a sheet of thickness  $\ell$  around the Austenite-Martensite interface. Hence one would expect that the minimal energy scales as

$$(\kappa L^2)^{1/2} (\epsilon^2 \mu L^3)^{1/2},$$

see [8, Section 3.1]. This however is *not* the optimal energy scaling: A small twin width  $\ell$  is only required near the Austenite-Martensite interface. Hence it seems advantageous to refine  $\ell$  by twin branching when approaching this interface (see for instance [1, Plate 2b, Fig 1.2 (b)] for experimental pictures of twin branching). However, the branching of twins requires slight deviations from the twin plane and thus also generates stresses. The reduced model of Kohn & Müller allows to assess the elastic energy generated by branching; they construct a branched configuration with energy scaling as

$$(\kappa L^2)^{2/3} (\epsilon^2 \mu L^3)^{1/3}, \tag{6}$$

see [8, Section 3.2]. Kohn & Müller also prove [9, Theorem 1.1] that their construction *cannot be improved in terms of scaling*, provided the non-dimensional number  $\frac{\kappa L^2}{\epsilon^2 \mu L^3} = \frac{\kappa}{\epsilon^2 \mu L}$  is sufficiently small (and that the neglected Lamé constant  $\lambda$  is not much larger than  $\mu$ ). It is the main result of our paper, Theorem 1, that this lower bound also holds for the full model  $E_{interf} + E_{elast}$ .

Hence, loosely speaking, we may say that our analysis combines the qualitative treatment (rigidity) of the full model in [5] with the quantitative treatment (Ansatz-free lower bounds optimal in scaling) of the reduced model in [9].

This paper is an extension of [2]. The extension is twofold: Here, we treat the three Martensitic variants *plus* the Austenite, whereas in [2], no Austenite was allowed (i. e.  $\chi_0 \equiv 0$ ). In [2], we considered the (somewhat artificial) case of periodic boundary conditions (in particular,  $B$  was replaced by  $(0, L)^3$ ), whereas here, we obtain a *local* result. The latter in particular rules out Fourier methods which were extensively used in [2].

## 1.5 The geometrically nonlinear case

The situation in the geometrically nonlinear case is different: Müller and Sverak have constructed *exactly* stress-free configurations which involve two Martensitic variants in two dimensions that are *not* twins [11, Example c].

Extending these ideas, Conti, Dolzmann & Kirchheim have constructed exactly stress-free configurations which involve *all* three Martensitic variants [4, Theorem 1.1]. In particular, their result yields an exactly stress-free configuration that involves Austenite *and* Martensite. These configurations are constructed by an iterative method (Gromov’s “convex integration”) and thus have a complicated phase distribution. Indeed, Dolzmann & Müller [6, Theorem 1.1] have shown that also in the geometrically nonlinear case, a mixture of two Martensitic variants in two dimensions with *finite* interfacial energy must be (locally) twins. Kirchheim has extended this result to three Martensitic variants in three dimensions [7, Theorem 1.2]. Hence in the geometrically nonlinear case, the finiteness of surface energy is needed even for the qualitative rigidity result.

## 2 Nondimensionalization, symmetry and main result

We nondimensionalize and rescale the model as follows:

- We measure energy density in units of  $2\mu\epsilon$ ,
- we measure length in units of the sample diameter  $L$ ,
- we measure strains in units of  $\epsilon$ .

Hence we are left with the single nondimensional parameter

$$\eta := \frac{\kappa L^2}{2\epsilon^2 \mu L^3} = \frac{\kappa}{2\epsilon^2 \mu L}.$$

**Definition 1.** *The objects we consider are*

- i) the Austenite and three Martensitic phases described by their characteristic functions, i. e.*

$$\chi_0, \chi_1, \chi_2, \chi_3 \in \{0, 1\} \quad \text{and} \quad \chi_0 + \chi_1 + \chi_2 + \chi_3 = 1. \quad (7)$$

- ii) the strain described by*

$$e \in \text{symmetric } 3 \times 3 \text{ tensors} \\ \text{such that there exists a field } u \in \mathbb{R}^3 \text{ with } e = \frac{1}{2}(\nabla u + \nabla^t u). \quad (8)$$



iii) the elastic and interfacial energies in the ball  $B_1$  of radius 1 given by

$$\begin{aligned}
E_{elast} &:= \int_{B_1} \left| e - \chi_1 \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right. \\
&\quad \left. - \chi_2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \chi_3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \right|^2 dx, \\
E_{interf} &:= \int_{B_1} (|\nabla\chi_0| + |\nabla\chi_1| + |\nabla\chi_2| + |\nabla\chi_3|) dx.
\end{aligned} \tag{9}$$

It will turn out that the following axes and planes will play a special role in this cubic-to-tetragonal phase transition:

- Three cubic axes. These three cubic axes correspond to the three martensitic variants  $\chi_1, \chi_2, \chi_3$ . Note that the model is invariant under the permutations and reflections w. r. t. the cubic axes; those are generated by  $(x_1, x_2, x_3) \rightsquigarrow (x_2, x_1, x_3)$ ,  $(x_1, x_2, x_3) \rightsquigarrow (x_3, x_2, x_1)$ , and  $(x_1, x_2, x_3) \rightsquigarrow (x_1, x_2, -x_3)$ .
- Six planes  $(011)$ ,  $(01\bar{1})$ ,  $(101)$ ,  $(\bar{1}01)$ ,  $(110)$ ,  $(1\bar{1}0)$ , which by definition of crystallographic notation are the planes defined by  $x_2 + x_3 = \text{const}$ ,  $x_2 - x_3 = \text{const}$ ,  $x_1 + x_3 = \text{const}$ ,  $-x_1 + x_3 = \text{const}$ ,  $x_1 + x_2 = \text{const}$ ,  $x_1 - x_2 = \text{const}$ , respectively. These are the twin planes of the cubic-to-tetragonal phase transition. Each unordered pair of Martensite variants admits two twin planes. For instance, the (unordered) pair of Martensitic variants  $\chi_1, \chi_2$  admits the two twin planes  $(110)$  and  $(1\bar{1}0)$ . We occasionally identify these six planes with the *dual* vectors with the coordinates  $(0, 1, 1)$ ,  $(0, 1, -1)$ ,  $(1, 0, 1)$ ,  $(-1, 0, 1)$ ,  $(1, 1, 0)$ ,  $(1, -1, 0)$ , respectively. These planes first appear in Proposition 1.
- Four space diagonals  $\{[111], [1\bar{1}\bar{1}], [\bar{1}\bar{1}1], [\bar{1}11]\}$ . We occasionally identify these four axes with the *primal* vectors with the coordinates  $(1, 1, 1)$ ,  $(1, 1, -1)$ ,  $(1, -1, 1)$ ,  $(-1, 1, 1)$ , respectively. These axes appear first in Lemma 1 and mediate between the cubic axes and the twin planes.

**Theorem 1.** *There exists a small but universal radius  $r > 0$  such that we have for any parameter  $\eta \leq r$ : If  $\chi_0, \chi_1, \chi_2, \chi_3, e, E_{interf}$  and  $E_{elast}$  are as in Definition 1 and if we abbreviate the rescaled energy in  $B_1$  as*

$$E := \eta^{1/3} E_{interf} + \eta^{-2/3} E_{elast},$$

*then the following three dichotomies hold:*

i) We have

$$\int_{B_r} |1 - \chi_0| dx \lesssim E \quad \underline{\text{or}} \quad \int_{B_r} |\chi_0| dx \lesssim E. \quad (10)$$

ii) In case of  $\int_{B_r} |\chi_0| dx \lesssim E$ , we have

$$\int_{B_r} |\chi_1| dx \lesssim E^{1/2} \quad \underline{\text{or}} \quad \int_{B_r} |\chi_2| dx \lesssim E^{1/2} \quad \underline{\text{or}} \quad \int_{B_r} |\chi_3| dx \lesssim E^{1/2}.$$

iii) In case of  $\int_{B_r} |\chi_3| dx \lesssim E^{1/2}$ , there exist functions  $f_{(110)}$ ,  $f_{(1\bar{1}0)}$  such that we have

$$\int_{B_r} |\chi_1 - f_{(110)}| dx \lesssim E^{1/4} \quad \underline{\text{or}} \quad \int_{B_r} |\chi_1 - f_{(1\bar{1}0)}| dx \lesssim E^{1/4},$$

where  $f_{(110)}$ ,  $f_{(1\bar{1}0)}$  denote functions only depending on  $x \cdot (110) = x_1 + x_2$  and  $x \cdot (1\bar{1}0) = x_1 - x_2$ , respectively. In the two other cases, the analogue statement holds.

We close this section with three remarks: The first remark is that Theorem 1 i) and ii) can be reformulated in terms of the volume fractions  $\theta_0 := \int_{B_r} \chi_0 dx$ ,  $\theta_1 := \int_{B_r} \chi_1 dx$ ,  $\theta_2 := \int_{B_r} \chi_2 dx$ , and  $\theta_3 := \int_{B_r} \chi_3 dx$  as follows:

$$\begin{aligned} (1 - \theta_0) \theta_0 &\lesssim E, \\ \theta_1 \theta_2 \theta_3 &\lesssim E^{1/2}. \end{aligned}$$

The second remark is that it is possible to reformulate Theorem 1 in terms of the strain  $e$  alone by introducing

$$\bar{E} := \min_{\chi_0, \chi_1, \chi_2, \chi_3 \text{ satisfy (7)}} E. \quad (11)$$

Indeed, provided  $\bar{E} \leq 1$ , Theorem 1 assumes the form

i) We have

$$\begin{aligned} \int_{B_r} |e|^2 dx &\lesssim \bar{E} \quad \underline{\text{or}} \\ \int_{B_r} \min\{|e - e^{(1)}|^2, |e - e^{(2)}|^2, |e - e^{(3)}|^2\} dx &\lesssim \bar{E}, \end{aligned} \quad (12)$$

where we use the notation (2) with  $\epsilon = 1$ .

ii) In the last case, we have

$$\begin{aligned}
\int_{B_r} \min\{|e - e^{(2)}|^2, |e - e^{(3)}|^2\} dx &\lesssim \bar{E}^{1/2} \quad \underline{\text{or}} \\
\int_{B_r} \min\{|e - e^{(1)}|^2, |e - e^{(3)}|^2\} dx &\lesssim \bar{E}^{1/2} \quad \underline{\text{or}} \\
\int_{B_r} \min\{|e - e^{(1)}|^2, |e - e^{(2)}|^2\} dx &\lesssim \bar{E}^{1/2}. \tag{13}
\end{aligned}$$

iii) In the last case, we have

$$\begin{aligned}
\int_{B_r} |e - f_{(110)}e^{(1)} - (1 - f_{(110)})e^{(2)}|^2 dx &\lesssim \bar{E}^{1/4} \quad \underline{\text{or}} \\
\int_{B_r} |e - f_{(1\bar{1}0)}e^{(1)} - (1 - f_{(1\bar{1}0)})e^{(2)}|^2 dx &\lesssim \bar{E}^{1/4}, \tag{14}
\end{aligned}$$

where  $f_{(110)}$ ,  $f_{(1\bar{1}0)}$  denote  $\{0, 1\}$ -valued functions only depending on  $x \cdot (110) = x_1 + x_2$  and  $x \cdot (1\bar{1}0) = x_1 - x_2$ , respectively.

These estimates follow immediately from those of Theorem 1 as we shall argue at the end of the proof of the latter.

The third remark points out the limitations of Theorem 1. The work by Zhang, James, and Müller [12] prompted some interest in the nucleation of Martensite in Austenite. One natural approach to characterize the saddle point in the energy landscape is to determine the minimal energy given the volume of the Martensite (here, we think of the Martensite embedded in an infinite matrix of Austenite). By summation of translates of balls, the local estimate (10) would yield a lower bound on  $E$  (with  $B_1$  replaced by  $\mathbb{R}^3$ ) in terms of the volume  $\int_{\mathbb{R}^3} (1 - \chi_0) dx$ . However, this lower bound does not have the optimal scaling in the volume  $\int_{\mathbb{R}^3} (1 - \chi_0) dx$ . In fact, a more detailed argument within our approach (in particular a quantification of the scaling of estimate (31) in  $|h| + |h'| \ll 1$ ) would already lead to an improved scaling. Hence Theorem 1 i), as it stands, does not contribute to the problem of nucleation, although we believe that some of the methods may.

## 3 Structure of the proof and auxiliary lemmas

### 3.1 Lemma 1

The form of the elastic energy makes it convenient to introduce the following three modified characteristic functions:

**Definition 2.** We set

$$\tilde{\chi}_i := \chi_0 + 3\chi_i - 1 \quad \text{for } i = 1, 2, 3 \quad (15)$$

and note that

$$\tilde{\chi}_1 + \tilde{\chi}_2 + \tilde{\chi}_3 = 0 \quad (16)$$

and that the elastic energy in the ball  $B_1$  can be reformulated as

$$E_{\text{elast}} = \int_{B_1} \left| e + \begin{pmatrix} \tilde{\chi}_1 & 0 & 0 \\ 0 & \tilde{\chi}_2 & 0 \\ 0 & 0 & \tilde{\chi}_3 \end{pmatrix} \right|^2 dx. \quad (17)$$

In view of (17), the compatibility constraints (8) for the symmetric strain tensor  $e$  together with a control of the elastic energy  $E_{\text{elast}}$  yields some control on the three functions  $\tilde{\chi}_1, \tilde{\chi}_2, \tilde{\chi}_3$  — this is the content of the upcoming Lemma 1. Since the compatibility constraints on  $e$  can be characterized as linear *second-order* differential relations this control will be formulated in terms of linear second-order differential operators. Indeed, the gradient  $\nabla u$  of some displacement field  $u$  is characterized by linear *first-order* differential constraints; the strain tensor  $e$ , which is the symmetrized gradient  $\frac{1}{2}(\nabla u + \nabla^t u)$  in our geometrically linear setting, is characterized by linear *second-order* differential relations.

Let us first discuss how Lemma 1 relates to the symmetry of the cubic-to-tetragonal phase transition by drawing its conclusion in case of *vanishing* elastic energy. In this extreme case, Lemma 1 states that certain *mixed* second derivatives of  $\tilde{\chi}_1, \tilde{\chi}_2, \tilde{\chi}_3$  vanish. More precisely, these derivatives are in direction of the four space diagonals  $\{[111], [1\bar{1}\bar{1}], [\bar{1}\bar{1}1], [\bar{1}11]\}$  and the combinatorics are as follows: For  $\tilde{\chi}_1$ , which is associated with the  $[100]$  axis, *two* mixed derivatives, namely  $\partial_{[111]}\partial_{[\bar{1}\bar{1}\bar{1}]}$  and  $\partial_{[1\bar{1}\bar{1}]}\partial_{[\bar{1}11]}$ , vanish. Let us argue that this is exactly the result we expect qualitatively: By an elementary argument (see statement iii) before Lemma 4), the vanishing of these two mixed derivatives implies the decomposition of  $\tilde{\chi}_1$  into four functions of a *single* variable each. More precisely,

$$\tilde{\chi}_1 = f_{(101)} + f_{(\bar{1}01)} + f_{(110)} + f_{(1\bar{1}0)}, \quad (18)$$

where  $f_{(101)}, f_{(\bar{1}01)}, f_{(110)}, f_{(1\bar{1}0)}$  denote functions which are constant along planes perpendicular to  $(101), (\bar{1}01), (110), (1\bar{1}0)$ , respectively (that is, functions that only depend on the coordinates  $(101) \cdot x, (\bar{1}01) \cdot x, (110) \cdot x, (1\bar{1}0) \cdot x$ ,

respectively). With  $\tilde{\chi}_1$  replaced by  $\chi_1$ , (18) is exactly what we expect qualitatively: (110) and (1 $\bar{1}$ 0) are the two twin planes the Martensitic variant  $\chi_1$  can form with  $\chi_2$ ; (101) and ( $\bar{1}$ 01) are the two twin planes it can form with  $\chi_3$ .

Let us now briefly discuss the statement of the upcoming Lemma 1 more quantitatively. Since the elastic energy is an  $L^2$ -expression in the functions  $\tilde{\chi}_1, \tilde{\chi}_2, \tilde{\chi}_3$ , see (17), the above-mentioned mixed second derivatives of these functions are controlled in the negative Sobolev space  $H^{-2}$ . This means that they can be written as a linear combination of second derivatives of functions (cf. (19)) that are controlled in  $L^2$  (cf. (20)). However, it will be important for Lemma 2 that on the r. h. s. of (19), *only* the three combinations of the two directional derivatives that appear on the l. h. s. (that is,  $\partial_{[111]}, \partial_{[\bar{1}\bar{1}\bar{1}]}$ ) occur.

**Lemma 1.** *There exist three functions  $\rho_{[111],[111]}, \rho_{[111],[\bar{1}\bar{1}\bar{1}]}, \rho_{[\bar{1}\bar{1}\bar{1}],[\bar{1}\bar{1}\bar{1}]}$  with*

$$\begin{aligned} & \partial_{[111]}\partial_{[\bar{1}\bar{1}\bar{1}]} \tilde{\chi}_1 & (19) \\ & = \partial_{[111]}\partial_{[111]}\rho_{[111],[111]} + 2\partial_{[111]}\partial_{[\bar{1}\bar{1}\bar{1}]}\rho_{[111],[\bar{1}\bar{1}\bar{1}]} + \partial_{[\bar{1}\bar{1}\bar{1}]}\partial_{[\bar{1}\bar{1}\bar{1}]}\rho_{[\bar{1}\bar{1}\bar{1}],[\bar{1}\bar{1}\bar{1}]}, \end{aligned}$$

such that

$$\int_{B_1} (\rho_{[111],[111]}^2 + \rho_{[111],[\bar{1}\bar{1}\bar{1}]}^2 + \rho_{[\bar{1}\bar{1}\bar{1}],[\bar{1}\bar{1}\bar{1}]}^2) dx \lesssim E_{elast}. \quad (20)$$

The same statement holds if the pair of axes  $\{[111], [\bar{1}\bar{1}\bar{1}]\}$  is replaced by what we obtain by  $(x_1, x_2, x_3) \rightsquigarrow (x_1, x_2, -x_3)$ , that is  $\{[11\bar{1}], [1\bar{1}\bar{1}]\}$ . The same statement also holds with  $\tilde{\chi}_1$  and  $\{[111], [\bar{1}\bar{1}\bar{1}]\}, \{[11\bar{1}], [1\bar{1}\bar{1}]\}$  replaced by what we obtain from  $(x_1, x_2, x_3) \rightsquigarrow (x_2, x_1, x_3)$  and  $(x_1, x_2, x_3) \rightsquigarrow (x_3, x_2, x_1)$ , that is

- $\tilde{\chi}_2$  and the two pairs of axes  $\{[111], [1\bar{1}\bar{1}]\}$  and  $\{[11\bar{1}], [\bar{1}\bar{1}\bar{1}]\}$ ,
- $\tilde{\chi}_3$  and the two pairs of axes  $\{[111], [11\bar{1}]\}$  and  $\{[1\bar{1}\bar{1}], [\bar{1}\bar{1}\bar{1}]\}$ .

### 3.2 Lemma 2

The following Lemma 2 encodes the information given in Lemma 1 in terms of *finite* differences instead of *infinitesimal* derivatives. As Corollary 1 will show, the passage from derivatives to finite differences is not just technical, but allows to connect in a natural way to the pointwise nonlinear constraint that we are dealing with characteristic functions.

**Definition 3.** For any function  $f$ , any vector  $a$  and scalar  $h$  we define the function  $\partial_a^h f$  via

$$(\partial_a^h f)(x) = f(x + ha) - f(x).$$

Note that we do not divide by the step size  $h$ .

Naively, one might expect that since the mixed second derivative  $\partial_{[111]}\partial_{[\bar{1}\bar{1}\bar{1}]}\tilde{\chi}_1$  is controlled in  $H^{-2}$ , the corresponding mixed finite differences  $\partial_{[111]}^h\partial_{[\bar{1}\bar{1}\bar{1}]}^{h'}\tilde{\chi}_1$  are controlled in  $L^2$  for any shifts  $h$  and  $h'$ . This would be true in the sense that if *all* second derivatives are controlled in  $H^{-2}$ , then all second finite differences are controlled in  $L^2$ . It is *not* true for the just two mixed derivatives  $\partial_{[111]}\partial_{[\bar{1}\bar{1}\bar{1}]}$  and  $\partial_{[1\bar{1}\bar{1}]}\partial_{[\bar{1}\bar{1}\bar{1}]}$ .

In case of periodic boundary conditions or of a formulation in the whole space, a reformulation in Fourier space shows that this loss of *exactly one entire* derivative is unavoidable. Indeed, the Fourier multiplier of  $\tilde{\chi}_1$  associated with the control of  $\partial_{[111]}\partial_{[\bar{1}\bar{1}\bar{1}]}$  and  $\partial_{[1\bar{1}\bar{1}]}\partial_{[\bar{1}\bar{1}\bar{1}]}$  in  $H^{-2}$  (in the stronger sense of (19)) is given by

$$\frac{(k \cdot [111])^2(k \cdot [\bar{1}\bar{1}\bar{1}])^2}{((k \cdot [111])^2 + (k \cdot [\bar{1}\bar{1}\bar{1}])^2)^2} + \frac{(k \cdot [1\bar{1}\bar{1}])^2(k \cdot [1\bar{1}\bar{1}])^2}{((k \cdot [1\bar{1}\bar{1}])^2 + (k \cdot [1\bar{1}\bar{1}])^2)^2}. \quad (21)$$

On the other hand, the control of, say, the finite differences  $\partial_{[111]}^h\partial_{[\bar{1}\bar{1}\bar{1}]}^{h'}\tilde{\chi}_1$  in  $H^{-\alpha}$  for some exponent  $\alpha$  leads to the Fourier multiplier

$$\frac{|e^{-ikh \cdot [111]} - 1|^2 |e^{-ih'k \cdot [\bar{1}\bar{1}\bar{1}]} - 1|^2}{|k|^{2\alpha}}. \quad (22)$$

Morally speaking, we will now probe these two Fourier multipliers with a layer-within-layer construction, where the outer layer can be thought to consist just in a phase shift of the inner layer. Let us denote by

$$\ell_{in} \ll \ell_{out} \ll 1 \quad (23)$$

the periods of the inner and outer layers, respectively. The two families of planes, along which the inner and the outer layer are constant, have to be chosen with care:

- The plane for the inner layer has to contain at least one of the axes  $[111]$ ,  $[\bar{1}\bar{1}\bar{1}]$  and at least one of the axes  $[1\bar{1}\bar{1}]$ ,  $[\bar{1}\bar{1}1]$ , and has to be transversal to one of the axes  $[111]$ ,  $[\bar{1}\bar{1}\bar{1}]$ . Let us pick  $(110)$ .
- The plane for the outer layer also has to contain at least one of the axes  $[111]$ ,  $[\bar{1}\bar{1}\bar{1}]$  and at least one of the axes  $[1\bar{1}\bar{1}]$ ,  $[\bar{1}\bar{1}1]$ , but has to be transversal to the *other* of the two axes  $[111]$ ,  $[\bar{1}\bar{1}\bar{1}]$ . Let us pick  $(1\bar{1}0)$ .

This choice of scales and planes amounts to evaluating the Fourier multipliers at wave vectors  $k$  of the form

$$k = \frac{2\pi}{\ell_{out}}(1\bar{1}0) + \frac{2\pi}{\ell_{in}}(110). \quad (24)$$

For maximal effect, we probe the layers by shifts  $h, h'$  in transversal directions which correspond to a quarter of the layer period, that is

$$h = \frac{\ell_{in}}{4} \quad \text{and} \quad h' = \frac{\ell_{out}}{4}. \quad (25)$$

Note that

$$\begin{aligned} k \cdot [111] &= \frac{4\pi}{\ell_{in}}, & k \cdot [\bar{1}11] &= -\frac{4\pi}{\ell_{out}}, \\ k \cdot [11\bar{1}] &= \frac{4\pi}{\ell_{in}}, & k \cdot [1\bar{1}1] &= \frac{4\pi}{\ell_{out}}. \end{aligned} \quad (26)$$

Using the separation of scales (23), this yields for the Fourier multiplier (21):

$$\frac{(k \cdot [111])^2 (k \cdot [\bar{1}11])^2}{((k \cdot [111])^2 + (k \cdot [\bar{1}11])^2)^2} + \frac{(k \cdot [11\bar{1}])^2 (k \cdot [1\bar{1}1])^2}{((k \cdot [11\bar{1}])^2 + (k \cdot [1\bar{1}1])^2)^2} \approx 2 \frac{\ell_{in}^2}{\ell_{out}^2}. \quad (27)$$

We now turn to the Fourier multiplier (22). The choice (25) of the shifts is such that in conjunction with (26) we obtain

$$hk \cdot [111] = \pi \quad \text{and} \quad h'k \cdot [\bar{1}11] = -\pi,$$

so that

$$\frac{|e^{-ihk \cdot [111]} - 1|^2 |e^{-ih'k \cdot [\bar{1}11]} - 1|^2}{|k|^{2\alpha}} \approx \frac{16}{(2\pi)^{2\alpha}} (\ell_{in}^2)^\alpha, \quad (28)$$

where we have used (24) and (23) to approximate the denominator. Now a comparison of (27) and (28) shows that in case of  $\alpha < 1$ , the Fourier multiplier (21) can be made much smaller than the Fourier multiplier (22) while preserving (23).

In the theory of partial differential equations, this loss of one derivative (with respect to the maximal regularity) is well-known for the (one dimensional) wave equation, where the D'Alembertian can also be written as a single mixed second derivative:  $\partial_t \partial_t - \partial_x \partial_x = (\partial_t + \partial_x)(\partial_t - \partial_x)$ .

**Lemma 2.** *There exists a possible small but universal radius  $r > 0$  such that for any shifts  $|h|, |h'| \leq r$  there exist three functions  $j_{[111]}, j_{[\bar{1}11]}, j$  in  $B_r$  with*

$$\partial_{[111]}^h \partial_{[\bar{1}11]}^{h'} \tilde{\chi}_1 = \partial_{[111]} j_{[111]} + \partial_{[\bar{1}11]} j_{[\bar{1}11]} + j \quad \text{in } B_r \quad (29)$$

such that

$$\int_{B_r} (j_{[111]}^2 + j_{[\bar{1}\bar{1}\bar{1}]}^2 + j^2) dx \lesssim E_{elast}.$$

As for Lemma 1, the same statement holds with the pair of axes  $\{[111], [\bar{1}\bar{1}\bar{1}]\}$  replaced by  $\{[11\bar{1}], [1\bar{1}\bar{1}]\}$ . The same statement also holds with  $\tilde{\chi}_1$  and  $\{[111], [\bar{1}\bar{1}\bar{1}]\}$ ,  $\{[11\bar{1}], [1\bar{1}\bar{1}]\}$  replaced by  $\tilde{\chi}_2$  and  $\{[111], [1\bar{1}\bar{1}]\}$ ,  $\{[11\bar{1}], [\bar{1}\bar{1}\bar{1}]\}$ , and by  $\tilde{\chi}_3$  and  $\{[111], [11\bar{1}]\}$ ,  $\{[1\bar{1}\bar{1}], [\bar{1}\bar{1}\bar{1}]\}$ .

### 3.3 Lemma 3

In order to proceed, we have to convert the weak  $H^{-1}$ -control on the mixed derivative  $\partial_{[111]}^h \partial_{[\bar{1}\bar{1}\bar{1}]}^{h'} \tilde{\chi}$  stated in Lemma 2 into strong  $L^2$ -control (which coincides with strong  $L^1$ -control, since  $\partial_{[111]}^h \partial_{[\bar{1}\bar{1}\bar{1}]}^{h'} \tilde{\chi}$  only assumes a fixed set of finite values). Only on the level of a strong control are we able to connect to the additional information that  $\chi_0, \chi_1, \chi_2$ , and  $\chi_3$  are characteristic functions. This passage from a weak to a strong topology can only be achieved with help of the interfacial energy. The scaling of  $(\eta^{1/3}, \eta^{-2/3})$ , cf. (31), in the parameter  $\eta$  is dictated by the fact that according to Lemma 2, the elastic energy controls the *square* of a *negative* gradient of  $\partial_{[111]}^h \partial_{[\bar{1}\bar{1}\bar{1}]}^{h'} \tilde{\chi}_1$  (the square of the  $H^{-1}$ -norm of  $\partial_{[111]}^h \partial_{[\bar{1}\bar{1}\bar{1}]}^{h'} \tilde{\chi}_1$ ) whereas the interfacial energy controls (the  $L^1$ -norm of ) a single gradient of  $\tilde{\chi}_1$  (and thus also of its finite differences  $\partial_{[111]}^h \partial_{[\bar{1}\bar{1}\bar{1}]}^{h'} \tilde{\chi}_1$ ).

In the proof of Lemma 3, the above scaling argument is made rigorous by the application of the interpolation estimate

$$\begin{aligned} \int \phi^2 dx &\lesssim \left( \int |\nabla \phi| dx \sup |\phi| \right)^{2/3} \left( \int |j|^2 dx \right)^{1/3} \\ &\lesssim \eta^{1/3} \int |\nabla \phi| dx \sup |\phi| + \eta^{-2/3} \int |j|^2 dx, \end{aligned} \quad (30)$$

where the field  $j$  in last r. h. s. factor is related to  $\phi$  via  $\nabla \cdot j = \phi$ , so that the last factor defines the  $H^{-1}$ -norm of  $\phi$ . This interpolation estimate was identified as crucial in [3, Lemma 2.3 (2.3)] for the analysis of branching phenomena.

**Lemma 3.** *There exists a universal radius  $r > 0$  such that for all parameters  $\eta \leq r$  and shifts  $|h|, |h'| \leq r$  we have*

$$\int_{B_r} |\partial_{[111]}^h \partial_{[\bar{1}\bar{1}\bar{1}]}^{h'} \tilde{\chi}_1| dx \lesssim \eta^{1/3} E_{interf} + \eta^{-2/3} E_{elast}. \quad (31)$$



As before, the same statement holds with  $\{[111], [\bar{1}\bar{1}\bar{1}]\}$  replaced by  $\{[11\bar{1}], [1\bar{1}\bar{1}]\}$ . The same statement also holds with  $\tilde{\chi}_1$  and  $\{[111], [\bar{1}\bar{1}\bar{1}]\}$ ,  $\{[11\bar{1}], [1\bar{1}\bar{1}]\}$  replaced by  $\tilde{\chi}_2$  and  $\{[111], [1\bar{1}\bar{1}]\}$ ,  $\{[11\bar{1}], [\bar{1}\bar{1}\bar{1}]\}$ , and by  $\tilde{\chi}_3$  and  $\{[111], [11\bar{1}]\}$ ,  $\{[1\bar{1}\bar{1}], [\bar{1}\bar{1}\bar{1}]\}$ .

### 3.4 Corollary 1

Working with (second-order) finite differences has many advantages:

- It allows to convert the  $H^{-2}$ -control of Lemma 1 into  $H^{-1}$ -control, cf. Lemma 2.
- It allows to transfer the  $BV$ -control given by the interfacial energy from the characteristic functions themselves to their derivatives, cf. the proof of Lemma 3.
- It allows to make efficient use of the fact that we are taking second derivatives of *characteristic* functions.

The last point becomes apparent in the following Corollary 1 to Lemma 3.

**Corollary 1.** *There exists a universal radius  $r > 0$  such that for all parameters  $\eta \leq r$  and shifts  $|h|, |h'| \leq r$  we have*

$$\begin{aligned} & \int_{B_r} |\partial_{[111]}^h \partial_{[\bar{1}\bar{1}\bar{1}]}^{h'} \chi_0| dx + \int_{B_r} |\partial_{[111]}^h \partial_{[\bar{1}\bar{1}\bar{1}]}^{h'} \chi_1| dx \\ & \lesssim \eta^{1/3} E_{interf} + \eta^{-2/3} E_{elast}. \end{aligned}$$

As before, the same statement holds with  $\{[111], [\bar{1}\bar{1}\bar{1}]\}$  replaced by  $\{[11\bar{1}], [1\bar{1}\bar{1}]\}$ . The same statement also holds with the two characteristic functions  $\chi_0, \chi_1$  and the two sets of pairs of axes  $\{[111], [\bar{1}\bar{1}\bar{1}]\}$ ,  $\{[11\bar{1}], [1\bar{1}\bar{1}]\}$  replaced by  $\chi_0, \chi_2$  and  $\{[111], [1\bar{1}\bar{1}]\}$ ,  $\{[11\bar{1}], [\bar{1}\bar{1}\bar{1}]\}$ ,  $\chi_0, \chi_3$  and  $\{[111], [11\bar{1}]\}$ ,  $\{[1\bar{1}\bar{1}], [\bar{1}\bar{1}\bar{1}]\}$ .

Note that for the characteristic function  $\chi_0$  of the Austenite, Corollary 1 provides the strong information that the six mixed finite differences

$$\partial_{[111]}^h \partial_{[1\bar{1}\bar{1}]}^{h'}, \partial_{[111]}^h \partial_{[1\bar{1}\bar{1}]}^{h'}, \partial_{[111]}^h \partial_{[\bar{1}\bar{1}\bar{1}]}^{h'}, \partial_{[11\bar{1}]}^h \partial_{[\bar{1}\bar{1}\bar{1}]}^{h'}, \partial_{[11\bar{1}]}^h \partial_{[\bar{1}\bar{1}\bar{1}]}^{h'}, \partial_{[1\bar{1}\bar{1}]}^h \partial_{[\bar{1}\bar{1}\bar{1}]}^{h'}$$

are locally controlled in  $L^1$ . Suppose for a moment that they vanish (because for instance the elastic energy vanishes). Then also the corresponding six mixed derivatives vanish. Note that the projection of the six  $3 \times 3$ -tensors

$$[111] \otimes [1\bar{1}\bar{1}], [111] \otimes [1\bar{1}\bar{1}], [111] \otimes [\bar{1}\bar{1}\bar{1}], [11\bar{1}] \otimes [1\bar{1}\bar{1}], [11\bar{1}] \otimes [\bar{1}\bar{1}\bar{1}], [1\bar{1}\bar{1}] \otimes [\bar{1}\bar{1}\bar{1}]$$

onto the space of symmetric tensors *spans* the space of symmetric tensors (since this space has dimension six, the number is minimal). This would yield that *all* second derivatives vanish. Hence  $\chi_0$  would be affine. But an affine characteristic function is constant. This is the basis of the proof of Theorem 1 i).

### 3.5 Lemma 4

From the Martensitic part of Corollary 1 we would like to draw a quantitative version of (18) (with  $\tilde{\chi}_1$  replaced by  $\chi_1$ ). Since we don't know how to quantify (33), we follow an algebraically more involved strategy somewhat similar to the argument for vanishing elastic energy in [5, Lemma 3.2]. In order to do so, we need a quantification of two elementary statements that are of the form: If certain (second mixed) derivatives of a function  $f$  of three variables vanish, then  $f$  can be written as the sum of functions which depend on one or two variables only.

In order to be more precise, we fix a basis  $\{a, b, c\}$  of  $\mathbb{R}^3$  and denote by  $\{a^*, b^*, c^*\}$  its dual basis. In the applications, the basis will be formed by three out of the four space diagonals  $\{[111], [11\bar{1}], [1\bar{1}1], [\bar{1}11]\}$ , so that the dual basis is (up to scalar factors) always formed by three linear independent vectors out of the six face diagonals  $\{(011), (01\bar{1}), (101), (\bar{1}01), (110), (1\bar{1}0)\}$  (for instance, the dual basis of  $\{[111], [11\bar{1}], [1\bar{1}1]\}$  is, up to scalar factors,  $\{(011), (\bar{1}01), (1\bar{1}0)\}$ ). The two elementary statements we refer to are the following:

i) We have

$$\partial_a \partial_b f \equiv 0 \implies f = f_a + f_b,$$

where  $f_a$  and  $f_b$  are constant along the lines parallel to  $a$  and  $b$ , respectively, i. e.  $\partial_a f_a = 0$  and  $\partial_b f_b = 0$ .

ii) We have

$$\partial_a \partial_b f \equiv \partial_c f \equiv 0 \implies f = f_{a^*} + f_{b^*},$$

where  $f_{a^*}$  and  $f_{b^*}$  are constant along the planes perpendicular to  $a^*$  and  $b^*$  respectively, i. e.  $f_{a^*}$  and  $f_{b^*}$  are function of  $a^* \cdot x$  and  $b^* \cdot x$ , respectively.

The following Lemma 4 quantifies and localizes these statements on the level of  $L^1$ -estimates of finite differences.

As mentioned above, it would be desirable for the treatment of Martensite to have a quantification of (18). Let us put (18) into the same form as i) and ii):

iii) We have

$$\begin{aligned}\partial_{[111]}\partial_{[\bar{1}\bar{1}\bar{1}]}f &\equiv \partial_{[11\bar{1}]}\partial_{[1\bar{1}\bar{1}]}f \equiv 0 \\ \implies f &= f_{(110)} + f_{(1\bar{1}0)} + f_{(101)} + f_{(\bar{1}01)}.\end{aligned}\quad (32)$$

Note that the decomposition result (32) formally has the pleasing algebraic form

$$\begin{aligned}\ker(\partial_{[111]}\partial_{[\bar{1}\bar{1}\bar{1}]}) \cap \ker(\partial_{[11\bar{1}]}\partial_{[1\bar{1}\bar{1}]}) \\ = \ker(\partial_{[\bar{1}\bar{1}\bar{1}]}) \cap \ker(\partial_{[11\bar{1}]}) + \ker(\partial_{[111]}) \cap \ker(\partial_{[1\bar{1}\bar{1}]}) \\ + \ker(\partial_{[\bar{1}\bar{1}\bar{1}]}) \cap \ker(\partial_{[11\bar{1}]}) + \ker(\partial_{[111]}) \cap \ker(\partial_{[1\bar{1}\bar{1}]}).\end{aligned}\quad (33)$$

We observe that iii) follows easily from applying ii) twice: Note that  $g := \partial_{[111]}f$  satisfies  $\partial_{[\bar{1}\bar{1}\bar{1}]}g = \partial_{[11\bar{1}]}\partial_{[1\bar{1}\bar{1}]}g = 0$ . Hence we may apply ii) to  $g$  and  $a = [11\bar{1}]$ ,  $b = [\bar{1}\bar{1}\bar{1}]$ , and  $c = [\bar{1}\bar{1}\bar{1}]$  yielding the decomposition

$$g = g_{(110)} + g_{(101)}.$$

The form of this decomposition lifts to the anti-derivative  $f$  at the expense of an additional  $[111]$ -independent term:

$$f = f_{(110)} + f_{(101)} + f_{[111]}.\quad (34)$$

Applying  $\partial_{[11\bar{1}]}\partial_{[1\bar{1}\bar{1}]}$  to the last identity, we find (since by assumption  $f$ , and by construction  $f_{(110)}$  and  $f_{(101)}$  vanish under this operator),

$$\partial_{[11\bar{1}]}\partial_{[1\bar{1}\bar{1}]}f_{[111]} = 0.$$

Since by construction  $\partial_{[111]}f_{[111]} = 0$ , we may apply ii) once more, this time with  $f$  replaced by  $f_{[111]}$  and  $a = [11\bar{1}]$ ,  $b = [1\bar{1}\bar{1}]$ , and  $c = [111]$ . This application yields

$$f_{[111]} = f_{(\bar{1}01)} + f_{(1\bar{1}0)}.\quad (35)$$

Inserting (35) as desired into (34), we obtain as desired the r. h. s. of (32).

We were not able to quantify iii) as we did for i), ii). The proof of parts i) and ii) of Lemma 4 relies on the fact that *three* (independent) vectors  $a$ ,  $b$ , and  $c$  in  $\mathbb{R}^3$  come with a “fundamental domain”, the parallelepiped spanned by these vectors. There is no suitable equivalent for *four* vectors like our  $[111]$ ,  $[11\bar{1}]$ ,  $[1\bar{1}\bar{1}]$ ,  $[\bar{1}\bar{1}\bar{1}]$ . Likewise, it does not seem straightforward to mimic the above derivation of iii) from ii) on the level of *finite* differences (controlled in  $L^1$ ) instead of (vanishing) *infinitesimal* derivatives: The operation of taking *anti-derivatives* as in (34) is ill-defined on the discrete level in the sense that the anti-derivative depends on the shift  $h$ .

**Lemma 4.** *There exists a possible small but universal radius  $r > 0$  such that for any function  $f$  on  $B_1$*

*i) there exist two functions  $f_a$  and  $f_b$  on  $B_r$  constant along the lines parallel to  $a$  and  $b$ , respectively, such that*

$$\int_{B_r} |f - f_a - f_b| dx \lesssim \sup_{|h_a|, |h_b| \leq 1} \int_{B_1} |\partial_a^{h_a} \partial_b^{h_b} f| dx;$$

*ii) there exist two functions  $f_{a^*}$  and  $f_{b^*}$  on  $B_r$  constant along the planes orthogonal to  $a^*$  and  $b^*$ , respectively, such that*

$$\int_{B_r} |f - f_{a^*} - f_{b^*}| dx \lesssim \sup_{|h_a|, |h_b|, |h_c| \leq 1} \int_{B_1} (|\partial_a^{h_a} \partial_b^{h_b} f| + |\partial_c^{h_c} f|) dx.$$

### 3.6 Lemma 5

The second ingredient is the quantification of a uniqueness statement in decompositions of the type occurring in Lemma 4. The uniqueness statement is the following: Suppose that a function  $f$  of three variables can locally be written as the sum of six functions of a single variable each, more precisely,

$$f = f_{(011)} + f_{(01\bar{1})} + f_{(101)} + f_{(\bar{1}01)} + f_{(110)} + f_{(1\bar{1}0)},$$

where, following the notational convention of Lemma 4,  $f_{(011)}$ ,  $f_{(01\bar{1})}$ ,  $f_{(101)}$ ,  $f_{(\bar{1}01)}$ ,  $f_{(110)}$ ,  $f_{(1\bar{1}0)}$  are functions which are constant along planes orthogonal to the vectors  $(011)$ ,  $(01\bar{1})$ ,  $(101)$ ,  $(\bar{1}01)$ ,  $(110)$ ,  $(1\bar{1}0)$ , respectively. Then the uniqueness statement is actually a uniqueness result *modulo affine functions* and reads as follows:

$$f \text{ is affine} \implies f_{(011)}, \dots, f_{(1\bar{1}0)} \text{ are affine.} \quad (36)$$

Since affine functions are characterized by the vanishing of all second derivatives  $\partial_a \partial_b$ , (36) can be reformulated as

$$\begin{aligned} \forall a, b \in \mathbb{R}^3 \quad \partial_a \partial_b f &\equiv 0 \\ \implies \forall a, b \in \mathbb{R}^3 \quad \partial_a \partial_b f_{(011)} &\equiv \dots \equiv \partial_a \partial_b f_{(1\bar{1}0)} \equiv 0. \end{aligned} \quad (37)$$

The statement(37) reflects the fact that the 6 symmetric  $3 \times 3$ -tensors

$$\begin{aligned} (011) \otimes (011), \quad (101) \otimes (101), \quad (110) \otimes (110), \\ (01\bar{1}) \otimes (01\bar{1}), \quad (\bar{1}01) \otimes (\bar{1}01), \quad (1\bar{1}0) \otimes (1\bar{1}0) \end{aligned}$$

are linearly independent (note that 6 is the maximal number of linearly independent symmetric  $3 \times 3$ -tensors). The following Lemma 5 is a localization and quantification of (37) on the level of finite differences in the  $L^1$ -topology.

**Lemma 5.** *There exists a possibly small but universal radius  $r > 0$  with the following property: If the functions  $f_{(011)}$ ,  $f_{(01\bar{1})}$ ,  $f_{(101)}$ ,  $f_{(\bar{1}01)}$ ,  $f_{(110)}$ , and  $f_{(\bar{1}\bar{1}0)}$  are constant along the planes  $(011)$ ,  $(01\bar{1})$ ,  $(101)$ ,  $(\bar{1}01)$ ,  $(110)$ , and  $(\bar{1}\bar{1}0)$ , respectively, and are related to the function  $f$  by*

$$f = f_{(011)} + f_{(01\bar{1})} + f_{(101)} + f_{(\bar{1}01)} + f_{(110)} + f_{(\bar{1}\bar{1}0)} \quad \text{on } B_1, \quad (38)$$

then we have

$$\begin{aligned} & \sup_{\substack{a,b \in S^2 \\ |h_a|, |h_b| \leq r}} \int_{B_r} |\partial_a^{h_a} \partial_b^{h_b} f_{(011)}| dx + \cdots + \sup_{\substack{a,b \in S^2 \\ |h_a|, |h_b| \leq r}} \int_{B_r} |\partial_a^{h_a} \partial_b^{h_b} f_{(\bar{1}\bar{1}0)}| dx \\ & \lesssim \sup_{\substack{a,b \in S^2 \\ |h_a|, |h_b| \leq 1}} \int_{B_1} |\partial_a^{h_a} \partial_b^{h_b} f| dx. \end{aligned}$$

### 3.7 Proposition 1

Bringing together Corollary 1 and the two auxiliary Lemmas 4 and 5 on the existence and uniqueness of decompositions, we are now in the position to establish an important intermediary result. Loosely speaking, it states that *up to affine functions* (which are factored out by second-order finite differences), and *up to a controlled error in  $L^1(B_r)$* , the four characteristic functions behave as

$$\begin{aligned} \chi_0 &= 0, \\ \chi_1 &= -f_{(101)} - f_{(\bar{1}01)} + f_{(110)} + f_{(\bar{1}\bar{1}0)}, \\ \chi_2 &= f_{(011)} + f_{(01\bar{1})} - f_{(110)} - f_{(\bar{1}\bar{1}0)}, \\ \chi_3 &= -f_{(011)} - f_{(01\bar{1})} + f_{(101)} + f_{(\bar{1}01)}. \end{aligned}$$

Loosely speaking, Proposition 1 is the counterpart of [5, Lemma 3.2] in case of vanishing elastic energy.

**Proposition 1.** *There exists a small but universal radius  $r > 0$  such that for sufficiently small parameters  $\eta \leq r$ , there exist functions  $f_{(011)}$ ,  $f_{(01\bar{1})}$ ,  $f_{(101)}$ ,  $f_{(\bar{1}01)}$ ,  $f_{(110)}$ , and  $f_{(\bar{1}\bar{1}0)}$  constant along the planes  $(011)$ ,  $(01\bar{1})$ ,  $(101)$ ,  $(\bar{1}01)$ ,  $(110)$ , and  $(\bar{1}\bar{1}0)$ , respectively, and are related to the function  $f$  by*

(110), and (1 $\bar{1}$ 0), respectively, such that

$$\begin{aligned}
& \sup_{\substack{a,b \in S^2 \\ |h_a|, |h_b| \leq r}} \int_{B_r} \left( |\partial_a^{h_a} \partial_b^{h_b} \chi_0| \right. \\
& \quad + |\partial_a^{h_a} \partial_b^{h_b} (\chi_1 + f_{(101)} + f_{(\bar{1}01)} - f_{(110)} - f_{(1\bar{1}0)})| \\
& \quad + |\partial_a^{h_a} \partial_b^{h_b} (\chi_2 - f_{(011)} - f_{(01\bar{1})} + f_{(110)} + f_{(1\bar{1}0)})| \\
& \quad \left. + |\partial_a^{h_a} \partial_b^{h_b} (\chi_3 + f_{(011)} + f_{(01\bar{1})} - f_{(101)} - f_{(\bar{1}01)})| \right) dx \\
& \lesssim \eta^{1/3} E_{interf} + \eta^{-2/3} E_{elast}. \tag{39}
\end{aligned}$$

### 3.8 Lemma 6

In Proposition 1, when using the Lemmas 4 and 5, which are of rather algebraic character, we have not yet leveraged the fact that we apply them to characteristic functions. However, the non-convexity of the set of values of characteristic functions has already been crucially used in Corollary 1: Corollary 1 separated the amalgated information on  $\tilde{\chi}_1, \tilde{\chi}_2, \tilde{\chi}_3$  into information on the characteristic function  $\chi_0$  of the Austenite on the one hand and on the characteristic functions  $\chi_1, \chi_2, \chi_3$  of the Martensite on the other hand.

In the following Lemma 6, we again crucially use the property of a characteristic function. Lemma 6 is the quantification of the following fact (in two space dimensions): Let  $\chi$  be  $\{0, 1\}$ -valued function and  $f_{1^*}(x_1), f_{2^*}(x_2)$  be functions of  $x_1$  and  $x_2$ , respectively, and let  $\ell$  be an affine function. Suppose that they are related by  $\chi = f_{1^*} + f_{2^*} + \ell$ . Then  $f_{1^*}$  or  $f_{2^*}$  are affine. Avoiding the notion of ‘‘affine’’ at the expense of second derivatives, this can also be formulated as follows:

$$\begin{aligned}
\partial_1 \partial_1 (\chi - f_{1^*} - f_{2^*}) &\equiv \partial_1 \partial_2 (\chi - f_{1^*} - f_{2^*}) \equiv \partial_2 \partial_2 (\chi - f_{1^*} - f_{2^*}) \equiv 0 \\
&\implies \\
\partial_1 \partial_1 f_{1^*} &\equiv 0 \quad \underline{\text{or}} \quad \partial_2 \partial_2 f_{2^*} \equiv 0.
\end{aligned}$$

Like the previous lemmas, Lemma 6 localizes and quantifies this fact on the level of second-order finite differences in the  $L^1$ -topology.

In the same sense that Corollary 1 is at the core of the dichotomy for the Austenite (Theorem 1 i)), Lemma 6 is at the core of the dichotomy (by a slight misuse of language) for the Martensite (Theorem 1 ii) and iii)).

**Lemma 6.** *There exists a small but universal radius  $r > 0$  with the following property: If  $\chi \in \{0, 1\}$  is a function of  $(x_1, x_2)$ ,  $f_{1^*}$  a function of  $x_1$ , and  $f_{2^*}$  a function of  $x_2$ , then*

$$\begin{aligned} & \min \left\{ \sup_{|h|, |h'| \leq r} \int_{(-r, r)} |\partial_1^h \partial_1^{h'} f_{1^*}| dx_1, \sup_{|h|, |h'| \leq r} \int_{(-r, r)} |\partial_2^h \partial_2^{h'} f_{2^*}| dx_2 \right\} \\ & \lesssim \left( \sup_{|h|, |h'| \leq \frac{1}{2}} \int_{(-1, 1)^2} (|\partial_1^h \partial_1^{h'} (\chi - f_{1^*} - f_{2^*})| + |\partial_1^h \partial_2^{h'} (\chi - f_{1^*} - f_{2^*})| \right. \\ & \quad \left. + |\partial_2^h \partial_2^{h'} (\chi - f_{1^*} - f_{2^*})|) dx_1 dx_2 \right)^{1/2}, \end{aligned} \quad (40)$$

provided the r. h. s. is  $\leq 1$ .

The following example shows that the loss of homogeneity through the square root on the r. h. s. of (40) is unavoidable. For  $\epsilon \ll 1$  we consider the three functions:

- Let  $f_{1^*}$  be the characteristic function of the small interval  $x_1 \in (-\epsilon, \epsilon)$ .
- Let  $f_{2^*}$  be the characteristic function of the small interval  $x_2 \in (-\epsilon, \epsilon)$ .
- Let  $\chi$  be the characteristic function of the thin cross  $(x_1, x_2) \in ((-\epsilon, \epsilon) \times \{0\}) \cup (\{0\} \times (-\epsilon, \epsilon))$ .

Then we have on the one hand for  $r \gg \epsilon$ :

$$\sup_{|h|, |h'| \leq r} \int_{(-r, r)} |\partial_1^h \partial_1^{h'} f_{1^*}| dx_1 \sim \sup_{|h|, |h'| \leq r} \int_{(-r, r)} |\partial_2^h \partial_2^{h'} f_{2^*}| dx_2 \sim \epsilon,$$

but on the other hand

$$\int_{(-\frac{1}{2}, \frac{1}{2})^2} |\chi - f_{1^*} - f_{2^*}| dx_1 dx_2 = 4\epsilon^2,$$

and thus (at most)  $\epsilon^2$ -scaling also for the  $L^1$ -norm of the second-order finite differences of  $\chi - f_{1^*} - f_{2^*}$ .

### 3.9 Lemma 7

Suppose that a function  $f$  of one variable is periodic with *two* periods  $h_1, h_2$  that are *irrationally* related. Then  $f$  is constant. This fact can be easily proved by representing  $f$  with help of Fourier series based one of the periods ( $h_1$ , say) and then expressing the other periodicity in that framework.

This fact can also be expressed on the level of first-order finite differences as follows: Provided a set  $S$  of shifts contains irrationally related numbers then

$$\forall h \in S \quad \partial_1^h f \equiv 0 \quad \implies \quad \forall h \in \mathbb{R} \quad \partial_1^h f \equiv 0.$$

As always, we need a localization and a quantification in terms of the  $L^1$ -topology. On the other hand, we may be generous on the cardinality of the set  $S$  and assume that it is of positive Lebesgue measure. Lemma 7 will be extremely convenient in the proof of Proposition 2.

**Lemma 7.** *There exists a universal radius  $r > 0$  with the following property: For any set  $S \subset [-r, r]$  and function  $f$  of a single variable  $x_1$  we have*

i)

$$\sup_{|h| \leq r} \int_{(-r, r)} |\partial_1^h f| dx_1 \lesssim (\mathcal{L}^1(S))^{-2} \sup_{h \in S} \int_{(-\frac{1}{2}, \frac{1}{2})} |\partial_1^h f| dx_1, \quad (41)$$

ii)

$$\sup_{|h| \leq r} \int_{(-r, r)} |\partial_1^h f| dx_1 \lesssim (\mathcal{L}^1(S))^{-3} \int_{(-r, r)} \int_{(-\frac{1}{2}, \frac{1}{2})} |\partial_1^h f| dx_1 dh, \quad (42)$$

iii)

$$\int_{(-r, r)} \int_{(-r, r)} |\partial_1^h f| dx_1 dh \lesssim (\mathcal{L}^1(S))^{-3} \int_S \int_{(-\frac{1}{2}, \frac{1}{2})} |\partial_1^h f| dx_1 dh. \quad (43)$$

We would not be surprised if such a result can be found in the literature; we do not claim that the exponent of  $\mathcal{L}^1(S)$  is optimal.

### 3.10 Proposition 2

We are now in the position to establish a prototype of the dichotomy also for the Martensitic part, that is, of Theorem 1 ii). By prototype we mean that Proposition 2 just states that one of the three characteristic functions  $\chi_1, \chi_2, \chi_3$  is close to a *quadratic* function, more precisely that its *third-order* finite differences are controlled in  $L^1$ . Hence with respect to the Austenitic part of Proposition 1, the kernel of the finite differences increases from *affine* functions to *quadratic* functions. Since we are dealing with characteristic functions, this does not matter, as Lemma 8 will show.



**Proposition 2.** *There exists a universal radius  $r > 0$  such that for all sufficiently small parameters  $\eta \leq r$  we have*

$$\begin{aligned}
& \min \left\{ \sup_{\substack{a,b,c \in S^2 \\ |h_a|, |h_b|, |h_c| \leq r}} \int_{B_r} |\partial_a^{h_a} \partial_b^{h_b} \partial_c^{h_c} \chi_1| dx, \right. \\
& \quad \sup_{\substack{a,b,c \in S^2 \\ |h_a|, |h_b|, |h_c| \leq r}} \int_{B_r} |\partial_a^{h_a} \partial_b^{h_b} \partial_c^{h_c} \chi_2| dx, \\
& \quad \left. \sup_{\substack{a,b,c \in S^2 \\ |h_a|, |h_b|, |h_c| \leq r}} \int_{B_r} |\partial_a^{h_a} \partial_b^{h_b} \partial_c^{h_c} \chi_3| dx \right\} \\
& \lesssim (\eta^{1/3} E_{interf} + \eta^{-2/3} E_{elast})^{1/2}. \tag{44}
\end{aligned}$$

### 3.11 Lemma 8

Finally, we need a quantification (and localization) of the fact that a characteristic function  $\chi$  for which all (distributional) second-order derivatives vanish, must be constant, in particular  $\chi \equiv 0$  or  $\chi \equiv 1$ . As in the previous lemmas, we quantify by measuring second-order finite differences in  $L^1$ . Of course, it is this lemma which ultimately yields the dichotomy for both Austenite and Martensite, that is, both parts i) and ii) of Theorem 1.

**Lemma 8.** *There exists a small but universal radius  $r > 0$  such that for any  $\{0, 1\}$ -valued function  $\chi$  we have*

$$\min \left\{ \int_{B_r} \chi dx, \int_{B_r} (1 - \chi) dx \right\} \lesssim \sup_{\substack{a,b,c \in S^2 \\ |h|, |h'|, |h''| \leq 1}} \int_{B_1} |\partial_a^h \partial_b^{h'} \partial_c^{h''} \chi| dx. \tag{45}$$

Alternatively, we could have stated Lemma 8 in two steps: 1) The statement that smallness of third-order finite differences implies closeness to quadratic functions, more precisely, that there exists a quadratic function  $q$  such that

$$\int_{B_r} |f - q| dx \lesssim \sup_{\substack{a,b,c \in S^2 \\ |h|, |h'|, |h''| \leq 1}} \int_{B_1} |\partial_a^h \partial_b^{h'} \partial_c^{h''} f| dx,$$

and 2) the statement that characteristic functions that are close to quadratic are close to constant, more precisely, that for any function  $\chi \in \{0, 1\}$  and quadratic function  $q$

$$\min \left\{ \int_{B_r} \chi dx, \int_{B_r} (1 - \chi) dx \right\} \lesssim \int_{B_1} |\chi - q| dx.$$

However, the proof in one step is shorter.

## 4 Proofs

PROOF OF LEMMA 1.

Let us start with a symmetry consideration. We remark that it is enough to establish the result for  $\tilde{\chi}_1$ . Indeed, the property of being a strain (8) and the elastic energy (17) is invariant under permutation of the cubic axes, so that the results for  $\tilde{\chi}_2$  and  $\tilde{\chi}_3$  follow by symmetry. We also note that it is enough to establish the result for the pair of axes  $\{[111], [\bar{1}\bar{1}\bar{1}]\}$ . Indeed, the second pair of (non-oriented) axes,  $\{[11\bar{1}], [1\bar{1}\bar{1}]\}$ , can also be written as  $\{[11\bar{1}], [\bar{1}\bar{1}\bar{1}]\}$ . Hence it remains to observe that under the change of variables  $x_3 \rightsquigarrow -x_3$  (and likewise  $u_3 \rightsquigarrow -u_3$  for the displacement field), strains transform as

$$\begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{12} & e_{22} & e_{23} \\ e_{13} & e_{23} & e_{33} \end{pmatrix} \rightsquigarrow \begin{pmatrix} e_{11} & e_{12} & -e_{13} \\ e_{12} & e_{22} & -e_{23} \\ -e_{13} & -e_{23} & e_{33} \end{pmatrix}$$

In particular, the elastic energy (17) is invariant under this change of coordinates.

The main step of this proof consists in rewriting the compatibility conditions on the strain  $e$  as

$$\begin{aligned} & (\partial_1 + (\partial_2 + \partial_3))(-\partial_1 + (\partial_2 + \partial_3))\tilde{\chi}_1 \\ &= -\partial_1\partial_1(e_{22} + 2e_{23} + e_{33} + \tilde{\chi}_1) \\ & \quad + 2\partial_1(\partial_2 + \partial_3)(e_{12} + e_{13}) \\ & \quad - (\partial_2 + \partial_3)(\partial_2 + \partial_3)(e_{11} + \tilde{\chi}_2 + \tilde{\chi}_3). \end{aligned} \quad (46)$$

Before establishing formula (46), let us show how it yields (19) and (20). Expressing the directional derivatives  $\partial_1$  and  $\partial_2 + \partial_3$  into directional derivatives  $\partial_{[111]}$  and  $\partial_{[\bar{1}\bar{1}\bar{1}]}$  according to  $\partial_{[111]} = \partial_1 + (\partial_2 + \partial_3)$ ,  $\partial_{[\bar{1}\bar{1}\bar{1}]} = -\partial_1 + (\partial_2 + \partial_3)$ ,  $\partial_1 = \frac{1}{2}(\partial_{[111]} - \partial_{[\bar{1}\bar{1}\bar{1}]})$  and  $\partial_2 + \partial_3 = \frac{1}{2}(\partial_{[111]} + \partial_{[\bar{1}\bar{1}\bar{1}]})$ , (46) turns into

$$\begin{aligned} & -4\partial_{[111]}\partial_{[\bar{1}\bar{1}\bar{1}]} \tilde{\chi} \\ &= (\partial_{[111]} - \partial_{[\bar{1}\bar{1}\bar{1}]}) (\partial_{[111]} - \partial_{[\bar{1}\bar{1}\bar{1}]}) (e_{11} + \tilde{\chi}_1) \\ & \quad - 2(\partial_{[111]} - \partial_{[\bar{1}\bar{1}\bar{1}]}) (\partial_{[111]} + \partial_{[\bar{1}\bar{1}\bar{1}]}) (e_{12} + e_{13}) \\ & \quad + (\partial_{[111]} + \partial_{[\bar{1}\bar{1}\bar{1}]}) (\partial_{[111]} + \partial_{[\bar{1}\bar{1}\bar{1}]}) ((e_{22} + \tilde{\chi}_2) + 2e_{23} + (e_{33} + \tilde{\chi}_3)). \end{aligned} \quad (47)$$

Factorizing the products of sums of derivatives shows that (47) turns into (19) with the functions  $\rho_{[111],[111]}$ ,  $\rho_{[111],[\bar{1}11]}$ ,  $\rho_{[\bar{1}11],[\bar{1}11]}$  given by

$$\begin{aligned} -4\rho_{[111],[111]} &= (e_{11} + \tilde{\chi}_1) - 2(e_{12} + e_{13}) + ((e_{22} + \tilde{\chi}_2) \\ &\quad + 2e_{23} + (e_{33} + \tilde{\chi}_3)), \\ -4\rho_{[111],[\bar{1}11]} &= -(e_{11} + \tilde{\chi}_1) + ((e_{22} + \tilde{\chi}_2) + 2e_{23} + (e_{33} + \tilde{\chi}_3)), \\ -4\rho_{[\bar{1}11],[\bar{1}11]} &= (e_{11} + \tilde{\chi}_1) + 2(e_{12} + e_{13}) \\ &\quad + ((e_{22} + \tilde{\chi}_2) + 2e_{23} + (e_{33} + \tilde{\chi}_3)). \end{aligned}$$

Hence the functions  $\rho_{[111],[111]}$ ,  $\rho_{[111],[\bar{1}11]}$ ,  $\rho_{[\bar{1}11],[\bar{1}11]}$  are linear combinations of the entries of the  $3 \times 3$ -tensor

$$e + \begin{pmatrix} \tilde{\chi}_1 & 0 & 0 \\ 0 & \tilde{\chi}_2 & 0 \\ 0 & 0 & \tilde{\chi}_3 \end{pmatrix} = \begin{pmatrix} e_{11} + \tilde{\chi}_1 & e_{12} & e_{13} \\ e_{12} & e_{22} + \tilde{\chi}_2 & e_{23} \\ e_{13} & e_{23} & e_{33} + \tilde{\chi}_3 \end{pmatrix}.$$

The Frobenius norm of the latter is the elastic energy density, cf. (17). This observation implies (20).

We now argue that (47) is indeed just a reformulation of the compatibility constraint

$$\partial_1 \partial_1 (e_{22} + 2e_{23} + e_{33}) - 2\partial_1 (\partial_2 + \partial_3)(e_{12} + e_{13}) + (\partial_2 + \partial_3)(\partial_2 + \partial_3)e_{11} = 0 \quad (48)$$

on the strain tensor  $e = \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{12} & e_{22} & e_{23} \\ e_{13} & e_{23} & e_{33} \end{pmatrix}$ . Indeed, this follows from the identity

$$\begin{aligned} &(\partial_1 + (\partial_2 + \partial_3))(-\partial_1 + (\partial_2 + \partial_3))\tilde{\chi}_1 \\ &= (-\partial_1 \partial_1 + (\partial_2 + \partial_3)(\partial_2 + \partial_3))\tilde{\chi}_1 \\ &\stackrel{(16)}{=} -\partial_1 \partial_1 \tilde{\chi}_1 - (\partial_2 + \partial_3)(\partial_2 + \partial_3)(\tilde{\chi}_2 + \tilde{\chi}_3), \end{aligned}$$

which shows that the functions  $\tilde{\chi}_1, \tilde{\chi}_2, \tilde{\chi}_3$  drop out of (46).

For the convenience of the reader, we derive the compatibility relation (48) from (8). We first note that a *two-dimensional* strain tensor  $\begin{pmatrix} e_{11} & e_{12} \\ e_{12} & e_{22} \end{pmatrix}$  is characterized by the linear second-order constraint

$$\partial_1 \partial_1 e_{22} - 2\partial_1 \partial_2 e_{12} + \partial_2 \partial_2 e_{11} = 0, \quad (49)$$

which has the same structure as (48). Indeed: If the strain tensor comes from the *two-dimensional* displacement field  $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ , i. e.

$$\begin{pmatrix} e_{11} & e_{12} \\ e_{12} & e_{22} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \partial_1 u_1 & \partial_2 u_1 \\ \partial_1 u_2 & \partial_2 u_2 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \partial_1 u_1 & \partial_1 u_2 \\ \partial_2 u_1 & \partial_2 u_2 \end{pmatrix},$$

we have

$$\begin{aligned} & \partial_1 \partial_1 e_{22} - 2\partial_1 \partial_2 e_{12} + \partial_2 \partial_2 e_{11} \\ &= \partial_1 \partial_1 \partial_2 u_2 - \partial_1 \partial_2 (\partial_1 u_2 + \partial_2 u_1) + \partial_2 \partial_2 \partial_1 u_1 = 0, \end{aligned}$$

since partial derivatives commute.

Next, we note that if  $\begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{12} & e_{22} & e_{23} \\ e_{13} & e_{23} & e_{33} \end{pmatrix}$  is a *three-dimensional* strain, then

$\begin{pmatrix} e_{11} & e_{12} + e_{13} \\ e_{12} + e_{13} & e_{22} + 2e_{23} + e_{33} \end{pmatrix}$  is a two-dimensional strain. Indeed, let the

vector field  $\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$  be a three-dimensional displacement field corresponding

to the three-dimensional strain. Then  $\begin{pmatrix} u_1 \\ u_2 + u_3 \end{pmatrix}$ , seen as a function of  $x_1$  and  $x_2 + x_3$ , is a displacement field corresponding to the two-dimensional strain:

$$\begin{aligned} \partial_1(u_2 + u_3) + (\partial_2 + \partial_3)u_1 &= (\partial_1 u_2 + \partial_2 u_1) + (\partial_1 u_3 + \partial_3 u_1) \\ &= 2(e_{12} + e_{13}), \\ (\partial_2 + \partial_3)(u_2 + u_3) &= \partial_2 u_2 + (\partial_2 u_3 + \partial_3 u_2) + \partial_3 u_3 \\ &= e_{22} + 2e_{23} + e_{33}. \end{aligned}$$

Thus, in view of (49), our three-dimensional strain satisfies (48).

#### PROOF OF LEMMA 2.

Let  $a, b$  be two linearly independent vectors out of the four  $\{[111], [11\bar{1}], [1\bar{1}1], [\bar{1}11]\}$ . Let  $\chi$  denote one of the three functions  $\tilde{\chi}_1, \tilde{\chi}_2, \tilde{\chi}_3$ . Lemma 1 provides the following type of information: There exist three functions  $\eta_{aa}, \eta_{ab}, \eta_{bb}$  such that

$$\partial_a \partial_b \chi = \partial_a \partial_a \eta_{aa} + 2\partial_a \partial_b \eta_{ab} + \partial_b \partial_b \eta_{bb} \quad \text{in } B_1. \quad (50)$$

Let  $0 < r \ll 1$  denote a generic but universal radius (“generic” means that its value may change from formula to formula). In order to prove Lemma 2,

we have to show that for any shifts  $|h_a|, |h_b| \leq r$  there exist three functions  $j_a, j_b, j$  with

$$\partial_a^{h_a} \partial_b^{h_b} \chi = \partial_a j_a + \partial_b j_b + j \quad \text{in } B_r \quad (51)$$

and

$$\int_{B_r} (j_a^2 + j_b^2 + j^2) dx \lesssim \int_{B_1} (\eta_{aa}^2 + \eta_{ab}^2 + \eta_{bb}^2) dx. \quad (52)$$

We reformulate (50) as

$$\partial_a \partial_b (\chi - 2\eta_{ab}) = \partial_a \partial_a \eta_{aa} + \partial_b \partial_b \eta_{bb} \quad \text{in } B_1,$$

evaluate at  $x + h'_a a + h'_b b$  (for  $|x| \leq r$ ), i. e.

$$\begin{aligned} & \frac{d}{dh'_a} \frac{d}{dh'_b} (\chi - 2\eta_{ab})(x + h'_a a + h'_b b) \\ &= \frac{d}{dh'_a} \partial_a \eta_{aa}(x + h'_a a + h'_b b) + \frac{d}{dh'_b} \partial_b \eta_{bb}(x + h'_a a + h'_b b), \end{aligned}$$

and integrate over  $\int_0^{h_a} dh'_a \int_0^{h_b} dh'_b$ , i. e.

$$\begin{aligned} & \partial_a^{h_a} \partial_b^{h_b} (\chi - 2\eta_{ab})(x) \\ &= (\chi - 2\eta_{ab})(x + h_a a + h_b b) - (\chi - 2\eta_{ab})(x + h_a a) \\ & \quad - (\chi - 2\eta_{ab})(x + h_b b) + (\chi - 2\eta_{ab})(x) \\ &= \partial_a \left( \int_0^{h_b} \eta_{aa}(x + h_a a + h'_b b) dh'_b - \int_0^{h_b} \eta_{aa}(x + h'_b b) dh'_b \right) \\ & \quad + \partial_b \left( \int_0^{h_a} \eta_{bb}(x + h'_a a + h_b b) dh'_a - \int_0^{h_a} \eta_{bb}(x + h'_a a) dh'_a \right). \end{aligned}$$

Hence the functions satisfying (51) are seen to be

$$\begin{aligned} j_a(x) &:= \int_0^{h_b} \eta_{aa}(x + h_a a + h'_b b) dh'_b - \int_0^{h_b} \eta_{aa}(x + h'_b b) dh'_b, \\ j_b(x) &:= \int_0^{h_a} \eta_{bb}(x + h'_a a + h_b b) dh'_a - \int_0^{h_a} \eta_{bb}(x + h'_a a) dh'_a, \\ j(x) &:= -2\partial_a^{h_a} \partial_b^{h_b} \eta_{ab}(x) \\ &= -2(\eta_{ab}(x + h_a a + h_b b) - \eta_{ab}(x + h_a a) - \eta_{ab}(x + h_b b) + \eta_{ab}(x)). \end{aligned}$$

Estimate (52) follows immediately from this representation.

**PROOF OF LEMMA 3.**

Let  $0 < r' \ll r \ll 1$  denotes generic, but universal radii (generic means that the particular value may change from estimate to estimate). We fix the shifts  $|h|, |h'| \leq r$  and set for abbreviation:

$$\phi := \partial_{[111]}^h \partial_{[\bar{1}\bar{1}\bar{1}]}^{h'} \tilde{\chi}_1 \stackrel{(15)}{=} \partial_{[111]}^h \partial_{[\bar{1}\bar{1}\bar{1}]}^{h'} (\chi_0 + 3\chi_1).$$

Since  $\chi_0$  and  $\chi_1$  are  $\{0, 1\}$ -valued, and the second finite differences are a fixed combination of values at four points,  $\phi$  has values in a fixed finite set. Therefore, we have two estimates that defy homogeneity in both directions:

$$\int_{B_r} |\phi| dx \lesssim \int_{B_r} \phi^2 dx \quad \text{and} \quad \sup_{B_1} |\phi| \lesssim 1. \quad (53)$$

Moreover, since  $\phi$  is a linear combination of translations of  $\chi_0$  and  $\chi_1$ , we have that the  $BV$ -norm of  $\phi$  is controlled by the interfacial energy. More precisely, since  $|h|, |h'| \leq r$ , we have

$$\int_{B_r} |\nabla \phi| dx \lesssim E_{interf}(B_1). \quad (54)$$

Finally, we note that Lemma 2 translates into the  $H^{-1}$ -control of  $\phi$  by the elastic energy:

$$\phi = \nabla \cdot j + j_0 \quad \text{in } B_r \quad \text{with} \quad \int_{B_r} (|j|^2 + j_0^2) dx \lesssim E_{elast}(B_1). \quad (55)$$

Lemma 3 now follows from the interpolation estimate (30) in its local version

$$\int_{B_{r'}} \phi^2 dx \lesssim \eta^{1/3} \int_{B_r} |\nabla \phi| dx \sup_{B_r} |\phi| + \eta^{-2/3} \int_{B_r} (|j|^2 + j_0^2) dx, \quad (56)$$

which in view of  $\eta \ll 1$  holds for  $0 < r' \ll r$ . Indeed, in view of the first estimate in (53), the l. h. s. of (31) can be estimated by the l. h. s. of (56). On the other hand, because of the second estimate in (53), and the estimates in (54) and (55), the r. h. s. of (56) can be controlled by the total energy, that is, the r. h. s. of (31).

For the convenience of the reader, we reproduce the easy proof of (56). By rescaling (with the universal radius  $r$ ), we may w. l. o. g. assume that  $r = 1$ , and rename  $r'$  to  $r$ . Let  $f_\epsilon$  denote the convolution of the function  $f$  with a universal Dirac kernel of scale  $\epsilon$ . We recall the convolution estimates for

$\epsilon \ll 1$ :

$$\sup_{B_r} |\phi_\epsilon| \leq \sup_{B_1} |\phi|, \quad (57)$$

$$\int_{B_r} j_{0,\epsilon}^2 dx \leq \int_{B_1} j_0^2 dx, \quad (58)$$

$$\int_{B_r} |\phi_\epsilon - \phi| dx \lesssim \epsilon \int_{B_1} |\nabla \phi| dx, \quad (59)$$

$$\int_{B_r} (\nabla \cdot j_\epsilon)^2 dx \lesssim \epsilon^{-2} \int_{B_1} |j|^2 dx. \quad (60)$$

We also note that the identity in (55) turns into

$$\phi_\epsilon = \nabla \cdot j_\epsilon + j_{0,\epsilon} \text{ in } B_r. \quad (61)$$

We collect these results in

$$\begin{aligned} & \int_{B_r} \phi^2 dx \\ & \lesssim \int_{B_r} (\phi_\epsilon - \phi)^2 dx + \int_{B_r} \phi_\epsilon^2 dx \\ & \stackrel{(61)}{\lesssim} \left( \sup_{B_r} |\phi_\epsilon| + \sup_{B_r} |\phi| \right) \int_{B_r} |\phi_\epsilon - \phi| dx \\ & \quad + \int_{B_r} (\nabla \cdot j_\epsilon)^2 dx + \int_{B_r} j_{0,\epsilon}^2 dx \\ & \stackrel{(57),(59),(60),(58)}{\lesssim} \epsilon \sup_{B_1} |\phi| \int_{B_1} |\nabla \phi| dx + \epsilon^{-2} \int_{B_1} |j|^2 dx + \int_{B_1} j_0^2 dx. \end{aligned}$$

We now see that the choice of  $\epsilon = \eta^{1/3} \leq 1$  yields (56).

PROOF OF COROLLARY 1.

In order to pass from Lemma 3 to Corollary 1, it suffices to argue that for two characteristic functions  $\chi_0, \chi$ , for two vectors  $a, b$  and for two shifts  $h_a, h_b$  one always has

$$|\partial_a^{h_a} \partial_b^{h_b} \chi_0| + |\partial_a^{h_a} \partial_b^{h_b} \chi| \lesssim |\partial_a^{h_a} \partial_b^{h_b} (\chi_0 + 3\chi)|. \quad (62)$$

In fact, this is straightforward: For any  $\{0, 1\}$ -valued function  $\chi$ , its second-order finite differences only can assume the following values

$$\begin{aligned} \partial_a^{h_a} \partial_b^{h_b} \chi(x) &= \chi(x + h_a a + h_b b) - \chi(x + h_a a) \\ &\quad - \chi(x + h_b b) + \chi(x) \in \{-2, -1, 0, 1, 2\}. \end{aligned}$$

In particular, for the r. h. s. of (62), that is for

$$\partial_a^{h_a} \partial_b^{h_b} (\chi_0 + 3\chi) = \partial_a^{h_a} \partial_b^{h_b} \chi_0 + 3\partial_a^{h_a} \partial_b^{h_b} \chi,$$

the first r. h. s. terms can only assume the values  $\{-2, -1, 0, 1, 2\}$ , whereas the second r. h. s. term can only assume the values  $= 3\{-2, -1, 0, 1, 2\} = \{-6, -3, 0, 3, 6\}$ . Hence the sum of both terms vanishes only if both terms vanish individually. Since there is only a finite set of values involved, this qualitative statement implies the quantitative statement (62).

PROOF OF LEMMA 4.

After an affine change of variables, we may assume that  $\{a, b, c\} = \{a^*, b^*, c^*\}$  is the standard basis of  $\mathbb{R}^3$  (the Jacobian cannot degenerate since before transformation, the basis  $\{a, b, c\}$  is chosen from the fixed set  $\{[111], [11\bar{1}], [1\bar{1}1], [\bar{1}11]\}$ ). At the expense of reducing  $r$ , we may replace balls by cubes so that the statement of Lemma 4 follows from

i) There exist functions  $f_1(x_2, x_3), f_2(x_1, x_3)$  such that

$$\int_{(-\frac{1}{2}, \frac{1}{2})^3} |f - f_1(x_2, x_3) - f_2(x_1, x_3)| dx \lesssim \sup_{|h_1|, |h_2| \leq 1} \int_{(-1, 1)^3} |\partial_1^{h_1} \partial_2^{h_2} f| dx, \quad (63)$$

ii) There exist functions  $f_{1^*}(x_1), f_{2^*}(x_2)$  such that

$$\begin{aligned} & \int_{(-\frac{1}{2}, \frac{1}{2})^3} |f - f_{1^*}(x_1) - f_{2^*}(x_2)| dx \\ & \lesssim \sup_{|h_1|, |h_2|, |h_3| \leq 1} \int_{(-1, 1)^3} (|\partial_1^{h_1} \partial_2^{h_2} f| + |\partial_3^{h_3} f|) dx, \end{aligned} \quad (64)$$



We start with (63) and note that

$$\begin{aligned}
& \sup_{|h_1|, |h_2| \leq 1} \int_{(-1,1)^3} |\partial_1^{h_1} \partial_2^{h_2} f| dx \\
&= \sup_{|h_1|, |h_2| \leq 1} \int_{(-1,1)^3} \left| f(x_1 + h_1, x_2 + h_2, x_3) - f(x_1, x_2 + h_2, x_3) \right. \\
&\quad \left. - f(x_1 + h_1, x_2, x_3) + f(x_1, x_2, x_3) \right| dx \\
&\gtrsim \int_{(-1,1)^2} \int_{(-1,1)^3} \left| f(x_1 + h_1, x_2 + h_2, x_3) - f(x_1, x_2 + h_2, x_3) \right. \\
&\quad \left. - f(x_1 + h_1, x_2, x_3) + f(x_1, x_2, x_3) \right| dx dh_1 dh_2 \\
&\geq \int_{(-\frac{1}{2}, \frac{1}{2})^2} \int_{(-\frac{1}{2}, \frac{1}{2})^3} \left| f(x'_1, x'_2, x_3) - f(x_1, x'_2, x_3) \right. \\
&\quad \left. - f(x'_1, x_2, x_3) + f(x_1, x_2, x_3) \right| dx_1 dx_2 dx_3 dx'_1 dx'_2 \tag{65}
\end{aligned}$$

$$\begin{aligned}
&\geq \int_{(-\frac{1}{2}, \frac{1}{2})^3} \left| \int_{(-\frac{1}{2}, \frac{1}{2})^2} f(x'_1, x'_2, x_3) dx'_1 dx'_2 - \int_{(-\frac{1}{2}, \frac{1}{2})} f(x_1, x'_2, x_3) dx'_2 \right. \\
&\quad \left. - \int_{(-\frac{1}{2}, \frac{1}{2})} f(x'_1, x_2, x_3) dx'_1 + f(x_1, x_2, x_3) \right| dx_1 dx_2 dx_3, \tag{66}
\end{aligned}$$

where we have used the change of variables  $(x'_1, x'_2) = (x_1 + h_1, x_2 + h_2)$  in line (65) and Jensen's inequality in line (66). Hence we obtain (63) with

$$\begin{aligned}
f_1(x_2, x_3) &:= \int_{(-\frac{1}{2}, \frac{1}{2})} f(x'_1, x_2, x_3) dx'_1 - \frac{1}{2} \int_{(-\frac{1}{2}, \frac{1}{2})^2} f(x'_1, x'_2, x_3) dx'_1 dx'_2, \\
f_2(x_1, x_3) &:= \int_{(-\frac{1}{2}, \frac{1}{2})} f(x_1, x'_2, x_3) dx'_2 - \frac{1}{2} \int_{(-\frac{1}{2}, \frac{1}{2})^2} f(x'_1, x'_2, x_3) dx'_1 dx'_2 \tag{67}
\end{aligned}$$

We now turn to (64). In view of (63) and the triangle inequality in  $L^1$ , it suffices to show

$$\begin{aligned}
& \int_{(-\frac{1}{2}, \frac{1}{2})^2} |f_1(x_2, x_3) - f_{2^*}(x_2)| dx_2 dx_3 + \int_{(-\frac{1}{2}, \frac{1}{2})^2} |f_2(x_1, x_3) - f_{1^*}(x_1)| dx_1 dx_3 \\
&\lesssim \sup_{|h_3| \leq 1} \int_{(-1,1)^3} |\partial_3^{h_3} f| dx, \tag{68}
\end{aligned}$$

where we define  $f_{1^*}, f_{2^*}$  via averaging in  $x_3 \in (-\frac{1}{2}, \frac{1}{2})$ :

$$f_{1^*}(x_1) := \int_{(-\frac{1}{2}, \frac{1}{2})} f_2(x_1, x_3) dx_3 \quad \text{and} \quad f_{2^*}(x_2) := \int_{(-\frac{1}{2}, \frac{1}{2})} f_1(x_2, x_3) dx_3. \tag{69}$$

Let us show the second half of (68):

$$\begin{aligned}
& \int_{(-\frac{1}{2}, \frac{1}{2})^2} |f_{1^*}(x_1) - f_2(x_1, x_3)| dx_1 dx_3 \\
& \stackrel{(69)}{\leq} \int_{(-\frac{1}{2}, \frac{1}{2})} \int_{(-\frac{1}{2}, \frac{1}{2})^2} |f_2(x_1, x'_3) - f_2(x_1, x_3)| dx_1 dx_3 dx'_3 \\
& \stackrel{(67)}{\leq} \frac{3}{2} \int_{(-\frac{1}{2}, \frac{1}{2})} \int_{(-\frac{1}{2}, \frac{1}{2})^3} |f(x_1, x_2, x'_3) - f(x_1, x_2, x_3)| dx_1 dx_2 dx_3 dx'_3 \\
& \leq \frac{3}{2} \int_{(-1, 1)} \int_{(-1, 1)^3} |f(x_1, x_2, x_3 + h_3) - f(x_1, x_2, x_3)| dx_1 dx_2 dx_3 dh_3 \\
& \leq 3 \sup_{|h_3| \leq 1} \int_{(-1, 1)^3} |f(x_1, x_2, x_3 + h_3) - f(x_1, x_2, x_3)| dx_1 dx_2 dx_3 \\
& = 3 \sup_{|h_3| \leq 1} \int_{(-1, 1)^3} |\partial_3^{h_3} f| dx.
\end{aligned}$$

PROOF OF LEMMA 5.

We first note that by symmetry, it is sufficient to show

$$\sup_{\substack{a, b \in S^2 \\ |h_a|, |h_b| \leq r}} \int_{B_r} |\partial_a^{h_a} \partial_b^{h_b} f_{(011)}| dx \lesssim \sup_{\substack{a, b \in S^2 \\ |h_a|, |h_b| \leq 1}} \int_{B_1} |\partial_a^{h_a} \partial_b^{h_b} f| dx. \quad (70)$$

Next, we note that for a sufficiently small universal  $r > 0$  for all  $|h|, |h'| \leq r$

$$\int_{B_r} |\partial_{[111]}^h \partial_{[\bar{1}\bar{1}\bar{1}]}^{h'} f_{(011)}| dx \leq \int_{B_r} |\partial_{[111]}^h \partial_{[\bar{1}\bar{1}\bar{1}]}^{h'} f| dx \quad (71)$$

(in fact, we have equality). Indeed, we apply the second-order finite differences  $\partial_{[111]}^h \partial_{[\bar{1}\bar{1}\bar{1}]}^{h'}$  to the identity (38) (because of the shifts, the identity on  $B_1$  deteriorates into an identity on the smaller ball  $B_r$ ) and note that  $\partial_{[111]}^h \partial_{[\bar{1}\bar{1}\bar{1}]}^{h'}$  makes the five remaining functions  $f_{(01\bar{1})}, f_{(101)}, f_{(\bar{1}01)}, f_{(110)}, f_{(1\bar{1}0)}$  vanish. Indeed:

- Like its continuous counterpart  $\partial_{[111]}$ , the first-order finite differences  $\partial_{[111]}^h$  make three of the five functions vanish:  $\partial_{[111]}^h f_{(01\bar{1})} = \partial_{[111]}^h f_{(\bar{1}01)} = \partial_{[111]}^h f_{(1\bar{1}0)} = 0$ .
- The operator  $\partial_{[\bar{1}\bar{1}\bar{1}]}^{h'}$  makes three of the five functions vanish:  $\partial_{[\bar{1}\bar{1}\bar{1}]}^{h'} f_{(01\bar{1})} = \partial_{[\bar{1}\bar{1}\bar{1}]}^{h'} f_{(101)} = \partial_{[\bar{1}\bar{1}\bar{1}]}^{h'} f_{(110)} = 0$ . Both cover the remaining five functions.

It remains to integrate over  $B_r$ .

We finally argue how (71) implies (70). Obviously, the r. h. s. of (71) is estimated by the r. h. s. of (70). We turn to the integrand of the l. h. s. of (70) and note that for all vectors  $a, b \in \mathbb{R}$  we have by definition

$$\begin{aligned} \partial_a^{h_a} \partial_b^{h_b} f_{(011)}(x) &= f_{(011)}(x \cdot (011) + h_a a \cdot (011) + h_b b \cdot (011)) \\ &\quad - f_{(011)}(x \cdot (011) + h_a a \cdot (011)) \\ &\quad - f_{(011)}(x \cdot (011) + h_b b \cdot (011)) + f_{(011)}(x \cdot (011)), \end{aligned} \quad (72)$$

where we identify  $f_{(011)}(x) = f_{(011)}(x \cdot (011))$ . In particular, with the choice of  $a = [111]$ ,  $b = [\bar{1}11]$ ,  $h_a = h$ ,  $h' = h_b$ , equation (72) turns into a representation of the integrand of the l. h. s. of (71):

$$\begin{aligned} &\partial_{[111]}^h \partial_{[\bar{1}11]}^{h'} f_{(011)}(x) \\ &= f_{(011)}(x \cdot (011) + 2(h + h')) - f_{(011)}(x \cdot (011) + 2h) \\ &\quad - f_{(011)}(x \cdot (011) + 2h') + f_{(011)}(x \cdot (011)). \end{aligned} \quad (73)$$

From (72) and (73) we see that *any* second-order finite differences  $\partial_a^{h_a} \partial_b^{h_b}$  of the single-variable function  $f_{(011)}$  can be expressed in terms of the particular second-order finite differences  $\partial_{[111]}^h \partial_{[\bar{1}11]}^{h'}$ :

$$\begin{aligned} \partial_a^{h_a} \partial_b^{h_b} f_{(011)} &= \partial_{[111]}^h \partial_{[\bar{1}11]}^{h'} f_{(011)} \\ \text{for the choice of } h &= \frac{1}{2} a \cdot (011) h_a \text{ and } h' = \frac{1}{2} b \cdot (011) h_b. \end{aligned}$$

In particular, the l. h. s. of (70) can be expressed in terms of the l. h. s. of (71). This completes the argument that (71) implies (70).

#### PROOF OF PROPOSITION 1.

Let  $0 < r'' \ll r' \ll r \ll 1$  denote generic, but universal radii. We use the short-hand notation introduced before Lemma 4:

- For any vector  $a$  (like  $[111]$ ,  $[11\bar{1}]$ ,  $[1\bar{1}1]$ ,  $[\bar{1}11]$ ),  $f_a$  denotes a function in  $B_r$  that is constant along axes parallel to  $a$ .
- For any dual vector  $a^*$  (like  $(011)$ ,  $(01\bar{1})$ ,  $(101)$ ,  $(\bar{1}01)$ ,  $(110)$ ,  $(1\bar{1}0)$ ),  $f_{a^*}$  denotes a function in  $B_r$  that is constant along planes perpendicular to  $a^*$ .

Let us denote by  $E$  the quantity controlled by the rescaled energy according to Corollary 1, that is,

$$\begin{aligned} E &:= \max_{(a,b)} \sup_{|h_a|, |h_b| \leq r} \int_{B_r} \left( |\partial_a^{h_a} \partial_b^{h_b} \chi_0| + |\partial_a^{h_a} \partial_b^{h_b} \chi_1| \right. \\ &\quad \left. + |\partial_a^{h_a} \partial_b^{h_b} \chi_2| + |\partial_a^{h_a} \partial_b^{h_b} \chi_3| \right) dx, \end{aligned} \quad (74)$$

where the maximum is taken over the six pairs of axes

$$(a, b) \in \{([111], [\bar{1}\bar{1}\bar{1}]), ([11\bar{1}], [1\bar{1}1]), ([111], [1\bar{1}\bar{1}]), ([11\bar{1}], [\bar{1}11]), ([111], [11\bar{1}]), ([\bar{1}\bar{1}\bar{1}], [\bar{1}11])\}.$$

It suffices to show there exist single variable functions  $f_{(110)}$ ,  $f_{(01\bar{1})}$ ,  $f_{(101)}$ ,  $f_{(\bar{1}01)}$ ,  $f_{(110)}$ ,  $f_{(1\bar{1}0)}$ , such that we have for the Austenite

$$\sup_{\substack{a, b \in S^2 \\ |h_a|, |h_b| \leq r'}} \int_{B_{r'}} |\partial_a^{h_a} \partial_b^{h_b} \chi_0| \lesssim E, \quad (75)$$

and for the Martensite

$$\begin{aligned} & \sup_{\substack{a, b \in S^2 \\ |h_a|, |h_b| \leq r'}} \int_{B_{r'}} |\partial_a^{h_a} \partial_b^{h_b} (\chi_1 + f_{(101)} + f_{(\bar{1}01)} - f_{(110)} - f_{(1\bar{1}0)})| dx \\ & + \sup_{\substack{a, b \in S^2 \\ |h_a|, |h_b| \leq r'}} \int_{B_{r'}} |\partial_a^{h_a} \partial_b^{h_b} (\chi_2 - f_{(011)} - f_{(01\bar{1})} + f_{(110)} + f_{(1\bar{1}0)})| dx \\ & + \sup_{\substack{a, b \in S^2 \\ |h_a|, |h_b| \leq r'}} \int_{B_{r'}} |\partial_a^{h_a} \partial_b^{h_b} (\chi_3 + f_{(011)} + f_{(01\bar{1})} - f_{(101)} - f_{(\bar{1}01)})| dx \\ & \lesssim E. \end{aligned} \quad (76)$$

We first establish the Austenitic result (75), which amounts to the statement that *all* second-order finite differences can be controlled by our *six* mixed second-order finite differences (note that 6 is the minimal number for such a result, since it is the dimension of the space of symmetric  $3 \times 3$ -tensors). To that purpose, we need the auxiliary result that all *first-order* finite differences can be controlled by *three* first-order finite differences. More precisely: For any basis  $\{a, b, c\}$  formed out of the four vectors  $[111]$ ,  $[11\bar{1}]$ ,  $[1\bar{1}\bar{1}]$ , and  $[\bar{1}\bar{1}\bar{1}]$ , and for any function  $f$  on  $B_r$ , we have

$$\sup_{\substack{d \in S^2 \\ |h| \leq r'}} \int_{B_{r'}} |\partial_d^h f| dx \lesssim \sup_{|h| \leq r} \int_{B_r} (|\partial_a^h f| + |\partial_b^h f| + |\partial_c^h f|) dx. \quad (77)$$

The auxiliary result is easy to establish: Let a direction  $d \in S^2$  and a shift with  $|h| \leq r'$  be given. Since  $\{a, b, c\}$  is a basis, there exist scalars  $\alpha, \beta, \gamma$  such that that  $d$  can be written as a linear combination:

$$d = \alpha a + \beta b + \gamma c. \quad (78)$$

Since  $\{a, b, c\}$  is one of 4 possible bases, the coefficients cannot blow up:

$$|\alpha|, |\beta|, |\gamma| \lesssim 1. \quad (79)$$

We first note that (78) implies

$$\begin{aligned} & (\partial_d^h f)(x) \\ &= f(x + hd) - f(x) \\ &= (f(x + \alpha ha + \beta hb + \gamma hc) - f(x + \alpha ha + \beta hb)) \\ &\quad + (f(x + \alpha ha + \beta hb) - f(x + \alpha ha)) + (f(x + \alpha ha) - f(x)) \\ &= (\partial_c^{\gamma h} f)(x + \alpha ha + \beta hb) + (\partial_b^{\beta h} f)(x + \alpha ha) + (\partial_a^{\alpha h} f)(x). \end{aligned} \quad (80)$$

We then note that because of (79), the formula (80) yields the estimate (77).

We now can establish the second-order result (75) by applying the first-order result (77) *twice*. Let the two directions  $a, b \in S^2$  and the two shifts with  $|h_a|, |h_b| \leq r''$  be given. We apply (77) to  $\{a, b, c\}$  replaced by  $\{[11\bar{1}], [1\bar{1}1], [\bar{1}11]\}$ , to  $f = \partial_b^{h_b} \chi_0$ , to  $d$  replaced by  $a$  and  $h$  replaced by  $h_a$ :

$$\begin{aligned} & \int_{B_{r''}} |\partial_a^{h_a} \partial_b^{h_b} \chi_0| dx \\ & \lesssim \sup_{|h| \leq r'} \int_{B_{r'}} \left( |\partial_{[11\bar{1}]}^h \partial_b^{h_b} \chi_0| + |\partial_{[1\bar{1}1]}^h \partial_b^{h_b} \chi_0| + |\partial_{[\bar{1}11]}^h \partial_b^{h_b} \chi_0| \right) dx \\ & = \sup_{|h| \leq r'} \int_{B_{r'}} \left( |\partial_b^{h_b} \partial_{[11\bar{1}]}^h \chi_0| + |\partial_b^{h_b} \partial_{[1\bar{1}1]}^h \chi_0| + |\partial_b^{h_b} \partial_{[\bar{1}11]}^h \chi_0| \right) dx. \end{aligned} \quad (81)$$

Let us treat the first term on the r. h. s. of (81) (the two others are treated likewise). We apply (77) to  $\{a, b, c\}$  replaced by  $\{[111], [1\bar{1}1], [\bar{1}11]\}$ , to  $f = \partial_{[11\bar{1}]}^h \chi_0$ , to  $d$  replaced by  $b$  and  $h$  replaced by  $h_b$ :

$$\begin{aligned} & \int_{B_{r'}} |\partial_b^{h_b} \partial_{[11\bar{1}]}^h \chi_0| dx \\ & \lesssim \sup_{|h'| \leq r} \int_{B_r} \left( |\partial_{[111]}^{h'} \partial_{[11\bar{1}]}^h \chi_0| + |\partial_{[1\bar{1}1]}^{h'} \partial_{[11\bar{1}]}^h \chi_0| + |\partial_{[\bar{1}11]}^{h'} \partial_{[11\bar{1}]}^h \chi_0| \right) dx. \end{aligned} \quad (82)$$

Inserting (82) into (81) yields (75).

We now turn to the Martensitic result (76). To curtail notation, we introduce two equivalence relations:

- For two functions  $f$  and  $g$ , we write

$$f \approx_{0,E} g,$$

provided for some  $r > 0$ , the  $L^1(B_r)$  norm of  $f - g$  is estimated by  $E$  (as usual, with a *universal* multiplicative constant):

$$\int_{B_r} |f - g| dx \lesssim E.$$

- For two functions  $f$  and  $g$ , we write

$$f \approx_{2,E} g,$$

provided for some  $r > 0$ , all second-order finite differences of  $f - g$  in the  $L^1(B_r)$ -norm are estimated by  $E$

$$\sup_{\substack{a,b \in S^2 \\ |h_a|, |h_b| \leq r}} \int_{B_r} |\partial_a^{h_a} \partial_b^{h_b} (f - g)| dx \lesssim E$$

Morally speaking,  $f \approx_{2,E} g$  means that  $f$  and  $g$  are  $L^1$ -close up to affine functions.

In this language, the statement of Proposition 1 takes on the compact form of

$$\begin{aligned} \chi_0 &\approx_{2,E} 0, \\ \chi_1 &\approx_{2,E} -f_{(101)} - f_{(\bar{1}01)} + f_{(110)} + f_{(\bar{1}\bar{1}0)}, \\ \chi_2 &\approx_{2,E} f_{(011)} + f_{(01\bar{1})} - f_{(110)} - f_{(\bar{1}\bar{1}0)}, \\ \chi_3 &\approx_{2,E} -f_{(011)} - f_{(01\bar{1})} + f_{(101)} + f_{(\bar{1}01)}. \end{aligned} \quad (83)$$

We will use the just established Austenitic result in form of

$$\chi_1 + \chi_2 + \chi_3 \stackrel{(7)}{=} 1 - \chi_0 \approx_{2,E} 0. \quad (84)$$

We start by applying Lemma 4 i) to  $f$ ,  $a$ , and  $b$  replaced by

$$\begin{aligned} \chi_1, \quad a &= [111], \quad b = [\bar{1}11], \\ \chi_1, \quad a &= [11\bar{1}], \quad b = [1\bar{1}1], \\ \chi_2, \quad a &= [111], \quad b = [1\bar{1}\bar{1}], \\ \chi_2, \quad a &= [11\bar{1}], \quad b = [\bar{1}\bar{1}1], \\ \chi_3, \quad a &= [111], \quad b = [11\bar{1}], \\ \chi_3, \quad a &= [1\bar{1}\bar{1}], \quad b = [\bar{1}11]. \end{aligned}$$

This yields  $2 \times 3 = 6$  relations

$$\begin{aligned} \chi_1 &\approx_{0,E} f_{1,[111]} + f_{1,[\bar{1}11]}, \\ \chi_1 &\approx_{0,E} f_{1,[11\bar{1}]} + f_{1,[1\bar{1}1]}, \\ \chi_2 &\approx_{0,E} f_{2,[111]} + f_{2,[1\bar{1}\bar{1}]}, \\ \chi_2 &\approx_{0,E} f_{2,[11\bar{1}]} + f_{2,[\bar{1}\bar{1}1]}, \\ \chi_3 &\approx_{0,E} f_{3,[111]} + f_{3,[11\bar{1}]}, \\ \chi_3 &\approx_{0,E} f_{3,[1\bar{1}\bar{1}]} + f_{3,[\bar{1}11]}. \end{aligned} \quad (85)$$

We now insert these  $2 \times 3 = 6$  relations into the Austenitic result (84), giving us potentially  $2^3 = 8$  new relations. But we are only interested in those new relations where the functions  $f_{\cdot}$  involved depend only on three out of the 4 possible axes  $[111]$ ,  $[\bar{1}\bar{1}\bar{1}]$ ,  $[1\bar{1}\bar{1}]$ , and  $[11\bar{1}]$ . There are 4 of these relevant relations, which can be thought of being parameterized by the axis they *omit*. In the following enumeration, the first relation omits  $[111]$ , the second omits  $[\bar{1}\bar{1}\bar{1}]$ , the third omits  $[1\bar{1}\bar{1}]$ , and the fourth omits  $[11\bar{1}]$ :

$$\begin{aligned}
0 &\approx_{2,E} (f_{1,[11\bar{1}]} + f_{1,[\bar{1}\bar{1}\bar{1}]}) + (f_{2,[11\bar{1}]} + f_{2,[\bar{1}\bar{1}\bar{1}]}) + (f_{3,[1\bar{1}\bar{1}]} + f_{3,[\bar{1}\bar{1}\bar{1}]}) \\
&= (f_{1,[11\bar{1}]} + f_{2,[11\bar{1}]} + (f_{3,[1\bar{1}\bar{1}]} + f_{1,[\bar{1}\bar{1}\bar{1}]}) + (f_{2,[\bar{1}\bar{1}\bar{1}]} + f_{3,[\bar{1}\bar{1}\bar{1}]}), \\
0 &\approx_{2,E} (f_{1,[11\bar{1}]} + f_{1,[\bar{1}\bar{1}\bar{1}]}) + (f_{2,[111]} + f_{2,[1\bar{1}\bar{1}]}) + (f_{3,[111]} + f_{3,[1\bar{1}\bar{1}]}) \\
&= (f_{2,[111]} + f_{3,[111]} + (f_{3,[11\bar{1}]} + f_{1,[1\bar{1}\bar{1}]}) + (f_{1,[\bar{1}\bar{1}\bar{1}]} + f_{2,[\bar{1}\bar{1}\bar{1}]}), \\
0 &\approx_{2,E} (f_{1,[111]} + f_{1,[\bar{1}\bar{1}\bar{1}]}) + (f_{2,[11\bar{1}]} + f_{2,[\bar{1}\bar{1}\bar{1}]}) + (f_{3,[111]} + f_{3,[1\bar{1}\bar{1}]}) \\
&= (f_{3,[111]} + f_{1,[111]} + (f_{2,[11\bar{1}]} + f_{3,[1\bar{1}\bar{1}]}) + (f_{1,[\bar{1}\bar{1}\bar{1}]} + f_{2,[\bar{1}\bar{1}\bar{1}]}), \\
0 &\approx_{2,E} (f_{1,[111]} + f_{1,[\bar{1}\bar{1}\bar{1}]}) + (f_{2,[111]} + f_{2,[1\bar{1}\bar{1}]}) + (f_{3,[1\bar{1}\bar{1}]} + f_{3,[\bar{1}\bar{1}\bar{1}]}) \\
&= (f_{1,[111]} + f_{2,[111]} + (f_{2,[1\bar{1}\bar{1}]} + f_{3,[1\bar{1}\bar{1}]}) + (f_{3,[\bar{1}\bar{1}\bar{1}]} + f_{1,[\bar{1}\bar{1}\bar{1}]}).
\end{aligned} \tag{86}$$

We now probe  $\approx_{2,E}$  in (86) by our second-order finite differences  $\partial_a^h \partial_b^{h'}$  with  $(a, b)$  one of the 6 pairs  $([111], [\bar{1}\bar{1}\bar{1}])$ ,  $([11\bar{1}], [1\bar{1}\bar{1}])$ ,  $([111], [1\bar{1}\bar{1}])$ ,  $([11\bar{1}], [\bar{1}\bar{1}\bar{1}])$ ,  $([111], [11\bar{1}])$ ,  $([1\bar{1}\bar{1}], [\bar{1}\bar{1}\bar{1}])$ . Since there are 4 relations and 6 operators this potentially yields  $6 \times 4 = 24$  new relations. However, of interest are only those relations which arise when  $\partial_a^h \partial_b^{h'}$  falls onto one of those 4 relation in (86) where the omitted axis is neither  $a$  nor  $b$ , since in this case, two out of three of the parentheses on the r. h. s. of (86) vanish. These leaves  $3 \times 4 = 12$

relations of interest:

$$\begin{aligned}
\partial_{[\bar{1}\bar{1}\bar{1}]}^h \partial_{[\bar{1}\bar{1}\bar{1}]}^{h'} (f_{1,[11\bar{1}]} + f_{2,[11\bar{1}]}) &\approx_{0,E} 0, \\
\partial_{[11\bar{1}]}^h \partial_{[11\bar{1}]}^{h'} (f_{3,[1\bar{1}\bar{1}]} + f_{1,[1\bar{1}\bar{1}]}) &\approx_{0,E} 0, \\
\partial_{[11\bar{1}]}^h \partial_{[11\bar{1}]}^{h'} (f_{2,[\bar{1}\bar{1}\bar{1}]} + f_{3,[\bar{1}\bar{1}\bar{1}]}) &\approx_{0,E} 0, \\
\partial_{[11\bar{1}]}^h \partial_{[11\bar{1}]}^{h'} (f_{2,[111]} + f_{3,[111]}) &\approx_{0,E} 0, \\
\partial_{[111]}^h \partial_{[111]}^{h'} (f_{3,[11\bar{1}]} + f_{1,[11\bar{1}]}) &\approx_{0,E} 0, \\
\partial_{[111]}^h \partial_{[111]}^{h'} (f_{1,[1\bar{1}\bar{1}]} + f_{2,[1\bar{1}\bar{1}]}) &\approx_{0,E} 0, \\
\partial_{[11\bar{1}]}^h \partial_{[11\bar{1}]}^{h'} (f_{3,[111]} + f_{1,[111]}) &\approx_{0,E} 0, \\
\partial_{[111]}^h \partial_{[111]}^{h'} (f_{2,[11\bar{1}]} + f_{3,[11\bar{1}]}) &\approx_{0,E} 0, \\
\partial_{[111]}^h \partial_{[111]}^{h'} (f_{1,[\bar{1}\bar{1}\bar{1}]} + f_{2,[\bar{1}\bar{1}\bar{1}]}) &\approx_{0,E} 0, \\
\partial_{[\bar{1}\bar{1}\bar{1}]}^h \partial_{[\bar{1}\bar{1}\bar{1}]}^{h'} (f_{1,[111]} + f_{2,[111]}) &\approx_{0,E} 0, \\
\partial_{[111]}^h \partial_{[111]}^{h'} (f_{2,[1\bar{1}\bar{1}]} + f_{3,[1\bar{1}\bar{1}]}) &\approx_{0,E} 0, \\
\partial_{[111]}^h \partial_{[111]}^{h'} (f_{3,[\bar{1}\bar{1}\bar{1}]} + f_{1,[\bar{1}\bar{1}\bar{1}]}) &\approx_{0,E} 0.
\end{aligned}$$

An application of Lemma 4 ii) to  $f$ ,  $a$ ,  $b$ , and  $c$  replaced by

$$\begin{aligned}
&f_{1,[11\bar{1}]} + f_{2,[11\bar{1}]}, \quad [1\bar{1}\bar{1}], \quad [\bar{1}\bar{1}\bar{1}], \quad [11\bar{1}], \\
&f_{3,[1\bar{1}\bar{1}]} + f_{1,[1\bar{1}\bar{1}]}, \quad [11\bar{1}], \quad [\bar{1}\bar{1}\bar{1}], \quad [1\bar{1}\bar{1}], \\
&f_{2,[\bar{1}\bar{1}\bar{1}]} + f_{3,[\bar{1}\bar{1}\bar{1}]}, \quad [11\bar{1}], \quad [1\bar{1}\bar{1}], \quad [\bar{1}\bar{1}\bar{1}], \\
&f_{2,[111]} + f_{3,[111]}, \quad [11\bar{1}], \quad [1\bar{1}\bar{1}], \quad [111], \\
&f_{3,[11\bar{1}]} + f_{1,[11\bar{1}]}, \quad [111], \quad [1\bar{1}\bar{1}], \quad [11\bar{1}], \\
&f_{1,[1\bar{1}\bar{1}]} + f_{2,[1\bar{1}\bar{1}]}, \quad [111], \quad [11\bar{1}], \quad [1\bar{1}\bar{1}], \\
&f_{3,[111]} + f_{1,[111]}, \quad [11\bar{1}], \quad [\bar{1}\bar{1}\bar{1}], \quad [111], \\
&f_{2,[11\bar{1}]} + f_{3,[11\bar{1}]}, \quad [111], \quad [\bar{1}\bar{1}\bar{1}], \quad [11\bar{1}], \\
&f_{1,[\bar{1}\bar{1}\bar{1}]} + f_{2,[\bar{1}\bar{1}\bar{1}]}, \quad [111], \quad [11\bar{1}], \quad [\bar{1}\bar{1}\bar{1}], \\
&f_{1,[111]} + f_{2,[111]}, \quad [1\bar{1}\bar{1}], \quad [\bar{1}\bar{1}\bar{1}], \quad [111], \\
&f_{2,[1\bar{1}\bar{1}]} + f_{3,[1\bar{1}\bar{1}]}, \quad [111], \quad [\bar{1}\bar{1}\bar{1}], \quad [1\bar{1}\bar{1}], \\
&f_{3,[\bar{1}\bar{1}\bar{1}]} + f_{1,[\bar{1}\bar{1}\bar{1}]}, \quad [111], \quad [1\bar{1}\bar{1}], \quad [\bar{1}\bar{1}\bar{1}]
\end{aligned}$$



yields the 12 relations

$$\begin{aligned}
f_{1,[11\bar{1}]} + f_{2,[11\bar{1}]} &\approx_{0,E} g_{(011)} + g_{(101)}, \\
f_{3,[1\bar{1}1]} + f_{1,[1\bar{1}1]} &\approx_{0,E} g_{(011)} + g_{(110)}, \\
f_{2,[\bar{1}11]} + f_{3,[\bar{1}11]} &\approx_{0,E} g_{(101)} + g_{(110)}, \\
f_{2,[111]} + f_{3,[111]} &\approx_{0,E} g_{(1\bar{1}0)} + g_{(\bar{1}01)}, \\
f_{3,[11\bar{1}]} + f_{1,[11\bar{1}]} &\approx_{0,E} g_{(1\bar{1}0)} + g_{(011)}, \\
f_{1,[1\bar{1}1]} + f_{2,[1\bar{1}1]} &\approx_{0,E} g_{(\bar{1}01)} + g_{(011)}, \\
f_{3,[111]} + f_{1,[111]} &\approx_{0,E} g_{(1\bar{1}0)} + g_{(01\bar{1})}, \\
f_{2,[11\bar{1}]} + f_{3,[11\bar{1}]} &\approx_{0,E} g_{(1\bar{1}0)} + g_{(101)}, \\
f_{1,[\bar{1}11]} + f_{2,[\bar{1}11]} &\approx_{0,E} g_{(01\bar{1})} + g_{(101)}, \\
f_{1,[111]} + f_{2,[111]} &\approx_{0,E} g_{(\bar{1}01)} + g_{(01\bar{1})}, \\
f_{2,[1\bar{1}1]} + f_{3,[1\bar{1}1]} &\approx_{0,E} g_{(\bar{1}01)} + g_{(110)}, \\
f_{3,[\bar{1}11]} + f_{1,[\bar{1}11]} &\approx_{0,E} g_{(01\bar{1})} + g_{(110)}.
\end{aligned}$$

Here and till the end of this proof, the symbol  $g_{a^*}$  will denote a *generic* function that is constant along the planes orthogonal to the dual vector  $a^*$ . We now consider all 4 triplets  $\{f_{1,a}, f_{2,a}, f_{3,a}\}$  with one of the 4 axis  $a \in \{[111], [11\bar{1}], [1\bar{1}1], [\bar{1}11]\}$  *fixed*. In the above relations, the elements of a given triplet appear in the 3 forms of  $f_{1,a} + f_{2,a}$ ,  $f_{2,a} + f_{3,a}$ ,  $f_{3,a} + f_{1,a}$ . Since the  $3 \times 3$ -matrix

$$\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
1 & 0 & 1
\end{pmatrix}$$

is non-singular, the above  $3 \times 4$  relations yield new  $3 \times 4$  relations. These we

order by the axes  $a$  in the usual order  $[111]$ ,  $[11\bar{1}]$ ,  $[\bar{1}11]$ ,  $[\bar{1}\bar{1}1]$ :

$$\begin{aligned}
f_{1,[111]} &\approx_{0,E} g_{(01\bar{1})} + g_{(\bar{1}01)} + g_{(1\bar{1}0)}, \\
f_{2,[111]} &\approx_{0,E} g_{(01\bar{1})} + g_{(\bar{1}01)} + g_{(1\bar{1}0)}, \\
f_{3,[111]} &\approx_{0,E} g_{(01\bar{1})} + g_{(\bar{1}01)} + g_{(1\bar{1}0)}, \\
f_{1,[11\bar{1}]} &\approx_{0,E} g_{(011)} + g_{(101)} + g_{(1\bar{1}0)}, \\
f_{2,[11\bar{1}]} &\approx_{0,E} g_{(011)} + g_{(101)} + g_{(1\bar{1}0)}, \\
f_{3,[11\bar{1}]} &\approx_{0,E} g_{(011)} + g_{(101)} + g_{(1\bar{1}0)}, \\
f_{1,[\bar{1}11]} &\approx_{0,E} g_{(011)} + g_{(\bar{1}01)} + g_{(110)}, \\
f_{2,[\bar{1}11]} &\approx_{0,E} g_{(011)} + g_{(\bar{1}01)} + g_{(110)}, \\
f_{3,[\bar{1}11]} &\approx_{0,E} g_{(011)} + g_{(\bar{1}01)} + g_{(110)}, \\
f_{1,[\bar{1}\bar{1}1]} &\approx_{0,E} g_{(01\bar{1})} + g_{(101)} + g_{(110)}, \\
f_{2,[\bar{1}\bar{1}1]} &\approx_{0,E} g_{(01\bar{1})} + g_{(101)} + g_{(110)}, \\
f_{3,[\bar{1}\bar{1}1]} &\approx_{0,E} g_{(01\bar{1})} + g_{(101)} + g_{(110)}.
\end{aligned} \tag{87}$$

After this “unfolding” to 12 relations, we “contract” to 6 relations by inserting (87) into (85). We stick to our notational convention that  $g_{a^*}$  denotes a generic function, whereas  $f_{a^*}$  denotes a function we — momentarily — want to give a specific name:

$$\begin{aligned}
\chi_1 &\approx_{0,E} f_{(01\bar{1})} + g_{(101)} + g_{(\bar{1}01)} + g_{(110)} + g_{(1\bar{1}0)}, \\
\chi_1 &\approx_{0,E} f_{(011)} + g_{(101)} + g_{(\bar{1}01)} + g_{(110)} + g_{(1\bar{1}0)}, \\
\chi_2 &\approx_{0,E} g_{(011)} + g_{(01\bar{1})} + f_{(\bar{1}01)} + g_{(110)} + g_{(1\bar{1}0)}, \\
\chi_2 &\approx_{0,E} g_{(011)} + g_{(01\bar{1})} + f_{(101)} + g_{(110)} + g_{(1\bar{1}0)}, \\
\chi_3 &\approx_{0,E} g_{(011)} + g_{(01\bar{1})} + g_{(101)} + g_{(\bar{1}01)} + f_{(1\bar{1}0)}, \\
\chi_3 &\approx_{0,E} g_{(011)} + g_{(01\bar{1})} + g_{(101)} + g_{(\bar{1}01)} + f_{(110)}.
\end{aligned} \tag{88}$$

Contracting once more from 6 to 3 by taking the difference of an even and the following odd row in (88) we obtain the 3 relations

$$\begin{aligned}
0 &\approx_{0,E} f_{(011)} - f_{(01\bar{1})} + g_{(101)} + g_{(\bar{1}01)} + g_{(110)} + g_{(1\bar{1}0)}, \\
0 &\approx_{0,E} g_{(011)} + g_{(01\bar{1})} + f_{(101)} - f_{(\bar{1}01)} + g_{(110)} + g_{(1\bar{1}0)}, \\
0 &\approx_{0,E} g_{(011)} + g_{(01\bar{1})} + g_{(101)} + g_{(\bar{1}01)} + f_{(110)} - f_{(1\bar{1}0)}.
\end{aligned}$$

According to the approximate uniqueness statement modulo affine functions in Lemma 5, this yields for our “named” functions (we are interested only in

these)

$$\begin{aligned} f_{(011)} &\approx_{2,E} f_{(01\bar{1})} \approx_{2,E} 0, \\ f_{(101)} &\approx_{2,E} f_{(\bar{1}01)} \approx_{2,E} 0, \\ f_{(110)} &\approx_{2,E} f_{(1\bar{1}0)} \approx_{2,E} 0. \end{aligned}$$

With this information, (88) can be updated to

$$\begin{aligned} \chi_1 &\approx_{2,E} f_{1,(101)} + f_{1,(\bar{1}01)} + f_{1,(110)} + f_{1,(1\bar{1}0)}, \\ \chi_2 &\approx_{2,E} f_{2,(011)} + f_{2,(01\bar{1})} + f_{2,(110)} + f_{2,(1\bar{1}0)}, \\ \chi_3 &\approx_{2,E} f_{3,(011)} + f_{3,(01\bar{1})} + f_{3,(101)} + f_{3,(\bar{1}01)}, \end{aligned} \quad (89)$$

where we gave the remaining functions specific names.

Using once more the Austenitic result (84), we obtain by contracting (89) the following relation

$$\begin{aligned} 0 &\approx_{2,E} f_{1,(101)} + f_{1,(\bar{1}01)} + f_{1,(110)} + f_{1,(1\bar{1}0)} \\ &\quad + f_{2,(011)} + f_{2,(01\bar{1})} + f_{2,(110)} + f_{2,(1\bar{1}0)} \\ &\quad + f_{3,(011)} + f_{3,(01\bar{1})} + f_{3,(101)} + f_{3,(\bar{1}01)}. \end{aligned} \quad (90)$$

Applying once again Lemma 5, we obtain the six relations

$$\begin{aligned} 0 &\approx_{2,E} f_{2,(011)} + f_{3,(011)}, \\ 0 &\approx_{2,E} f_{2,(01\bar{1})} + f_{3,(01\bar{1})}, \\ 0 &\approx_{2,E} f_{1,(101)} + f_{3,(101)}, \\ 0 &\approx_{2,E} f_{1,(\bar{1}01)} + f_{3,(\bar{1}01)}, \\ 0 &\approx_{2,E} f_{1,(110)} + f_{2,(110)}, \\ 0 &\approx_{2,E} f_{1,(1\bar{1}0)} + f_{2,(1\bar{1}0)}. \end{aligned}$$

Inserting this into (89) yields the Martensitic part of (83).

PROOF OF LEMMA 6.

We first establish Lemma 6 in a version without the affine function, that is, in the case where there are no *second*-order difference quotients: For two functions  $g_1^*$  and  $g_2^*$  only depending on  $x_1$  and  $x_2$ , respectively, we claim

$$\min \left\{ \sup_{|h| \leq \frac{1}{2}} \int_{(-\frac{1}{2}, \frac{1}{2})} |\partial_1^h g_1^*| dx_1, \sup_{|h| \leq \frac{1}{2}} \int_{(-\frac{1}{2}, \frac{1}{2})} |\partial_2^h g_2^*| dx_2 \right\}$$

$$\lesssim \left( \int_{(-1,1)^2} |\chi - g_{1^*} - g_{2^*}| dx_1 dx_2 \right)^{1/2}, \quad (91)$$

under the assumption that the r. h. s. of (91) is  $\leq 1$ . We start by arguing that we may assume in addition that there exist values  $a_1, a_2$  such that

$$g_{1^*} \in \{a_1, a_1 + 1\}, \quad \text{and} \quad g_{2^*} \in \{a_2, a_2 + 1\}. \quad (92)$$

Let us give the argument for the first half of (92) (the argument for the second half is the same). Since there always exists a point where a function is below its average, there exists an  $x_2^* \in (-1, 1)$  such that

$$\begin{aligned} & \int_{(-1,1)} |\chi(x_1, x_2^*) - g_{1^*}(x_1) - g_{2^*}(x_2^*)| dx_1 \\ & \leq \frac{1}{2} \int_{(-1,1)^2} |\chi(x_1, x_2) - g_{1^*}(x_1) - g_{2^*}(x_2)| dx_1 dx_2. \end{aligned}$$

Hence the  $L^1((-1, 1))$ -distance of  $g_{1^*}$  to the function

$$\tilde{g}_{1^*}(x_1) := \chi(x_1, x_2^*) - g_{2^*}(x_2^*)$$

is controlled by the r. h. s. of (91) (since it is at most 1, the square root does not matter here). Clearly, the  $g_{1^*}$ -term on l. h. s. of (91) is Lipschitz continuous w. r. t. the  $L^1((-1, 1))$ -distance. So  $g_{1^*}$  may indeed be replaced by  $\tilde{g}_{1^*}$ , which because of  $\chi \in \{0, 1\}$  only assumes the two values  $-g_{2^*}(x_2^*)$  and  $1 - g_{2^*}(x_2^*)$ , so that the first part of (92) can be assumed to hold (with  $a_1 = -g_{2^*}(x_2^*)$ ).

We now turn to the proof of (91) proper. We introduce the volume fractions

$$\begin{aligned} \lambda_1 & := \frac{1}{2} \mathcal{L}^1(\{x_1 \in (-1, 1) | g_{1^*}(x_1) = a_1\}), \\ \lambda_2 & := \frac{1}{2} \mathcal{L}^1(\{x_2 \in (-1, 1) | g_{2^*}(x_2) = a_2\}) \end{aligned} \quad (93)$$

and observe that because  $\chi \in \{0, 1\}$

$$\begin{aligned} \epsilon & := \int_{(-1,1)^2} |\chi - g_{1^*} - g_{2^*}| dx_1 dx_2 \\ & \stackrel{(93),(92)}{\geq} 4\lambda_1 \lambda_2 \text{dist}(a_1 + a_2, \{0, 1\}) \\ & \quad + 4(1 - \lambda_1)(1 - \lambda_2) \text{dist}(a_1 + a_2 + 2, \{0, 1\}). \end{aligned} \quad (94)$$

Since the two factors  $\text{dist}(a_1 + a_2, \{0, 1\})$  and  $\text{dist}(a_1 + a_2 + 2, \{0, 1\})$  cannot *both* be small in view of

$$\begin{aligned} & \max \{ \text{dist}(a_1 + a_2, \{0, 1\}), \text{dist}(a_1 + a_2 + 2, \{0, 1\}) \} \\ &= \max \{ \min \{ |a_1 + a_2|, |a_1 + a_2 - 1| \}, \min \{ |a_1 + a_2 + 2|, |a_1 + a_2 + 1| \} \} \\ &\geq \min \{ \max \{ |a_1 + a_2|, |a_1 + a_2 + 2| \}, \max \{ |a_1 + a_2|, |a_1 + a_2 + 1| \}, \\ &\quad \max \{ |a_1 + a_2 - 1|, |a_1 + a_2 + 2| \}, \max \{ |a_1 + a_2 - 1|, |a_1 + a_2 + 1| \} \} \\ &\geq \frac{1}{2}, \end{aligned}$$

we obtain from (94) one of the two cases

$$\lambda_1 \lambda_2 \leq \frac{\epsilon}{2} \quad \underline{\text{or}} \quad (1 - \lambda_1)(1 - \lambda_2) \leq \frac{\epsilon}{2}.$$

This yields one of the four cases

$$\lambda_1 \leq \sqrt{\frac{\epsilon}{2}} \quad \underline{\text{or}} \quad \lambda_2 \leq \sqrt{\frac{\epsilon}{2}} \quad \underline{\text{or}} \quad 1 - \lambda_1 \leq \sqrt{\frac{\epsilon}{2}} \quad \underline{\text{or}} \quad 1 - \lambda_2 \leq \sqrt{\frac{\epsilon}{2}}.$$

Let us just treat the first case (the argument in the other three cases is the same); in this case we have

$$\int_{(-1,1)} |g_{1^*} - (a_1 + 1)| dx_1 \stackrel{(93),(92)}{=} 2\lambda_1 \leq \sqrt{2\epsilon}.$$

This fact that  $g_{1^*}$  is  $L^1$ -close to a constant yields as desired that the finite differences are small in  $L^1$  on a smaller interval:

$$\sup_{|h| \leq \frac{1}{2}} \int_{(-\frac{1}{2}, \frac{1}{2})} |\partial_1^h g_{1^*}| dx_1 \leq 2\sqrt{2\epsilon}.$$

We now turn to Lemma 6 in its original version (i. e. with affine functions). We shall derive it by post-processing (91). Let  $0 < r' \ll r \ll 1$  be generic, but universal radii. Let  $E$  denote the r. h. s. of (40)

$$\begin{aligned} E := & \sup_{|h|, |h'| \leq \frac{1}{2}} \int_{(-1,1)^2} (|\partial_1^h \partial_1^{h'} (\chi - f_{1^*} - f_{2^*})| + |\partial_1^h \partial_2^{h'} (\chi - f_{1^*} - f_{2^*})| \\ & + |\partial_2^h \partial_2^{h'} (\chi - f_{1^*} - f_{2^*})|) dx_1 dx_2. \end{aligned} \quad (95)$$

Note that since  $\partial_1^h \partial_2^{h'} (f_{1^*} + f_{2^*}) = 0$  we have in particular for the mixed derivatives

$$\sup_{|h|, |h'| \leq \frac{1}{2}} \int_{(-1,1)^2} |\partial_1^h \partial_2^{h'} \chi| dx_1 dx_2 \leq E.$$

Hence by Lemma 4 ii) (to be more precise, we introduce  $x_3$  as a dummy variable and apply Lemma 4 ii) to  $a = [100]$ ,  $b = [010]$ , and  $c = [001]$ ), there exist functions  $g_{1^*}$  and  $g_{2^*}$  of  $x_1$  and  $x_2$ , respectively, such that

$$\int_{(-r,r)^2} |\chi - g_{1^*} - g_{2^*}| dx_1 dx_2 \lesssim E. \quad (96)$$

We now apply (91) and obtain because of (96) and  $E \leq 1$

$$\min \left\{ \sup_{|h| \leq r} \int_{(-r,r)} |\partial_1^h g_{1^*}| dx_1, \sup_{|h| \leq r} \int_{(-r,r)} |\partial_2^h g_{2^*}| dx_2 \right\} \lesssim E^{1/2}.$$

Say that

$$\sup_{|h| \leq r} \int_{(-r,r)} |\partial_1^h g_{1^*}| dx_1 \lesssim E^{1/2}.$$

Hence by (96),  $\partial_1^h g_{2^*} = 0$ , and  $E \leq 1$  we have

$$\sup_{|h| \leq r} \int_{(-r,r)^2} |\partial_1^h \chi| dx \lesssim E^{1/2}. \quad (97)$$

Using  $\partial_1^h \partial_1^{h'} f_{2^*} = 0$  and  $E \leq 1$ , we obtain the desired result:

$$\begin{aligned} & \sup_{|h|, |h'| \leq r'} \int_{(-r',r')} |\partial_1^h \partial_1^{h'} f_{1^*}| dx_1 \\ & \stackrel{(95)}{\lesssim} E + \sup_{|h|, |h'| \leq r'} \int_{(-r',r')^2} |\partial_1^h \partial_1^{h'} \chi| dx \\ & \lesssim E + \sup_{|h| \leq r} \int_{(-r,r)^2} |\partial_1^h \chi| dx \\ & \stackrel{(97)}{\lesssim} E^{1/2}. \end{aligned} \quad (98)$$

PROOF OF LEMMA 7.

Let us drop the subscript 1 of the variable  $x_1$  and the differential  $\partial_1$ .

The main part of Lemma 7 is part i). We first prove an  $L^2$ -version of (41), that is

$$\sup_{|h| \leq \frac{1}{4}} \int_{(-\frac{1}{4}, \frac{1}{4})} |\partial^h f|^2 dx \lesssim (\mathcal{L}^1(S))^{-2} \sup_{h \in S} \int_{(-\frac{1}{2}, \frac{1}{2})} |\partial^h f|^2 dx. \quad (99)$$

The advantage of this  $L^2$ -version is that it is amenable to Fourier series techniques. We first prove (99) under the additional assumption that the set

$S \subset [-1, 1]$  actually reaches as far out as possible, i. e.  $1 \in S$  or  $-1 \in S$ . By symmetry, we may w. l. o. g. assume

$$1 \in S \subset [-1, 1]. \quad (100)$$

We now give the argument for (99) under the assumption (100). Despite the fact that  $f$  is not (exactly) 1-periodic, we introduce its Fourier coefficients on the interval  $(-\frac{1}{2}, \frac{1}{2})$  (which is of length 1):

$$\mathcal{F}f(k) := \int_{(-\frac{1}{2}, \frac{1}{2})} e^{ikx} f(x) dx \quad \text{for } k \in 2\pi\mathbb{Z}.$$

We will re-express both sides of (99) in terms of the Fourier coefficients of  $f$ . We shall first argue that the r. h. s. of (99) can be minorated in terms of the Fourier coefficients. More precisely, for any  $|h| \leq 1$  (hence in particular for  $h \in S$ )

$$\sum_{k \in 2\pi\mathbb{Z}} |(1 - e^{-ikh})\mathcal{F}f(k)|^2 \lesssim \int_{(-\frac{1}{2}, \frac{1}{2})} |\partial^h f|^2 dx + \int_{(-\frac{1}{2}, \frac{1}{2})} |\partial^1 f|^2 dx. \quad (101)$$

Let us give the argument for (101). By symmetry, we may assume  $h \in [0, 1]$ . Momentarily introducing  $f_{per}$ , the 1-periodic extension of  $f|_{(-\frac{1}{2}, \frac{1}{2})}$ , we see that the l. h. s. of (101) can be expressed as

$$\begin{aligned} & \sum_{k \in 2\pi\mathbb{Z}} |(1 - e^{-ikh})\mathcal{F}f(k)|^2 \\ &= \int_{(-\frac{1}{2}, \frac{1}{2})} |f_{per}(x+h) - f_{per}(x)|^2 dx \\ &= \int_{(-\frac{1}{2}, \frac{1}{2}-h)} |f(x+h) - f(x)|^2 dx + \int_{(\frac{1}{2}-h, \frac{1}{2})} |f(x+h-1) - f(x)|^2 dx. \end{aligned}$$

By the triangle inequality in  $L^2((\frac{1}{2}-h, \frac{1}{2}))$  and a change of variables, this

yields as desired (101):

$$\begin{aligned}
& \sum_{k \in 2\pi\mathbb{Z}} |(1 - e^{-ikh})\mathcal{F}f(k)|^2 \\
& \leq \int_{(-\frac{1}{2}, \frac{1}{2}-h)} |f(x+h) - f(x)|^2 dx \\
& \quad + 2 \int_{(\frac{1}{2}-h, \frac{1}{2})} |f(x+h) - f(x)|^2 dx \\
& \quad + 2 \int_{(\frac{1}{2}-h, \frac{1}{2})} |f(x+h) - f(x+h-1)|^2 dx \\
& = \int_{(-\frac{1}{2}, \frac{1}{2}-h)} |f(x+h) - f(x)|^2 dx \\
& \quad + 2 \int_{(\frac{1}{2}-h, \frac{1}{2})} |f(x+h) - f(x)|^2 dx \\
& \quad + 2 \int_{(-\frac{1}{2}, -\frac{1}{2}+h)} |f(x+1) - f(x)|^2 dx \\
& \leq 2 \int_{(-\frac{1}{2}, \frac{1}{2})} |f(x+h) - f(x)|^2 dx + 2 \int_{(-\frac{1}{2}, \frac{1}{2})} |f(x+1) - f(x)|^2 dx.
\end{aligned}$$

We now argue that also the l. h. s. of (99) can be majorated in terms of the Fourier coefficients of  $f$ . Indeed, we have for all  $|h| \leq \frac{1}{4}$

$$\begin{aligned}
\int_{(-\frac{1}{4}, \frac{1}{4})} |\partial^h f|^2 dx & = \int_{(-\frac{1}{4}, \frac{1}{4})} |f(x+h) - f(x)|^2 dx \\
& \leq 2 \int_{(-\frac{1}{4}, \frac{1}{4})} |f(x+h) - \int_{(-\frac{1}{2}, \frac{1}{2})} f(x') dx'|^2 dx \\
& \quad + 2 \int_{(-\frac{1}{4}, \frac{1}{4})} |f(x) - \int_{(-\frac{1}{2}, \frac{1}{2})} f(x') dx'|^2 dx \\
& \stackrel{|h| \leq \frac{1}{4}}{\leq} 4 \int_{(-\frac{1}{2}, \frac{1}{2})} |f(x) - \int_{(-\frac{1}{2}, \frac{1}{2})} f(x') dx'|^2 dx \\
& = 4 \sum_{\substack{k \in 2\pi\mathbb{Z} \\ k \neq 0}} |\mathcal{F}f(k)|^2. \tag{102}
\end{aligned}$$

In view of (101), (100), and (102), statement (99) follows from

$$\sum_{\substack{k \in 2\pi\mathbb{Z} \\ k \neq 0}} |\mathcal{F}f(k)|^2 \lesssim (\mathcal{L}^1(S))^{-2} \sup_{h \in S} \sum_{k \in 2\pi\mathbb{Z}} |(1 - e^{-ikh})\mathcal{F}f(k)|^2,$$



which in turn follows from its averaged version

$$\sum_{\substack{k \in 2\pi\mathbb{Z} \\ k \neq 0}} |\mathcal{F}f(k)|^2 \lesssim (\mathcal{L}^1(S))^{-3} \int_S \sum_{k \in 2\pi\mathbb{Z}} |(1 - e^{-ikh})\mathcal{F}f(k)|^2 dh.$$

This formulation in terms of Fourier series has the advantage that we now see that it reduces to the following property of the set  $S$ :

$$\int_S |1 - e^{-ikh}|^2 dh \gtrsim \mathcal{L}^1(S)^3 \quad \text{for all non-zero } k \in 2\pi\mathbb{Z}. \quad (103)$$

Let us establish (103) and first argue that it is sufficient to consider the case of  $k = 2\pi$ . Indeed, consider a general nonzero  $k \in 2\pi\mathbb{Z}$ . Given the measure  $\mathcal{L}^1(S)$  of the set  $S \subset [-1, 1]$ , the minimum value of  $\int_S |1 - e^{-ikh}|^2 dh$  only depends on the distribution of values of the function  $g: [-1, 1] \ni h \mapsto |1 - e^{-ikh}|^2$ . In fact, the best choice is always of the form  $S = \{h \in [-1, 1] \mid |1 - e^{-ikh}|^2 < \alpha\}$ . Note that  $g$  is a rescaling of the 1-periodic function  $\hat{g}: [-1, 1] \ni h \mapsto |1 - e^{-2\pi ih}|^2$  by an integer and thus has the same distribution function as  $\hat{g}$ . Therefore, it indeed suffices to consider the case of  $k = 2\pi$  in (103), that is, it suffices to show

$$\int_S |1 - e^{-2\pi ih}|^2 dh \gtrsim \mathcal{L}^1(S)^3. \quad (104)$$

Estimate (104) is easily seen to be true: The only zeros of the function  $[-1, 1] \ni h \mapsto |1 - e^{-2\pi ih}|^2$  are  $h \in \{-1, 0, 1\}$  and those zeros are of first order. Hence the best strategy to minimize  $\int_S |1 - e^{-2\pi ih}|^2 dh$  given  $\mathcal{L}^1(S)$ , namely sets  $S$  of the form  $S_\alpha = \{h \in [-1, 1] \mid |1 - e^{-2\pi ih}|^2 < \alpha\}$ , lead to

$$\mathcal{L}^1(S_\alpha) \sim \alpha^{1/2} \quad \text{and} \quad \int_{S_\alpha} |1 - e^{-2\pi ih}|^2 dh \sim \alpha^{3/2}$$

for  $\alpha \ll 1$ . This yields (104).

We now derive part i) in its  $L^1$ -version, that is

$$\sup_{|h| \leq \frac{1}{16}} \int_{(-\frac{1}{16}, \frac{1}{16})} |\partial^h f| dx \lesssim (\mathcal{L}^1(S))^{-3/2} \sup_{h \in S} \int_{(-\frac{1}{2}, \frac{1}{2})} |\partial^h f| dx, \quad (105)$$

from the  $L^2$ -version (99). As for (99), we first prove (105) under the additional assumption that the set  $S \subset [-\frac{1}{4}, \frac{1}{4}]$  actually reaches as far out as possible, i. e.  $\frac{1}{4} \in S$  or  $-\frac{1}{4} \in S$ . By symmetry, we may w. l. o. g. assume

$$\frac{1}{4} \in S \subset [-\frac{1}{4}, \frac{1}{4}]. \quad (106)$$

In fact, we will apply (99) not to  $f$ , but to its convolution  $f * \phi$  with the function  $\phi$  that corresponds to the average over  $S$ , i. e.

$$\phi(h) = \begin{cases} (\mathcal{L}^1(S))^{-1} & \text{for } h \in S \\ 0 & \text{for } h \notin S \end{cases}. \quad (107)$$

Moreover, we will apply (99) in its rescaled version by a factor of 4 (to fit (106)) yielding

$$\sup_{|h| \leq \frac{1}{16}} \int_{(-\frac{1}{16}, \frac{1}{16})} |\partial^h(f * \phi)|^2 dx \lesssim (\mathcal{L}^1(S))^{-2} \sup_{h \in S} \int_{(-\frac{1}{8}, \frac{1}{8})} |\partial^h(f * \phi)|^2 dx. \quad (108)$$

We now argue that we can trade in the convolution against a passage from  $L^2$  to  $L^1$  on both sides of (108).

We start with the r. h. s. of (108), where the convolution (with some  $L^2$ -function) is necessary to replace the  $L^2$ -norm by the  $L^1$ -norm. Indeed, we have by the standard convolution estimate for any  $h \in [-\frac{1}{4}, \frac{1}{4}]$  (and a fortiori in  $S$ )

$$\begin{aligned} & \int_{(-\frac{1}{8}, \frac{1}{8})} |\partial^h(f * \phi)|^2 dx \\ &= \int_{(-\frac{1}{8}, \frac{1}{8})} |(\partial^h f) * \phi|^2 dx \\ &\leq \int_{\mathbb{R}} |\phi|^2 dh \left( \int_{(-\frac{1}{2}, \frac{1}{2})} |\partial^h f| dx \right)^2 \\ &\stackrel{(107)}{=} (\mathcal{L}^1(S))^{-1} \left( \int_{(-\frac{1}{2}, \frac{1}{2})} |\partial^h f| dx \right)^2. \end{aligned} \quad (109)$$

We now turn to the l. h. s. of (108), where the convolution (with a function supported on  $S$ ) does not hurt as we shall see. Indeed, we have for all  $|h| \leq \frac{1}{16}$ :

$$\begin{aligned} & \int_{(-\frac{1}{16}, \frac{1}{16})} |\partial^h f| dx \\ &\stackrel{|h| \leq \frac{1}{16}}{\leq} \int_{(-\frac{1}{16}, \frac{1}{16})} |\partial^h(f * \phi)| dx + 2 \int_{(-\frac{1}{8}, \frac{1}{8})} |f * \phi - f| dx \\ &\stackrel{(107)}{\leq} \int_{(-\frac{1}{16}, \frac{1}{16})} |\partial^h(f * \phi)| dx + 2(\mathcal{L}^1(S))^{-1} \int_S \int_{(-\frac{1}{8}, \frac{1}{8})} |\partial^{h'} f| dx dh' \\ &\leq 8^{-\frac{1}{2}} \left( \int_{(-\frac{1}{16}, \frac{1}{16})} |\partial^h(f * \phi)|^2 dx \right)^{1/2} + 2 \sup_{h' \in S} \int_{(-\frac{1}{2}, \frac{1}{2})} |\partial^{h'} f| dx. \end{aligned} \quad (110)$$

Now inserting the square of (110) into (108) and then into (109) yields

$$\begin{aligned} & \left( \sup_{|h| \leq \frac{1}{16}} \int_{(-\frac{1}{16}, \frac{1}{16})} |\partial^h f| dx \right)^2 \\ & \lesssim ((\mathcal{L}^1(S))^{-3} + 1) \left( \sup_{h \in S} \int_{(-\frac{1}{2}, \frac{1}{2})} |\partial^h f| dx \right)^2, \end{aligned}$$

which clearly implies (105).

Finally, we have to get rid of the constraint (106). We first note that if we replace (106) by

$$S \subset [-h_*, h_*] \quad \text{and} \quad (h_* \in S \text{ or } -h_* \in S) \quad (111)$$

for some  $h_* \in (0, \frac{1}{4}]$ , (105) turns into

$$\sup_{|h| \leq \frac{h_*}{4}} \int_{(-\frac{h_*}{4}, \frac{h_*}{4})} |\partial^h f| dx \lesssim h_*^{3/2} (\mathcal{L}^1(S))^{-3/2} \sup_{h \in S} \int_{(-2h_*, 2h_*)} |\partial^h f| dx, \quad (112)$$

as can be seen by rescaling all length with  $4h_*$ . We now will get rid of the possibly small  $h_*$  on the l. h. s. of (112) in order to obtain (41).

We start with the  $h_*$  in  $\int_{(-\frac{h_*}{4}, \frac{h_*}{4})}$  of (112). Note that the interval  $(-\frac{1}{16}, \frac{1}{16})$  can be covered by the order of  $h_*^{-1}$  (more precisely,  $\frac{1}{4h_*}$  many) translates of the interval  $(-\frac{h_*}{4}, \frac{h_*}{4})$ . Because of  $h_* \leq \frac{1}{4}$ , the translates of the enlarged interval  $(-2h_*, 2h_*)$  are all contained in  $(-\frac{1}{2}, \frac{1}{2})$ . Hence summing these translates of (112) we obtain

$$\sup_{|h| \leq \frac{h_*}{4}} \int_{(-\frac{1}{16}, \frac{1}{16})} |\partial^h f| dx \lesssim h_*^{1/2} (\mathcal{L}^1(S))^{-3/2} \sup_{h \in S} \int_{(-\frac{1}{2}, \frac{1}{2})} |\partial^h f|^2 dx. \quad (113)$$

We now turn to the  $h_*$  in  $\sup_{|h| \leq \frac{h_*}{4}}$  of (112). Since finite differences  $\partial^h f$  of  $f$  with shift  $h$  of order at most one (more precisely  $|h| \leq \frac{1}{32}$ ) can be written as the sum of the order of  $h_*^{-1}$  terms (more precisely, at most  $\frac{1}{8h_*}$  terms), each of these terms being a translate of the finite differences  $\partial^{h'} f$  (more precisely translated by at most  $\frac{1}{32}$ ) with a shift at most of order  $h_*$  (more precisely  $|h'| \leq \frac{h_*}{4}$ ), we can pass from shifts of order  $h_*$  to shifts of order one. More

precisely, we obtain from (113) by the triangle inequality in  $L^1$ :

$$\begin{aligned}
& \sup_{|h| \leq \frac{1}{32}} \int_{(-\frac{1}{32}, \frac{1}{32})} |\partial^h f| dx \\
& \leq h_*^{-1/2} (\mathcal{L}^1(S))^{-3/2} \sup_{h \in S} \int_{(-\frac{1}{2}, \frac{1}{2})} |\partial^h f| dx \\
& \stackrel{(111)}{\lesssim} (\mathcal{L}^1(S))^{-2} \sup_{h \in S} \int_{(-\frac{1}{2}, \frac{1}{2})} |\partial^h f| dx. \tag{114}
\end{aligned}$$

It remains to argue that an  $h_* \in (0, \frac{1}{4}]$  with (111) exists. Indeed, by continuity in  $h$  of the l. h. s. of (41), we may assume that  $S$  is closed. Choosing  $r \leq \frac{1}{4}$  in the statement of Lemma 7 appropriately, we may also assume that  $S \subset [-\frac{1}{4}, \frac{1}{4}]$ . If then we set

$$h_* = \max |S| = \max\{|h| | h \in S\},$$

we automatically have (111) and  $h_* \leq \frac{1}{4}$ .

Finally, we address parts ii) and iii) of Lemma 7 which will be easy consequences of part i). Since we trivially have

$$\begin{aligned}
\int_S \int_{(-\frac{1}{2}, \frac{1}{2})} |\partial^h f| dx dh & \leq \int_{(-r, r)} \int_{(-\frac{1}{2}, \frac{1}{2})} |\partial^h f| dx dh, \\
\int_{(-r, r)} \int_{(-r, r)} |\partial^h f| dx dh & \leq \sup_{|h| \leq r} \int_{(-r, r)} |\partial^h f| dx,
\end{aligned}$$

it just remains to show that

$$\sup_{|h| \leq r} \int_{(-r, r)} |\partial^h f| dx \lesssim (\mathcal{L}^1(S))^{-3} \int_S \int_{(-\frac{1}{2}, \frac{1}{2})} |\partial^h f| dx dh. \tag{115}$$

In order to show (115), we introduce a density  $\rho$  and its average  $M$  on  $S$ :

$$\rho(h) := \int_{(-\frac{1}{2}, \frac{1}{2})} |\partial^h f| dx, \quad M := (\mathcal{L}^1(S))^{-1} \int_S \rho(h) dh. \tag{116}$$

Consider the subset  $\tilde{S}$  of  $S$  of shifts with not too much above-average density:

$$\tilde{S} := \{h \in S | \rho(h) \leq 2M\}. \tag{117}$$

The set  $\tilde{S}$  has measure comparable to  $S$ . Indeed, because of

$$\begin{aligned}
\mathcal{L}^1(S - \tilde{S}) & = \mathcal{L}^1(\{h \in S | \rho(h) > 2M\}) \\
& \leq (2M)^{-1} \int_S \rho(h) dh \stackrel{(116)}{=} \frac{1}{2} \mathcal{L}^1(S),
\end{aligned}$$

we get

$$\mathcal{L}^1(\tilde{S}) \geq \frac{1}{2}\mathcal{L}^1(S). \quad (118)$$

We now obtain (115) with help of part i) of Lemma 7, applied to  $\tilde{S}$ :

$$\begin{aligned} \sup_{|h| \leq \frac{1}{32}} \int_{(-\frac{1}{32}, \frac{1}{32})} |\partial^h f| dx &\stackrel{(114)}{\lesssim} (\mathcal{L}^1(\tilde{S}))^{-2} \sup_{h \in \tilde{S}} \int_{(-\frac{1}{2}, \frac{1}{2})} |\partial^h f| dx \\ &\stackrel{(116)}{=} (\mathcal{L}^1(\tilde{S}))^{-2} \sup_{h \in \tilde{S}} \rho(h) \\ &\stackrel{(117)}{\leq} (\mathcal{L}^1(\tilde{S}))^{-2} 2M \\ &\stackrel{(116)}{=} (\mathcal{L}^1(\tilde{S}))^{-2} 2 (\mathcal{L}^1(S))^{-1} \int_S \int_{(-\frac{1}{2}, \frac{1}{2})} |\partial^h f| dx dh \\ &\stackrel{(118)}{\lesssim} (\mathcal{L}^1(S))^{-3} \int_S \int_{(-\frac{1}{2}, \frac{1}{2})} |\partial^h f| dx dh. \end{aligned}$$

#### PROOF OF PROPOSITION 2.

Let  $0 < r''' \ll r'' \ll r' \ll r \ll 1$  denote generic, but universal radii (by convention ordered in the above fashion). We abbreviate the (rescaled) energy as

$$E := \eta^{1/3} E_{intef}(B_1) + \eta^{-2/3} E_{elast}(B_1)$$

and observe that we may assume that

$$E \leq 1, \quad (119)$$

since the estimate (44) holds trivially for  $E \geq 1$  because the l. h. s. of (44) (finite differences of characteristic functions in  $L^1$ ) is at most of order 1. To ease the burden of notation, we introduce an equivalence relation similar to the ones in Proposition 1: For two functions  $f$  and  $g$  defined on  $B_r$ , we write

$$f \approx_{3, E^{1/2}} g,$$

provided all *third-order* finite differences of  $f - g$  are controlled in  $L^1(B_r)$  for some  $r$  by  $E^{1/2}$ :

$$\sup_{\substack{a, b, c \in S^2 \\ |h_a|, |h_b|, |h_c| \leq r}} \int_{B_r} |\partial_a^{h_a} \partial_b^{h_b} \partial_c^{h_c} (f - g)| dx \lesssim E^{1/2}.$$

The starting point for Proposition 2 is the Martensitic part of Proposition 1, which in particular (in view of (119),  $\approx_{2,E}$  entails  $\approx_{3,E^{1/2}}$ ) yields

$$\begin{aligned} \chi_1 &\approx_{3,E^{1/2}} & -f_{(101)} & -f_{(\bar{1}01)} & +f_{(110)} & +f_{(1\bar{1}0)}, \\ \chi_2 &\approx_{3,E^{1/2}} & f_{(011)} & +f_{(01\bar{1})} & -f_{(110)} & -f_{(1\bar{1}0)}, \\ \chi_3 &\approx_{3,E^{1/2}} & -f_{(011)} & -f_{(01\bar{1})} & +f_{(101)} & +f_{(\bar{1}01)}. \end{aligned} \quad (120)$$

The main step of the proof consists in using Lemma 6 (and Lemma 7) to establish the three alternatives

$$f_{(101)} \approx_{3,E^{1/2}} f_{(\bar{1}01)} \approx_{3,E^{1/2}} 0 \quad \underline{\text{or}} \quad f_{(110)} \approx_{3,E^{1/2}} f_{(1\bar{1}0)} \approx_{3,E^{1/2}} 0, \quad (121)$$

$$f_{(011)} \approx_{3,E^{1/2}} f_{(01\bar{1})} \approx_{3,E^{1/2}} 0 \quad \underline{\text{or}} \quad f_{(110)} \approx_{3,E^{1/2}} f_{(1\bar{1}0)} \approx_{3,E^{1/2}} 0, \quad (122)$$

$$f_{(011)} \approx_{3,E^{1/2}} f_{(01\bar{1})} \approx_{3,E^{1/2}} 0 \quad \underline{\text{or}} \quad f_{(101)} \approx_{3,E^{1/2}} f_{(\bar{1}01)} \approx_{3,E^{1/2}} 0. \quad (123)$$

Before establishing (121)-(123), let us show how they imply the statement of Proposition 2. Since none of the 3 pairs  $(f_{(011)}, f_{(01\bar{1})})$ ,  $(f_{(101)}, f_{(\bar{1}01)})$ , and  $(f_{(110)}, f_{(1\bar{1}0)})$ , appears in all 3 lines (121)-(123) simultaneously, at least *two* of these pairs must be controlled. Hence one of the following 3 cases must occur

$$\begin{aligned} (f_{(101)} \approx_{3,E^{1/2}} f_{(\bar{1}01)} \approx_{3,E^{1/2}} f_{(110)} \approx_{3,E^{1/2}} f_{(1\bar{1}0)} \approx_{3,E^{1/2}} 0) &\quad \underline{\text{or}} \\ (f_{(011)} \approx_{3,E^{1/2}} f_{(01\bar{1})} \approx_{3,E^{1/2}} f_{(110)} \approx_{3,E^{1/2}} f_{(1\bar{1}0)} \approx_{3,E^{1/2}} 0) &\quad \underline{\text{or}} \\ (f_{(011)} \approx_{3,E^{1/2}} f_{(01\bar{1})} \approx_{3,E^{1/2}} f_{(101)} \approx_{3,E^{1/2}} f_{(\bar{1}01)} \approx_{3,E^{1/2}} 0). & \end{aligned}$$

A glance back to (120) reveals that these 3 cases translate one-by-one into

$$\chi_1 \approx_{3,E^{1/2}} 0 \quad \underline{\text{or}} \quad \chi_2 \approx_{3,E^{1/2}} 0 \quad \underline{\text{or}} \quad \chi_3 \approx_{3,E^{1/2}} 0,$$

which is just our compact reformulation of the statement of Proposition 2.

We now turn to the proof of (121)-(123). By symmetry, it is sufficient to treat (121). In this process, it is helpful to combine the pairs according to

$$f_{[010]} := -f_{(101)} - f_{(\bar{1}01)} \quad \text{and} \quad f_{[001]} := f_{(110)} + f_{(1\bar{1}0)}.$$

Note that this notation is in line with our usual convention:  $f_{[010]}$  is a function constant in  $x_2$  and  $f_{[001]}$  is a function constant in  $x_3$ . We start from the statement on  $\chi_1$  in Proposition 1 but use only the 3 second-order finite differences in direction of  $x_2$  and  $x_3$ :

$$\begin{aligned} &\sup_{|h|,|h'|\leq r} \left( |\partial_2^h \partial_2^{h'} (\chi_1 - f_{[010]} - f_{[001]})| \right. \\ &+ |\partial_2^h \partial_3^{h'} (\chi_1 - f_{[010]} - f_{[001]})| \\ &\left. + |\partial_3^h \partial_3^{h'} (\chi_1 - f_{[010]} - f_{[001]})| \right) dx \lesssim E. \end{aligned} \quad (124)$$

In order to prepare the application of Lemma 6 to the variables  $(x_2, x_3)$ , we now show with help of Lemma 7 ii) that (124) can be strengthened to

$$\begin{aligned} & \int_{(-r,r)} \sup_{|h|,|h'|\leq r} \int_{(-r,r)^2} \left( |\partial_2^h \partial_2^{h'} (\chi_1 - f_{[010]} - f_{[001]})| \right. \\ & \quad + |\partial_2^h \partial_3^{h'} (\chi_1 - f_{[010]} - f_{[001]})| \\ & \quad \left. + |\partial_3^h \partial_3^{h'} (\chi_1 - f_{[010]} - f_{[001]})| \right) dx_2 dx_3 dx_1 \lesssim E. \end{aligned} \quad (125)$$

Indeed, introducing the abbreviation  $g := \chi_1 - f_{[010]} - f_{[001]}$ , we shall show that Lemma 7 ii) yields the “inverse estimate”

$$\sup_{|h|,|h'|\leq r} \int_{B_r} |\partial_2^h \partial_2^{h'} g| dx \gtrsim \int_{(-r',r')} \sup_{|h|,|h'|\leq r'} \int_{(-r',r')^2} |\partial_2^h \partial_2^{h'} g| dx_2 dx_3 dx_1. \quad (126)$$

The passage of the second row and third row of (124) to the second row and third row of (125), respectively, follows by the same arguments. Here comes the string of inequalities that establishes (126):

$$\begin{aligned} & \sup_{|h|,|h'|\leq r} \int_{B_r} |\partial_2^h \partial_2^{h'} g| dx \\ & \gtrsim \int_{(-r',r')} \int_{(-r',r')} \int_{(-r',r')} \int_{(-r',r')} \int_{(-r',r')} |\partial_2^h \partial_2^{h'} g| dx_2 dh dh' dx_3 dx_1 \\ & \stackrel{(42)}{\gtrsim} \int_{(-r',r')} \int_{(-r',r')} \int_{(-r',r')} \sup_{|h|\leq r''} \int_{(-r'',r'')} |\partial_2^h \partial_2^{h'} g| dx_2 dh' dx_3 dx_1 \end{aligned} \quad (127)$$

$$\begin{aligned} & \gtrsim \int_{(-r',r')} \sup_{|h|\leq r''} \int_{(-r',r')} \int_{(-r',r')} \int_{(-r'',r'')} |\partial_2^{h'} \partial_2^h g| dx_2 dh' dx_3 dx_1 \\ & \stackrel{(42)}{\gtrsim} \int_{(-r',r')} \sup_{|h|\leq r''} \int_{(-r',r')} \sup_{|h'|\leq r'''} \int_{(-r''',r''')} |\partial_2^{h'} \partial_2^h g| dx_2 dx_3 dx_1 \end{aligned} \quad (128)$$

$$\gtrsim \int_{(-r',r')} \sup_{|h|,|h'|\leq r'''} \int_{(-r''',r''')^2} |\partial_2^h \partial_2^{h'} g| dx_2 dx_3 dx_1. \quad (129)$$

In order to get to (127), we have applied (42) to the function  $x_2 \mapsto \partial_2^{h'} g(x_1, x_2, x_3)$  and the shift  $h$ ; in order to get to (128), we have applied (42) to the function  $x_2 \mapsto \partial_2^h g(x_1, x_2, x_3)$  and the shift  $h'$ .

Now comes the central step of the proof: We consider  $x_1$  as a parameter and apply Lemma 6 to the characteristic function  $(x_2, x_3) \mapsto \chi_1(x_1, x_2, x_3)$  and the “single-variable” functions  $x_2 \mapsto f_{[001]}(x_1, x_2)$  and  $x_3 \mapsto f_{[010]}(x_1, x_3)$ . We

obtain

$$\begin{aligned}
& \min \left\{ \sup_{|h|, |h'| \leq r'} \int_{(-r', r')} |\partial_2^h \partial_2^{h'} f_{[001]}| dx_2, \right. \\
& \quad \left. \sup_{|h|, |h'| \leq r'} \int_{(-r', r')} |\partial_3^h \partial_3^{h'} f_{[010]}| dx_3 \right\} \\
& \lesssim \left( \sup_{|h|, |h'| \leq r} \int_{(-r, r)^2} \left( |\partial_2^h \partial_2^{h'} (\chi_1 - f_{[010]} - f_{[001]})| \right. \right. \\
& \quad \left. \left. + |\partial_2^h \partial_3^{h'} (\chi_1 - f_{[010]} - f_{[001]})| \right. \right. \\
& \quad \left. \left. + |\partial_3^h \partial_3^{h'} (\chi_1 - f_{[010]} - f_{[001]})| \right) dx_2 dx_3 \right)^{1/2}.
\end{aligned}$$

We now integrate this estimate over  $x_1 \in (-r', r')$ , use Cauchy Schwarz' inequality on the r. h. s. and combine with (125) to obtain

$$\begin{aligned}
& \int_{(-r, r)} \min \left\{ \sup_{|h|, |h'| \leq r} \int_{(-r, r)} |\partial_2^h \partial_2^{h'} f_{[001]}| dx_2, \right. \\
& \quad \left. \sup_{|h|, |h'| \leq r} \int_{(-r, r)} |\partial_3^h \partial_3^{h'} f_{[010]}| dx_3 \right\} dx_1 \lesssim E^{1/2}.
\end{aligned}$$

We use this estimate only in its weaker form

$$\begin{aligned}
& \int_{(-r, r)} \min \left\{ \int_{(-r, r)^2} \int_{(-r, r)} |\partial_2^h \partial_2^{h'} f_{[001]}| dx_2 dh dh', \right. \\
& \quad \left. \int_{(-r, r)^2} \int_{(-r, r)} |\partial_3^h \partial_3^{h'} f_{[010]}| dx_3 dh dh' \right\} dx_1 \lesssim E^{1/2}. \quad (130)
\end{aligned}$$

Hence there exists a set  $I \subset [-r_0, r_0]$  (where  $r_0$  is a small universal radius chosen at the end of the proof) of  $x_1$ -coordinates with the property that

$$\mathcal{L}^1(I) \sim 1$$

and such that the following alternative holds

$$\begin{aligned}
& \int_I \int_{(-r, r)^2} \int_{(-r, r)} |\partial_2^h \partial_2^{h'} f_{[001]}| dx_2 dh dh' dx_1 \lesssim E^{1/2} \quad \underline{\text{or}} \quad (131) \\
& \int_I \int_{(-r, r)^2} \int_{(-r, r)} |\partial_3^h \partial_3^{h'} f_{[010]}| dx_3 dh dh' dx_1 \lesssim E^{1/2}.
\end{aligned}$$

By symmetry, it is enough to consider the first case and to show that it implies the second case in (121), which we can also formulate as

$$\sup_{|h|, |h'|, |h''| \leq r} \int_{(-r, r)} \left( |\partial_s^h \partial_s^{h'} \partial_s^{h''} f_{(110)}(s)| + |\partial_s^h \partial_s^{h'} \partial_s^{h''} f_{(1\bar{1}0)}(s)| \right) ds \lesssim E^{1/2}.$$



By symmetry, it suffices to show the first part of the preceding estimate, that is,

$$\sup_{|h|, |h'|, |h''| \leq r} \int_{(-r, r)} |\partial_s^h \partial_s^{h'} \partial_s^{h''} f_{(110)}(s)| ds \lesssim E^{1/2}. \quad (132)$$

The rest of the proof is devoted to deducing (132) from (131).

The main step in passing from (131) to (132) consists in showing that (131) implies

$$\int_{(-r, r)^3} \int_{(-r, r)} |\partial_s^h \partial_s^{h'} \partial_s^{h''} f_{(110)}(s)| ds dh dh' dh'' \lesssim E^{1/2}; \quad (133)$$

the passage from (133) (average) to (132) (supremum) is again an easy application of Lemma 7 ii), as we shall see later. Note that in view of the definition of  $f_{[001]}$ , (131) can be reformulated as

$$\begin{aligned} \int_{(-r, r)^2} \int_I \int_{(-r, r)} |\partial_s^h \partial_s^{h'} f_{(110)}(x_1 + x_2) + \partial_s^h \partial_s^{h'} f_{(1\bar{1}0)}(x_1 - x_2)| dx_2 dx_1 dh dh' \\ \lesssim E^{1/2}. \end{aligned} \quad (134)$$

Hence in order to pass from (134) to (133) it suffices to show that

$$\begin{aligned} \int_{(-r', r')^3} \int_{(-r', r')} |\partial_s^{h''} \partial_s^h \partial_s^{h'} f_{(110)}(s)| ds dh dh' dh'' \\ \lesssim \int_{(-r, r)^2} \int_I \int_{(-r, r)} |\partial_s^h \partial_s^{h'} f_{(110)}(x_1 + x_2) \\ + \partial_s^h \partial_s^{h'} f_{(1\bar{1}0)}(x_1 - x_2)| dx_2 dx_1 dh dh'. \end{aligned} \quad (135)$$

We now see that  $(h, h')$  can be treated as a parameter so that (135) follows from

$$\begin{aligned} \int_{(-r', r')} \int_{(-r', r')} |\partial_s^{h''} g_{(110)}(s)| ds dh'' \\ \lesssim \int_I \int_{(-r, r)} |g_{(110)}(x_1 + x_2) + g_{(1\bar{1}0)}(x_1 - x_2)| dx_2 dx_1, \end{aligned} \quad (136)$$

where we have used the abbreviations

$$g_{(110)} := \partial_s^h \partial_s^{h'} f_{(110)} \quad \text{and} \quad g_{(1\bar{1}0)} := \partial_s^h \partial_s^{h'} f_{(1\bar{1}0)}.$$

Let us now address (136), which we will show with help of Lemma 7 iii). At first, we have to eliminate  $g_{(1\bar{1}0)}$ . To this purpose, we write down two copies

of the r. h. s. of (136)

$$\begin{aligned} & \int_I \int_{(-r,r)} |g_{(110)}(x_1 + x_2) + g_{(1\bar{1}0)}(x_1 - x_2)| dx_2 dx_1, \\ & \int_I \int_{(-r,r)} |g_{(110)}(x'_1 + x'_2) + g_{(1\bar{1}0)}(x'_1 - x'_2)| dx'_2 dx'_1, \end{aligned}$$

which we integrate over a dummy variable over  $I$  (since  $I \subset [-r_0, r_0]$  this does not increase the value):

$$\int_I \int_I \int_{(-r,r)} |g_{(110)}(x_1 + x_2) + g_{(1\bar{1}0)}(x_1 - x_2)| dx_2 dx_1 dx'_1, \quad (137)$$

$$\int_I \int_I \int_{(-r,r)} |g_{(110)}(x'_1 + x'_2) + g_{(1\bar{1}0)}(x'_1 - x'_2)| dx'_2 dx_1 dx'_1. \quad (138)$$

We now change the inner variables

- in (137) from  $x_2$  to  $s$  such that  $x_1 + x_2 = s$  with the effect that  $x_1 - x_2 = 2x_1 - s$ ,
- in (138) from  $x'_2$  to  $s$  such that  $x'_1 - x'_2 = 2x_1 - s$  with the effect that  $x'_1 + x'_2 = s + 2(x'_1 - x_1)$ .

This yields

$$\begin{aligned} & \int_I \int_I \int_{(-r',r')} |g_{(110)}(s) + g_{(1\bar{1}0)}(2x_1 - s)| ds dx_1 dx'_1, \\ & \int_I \int_I \int_{(-r',r')} |g_{(110)}(s + 2(x'_1 - x_1)) + g_{(1\bar{1}0)}(2x_1 - s)| ds dx_1 dx'_1. \end{aligned}$$

By the above derivation, both terms are dominated by the r. h. s. of (136), hence also the modulus of their difference. Hence we obtain by the triangle inequality in  $L^1(I \times I \times (-r', r'))$

$$\begin{aligned} & \int_I \int_I \int_{(-r',r')} |g_{(110)}(s + 2(x'_1 - x_1)) - g_{(110)}(s)| ds dx_1 dx'_1 \\ & \lesssim \int_I \int_{(-r,r)} |g_{(110)}(x_1 + x_2) + g_{(1\bar{1}0)}(x_1 - x_2)| dx_2 dx_1. \quad (139) \end{aligned}$$

With (139) we have successfully eliminated  $g_{(1\bar{1}0)}$  and thus reduced our goal (136) to the following statement on a single function  $g_{(110)}$  of a single variable and on a set  $I$  of full measure:

$$\begin{aligned} & \int_{(-r',r')} \int_{(-r',r')} |\partial_s^{h''} g_{(110)}(s)| ds dh'' \\ & \lesssim \int_I \int_I \int_{(-r',r')} |g_{(110)}(s + 2(x'_1 - x_1)) - g_{(110)}(s)| ds dx_1 dx'_1. \quad (140) \end{aligned}$$

We finally show how to reduce estimate (140) from Lemma 7 iii). Clearly, we have to understand the shifts produced by  $2(x'_1 - x_1)$ . We note that the r. h. s. of (140) can be written as

$$\begin{aligned} & \int_I \int_I \int_{(-r', r')} |g_{(110)}(s + 2(x'_1 - x_1)) - g_{(110)}(s)| ds dx_1 dx'_1 \\ &= \int_{\mathbb{R}} \int_{(-r', r')} |\partial_s^{h''} g_{(110)}(s)| ds \omega(h'') dh'', \end{aligned} \quad (141)$$

where the weight function  $\omega$  can be computed to be the rescaled convolution of the characteristic function  $\chi_I$  of  $I$  with the characteristic function of  $-I$ :

$$\omega(h'') = \frac{1}{2} (\chi_I * \chi_{-I})\left(\frac{h''}{2}\right).$$

This implies

$$\omega \leq \frac{1}{2} \quad \text{and} \quad \int_{\mathbb{R}} \omega dh'' = (\mathcal{L}^1(I))^2.$$

Moreover, since  $I \subset [-r_0, r_0]$  and  $\mathcal{L}^1(I) \sim 1$ , we have

$$\text{supp} \omega \subset [-4r_0, 4r_0], \quad \omega \lesssim 1, \quad \int_{\mathbb{R}} \omega dh'' \sim 1,$$

so that there exists a set  $S$  of shifts with

$$\omega \gtrsim \chi_S, \quad S \subset [-4r_0, 4r_0], \quad \mathcal{L}^1(S) \sim 1.$$

In view of this, (141) yields

$$\begin{aligned} & \int_I \int_I \int_{(-r', r')} |g_{(110)}(s + 2(x'_1 - x_1)) - g_{(110)}(s)| ds dx_1 dx'_1 \\ & \gtrsim \int_S \int_{(-r', r')} |\partial_s^{h''} g_{(110)}(s)| ds dh''. \end{aligned} \quad (142)$$

If we now choose  $r_0$  sufficiently small w. r. t. the generic but universal  $r'$  in (142) (so small that  $4r_0 \leq r_1 r'$ , where  $r_1$  is the small universal radius from Lemma 7), we may apply Lemma 7 iii)(rescaled by the universal factor  $2r'$ ):

$$\int_S \int_{(-r', r')} |\partial_s^{h''} g_{(110)}(s)| ds dh'' \gtrsim \int_{(-r'', r'')} \int_{(-r'', r'')} |\partial_s^{h''} g_{(110)}(s)| ds dh''. \quad (143)$$

The combination of (142) and (143) gives (140). This establishes (133).

Finally, we have to argue how Lemma 7 ii) allows us to pass from (133) to (132). What separates (133) from (132) is the estimate

$$\begin{aligned} & \sup_{|h|, |h'|, |h''| \leq r'} \int_{(-r', r')} |\partial_s^h \partial_s^{h'} \partial_s^{h''} f_{(110)}(s)| ds \\ & \lesssim \int_{(-r, r)^3} \int_{(-r, r)} |\partial_s^h \partial_s^{h'} \partial_s^{h''} f_{(110)}(s)| ds dh dh' dh''. \end{aligned}$$

Like for (127), this estimate can be derived by iteratively (or rather inductively over the dimension) applying Lemma 7 ii):

$$\begin{aligned} & \int_{(-r, r)} \int_{(-r, r)} \int_{(-r, r)} \int_{(-r, r)} |\partial_s^h \partial_s^{h'} \partial_s^{h''} f_{(110)}(s)| ds dh dh' dh'' \\ & \stackrel{(42)}{\gtrsim} \int_{(-r, r)} \int_{(-r, r)} \sup_{|h| \leq r'} \int_{(-r', r')} |\partial_s^h \partial_s^{h'} \partial_s^{h''} f_{(110)}(s)| ds dh' dh'' \\ & \gtrsim \sup_{|h| \leq r'} \int_{(-r, r)} \int_{(-r, r)} \int_{(-r', r')} |\partial_s^{h'} \partial_s^h \partial_s^{h''} f_{(110)}(s)| ds dh' dh'' \\ & \stackrel{(42)}{\gtrsim} \sup_{|h| \leq r'} \int_{(-r, r)} \sup_{|h'| \leq r''} \int_{(-r'', r'')} |\partial_s^{h'} \partial_s^h \partial_s^{h''} f_{(110)}(s)| ds dh'' \\ & \gtrsim \sup_{|h| \leq r'} \sup_{|h'| \leq r''} \int_{(-r, r)} \int_{(-r'', r'')} |\partial_s^{h''} \partial_s^{h'} \partial_s^h f_{(110)}(s)| ds dh'' \\ & \stackrel{(42)}{\gtrsim} \sup_{|h| \leq r} \sup_{|h'| \leq r''} \sup_{|h''| \leq r'''} \int_{(-r''', r''')} |\partial_s^{h''} \partial_s^{h'} \partial_s^h f_{(110)}(s)| ds. \end{aligned}$$

#### PROOF OF LEMMA 8.

The crucial object will be the *unsigned* moving average  $\phi * \chi$  of the function  $\chi$  defined via

$$(\phi * \chi)(x) := \frac{4}{3} (\mathcal{L}^3(B_1))^{-1} \int_{B_1(x)} \chi(x') dx' - \frac{1}{3} (\mathcal{L}^3(B_2))^{-1} \int_{B_2(x)} \chi(x') dx'. \quad (144)$$

As the notation suggests, we also think of it as the convolution with a radially symmetric piecewise constant function  $\phi$ .

We start by arguing that

$$\int_{B_1} |\phi * \chi - \chi| dx \leq \frac{1}{3} \sup_{\substack{a, b, c \in S^2 \\ |h|, |h'|, |h''| \leq 1}} \int_{B_1} |\partial_a^h \partial_b^{h'} \partial_c^{h''} \chi| dx. \quad (145)$$

Spelling out the third-order differences:

$$\begin{aligned}
& (\partial_a^h \partial_b^{h'} \partial_c^{h''} \chi)(x) \\
&= \left\{ \begin{array}{ll} \chi(x + ha + h'b + h''c) & - \chi(x + ha + h'b) \\ - \chi(x + ha + h''c) & + \chi(x + ha) \\ - \chi(x + h'b + h''c) & + \chi(x + h'b) \\ + \chi(x + h''c) & - \chi(x) \end{array} \right\}, \tag{146}
\end{aligned}$$

we see that for  $a = b = c$  and  $h = -h' = -h''$ , the third-order finite differences collapses to

$$(\partial_a^h \partial_a^{-h} \partial_a^{-h} \chi)(x) = -\chi(x - 2ha) + 3\chi(x - ha) - 3\chi(x) + \chi(x + ha).$$

Averaging this sub-family of third-order finite differences over  $ha \in B_1$ , we obtain  $\phi * \chi$ :

$$\begin{aligned}
& \left(\frac{1}{3} \mathcal{H}^2(S^2)\right)^{-1} \int_{(0,1)} \int_{S^2} (\partial_a^h \partial_a^{-h} \partial_a^{-h} \chi)(x) \mathcal{H}^2(da) h^2 dh \\
&= -(\mathcal{L}^3(B_2))^{-1} \int_{B_2(x)} \chi(x') dx' + 3(\mathcal{L}^3(B_1))^{-1} \int_{B_1(x)} \chi(x') dx' \\
&\quad - 3\chi(x) + (\mathcal{L}^3(B_1))^{-1} \int_{B_1(x)} \chi(x') dx' \\
&= 3((\phi * \chi)(x) - \chi(x)).
\end{aligned}$$

This representation of  $\phi * \chi - \chi$  directly implies (145).

We now turn to the proof of Lemma 8 proper. By the symmetry  $\chi \rightsquigarrow 1 - \chi$ , we may assume that  $\chi$  has below-average volume fraction in  $B_1$ :

$$(\mathcal{L}^3(B_1))^{-1} \int_{B_1} \chi dx \leq \frac{1}{2}, \tag{147}$$

in which case we will show for some small but universal radius  $r > 0$

$$\int_{B_r} \chi dx \lesssim \sup_{\substack{a,b,c \in S^2 \\ |h|,|h'|,|h''| \leq 1}} \int_{B_1} |\partial_a^h \partial_b^{h'} \partial_c^{h''} \chi| dx.$$

In view of (145), it is enough to show

$$\int_{B_r} \chi dx \lesssim \int_{B_1} |\phi * \chi - \chi| dx. \tag{148}$$

The argument for (148) is easy. In view of the definition (144), (147) and  $\chi \geq 0$  imply that  $\phi * \chi$  is strictly less than 1 at the origin:

$$(\phi * \chi)(0) \leq \frac{4}{3} \frac{1}{2} = \frac{2}{3}. \quad (149)$$

Note that  $\phi * \chi$  is Lipschitz continuous with universal constant. (This can for instance be seen from the convolution estimate

$$\sup |\nabla(\phi * \chi)| \leq \int |\nabla\phi| dx \sup |\chi| = \int |\nabla\phi| dx \lesssim 1,$$

using the fact that the three-valued radially symmetry  $\phi$  has automatically finite BV-norm  $\int |\nabla\phi| dx$ .) Hence there exists a possibly small but universal radius  $r > 0$  so that the boundedness (149) away from 1 is preserved in the small ball  $B_r$ :

$$\phi * \chi \leq \frac{3}{4} \quad \text{in } B_r. \quad (150)$$

Hence we obtain, finally using the non-convex constraint  $\chi \in \{0, 1\}$ ,

$$\begin{aligned} \int_{B_1} |\phi * \chi - \chi| dx &\geq \int_{B_r \cap \{\chi=1\}} |\phi * \chi - 1| dx \\ &\stackrel{(150)}{\geq} \frac{1}{4} \mathcal{L}^3(B_r \cap \{\chi = 1\}) \\ &\stackrel{\chi \in \{0,1\}}{\geq} \frac{1}{4} \int_{B_r} \chi dx, \end{aligned} \quad (151)$$

which is the desired (148).

#### PROOF OF THEOREM 1.

As we are dealing with characteristic functions we can w.l.o.g. assume  $E \leq 1$ . Let  $0 < r \ll 1$  be a generic but universal radius. We extend our short-hand notation from Proposition 1 and 2: For two functions  $f$  and  $g$ , we write

$$f \approx_{i, E^\alpha} g,$$

provided for some  $r > 0$ , all  $i$ -th order finite differences of  $f - g$  with shifts  $|h| \leq r$  in the  $L^1(B_r)$ -norm are estimated by  $E^\alpha$ . In this compact notation, the statement of Theorem 1 consists of the following three claims:

i)

$$\chi_0 \approx_{0, E} 1 \quad \underline{\text{or}} \quad \chi_0 \approx_{0, E} 0. \quad (152)$$

ii) In case of  $\chi_0 \approx_{0,E} 0$ ,

$$\chi_1 \approx_{0,E^{1/2}} 0 \quad \underline{\text{or}} \quad \chi_2 \approx_{0,E^{1/2}} 0 \quad \underline{\text{or}} \quad \chi_3 \approx_{0,E^{1/2}} 0. \quad (153)$$

iii) In case of  $\chi_3 \approx_{0,E^{1/2}} 0$ ,

$$\chi_1 \approx_{0,E^{1/4}} f_{(110)} \quad \underline{\text{or}} \quad \chi_1 \approx_{0,E^{1/4}} f_{(1\bar{1}0)}. \quad (154)$$

Claim (152) is at hand: By the Austenitic part of Proposition 1, we have

$$\chi_0 \approx_{2,E} 0.$$

In particular, this yields  $\chi_0 \approx_{3,E} 0$ , so that (152) follows via Lemma 8. Claim (153) is also at hand: By Proposition 2 we have

$$\chi_1 \approx_{3,E^{1/2}} 0 \quad \underline{\text{or}} \quad \chi_2 \approx_{3,E^{1/2}} 0 \quad \underline{\text{or}} \quad \chi_3 \approx_{3,E^{1/2}} 0.$$

Hence (153) follows via Lemma 8.

Claim (154) requires a bit more care: We recall the Martensitic part of Proposition 1,

$$\begin{array}{l} \chi_1 \approx_{2,E} \quad \quad \quad -f_{(101)} \quad -f_{(\bar{1}01)} \quad +f_{(110)} \quad +f_{(1\bar{1}0)}, \\ \chi_2 \approx_{2,E} \quad f_{(011)} \quad +f_{(01\bar{1})} \quad \quad \quad -f_{(110)} \quad -f_{(1\bar{1}0)}, \\ \chi_3 \approx_{2,E} \quad -f_{(011)} \quad -f_{(01\bar{1})} \quad +f_{(101)} \quad +f_{(\bar{1}01)}. \end{array} \quad (155)$$

Because of our assumption that  $\chi_3 \approx_{0,E^{1/2}} 0$ , the last line of (155) turns into

$$0 \approx_{2,E^{1/2}} -f_{(011)} - f_{(01\bar{1})} + f_{(101)} + f_{(\bar{1}01)}.$$

Because of Lemma 5, this yields

$$0 \approx_{2,E^{1/2}} f_{(011)} \approx_{2,E^{1/2}} f_{(01\bar{1})} \approx_{2,E^{1/2}} f_{(101)} \approx_{2,E^{1/2}} f_{(\bar{1}01)}.$$

Therefore, the first line of (155) reduces to

$$\chi_1 \approx_{2,E^{1/2}} f_{(110)} + f_{(1\bar{1}0)}. \quad (156)$$

Like in Proposition 2, we use Lemma 6 (with a bit of help of Lemma 7 ii)) to deduce

$$\chi_1 \approx_{2,E^{1/4}} f_{(110)} \quad \underline{\text{or}} \quad \chi_1 \approx_{2,E^{1/4}} f_{(1\bar{1}0)} \quad (157)$$

from (156). Indeed, we probe (156) by second-order finite differences in directions orthogonal to  $x_3$  only:

$$\begin{aligned} \forall a, b \in \{[110], [1\bar{1}0]\} \\ \sup_{|h|, |h'| \leq r} \int_{B_r} |\partial_a^h \partial_b^{h'} (\chi_1 - f_{(110)} - f_{(1\bar{1}0)})| dx \lesssim E^{1/2}. \end{aligned} \quad (158)$$

We now appeal to the ‘‘inverse estimate’’ (126) of Proposition 2 (which is a consequence of Lemma 7 ii)) to upgrade (158) to

$$\begin{aligned} \forall a, b \in \{[110], [1\bar{1}0]\} \\ \int_{(-r, r)} \sup_{|h|, |h'| \leq r} \int_{(-r, r)^2} |\partial_a^h \partial_b^{h'} (\chi_1 - f_{(110)} - f_{(1\bar{1}0)})| dx_1 dx_2 dx_3 \lesssim E^{1/2}. \end{aligned}$$

In particular, there exists an  $x_3^* \in (-r, r)$  such that

$$\begin{aligned} \forall a, b \in \{[110], [1\bar{1}0]\} \\ \sup_{|h|, |h'| \leq r} \int_{(-r, r)^2} |\partial_a^h \partial_b^{h'} (\chi_1(\cdot, x_3^*) - f_{(110)} - f_{(1\bar{1}0)})| dx_1 dx_2 \lesssim E^{1/2}. \end{aligned}$$

Hence we obtain by Lemma 6 applied to  $(x_1, x_2) \mapsto \chi_1(x_1, x_2, x_3^*)$ , to  $(x_1, x_2) \mapsto f_{(110)}(x_1 + x_2)$  and  $(x_1, x_2) \mapsto f_{(1\bar{1}0)}(x_1 - x_2)$  and the coordinates  $(s, \bar{s}) = (x_1 + x_2, x_1 - x_2)$ :

$$\begin{aligned} \min \left\{ \sup_{|h|, |h'| \leq r} \int_{(-r, r)} |\partial_s^h \partial_{\bar{s}}^{h'} f_{(110)}(s)| ds, \right. \\ \left. \sup_{|h|, |h'| \leq r} \int_{(-r, r)} |\partial_{\bar{s}}^h \partial_s^{h'} f_{(1\bar{1}0)}(\bar{s})| d\bar{s} \right\} \lesssim E^{1/4}. \end{aligned} \quad (159)$$

Since  $f_{(110)}$  and  $f_{(1\bar{1}0)}$  are single-variable functions, this just means

$$f_{(110)} \approx_{2, E^{1/4}} 0 \quad \underline{\text{or}} \quad f_{(1\bar{1}0)} \approx_{2, E^{1/4}} 0. \quad (160)$$

Inserting this into (156), we obtain (157).

Finally, we want to upgrade (157) to (154) with possibly modified single-variable functions:

$$\chi_1 \approx_{0, E^{1/4}} g_{(110)} \quad \underline{\text{or}} \quad \chi_1 \approx_{0, E^{1/4}} g_{(1\bar{1}0)}. \quad (161)$$

Let us assume that the first alternative of (157) holds. We probe it only with second-order finite difference quotients along directions in the plane (110) because they make the  $f_{(110)}$ -term vanish:

$$\forall a, b \in \{[1\bar{1}1], [\bar{1}11]\} \quad \sup_{|h|, |h'| \leq r} \int_{B_r} |\partial_a^h \partial_b^{h'} \chi_1| dx \lesssim E^{1/4}.$$



We can add another finite difference at no cost:

$$\forall a, b, c \in \{[1\bar{1}1], [\bar{1}11]\} \\ \sup_{|h|, |h'|, |h''| \leq r} \int_{B_r} |\partial_a^h \partial_b^{h'} \partial_c^{h''} \chi_1| dx \lesssim E^{1/4}.$$

Once again, we appeal to the “inverse estimate” (126) to upgrade the above to

$$\forall a, b, c \in \{[1\bar{1}1], [\bar{1}11]\} \\ \int_{(-r,r)} \sup_{|h|, |h'|, |h''| \leq r} \int_{(-r,r)^2} |\partial_a^h \partial_b^{h'} \partial_c^{h''} \chi_1| dx' ds \lesssim E^{1/4},$$

where  $s = (110) \cdot x$  is the coordinate perpendicular to the plane (110) and  $x'$  stand for two coordinates in the plane (for instance in direction of  $[1\bar{1}1], [\bar{1}11]$ ). We may now apply Lemma 8 (in two instead of three space dimensions) to  $x' \mapsto \chi_1(x', s)$ :

$$\int_{(-r,r)} \min \left\{ \int_{(-r,r)^2} |\chi_1| dx', \int_{(-r,r)^2} |1 - \chi_1| dx' \right\} ds \lesssim E^{1/4}.$$

This statement has exactly the same content as the first alternative of (153).

We finally give the argument for (12)-(14). Let  $\chi_0, \chi_1, \chi_2$ , and  $\chi_3$  be minimizers in (11) (the infimum is attained because  $BV \cap L^\infty$  embeds compactly into  $L^2$ ; working with a minimal sequence would suffice for the following argument). We first turn to (12) and argue that the first alternative of Theorem 1 i) implies the first alternative of (12). By the triangle inequality in  $L^2(B_r)$ , we have

$$\int_{B_r} |e|^2 dx \lesssim \int_{B_r} |e - \chi_1 e^{(1)} - \chi_2 e^{(2)} - \chi_3 e^{(3)}|^2 dx + \int_{B_r} (\chi_1^2 + \chi_2^2 + \chi_3^2) dx.$$

The second r. h. s. term is estimated since we assume the first alternative of Theorem 1 i):

$$\int_{B_r} (\chi_1^2 + \chi_2^2 + \chi_3^2) dx \stackrel{(I.1)}{\leq} \int_{B_r} (\chi_1 + \chi_2 + \chi_3) dx \stackrel{(I.1)}{=} \int_{B_r} (1 - \chi_0) dx \lesssim E.$$

Hence we obtain as desired

$$\int_{B_r} |e|^2 dx \lesssim E_{elast} + E \stackrel{\eta \leq 1}{\lesssim} E = \bar{E}.$$

We now argue that the second alternative of Theorem 1 i) implies the second alternative of (12). We note that (7) implies

$$\begin{aligned}
& \min\{|e - e^{(1)}|, |e - e^{(2)}|, |e - e^{(3)}|\} \\
& \leq (1 - \chi_0)|e - \chi_1 e^{(1)} - \chi_2 e^{(2)} - \chi_3 e^{(3)}| \\
& \quad + \chi_0 \min\{|e - e^{(1)}|, |e - e^{(2)}|, |e - e^{(3)}|\} \\
& \leq |e - \chi_1 e^{(1)} - \chi_2 e^{(2)} - \chi_3 e^{(3)}| + \chi_0 \min\{|e^{(1)}|, |e^{(2)}|, |e^{(3)}|\}.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
& \int_{B_r} \min\{|e - e^{(1)}|^2, |e - e^{(2)}|^2, |e - e^{(3)}|^2\} dx \\
& \lesssim \int_{B_r} |e - \chi_1 e^{(1)} - \chi_2 e^{(2)} - \chi_3 e^{(3)}|^2 dx + \int_{B_r} \chi_0^2 dx.
\end{aligned}$$

The second r. h. s. term is estimated by  $E$  since we assume the second alternative of Theorem 1 i). Hence we have as for the first alternative

$$\int_{B_r} \min\{|e - e^{(1)}|^2, |e - e^{(2)}|^2, |e - e^{(3)}|^2\} dx \lesssim E_{elast} + E \stackrel{\eta \leq 1}{\lesssim} E = \bar{E}.$$

We now argue that the last alternative in Theorem 1 ii) implies the last alternative in (13). In the same spirit as above, we use (7) to get

$$\begin{aligned}
& \min\{|e - e^{(1)}|, |e - e^{(2)}|\} \\
& \leq (\chi_1 + \chi_2)|e - \chi_1 e^{(1)} - \chi_2 e^{(2)}| \\
& \quad + (\chi_0 + \chi_3) \min\{|e - e^{(1)}|, |e - e^{(2)}|\} \\
& \leq |e - \chi_1 e^{(1)} - \chi_2 e^{(2)}| + (\chi_0 + \chi_3) \min\{|e^{(1)}|, |e^{(2)}|\} \\
& \leq |e - \chi_1 e^{(1)} - \chi_2 e^{(2)} - \chi_3 e^{(3)}| + \chi_3 |e^{(3)}| + (\chi_0 + \chi_3) \min\{|e^{(1)}|, |e^{(2)}|\}.
\end{aligned}$$

This implies by the triangle inequality in  $L^2(B_r)$

$$\begin{aligned}
& \int_{B_r} \min\{|e - e^{(1)}|^2, |e - e^{(2)}|^2\} dx \\
& \lesssim \int_{B_r} |e - \chi_1 e^{(1)} - \chi_2 e^{(2)} - \chi_3 e^{(3)}|^2 dx + \int_{B_r} \chi_0^2 dx + \int_{B_r} \chi_3^2 dx.
\end{aligned}$$

By the last alternative in Theorem 1 ii), the two last terms on the r. h. s. are estimated by  $E$  and  $E^{1/2}$ , respectively. Hence we obtain

$$\begin{aligned}
& \int_{B_r} \min\{|e - e^{(1)}|^2, |e - e^{(2)}|^2\} dx \\
& \lesssim E_{elast} + E + E^{1/2} \stackrel{\eta \leq 1}{\lesssim} E + E^{1/2} = \bar{E} + \bar{E}^{1/2} \stackrel{\bar{E} \leq 1}{\lesssim} \bar{E}^{1/2}.
\end{aligned}$$

We finally argue that the first alternative in Theorem 1 iii) implies the first alternative in (14). We first note that w. l. o. g. , we may assume that  $f_{(110)}$  is  $\{0, 1\}$ -valued. Indeed, we can replace  $f_{(110)}$  by

$$\tilde{f}_{(110)} := \begin{cases} 1 & \text{for } f_{(110)} \geq \frac{1}{2} \\ 0 & \text{for } f_{(110)} < \frac{1}{2} \end{cases} .$$

Clearly,  $\tilde{f}_{(110)}$  inherits from  $f_{(110)}$  the property of only depending on  $x_1 + x_2$  and it satisfies  $|\chi_1 - \tilde{f}_{(110)}| \leq 2|\chi_1 - f_{(110)}|$ . We next note that according to (7), and the last alternatives in Theorem 1 ii) & i), we have  $\int_{B_r} |\chi_1 + \chi_2 - 1| dx = \int_{B_r} \chi_0 dx + \int_{B_r} \chi_3 dx \lesssim E + E^{1/2}$ . Hence we may upgrade the first alternative in Theorem 1 iii) to

$$\int_{B_r} (|\chi_1 - f_{(110)}| + |\chi_2 - (1 - f_{(110)})| + |\chi_3|) dx \lesssim E + E^{1/2} + E^{1/4} .$$

Since all functions are  $\{0, 1\}$ -valued, we may reformulate this as

$$\int_{B_r} (|\chi_1 - f_{(110)}|^2 + |\chi_2 - (1 - f_{(110)})|^2 + |\chi_3|^2) dx \lesssim E + E^{1/2} + E^{1/4} .$$

In view of the triangle inequality in  $L^2(B_r)$ , this yields as desired

$$\begin{aligned} & \int_{B_r} |e - f_{(110)}e^{(1)} - (1 - f_{(110)})e^{(2)}|^2 dx \\ & \lesssim E_{elast} + E + E^{1/2} + E^{1/4} \stackrel{\eta \leq 1}{\lesssim} E + E^{1/2} + E^{1/4} \\ & = \bar{E} + \bar{E}^{1/2} + \bar{E}^{1/4} \stackrel{\bar{E} \leq 1}{\lesssim} \bar{E}^{1/4} . \end{aligned}$$

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## References

- [1] K. Bhattacharya, Microstructure of martensite. Why it forms and how it gives rise to the shape-memory effect? Oxford Series on Materials Modelling. Oxford University Press, Oxford, (2003)

- [2] A. Capella-Kort, F. Otto, A rigidity result for a perturbation of the geometrically linear three-well problem, *Comm. Pure Applied Math.* **62** (12) (2009)
- [3] R. Choksi, R. V. Kohn, F. Otto, Domain branching in uniaxial ferromagnetic: A scaling law for the minimum energy, *Comm. Math. Phys.* **201**, 61-79 (1999)
- [4] S. Conti, G. Dolzmann, B. Kirchheim, Existence of Lipschitz minimizers for the three-well problem in solid-solid phase transitions, *Ann. I. H. Poincaré AN* **24**, 953-962 (2007)
- [5] G. Dolzmann, S. Müller, The influence of surface energy on stress-free microstructures in Shape Memory Alloys, *Meccanica* **30**, 527-539 (1995)
- [6] G. Dolzmann, S. Müller, Microstructure with finite surface energy: the two-well problem, *Arch. Rational; Mech. Anal.* **132**, 101-141 (1995)
- [7] B. Kirchheim, Lipschitz minimizers of the 3-well problem having gradients of bounded variation, *preprint 12*, Max-Planck-Institut für Mathematik in den Naturwissenschaften, Leipzig, 1998
- [8] R. V. Kohn, S. Müller, Branching of twins near an austenite-twinned-martensite interface, *Philosophical Magazine A*, **66** (5), 697-715 (1992)
- [9] R. V. Kohn, S. Müller, Surface energy and microstructure in coherent phase transitions, *Comm. Pure Applied Math.*, **47**, 405-435 (1994)
- [10] S. Müller, private communication
- [11] S. Müller, V. Šverák, Unexpected solutions of first and second order partial differential equations, *Documenta Mathematica*, Extra volume ICM 1998 II, 691-702 (1998)
- [12] Zhiyong Zhang, R. D. James, S. Müller, Energy barriers and hysteresis in martensitic phase transformations, *Acta Materialia* **57** (15) 4332-4352 (2009)

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