Critical fields in ferromagnetic thin films: Identification of four regimes

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Abstract

We are interested in critical fields for ferromagnetic elements: At which strength of the external field does a branch of stationary magnetizations become unstable and what is the unstable mode?

We consider samples which are infinite in the direction of the external field and have a rectangular cross section, of much smaller thickness than width, as an idealization of a thin film element.

For this geometry Aharoni, [1], claims that there are only three different regimes: The unstable mode is either of coherent rotation type, of buckling type or of

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curling type. We discover a large fourth parameter regime with an unstable mode displaying an oscillation in the infinite direction.

We prove the existence of exactly four regimes by rigorously analyzing the scaling of the Rayleigh quotient of the Hessian of the energy functional. The parameters are the film width, the film thickness and the exchange length.

Mathematics Subject Classification: 78A99, 49K20, 74G60

1 Motivation

The micromagnetics of ferromagnetic thin–film elements is a paradigm for a multi–scale pattern–forming system. On one hand, there is a material length scale which is the effective range of the attractive spin–spin interaction (5 nm for Permalloy). On the other hand, there are the sample dimensions (thicknesses typically hundreds of nm, widths typically several μm). Furthermore there is a long–range spin–spin interaction expressed by the stray field.

There is a well–accepted continuum model for the magnetization $m(x)$ for temperatures well below the Curie temperature. In its static version it comes as a variational problem for m. In recent years, several reduced models, suited for specific thin–film regimes, have been derived. They have been derived based on scale separation, within the framework of Γ–convergence [12, 9, 14]. This means that they are relevant for ground states and metastable states with energies close to the ground state energy.

For the technologically important switching under external fields, however, metastable states with energies far from the ground state are important. In a certain sense, an external field probes the complex energy landscape with its many wells. In the case of small samples, all stationary points of the energy have recently been mapped numerically [10].

A more traditional analytical approach to switching is “nucleation” theory. One envisions a ferromagnetic sample which is saturated by a strong external field. As one reduces the external field, an instability eventually occurs. The corresponding field is called the critical field. This first instability of the saturation branch is called nucleation. This may or may not be related to an irreversible event, i. e. switching, see [15]. This depends on the type of bifurcation.

Mathematically speaking, the critical field is the value of the external field at which the Hessian of the micromagnetic energy functional ceases to be positive definite. The degenerate subspace consists of the “unstable modes”. The related eigenvalue problem has been explicitly and completely solved for special geometries like ellipsoids of revolution [5, 11, 2]. As a consequence of the multiscale nature of the problem, there are different types of unstable modes, depending on and defining parameter regimes. For instance, for sufficiently small samples, the unstable mode corresponds to a coherent rotation of the magnetization, as in the Stoner–Wohlfarth model [17]. For sufficiently large samples, the unstable mode corresponds to a curling of m which does not generate a stray field [5]. A third mode, which corresponds to a buckling of the magnetization, has been found numerically [11].

In this paper, we revisit the nucleation problem for a cylindrical geometry which mimics an elongated thin–film element. There are only partial results for cylindrical geometries. We identify exactly four scaling regimes in the two non–dimensional parameters. One of these regimes displays an oscillatory buckling mode and is novel in the sense that the period of oscillation is determined by a subtle interaction of geometry and material length scale. It is noteworthy that this thin–film buckling regime stretches over a wide range in parameter space. This is in contrast with Aharoni's claim that buckling plays only a minor role [2, p. 202].

Our analysis is coarse as it identifies only the scaling of the critical field. As opposed to the more traditional treatment, it is not based on the explicit solution of the eigenvalue problem. This allows us to derive a complete but qualitative picture.

Both guidance and motivation for our work has been the ubiquitous concertina pattern in soft (i. e. low crystalline anisotropy) thin–film elements.

Figure 1

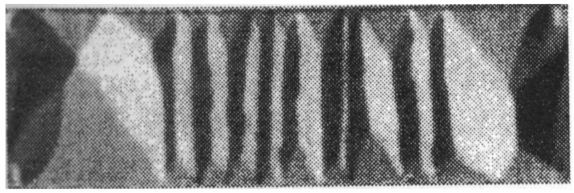


Figure 1: The concertina pattern

shows such a pattern seen from above for an elongated Permalloy thin–film element of $300nm$ thickness and $18\mu m$ width. The concertina pattern is the almost periodic microstructure in the center of the element's cross–section. It is formed by stripe–like domains separated by walls.

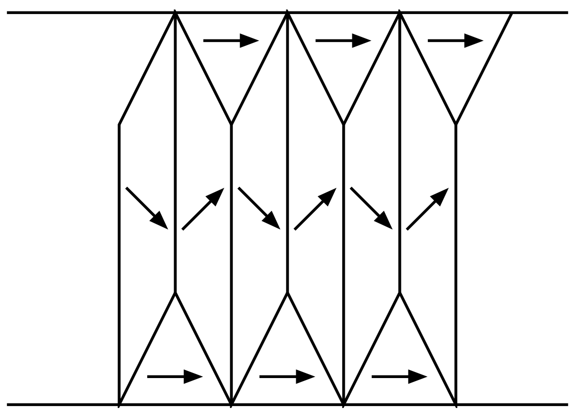


Figure 2: Mesoscopic magnetization

Figure 2 gives a sketch of the mesoscopic magnetization in the concertina pattern.

This pattern is experimentally generated as follows: First, the element is saturated along the long axis, then the external field is slowly reduced, eventually reversed. At some field strength, the uniform magnetization buckles into the concertina pattern. It is therefore tempting to make the following hypothesis: The period of the concertina pattern is the frozen–in period of the oscillatory buckling mode discovered here. In fact, this is the first article in a short series of papers in support of this hypothesis.

- In this paper we rigorously identify all the regimes for nucleation. The analysis is based on a variational characterization of the critical field in terms of the Rayleigh quotient of the Hessian at zero external field. We give upper and lower bounds, which match in terms of scaling in the non–dimensional parameters.

- In a companion paper, we asymptotically identify the unstable mode in the oscillatory buckling regime. This is done by identifying the Γ–limit of the Rayleigh quotient of the Hessian. We obtain a formula for the asymptotic oscillation period which allows for a quantitative comparison with concertina experiments.

- In a third paper, [6], we will examine the bifurcation in the oscillatory buckling regime more closely. Though it is a subcritical bifurcation, local minimizers for the energy can nevertheless still be found near the original groundstate. This can be ascertained by identifying the Γ-limit of a suitably renormalized version of the energy. As the resulting Γ-limit is coercive, no large shifts in the wavenumber can occur. Thus, in a certain sense we can expect the period of the unstable mode to be passed on to the concertina pattern.

Let us give a short synopsis of this paper's content. In an introductory section we review the basic notions central to our article. The subsequent section states the main result and gives a heuristic interpretation. Finally we give the proof of our main result in two sections – first we prove the upper bounds, then the lower bounds.

2 Introduction

2.1 The micromagnetic model

Our analysis is based on the micromagnetic model. Let $\Omega \subset \mathbb{R}^3$ describe the sample geometry. The micromagnetic model states that an experimentally observed magnetization $m\colon \Omega \to \mathbb{R}^3$ is a local minimum of the micromagnetic energy

$$
\begin{aligned}
E(m) &= d^2 \int_\Omega |\nabla m|^2 \, d^3x && (1)\\
&+ \int_{\mathbb{R}^3} |\nabla u_m|^2 \, d^3x && (2)\\
&- 2 \int_\Omega H_{ext} \cdot m \, d^3x && (3)
\end{aligned}
$$

among all m which satisfy the saturation constraint

$$|m|^2 \;=\; 1 \quad \text{in } \Omega. \tag{4}$$

This version of the model is partially non–dimensionalized: The magnetization m, the external field H_{ext}, the stray field $-\nabla u_m$, and the energy density are non–dimensional. Length on the other hand is dimensional.

Contribution (1) is the exchange energy, which is of quantum mechanical origin; d is the exchange length. Contribution (2) is the energy of the stray field $-\nabla u_m$. The stray field is determined by the static version of Maxwell's equations. They are conveniently stated in a distributional form:

$$\int_{\mathbb{R}^3} \nabla u_m \cdot \nabla\varphi \, d^3x \;=\; \int_\Omega m \cdot \nabla\varphi \, d^3x \quad \text{for all test functions } \varphi. \tag{5}$$

As can be seen from (5), there are two types of "magnetic charges" which give rise to a stray field:

$$\begin{array}{lll} \text{volume charges} & -\nabla \cdot m & \text{in } \Omega, \\ \text{surface charges} & \nu \cdot m & \text{on } \partial\Omega. \end{array}$$

Contribution (3) is the Zeeman term which models the interaction with the external field H_{ext}.

2.2 Geometry

We will work with the following sample geometry

$$\Omega \;=\; \mathbb{R} \times (0, \ell) \times (-\frac{t}{2}, \frac{t}{2}),$$

see Figure 3. The reasons for this choice are the following:

- Ω mimics an elongated thin–film element of thickness t and width $\ell \gg t$.

- Due to the translation invariance in x_1, Ω admits $m^* = (1,0,0)$ as a stationary point for *all* external fields of the form $H_{ext} = (-h_{ext}, 0, 0),\ h_{ext} \in \mathbb{R}$.

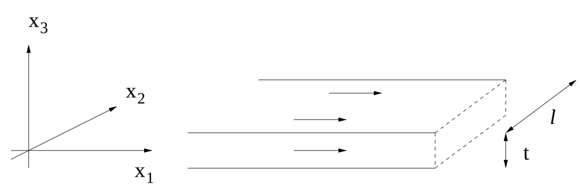


Figure 3: The geometry

2.3 Hessian

Due to the constraint (4), infinitesimal perturbations of $m^* = (1,0,0)$ are of the form

$$\zeta \;=\; (0, \zeta_2, \zeta_3), \quad \zeta \;=\; \zeta(x_1, x_2, x_3) \;\in\; \mathbb{R}^2. \tag{6}$$

An easy calculation shows that the Hessian $\mathrm{Hess}E(m^*)$ of E in m^* is given by

$$
\begin{aligned}
\frac{1}{2}\,\mathrm{Hess}E(m^*)(\zeta,\zeta) &= d^2\int_\Omega |\nabla\zeta|^2\,d^3x \\
&+ \int_{\mathbb{R}^3} |\nabla u_\zeta|^2\,d^3x \\
&- h_{ext}\int_\Omega |\zeta|^2\,d^3x, \qquad (7)
\end{aligned}
$$

where u_ζ is determined by ζ through (5).

2.4 Unstable mode / critical field

The critical field h_{crit} is the smallest h_{ext} for which $\mathrm{Hess}E(m^*)$ ceases to be positive definite. The unstable modes are the elements of the degenerate subspace of $\mathrm{Hess}E(m^*)$ for $h_{ext} = h_{crit}$. In the jargon of micromagnetics, this bifurcation is called nucleation. A variational characterization of both can be inferred from (7). Indeed, with the abbreviation

$$
\mathcal{R}(\zeta) \;=\; d^2\int_\Omega |\nabla\zeta|^2\,d^3x + \int_{\mathbb{R}^3} |\nabla u_\zeta|^2\,d^3x \qquad (8)
$$

we have

$$
\begin{aligned}
&\text{critical field} \\
&= \min\left\{\mathcal{R}(\zeta)\,\middle|\,\zeta \text{ as in (6) with } \int_\Omega |\zeta|^2 d^3x = 1\right\}, \qquad (9) \\
&\text{normalized unstable modes} \\
&= \operatorname{argmin}\left\{\mathcal{R}(\zeta)\,\middle|\,\zeta \text{ as in (6) with } \int_\Omega |\zeta|^2 d^3x = 1\right\}.
\end{aligned}
$$

2.5 Different types of modes

The critical field and the unstable modes can also be seen as the ground state for the operator

$$\mathcal{L}\zeta \;=\; -d^2\triangle_{\text{Neumann}}\zeta - \begin{pmatrix}\partial_2\\ \partial_3\end{pmatrix} u_\zeta. \tag{10}$$

Following [3], it is helpful to distinguish between "models" and "modes". A model is a special ansatz for an infinitesimal perturbation ζ. In view of (9), it gives an upper bound for h_{crit}. A mode is an eigenfunction of (10) — but only the eigenfunction corresponding to the smallest eigenvalue yields h_{crit} and the degenerate subspace.

We now discuss the physics literature. There, next to infinite prisms like our Ω, also ellipsoids have been considered, since they also allow for constant stationary states m^*. Brown [5] found two modes for ellipsoids of rotation: The first mode corresponds to a coherent rotation, the second mode corresponds to a curling of the magnetization. The characterizing feature of the curling mode is the complete absence of a stray field, i. e., no surface or volume charges are generated by this mode. Brown also found that for sufficiently small samples (w. r. t. d), the coherent mode has a lower eigenvalue than the curling mode.

In [11], three models are compared for an infinite circular cylinder: Coherent rotation, Brown's curling mode and a model for buckling. It is found that only in a small size range the buckling model beats the two modes. In [4], the infinite cylinder is investigated more systematically: Cylindrical coordinates (x_1, r, ϕ) reduce (10) to a 1–d problem in r parametrized by $(k_1, n) \in \mathbb{R} \times \mathbb{Z}$. The case $n = 0$ is treated completely, the lowest eigenvalue occurs for $k_1 = 0$ and corresponds to the curling mode. By a lower bound estimate, the cases $n \geq 2$ are discarded for nucleation. The case $n = 1$ is treated numerically, the buckling model of [11] is found to be close to an actual mode.

In [1], an infinite prism with rectangular cross–section is considered. This is the geometry considered by us, but in [1], no consideration was given to extreme aspect ratios of the rectangular cross–section, i. e. $t \ll \ell$. Guided by [4], modes and models which depend on the infinite direction z are ignored. Upper and lower bounds for h_{crit} are given.

Based on these works, Aharoni [3, p. 202] emphatically rules out any other type of unstable mode besides coherent rotation, buckling and curling.

3 The main result

3.1 Method

Despite the Cartesian setting, the Hessian operator $\mathcal{L}$ coming from (10) does not seem to be completely diagonalizable in an explicit way. Of course, it factorizes when Fourier transformed in the "infinite" direction x_1, a fact we will use in Subsection 4.2. But in the "finite" directions x_2 and x_3, the magnetostatic field contribution is diagonalized by Fourier transform whereas the exchange contribution is diagonalized by Fourier cosine series. This discrepancy is due to the fact that the non–local magnetostatic field contribution comes from Maxwell's equations in the entire space $\mathbb{R}^3$, whereas the local exchange contribution only "lives" in the sample Ω.

We take the approach of establishing upper and lower bounds for the critical field. This analysis is based on the Rayleigh quotient representation (9) rather than the diagonalization of $\mathcal{L}$. In this paper, it is performed in a way that lies between the concepts of "models" and "modes", as we shall explain now. Exact matching of "model"–induced upper bounds and ansatz–free lower bounds would give the exact value of $h_{crit.}$ and the corresponding "modes". Our approach is much more modest: We will match upper and lower bounds only in terms of scaling. By dimensional analysis, $h_{crit.}$ is a universal function of the two non–dimensional parameters $\frac{\ell}{d}$ and $\frac{t}{d}$ only:

$$h_{crit.} \;=\; h_{crit.}(\frac{\ell}{d},\frac{t}{d}).$$

This function has different scalings in $(\frac{\ell}{d},\frac{t}{d})$ in different regimes. We identify *all* scaling regimes for the critical field. In this sense, our analysis lies between the concepts of "models" and "modes".

The robust approach of proving upper and lower bounds which match just in terms of scaling has been quite useful to identify different regimes in micromagnetism and related problems [7, 8, 16]. So far, this strategy has been applied to the energy E itself, yielding results on the ground state. Here instead, it is applied to the Rayleigh quotient of the Hessian, yielding results on metastable states. The proof of the lower bounds is the challenging part from the point of view of analysis. It relies on appropriate interpolation inequalities which express the leading order competition between two energetic contributions. The relevant interpolation inequalities of course depend on the regime.

3.2 Statement of rigorous result

Theorem 1.

Provided $d, t \ll \ell$ *we have*

$$h_{crit.} \sim \left\{ \begin{array}{lll} \frac{t}{\ell} \ln\left(\frac{\ell}{t}\right) & \textit{for} & t \leq \frac{d^2}{\ell} \ln^{-1}\left(\frac{\ell}{d}\right) \\ \\ \left(\frac{d}{\ell}\right)^2 & \textit{for} & \frac{d^2}{\ell} \ln^{-1}(\frac{\ell}{d}) \leq t \leq \frac{d^2}{\ell} \\ \\ \left(\frac{dt}{\ell^2}\right)^{2/3} & \textit{for} & \frac{d^2}{\ell} \leq t \leq (d\ell)^{1/2} \\ \\ \left(\frac{d}{t}\right)^2 & \textit{for} & (d\ell)^{1/2} \leq t \end{array} \right\}.$$

Remark 1.

Theorem 1 is stated in a short formulation which we will use throughout the paper. Its long version is the following: There exists a universal constant $0 < C < \infty$ *such that whenever* $d, t \leq \frac{1}{C}\ell$*, we have*

$$h_{crit.} \leq C \left\{ \begin{array}{lll} \frac{t}{\ell} \ln\left(\frac{\ell}{t}\right) & \textit{for} & t \leq \frac{d^2}{\ell} \ln^{-1}\left(\frac{\ell}{d}\right) \\ \\ \left(\frac{d}{\ell}\right)^2 & \textit{for} & \frac{d^2}{\ell} \ln^{-1}(\frac{\ell}{d}) \leq t \leq \frac{d^2}{\ell} \\ \\ \left(\frac{dt}{\ell^2}\right)^{2/3} & \textit{for} & \frac{d^2}{\ell} \leq t \leq (d\ell)^{1/2} \\ \\ \left(\frac{d}{t}\right)^2 & \textit{for} & (d\ell)^{1/2} \leq t \end{array} \right\}$$

and

$$h_{crit.} \geq \frac{1}{C} \left\{ \begin{array}{lll} \frac{t}{\ell} \ln\left(\frac{\ell}{t}\right) & for & t \leq \frac{d^2}{\ell} \ln^{-1}\left(\frac{\ell}{d}\right) \\ \\ \left(\frac{d}{\ell}\right)^2 & for & \frac{d^2}{\ell} \ln^{-1}(\frac{\ell}{d}) \leq t \leq \frac{d^2}{\ell} \\ \\ \left(\frac{dt}{\ell^2}\right)^{2/3} & for & \frac{d^2}{\ell} \leq t \leq (\ell d)^{1/2} \\ \\ \left(\frac{d}{t}\right)^2 & for & (\ell d)^{1/2} \leq t \end{array} \right\}.$$

Observe that the expressions for $h_{crit.}$ match continuously (in terms of scaling).

A $(\frac{t}{d}, \frac{\ell}{d})$ – phase diagram for the scaling of $h_{crit.}(\frac{t}{d}, \frac{\ell}{d})$ is a more intuitive way to present the result, see Figure 4. This result is in contradiction with Aharoni's claim that there are at most three regimes. We obtain four regimes, as there are two different buckling regimes. The second of these regimes (Regime III) is characterized by an internal length scale λ in the infinite direction x_1, as we shall discuss in Subsection 3.3.

Theorem 1 is a consequence of the following two theorems:

Theorem 2. *(Upper bounds)*

In the regime $d, t \ll \ell$ it holds

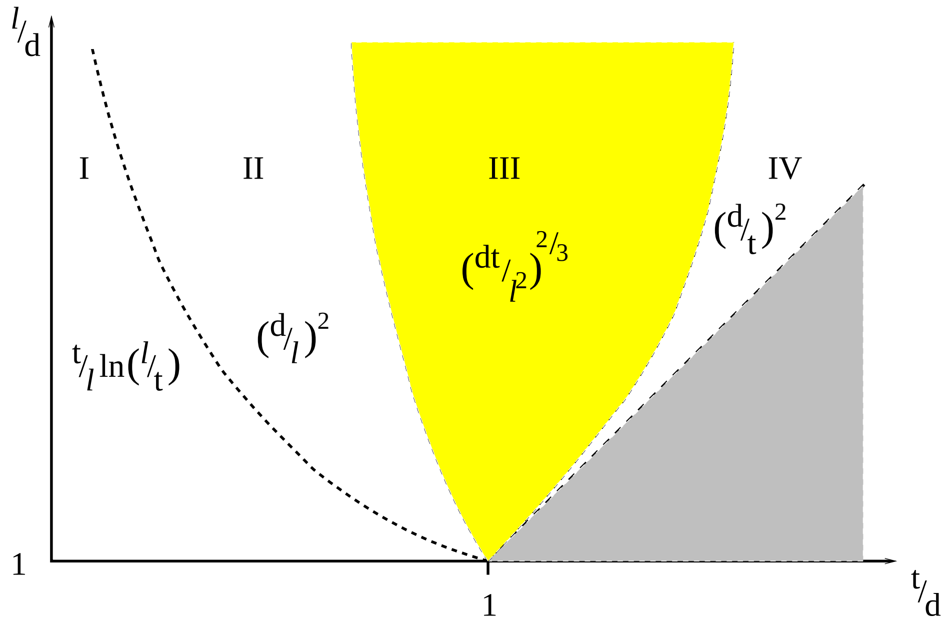


Figure 4: Graphical representation of Theorem 1

$$h_{crit.} \lesssim \min\left\{\frac{t}{\ell}\ln\left(\frac{\ell}{t}\right), \max\left\{\left(\frac{d}{\ell}\right)^2, \left(\frac{dt}{\ell^2}\right)^{2/3}\right\}, \left(\frac{d}{t}\right)^2\right\}.$$

Theorem 3. *(Lower bounds)*

In the regime $d, t \ll \ell$ it holds

$$h_{crit.} \gtrsim \left\{\begin{array}{lll} \min\left\{\frac{t}{\ell}\ln\left(\frac{\ell}{t}\right), \left(\frac{d}{\ell}\right)^2\right\} & \mathit{for} & t \leq \frac{d^2}{\ell} \\ \min\left\{\left(\frac{dt}{\ell^2}\right)^{2/3}, \left(\frac{d}{t}\right)^2\right\} & \mathit{for} & t \geq \frac{d^2}{\ell} \end{array}\right\}.$$

3.3 Heuristic interpretation

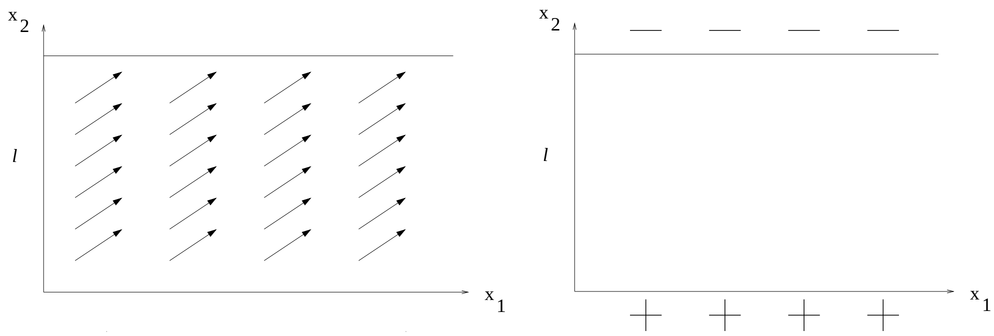
We now interpret these scaling results. This is done by discussion of the models which lead to the (optimal) upper bounds in Theorem 2. We go through the four regimes in the order of increasing thickness t (hence from Regime I to Regime IV). The models are determined by a subtle balance between the exchange energy and the magnetostatic energy. With increasing sample size the magnetostatic energy becomes more dominant: Regime I is completely dominated by exchange, Regime IV entirely by magnetostatics. The relative importance of surface charges and volume charges (which are at the origin of the magnetostatic energy) depends on size too: With increasing thickness the volume charges become more important. An interesting feature is dimensional reduction: The model in Regime I is constant, in Regime II it depends on x_2, in Regime III it depends on x_1 and x_2, whereas in Regime IV it depends on x_2 and x_3.

3.3.1 Regime I: Coherent rotation

This regime is driven by the avoidance of an exchange contribution. The exchange energy favors a spatially constant perturbation. But the coherent rotation necessarily creates a non–tangential magnetization at the sample edges $\partial\Omega$, which is penalized by magnetostatics, see (5). A coherent rotation in the film plane (the x_1, x_2–plane) only creates surface charges at the small lateral edges $\mathbb{R} \times \{0, \ell\} \times (-\frac{t}{2}, \frac{t}{2})$. Hence the model is of the form:

$$\zeta_2 \;\equiv\; (\ell t)^{-1/2}, \quad \zeta_3 \;\equiv\; 0,$$

where the constant $(\ell t)^{-1/2}$ is chosen such that the constraint in (9) is satisfied. The corresponding finite perturbation is sketched in Figure 5.



$h_{crit.} \sim \frac{t}{\ell} \ln\left(\frac{\ell}{t}\right)$ Surface charges

Figure 5: Coherent rotation

Let us comment on the scaling of $h_{crit.}$. There is no exchange contribution; there is no volume charge. Hence the only contribution to $\mathrm{Hess}E_0(m^*)$ comes from the surface charges. The surface charge at the two lateral edges $\mathbb{R} \times \{0\} \times (-\frac{t}{2}, \frac{t}{2})$ and $\mathbb{R} \times \{\ell\} \times (-\frac{t}{2}, \frac{t}{2})$ has density $(\ell t)^{-1/2}$ resp. $-(\ell t)^{-1/2}$. On length scales larger than t, the surface charge behaves like a *line* charge on $\mathbb{R} \times \{0\} \times \{0\}$ and $\mathbb{R} \times \{\ell\} \times \{0\}$ with density $\ell^{-1/2}t^{1/2}$ resp. $-\ell^{-1/2}t^{1/2}$. Hence the magnetostatic contribution in this two–dimensional setting (i.e. per unit length in x_1) scales as

$$\int_{\mathbb{R}^3} |\nabla U_\zeta|^2 \, dx \;\sim\; (\ln \frac{\ell}{t})(\ell^{-1/2}t^{1/2})^2 \;=\; \frac{t}{\ell} \ln \frac{\ell}{t}.$$

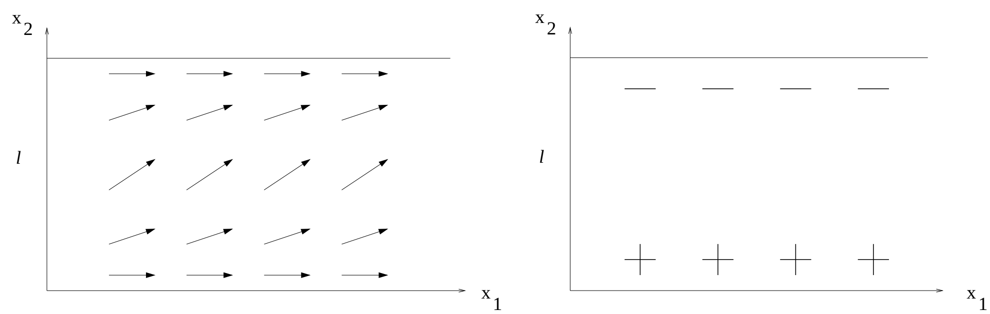
The argument of the logarithm is $\frac{\ell}{t}$ since t is the small scale cut-off and ℓ is the large scale cut-off. This yields the scaling of $h_{crit.}$.

3.3.2 Regime II: Non-oscillatory buckling

This regime is driven by the avoidance of surface charges: The magnetostatic influence is already strong enough to suppress any normal component at $\partial\Omega$. The exchange energy is still sufficiently strong to suppress variations in x_1- and x_3-directions. Hence we choose the model

$$\zeta_2 \;=\; \sqrt{2}(\ell t)^{-1/2}\sin(\frac{\pi x_2}{\ell}) \quad \text{and} \quad \zeta_3 \;\equiv\; 0.$$

Figure 6 shows the corresponding finite perturbation, which displays “Edge–pinning”.



$h_{crit.} \sim \left(\frac{d}{\ell}\right)^2$ Volume charges

Figure 6: Buckling

Let us again comment on the scaling of $h_{crit.}$. Surface charges are completely suppressed, but there are volume charges:

$$-\nabla\cdot\zeta \;=\; -\partial_2\zeta_2 \;=\; -\sqrt{2}\pi\ell^{-3/2}t^{-1/2}\cos(\frac{\pi x_2}{\ell}).$$

We assess their contribution: The volume charge density in $\mathbb{R}\times(0,\ell)\times(-\frac{t}{2},\frac{t}{2})$ scales as $\ell^{-3/2}t^{-1/2}$. For $t \ll \ell$, these charges behave like *surface* charges on

$\mathbb{R} \times (0, \ell) \times \{0\}$ with a density scaling as $\ell^{-3/2} t^{1/2}$. The energy of a surface charge on $\mathbb{R} \times (0, \ell) \times \{0\}$ with unit density would scale as ℓ^2 in our two–dimensional setting (i.e. per unit length in x_1). Hence we obtain for the magnetostatic contribution

$$\int_{\mathbb{R}^3} |\nabla U_\zeta|^2 \, dx \;\sim\; \ell^2 (t^{1/2} \ell^{-3/2})^2 \;=\; \frac{t}{\ell}.$$

On the other hand, the exchange contribution scales as

$$d^2 \int_\Omega |\nabla \zeta|^2 \, dx \;\sim\; \left(\frac{d}{\ell}\right)^2.$$

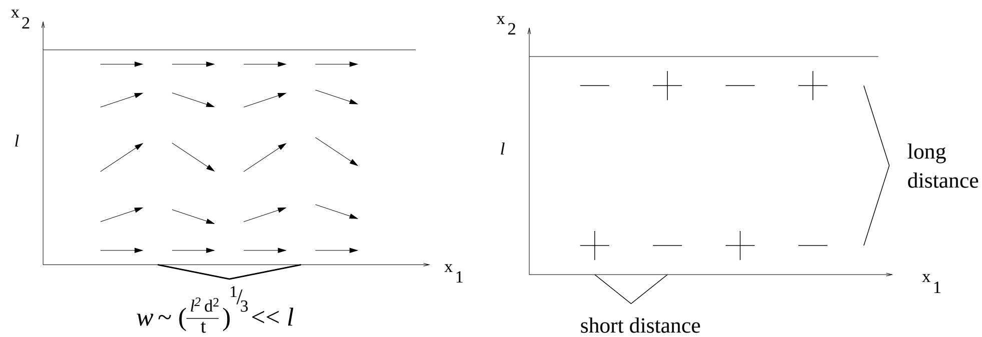
Hence in Regime II (i. e. $t \leq \frac{d^2}{\ell}$), the exchange contribution dominates the magnetostatic contribution and sets h_{crit}.

3.3.3 Regime III: Oscillatory buckling

This regime is driven by the competition of volume charges and exchange energy. As in Regime II, surface charges are suppressed at the expense of the exchange energy (edge pinning). This generates volume charges. As opposed to Regime II, volume charges do matter. Volume charges can be reduced by modulating the model from Regime II in the x_1–direction, see Figure 7. If the length scale w of this modulation is much smaller than ℓ, volume charges cancel over a length scale of w instead of ℓ, cf. Figure 7. We choose the following model:

$$\zeta_2 \;=\; 2(\ell t)^{-1/2} \cos(\frac{2\pi x_1}{w}) \sin(\frac{\pi x_2}{\ell}) \quad \text{and} \quad \zeta_3 \;\equiv\; 0.$$

Hence w is the period of oscillation in x_1.



$h_{crit.} \sim \left(\frac{dt}{\ell^2}\right)^{2/3}$ Volume charges

Figure 7: Oscillation

We now address the scaling of w and $h_{crit.}$. Provided $w \ll \ell$, the dominant part of the exchange energy comes from

$$\partial_1 \zeta_2 \;=\; 4\pi(\ell t)^{-1/2} w^{-1} \, \sin(\frac{2\pi x_1}{w}) \, \sin(\frac{\pi x_2}{\ell}).$$

Thus the exchange contribution scales as

$$d^2 \int_\Omega |\nabla \zeta|^2 \, dx \;\sim\; \left(\frac{d}{w}\right)^2. \tag{11}$$

There are no surface charges. The volume charge density in $\mathbb{R} \times (0,\ell) \times (-\frac{t}{2}, \frac{t}{2})$ is of the form

$$-\nabla \cdot \zeta \;=\; -\partial_2 \zeta_2 \;=\; -2\pi \ell^{-3/2} t^{-1/2} \, \cos(\frac{2\pi x_1}{w}) \, \cos(\frac{\pi x_2}{\ell}).$$

Provided $w \gg t$, this behaves like a *surface* charge density on $\mathbb{R} \times (0,\ell) \times \{0\}$ of the form

$$-2\pi \, \ell^{-3/2} t^{1/2} \, \cos(\frac{2\pi x_1}{w}) \, \cos(\frac{\pi x_2}{\ell}). \tag{12}$$

To leading order in $w \ll \ell$, the magnetostatic potential U_ζ generated by this surface charge density is of the form

$$U_\zeta \;\approx\; \frac{1}{2}\,\ell^{-3/2} t^{1/2} w\,\cos(\frac{2\pi x_1}{w})\,\cos(\frac{\pi x_2}{\ell})\,\exp(-\frac{2\pi|x_3|}{w}). \tag{13}$$

Hence the magnetostatic contribution $\int_{\mathbb{R}^3} |\nabla U_\zeta|^2\,dx$, which can be computed as the integral of the product of (12) and (13) over $\mathbb{R}\times(0,\ell)\times\{0\}$, scales as follows:

$$\int_{\mathbb{R}^3} |\nabla U_\zeta|^2\,dx \;\sim\; \frac{tw}{\ell}. \tag{14}$$

We see that the sum of exchange contribution (11) and magnetostatic contribution (14) is minimized for

$$w \;\sim\; \left(\frac{d^2\ell^2}{t}\right)^{1/3}.$$

For this choice we have

$$\mathcal{R}(\zeta) \;\sim\; \left(\frac{dt}{\ell^2}\right)^{\frac{2}{3}}. \tag{15}$$

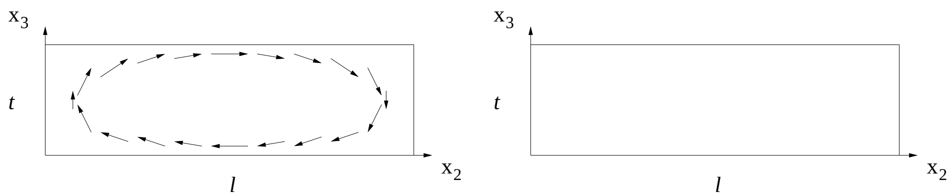
Note that the buckling model described in [11] exhibits oscillations with a dimensionless wavelength, as there is only one reduced parameter $\frac{r}{d}$ for the cylinder. In contrast, the wavelength in the present case is of the right dimensionality.

3.3.4 Regime IV: Curling

This regime is driven by the avoidance of both surface and volume charges. Hence the model ζ should be tangential to $\partial\Omega$ and divergence–free in Ω. We make the following choice:

$$\zeta_2 \;=\; 2(\ell t)^{-1/2}\sin(\frac{\pi x_2}{\ell})\,\sin(\frac{\pi\,x_3}{t}) \quad \text{and} \quad \zeta_3 \;=\; 2(\ell t)^{-1/2}\frac{t}{\ell}\,\cos(\frac{\pi x_2}{\ell})\,\cos(\frac{\pi\,x_3}{t}). \tag{16}$$

The resulting finite perturbations are helicoidal in nature, see Figure 8.



$h_{crit.} \sim \left(\frac{d}{t}\right)^2$ Charge-free

Figure 8: Curling

The scaling of $h_{crit.}$ is easy to explain: There is no contribution from the magnetostatic energy. In the exchange energy, the contribution from $\partial_3\zeta_2$ is dominant and scales as $(d/t)^2$, which yields the scaling of $h_{crit.}$.

4 Proof of Theorem 2: Upper bounds

In this section we argue that the models introduced in Subsection 3.3 indeed lead to the upper bounds stated in Theorem 2.

4.1 Magnetostatics in Fourier space

As can be easily deduced from the variational formulation (5), the magnetostatic contribution to the Hessian can be expressed in Fourier space as

$$\int_{\mathbb{R}^3} |\nabla U_\zeta|^2 \, dx \;=\; \int_{\mathbb{R}^3} \frac{|\xi \cdot \hat{\zeta}|^2}{|\xi|^2} \, d\xi. \tag{17}$$

Note that $\hat{\zeta} = \hat{\zeta}(\xi)$ is the Fourier transform of the ζ when continued by zero on Ω^c.

4.2 Perturbations of fixed wave number

The models described in Subsection 3.3 have one feature in common: They are either constant or of prescribed wavenumber in the infinite direction x_1. As such, they are not in $L^2(\Omega)$. The next lemma deals with this technicality.

Definition 1. *(Reduced Hessian)*

Let $\tilde{x}$ and its Fourier dual $\tilde{\xi}$ denote the "finite" coordinates, i. e.

$$\tilde{x} \;=\; (x_2, x_3)^T \quad \textit{resp.} \quad \tilde{\xi} \;=\; (\xi_2, \xi_3)^T, \quad \tilde{\Omega} \;=\; (0, \ell) \times (-\frac{t}{2}, \frac{t}{2}).$$

For fixed wave number ξ_1^0 and for given $\tilde{\zeta} = (\tilde{\zeta}_2, \tilde{\zeta}_3)^T : \tilde{\Omega} \to \mathbb{R}^2$ we define the reduced Hessian:

$$\begin{aligned}
&\frac{1}{2}\mathrm{Hess} E_0^{\xi_1^0}(m^*)(\tilde{\zeta}, \tilde{\zeta}) \\
&\quad = \; d^2 \int_{\tilde{\Omega}} \left(|\tilde{\nabla}\tilde{\zeta}|^2 + |\xi_1^0|^2 |\tilde{\zeta}|^2 \right) d\tilde{x} + \int_{\mathbb{R}^2} \frac{|\tilde{\xi} \cdot \hat{\tilde{\zeta}}|^2}{|\xi_1^0|^2 + |\tilde{\xi}|^2} \, d\tilde{\xi}.
\end{aligned}$$

Lemma 1. *(Reduced Rayleigh quotient)*

We have for any ξ_1^0*:*

$$\inf\left\{\frac{\mathrm{Hess}E_0(m^*)(\zeta,\zeta)}{2\int_\Omega|\zeta|^2\,dx}\;\Big|\;\zeta\;=\;(\zeta_2,\zeta_3)^T:\Omega\to\mathbb{R}^2\right\}$$
$$\leq\;\inf\left\{\frac{\mathrm{Hess}E_0^{\xi_1^0}(m^*)(\tilde\zeta,\tilde\zeta)}{2\int_{\tilde\Omega}|\tilde\zeta|^2\,d\tilde x}\;\Big|\;\tilde\zeta\;=\;(\tilde\zeta_2,\tilde\zeta_3)^T:\tilde\Omega\to\mathbb{R}^2\right\}.$$

Proof of Lemma 1. Let a test function $\tilde\zeta=\tilde\zeta(\tilde x)$ for the right hand side be given. Since $e^{i\xi_1^0x_1}\tilde\zeta(\tilde x)$ is not in $L^2(\Omega)$, we cut it off for $|x_1|\gg R$:

$$\zeta_R(x_1,\tilde x)\;=\;\eta\left(\frac{x_1}{R}\right)e^{i\xi_1^0x_1}\tilde\zeta(\tilde x),$$

where $\eta\in C_0^\infty(\mathbb{R})$ is a fixed cut–off function. We will show that this cut–off does not interfere in the limit $R\to\infty$:

$$\lim_{R\to\infty}\frac{\mathrm{Hess}E_0(m^*)(\zeta_R,\zeta_R)}{\int_\Omega|\zeta_R|^2\,dx}\;=\;\frac{\mathrm{Hess}E_0^{\xi_1^0}(m^*)(\tilde\zeta,\tilde\zeta)}{\int_{\tilde\Omega}|\tilde\zeta|^2\,d\tilde x},\tag{18}$$

which obviously entails the claim of this lemma. With the notation $x_1=R\hat x_1$ we have

$$\int_\Omega|\zeta_R|^2\,dx\;=\;R\int_{\mathbb{R}}\eta^2\,d\hat x_1\int_{\tilde\Omega}|\tilde\zeta|^2\,d\tilde x.\tag{19}$$

Since

$$\nabla\zeta_R\;=\;\left(\begin{array}{c}\frac{1}{R}\frac{d\eta}{d\hat x_1}e^{i\xi_1^0x_1}\tilde\zeta+i\,\xi_1^0\eta e^{i\xi_1^0x_1}\tilde\zeta\\[2ex]\eta e^{i\xi_1^0x_1}\tilde\nabla\tilde\zeta\end{array}\right),$$

we obtain for the exchange contribution (without d^2)

$$
\begin{aligned}
&\int_\Omega |\nabla \zeta_R|^2\, dx \\
&\quad = \frac{1}{R}\int_{\mathbb{R}} \left(\frac{d\eta}{d\hat{x}_1}\right)^2 d\hat{x}_1 \int_{\tilde{\Omega}} |\tilde{\zeta}|^2\, d\tilde{x} + R|\xi_1^0|^2 \int_{\mathbb{R}} \eta^2\, d\hat{x}_1 \int_{\tilde{\Omega}} |\tilde{\zeta}|^2\, d\tilde{x} \\
&\quad + R\int_{\mathbb{R}} \eta^2\, d\hat{x}_1 \int_{\tilde{\Omega}} |\tilde{\nabla}\tilde{\zeta}|^2\, d\tilde{x} \\
&\quad = R\int_{\mathbb{R}} \eta^2\, d\hat{x}_1 \left(\int_{\tilde{\Omega}} \left(|\tilde{\nabla}\tilde{\zeta}|^2 + |\xi_1^0|^2|\tilde{\zeta}|^2\right) d\tilde{x} + O(\frac{1}{R^2}) \int_{\tilde{\Omega}} |\tilde{\zeta}|^2\, d\tilde{x}\right). \quad (20)
\end{aligned}
$$

For the magnetostatic contribution we use the Fourier space representation (17). Since the Fourier transform of ζ_R factorizes, we have

$$
\begin{aligned}
\int_{\mathbb{R}^3} \frac{|\xi\cdot\widehat{\zeta_R}|^2}{|\xi|^2}\, d\xi &= \int_{\mathbb{R}}\int_{\mathbb{R}^2} \frac{|\widehat{\left(\eta(\frac{x_1}{R})e^{i\xi_1^0 x_1}\right)}(\xi_1)|^2 |\tilde{\xi}\cdot\hat{\tilde{\zeta}}|^2}{|\xi_1|^2 + |\tilde{\xi}|^2}\, d\tilde{\xi}\, d\xi_1 \\
&= R\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} \frac{R\,|\hat{\eta}(R(\xi_1+\xi_1^0))|^2}{|\xi_1|^2+|\tilde{\xi}|^2}\, d\xi_1\right) |\tilde{\xi}\cdot\hat{\tilde{\zeta}}|^2\, d\tilde{\xi}.
\end{aligned}
$$

Observe that the inner integral

$$
\int_{\mathbb{R}} \frac{R\,|\hat{\eta}(R(\xi_1+\xi_1^0))|^2}{|\xi_1|^2+|\tilde{\xi}|^2}\, d\xi_1 = \int_{\mathbb{R}} \frac{R\,|\hat{\eta}(R(\xi_1-\xi_1^0))|^2}{|\xi_1|^2+|\tilde{\xi}|^2}\, d\xi_1
$$

is a convolution in ξ_1 with the kernel

$$
\xi_1 \mapsto R\,|\hat{\eta}\,(R\xi_1)\,|^2.
$$

Hence we have for all $\tilde{\xi} \neq 0$

$$
\begin{aligned}
\lim_{R\to\infty} \int_{\mathbb{R}} \frac{R\,|\hat{\eta}(R(\xi_1+\xi_1^0))|^2}{|\xi_1|^2+|\tilde{\xi}|^2}\, d\xi_1 &= \int_{\mathbb{R}} |\hat{\eta}(\hat{\xi}_1)|^2\, d\hat{\xi}_1 \frac{1}{|\xi_1^0|^2+|\tilde{\xi}|^2} \\
&= \int_{\mathbb{R}} \eta(\hat{x}_1)^2\, d\hat{x}_1 \frac{1}{|\xi_1^0|^2+|\tilde{\xi}|^2}.
\end{aligned}
$$

On the other hand we have

$$\left|\int_{\mathbb{R}} \frac{R\,|\hat\eta(R(\xi_1+\xi_1^0))|^2}{|\xi_1|^2+|\tilde\xi|^2}\,d\xi_1\right| \;\le\; \left|\int_{\mathbb{R}} \frac{R\,|\hat\eta(R(\xi_1+\xi_1^0))|^2}{|\tilde\xi|^2}\,d\xi_1\right| \;=\; \int_{\mathbb{R}} \eta(\hat x_1)^2\,d\hat x_1 \frac{1}{|\tilde\xi|^2}.$$

Hence by the dominated convergence theorem

$$\lim_{R\to\infty}\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}} \frac{R\,|\hat\eta(R(\xi_1+\xi_1^0))|^2}{|\xi_1|^2+|\tilde\xi|^2}\,d\xi_1\right)|\tilde\xi\cdot\hat{\tilde\zeta}|^2\,d\tilde\xi = \int_{\mathbb{R}} \eta(\hat x_1)^2\,d\hat x_1 \int_{\mathbb{R}^2}\frac{|\tilde\xi\cdot\hat{\tilde\zeta}|^2}{|\xi_1^0|^2+|\tilde\xi|^2}\,d\tilde\xi,$$

which yields

$$\int_{\mathbb{R}^3}\frac{|\tilde\xi\cdot\hat\zeta_R|^2}{|\xi|^2}\,d\xi \;=\; R\int_{\mathbb{R}}\eta^2\,d\hat x_1\int_{\mathbb{R}^2}\frac{|\tilde\xi\cdot\hat{\tilde\zeta}|^2}{|\xi_1^0|^2+|\tilde\xi|^2}\,d\tilde\xi + o(R). \tag{21}$$

Now (19), (20) and (21) imply (18).

q.e.d.

Note that the exponential phase factor was chosen for simplicity. An ansatz with a cosine modulation would give the same result, without introducing a complex–valued ζ.

4.3 Upper bounds

Proposition 1. *(Coherent rotation) For $t \ll \ell$ we have*

$$h_{crit.} \;\lesssim\; \frac{t}{\ell}\ln\left(\frac{\ell}{t}\right).$$

Proof of Proposition 1. In view of Lemma 1 it suffices to construct a $\tilde{\zeta} = (\tilde{\zeta}_2, \tilde{\zeta}_3)^T : \tilde{\Omega} \to \mathbb{R}^2$ such that

$$\mathrm{Hess} E_0^0(\tilde{\zeta}, \tilde{\zeta}) \;=\; d^2 \int_{\tilde{\Omega}} |\tilde{\nabla}\tilde{\zeta}|^2 \, d\tilde{x} + \int_{\mathbb{R}^2} \frac{|\tilde{\xi} \cdot \widehat{\tilde{\zeta}}|^2}{|\tilde{\xi}|^2} \, d\tilde{\xi} \;\lesssim\; \frac{t}{\ell} \ln(\frac{\ell}{t}) \int_{\tilde{\Omega}} |\tilde{\zeta}|^2 \, d\tilde{x}. \quad (22)$$

Following Subsection 3.3.1, the model shall be of the following form:

$$\tilde{\zeta}_2 \;\equiv\; (\ell t)^{-1/2} \quad \text{and} \quad \tilde{\zeta}_3 \;\equiv\; 0.$$

The exchange contribution vanishes for this $\tilde{\zeta}$. For the magnetostatic part we compute the Fourier transform of the trivially extended $\tilde{\zeta}$:

$$\begin{aligned} \frac{|\xi \cdot \widehat{\tilde{\zeta}}|^2}{|\tilde{\xi}|^2} \;&=\; (\ell t)^{-1} \frac{|\xi_2|^2}{|\tilde{\xi}|^2} \, |\widehat{\chi_{(0,\ell)}}(\xi_2)|^2 \, |\widehat{\chi_{(-\frac{t}{2},\frac{t}{2})}}(\xi_3)|^2 \\ &=\; (\ell t)^{-1} \frac{|\xi_2|^2}{|\tilde{\xi}|^2} \, \frac{\sin^2(\frac{\ell}{2}\xi_2)}{|\xi_2|^2} \, \frac{\sin^2(\frac{t}{2}\xi_3)}{|\xi_3|^2} \\ &\lesssim\; (\ell t)^{-1} \min\left\{1, \frac{|\xi_2|^2}{|\xi_3|^2}\right\} \min\left\{\ell^2, \frac{1}{|\xi_2|^2}\right\} \min\left\{t^2, \frac{1}{|\xi_3|^2}\right\}. \quad (23) \end{aligned}$$

We first carry out the integration in ξ_3:

$$
\begin{aligned}
&\int_{\mathbb{R}} \min\left\{1, \frac{|\xi_2|^2}{|\xi_3|^2}\right\} \min\left\{t^2, \frac{1}{|\xi_3|^2}\right\} \, d\xi_3 \\
&\quad \sim \left\{ \begin{array}{ll} \int_0^{|\xi_2|} t^2 \, d|\xi_3| + \int_{|\xi_2|}^{\frac{1}{t}} \frac{|\xi_2|^2}{|\xi_3|^2} t^2 \, d|\xi_3| + \int_{\frac{1}{t}}^{\infty} \frac{|\xi_2|^2}{|\xi_3|^2} \frac{1}{|\xi_3|^2} \, d|\xi_3| & \text{for} \quad |\xi_2| \le \frac{1}{t} \\ \int_0^{\frac{1}{t}} t^2 \, d|\xi_3| + \int_{\frac{1}{t}}^{|\xi_2|} \frac{1}{|\xi_3|^2} \, d|\xi_3| + \int_{|\xi_2|}^{\infty} \frac{|\xi_2|^2}{|\xi_3|^2} \frac{1}{|\xi_3|^2} \, d|\xi_3| & \text{for} \quad |\xi_2| \ge \frac{1}{t} \end{array} \right\} \\
&\quad \sim \left\{ \begin{array}{ll} t^2|\xi_2| + t^3|\xi_2|^2 & \text{for} \quad |\xi_2| \le \frac{1}{t} \\ t + \frac{1}{|\xi_2|} & \text{for} \quad |\xi_2| \ge \frac{1}{t} \end{array} \right\} \\
&\quad \sim \left\{ \begin{array}{ll} t^2|\xi_2| & \text{for} \quad |\xi_2| \le \frac{1}{t} \\ t & \text{for} \quad |\xi_2| \ge \frac{1}{t} \end{array} \right\} \\
&\quad \sim t \min\{t|\xi_2|, 1\} . \qquad (24)
\end{aligned}
$$

We thus obtain from (23) and (24)

$$
\begin{aligned}
&\int_{\mathbb{R}^2} \frac{|\xi \cdot \widehat{\tilde{\zeta}}|^2}{|\tilde{\xi}|^2} \, d\tilde{\xi} \\
&\quad \lesssim \quad \ell^{-1} \int_{\mathbb{R}} \min\left\{\ell^2, \frac{1}{|\xi_2|^2}\right\} \min\{t|\xi_2|, 1\} \, d\xi_2 \\
&\quad \overset{t \le \ell}{\approx} \quad \ell^{-1} \left(\int_0^{\frac{1}{\ell}} \ell^2 t|\xi_2| \, d|\xi_2| + \int_{\frac{1}{\ell}}^{\frac{1}{t}} \frac{1}{|\xi_2|^2} t|\xi_2| \, d|\xi_2| + \int_{\frac{1}{t}}^{\infty} \frac{1}{|\xi_2|^2} \, d|\xi_2| \right) \\
&\quad \sim \quad \ell^{-1} \left(t + t \ln \frac{\ell}{t} + t \right) \\
&\quad \overset{t \ll \ell}{\sim} \quad \frac{t}{\ell} \ln \frac{\ell}{t} .
\end{aligned}
$$

q.e.d.

The Regimes II and III can be handled simultaneously.

Proposition 2. *(Buckling and oscillations)*

$$h_{crit.} \lesssim \max\left\{ \left(\frac{d}{\ell}\right)^2, \left(\frac{dt}{\ell^2}\right)^{2/3} \right\}.$$

PROOF OF PROPOSITION 2. We will apply Lemma 1 to $\xi_1^0 = \frac{2\pi}{w}$, where the wave length w will be optimized later. As in Subsection (3.3.3), we choose

$$\tilde{\zeta}_2(x_2) \;=\; \sqrt{2}(\ell t)^{-1/2} \sin(\frac{\pi x_2}{\ell}) \quad \text{and} \quad \tilde{\zeta}_3 \;\equiv\; 0.$$

We have to show that

$$\begin{aligned} d^2 \int_{\tilde{\Omega}} \left(|\tilde{\nabla}\tilde{\zeta}|^2 + \left(\frac{2\pi}{w}\right)^2 |\tilde{\zeta}|^2 \right) d\tilde{x} + \int_{\mathbb{R}^2} \frac{|\xi \cdot \hat{\tilde{\zeta}}|^2}{\left(\frac{2\pi}{w}\right)^2 + |\tilde{\xi}|^2} \, d\tilde{\xi} \\ \lesssim \max\left\{ \left(\frac{d}{\ell}\right)^2, \left(\frac{dt}{\ell^2}\right)^{2/3} \right\}. \end{aligned} \tag{25}$$

The exchange contribution scales as

$$d^2 \int_{\tilde{\Omega}} \left(|\tilde{\nabla}\tilde{\zeta}|^2 + \left(\frac{2\pi}{w}\right)^2 |\tilde{\zeta}|^2 \right) d\tilde{x} \;\sim\; d^2 \left(\frac{1}{\ell^2} + \frac{1}{w^2} \right). \tag{26}$$

For the magnetostatic contribution we observe

$$\frac{|\xi \cdot \widehat{\tilde{\zeta}}|^2}{(\frac{2\pi}{w})^2 + |\tilde{\xi}|^2} \;=\; (\ell t)^{-1} \frac{|\xi_2|^2}{(\frac{2\pi}{w})^2 + |\tilde{\xi}|^2} \, |\widehat{\chi_{(0,\ell)}(x_2) \sin(\frac{\pi x_2}{\ell})}(\xi_2)|^2 \, |\widehat{\chi_{(-\frac{t}{2},\frac{t}{2})}}(\xi_3)|^2. \tag{27}$$

In order to estimate the Fourier transform of $\chi_{(0,\ell)}(x_2)\sin(\frac{\pi x_2}{\ell})$ we notice

$$\begin{aligned} &\frac{d^2}{dx_2^2}\left[\chi_{(0,\ell)}(x_2)\sin(\frac{\pi x_2}{\ell})\right] \\ &\quad = -\left(\frac{\pi}{\ell}\right)^2 \chi_{(0,\ell)}(x_2)\sin(\frac{\pi x_2}{\ell}) - \frac{\pi}{\ell}(\delta(x_2) + \delta(x_2 - \ell)). \end{aligned}$$

Hence the large frequency range $|\xi_2| \gg \frac{1}{\ell}$ is dominated by the Dirac contribution $\frac{\pi}{\ell}(\delta(x_2) + \delta(x_2 - \ell))$. Thus we have

$$|\widehat{\chi_{(0,\ell)}(x_2)\sin(\frac{\pi x_2}{\ell})}(\xi_2)|^2 \;\lesssim\; \min\left\{\ell^2, \frac{1}{\ell^2|\xi_2|^4}\right\}. \tag{28}$$

We obtain from (27) and (28)

$$\begin{aligned}
&\frac{|\xi\cdot\widehat{\tilde{\zeta}}|^2}{(\frac{2\pi}{w})^2 + |\tilde{\xi}|^2} \\
&\quad\lesssim\; (\ell t)^{-1}\min\left\{1, w^2|\xi_2|^2, \frac{|\xi_2|^2}{|\xi_3|^2}\right\}\min\left\{\ell^2, \frac{1}{\ell^2|\xi_2|^4}\right\}\min\left\{t^2, \frac{1}{|\xi_3|^2}\right\} \\
&\quad=\; (\ell t)^{-1}\left(\min\{1, w|\xi_2|\}\right)^2\min\left\{\ell^2, \frac{1}{\ell^2|\xi_2|^4}\right\} \\
&\quad\;\;\times\min\left\{1, \left(\max\left\{\frac{1}{w}, |\xi_2|\right\}\right)^2\frac{1}{|\xi_3|^2}\right\}\min\left\{t^2, \frac{1}{|\xi_3|^2}\right\}.
\end{aligned} \tag{29}$$

As in (24) (with $|\xi_2|$ replaced by $\max\left\{\frac{1}{w}, |\xi_2|\right\}$) we have

$$\begin{aligned}
&\int_{\mathbb{R}}\min\left\{1, \left(\max\left\{\frac{1}{w}, |\xi_2|\right\}\right)^2\frac{1}{|\xi_3|^2}\right\}\min\left\{t^2, \frac{1}{|\xi_3|^2}\right\}\,d\xi_3 \\
&\quad\sim\; t\min\left\{t\max\left\{\frac{1}{w}, |\xi_2|\right\}, 1\right\} \\
&\quad\leq\; t^2\max\left\{\frac{1}{w}, |\xi_2|\right\} \\
&\quad=\; t^2|\xi_2|\left(\min\{1, w|\xi_2|\}\right)^{-1}.
\end{aligned} \tag{30}$$

From (29) and (30) we gather for the magnetostatic contribution

$$\begin{aligned}
&\int_{\mathbb{R}^2} \frac{|\xi \cdot \hat{\tilde{\zeta}}|^2}{(\frac{2\pi}{w})^2 + |\tilde{\xi}|^2}\, d\tilde{\xi} \\
&\quad \lesssim \frac{t}{\ell} \int_{\mathbb{R}} |\xi_2| \min\{1, w|\xi_2|\} \min\left\{\ell^2, \frac{1}{\ell^2|\xi_2|^4}\right\} d\xi_2 \\
&\quad \leq \frac{tw}{\ell} \int_{\mathbb{R}} |\xi_2|^2 \min\left\{\ell^2, \frac{1}{\ell^2|\xi_2|^4}\right\} d\xi_2 \\
&\quad \sim \frac{tw}{\ell} \left(\int_0^{\frac{1}{\ell}} \ell^2|\xi_2|^2\, d|\xi_2| + \int_{\frac{1}{\ell}}^{\infty} \frac{1}{\ell^2|\xi_2|^2}\, d|\xi_2|\right) \\
&\quad \sim \frac{tw}{\ell^2}.
\end{aligned} \tag{31}$$

We conclude from (26) and (31)

$$\begin{aligned}
d^2 \int_{\tilde{\Omega}} \left(|\tilde{\nabla}\tilde{\zeta}|^2 + \left(\frac{2\pi}{w}\right)^2 |\tilde{\zeta}|^2\right) d\tilde{x} + \int_{\mathbb{R}^2} \frac{|\xi \cdot \hat{\tilde{\zeta}}|^2}{\left(\frac{2\pi}{w}\right)^2 + |\tilde{\xi}|^2}\, d\tilde{\xi} \\
\lesssim \frac{d^2}{\ell^2} + \frac{d^2}{w^2} + \frac{tw}{\ell^2}.
\end{aligned}$$

This convex expression in w is optimized (in terms of scaling) for

$$w = \left(\frac{d^2\ell^2}{t}\right)^{1/3},$$

yielding

$$\begin{aligned}
d^2 \int_{\tilde{\Omega}} \left(|\tilde{\nabla}\tilde{\zeta}|^2 + \left(\frac{2\pi}{w}\right)^2 |\tilde{\zeta}|^2\right) d\tilde{x} + \int_{\mathbb{R}^2} \frac{|\xi \cdot \hat{\tilde{\zeta}}|^2}{\left(\frac{2\pi}{w}\right)^2 + |\tilde{\xi}|^2}\, d\tilde{\xi} \\
\lesssim \frac{d^2}{\ell^2} + \frac{t^{2/3}d^{2/3}}{\ell^{4/3}} = \left(\left(\frac{d}{\ell}\right)^2 + \left(\frac{dt}{\ell^2}\right)^{2/3}\right).
\end{aligned} \tag{32}$$

q.e.d.

Proposition 3. *(Curling)*

$$h_{crit.} \lesssim \left(\frac{d}{t}\right)^2.$$

Proof of Proposition 3. We use Lemma 1 with $\xi_1^0 = 0$. We take the charge–free model given in (16), which comes from the stream function

$$\tilde{\Psi} = \frac{2\sqrt{t}}{\pi\sqrt{\ell}} \sin(\frac{\pi x_2}{\ell}) \cos(\frac{\pi x_3}{t}).$$

Indeed,

$$\tilde{\zeta} = \begin{pmatrix} -\partial_3 \tilde{\Psi} \\ \partial_2 \tilde{\Psi} \end{pmatrix} = \begin{pmatrix} 2(\ell t)^{-1/2} \sin(\frac{\pi x_2}{\ell}) \sin(\frac{\pi x_3}{t}) \\ 2(\ell t)^{-1/2} \frac{t}{\ell} \cos(\frac{\pi x_2}{\ell}) \cos(\frac{\pi x_3}{t}) \end{pmatrix}.$$

By construction, $\tilde{\zeta}$ is divergence–free. Since Ψ vanishes on $\partial\tilde{\Omega}$, the normal component of $\tilde{\zeta}$ vanishes there too. Hence $\tilde{\zeta}$ is charge–free. Therefore $\mathrm{Hess}E_0^0(m^*)(\tilde{\zeta}, \tilde{\zeta})$ has only the exchange contribution:

$$\begin{aligned} \mathrm{Hess}E_0^0(\tilde{\zeta}, \tilde{\zeta}) &= 2d^2 \int_{\tilde{\Omega}} |\tilde{\nabla}\tilde{\zeta}|^2 \, d\tilde{x} \\ &\sim d^2 \left(\frac{1}{\ell^2} + \frac{1}{t^2} + \left(\frac{t}{\ell}\right)^2 \frac{1}{\ell^2} + \left(\frac{t}{\ell}\right)^2 \frac{1}{t^2} \right) \\ &\overset{t \ll \ell}{\sim} \frac{d^2}{t^2}. \end{aligned}$$

On the other hand, we have

$$\int_{\tilde{\Omega}} |\tilde{\zeta}|^2 \, d\tilde{x} \sim \left(1 + \left(\frac{t}{\ell}\right)^2\right) \overset{t \ll \ell}{\sim} 1.$$

q.e.d.

Theorem 2 then follows by combining Propositions 1, 2 and 3.

5 Proof of Theorem 3: Lower bounds

We recall the representation (9) for the critical field

$$h_{crit.} \;=\; \min\left\{ \mathcal{R}(\zeta) \,\middle|\, \zeta \text{ as in (6) with } \int_\Omega |\zeta|^2 d^3x = 1 \right\}$$

and of the Hessian

$$\frac{1}{2}\mathrm{Hess}E_0(m^*)(\zeta,\zeta) \;=\; d^2 \int_\Omega |\nabla\zeta|^2\, dx + \int_{\mathbb{R}^3} |\nabla U_\zeta|^2\, dx.$$

Therefore $\mathrm{Hess}E_0(m^*)(\zeta,\zeta)$ has to be bounded below by $\int_\Omega |\zeta|^2\, dx$ for any admissible ζ.

Throughout the proofs, we will use the fact that Ω is a thin film. Therefore, it is convenient to combine the horizontal variables notation-wise.

Definition 2.

Let x' and its Fourier–dual ξ' denote the "horizontal" variables:

$$x' \;=\; (x_1, x_2)^T \quad \textit{resp.} \quad \xi' \;=\; (\xi_1, \xi_2)^T, \quad \Omega' \;=\; \mathbb{R} \times (0, \ell).$$

Let $\langle\zeta_i\rangle_3$ denote the average of ζ_i in the vertical variable:

$$\langle\zeta_i\rangle_3 \;:=\; \frac{1}{t}\int_{-\frac{t}{2}}^{\frac{t}{2}} \zeta_i\, dx_3.$$

The first step is to show in Lemma 2 that the magnetostatic part $\int_{\mathbb{R}^3} |\nabla U_\zeta|^2 dx$ controls $\langle\zeta_2\rangle_3$. The intuition is the following: On one hand $\int_{\mathbb{R}^3} |\nabla U_\zeta|^2 dx$

controls

$$\begin{aligned} \nabla\cdot\zeta &= \partial_2\zeta_2+\partial_3\zeta_3 \quad \text{for} \quad |x_3| \;<\; \tfrac{t}{2} \quad \text{and} \\ \nu\cdot\zeta &= \pm\zeta_3 \qquad\qquad \text{for} \quad x_3 \;=\; \pm\tfrac{t}{2}. \end{aligned}$$

On the other hand, we have

$$\partial_2\langle\zeta_2\rangle_3 \quad = \quad \langle\partial_2\zeta_2+\partial_3\zeta_3\rangle_3-\frac{1}{t}\left[\zeta_3\right]_{x_3=-\frac{t}{2}}^{x_3=\frac{t}{2}}.$$

This shows that $\int_{\mathbb{R}^3}|\nabla U_\zeta|^2dx$ controls $\langle\zeta_2\rangle_3$. The control is expressed in terms of a Fourier–based norm. Not surprisingly, the symbol which defines the norm is not homogeneous, but displays a cross–over at length scales of the order of t. Lemma 3 establishes the less subtle fact that $\langle\zeta_3\rangle_3$ can be estimated in terms of the magnetostatic part and the *full* ζ_2–component.

Lemma 2. *(Magnetostatic estimate of* $\langle\zeta_2\rangle_3$*)*

We have for any admissible ζ

$$t\int_{\mathbb{R}^2}\min\left\{\frac{t}{|\xi'|},\frac{1}{|\xi'|^2}\right\}|\xi_2\widehat{\langle\zeta_2\rangle}_3|^2\,d\xi' \;\lesssim\; \int_{\mathbb{R}^3}|\nabla U_\zeta|^2\,dx.$$

Lemma 3. *(Magnetostatic estimate of* $\langle\zeta_3\rangle_3$*)*

We have for any admissible ζ

$$t\int_{\mathbb{R}^2}\min\left\{1,\frac{1}{t^2|\xi'|^2}\right\}|\widehat{\langle\zeta_3\rangle}_3|^2\,d\xi' \;\lesssim\; \int_{\mathbb{R}^3}|\nabla U_\zeta|^2\,dx+\int_\Omega|\zeta_2|^2\,dx.$$

Proof of Lemma 2. We start by constructing a test function $\varphi=\varphi(x)$ for (5). Be $\varphi'=\varphi'(x')$ arbitrary and for the time being fixed. Let $\varphi=\varphi(x)$ be the harmonic extension of φ' beyond $\mathbb{R}^2 \times (-\frac{t}{2}, \frac{t}{2})$, that is, φ is continuous and satisfies

$$\begin{aligned} \varphi(x', x_3) &= \varphi'(x') \quad \text{for } |x_3| \leq \tfrac{t}{2}, \\ \Delta\varphi &= 0 \qquad\quad\ \ \text{for } |x_3| \geq \tfrac{t}{2}. \end{aligned}$$

We use φ as test function in (5):

$$\begin{aligned} \int_{\mathbb{R}^3} \nabla U_\zeta \cdot \nabla\varphi\, dx &= \int_{\mathbb{R}^2\times(-\frac{t}{2},\frac{t}{2})} \zeta \cdot \nabla\varphi\, dx \\ &= \int_{\mathbb{R}^2\times(-\frac{t}{2},\frac{t}{2})} \zeta_2\, \partial_2\varphi'\, dx \\ &= t \int_{\mathbb{R}^2} \langle\zeta_2\rangle_3\, \partial_2\varphi'\, dx'. \end{aligned} \tag{33}$$

We estimate the left hand side of (33) by Cauchy–Schwarz:

$$\begin{aligned} &\left|\int_{\mathbb{R}^3} \nabla U_\zeta \cdot \nabla\varphi\, dx\right| \\ &\quad\leq \left(\int_{\mathbb{R}^3} |\nabla U_\zeta|^2\, dx \int_{\mathbb{R}^3} |\nabla\varphi|^2\, dx\right)^{1/2} \\ &\quad= \left(\int_{\mathbb{R}^3} |\nabla U_\zeta|^2\, dx\right)^{1/2} \left(\int_{\mathbb{R}^2\times(\mathbb{R}\setminus(-\frac{t}{2},\frac{t}{2}))} |\nabla\varphi|^2\, dx + t\int_{\mathbb{R}^2} |\nabla'\varphi'|^2\, dx'\right)^{1/2}, \end{aligned}$$

so that (33) turns into the estimate

$$\begin{aligned} &\left|t\int_{\mathbb{R}^2} \langle\zeta_2\rangle_3\, \partial_2\varphi'\, dx'\right| \\ &\quad\leq \left(\int_{\mathbb{R}^3} |\nabla U_\zeta|^2\, dx\right)^{1/2} \\ &\qquad\times \left(\int_{\mathbb{R}^2\times(\mathbb{R}\setminus(-\frac{t}{2},\frac{t}{2}))} |\nabla\varphi|^2\, dx + t\int_{\mathbb{R}^2} |\nabla'\varphi'|^2\, dx'\right)^{1/2}. \end{aligned} \tag{34}$$

We now express this estimate in terms of the Fourier transform $\widehat{\varphi'}$ of φ'. The Dirichlet integral of the harmonic extension of φ' can be expressed in terms of $\widehat{\varphi}'$ as follows:

$$\int_{\mathbb{R}^2\times(\mathbb{R}\setminus(-\frac{t}{2},\frac{t}{2}))} |\nabla\varphi|^2\,dx \;\sim\; \int_{\mathbb{R}^2} |\xi'||\widehat{\varphi}'|^2\,d\xi'. \tag{35}$$

Furthermore, we have

$$t\int_{\mathbb{R}^2} |\nabla'\varphi'|^2\,dx' \;=\; t\int_{\mathbb{R}^2} |\xi'|^2|\widehat{\varphi}'|^2\,d\xi' \tag{36}$$

and

$$t\int_{\mathbb{R}^2} \langle\zeta_2\rangle_3\,\partial_2\varphi'\,dx' \;=\; t\int_{\mathbb{R}^2} \widehat{\langle\zeta_2\rangle_3}\,(-i\xi_2)\widehat{\varphi}'^{\,*}\,d\xi', \tag{37}$$

where * denotes complex conjugation. In view of (35), (36) and (37), the estimate (34) turns into

$$\begin{aligned} t\,&\left|\int_{\mathbb{R}^2} \xi_2\,\widehat{\langle\zeta_2\rangle_3}\,\widehat{\varphi}'^{\,*}\,d\xi'\right| \\ &\lesssim \left(\int_{\mathbb{R}^3} |\nabla U_\zeta|^2\,dx\right)^{1/2} \left(\int_{\mathbb{R}^2} (|\xi'|+t|\xi'|^2)\,|\widehat{\varphi}'|^2\,d\xi'\right)^{1/2}. \end{aligned}$$

The choice

$$\widehat{\varphi}'(\xi') \;=\; \frac{\xi_2\,\widehat{\langle\zeta_2\rangle_3}}{|\xi'|+t|\xi'|^2}$$

leads to the estimate

$$t^2\int_{\mathbb{R}^2} \frac{1}{|\xi'|+t|\xi'|^2}|\xi_2\widehat{\langle\zeta_2\rangle_3}|^2\,d\xi' \;\lesssim\; \int_{\mathbb{R}^3} |\nabla U_\zeta|^2\,dx.$$

The relation

$$\frac{1}{|\xi'|+t|\xi'|^2} \;\sim\; \min\left\{\frac{1}{|\xi'|},\frac{1}{t|\xi'|^2}\right\}$$

completes the proof.

q.e.d.

Proof of Lemma 3. Let again $\varphi' = \varphi'(x')$ be arbitrary and for the time being fixed. Let $\varphi = \varphi(x)$ be the harmonic extension of $x_3\,\varphi'(x')$ beyond $\mathbb{R}^2 \times (-\frac{t}{2}, \frac{t}{2})$, i. e. φ is continuous and satisfies

$$\begin{aligned} \varphi(x', x_3) &= x_3\,\varphi'(x') \quad \text{for } |x_3| \leq \tfrac{t}{2}, \\ \Delta\varphi &= 0, \quad\quad\quad\quad \text{for } |x_3| \geq \tfrac{t}{2}. \end{aligned}$$

We use φ as a test function in (5):

$$\begin{aligned} \int_{\mathbb{R}^3} \nabla U_\zeta \cdot \nabla\varphi\,dx &= \int_{\mathbb{R}^2\times(-\frac{t}{2},\frac{t}{2})} \zeta \cdot \nabla\varphi\,dx \\ &= \int_{\mathbb{R}^2\times(-\frac{t}{2},\frac{t}{2})} \zeta_2\,x_3\,\partial_2\varphi'\,dx + \int_{\mathbb{R}^2\times(-\frac{t}{2},\frac{t}{2})} \zeta_3\,\varphi'\,dx \\ &= \int_{\mathbb{R}^2\times(-\frac{t}{2},\frac{t}{2})} \zeta_2\,x_3\,\partial_2\varphi'\,dx + t\int_{\mathbb{R}^2} \langle\zeta_3\rangle_3\,\varphi'\,dx'. \end{aligned} \tag{38}$$

We estimate the left hand side of (38) as follows

$$\begin{aligned} &\left|\int_{\mathbb{R}^3} \nabla U_\zeta \cdot \nabla\varphi\,dx\right| \\ &\quad\leq \left(\int_{\mathbb{R}^3} |\nabla U_\zeta|^2\,dx \int_{\mathbb{R}^3} |\nabla\varphi|^2\,dx\right)^{1/2} \\ &\quad\leq \left(\int_{\mathbb{R}^3} |\nabla U_\zeta|^2\,dx\right)^{1/2} \\ &\quad\quad\times\left(t\int_{\mathbb{R}^2} \left(t^2|\nabla'\varphi'|^2 + |\varphi'|^2\right)\,dx' + \int_{\mathbb{R}^2\times(\mathbb{R}\setminus(-\frac{t}{2},\frac{t}{2}))} |\nabla\varphi|^2\,dx\right)^{1/2}. \end{aligned}$$

The first term on the right hand side of (38) is bounded by

$$\left|\int_{\mathbb{R}^2\times(-\frac{t}{2},\frac{t}{2})} \zeta_2\,x_3\,\partial_2\varphi'\,dx\right| \leq \left(\int_{\mathbb{R}^2\times(-\frac{t}{2},\frac{t}{2})} |\zeta_2|^2\,dx\;\; t^3\int_{\mathbb{R}^2} |\nabla'\varphi'|^2\,dx'\right)^{1/2}.$$

Hence (38) turns into the estimate

$$
\begin{aligned}
t & \left| \int_{\mathbb{R}^2} \langle \zeta_3 \rangle_3 \, \varphi' \, dx' \right| \\
& \le \left(\int_{\mathbb{R}^3} |\nabla U_\zeta|^2 \, dx + \int_{\mathbb{R}^2 \times (-\frac{t}{2}, \frac{t}{2})} |\zeta_2|^2 \, dx \right)^{1/2} \\
& \quad \times \left(t \int_{\mathbb{R}^2} \left(t^2 |\nabla' \varphi'|^2 + |\varphi'|^2 \right) \, dx' + \int_{\mathbb{R}^2 \times (\mathbb{R} \setminus (-\frac{t}{2}, \frac{t}{2}))} |\nabla \varphi|^2 \, dx \right)^{1/2} . \quad (39)
\end{aligned}
$$

As in the last lemma, we express this estimate in terms of $\widehat{\varphi}'$. Note that now, φ is the harmonic extension of $\frac{t}{2}\varphi'$ resp. of $-\frac{t}{2}\varphi'$ so that

$$
\int_{\mathbb{R}^2 \times (\mathbb{R} \setminus (-\frac{t}{2}, \frac{t}{2}))} |\nabla \varphi|^2 \, dx \;\sim\; t^2 \int_{\mathbb{R}^2} |\xi'| |\widehat{\varphi}'|^2 \, d\xi' \;\lesssim\; t \int_{\mathbb{R}^2} (t^2 |\xi'|^2 + 1) |\widehat{\varphi}'|^2 \, d\xi'.
$$

Hence we obtain from (39)

$$
\begin{aligned}
t & \left| \int_{\mathbb{R}^2} \widehat{\langle \zeta_3 \rangle_3} \, \widehat{\varphi}'^{*} \, d\xi' \right| \\
& \lesssim \left(\int_{\mathbb{R}^3} |\nabla U_\zeta|^2 \, dx + \int_{\mathbb{R}^2 \times (-\frac{t}{2}, \frac{t}{2})} |\zeta_2|^2 \, dx \right)^{1/2} \\
& \quad \times \left(t \int_{\mathbb{R}^2} \left(t^2 |\xi'|^2 + 1 \right) |\widehat{\varphi}'|^2 \, d\xi' \right)^{1/2} .
\end{aligned}
$$

We choose φ' such that

$$
\widehat{\varphi}'(\xi') \;=\; \frac{\widehat{\langle \zeta_3 \rangle_3}}{t^2 |\xi'|^2 + 1},
$$

so that the last estimate turns into

$$
t \int_{\mathbb{R}^2} \frac{1}{t^2 |\xi'|^2 + 1} |\widehat{\langle \zeta_3 \rangle_3}|^2 \, d\xi' \;\lesssim\; \int_{\mathbb{R}^3} |\nabla U_\zeta|^2 \, dx + \int_{\mathbb{R}^2 \times (-\frac{t}{2}, \frac{t}{2})} |\zeta_2|^2 \, dx.
$$

The relation

$$\frac{1}{t^2|\xi'|^2+1} \;\sim\; \min\left\{1, \frac{1}{t^2|\xi'|^2}\right\}$$

completes the proof.

q.e.d.

Lemma 2 and Lemma 3 used the fact that the variation ζ has finite support in x_3. This brought in the length scale t. Lemma 4 and Lemma 5 exploit the fact that ζ has finite support in x_2. This will introduce the length scale ℓ.

Lemma 4.

Let ζ be admissible. Then

$$\int_{\Omega'} |\langle\zeta_2\rangle_3|^2\,dx' \;\lesssim\; \int_{\mathbb{R}^2} \min\left\{\ell^2|\xi_2|^2, 1\right\} |\widehat{\langle\zeta_2\rangle_3}|^2\,d\xi'.$$

Lemma 5.

Let ζ be admissible. Then we have for a length scale $\tau \ll \ell$:

$$\int_{\Omega'} |\langle\zeta_3\rangle_3|^2\,dx' \;\lesssim\; \int_{|\xi_2|\le\frac{1}{\tau}} |\widehat{\langle\zeta_3\rangle_3}|^2\,d\xi' + \tau^2\int_{\Omega'} |\partial_2\langle\zeta_3\rangle_3|^2\,dx'.$$

Proof of Lemma 4. This 2–d estimate reduces to the 1–d estimate

$$\int_0^\ell |\zeta|^2\,dx_2 \;\lesssim\; \int_{\mathbb{R}} \min\{\ell^2|\xi_2|^2, 1\}|\hat{\zeta}|^2\,d\xi_2$$

for any $\zeta = \zeta(x_2)$ with $\mathrm{supp}\zeta \subset (0,\ell)$. This estimate is easy to see:

$$\begin{aligned}\int_0^\ell |\zeta|^2\,dx_2 &= \frac{1}{2}\int_{\mathbb{R}} |\zeta(x_2+\ell)-\zeta(x_2)|^2\,dx_2 \\ &= \frac{1}{2}\int_{\mathbb{R}} |e^{i\xi_2\ell}-1|^2|\hat{\zeta}|^2\,d\xi_2 \\ &\lesssim \int_{\mathbb{R}} \min\left\{\ell^2|\xi_2|^2, 1\right\}|\hat{\zeta}|^2\,d\xi_2.\end{aligned}$$

q.e.d.

Proof of Lemma 5. This 2–d estimate is a consequence of the 1–d estimate

$$\int_0^\ell |\zeta|^2\,dx_2 \;\lesssim\; \int_{|\xi_2|\le\frac{1}{\tau}} |\hat{\zeta}|^2\,d\xi_2 + \tau^2\int_0^\ell |\partial_2\zeta|^2\,dx_2$$

for $\zeta = \zeta(x_2)$ with $\mathrm{supp}\zeta \subset (0,\ell)$. By Plancherel, it is sufficient to show

$$\int_{|\xi_2|\ge\frac{1}{\tau}} |\hat{\zeta}|^2\,d\xi_2 \;\le\; \frac{1}{2}\int_0^\ell |\zeta|^2\,dx_2 + C\tau^2\int_0^\ell |\partial_2\zeta|^2\,dx_2 \tag{40}$$

with some universal constant C.

We notice that since $\partial_2|\zeta|^2 = 2\zeta\partial_2\zeta$, we have

$$\begin{aligned}\sup_{x_2\in(0,\ell)} |\zeta|^2 &\lesssim \frac{1}{\ell}\int_0^\ell |\zeta|^2\,dx_2 + \int_0^\ell |\zeta\partial_2\zeta|\,dx_2 \\ &\lesssim \frac{1}{\ell}\int_0^\ell |\zeta|^2\,dx_2 + \left(\int_0^\ell |\zeta|^2\,dx_2\int_0^\ell |\partial_2\zeta|^2\,dx_2\right)^{1/2}.\end{aligned}$$

In particular

$$|\zeta(0)|^2 + |\zeta(\ell)|^2 \;\lesssim\; \frac{1}{\ell}\int_0^\ell |\zeta|^2\,dx_2 + \left(\int_0^\ell |\zeta|^2\,dx_2\int_0^\ell |\partial_2\zeta|^2\,dx_2\right)^{1/2}. \tag{41}$$

We split $\hat{\zeta}$ according to

$$\begin{aligned}\hat{\zeta}(\xi_2) &= \int_0^\ell e^{i\xi_2 x_2}\zeta\, dx_2 \\ &= \frac{i}{\xi_2}\int_0^\ell e^{i\xi_2 x_2}\partial_2\zeta\, dx_2 - \frac{i}{\xi_2}\left[e^{i\xi_2 x_2}\zeta\right]_{x_2=0}^{x_2=\ell} \\ &= \hat{\zeta}^{(1)}(\xi_2) + \hat{\zeta}^{(2)}(\xi_2).\end{aligned}$$

We notice that $\frac{\xi_2}{i}\hat{\zeta}^{(1)}$ is just the Fourier transform of $\partial_2\zeta$ (extended by zero on $\mathbb{R}$). Thus, on one hand, we have by Plancherel's identity

$$\int_{|\xi_2|\geq\frac{1}{\tau}} |\hat{\zeta}^{(1)}|^2\, d\xi_2 \;\leq\; \tau^2\int_{\mathbb{R}} |\xi_2\hat{\zeta}^{(1)}|^2\, d\xi_2 \;=\; \tau^2\int_0^\ell |\partial_2\zeta|^2\, dx_2. \tag{42}$$

On the other hand, we have

$$\begin{aligned}&\int_{|\xi_2|\geq\frac{1}{\tau}} |\hat{\zeta}^{(2)}|^2\, d\xi_2 \\ &\quad\lesssim \int_{|\xi_2|\geq\frac{1}{\tau}} \frac{1}{|\xi_2|^2}\, d\xi_2\left(|\zeta(0)|^2 + |\zeta(\ell)|^2\right) \\ &\quad\overset{(41)}{\lesssim} \tau\left(\frac{1}{\ell}\int_0^\ell |\zeta|^2\, dx_2 + \left(\int_0^\ell |\zeta|^2\, dx_2\int_0^\ell |\partial_2\zeta|^2\, dx_2\right)^{1/2}\right).\end{aligned} \tag{43}$$

Combining (42) and (43), we obtain

$$\begin{aligned}&\int_{|\xi_2|\geq\frac{1}{\tau}} |\hat{\zeta}|^2\, d\xi_2 \\ &\quad\lesssim \frac{\tau}{\ell}\int_0^\ell |\zeta|^2\, dx_2 + \left(\tau^2\int_0^\ell |\zeta|^2\, dx_2\int_0^\ell |\partial_2\zeta|^2\, dx_2\right)^{1/2} + \tau^2\int_0^\ell |\partial_2\zeta|^2\, dx_2.\end{aligned}$$

This implies (40) by Young's inequality and because of $\tau \ll \ell$.

q.e.d.

We now can establish the lower bound in the regimes III and IV.

Proposition 4.

Assume $d, t \ll \ell$ and

$$t \;\geq\; \frac{d^2}{\ell}. \tag{44}$$

Then we have for any admissible ζ

$$d^2 \int_{\Omega} |\nabla\zeta|^2\, dx + \int_{\mathbb{R}^3} |\nabla U_\zeta|^2\, dx \;\gtrsim\; \min\left\{\left(\frac{dt}{\ell^2}\right)^{2/3}, \left(\frac{d}{t}\right)^2\right\} \int_{\Omega} |\zeta|^2\, dx.$$

PROOF OF PROPOSITION 4. We start by an estimate of $\int_{\Omega'} |\langle\zeta_2\rangle_3|^2 dx'$. According to Lemma 4, we don't need an estimate of the entire spectrum:

$$\int_{\Omega'} |\langle\zeta_2\rangle_3|^2\, dx' \;\lesssim\; \int_{\mathbb{R}^2} \min\{\ell^2|\xi_2|^2, 1\} |\widehat{\langle\zeta_2\rangle_3}|^2\, d\xi'. \tag{45}$$

Inspired by the model in Regime III, we introduce a wave length w with

$$t \;\leq\; w \;\leq\; \ell \tag{46}$$

in x_1–direction. w will be optimized later. Starting from (45), we split the frequency domain into $|\xi_1| \leq \frac{1}{w}$ and $|\xi_1| \geq \frac{1}{w}$:

$$\begin{aligned}\int_{\Omega'} |\langle\zeta_2\rangle_3|^2\, dx' \;&\lesssim\; \int_{|\xi_1| \leq \frac{1}{w}} \min\{\ell^2|\xi_2|^2, 1\} |\widehat{\langle\zeta_2\rangle_3}|^2\, d\xi' \\ &\quad + \int_{|\xi_1| \geq \frac{1}{w}} \min\{\ell^2|\xi_2|^2, 1\} |\widehat{\langle\zeta_2\rangle_3}|^2\, d\xi'.\end{aligned} \tag{47}$$

The high frequencies are easily estimated by exchange

$$
\begin{aligned}
\int_{|\xi_1|\geq\frac{1}{w}} \min\{\ell^2|\xi_2|^2,1\}|\widehat{\langle\zeta_2\rangle_3}|^2\,d\xi' &\leq \int_{|\xi_1|\geq\frac{1}{w}} |\widehat{\langle\zeta_2\rangle_3}|^2\,d\xi' \\
&\leq w^2\int_{\mathbb{R}^2}|\xi_1|^2|\widehat{\langle\zeta_2\rangle_3}|^2\,d\xi' \\
&= w^2\int_{\Omega'}|\partial_1\langle\zeta_2\rangle_3|^2\,dx'. \qquad (48)
\end{aligned}
$$

In order to estimate the low frequency part in (47) by the magnetostatic contribution via Lemma 2, we need the following lemma on the Fourier multiplier.

Lemma 6.

Under the assumptions of Proposition 4 and for w as in (46) we have

$$
\min\left\{\ell^2|\xi_2|^2,1\right\} \lesssim \frac{\ell^2}{tw}\min\left\{\frac{t}{|\xi'|},\frac{1}{|\xi'|^2}\right\}|\xi_2|^2 \quad \textit{for} \quad |\xi_1| \leq \frac{1}{w}.
$$

Before proving Lemma 6, we proceed with the proof of Proposition 4. According to Lemma 2 and Lemma 6, the low frequencies in (47) are estimated as follows

$$
\begin{aligned}
& t\int_{|\xi_1|\leq\frac{1}{w}} \min\{\ell^2|\xi_2|^2,1\}|\widehat{\langle\zeta_2\rangle_3}|^2\,d\xi' \\
&\quad\lesssim \frac{\ell^2}{tw}t\int_{\mathbb{R}^2}\min\left\{\frac{t}{|\xi'|},\frac{1}{|\xi'|^2}\right\}|\xi_2\widehat{\langle\zeta_2\rangle_3}|^2\,d\xi' \\
&\quad\lesssim \frac{\ell^2}{tw}\int_{\mathbb{R}^3}|\nabla U_\zeta|^2\,dx. \qquad (49)
\end{aligned}
$$

Now (47), (48) and (49) combine to

$$t \int_{\Omega'} |\langle \zeta_2 \rangle_3|^2 \, dx' \;\lesssim\; \frac{\ell^2}{tw} \int_{\mathbb{R}^3} |\nabla U_\zeta|^2 \, dx + w^2 t \int_{\Omega'} |\partial_1 \langle \zeta_2 \rangle_3|^2 \, dx'. \tag{50}$$

We now have to pass from the x_3–averaged quantities to the original ones. To this purpose, we will repeatedly use Poincaré's estimate in form of

$$\int_\Omega |\zeta_i|^2 \, dx \;\lesssim\; t \int_{\Omega'} |\langle \zeta_i \rangle_3|^2 \, dx' + t^2 \int_\Omega |\partial_3 \zeta_i|^2 \, dx \quad \text{for } i = 2, 3 \tag{51}$$

and Jensen's inequality in form of

$$t \int_{\Omega'} |\partial_j \langle \zeta_i \rangle_3|^2 \, dx' \;\leq\; \int_\Omega |\nabla \zeta|^2 \, dx \quad \text{for } i = 2, 3 \text{ and } j = 1, 2. \tag{52}$$

Applying (51) and (52) to (50) we obtain

$$\int_\Omega |\zeta_2|^2 \, dx \;\lesssim\; \frac{\ell^2}{tw} \int_{\mathbb{R}^3} |\nabla U_\zeta|^2 \, dx + \max\left\{ \left(\frac{w}{d}\right)^2, \left(\frac{t}{d}\right)^2 \right\} d^2 \int_\Omega |\nabla \zeta|^2 \, dx. \tag{53}$$

As in Subsection 3.3.3, we balance magnetostatic and exchange contributions by choosing

$$w \;=\; \left\{ \begin{array}{ll} \left(\frac{\ell^2 d^2}{t}\right)^{1/3} & \text{in Regime III, i. e. } \; t \;\leq\; (\ell d)^{1/2} \\ t & \text{in Regime IV, i. e. } \; t \;\geq\; (\ell d)^{1/2} \end{array} \right\}.$$

Notice that $w \geq t$ by construction and $w \leq \ell$ thanks to (44). Hence the constraint (46) is satisfied. Observe that in Regime III we have

$$\frac{\ell^2}{tw} \;=\; \left(\frac{w}{d}\right)^2 \;=\; \left(\frac{\ell^2}{td}\right)^{2/3},$$

whereas in Regime IV,

$$\frac{\ell^2}{tw} \ \le \ \left(\frac{w}{d}\right)^2 \ = \ \left(\frac{t}{d}\right)^2.$$

Therefore, (53) turns into the desired

$$\begin{aligned}&\int_\Omega |\zeta_2|^2\,dx \\ &\qquad \lesssim \ \max\left\{\left(\frac{\ell^2}{dt}\right)^{2/3}, \left(\frac{t}{d}\right)^2\right\}\left(\int_{\mathbb{R}^3} |\nabla U_\zeta|^2\,dx + d^2\int_\Omega |\nabla\zeta|^2\,dx\right). \quad (54)\end{aligned}$$

We now turn to ζ_3 and start with $\langle\zeta_3\rangle_3$. According to Lemma 5, with $\tau := t \ll \ell$, it is enough to estimate the frequencies $|\xi_2| \le \frac{1}{t}$:

$$\int_{\Omega'} |\langle\zeta_3\rangle_3|^2\,dx' \ \lesssim \ \int_{|\xi_2|\le\frac{1}{t}} |\widehat{\langle\zeta_3\rangle_3}|^2\,d\xi' + t^2\int_{\Omega'} |\partial_2\langle\zeta_3\rangle_3|^2\,dx'. \qquad (55)$$

In order to estimate the first term on the right hand side of (55) by magnetostatics via Lemma 3, we again need an estimate on Fourier multipliers stated in the following lemma.

Lemma 7.

Under the assumptions of Proposition 4, we have

$$1 \ \lesssim \ \min\left\{1, \frac{1}{t^2|\xi'|^2}\right\} + t^2|\xi_1|^2 \quad \textit{for} \quad |\xi_2| \ \le \ \frac{1}{t}.$$

Before proving Lemma 7, we continue with the proof of Proposition 4. With help of Lemma 7 and Lemma 3, we can proceed with (55) as follows

$$\begin{aligned} t\int_{\Omega'} |\langle\zeta_3\rangle_3|^2\,dx' &\lesssim t\int_{\mathbb{R}^2} \min\left\{1, \frac{1}{t^2|\xi'|^2}\right\} |\widehat{\langle\zeta_3\rangle_3}|^2 d\xi' \\ &+ t^3 \int_{\mathbb{R}^2} |\xi_1|^2 |\widehat{\langle\zeta_3\rangle_3}|^2 d\xi' \\ &+ t^3 \int_{\Omega'} |\partial_2\langle\zeta_3\rangle_3|^2\,dx' \\ &\lesssim \int_{\mathbb{R}^3} |\nabla U_\zeta|^2\,dx + \int_\Omega |\zeta_2|^2\,dx \\ &+ t^3\int_{\Omega'} |\nabla'\langle\zeta_3\rangle_3|^2\,dx'. \end{aligned} \tag{56}$$

Applying (51) and (52) to (56), we obtain

$$\int_\Omega |\zeta_3|^2\,dx \lesssim \int_{\mathbb{R}^3} |\nabla U_\zeta|^2\,dx + \int_\Omega |\zeta_2|^2\,dx + t^2\int_\Omega |\nabla\zeta|^2\,dx. \tag{57}$$

We combine this with (54) to obtain

$$\int_\Omega |\zeta_3|^2\,dx \lesssim \max\left\{\left(\frac{\ell^2}{dt}\right)^{2/3}, \left(\frac{t}{d}\right)^2, 1\right\} \left(\int_{\mathbb{R}^3} |\nabla U_\zeta|^2\,dx + d^2\int_\Omega |\nabla\zeta|^2\,dx\right).$$

Since

$$\max\left\{\left(\frac{\ell^2}{dt}\right)^{2/3}, \left(\frac{t}{d}\right)^2\right\} \sim \frac{\ell}{d} \gg 1,$$

this yields the desired result.

q.e.d.

Proof of Lemma 6. We start by noticing that for $|\xi_1| \le \frac{1}{w}$,

$$\begin{aligned} \min\left\{\frac{t}{|\xi'|}, \frac{1}{|\xi'|^2}\right\} |\xi_2|^2 &\gtrsim \min\left\{\frac{t}{\frac{1}{w}+|\xi_2|}, \frac{1}{\frac{1}{w^2}+|\xi_2|^2}\right\} |\xi_2|^2 \\ &\sim \min\left\{tw, \frac{t}{|\xi_2|}, w^2, \frac{1}{|\xi_2|^2}\right\} |\xi_2|^2. \end{aligned}$$

Since by assumption (46), we have $w \geq t$, this implies

$$\min\left\{\frac{t}{|\xi'|}, \frac{1}{|\xi'|^2}\right\}|\xi_2|^2 \quad \gtrsim \quad \min\left\{tw|\xi_2|^2, t|\xi_2|, 1\right\}.$$

Hence we have to compare

$$m_1(|\xi_2|) \ = \ \min\left\{tw|\xi_2|^2, t|\xi_2|, 1\right\} \quad \text{and} \quad m_2(|\xi_2|) \ = \ \min\left\{\ell^2|\xi_2|^2, 1\right\}.$$

Notice that by (46) the inverse length scales are ordered as

$$\frac{1}{\ell} \ \leq \ \frac{1}{w} \ \leq \ \frac{1}{t}.$$

Therefore

$$m_1(|\xi_2|) \quad = \quad \left\{\begin{array}{ccccc} tw|\xi_2|^2 & \text{for} & & |\xi_2| & \leq \ \frac{1}{w} \\ t|\xi_2| & \text{for} & \frac{1}{w} \ \leq & |\xi_2| & \leq \ \frac{1}{t} \\ 1 & \text{for} & \frac{1}{t} \ \leq & |\xi_2| & \end{array}\right\},$$

$$m_2(|\xi_2|) \quad = \quad \left\{\begin{array}{ccccc} \ell^2|\xi_2|^2 & \text{for} & & |\xi_2| & \leq \ \frac{1}{\ell} \\ 1 & \text{for} & \frac{1}{\ell} \ \leq & |\xi_2| & \end{array}\right\}.$$

We conclude that

$$m_2(|\xi_2|) \ \lesssim \ \frac{\ell^2}{tw} m_1(|\xi_2|).$$

q.e.d.

Proof of Lemma 7. This inequality is easy to see:

$$
\begin{aligned}
&\min\left\{1,\frac{1}{t^2|\xi'|^2}\right\}+t^2|\xi_1|^2\\
&\quad\sim\quad \min\left\{1,\frac{1}{t^2|\xi_1|^2},\frac{1}{t^2|\xi_2|^2}\right\}+t^2|\xi_1|^2\\
&\quad\overset{|\xi_2|\le\frac{1}{t}}{=}\quad \min\left\{1,\frac{1}{t^2|\xi_1|^2}\right\}+t^2|\xi_1|^2\\
&\quad\ge\quad \min\left\{1,\frac{1}{t^2|\xi_1|^2}+t^2|\xi_1|^2\right\}\\
&\quad\gtrsim\quad 1.
\end{aligned}
$$

q.e.d.

We now prove the lower bound in the Regimes I and II. The analogue of Lemma 4 for these regimes is

Lemma 8.

Let ζ be admissible. Then

$$
\begin{aligned}
&\int_{\Omega'}|\langle\zeta_2\rangle_3|^2\,dx'\\
&\qquad\lesssim\ \frac{\ell}{t}\ln^{-1}\frac{\ell}{t}\int_{\mathbb{R}^2}\min\{t|\xi_2|,1\}\,|\widehat{\langle\zeta_2\rangle_3}|^2\,d\xi'+\ell^2\int_{\Omega'}|\partial_2\langle\zeta_2\rangle_3|^2\,dx'.
\end{aligned}
$$

Proof of Lemma 8. This 2–d estimate follows from the 1–d estimate

$$
\begin{aligned}
&\int_0^\ell|\zeta|^2\,dx_2\\
&\qquad\lesssim\ \frac{\ell}{t}\ln^{-1}\frac{\ell}{t}\int_{\mathbb{R}}\min\{t|\xi_2|,1\}\,|\widehat{\zeta}|^2\,d\xi_2+\ell^2\int_0^\ell|\partial_2\zeta|^2\,dx_2 \qquad (58)
\end{aligned}
$$

for any $\zeta=\zeta(x_2)$ with $\mathrm{supp}\zeta\subset(0,\ell)$.

As a first ingredient, we need the following estimate of the L^2–modulus of continuity by a Fourier–type norm:

$$\begin{aligned}
&\int_t^\infty \frac{1}{h}\int_{\mathbb{R}} |\zeta(x_2+h)-\zeta(x_2)|^2\,dx_2\,\frac{1}{h}\,dh \\
&\quad = \int_t^\infty \frac{1}{h}\int_{\mathbb{R}} |e^{ih\xi_2}-1|^2|\hat\zeta|^2\,d\xi_2\,\frac{1}{h}\,dh \\
&\quad \lesssim \int_t^\infty \frac{1}{h}\int_{\mathbb{R}} \min\left\{h^2|\xi_2|^2,1\right\}|\hat\zeta|^2\,d\xi_2\,\frac{1}{h}\,dh \\
&\quad = \int_{\mathbb{R}}\int_t^\infty \min\left\{|\xi_2|^2,\frac{1}{h^2}\right\}\,dh\,|\hat\zeta|^2\,d\xi_2 \\
&\quad \sim \frac{1}{t}\int_{\mathbb{R}} \min\left\{t|\xi_2|,1\right\}|\hat\zeta|^2\,d\xi_2,
\end{aligned} \tag{59}$$

where we have used

$$\begin{aligned}
&\int_t^\infty \min\left\{|\xi_2|^2,\frac{1}{h^2}\right\}dh \\
&= \left\{\begin{array}{ccc}
\int_t^{\frac{1}{|\xi_2|}} |\xi_2|^2\,dh + \int_{\frac{1}{|\xi_2|}}^\infty \frac{1}{h^2}\,dh & \text{for} & |\xi_2| \le \frac{1}{t} \\
\int_t^\infty \frac{1}{h^2}dh & \text{for} & |\xi_2| \ge \frac{1}{t}
\end{array}\right\} \\
&\sim \left\{\begin{array}{ccc}
|\xi_2| & \text{for} & |\xi_2| \le \frac{1}{t} \\
\frac{1}{t} & \text{for} & |\xi_2| \ge \frac{1}{t}
\end{array}\right\} \\
&\sim \frac{1}{t}\min\left\{t|\xi_2|,1\right\}.
\end{aligned}$$

As a second ingredient, we need the following Poincaré–type estimate:

$$\int_0^\ell |\zeta|^2\,dx_2 \;\lesssim\; \frac{\ell}{h}\int_{\ell-h}^\ell |\zeta|^2\,dx_2 + \ell^2\int_0^\ell |\partial_2\zeta|^2\,dx_2. \tag{60}$$

This can be shown as follows: Starting from

$$\zeta(y_2) \;=\; \zeta(x_2) + \int_{x_2}^{y_2} \partial_2\zeta(z_2)\,dz_2$$

we obtain

$$|\zeta(y_2)|^2 \;\lesssim\; |\zeta(x_2)|^2 + \ell \int_0^\ell |\partial_2\zeta(z_2)|^2\, dz_2.$$

We take the average over $x_2 \in (\ell - h, \ell)$,

$$|\zeta(y_2)|^2 \;\lesssim\; \frac{1}{h}\int_{\ell-h}^{\ell} |\zeta(x_2)|^2\, dx_2 + \ell \int_0^\ell |\partial_2\zeta(z_2)|^2\, dz_2,$$

and integrate over $y_2 \in (0, \ell)$. This yields (60).

Observe that because of $\operatorname{supp}\zeta \subset (0,\ell)$

$$\int_{\ell-h}^{\ell} |\zeta|^2\, dx_2 \;\le\; \int_{\mathbb{R}} |\zeta(x_2+h) - \zeta(x_2)|^2\, dx_2.$$

Hence, we obtain by (59)

$$\int_t^\ell \frac{1}{h}\int_{\ell-h}^{\ell} |\zeta|^2\, dx_2\, \frac{1}{h}\, dh \;\lesssim\; \frac{1}{t}\int_{\mathbb{R}} \min\{t|\xi_2|, 1\}\, |\widehat{\zeta}|^2\, d\xi_2. \tag{61}$$

We now perform $\int_t^\ell \cdot \frac{1}{h} dh$ on (60). This yields

$$\begin{aligned} &\ln\left(\frac{\ell}{t}\right) \int_0^\ell |\zeta|^2\, dx_2 \\ &\lesssim\; \ell \int_t^\ell \frac{1}{h}\int_{\ell-h}^{\ell} |\zeta|^2\, dx_2\, \frac{1}{h}\, dh + \ln\left(\frac{\ell}{t}\right) \ell^2 \int_0^\ell |\partial_2\zeta|^2\, dx_2. \end{aligned} \tag{62}$$

The combination of (61) and (62) gives

$$\ln\left(\frac{\ell}{t}\right) \int_0^\ell |\zeta|^2\, dx_2 \;\lesssim\; \frac{\ell}{t}\int_{\mathbb{R}} \min\{t|\xi_2|, 1\}\, |\widehat{\zeta}|^2\, d\xi_2 + \ln\left(\frac{\ell}{t}\right) \ell^2 \int_0^\ell |\partial_2\zeta|^2\, dx_2,$$

which yields (58).

q.e.d.

We can now treat Regimes I and II.

Proposition 5.

Assume $d, t \ll \ell$ and

$$t \;\leq\; \frac{d^2}{\ell}. \tag{63}$$

Then we have for any admissible ζ

$$d^2 \int_\Omega |\nabla\zeta|^2 \, dx + \int_{\mathbb{R}^3} |\nabla U_\zeta|^2 \, dx \;\gtrsim\; \min\left\{ \frac{t}{\ell} \ln\left(\frac{\ell}{t}\right), \left(\frac{d}{\ell}\right)^2 \right\} \int_\Omega |\zeta|^2 \, dx.$$

Proof of Proposition 5. We start by an estimate of $\int_{\Omega'} |\langle\zeta_2\rangle_3|^2 \, dx'$.

According to Lemma 8, we have

$$\begin{aligned} &\int_{\Omega'} |\langle\zeta_2\rangle_3|^2 \, dx' \\ &\qquad \lesssim\; \frac{\ell}{t} \ln^{-1}\frac{\ell}{t} \int_{\mathbb{R}^2} \min\{t|\xi_2|, 1\} \, |\widehat{\langle\zeta_2\rangle_3}|^2 \, d\xi' + \ell^2 \int_{\Omega'} |\partial_2\langle\zeta_2\rangle_3|^2 \, dx'. \end{aligned} \tag{64}$$

In order to make use of Lemma 2, we need the following multiplier estimate:

Lemma 9.

$$\min\{t|\xi_2|, 1\} \;\lesssim\; \min\left\{ \frac{t}{|\xi'|}, \frac{1}{|\xi'|^2} \right\} |\xi_2|^2 + d^2|\xi_1|^2 + \left(\frac{t}{d}\right)^2.$$

We postpone the proof of Lemma 9 and proceed with the proof of Proposition

5. Applying Lemma 9 to (64) we gather

$$\begin{aligned}\int_{\Omega'} |\langle\zeta_2\rangle_3|^2\,dx' &\lesssim \frac{\ell}{t}\ln^{-1}\frac{\ell}{t}\int_{\mathbb{R}^2}\min\left\{\frac{t}{|\xi'|},\frac{1}{|\xi'|^2}\right\}|\xi_2|^2|\widehat{\langle\zeta_2\rangle}_3|^2\,d\xi' \\ &+ \frac{\ell}{t}\ln^{-1}\frac{\ell}{t}\,d^2\int_{\Omega'}|\partial_1\langle\zeta_2\rangle_3|^2\,dx' \\ &+ \frac{t\ell}{d^2}\ln^{-1}\frac{\ell}{t}\int_{\Omega'}|\langle\zeta_2\rangle_3|^2\,dx' \\ &+ \ell^2\int_{\Omega'}|\partial_2\langle\zeta_2\rangle_3|^2\,dx'.\end{aligned}$$

By assumption (63), we are in the regime $\frac{t\ell}{d^2}\ln^{-1}\frac{\ell}{t}\ll 1$. Hence the above estimate yields

$$\begin{aligned}\int_{\Omega'} |\langle\zeta_2\rangle_3|^2\,dx' &\lesssim \frac{\ell}{t}\ln^{-1}\frac{\ell}{t}\int_{\mathbb{R}^2}\min\left\{\frac{t}{|\xi'|},\frac{1}{|\xi'|^2}\right\}|\xi_2|^2|\widehat{\langle\zeta_2\rangle}_3|^2\,d\xi' \\ &+ \max\left\{\frac{\ell}{t}\ln^{-1}\frac{\ell}{t},\left(\frac{\ell}{d}\right)^2\right\}d^2\int_{\Omega'}|\nabla'\langle\zeta_2\rangle_3|^2\,dx'.\end{aligned}$$

We now apply Lemma 2 and obtain

$$\begin{aligned}t\int_{\Omega'} |\langle\zeta_2\rangle_3|^2\,dx' &\lesssim \frac{\ell}{t}\ln^{-1}\frac{\ell}{t}\int_{\mathbb{R}^3}|\nabla U_\zeta|^2\,dx \\ &+ \max\left\{\frac{\ell}{t}\ln^{-1}\frac{\ell}{t},\left(\frac{\ell}{d}\right)^2\right\}d^2 t\int_{\Omega'}|\nabla'\langle\zeta_2\rangle_3|^2\,dx'.\end{aligned}$$

With help of Poincaré's estimate, cf. (51), and Jensen's inequality, cf. (52), this turns into

$$\begin{aligned}
&\int_\Omega |\zeta_2|^2\,dx \\
&\lesssim \quad \frac{\ell}{t}\ln^{-1}\frac{\ell}{t}\int_{\mathbb{R}^3}|\nabla U_\zeta|^2\,dx \\
&+ \quad \max\left\{\frac{\ell}{t}\ln^{-1}\frac{\ell}{t},\left(\frac{\ell}{d}\right)^2,\left(\frac{t}{d}\right)^2\right\} d^2\int_\Omega|\nabla\zeta|^2\,dx \\
&\overset{t\le\ell}{\le} \quad \max\left\{\frac{\ell}{t}\ln^{-1}\frac{\ell}{t},\left(\frac{\ell}{d}\right)^2\right\}\left(d^2\int_\Omega|\nabla\zeta|^2\,dx+\int_{\mathbb{R}^3}|\nabla U_\zeta|^2\,dx\right). \quad (65)
\end{aligned}$$

This treats the ζ_2–component.

We now turn to the ζ_3–component and start with $\langle\zeta_3\rangle_3$. According to Lemma 5 with $\tau = t \ll \ell$ we have

$$\begin{aligned}
\int_{\Omega'}|\langle\zeta_3\rangle_3|^2\,dx' \quad &\lesssim \quad \int_{|\xi_2|\le\frac{1}{t}}|\widehat{\langle\zeta_3\rangle_3}|^2\,d\xi' + t^2\int_{\Omega'}|\partial_2\langle\zeta_3\rangle_3|^2\,dx' \\
&\lesssim \quad \int_{|\xi'|\le\frac{1}{t}}|\widehat{\langle\zeta_3\rangle_3}|^2\,d\xi' + t^2\int_{\mathbb{R}^2}|\xi_1|^2|\widehat{\langle\zeta_3\rangle_3}|^2\,d\xi' \\
&+ \quad t^2\int_{\Omega'}|\partial_2\langle\zeta_3\rangle_3|^2\,dx' \\
&\le \quad \int_{\mathbb{R}^2}\min\left\{1,\frac{1}{t^2|\xi'|^2}\right\}|\widehat{\langle\zeta_3\rangle_3}|^2\,d\xi' + t^2\int_{\Omega'}|\nabla'\langle\zeta_3\rangle_3|^2\,dx'.
\end{aligned}$$

An application of Lemma 3 now yields

$$t\int_{\Omega'}|\langle\zeta_3\rangle_3|^2\,dx' \;\lesssim\; \int_{\mathbb{R}^3}|\nabla U_\zeta|^2\,dx + \int_\Omega|\zeta_2|^2\,dx + t^3\int_{\Omega'}|\nabla'\langle\zeta_3\rangle_3|^2\,dx'.$$

With help of Poincaré, cf. (51), and Jensen, cf. (52), this turns into

$$\int_\Omega|\zeta_3|^2\,dx \;\lesssim\; \int_{\mathbb{R}^3}|\nabla U_\zeta|^2\,dx + \int_\Omega|\zeta_2|^2\,dx + t^2\int_\Omega|\nabla\zeta|^2\,dx.$$

Combining this estimate with (65), we obtain

$$\begin{aligned}
&\int_\Omega |\zeta_3|^2\,dx \\
&\quad\lesssim \quad \max\left\{\frac{\ell}{t}\ln^{-1}\frac{\ell}{t}, \left(\frac{\ell}{d}\right)^2, 1, \left(\frac{t}{d}\right)^2\right\}\left(d^2\int_\Omega |\nabla\zeta|^2\,dx + \int_{\mathbb{R}^3}|\nabla U_\zeta|^2\,dx\right) \\
&\quad\overset{\ell\geq t,d}{=} \quad \max\left\{\frac{\ell}{t}\ln^{-1}\frac{\ell}{t}, \left(\frac{\ell}{d}\right)^2\right\}\left(d^2\int_\Omega |\nabla\zeta|^2\,dx + \int_{\mathbb{R}^3}|\nabla U_\zeta|^2\,dx\right).
\end{aligned}$$

This treats the ζ_3–component.

q.e.d.

Proof of Lemma 9. The case of $|\xi_1| \leq |\xi_2|$ is easy since then $|\xi'| \sim |\xi_2|$ and thus

$$\min\left\{\frac{t}{|\xi'|}, \frac{1}{|\xi'|^2}\right\}|\xi_2|^2 \ \sim\ \min\left\{t|\xi_2|, 1\right\}.$$

We now consider the case of $|\xi_1| \geq |\xi_2|$. We have to show that

$$\min\left\{t|\xi_2|, 1\right\} \ \lesssim\ \min\left\{\frac{t|\xi_2|^2}{|\xi_1|}, \frac{|\xi_2|^2}{|\xi_1|^2}\right\} + d^2|\xi_1|^2 + \left(\frac{t}{d}\right)^2. \tag{66}$$

By Cauchy–Schwarz we have

$$t|\xi_1| \ \leq\ d^2|\xi_1|^2 + \left(\frac{t}{d}\right)^2. \tag{67}$$

Therefore using (67) and the fact that $|\xi_1| \geq |\xi_2|$ we have

$$\begin{aligned}
\min\{t|\xi_2|, 1\} \ &\leq\ \min\left\{\frac{t|\xi_2|^2}{|\xi_1|}, \frac{|\xi_2|^2}{|\xi_1|^2}\right\} + t|\xi_1| \\
&\leq\ \min\left\{\frac{t|\xi_2|^2}{|\xi_1|}, \frac{|\xi_2|^2}{|\xi_1|^2}\right\} + d^2|\xi_1|^2 + \left(\frac{t}{d}\right)^2
\end{aligned}$$

q.e.d.

Theorem 3 follows by Propositions 4 and 5.

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