Oscillatory buckling mode in thin-film nucleation

Rubén Cantero-Álvarez, Felix Otto*

January 5, 2006

Abstract

We are interested in switching of elongated thin–film elements as described by the micromagnetic model. Nucleation occurs when the saturation branch becomes unstable at a critical field. It is characterized by a degeneracy of the Hessian of the micromagnetic energy. The degenerate subspace describes the unstable mode.

In a prior work, [1], we showed that there are four regimes for nucleation. We proved this by identifying the scaling of the critical field in the non–dimensional parameters. This contradicts a claim by Aharoni that there are at most three regimes.

Two of these regimes are buckling regimes, where the magnetization is pinned at the lateral facets of the sample and buckles in–plane. As shown in [1], these regimes differ in the scaling of the critical field. Here we show that also the unstable modes are qualitatively different: Only in one of the regimes they oscillate in the long direction.

The unstable modes are asymptotically identified by Γ–convergence, which is applied to the Rayleigh quotient of the Hessian.

Mathematics Subject Classification: 78A99, 49K20, 74G60

*Institute of Applied Mathematics, University of Bonn, Germany

1 Motivation

As in the companion paper [1], we are interested in the concertina pattern in soft (i. e. low crystalline anisotropy) ferromagnetic thin–film elements, see Figures 1, 2.

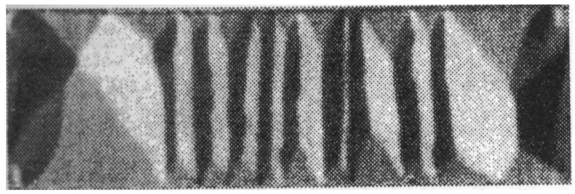

Figure 1: The concertina pattern

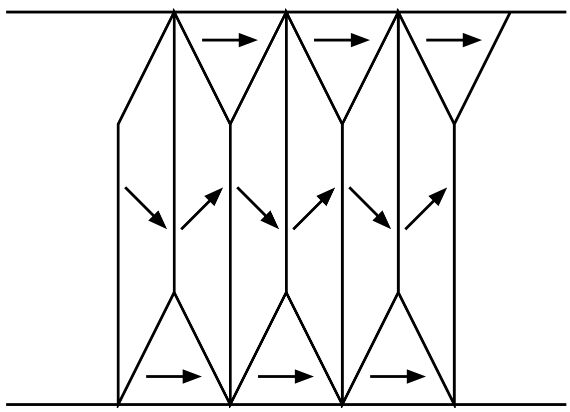

Figure 2: Mesoscopic magnetization

Experimentally, this pattern is generated by destabilizing the groundstate magnetization, which is uniform along the long axis, by applying a reverse external field. In order to understand this phenomenon, we start with nucleation theory for the micromagnetic model, which describes the onset of a bifurcation at a critical external field. For more motivation and background, we refer to the companion paper [1].

Let us give a short synopsis of this paper's content. In an introductory section we review the basic notions central to our article. The subsequent section states the two main theorems, section 4 compares the oscillation period in Regime III with experimental results for the concertina pattern. Sections 5 and 7 are concerned with the proof structure for both main theorems. The proofs are given in the last section. Section 6 mentions a link between the oscillation phenomenon in Regime III and the magnetization ripple.

2 Introduction

2.1 The setting

For a given sample geometry $\Omega \subset \mathbb{R}^3$, an experimentally observed magnetization $m\colon \Omega \to \mathbb{R}^3$ is a local minimum of the micromagnetic energy

$$E(m) \;=\; d^2 \int_\Omega |\nabla m|^2\, dx + \int_{\mathbb{R}^3} |\nabla u_m|^2\, dx - 2\int_\Omega H_{ext}\cdot m\, dx$$

among all m which satisfy the saturation constraint

$$|m|^2 \;=\; 1 \quad \text{in } \Omega. \tag{1}$$

The stray field $-\nabla u_m$ is determined by the static version of Maxwell's equations. They are conveniently stated in a distributional form:

$$\int_{\mathbb{R}^3} \nabla u_m \cdot \nabla\varphi\, dx \;=\; \int_\Omega m\cdot\nabla\varphi\, dx \quad \text{for all test functions } \varphi. \tag{2}$$

The sample geometry is

$$\Omega \;=\; \mathbb{R}\times(0,\ell)\times(-\frac{t}{2},\frac{t}{2}).$$

see Figure 3.

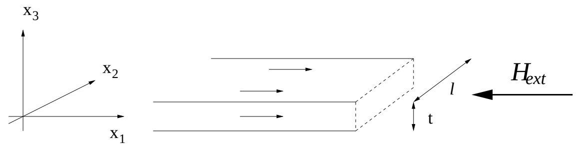

Figure 3: The geometry

2.2 Hessian

Due to the constraint (1), infinitesimal perturbations of $m^* = (1,0,0)$ are of the form

$$\zeta \;=\; (0,\zeta_2,\zeta_3), \quad \zeta \;=\; \zeta(x_1,x_2,x_3). \tag{3}$$

An easy calculation shows that the Hessian Hess$E(m^*)$ of E in m^* is given by

$$\begin{aligned} \frac{1}{2}\,\mathrm{Hess}E(m^*)(\zeta,\zeta) \;&=\; d^2\int_\Omega |\nabla\zeta|^2\,dx \\ &+\; \int_{\mathbb{R}^3} |\nabla u_\zeta|^2\,dx \\ &-\; h_{ext}\int_\Omega |\zeta|^2\,dx, \end{aligned} \tag{4}$$

where u_ζ is determined by ζ through (2).

2.3 Unstable mode / critical field

The critical field h_{crit} is the smallest h_{ext} for which Hess$E(m^*)$ ceases to be positive definite. The unstable modes are the elements of the degenerate subspace of Hess$E(m^*)$ for $h_{ext} = h_{crit}$. In the jargon of micromagnetics, this bifurcation is called nucleation. A variational characterization of both can be inferred from (4). Indeed, with the abbreviation

$$\mathcal{R}(\zeta) \;=\; d^2\int_\Omega |\nabla\zeta|^2\,dx + \int_{\mathbb{R}^3} |\nabla u_\zeta|^2\,dx \tag{5}$$

we have

$$\begin{aligned}
&\text{critical field} \\
&= \min\left\{\mathcal{R}(\zeta)\,\middle|\,\zeta \text{ as in (3) with } \int_\Omega |\zeta|^2 dx = 1\right\}, \qquad (6) \\
&\text{normalized unstable modes} \\
&= \operatorname{argmin}\left\{\mathcal{R}(\zeta)\,\middle|\,\zeta \text{ as in (3) with } \int_\Omega |\zeta|^2 dx = 1\right\}.
\end{aligned}$$

2.4 Factorization

It is natural to capitalize on the translation invariance of $\mathcal{R}$ in x_1 by a partial Fourier transform in that variable. We denote by k_1 the dual variable. We also introduce the notation

$$\tilde{x} := (x_2, x_3), \quad \tilde{\Omega} := (0,\ell) \times (-t/2, t/2).$$

The infinitesimal perturbations of mode k_1 are described by

$$\zeta \;=\; (0, \zeta_2, \zeta_3), \quad \zeta \;=\; \zeta(x_2, x_3). \tag{7}$$

The analogue of (5) is

$$\mathcal{R}(k_1, \zeta) \;=\; d^2 \int_{\tilde{\Omega}} (k_1^2 |\zeta|^2 + |\tilde{\nabla}\zeta|^2)\, d\tilde{x} + \int_{\mathbb{R}^2} (k_1^2 |u_\zeta|^2 + |\tilde{\nabla} u_\zeta|^2)\, d\tilde{x}, \tag{8}$$

where u_ζ is defined by

$$\int_{\mathbb{R}^2} (k_1^2 u_\zeta\, \overline{\varphi} + \tilde{\nabla} u_\zeta \cdot \overline{\tilde{\nabla}\varphi})\, d\tilde{x} \;=\; \int_{\tilde{\Omega}} \zeta \cdot \overline{\tilde{\nabla}\varphi}\, d\tilde{x}$$

for all test functions $\varphi(x_2, x_3) \in \mathbb{C}$.

We are interested in

$$\begin{aligned}
&\text{critical field}\\
&= \min\left\{\mathcal{R}(k_1,\zeta)\,\middle|\,k_1,\ \zeta \text{ as in (7) with } \int_{\tilde\Omega}|\zeta|^2\,d\tilde x = 1\right\}, \qquad (9)\\
&\text{normalized unstable modes}\\
&= \operatorname{argmin}\left\{\mathcal{R}(k_1,\zeta)\,\middle|\,k_1,\ \zeta \text{ as in (7) with } \int_{\tilde\Omega}|\zeta|^2\,d\tilde x = 1\right\}. \qquad (10)
\end{aligned}$$

3 Result

Our main result is the asymptotic identification of the unstable modes and the critical field in the new regime (III) and the adjacent one (II). The similarity of both regimes is given in the edge–pinning effect, which restricts both unstable modes to have zero boundary values in x_2. But the difference in the scaling of the critical field, and, more remarkably, in the k_1-dependence of the unstable mode clearly sets both regimes apart. Our argument is based on the variational characterization of both in (9) resp. (10). On this level, our result is formulated in the following two theorems.

Theorem 1. *Let (k_1^*, ζ^*) be a minimizer of*

$$\mathcal{R}(k_1,\zeta) \quad \textit{constrained by} \quad \int_{\tilde\Omega}|\zeta|^2 d\tilde x \;=\; 1.$$

In the regime

$$\frac{d^2}{\ell} \;\ll\; t \;\ll\; (d\ell)^{1/2} \qquad (11)$$

we have

$$\left(\frac{\ell^2}{dt}\right)^{2/3} \mathcal{R}(k_1^*,\zeta^*) \;\approx\; 3(\frac{\pi}{2})^{4/3}, \qquad (12)$$

$$\left(\frac{d^2\ell^2}{t}\right)^{1/3} |k_1^*| \;\approx\; (\frac{\pi}{2})^{2/3}, \tag{13}$$

$$\frac{1}{t\,\ell}\int_{\tilde\Omega}\left|\zeta^*(\tilde x) - \sqrt{2}\,c\,\sin(\pi\,x_2/\ell)\right|^2 d\tilde x \;\ll\; 1 \tag{14}$$

for some $c \in \mathbb{C}$ *with* $|c| = 1$.

From (12) we infer that the critical field is given by

$$h_{crit} \;\approx\; 3\left(\frac{\pi^2}{4}\,\frac{d\,t}{\ell^2}\right)^{2/3}.$$

In particular, our prior result in [1] shows that the regime (11) is optimal. From (13) and (14) we infer that the unstable subspace asymptotically consists of all perturbations ζ of the form

$$\zeta_2^* \;=\; c\,\cos(2\,\pi\,(x_1+\xi)/w^*)\,\sin(\pi\,x_2/\ell), \quad \zeta_3^* \;\equiv\; 0,$$

for some constants $c, \xi \in \mathbb{R}$, where the period $w^* = \frac{2\,\pi}{|k_1^*|}$ of oscillation in the infinite direction x_1 is given by

$$w^* \;=\; \left(32\pi\,\frac{d^2\,\ell^2}{t}\right)^{1/3}. \tag{15}$$

Theorem 2. *Let* (k_1^*, ζ^*) *be a minimizer of*

$$\mathcal{R}(k_1,\zeta) \quad \text{constrained by} \quad \int_{\tilde\Omega}|\zeta|^2 d\tilde x \;=\; 1.$$

In the regime

$$\frac{d^2}{\ell}\ln^{-1}\left(\frac{\ell}{d}\right) \;\ll\; t \;\ll\; \frac{d^2}{\ell} \tag{16}$$

we have

$$\left(\frac{\ell}{d}\right)^2 \mathcal{R}(k_1^*,\zeta^*) \;\approx\; \pi^2, \tag{17}$$

$$|k_1^*| \;=\; 0, \tag{18}$$

$$\frac{1}{t\,\ell} \int_{\tilde{\Omega}} \left| \zeta^*(\tilde{x}) - \sqrt{2}\, c\, \sin(\pi\, x_2/\ell) \right|^2 \, d\tilde{x} \;\ll\; 1 \tag{19}$$

for some $c \in \mathbb{C}$ *with* $|c| = 1$.

We remark that (16) with $x \;:=\; \frac{d^2}{\ell t}$ and $y \;:=\; \frac{\ell}{d}$ reads as

$$x \;\ll\; 1 \;\ll\; x\, \ln(y).$$

This, in turn, yields

$$|\ln(x)| \;=\; -\ln(x) \;\ll\; \ln(\ln(y)) \;\ll\; \ln(y),$$

whereby

$$\ln(y) \;\sim\; \ln(x\, y^2),$$

which can be used to state that

$$\frac{d^2}{\ell} \ln^{-1}\left(\frac{\ell}{d}\right) \;\sim\; \frac{d^2}{\ell} \ln^{-1}\left(\frac{\ell}{t}\right). \tag{20}$$

This reformulation will replace the l. h. s. term in (16) for all further purposes.

From (17) we infer that the critical field is given by

$$h_{crit} \;\approx\; \left(\pi\, \frac{d}{\ell}\right)^2.$$

In particular, our prior result in [1] shows that the regime (16) is optimal. From (18) and (19) we infer that the unstable subspace asymptotically consists of all perturbations ζ of the form

$$\zeta_2^* \;=\; c\, \sin(\pi\, x_2/\ell), \quad \zeta_3^* \;\equiv\; 0,$$

for some constant $c \in \mathbb{R}$.

4 The concertina pattern: Comparison with experiments

We return to the hypothesis that the observed period w_{exp} of the concertina pattern is the frozen–in length scale (15) of the unstable mode, see our introduction. We have compared w_{exp} to w for eight experiments pictured in [5]. These experiments cover a substantial range of the non–dimensional parameters t/d and ℓ/d, see Table 4. We find a deviation by a factor $0.5 - 0.7$, which warrants closer investigation in the future.

$\frac{t}{d}$	$\frac{\ell}{d}$	$\frac{w_{exp}}{d}$	$\frac{w_{theo}}{d}$	$\frac{w_{theo}}{w_{exp}}$
8	4000	1200	586	0.49
48	8000	700	512	0.73
60	3600	540	279	0.52
10	2000	500	347	0.69
8	2800	800	462	0.56
8	7000	1700	851	0.50

5 Structure of the proof of Theorem 1

Theorem 1 states in particular that the minimizer ζ^* becomes x_3–independent in the limiting regime. In Lemma 1, we will relate this to the magnetostatic contribution in two ways:

- In part i) of Lemma 1, we express the magnetostatic contribution for an infinitesimal perturbation which is x_3–independent, that is,

$$\zeta = \zeta(x_2),$$

in terms of the 1–d Fourier transform $\mathcal{F}\zeta$. When taking the Fourier transform, we always think of $\zeta(x_2)$ as being trivially extended beyond $x_2 \in (0, \ell)$.

- In part ii) of Lemma 1, we optimally bound the magnetostatic contribution by below in terms of the vertically averaged ζ, that is,

$$\langle\zeta\rangle(x_2) \;:=\; \frac{1}{t}\int_{-\frac{t}{2}}^{\frac{t}{2}} \zeta(x_2, x_3)\, dx_3.$$

Again, the result is expressed in terms of the Fourier transform $\mathcal{F}\langle\zeta\rangle$.

Lemma 1.

i) For admissible ζ with $\zeta = \zeta(x_2)$ we have

$$\begin{aligned}\int_{\mathbb{R}^2} \left(k_1^2|u_\zeta|^2 + |\tilde\nabla u_\zeta|^2\right) d^2\tilde x \;&=\; t\int_{\mathbb{R}} \frac{\exp(-t|k'|) - 1 + t|k'|}{(t|k'|)^3}\,(tk_2)^2\,|\mathcal{F}\zeta_2|^2\, dk_2 \\ &+\; t\int_{\mathbb{R}} \frac{1-\exp(-t|k'|)}{t|k'|}\,|\mathcal{F}\zeta_3|^2\, dk_2.\end{aligned}$$

ii) For any admissible ζ we have

$$\begin{aligned}&\int_{\mathbb{R}^2} \left(k_1^2|u_\zeta|^2 + |\tilde\nabla u_\zeta|^2\right) d^2\tilde x \\ &\geq\; t\int_{\mathbb{R}} \frac{1}{2\,t|k'| + (t|k'|)^2}\,(tk_2)^2\,|\mathcal{F}\langle\zeta_2\rangle|^2\, dk_2 \\ &+\; t\int_{\mathbb{R}} \frac{1}{1+\frac{1}{2}\,t|k'| + \frac{1}{12}\,(t|k'|)^2}|\mathcal{F}\langle\zeta_3\rangle - ik_2\mathcal{F}\langle x_3\zeta_2\rangle|^2\, dk_2.\end{aligned}$$

Here, we use the notation $k' = (k_1, k_2)$. Notice that the Fourier multipliers in Lemma 1 display a cross–over at $t|k'| \sim 1$, that is, at horizontal length scales which are of order of the thickness t. The Fourier multipliers asymptotically coincide in the regime of long horizontal length scales, as they should:

$$\left.\begin{array}{ccccc} \dfrac{\exp(-t|k'|) - 1 + t|k'|}{(t|k'|)^3} & \approx & \dfrac{1}{2\,t|k'| + (t|k'|)^2} & \approx & \dfrac{1}{2\,t|k'|}, \\ \dfrac{1 - \exp(-t|k'|)}{t|k'|} & \approx & \dfrac{1}{1 + \frac{1}{2}\,t|k'| + \frac{1}{12}\,(t|k'|)^2} & \approx & 1 \end{array}\right\} \quad \text{for } t|k'| \ll 1.$$

As Theorem 2 suggests, it will be convenient to introduce the following non–dimensional parameters and rescaled quantities:

Definition 1.

- *To a triple* (d, t, ℓ) *we associate*

$$\epsilon \;:=\; \left(\frac{d^2}{t\ell}\right)^{2/3} \quad \text{and} \quad \delta \;:=\; \left(\frac{t^2}{d\ell}\right)^{2/3}.$$

- *To* $(k_1, \zeta) \in \mathbb{R} \times L^2(\tilde{\Omega})$ *we associate* $(\hat{k}_1, \hat{\zeta}) \in \mathbb{R} \times L^2(\hat{\tilde{\Omega}})$ *via*

$$\left(\frac{d^2\ell^2}{t}\right)^{1/3} k_1 \;=\; \hat{k}_1 \quad \text{and} \quad (t\ell)^{1/2}\zeta(\ell\hat{x}_2, t\hat{x}_3) \;=\; \hat{\zeta}(\hat{x}_2, \hat{x}_3).$$

- *We write*

$$\left(\frac{\ell^2}{dt}\right)^{2/3} \mathcal{R}(k_1, \zeta) \;=\; \hat{\mathcal{R}}(k_1, \zeta).$$

We now express the bounds on the magnetostatic contribution from Lemma 1 in terms of these rescaled quantities, and combine them with the exchange contribution.

Corollary 1.

i) For any $\zeta \in L^2(\tilde{\Omega})$ *such that* $\zeta = \zeta(x_2)$, $\zeta_3 \equiv 0$ *with* $\hat{\zeta}_2 \in H_0^1((0,1))$, *we have*

$$\hat{\mathcal{R}}(k_1,\zeta) \;\le\; \hat{k}_1^2 \int_0^1 |\hat{\zeta}_2|^2\, d\hat{x}_2 + (\frac{1}{2\,|\hat{k}_1|} + \epsilon) \int_0^1 |\partial_2 \hat{\zeta}_2|^2\, d\hat{x}_2. \tag{21}$$

ii) For any $\zeta \in L^2(\tilde{\Omega})$ *we have*

$$\hat{\mathcal{R}}(k_1,\zeta) \;\ge\; \hat{k}_1^2 \int_0^1 |\langle\hat{\zeta}_2\rangle|^2\, d\hat{x}_2 + \int_{\mathbb{R}} \frac{1}{2\,(\hat{k}_1^2 + \epsilon\,\hat{k}_2^2)^{1/2} + \delta\,(\hat{k}_1^2 + \epsilon\,\hat{k}_2^2)}\,\hat{k}_2^2\,|\mathcal{F}\langle\hat{\zeta}_2\rangle|^2 d\hat{k}_2. \tag{22}$$

Here $H_0^1((0,1))$ denotes the Sobolev space of functions $\hat{\zeta}_2 \in L^2((0,1))$ whose distributional derivative $\partial_2\hat{\zeta}_2$ is in $L^2((0,1))$ and which vanishes at the boundary, i. e. $\hat{\zeta}_2(0) = \hat{\zeta}_2(1) = 0$. The weak boundary conditions can be characterized as follows

$$H_0^1((0,1)) \;=\; \left\{ \hat{\zeta}_{2|(0,1)} \,\middle|\, \hat{\zeta}_2 \in L^2(\mathbb{R}),\ \partial_2\hat{\zeta}_2 \in L^2(\mathbb{R}),\ \mathrm{supp}\hat{\zeta}_2 \subset [0,1] \right\}. \tag{23}$$

In order to relate the upper bound of Corollary 1 i) to the lower bound of Corollary 1 ii), we have to show the following: If the rescaled Hessian $\hat{\mathcal{R}}(k_1,\zeta)$ is bounded,

- the rescaled wave vector $\hat{k}_1$ is bounded,
- the rescaled perturbation $\hat{\zeta}$ is close to its vertically averaged horizontal component $\langle\hat{\zeta}_2\rangle$.

This is the purpose of the next lemma.

Lemma 2. *For any* $\zeta \in L^2(\tilde{\Omega})$ *with* $\int_{\tilde{\Omega}} |\zeta|^2\, d^2\tilde{x} = 1$ *we have*

$$\int_{\hat{\tilde{\Omega}}} |\hat{\zeta} - \langle\hat{\zeta}_2\rangle|^2\, d\hat{\tilde{x}} \;\le\; C\,(\delta^2 + \epsilon\delta)\,\hat{\mathcal{R}}(k_1,\zeta), \tag{24}$$

$$|\hat{k}_1|^2 \;\le\; \hat{\mathcal{R}}(k_1,\zeta) \tag{25}$$

for some universal constant $C < \infty$.

Our proof of Theorem 1 is qualitative and based on passing to the limit. The next corollary states the necessary relative compactness result.

Corollary 2. *Let $(d_\nu, t_\nu, \ell_\nu)_{\nu\uparrow\infty}$ be such that*

$$\epsilon_\nu,\ \delta_\nu\ \rightarrow\ 0. \tag{26}$$

Let $(k_{1,\nu}, \zeta_\nu)_{\nu\uparrow\infty}$ be such that

$$\int_{\tilde\Omega} |\zeta_\nu|^2\, d^2\tilde x\ =\ 1 \quad \textit{and} \quad \hat{\mathcal{R}}(k_{1,\nu}, \zeta_\nu)\ \textit{is bounded.} \tag{27}$$

Then $(\hat k_{1,\nu}, \hat\zeta_\nu)$ is relatively compact in $\mathbb{R}\times L^2(\widehat{\tilde\Omega})$. Moreover, any limit $\hat\zeta$ is of the form

$$\hat\zeta\ =\ \hat\zeta(\hat x_2) \quad \textit{and} \quad \hat\zeta_3\ \equiv\ 0 \quad \textit{with} \quad \hat\zeta_2 \in H^1_0((0,1)). \tag{28}$$

Instead of identifying the limit of the minimizers $(k^*_{1,\nu}, \zeta^*_\nu)$ directly, we characterize it as the minimizer of a limiting problem. The appropriate notion of the limiting variational problem is that of the Γ–limit [2].

Proposition 1. *Let $(d_\nu, t_\nu, \ell_\nu)_{\nu\uparrow\infty}$ be such that*

$$\epsilon_\nu,\ \delta_\nu\ \rightarrow\ 0.$$

Then the variational problem in $(k_1, \zeta)\in\mathbb{R}\times L^2(\tilde\Omega)$ of minimizing

$$\hat{\mathcal{R}}(k_1,\zeta) \quad \textit{constrained to} \quad \int_{\tilde\Omega} |\zeta|^2\, d\tilde x\ =\ 1$$

Γ–converges to the variational problem in $(\hat k_1, \hat\zeta_2)\in\mathbb{R}\times H^1_0((0,1))$ of minimizing

$$\hat k_1^2 \int_0^1 |\hat\zeta_2|^2\, d\hat x_2 + \frac{1}{2\,|\hat k_1|}\int_0^1 |\partial_2\hat\zeta_2|^2\, d\hat x_2 \quad \textit{constrained to} \quad \int_0^1 |\hat\zeta_2|^2\, d\hat x_2\ =\ 1 \tag{29}$$

under the following notion of convergence

$$\hat{k}_{1,\nu} \;\to\; \hat{k}_1 \quad \textit{and} \quad \int_{\widehat{\tilde{\Omega}}} |\hat{\zeta}_\nu(\widehat{\tilde{x}}) - \hat{\zeta}_2(\hat{x}_2)|^2 d\widehat{\tilde{x}} \;\to\; 0. \tag{30}$$

For $k_1 = 0$, we assign the value $+\infty$ to the prefactor $\frac{1}{2\,|k_1|}$ in (29).

In the last step we identify the minimizer of the limiting problem.

Lemma 3. *Any minimizer* $(\hat{k}_1^*, \hat{\zeta}_2^*) \in \mathbb{R} \times H_0^1((0,1))$ *of*

$$\hat{k}_1^2 \int_0^1 |\hat{\zeta}_2|^2\, d\hat{x}_2 + \frac{1}{2\,|\hat{k}_1|} \int_0^1 |\partial_2 \hat{\zeta}_2|^2\, d\hat{x}_2 \quad \textit{constrained to} \quad \int_0^1 |\hat{\zeta}_2|^2\, d\hat{x}_2 \;=\; 1$$

is of the form

$$|\hat{k}_1^*| \;=\; (\frac{\pi}{2})^{2/3}, \tag{31}$$

$$\hat{\zeta}_2^*(\hat{x}_2) \;=\; \sqrt{2}\, c\, \sin(\pi\, \hat{x}_2) \quad \textit{for some } |c| = 1. \tag{32}$$

The minimum value is given by

$$3\,(\frac{\pi}{2})^{4/3}.$$

6 The magnetization ripple

There is relation between the nucleation problem in Regime III and the “magnetization ripple”. The magnetization ripple is an oscillatory small–angle modulation of the magnetization with wave–vector in direction of the average magnetization, see e. g. [5, p.452]. See Figure 4 for a sketch.

The origin of this modulation is well–understood [4, 3]: It is triggered by crystalline anisotropy (the fact that the crystalline structure of the material favors a magnetization direction) in conjunction with the polycristalline nature of the sample material. The axis favored by the crystalline anisotropy fluctuates from grain to grain

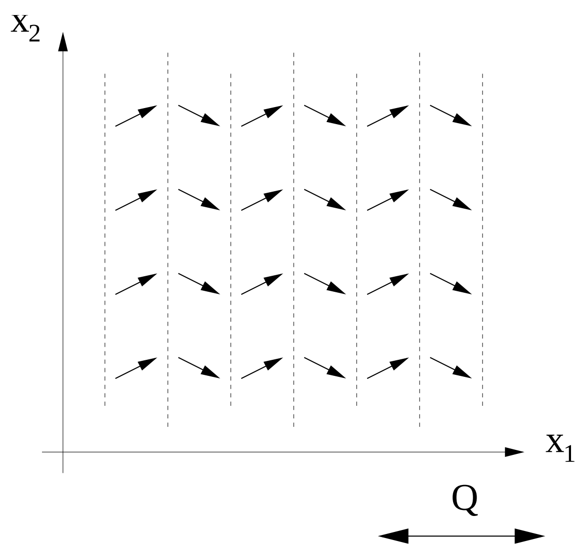

Figure 4: Magnetization ripple

around its average direction, say $(1,0,0)$. Mathematically, this has the same effect as a spatially heterogeneous external field which behaves like white noise on length scales large compared to the grain size. These isotropic fluctuations translate into strongly anisotropic fluctuations of the magnetization around its average direction. The reason is that the long–range magnetostatic interaction favors a longitudinal modulation over a transverse one.

This intuition can be quantified by linearizing around the constant magnetization $m^* = (1,0,0)$ favored by crystalline anisotropy in an infinite film of thickness t. The analysis shows that the relevant part of exchange and magnetostatic energy is given by the following contributions

$$\int (d^2 k_1^2 + \tfrac{1}{2}\, t |k_1|^{-1} k_2^2)\, |\mathcal{F}\langle\zeta_2\rangle|^2\, dk_1 dk_2, \tag{33}$$

where $\mathcal{F}\langle\zeta_2\rangle$ denotes the Fourier transform of the vertical average $\langle\zeta_2\rangle$ of the in–plane component ζ_2 of the infinitesimal perturbation ζ. Notice that after the rescaling from

Definition 1, the functional (33) is identical to (29). The only difference lies in the fact that (33) extends over an infinite plane whereas (29) corresponds to an infinite strip of finite width.

In order to predict the dominant ripple width w_{ripple}, one has to supplement (33) by the anisotropy energy with respect to the average direction favored by crystalline anisotropy, i. e. $\int Q\,\zeta_2^2\,dx_1dx_2 = \int Q\,|\mathcal{F}\langle\zeta_2\rangle|^2\,dk_1dk_2$. Here Q denotes the non–dimensional anisotropy material constant. For sufficiently small Q, this leads to

$$w_{ripple} \sim d\,Q^{-1/2}.$$

Let us compare nucleation in Regime III with the linear theory of the magnetization ripple. Although the forcing is different (constant external field plus pinning at the lateral facets vs. fluctuating effective field), the anisotropic response of the system is similar and to leading order described by the same Fourier symbol: $d^2k_1^2+\frac{1}{2}t|k_1|^{-1}k_2^2$. In both cases, it leads to the selection of a dominant longitudinal wave–length.

7 Structure of the proof of Theorem 2

Theorem 1 states in the same way as Theorem 2 that the minimizer ζ^* becomes x_3–independent in the limiting regime. Thus Lemma 1 sets a starting point here, too.

In the context of Theorem 1, the convenient non–dimensional parameters and rescaled quantities are:

Definition 2.

- *To a triple* (d,t,ℓ) *we associate*

$$\epsilon \;:=\; \frac{d^2}{t\ell}\ln^{-1}\left(\frac{\ell}{t}\right), \quad \delta \;:=\; \frac{t\ell}{d^2} \quad \textit{and} \quad \alpha := \frac{t}{\ell}.$$

- *To* $(k_1,\zeta)\in\mathbb{R}\times L^2(\tilde{\Omega})$ *we associate* $(\hat{k}_1,\hat{\zeta})\in\mathbb{R}\times L^2(\hat{\tilde{\Omega}})$ *via*

$$\ell\, k_1 \;=\; \hat{k}_1 \quad \textit{and} \quad (t\ell)^{1/2}\zeta(\ell\hat{x}_2,t\hat{x}_3) \;=\; \hat{\zeta}(\hat{x}_2,\hat{x}_3).$$

- *We write*

$$\left(\frac{\ell}{d}\right)^2\mathcal{R}(k_1,\zeta) \;=\; \hat{\mathcal{R}}(k_1,\zeta).$$

The analogue of Corollary 1 is

Corollary 3.

i) For any $\zeta\in L^2(\tilde{\Omega})$ *such that* $\zeta=\zeta(x_2)$*,* $\zeta_3\equiv 0$ *with* $\hat{\zeta}_2\in H_0^1((0,1))$*, we have*

$$\hat{\mathcal{R}}(k_1,\zeta) \;\le\; \int_0^1\left(\hat{k}_1^2\,|\hat{\zeta}_2|^2+|\partial_2\hat{\zeta}_2|^2\right)d\hat{x}_2+\frac{\delta}{2}\int_{\mathbb{R}}|\hat{k}_2||\mathcal{F}\hat{\zeta}_2|^2\,d\hat{k}_2. \tag{34}$$

ii) For any $\zeta\in L^2(\tilde{\Omega})$ *we have*

$$\hat{\mathcal{R}}(k_1,\zeta) \;\ge\; \int_0^1\left(\hat{k}_1^2\,|\langle\hat{\zeta}_2\rangle|^2+|\partial_2\langle\hat{\zeta}_2\rangle|^2\right)d\hat{x}_2. \tag{35}$$

The space $H_0^1((0,1))$ is defined as in (23).

In order to relate these bounds, again we have to show the following: If the rescaled Hessian $\hat{\mathcal{R}}(k_1,\zeta)$ is bounded,

- the rescaled wave vector $\hat{k}_1$ is bounded,

- the rescaled perturbation $\hat{\zeta}$ is close to its vertically averaged horizontal component $\langle\hat{\zeta}_2\rangle$.

This is the purpose of the next lemma.

Lemma 4. *For any* $\zeta \in L^2(\tilde{\Omega})$ *with* $\int_{\tilde{\Omega}} |\zeta|^2 \, d^2\tilde{x} = 1$ *we have*

$$\int_{\hat{\tilde{\Omega}}} |\hat{\zeta} - \langle\hat{\zeta}_2\rangle|^2 \, d\hat{\tilde{x}} \;\le\; C\,(\frac{\alpha}{\delta} + \alpha^2)\,\hat{\mathcal{R}}(k_1,\zeta), \tag{36}$$

$$|\hat{k}_1|^2 \;\le\; \hat{\mathcal{R}}(k_1,\zeta) \tag{37}$$

for some universal constant $C < \infty$. *Note that*

$$\frac{\alpha}{\delta} \;=\; \varepsilon\,\alpha\,\ln\left(\frac{1}{\alpha}\right).$$

Zero boundary conditions in the limit are provided by the following lemma.

Lemma 5. *For any* $\zeta \in H^1(\tilde{\Omega})$ *with* $\int_{\tilde{\Omega}} |\zeta|^2 \, d^2\tilde{x} = 1$ *we have*

$$|\langle\hat{\zeta}_2\rangle(0)|^2 + |\langle\hat{\zeta}_2\rangle(1)|^2 \;\le\; C\left(\varepsilon + \ln^{-1}\left(\frac{1}{\alpha}\right)\right)\hat{\mathcal{R}}(k_1,\zeta) \tag{38}$$

for some universal constant $C < \infty$.

The necessary relative compactness result for passing to the limit is

Corollary 4. *Let* $(d_\nu, t_\nu, \ell_\nu)_{\nu\uparrow\infty}$ *be such that*

$$\delta_\nu,\,\alpha_\nu \;\to\; 0 \quad (\Rightarrow\; \epsilon_\nu \;\to 0). \tag{39}$$

Let $(k_{1,\nu}, \zeta_\nu)_{\nu\uparrow\infty}$ *be such that*

$$\int_{\tilde{\Omega}} |\zeta_\nu|^2 \, d\tilde{x} \;=\; 1 \quad \textit{and} \quad \hat{\mathcal{R}}(k_{1,\nu},\zeta_\nu) \;\; \textit{is bounded.} \tag{40}$$

Then $(\hat{k}_{1,\nu}, \hat{\zeta}_\nu)$ *is relatively compact in* $\mathbb{R} \times L^2(\widehat{\tilde{\Omega}})$*. Moreover, any limit* $\hat{\zeta}$ *is of the form*

$$\hat{\zeta} \;=\; \hat{\zeta}(\hat{x}_2) \quad \textit{and} \quad \hat{\zeta}_3 \;\equiv\; 0 \quad \textit{with} \quad \hat{\zeta}_2 \in H^1_0((0,1)). \tag{41}$$

The Γ–limit is given in the following proposition.

Proposition 2. *Let* $(d_\nu, t_\nu, \ell_\nu)_{\nu\uparrow\infty}$ *be such that*

$$\delta_\nu,\, \alpha_\nu \;\to\; 0 \quad (\Rightarrow\; \epsilon_\nu \;\to 0)$$

Then the variational problem in $(k_1, \zeta) \in \mathbb{R} \times L^2(\tilde{\Omega})$ *of minimizing*

$$\hat{\mathcal{R}}(k_1, \zeta) \quad \textit{constrained to} \quad \int_{\tilde{\Omega}} |\zeta|^2\, d\tilde{x} \;=\; 1$$

Γ*–converges to the variational problem in* $(\hat{k}_1, \hat{\zeta}_2) \in \mathbb{R} \times H^1_0((0,1))$ *of minimizing*

$$\int_0^1 \left(\hat{k}_1^2\, |\hat{\zeta}_2|^2 + |\partial_2 \hat{\zeta}_2|^2 \right) d\hat{x}_2 \quad \textit{constrained to} \quad \int_0^1 |\hat{\zeta}_2|^2\, d\hat{x}_2 \;=\; 1 \tag{42}$$

under the following notion of convergence

$$\hat{k}_{1,\nu} \;\to\; \hat{k}_1 \quad \textit{and} \quad \int_{\widehat{\tilde{\Omega}}} |\hat{\zeta}_\nu(\widehat{\tilde{x}}) - \hat{\zeta}_2(\hat{x}_2)|^2 d\widehat{\tilde{x}} \;\to\; 0. \tag{43}$$

In the last step we identify the minimizer of the limiting problem.

Lemma 6. *Any minimizer* $(\hat{k}_1^*, \hat{\zeta}_2^*) \in \mathbb{R} \times H^1_0((0,1))$ *of*

$$\int_0^1 \left(\hat{k}_1^2\, |\hat{\zeta}_2|^2 + |\partial_2 \hat{\zeta}_2|^2 \right) d\hat{x}_2 \quad \textit{constrained to} \quad \int_0^1 |\hat{\zeta}_2|^2\, d\hat{x}_2 \;=\; 1$$

is of the form

$$\hat{k}_1^* \;=\; 0 \tag{44}$$

$$\hat{\zeta}_2^*(\hat{x}_2) \;=\; \sqrt{2}\, c\, \sin(\pi\, \hat{x}_2) \quad \textit{for some } |c| = 1. \tag{45}$$

The minimum value is given by

$$\pi^2.$$

8 Proofs

Proof of Lemma 1. W. l. o. g. we assume $t = 1$. Part i) is actually well–known. We reproduce its proof for the convenience of the reader and since the same approach is used for part ii).

The original problem is given by

$$-k_1^2 u_\zeta + \tilde{\Delta} u_\zeta \;=\; \partial_2\zeta_2 + \partial_3\zeta_3 \quad \text{distributionally.} \tag{46}$$

As $\zeta = \zeta(x_2)$, the Fourier transform of (46) in x_2 is:

$$\begin{aligned}
-|k'|^2\mathcal{F}u_\zeta + \partial_3^2\mathcal{F}u_\zeta &= 0 && \text{for } x_3 \notin (-\tfrac{1}{2}, \tfrac{1}{2}), && (47)\\
-|k'|^2\mathcal{F}u_\zeta + \partial_3^2\mathcal{F}u_\zeta &= ik_2\mathcal{F}\zeta_2 && \text{for } x_3 \in (-\tfrac{1}{2}, \tfrac{1}{2}), && (48)\\
\partial_3\mathcal{F}u_\zeta(k_2, \ \tfrac{1}{2}+) - \partial_3\mathcal{F}u_\zeta(k_2, \ \tfrac{1}{2}-) &= -\mathcal{F}\zeta_3(k_2) && \text{for } x_3 = \tfrac{1}{2}, && (49)\\
\partial_3\mathcal{F}u_\zeta(k_2, -\tfrac{1}{2}+) - \partial_3\mathcal{F}u_\zeta(k_2, -\tfrac{1}{2}-) &= \mathcal{F}\zeta_3(k_2) && \text{for } x_3 = \tfrac{1}{2}. && (50)
\end{aligned}$$

In view of (47) and (48), $\mathcal{F}u_\zeta$ must be of the form

$$\begin{aligned}
\mathcal{F}u_\zeta(k_2, x_3) \;=\; & \overline{u}(k_2) \times \left\{ \begin{array}{ll} \sinh(\frac{|k'|}{2})\exp((\frac{1}{2} - x_3)|k'|) & \frac{1}{2} \le x_3 \\ \exp(\frac{|k'|}{2}) - \cosh(|k'|x_3) & -\frac{1}{2} \le x_3 \le \frac{1}{2} \\ \sinh(\frac{|k'|}{2})\exp((\frac{1}{2} + x_3)|k'|) & x_3 \le -\frac{1}{2} \end{array} \right\} \\
& + \; \overline{v}(k_2) \times \left\{ \begin{array}{ll} \sinh(\frac{|k'|}{2})\exp((\frac{1}{2} - x_3)|k'|) & \frac{1}{2} \le x_3 \\ \sinh(|k'|x_3) & -\frac{1}{2} \le x_3 \le \frac{1}{2} \\ -\sinh(\frac{|k'|}{2})\exp((\frac{1}{2} + x_3)|k'|) & x_3 \le -\frac{1}{2} \end{array} \right\}.
\end{aligned} \tag{51}$$

Because of

$$
\begin{aligned}
\partial_3 \mathcal{F} u_\zeta(k_2, x_3) \quad &= \quad \overline{u}(k_2) \times \left\{ \begin{array}{ll} -|k'| \sinh(\frac{|k'|}{2}) \exp((\frac{1}{2} - x_3)|k'|) & \frac{1}{2} \leq x_3 \\ -|k'| \sinh(|k'| x_3) & -\frac{1}{2} \leq x_3 \leq \frac{1}{2} \\ |k'| \sinh(\frac{|k'|}{2}) \exp((\frac{1}{2} + x_3)|k'|) & x_3 \leq -\frac{1}{2} \end{array} \right\} \\
&+ \quad \overline{v}(k_2) \times \left\{ \begin{array}{ll} -|k'| \sinh(\frac{|k'|}{2}) \exp((\frac{1}{2} - x_3)|k'|) & \frac{1}{2} \leq x_3 \\ |k'| \cosh(|k'| x_3) & -\frac{1}{2} \leq x_3 \leq \frac{1}{2} \\ -|k'| \sinh(\frac{|k'|}{2}) \exp((\frac{1}{2} + x_3)|k'|) & x_3 \leq -\frac{1}{2} \end{array} \right\},
\end{aligned}
\tag{52}
$$

(49) & (50) turn both into

$$
\mathcal{F}\zeta_3(k_2) \quad = \quad |k'| \exp(\tfrac{|k'|}{2})\, \overline{v}(k_2), \tag{53}
$$

whereby

$$
\overline{v}(k_2) \;=\; \frac{\exp(-\frac{|k'|}{2})}{|k'|} \mathcal{F}\zeta_3(k_2). \tag{54}
$$

Obviously, by (48):

$$
\overline{u}(k_2) \;=\; \frac{-ik_2 \exp(-\frac{|k'|}{2})}{|k'|^2} \mathcal{F}\zeta_2(k_2). \tag{55}
$$

We obtain by Plancherel

$$
\begin{aligned}
&\int_{\mathbb{R}^2} \left(k_1^2 |u_\zeta|^2 + |\tilde{\nabla} u_\zeta|^2 \right) d\tilde{x} \\
&= \quad \int_{\mathbb{R}} \int_{\mathbb{R}} \left(|k'|^2 |u_\zeta|^2 + |\partial_3 u_\zeta|^2 \right) dx_3\, dk_2 \\
&\overset{(51),(52)}{=} \quad \int_{\mathbb{R}} |\overline{u}|^2 |k'| \left\{ 1 + \exp(|k'|)\,(-1 + |k'|) \right\} dk_2 + \int_{\mathbb{R}} |\overline{v}|^2 |k'| \left(\exp(|k'|) - 1 \right) dk_2 \\
&\overset{(54),(55)}{=} \quad \int_{\mathbb{R}} \frac{\exp(-|k'|) - 1 + |k'|}{|k'|^3} k_2^2\, |\mathcal{F}\zeta_2|^2\, dk_2 + \int_{\mathbb{R}} \frac{1 - \exp(-|k'|)}{|k'|} |\mathcal{F}\zeta_3|^2\, dk_2.
\end{aligned}
$$

We now address part ii). We will actually show

$$\min_{\zeta_3}\left\{\int_{\mathbb{R}^2}(k_1^2|u_\zeta|^2+|\tilde\nabla u_\zeta|^2)\,d\tilde x\,\middle|\,\langle\zeta_3\rangle\text{ given}\right\} \tag{56}$$
$$=\int_{\mathbb{R}}\frac{|k_2\mathcal{F}\langle\zeta_2\rangle|^2}{2|k'|+|k'|^2}\,dk_2+\int_{\mathbb{R}}\frac{|-ik_2\mathcal{F}\langle x_3\zeta_2\rangle+\mathcal{F}\langle\zeta_3\rangle|^2}{1+\frac{1}{2}|k'|+\frac{1}{12}|k'|^2}\,dk_2.$$

We first notice that (56) can be written as a saddle point problem. This is a consequence of the following representation of the stray field energy

$$\int_{\mathbb{R}^2}\left(k_1^2|u_\zeta|^2+|\tilde\nabla u_\zeta|^2\right)d\tilde x$$
$$=\max_v\left\{-\int_{\mathbb{R}^2}\left(k_1^2|v|^2+|\tilde\nabla v|^2\right)d\tilde x+2\int_{\tilde\Omega}\tilde\nabla v\cdot\zeta\,d\tilde x\right\}.$$

The first variation w. r. t. ζ_3 yields

$$\partial_3^2u_\zeta\quad=\quad 0\quad\text{for}\quad x_3\in(-\tfrac{1}{2},\tfrac{1}{2}). \tag{57}$$

Notice that since the average $\langle\zeta_3\rangle$ of ζ_3 in x_3 is prescribed, it is only the second derivative $\partial_3^2u_\zeta$ w. r. t. x_3 which vanishes and not the first. The first variation w. r. t. v yields

$$\begin{aligned}
-k_1^2u_\zeta+\tilde\Delta u_\zeta &= 0 && \text{for } x_3\notin(-\tfrac{1}{2},\tfrac{1}{2}),\\
-k_1^2u_\zeta+\tilde\Delta u_\zeta &= \partial_2\zeta_2+\partial_3\zeta_3 && \text{for } x_3\in(-\tfrac{1}{2},\tfrac{1}{2}),\\
\partial_3u_\zeta(x_2,\ \tfrac{1}{2}+)-\partial_3u_\zeta(x_2,\ \tfrac{1}{2}-) &= -\zeta_3(x_2,\ \tfrac{1}{2}) && \text{for } x_3=\tfrac{1}{2},\\
\partial_3u_\zeta(x_2,-\tfrac{1}{2}+)-\partial_3u_\zeta(x_2,-\tfrac{1}{2}-) &= \zeta_3(x_2,-\tfrac{1}{2}) && \text{for } x_3=\tfrac{1}{2}.
\end{aligned}$$

We take the Fourier transform in x_2 and obtain by use of (57):

$$\begin{aligned}
-|k'|^2\mathcal{F}u_\zeta + \partial_3^2\mathcal{F}u_\zeta &= 0 && \text{for } x_3 \notin (-\tfrac{1}{2}, \tfrac{1}{2}), && (58)\\
-|k'|^2\mathcal{F}u_\zeta &= ik_2\mathcal{F}\zeta_2 + \partial_3\mathcal{F}\zeta_3 && \text{for } x_3 \in (-\tfrac{1}{2}, \tfrac{1}{2}), && (59)\\
\partial_3\mathcal{F}u_\zeta(k_2, \ \tfrac{1}{2}+) - \partial_3\mathcal{F}u_\zeta(k_2, \ \tfrac{1}{2}-) &= -\mathcal{F}\zeta_3(k_2, \ \tfrac{1}{2}) && \text{for } x_3 = \tfrac{1}{2}, && (60)\\
\partial_3\mathcal{F}u_\zeta(k_2, -\tfrac{1}{2}+) - \partial_3\mathcal{F}u_\zeta(k_2, -\tfrac{1}{2}-) &= \mathcal{F}\zeta_3(k_2, -\tfrac{1}{2}) && \text{for } x_3 = \tfrac{1}{2}. && (61)
\end{aligned}$$

In view of (57) and (58), $\mathcal{F}u_\zeta$ must be of the form

$$\begin{aligned}
\mathcal{F}u_\zeta(k_2, x_3) &= \overline{u}(k_2) \times \left\{ \begin{array}{ll} \exp((\frac{1}{2} - x_3)|k'|) & \frac{1}{2} \le x_3 \\ 1 & -\frac{1}{2} \le x_3 \le \ \frac{1}{2} \\ \exp((\frac{1}{2} + x_3)|k'|) & x_3 \le -\frac{1}{2} \end{array} \right\} \\
&+ \ \overline{v}(k_2) \times \left\{ \begin{array}{ll} \frac{1}{2}\exp((\frac{1}{2} - x_3)|k'|) & \frac{1}{2} \le x_3 \\ x_3 & -\frac{1}{2} \le x_3 \le \ \frac{1}{2} \\ -\frac{1}{2}\exp((\frac{1}{2} + x_3)|k'|) & x_3 \le -\frac{1}{2} \end{array} \right\}. \qquad (62)
\end{aligned}$$

Because of

$$\begin{aligned}
\partial_3\mathcal{F}u_\zeta(k_2, x_3) &= \overline{u}(k_2) \times \left\{ \begin{array}{ll} -|k'|\exp((\frac{1}{2} - x_3)|k'|) & \frac{1}{2} \le x_3 \\ 0 & -\frac{1}{2} \le x_3 \le \ \frac{1}{2} \\ |k'|\exp((\frac{1}{2} + x_3)|k'|) & x_3 \le -\frac{1}{2} \end{array} \right\} \\
&+ \ \overline{v}(k_2) \times \left\{ \begin{array}{ll} -\frac{1}{2}|k'|\exp((\frac{1}{2} - x_3)|k'|) & \frac{1}{2} \le x_3 \\ 1 & -\frac{1}{2} \le x_3 \le \ \frac{1}{2} \\ -\frac{1}{2}|k'|\exp((\frac{1}{2} + x_3)|k'|) & x_3 \le -\frac{1}{2} \end{array} \right\}, \qquad (63)
\end{aligned}$$

(60) & (61) turn into

$$\begin{aligned}\mathcal{F}\zeta_3(k_2, \quad \tfrac{1}{2}) &= \quad |k'|\,\overline{u}(k_2) + (1+\tfrac{1}{2}|k'|)\overline{v}(k_2),\\ \mathcal{F}\zeta_3(k_2, -\tfrac{1}{2}) &= -|k'|\,\overline{u}(k_2) + (1+\tfrac{1}{2}|k'|)\overline{v}(k_2).\end{aligned} \tag{64}$$

We now determine $\overline{u}$ and $\overline{v}$. To this aim, we consider (59) which in view of (62) turns into

$$-|k'|^2\,(\overline{u}(k_2) + \overline{v}(k_2)\,x_3) \;=\; ik_2\mathcal{F}\zeta_2 + \partial_3\mathcal{F}\zeta_3. \tag{65}$$

For $\overline{u}$, we take the average in x_3 of (65)

$$\begin{aligned}-|k'|^2\overline{u}(k_2) &= ik_2\mathcal{F}\langle\zeta_2\rangle(k_2) + \big(\mathcal{F}\zeta_3(k_2,\tfrac{1}{2}) - \mathcal{F}\zeta_3(k_2,-\tfrac{1}{2})\big)\\ &\overset{(64)}{=} ik_2\mathcal{F}\langle\zeta_2\rangle(k_2) + 2|k'|\overline{u}(k_2),\end{aligned}$$

yielding

$$\int_{\tilde\Omega}|\zeta|^2\,d\tilde{x}, \overline{u}(k_2) \;=\; \frac{-ik_2\mathcal{F}\langle\zeta_2\rangle(k_2)}{2|k'|+|k'|^2}. \tag{66}$$

For $\overline{v}$, we multiply (65) with x_3 and take the average in x_3:

$$\begin{aligned}-\tfrac{1}{12}|k'|^2\,\overline{v}(k_2) &= -\langle x_3^2\rangle\,|k'|^2\,\overline{v}(k_2)\\ &= ik_2\mathcal{F}\langle x_3\zeta_2\rangle(k_2) + \mathcal{F}\langle\, x_3\,\partial_3\zeta_3\rangle(k_2)\\ &= ik_2\mathcal{F}\langle x_3\zeta_2\rangle(k_2) - \mathcal{F}\langle\zeta_3\rangle(k_2) + \tfrac{1}{2}\,\big(\mathcal{F}\zeta_3(k_2,\tfrac{1}{2}) + \mathcal{F}\zeta_3(k_2,-\tfrac{1}{2})\big)\\ &\overset{(64)}{=} ik_2\mathcal{F}\langle x_3\zeta_2\rangle(k_2) - \mathcal{F}\langle\zeta_3\rangle(k_2) + (1+\tfrac{1}{2}|k'|)\,\overline{v}(k_2),\end{aligned}$$

yielding

$$\overline{v}(k_2) \;=\; \frac{-ik_2\mathcal{F}\langle x_3\zeta_2\rangle(k_2) + \mathcal{F}\langle\zeta_3\rangle(k_2)}{1+\tfrac{1}{2}|k'|+\tfrac{1}{12}|k'|^2}. \tag{67}$$

Hence we obtain by Plancherel (mixed terms vanish because of different symmetry in x_3)

$$
\begin{aligned}
&\int_{\mathbb{R}^2} \left(k_1^2|u_\zeta|^2 + |\tilde{\nabla} u_\zeta|^2\right) d\tilde{x} \\
&= \quad \int_{\mathbb{R}}\int_{\mathbb{R}} \left(|k'|^2|\mathcal{F}u_\zeta|^2 + |\partial_3 \mathcal{F} u_\zeta|^2\right) dx_3\, dk_2 \\
&\overset{(62),(63)}{=} \int_{\mathbb{R}} |\overline{u}_\zeta|^2 \left(|k'|^2 + 2|k'|\right) dk_2 + \int_{\mathbb{R}} |\overline{v}|^2 \left(1 + \tfrac{1}{2}|k'| + \tfrac{1}{12}|k'|^2\right) dk_2 \\
&\overset{(66),(67)}{=} \int_{\mathbb{R}} \frac{k_2^2\,|\mathcal{F}\langle \zeta \int_{\tilde{\Omega}} |\zeta|^2\, d\tilde{x}_{,2}\rangle|^2}{2|k'| + |k'|^2}\, dk_2 + \int_{\mathbb{R}} \frac{|-ik_2 \mathcal{F}\langle x_3\zeta_2\rangle + \mathcal{F}\langle \zeta_3\rangle|^2}{1 + \frac{1}{2}|k'| + \frac{1}{12}|k'|^2}\, dk_2.
\end{aligned}
$$

q.e.d.

Proof of Corollary 1. We first address part i). For ζ with $\zeta = \zeta(x_2)$ and $\zeta_3 \equiv 0$ we have according to Lemma 1 i)

$$
\begin{aligned}
&\mathcal{R}(k_1, \zeta) \\
&= \; d^2\, t \int_0^\ell (k_1^2\, |\zeta_2|^2 + |\partial_2 \zeta_2|^2)\, dx_2 + t \int_{\mathbb{R}} \frac{\exp(-t|k'|) - 1 + t|k'|}{(t|k'|)^3}\, (tk_2)^2\, |\mathcal{F}\zeta_2|^2\, dk_2.
\end{aligned}
$$

Using the Fourier multiplier inequality

$$
\frac{\exp(-t|k'|) - 1 + t|k'|}{(t|k'|)^3} \;\le\; \frac{1}{2\,t|k'|} \;\le\; \frac{1}{2\,t|k_1|},
$$

the above turns into

$$
\mathcal{R}(k_1, \zeta) \;\le\; d^2\, t\, k_1^2 \int_0^\ell |\zeta_2|^2\, dx_2 + d^2\, t \int_0^\ell |\partial_2 \zeta_2|^2\, dx_2 + \frac{t^2}{2|k_1|} \int_{\mathbb{R}} k_2^2\, |\mathcal{F}\zeta_2|^2\, dk_2.
$$

Notice that since $\zeta_2 \in H_0^1((0,1))$ (in the definition of (23)) we have

$$
\int_{\mathbb{R}} k_2^2\, |\mathcal{F}\zeta_2|^2\, dk_2 \;=\; \int_0^\ell |\partial_2 \zeta_2|^2\, dx_2,
$$

such that the last inequality assumes the form

$$
\mathcal{R}(k_1, \zeta) \;\le\; d^2\, t\, k_1^2 \int_0^\ell |\zeta_2|^2\, dx_2 + (d^2\, t\, + \frac{t^2}{2|k_1|}) \int_0^\ell |\partial_2 \zeta_2|^2\, dx_2.
$$

Under the rescaling of Definition 1, this turns into (21).

We now address part ii). We have according to Lemma 1 i)

$$\begin{aligned} &\mathcal{R}(k_1,\zeta) \\ &\geq\; d^2 \int_{\tilde\Omega} (k_1^2\,|\zeta|^2 + |\tilde\nabla\zeta|^2)\,d\tilde x + t\int_{\mathbb{R}} \frac{1}{2\,t|k'| + (t|k'|)^2}\,(tk_2)^2\,|\mathcal{F}\langle\zeta_2\rangle|^2\,dk_2. \end{aligned}$$

With help of Jensen's inequality

$$\int_{\tilde\Omega} |\zeta|^2\,d\tilde x \;\geq\; t\int_0^\ell |\langle\zeta\rangle|^2\,dx_2 \;\geq\; t\int_0^\ell |\langle\zeta_2\rangle|^2\,dx_2$$

the above turns into

$$\begin{aligned} &\mathcal{R}(k_1,\zeta) \\ &\geq\; d^2\,t\,k_1^2 \int_0^\ell |\langle\zeta_2\rangle|^2\,dx_2 + t^2\int_{\mathbb{R}} \frac{1}{2\,(k_1^2+k_2^2)^{1/2} + t(k_1^2+k_2^2)^2}\,k_2^2\,|\mathcal{F}\langle\zeta_2\rangle|^2\,dk_2, \end{aligned}$$

which is the unrescaled version of (22).

q.e.d.

Proof of Lemma 2 and Lemma 4. Let $C<\infty$ denote a generic universal constant. We write

$$\int_{\tilde\Omega} |\zeta - \langle\zeta_2\rangle|^2\,d\tilde x \;\leq\; \int_{\tilde\Omega} |\zeta-\langle\zeta\rangle|^2\,d\tilde x + \int_{\tilde\Omega} |\langle\zeta_3\rangle|^2\,d\tilde x. \tag{68}$$

The first term on the r. h. s. in (68) is estimated by Poincaré's inequality

$$\int_{\tilde\Omega} |\zeta-\langle\zeta\rangle|^2\,d\tilde x \;\leq\; C\,t^2\int_{\tilde\Omega} |\partial_3\zeta|^2\,d\tilde x \;\leq\; C\left(\frac{t}{d}\right)^2 \mathcal{R}(k_1,\zeta). \tag{69}$$

For the second term in (68) we start by observing that because of $t\ll\ell$ we have

$$t\int_0^\ell |\langle\zeta_3\rangle|^2\,dx_2 \;\leq\; C\left(t\int_{t|k_2|\leq 1} |\mathcal{F}\langle\zeta_3\rangle|^2\,dk_2 + t^3\int_0^\ell |\partial_2\langle\zeta_3\rangle|^2\,dx_2\right). \tag{70}$$

For a proof of the elementary estimate (70) we refer to [1, Lemma 5]. The second term in (70) can be estimated by Jensen's inequality

$$t^3 \int_0^\ell |\partial_2 \langle \zeta_3 \rangle|^2 \, dx_2 \ \le \ t^2 \int_{\tilde\Omega} |\partial_2 \zeta_3|^2 \, d\tilde x \ \le \ \left(\frac{t}{d} \right)^2 \mathcal{R}(k_1, \zeta). \tag{71}$$

For the first term in (70), we appeal to the Fourier multiplier estimate

$$1 \ \le \ C \left(\min\{1, \frac{1}{t^2 |k'|^2}\} + t^2 \, k_1^2 \right) \qquad \text{for } t|k_2| \le 1, \tag{72}$$

an elementary inequality we established in [1, Lemma 7]. Thanks to (72) we have

$$t \int_{t|k_2|\le 1} |\mathcal{F}\langle \zeta_3 \rangle|^2 \, dk_2 \ \le \ C \left(t \int_{\mathbb{R}} \min\{1, \frac{1}{t^2|k'|^2}\} \, |\mathcal{F}\langle \zeta_3 \rangle|^2 \, dk_2 + t^3 \, k_1^2 \int_0^\ell |\langle \zeta_3 \rangle|^2 \, dx_2 \right). \tag{73}$$

For the last term in (73) we appeal once more to Jensen's inequality

$$t^3 \, k_1^2 \int_0^\ell |\langle \zeta_3 \rangle|^2 \, dx_2 \ \le \ t^2 \, k_1^2 \int_{\tilde\Omega} |\zeta_3|^2 \, d\tilde x \ \le \ \left(\frac{t}{d} \right)^2 \mathcal{R}(k_1, \zeta). \tag{74}$$

We now turn to the first r. h. s. term of (73). To this purpose, we appeal to Lemma 1 i), which yields in particular

$$t \int_{\mathbb{R}} \frac{1}{1 + \frac{1}{2} \, t|k'| + \frac{1}{12} \, (t|k'|)^2} |\mathcal{F}\langle \zeta_3 \rangle - i k_2 \mathcal{F}\langle x_3 \zeta_2 \rangle|^2 \, dk_2 \ \le \ \mathcal{R}(k_1, \zeta),$$

which we use in form of

$$\begin{aligned} t \int_{\mathbb{R}} \frac{1}{1 + \frac{1}{2} \, t|k'| + \frac{1}{12} \, (t|k'|)^2} |\mathcal{F}\langle \zeta_3 \rangle|^2 \, dk_2 \ &\le \ 2\mathcal{R}(k_1, \zeta) \\ &+ \ 2t \int_{\mathbb{R}} \frac{k_2^2}{1 + \frac{1}{2} \, t|k'| + \frac{1}{12} \, (t|k'|)^2} |\mathcal{F}\langle x_3 \zeta_2 \rangle|^2 \, dk_2 \end{aligned}$$

Since

$$\frac{12}{t^2 k_2^2} \ \ge \ \frac{1}{1 + \frac{1}{2} \, t|k'| + \frac{1}{12} \, (t|k'|)^2} \ \ge \ \frac{1}{C} \, \min\{1, \frac{1}{t^2|k'|^2}\},$$

the above yields

$$t\int_{\mathbb{R}} \min\{1,\frac{1}{t^2|k'|^2}\}\,|\mathcal{F}\langle\zeta_3\rangle|\,dk_2 \;\le\; C\left(\mathcal{R}(k_1,\zeta)+\frac{1}{t}\int_0^\ell |\langle x_3\,\zeta_2\rangle|^2\,dx_2\right). \tag{75}$$

The last term in (75) can be estimated via Cauchy–Schwarz and Poincaré:

$$\begin{aligned}
\frac{1}{t}\int_0^\ell |\langle x_3\,\zeta_2\rangle|^2\,dx_2 &= \frac{1}{t}\int_0^\ell |\langle x_3\,(\zeta_2-\langle\zeta_2\rangle)\rangle|^2\,dx_2 \\
&\le \frac{t}{12}\int_0^\ell \langle|\zeta_2-\langle\zeta_2\rangle|^2\rangle\,dx_2 \\
&= \frac{1}{12}\int_{\tilde\Omega} |\zeta_2-\langle\zeta_2\rangle|^2\,d\tilde x \\
&\le C\,t^2\int_{\tilde\Omega} |\partial_3\zeta_2|^2\,d\tilde x \\
&= C\left(\frac{t}{d}\right)^2\mathcal{R}(k_1,\zeta).
\end{aligned} \tag{76}$$

Inserting (76) into (75) yields

$$t\int_{\mathbb{R}} \min\{1,\frac{1}{t^2|k'|^2}\}\,|\mathcal{F}\langle\zeta_3\rangle|\,dk_2 \;\le\; C\left(1+\left(\frac{t}{d}\right)^2\right)\mathcal{R}(k_1,\zeta). \tag{77}$$

We now collect (69), (71), (74), and (77):

$$\int_{\tilde\Omega} |\zeta-\langle\zeta_2\rangle|^2\,d\tilde x \;\le\; C\left(1+\left(\frac{t}{d}\right)^2\right)\mathcal{R}(k_1,\zeta), \tag{78}$$

which is the unrescaled version of (24) and (36).

We finally adress (25) and (37). By (8) we have

$$\mathcal{R}(k_1,\zeta) \;\ge\; d^2\,k_1^2\int_{\tilde\Omega} |\zeta|^2\,d\tilde x,$$

which, by use of

$$\int_{\tilde\Omega} |\zeta|^2\,d\tilde x \;=\; 1$$

yields

$$\mathcal{R}(k_1,\zeta) \;\ge\; d^2\,k_1^2.$$

This is just an unrescaled version of (25) and (37).

q.e.d.

Proof of Corollary 2. The second estimate in Lemma 2 implies that

$$\{\hat k_{1,\nu}\}_{\nu\uparrow\infty} \quad \text{is bounded} \tag{79}$$

and thus relatively compact in $\mathbb{R}$. The first estimate in Lemma 2 implies that any $L^2(\widehat{\hat\Omega})$–limit $\hat\zeta$ of $\{\hat\zeta_\nu\}_{\nu\uparrow\infty}$ satisfies (28).

It remains to argue that $\{\langle\hat\zeta_{2,\nu}\rangle\}_{\nu\uparrow\infty}$ is relatively compact in $L^2((0,1))$. Since $(0,1)$ is bounded and since $\{\langle\hat\zeta_{2,\nu}\rangle\}_{\nu\uparrow\infty}$ is bounded in $L^2((0,1))$, it remains to show that the high frequencies $|k_2|\gg 1$ are uniformly small in the sense of

$$\lim_{M\uparrow\infty}\limsup_{\nu\uparrow\infty}\int_{|\hat k_2|\ge M}|\mathcal{F}\langle\hat\zeta_{2,\nu}\rangle|^2 d\hat k_2 \;=\; 0. \tag{80}$$

We will infer this from Corollary 1 ii). Indeed

$$\begin{aligned}
&\int_{|\hat k_2|\ge M}|\mathcal{F}\langle\hat\zeta_{2,\nu}\rangle|^2 d\hat k_2\\
\le\;& \left(\inf_{|\hat k_2|\ge M}\frac{\hat k_2^2}{2\,(\hat k_{1,\nu}^2+\epsilon_\nu\,\hat k_2^2)^{1/2}+\delta_\nu\,(\hat k_{1,\nu}^2+\epsilon_\nu\,\hat k_2^2)}\right)^{-1}\\
&\int_{\mathbb{R}}\frac{1}{2\,(\hat k_{1,\nu}^2+\epsilon_\nu\,\hat k_2^2)^{1/2}+\delta_\nu\,(\hat k_{1,\nu}^2+\epsilon_\nu\,\hat k_2^2)}\,\hat k_2^2\,|\mathcal{F}\langle\hat\zeta_{2,\nu}\rangle|^2 d\hat k_2\\
\le\;& \left(\frac{M^2}{2\,(\hat k_{1,\nu}^2+\epsilon_\nu\,M^2)^{1/2}+\delta_\nu\,(\hat k_{1,\nu}^2+\epsilon_\nu\,M^2)}\right)^{-1}\hat{\mathcal{R}}(k_{1,\nu},\zeta_\nu)\\
=\;& \left(\frac{1}{M}\Big(\Big(\frac{\hat k_{1,\nu}}{M}\Big)^2+\epsilon_\nu\Big)^{1/2}+\delta_\nu\Big(\Big(\frac{\hat k_{1,\nu}}{M}\Big)^2+\epsilon_\nu\Big)\right)\hat{\mathcal{R}}(k_{1,\nu},\zeta_\nu).
\end{aligned}$$

Hence we obtain from (26)

$$\limsup_{\nu\uparrow\infty}\int_{|\hat k_2|\ge M}|\mathcal{F}\langle\hat\zeta_{2,\nu}\rangle|^2 d\hat k_2 \;\le\; \frac{1}{M^2}\,\limsup_{\nu\uparrow\infty}|\hat k_{1,\nu}|\,\limsup_{\nu\uparrow\infty}\hat{\mathcal{R}}(k_{1,\nu},\zeta_\nu).$$

Thus (80) follows from (79) and (27). Finally, we show that the limit is in $H^1_0((0,1))$.

The notion (30) implies in particular

$$\langle\hat{\zeta}_{2,\nu}\rangle \;\to\; \hat{\zeta}_2 \quad \text{in } L^2((0,1)) \tag{81}$$

and thus by Plancherel

$$\mathcal{F}\langle\hat{\zeta}_{2,\nu}\rangle \;\to\; \mathcal{F}\hat{\zeta}_2 \quad \text{in } L^2((0,1))$$

Hence for a subsequence, which we do not distinguish in notation,

$$\mathcal{F}\langle\hat{\zeta}_{2,\nu}\rangle \;\to\; \mathcal{F}\hat{\zeta}_2 \quad \text{a. e. in } (0,1). \tag{82}$$

Again, by Corollary 1 ii), we have

$$\begin{aligned}
&\hat{\mathcal{R}}(k_{1,\nu},\zeta_\nu)\\
&\quad\geq\; \hat{k}^2_{1,\nu}\int_0^1 |\langle\hat{\zeta}_{2,\nu}\rangle|^2\,dx_2\\
&\quad\quad+\int_{\mathbb{R}}\frac{1}{2\,(\hat{k}^2_{1,\nu}+\epsilon_\nu\,\hat{k}^2_2)^{1/2}+\delta_\nu\,(\hat{k}^2_{1,\nu}+\epsilon_\nu\,\hat{k}^2_2)}\,\hat{k}^2_2\,|\mathcal{F}\langle\hat{\zeta}_{2,\nu}\rangle|^2 d\hat{k}_2.
\end{aligned}$$

We infer from $k_{1,\nu}\to k_1$, (81) and (82) (using Fatou's Lemma)

$$\liminf_{\nu\uparrow\infty}\hat{\mathcal{R}}(k_{1,\nu},\zeta_\nu)\;\geq\;\hat{k}^2_1\int_0^1|\langle\hat{\zeta}_2\rangle|^2\,dx_2+\int_{\mathbb{R}}\frac{1}{2\,|\hat{k}_1|}\,\hat{k}^2_2\,|\mathcal{F}\langle\hat{\zeta}_2\rangle|^2 d\hat{k}_2.$$

In particular

$$\int_{\mathbb{R}}\hat{k}^2_2\,|\mathcal{F}\langle\hat{\zeta}_2\rangle|^2\,d\hat{k}_2\;<\;\infty.$$

In view of the definition (23), this yields $\hat{\zeta}_2\in H^1_0((0,1))$, as $supp\,\hat{\zeta}_2\subset[0,1]$.

q.e.d.

Proof of Proposition 1. The proof of Γ-convergence consists of two parts:

- Construction. For any $(\hat{k}_1, \hat{\zeta}_2) \in \mathbb{R} \times H^1_0((0,1))$ with $\int_0^1 |\hat{\zeta}_2|^2\, d\hat{x}_2 = 1$ there exists a sequence $(k_{1,\nu}, \zeta_\nu) \in \mathbb{R} \times L^2(\tilde{\Omega})$ with $\int_{\tilde{\Omega}} |\zeta|^2\, d^2\tilde{x} = 1$ which converges in the sense of (30) such that

$$\limsup_{\nu\uparrow\infty} \hat{\mathcal{R}}(k_{1,\nu}, \zeta_\nu) \;\le\; \hat{k}_1^2 \int_0^1 |\hat{\zeta}_2|^2\, d\hat{x}_2 + \frac{1}{2\,|\hat{k}_1|} \int_0^1 |\partial_2\hat{\zeta}_2|^2\, d\hat{x}_2. \tag{83}$$

- Lower semicontinuity. For any sequence $(k_{1,\nu}, \zeta_\nu) \in \mathbb{R} \times L^2(\tilde{\Omega})$ with bounded $\hat{\mathcal{R}}(k_{1,\nu}, \zeta_\nu)$ which converges to a $(\hat{k}_1, \hat{\zeta}_2) \in \mathbb{R} \times L^2((0,1))$ in the sense of (30) one has

$$\hat{\zeta}_2 \;\in\; H^1_0((0,1))$$

and

$$\liminf_{\nu\uparrow\infty} \hat{\mathcal{R}}(k_{1,\nu}, \zeta_\nu) \;\ge\; \hat{k}_1^2 \int_0^1 |\hat{\zeta}_2|^2\, d\hat{x}_2 + \frac{1}{2\,|\hat{k}_1|} \int_0^1 |\partial_2\hat{\zeta}_2|^2\, d\hat{x}_2. \tag{84}$$

For the construction part, we define $(k_{1,\nu}, \zeta_\nu) \in \mathbb{R} \times L^2(\tilde{\Omega})$ according to Definition 1:

$$\left(\frac{d_\nu^2 \ell_\nu^2}{t_\nu}\right)^{1/3} k_{1,\nu} \;=\; \hat{k}_1 \quad\text{and}\quad (t\ell)^{1/2}\zeta_\nu(\ell\hat{x}_2, t\hat{x}_3) \;=\; \hat{\zeta}_2(\hat{x}_2).$$

Then (30) is trivially fulfilled. Furthermore, we have by part i) of Corollary 1

$$\hat{\mathcal{R}}(k_{1,\nu}, \zeta_\nu) \;\le\; \hat{k}_1^2 \int_0^1 |\hat{\zeta}_2|^2\, dx_2 + (\frac{1}{2\,|\hat{k}_1|} + \epsilon_\nu) \int_0^1 |\partial_2\hat{\zeta}_2|^2\, d\hat{x}_2.$$

This equality establishes (83).

We now address the lower semicontinuity part. As seen in the proof of Corollary 2,

$$\liminf_{\nu\uparrow\infty} \hat{\mathcal{R}}(k_{1,\nu}, \zeta_\nu) \;\ge\; \hat{k}_1^2 \int_0^1 |\langle\hat{\zeta}_2\rangle|^2\, dx_2 + \int_{\mathbb{R}} \frac{1}{2\,|\hat{k}_1|}\, \hat{k}_2^2\, |\mathcal{F}\langle\hat{\zeta}_2\rangle|^2 d\hat{k}_2,$$

which by $\hat{\zeta}_2 \in H^1_0((0,1))$ yields (84).

q.e.d.

Proof of Lemma 3. The statement reduces to two observations

- For any $\hat\zeta_2 \in H^1_0((0,1))$ we have

$$\int_0^1 |\partial_2\hat\zeta_2|^2\, d\hat x_2 \;\geq\; \pi^2 \int_0^1 |\hat\zeta_2|^2\, d\hat x_2$$

  with equality if and only if $\hat\zeta_2$ is of the form (32) for some $c \in \mathbb{C}$.

- It holds

$$\hat k_1^2 + \frac{\pi^2}{2\,|\hat k_1|} \;\geq\; 3\,(\frac{\pi}{2})^{4/3}$$

  with equality if and only if $\hat k_1$ satisfies (31).

q.e.d.

Proof of Theorem 1. Our argument is indirect. Assume the statement were not true. Then there exists a sequence numbers (d_ν, t_ν, ℓ_ν) such that (11) holds, and a corresponding sequence $(k^*_{1,\nu}, \zeta^*_\nu)$ of minimizers such that one of the following three conditions is violated

$$|k^*_{1,\nu}| \;\rightarrow\; (\frac{\pi}{2})^{2/3}, \tag{85}$$

$$\int_{\widehat\Omega} |\hat\zeta_\nu(\widehat x) - \sqrt{2}\, c\, \sin(\pi \hat x_2)|^2\, d\widehat x \;\rightarrow\; 0 \quad \text{for some } |c| = 1, \tag{86}$$

$$\hat{\mathcal{R}}(k^*_{1,\nu}, \zeta^*_\nu) \;\rightarrow\; 3(\frac{\pi}{2})^{4/3}. \tag{87}$$

Lemma 3 implies in particular that the minimum of the Γ–limit (29) is finite. According to the construction part of Proposition 1 it follows that $\limsup_{\nu\uparrow\infty} \hat{\mathcal{R}}(k^*_{1,\nu}, \zeta^*_\nu)$

is finite. According to Corollary 2, there thus exists $(\hat{k}_1^*, \hat{\zeta}_2^*) \in \mathbb{R} \times L^2((1,0))$ such that for a subsequence (which we do not distinguish in notation)

$$\hat{k}_{1,\nu}^* \;\to\; \hat{k}_1^*, \tag{88}$$

$$\int_{\hat{\widehat{\Omega}}} |\hat{\zeta}_\nu^*(\widehat{\hat{x}}) - \hat{\zeta}_2^*(\hat{x}_2)|^2 \, d\widehat{\hat{x}} \;\to\; 0, \tag{89}$$

which is just the notion of convergence (30) in the Γ–convergence.

The Γ–convergence of Proposition 1 implies that, cf. [2],

$$(\hat{k}_1^*, \hat{\zeta}_2^*) \in \mathbb{R} \times H_0^1((0,1)) \quad \text{is a minimizer of (29)} \tag{90}$$

and that

$$\hat{\mathcal{R}}(k_{1,\nu}^*, \zeta_\nu^*) \;\to\; \hat{k}_1^{*\,2} \int_0^1 |\hat{\zeta}_2^*|^2 \, d\hat{x}_2 + \frac{1}{2\,|\hat{k}_1^*|} \int_0^1 |\partial_2 \hat{\zeta}_2^*|^2 \, d\hat{x}_2. \tag{91}$$

Finally, we appeal to Lemma 3: (90) implies that

$$\begin{aligned}
|\hat{k}_1^*| &= (\frac{\pi}{2})^{2/3}, \\
\hat{\zeta}_2(\hat{x}_2) &= \sqrt{2}\, c \sin(\pi\, \hat{x}_2) \quad \text{for some } |c| = 1, \\
|\hat{k}_1^*|^2 \int_0^1 |\hat{\zeta}_2|^2 \, dx_2 + \frac{1}{2\,|\hat{k}_1^*|} \int_0^1 |\partial_2 \hat{\zeta}_2|^2 \, dx_2 &= 3\,(\frac{\pi}{2})^{4/3}.
\end{aligned}$$

In view of (88), (89) and (91) this is in contradiction to the assumption that one of the three properties (85), (86) resp. (87) is violated.

q.e.d.

Proof of Lemma 5. We first remark that C denotes a generic universal constant, whose actual value may change from line to line. We have for $x \leq 1$

$$\begin{aligned}
\langle \hat{\zeta}_2 \rangle(0) &= \langle \hat{\zeta}_2 \rangle(x) - \int_0^x \partial_2 \langle \hat{\zeta}_2 \rangle \, dx \\
|\langle \hat{\zeta}_2 \rangle(0)|^2 &\leq C \left(|\langle \hat{\zeta}_2 \rangle(x)|^2 + x \int_0^1 |\partial_2 \langle \hat{\zeta}_2 \rangle|^2 \, dx \right).
\end{aligned}$$

An averaged integration over the small interval $(0,h)$ yields

$$|\langle\hat{\zeta}_2\rangle(0)|^2 \;\le\; \frac{C}{h}\int_0^h |\langle\hat{\zeta}_2\rangle|^2\,dx + C\,h\int_0^1 |\partial_2\langle\hat{\zeta}_2\rangle|^2\,dx.$$

The next step is, by $\int_\alpha^1 \frac{1}{h}\cdot dh$,

$$\ln\left(\frac{1}{\alpha}\right)|\langle\hat{\zeta}_2\rangle(0)|^2 \;\le\; C\int_\alpha^1 \frac{1}{h^2}\int_0^h |\langle\hat{\zeta}_2\rangle|^2\,dx\,dh + C\int_0^1 |\partial_2\langle\hat{\zeta}_2\rangle|^2\,dx. \tag{92}$$

Now we invoke some elements of the proof of [1, Lemma 8], quoting and varying slightly.

$$\begin{aligned}
&\int_\alpha^1 \frac{1}{h^2}\int_0^h |\langle\hat{\zeta}_2\rangle|^2\,dx\,dh\\
&\le C\int_\alpha^1 \frac{1}{h^2}\int_{\mathbb{R}} |\langle\hat{\zeta}_2\rangle(x+h)-\langle\hat{\zeta}_2\rangle(x)|^2\,dx\,dh\\
&=\;C\int_\alpha^1 \frac{1}{h^2}\int_{\mathbb{R}} |e^{i\hat{k}_2 h}-1|^2\,|\mathcal{F}\langle\hat{\zeta}_2\rangle|^2\,d\hat{k}_2\,dh\\
&\le\;C\int_{\mathbb{R}}\int_\alpha^1 \min\left\{\hat{k}_2^2,\frac{1}{h^2}\right\}\,dh\,|\mathcal{F}\langle\hat{\zeta}_2\rangle|^2\,d\hat{k}_2\\
&\le\;C\int_{\mathbb{R}} \min\left\{\frac{1}{\alpha},|\hat{k}_2|,\hat{k}_2^2\right\}\,|\mathcal{F}\langle\hat{\zeta}_2\rangle|^2\,d\hat{k}_2,
\end{aligned}\tag{93}$$

where use was made of

$$\begin{aligned}
&\int_\alpha^1 \min\left\{\hat{k}_2^2,\frac{1}{h^2}\right\}\,dh\\
&=\left\{\begin{array}{ccc}
\int_\alpha^1 \hat{k}_2^2 dh & \text{for} & |\hat{k}_2|\;\le\;1\\
\int_\alpha^{\frac{1}{|\hat{k}_2|}} \hat{k}_2^2\,dh + \int_{\frac{1}{|\hat{k}_2|}}^1 \frac{1}{h^2}\,dh & \text{for} & 1\;\le\;|\hat{k}_2|\;\le\;\frac{1}{\alpha}\\
\int_\alpha^1 \frac{1}{h^2}\,dh & \text{for} & |\hat{k}_2|\;\ge\;\frac{1}{\alpha}
\end{array}\right\}\\
&\le\;C\left\{\begin{array}{ccc}
\hat{k}_2^2 & \text{for} & |\hat{k}_2|\;\le\;1\\
|\hat{k}_2| & \text{for} & 1\;\le\;|\hat{k}_2|\;\le\;\frac{1}{\alpha}\\
\frac{1}{\alpha} & \text{for} & |\hat{k}_2|\;\ge\;\frac{1}{\alpha}
\end{array}\right\}\\
&=\;C\;\min\left\{\frac{1}{\alpha},|\hat{k}_2|,\hat{k}_2^2\right\}.
\end{aligned}$$

On the other hand, when considering the lower bound on the magnetostatic contribution, as given in Lemma 1 and after rescaling, it is true that

$$\begin{aligned}
\hat{\mathcal{R}}(k_1,\zeta) &\geq \delta \int_{\mathbb{R}} \frac{\hat{k}_2^2}{2|\hat{k}'|+\alpha|\hat{k}'|^2} |\mathcal{F}\langle\hat{\zeta}_2\rangle|^2 \, d\hat{k}_2 \\
&\geq C\,\delta \int_{\mathbb{R}} \min\left\{\frac{1}{\alpha}, |\hat{k}_2|, \frac{\hat{k}_2^2}{|\hat{k}_1|}, \frac{\hat{k}_2^2}{\alpha \hat{k}_1^2}\right\} |\mathcal{F}\langle\hat{\zeta}_2\rangle|^2 \, d\hat{k}_2. \qquad (94)
\end{aligned}$$

$\mathcal{R}(k_1,\zeta)$ is considered to be bounded, thus $\hat{k}_1$ is bounded by Lemma 4 and

$$\begin{aligned}
&\int_\alpha^1 \frac{1}{h^2} \int_0^h |\langle\hat{\zeta}_2\rangle|^2 \, dx \, dh \\
&\overset{(93)}{\leq} C \int_{\mathbb{R}} \min\left\{\frac{1}{\alpha}, |\hat{k}_2|, \hat{k}_2^2\right\} |\mathcal{F}\langle\hat{\zeta}_2\rangle|^2 \, d\hat{k}_2 \\
&\leq C \int_{\mathbb{R}} \min\left\{\frac{1}{\alpha}, |\hat{k}_2|, \frac{\hat{k}_2^2}{|\hat{k}_1|}, \frac{\hat{k}_2^2}{\alpha \hat{k}_1^2}\right\} |\mathcal{F}\langle\hat{\zeta}_2\rangle|^2 \, d\hat{k}_2 \\
&\overset{(94)}{\leq} \frac{C}{\delta} \left(\delta \int_{\mathbb{R}} \frac{\hat{k}_2^2}{2|\hat{k}'|+\alpha|\hat{k}'|^2} |\mathcal{F}\langle\hat{\zeta}_2\rangle|^2 \, d\hat{k}_2\right). \qquad (95)
\end{aligned}$$

By combining (92) and (95), then

$$\begin{aligned}
\ln\left(\frac{1}{\alpha}\right)|\langle\hat{\zeta}_2\rangle(0)|^2 &\leq \frac{C}{\delta}\left(\delta \int_{\mathbb{R}} \frac{\hat{k}_2^2}{2|\hat{k}'|+\alpha|\hat{k}'|^2} |\mathcal{F}\langle\hat{\zeta}_2\rangle|^2 \, d\hat{k}_2\right) + \int_0^1 |\partial_2\langle\hat{\zeta}_2\rangle|^2 \, dx \\
&\leq C\left(\frac{1}{\delta}+1\right)\hat{\mathcal{R}}(k_1,\zeta),
\end{aligned}$$

and finally

$$\begin{aligned}
|\langle\hat{\zeta}_2\rangle(0)|^2 &\leq C \, \ln^{-1}\left(\frac{1}{\alpha}\right)\left(\frac{1}{\delta}+1\right)\hat{\mathcal{R}}(k_1,\zeta) \\
&= C\left(\varepsilon + \ln^{-1}\left(\frac{1}{\alpha}\right)\right)\hat{\mathcal{R}}(k_1,\zeta).
\end{aligned}$$

This holds analogously for $|\langle\hat{\zeta}_2\rangle(1)|^2$.

q.e.d.

Proof of Corollary 3. We first address part i). For ζ with $\zeta = \zeta(x_2)$ and $\zeta_3 \equiv 0$ we have according to Lemma 1 i)

$$\begin{aligned}
&\mathcal{R}(k_1, \zeta) \\
&= d^2\, t \int_0^\ell (k_1^2\, |\zeta_2|^2 + |\partial_2 \zeta_2|^2)\, dx_2 + t \int_{\mathbb{R}} \frac{\exp(-t|k'|) - 1 + t|k'|}{(t|k'|)^3}\, (tk_2)^2\, |\mathcal{F}\zeta_2|^2\, dk_2.
\end{aligned}$$

Using the Fourier multiplier inequality

$$\frac{\exp(-t|k'|) - 1 + t|k'|}{(t|k'|)^3} \leq \frac{1}{2\, t|k'|} \leq \frac{1}{2\, t|k_2|},$$

the above turns into

$$\mathcal{R}(k_1, \zeta) \leq d^2\, t \int_0^\ell \left(k_1^2\, |\zeta_2|^2 + |\partial_2 \zeta_2|^2\right)\, dx_2 + \frac{t^2}{2} \int_{\mathbb{R}} |k_2|\, |\mathcal{F}\zeta_2|^2\, dk_2.$$

Under the rescaling of Definition 2, this turns into (34).

We now address part ii). We have according to Lemma 1 ii)

$$\mathcal{R}(k_1, \zeta) \geq d^2 \int_{\tilde{\Omega}} (k_1^2\, |\zeta|^2 + |\tilde{\nabla} \zeta|^2)\, d\tilde{x} + t \int_{\mathbb{R}} \frac{1}{2\, t|k'| + (t|k'|)^2}\, (tk_2)^2\, |\mathcal{F}\langle \zeta_2 \rangle|^2\, dk_2.$$

With help of Jensen's inequality

$$\int_{\tilde{\Omega}} |\zeta|^2\, d\tilde{x} \geq t \int_0^\ell |\langle \zeta \rangle|^2\, dx_2 \geq t \int_0^\ell |\langle \zeta_2 \rangle|^2\, dx_2$$

the above turns into

$$\mathcal{R}(k_1, \zeta) \geq d^2\, t \int_0^\ell \left(k_1^2\, |\langle \zeta_2 \rangle|^2 + |\partial_2 \langle \zeta_2 \rangle|^2\right)\, dx_2,$$

which is the unrescaled version of (35).

q.e.d.

Proof of Corollary 4. The second estimate in Lemma 4 implies that

$$\{\hat{k}_{1,\nu}\}_{\nu\uparrow\infty} \quad \text{is bounded} \tag{96}$$

and thus relatively compact in $\mathbb{R}$. The first estimate in Lemma 4 implies that any $L^2(\widehat{\tilde{\Omega}})$–limit $\hat{\zeta}$ of $\{\hat{\zeta}_\nu\}_{\nu\uparrow\infty}$ satisfies (41).

It remains to argue that $\{\langle\hat{\zeta}_{2,\nu}\rangle\}_{\nu\uparrow\infty}$ is relatively compact in $L^2((0,1))$. Actually, for bounded

$$\hat{\mathcal{R}}(k_1,\zeta) \;\geq\; \int_0^1 \left(\hat{k}_1^2\,|\langle\hat{\zeta}_2\rangle|^2 + |\partial_2\langle\hat{\zeta}_2\rangle|^2\right)\,d\hat{x}_2,$$

both sequence and limit function are bounded in $H^1((0,1))$, and thus in $C^{0,\frac{1}{2}}([0,1])$, too. The Arzelà-Ascoli theorem asserts that there exists a $\langle\hat{\zeta}_2\rangle \in H^1((0,1)) \cap C^{0,\frac{1}{2}}([0,1])$ s. t. for a subsequence, which again we do not distinguish in notation,

$$\langle\hat{\zeta}_{2,\nu}\rangle(x_2) \;\to\; \hat{\zeta}_2(x_2) \quad \text{uniformly in } x_2 \in [0,1]. \tag{97}$$

With the zero boundary values from Lemma 5 we thus have

$$\hat{\zeta}_2 \in H^1_0((0,1)).$$

q.e.d.

Proof of Proposition 2. The proof of Γ-convergence consists of two parts:

- Construction. For any $(\hat{k}_1,\hat{\zeta}_2) \in \mathbb{R}\times H^1_0((0,1))$ with $\int_0^1 |\hat{\zeta}_2|^2\,d\hat{x}_2 = 1$ there exists a sequence $(k_{1,\nu},\zeta_\nu) \in \mathbb{R}\times L^2(\tilde{\Omega})$ with $\int_{\tilde{\Omega}} |\zeta|^2\,d^2\tilde{x} = 1$ which converges in the sense of (43) such that

$$\limsup_{\nu\uparrow\infty} \hat{\mathcal{R}}(k_{1,\nu},\zeta_\nu) \;\leq\; \int_0^1 \left(\hat{k}_1^2\,|\hat{\zeta}_2|^2 + |\partial_2\hat{\zeta}_2|^2\right)\,d\hat{x}_2. \tag{98}$$

- Lower semicontinuity. For any sequence $(k_{1,\nu},\zeta_\nu)\in\mathbb{R}\times L^2(\tilde\Omega)$ with bounded $\hat{\mathcal{R}}(k_{1,\nu},\zeta_\nu)$ which converges to a $(\hat k_1,\hat\zeta_2)\in\mathbb{R}\times L^2((0,1))$ in the sense of (43) one has

$$\hat\zeta_2 \;\in\; H_0^1((0,1))$$

and

$$\liminf_{\nu\uparrow\infty}\hat{\mathcal{R}}(k_{1,\nu},\zeta_\nu)\;\ge\;\int_0^1\left(\hat k_1^2\,|\hat\zeta_2|^2+|\partial_2\hat\zeta_2|^2\right)\,d\hat x_2 \tag{99}$$

For the construction part, we define $(k_{1,\nu},\zeta_\nu)\in\mathbb{R}\times L^2(\tilde\Omega)$ according to Definition 2:

$$\ell\,k_{1,\nu}\;=\;\hat k_1\quad\text{and}\quad(t\ell)^{1/2}\zeta_\nu(\ell\hat x_2,t\hat x_3)\;=\;\hat\zeta_2(\hat x_2).$$

Then (43) is trivially fulfilled. Furthermore, we have by part i) of Corollary 3

$$\hat{\mathcal{R}}(k_{1,\nu},\zeta_\nu)\;\le\;\int_0^1\left(\hat k_1^2\,|\hat\zeta_2|^2+|\partial_2\hat\zeta_2|^2\right)\,d\hat x_2+\frac{\delta_\nu}{2}\int_0^1|\hat k_2||\mathcal{F}\hat\zeta_2|^2\,d\hat k_2.$$

This inequality establishes (98).

We now address the lower semicontinuity part. The notion (43) implies in particular

$$\langle\hat\zeta_{2,\nu}\rangle\;\to\;\hat\zeta_2\quad\text{in }L^2((0,1)). \tag{100}$$

Appealing to part ii) in Corollary 3:

$$\hat{\mathcal{R}}(k_{1,\nu},\zeta_\nu)\;\ge\;\int_0^1\left(\hat k_{1,\nu}^2\,|\langle\hat\zeta_{2,\nu}\rangle|^2+|\partial_2\langle\hat\zeta_{2,\nu}\rangle|^2\right)\,dx_2,$$

we infer from $k_{1,\nu}\to k_1$, (100) and the lower semicontinuity of the H^1-norm:

$$\liminf_{\nu\uparrow\infty}\hat{\mathcal{R}}(k_{1,\nu},\zeta_\nu)\;\ge\;\int_0^1\left(\hat k_1^2\,|\langle\hat\zeta_2\rangle|^2+|\partial_2\langle\hat\zeta_2\rangle|^2\right)\,dx_2.$$

q.e.d.

Proof of Lemma 6. The statement reduces to two observations

- For any $\hat\zeta_2 \in H^1_0((0,1))$ we have

$$\int_0^1 |\partial_2\hat\zeta_2|^2\, d\hat x_2 \;\geq\; \pi^2 \int_0^1 |\hat\zeta_2|^2\, d\hat x_2$$

  with equality if and only if $\hat\zeta_2$ is of the form (45) for some $c \in \mathbb{C}$.

- It holds

$$\hat k_1^2 + \pi^2 \;\geq\; \pi^2$$

  with equality if and only if $\hat k_1 \;=\; 0$.

q.e.d.

Proof of Theorem 2. Our argument is indirect. Assume the statement were not true. Then there exists a sequence numbers (d_ν, t_ν, ℓ_ν) such that (39) holds, and a corresponding sequence $(k^*_{1,\nu}, \zeta^*_\nu)$ of minimizers such that one of the following three conditions is violated

$$|k^*_{1,\nu}| \;\to\; 0, \tag{101}$$

$$\int_{\hat{\tilde\Omega}} |\hat\zeta_\nu(\hat{\tilde x}) - \sqrt{2}\, c\, \sin(\pi\hat x_2)|^2\, d\hat{\tilde x} \;\to\; 0 \quad \text{for some } |c| = 1, \tag{102}$$

$$\hat{\mathcal{R}}(k^*_{1,\nu}, \zeta^*_\nu) \;\to\; \pi^2. \tag{103}$$

Lemma 6 implies in particular that the minimum of the Γ–limit (42) is finite. According to the construction part of Proposition 2 it follows that $\limsup_{\nu\uparrow\infty} \hat{\mathcal{R}}(k^*_{1,\nu}, \zeta^*_\nu)$ is finite. According to Corollary 4, there thus exists $(\hat k^*_1, \hat\zeta^*_2) \in \mathbb{R} \times L^2((1,0))$ such that for a subsequence (which we do not distinguish in notation)

$$\begin{aligned} \hat{k}^*_{1,\nu} &\to \hat{k}^*_1, \qquad (104) \\ \int_{\hat{\tilde{\Omega}}} |\hat{\zeta}^*_\nu(\widehat{\tilde{x}}) - \hat{\zeta}^*_2(\hat{x}_2)|^2 \, d\widehat{\tilde{x}} &\to 0, \qquad (105) \end{aligned}$$

which is just the notion of convergence (43) in the Γ–convergence.

The Γ–convergence of Proposition 2 implies that, cf. [2],

$$(\hat{k}^*_1, \hat{\zeta}^*_2) \in \mathbb{R} \times H^1_0((0,1)) \quad \text{is a minimizer of (42)} \qquad (106)$$

and that

$$\hat{\mathcal{R}}(k^*_{1,\nu}, \zeta^*_\nu) \to \hat{k}^{*\,2}_1 \int_0^1 |\hat{\zeta}^*_2|^2 \, d\hat{x}_2 + \int_0^1 |\partial_2 \hat{\zeta}^*_2|^2 \, d\hat{x}_2. \qquad (107)$$

Finally, we appeal to Lemma 6: (106) implies that

$$\begin{aligned} \hat{k}^*_1 &= 0 \\ \hat{\zeta}_2(\hat{x}_2) &= \sqrt{2}\, c \, \sin(\pi \, \hat{x}_2) \quad \text{for some } |c| = 1, \\ |\hat{k}^*_1|^2 \int_0^1 |\hat{\zeta}_2|^2 \, d\hat{x}_2 + \int_0^1 |\partial_2 \hat{\zeta}_2|^2 \, d\hat{x}_2 &= \pi^2. \end{aligned}$$

In view of (104), (105) and (107) this is in contradiction to the assumption that one of the three properties (101), (102) resp. (103) is violated.

q.e.d.

9 Acknowledgements

RCA acknowledges support by the German Science Foundation through SFB 611.

FO acknowledges partial support by the SFB 611.

References

[1] Cantero-Álvarez, R. and Otto, F.: Critical Fields in Ferromagnetic Thin Films: Identification of Four Regimes, *SFB Preprint* **128** (2004), see http://www-mathphys.iam.uni-bonn.de/∼cantero/publications/CAO04.ps

[2] Dal Maso,G.: *An introduction to* Γ*−convergence*, Birkhäuser Boston, 1993.

[3] K. J. Harte, Theory of Magnetization Ripple in Ferromagnetic Films, *J. Appl. Phys.* **39**, 3 (1968), 1503–1523.

[4] H. Hoffmann, Theory of Magnetization Ripple, *IEEE Trans. on Magnetics* **4**, 1 (1968), 32–38.

[5] Hubert, A. and Schäfer, R.: *Magnetic Domains*. Springer-Verlag Berlin, 2000