COARSENING RATES IN OFF-CRITICAL MIXTURES

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Abstract. We study coarsening of a binary mixture within the Mullins-Sekerka evolution in the regime where one phase has small volume fraction $\phi \ll 1$. Heuristic arguments suggest that the energy density, which represents the inverse of a typical length scale, decreases as $\phi t^{-1/3}$ as $t \to \infty$. We prove rigorously a corresponding weak lower bound. Moreover, we establish a stronger result for the two-dimensional case, where we find a lower bound of the form $\phi(\ln \phi^{-1})^{1/3} t^{-1/3}$. Our approach follows closely the analysis in [6], which exploits the relation between two suitable length scales. Our main contribution is an isoperimetric inequality in the two-dimensional case.

Key words. Mullins-Sekerka evolution, coarsening rates, isoperimetric inequalities

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1. Introduction. The Mullins-Sekerka model describes the late stage coarsening in the phase separation of a binary mixture. In this model the interface between two phases is characterized by the boundary $\partial\Omega$ of a region $\Omega \subset Q \subset I\!R^n$, where Ω denotes the region covered by one phase. The Mullins-Sekerka model is based on the assumption that the diffusion field, given by the negative gradient $-\nabla u$ of the chemical potential u, is in quasistationary equilibrium given the phase distribution and satisfies the Gibbs-Thomson condition of local equilibrium on the interface. That is u satisfies

$$-\Delta u = 0 \qquad \text{in } Q\backslash\partial\Omega, \tag{1.1}$$

$$u = \kappa \qquad \text{on } \partial\Omega, , \tag{1.2}$$

where κ denotes the mean curvature of the interface and is defined to be positive if Ω is a ball. Then the normal velocity v of the interface is given by the jump $[\nabla u \cdot \vec{n}]$ of the diffusion field across the interface, that is

$$v = [\nabla u \cdot \vec{n}] \qquad \text{on } \partial\Omega. \tag{1.3}$$

With periodic or Neumann boundary conditions the evolution 1.1-1.3 reduces the total surface area and keeps the volume fraction covered by each phase constant. In the following we consider a periodic setting with period cell $Q = [0, l)^n$, where l is the system size. This is not a significant restriction, since our results will be independent of the size l.

Equations 1.1-1.3 do in general not possess a global smooth solutions since the geometry might become singular, for example when two previously separated regions touch or when pinching occurs. We assume in the following that we are in a situation where a piecewise smooth solution of the Mullins-Sekerka problem exists and for which the evolution of $\partial\Omega$ is continuous for all times. For typical scenarios in the regime of small volume fraction this assumption is satisfied. In this regime the new phase nucleates in form of many small well separated regions which quickly become approximately radially symmetric. If such a configuration is taken as initial data for

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the Mullins-Sekerka evolution, then there exists a global solution in the described sense [1]. The different regions compete for surface area such that the smaller ones shrink and disappear while the larger ones grow.

We are interested in the coarsening rate within the Mullins-Sekerka evolution in the regime of small volume fraction, i.e. $\phi := |\Omega \cap Q|/|Q| \ll 1$. Here and in the following we denote by $|\omega| := \mathcal{L}^n(\omega)$ the n-dimensional Lebesgue measure of a set $\omega \subset I\!R^n$. The coarsening rate is described by the growth rate of a typical length scale $L = L(t)$. Since the Mullins-Sekerka evolution is scale invariant with respect to a rescaling $x \mapsto \lambda x, t \mapsto \lambda^3 t$, the only possible universal growth law is $L(t) \sim t^{1/3}$. This growth law has been rigorously established in form of a weak upper bound in [6] for the Cahn-Hilliard equation in the case of equal volume fractions.

How do we expect that this growth law depends on the volume fraction ϕ? To that aim consider as a natural inverse length scale the energy density

$$E = \text{ interfacial area per unit volume } = \frac{\mathcal{H}^{n-1}(\partial\Omega \cap Q)}{|Q|}. \tag{1.4}$$

If now, for example, the configuration consists of N approximately spherical domains of mean radius $\overline{R}(t)$, we expect

$$E(t) \sim \frac{1}{|Q|} N \overline{R}^{N-1}(t) \sim \phi \frac{1}{\overline{R}(t)} \tag{1.5}$$

A prediction of the scaling of $\overline{R}(t)$ goes back to the classical theory by Lifshitz, Slyozov and Wagner [8, 10] (see also [9] for more details on the two-dimensional case). To that aim consider again a collection of approximately spherical regions with radius R, total volume fraction ϕ, which are well separated by a typical distance of order $d \sim \frac{R}{\phi^{1/n}}$. Due to the clear separation of length scales, the potential u should approximately be close to a slowly varying field $\bar{u}$ away from the particles. Hence, due to 1.1 and 1.2 we expect that close to a particle with center x_0 the potential is well approximated by

$$u \sim \begin{cases} (1 - R\bar{u}) \frac{R^{n-3}}{|x-x_0|^{n-2}} & : \quad n \geq 3 \\ \left(\frac{1}{R} - \bar{u}\right) \frac{\ln \frac{d}{|x-x_0|}}{\ln\left(\frac{d}{R}\right)} & : \quad n = 2 \end{cases} .$$

Plugging the preceding formula into 1.3 gives

$$\dot{R}(t) \sim \begin{cases} \frac{C(n)}{R^2(t)}(R\bar{u} - 1) & : \quad n \geq 3 \\ \frac{1}{\ln\left(\frac{d}{R}\right)} \frac{1}{R^2(t)}(R\bar{u} - 1), & : \quad n = 2 \end{cases} . \tag{1.6}$$

Thus, the growth rate of a particle is to leading order independent of ϕ for $n \geq 3$ and depends only on $\ln\left(\frac{d}{R}\right) \sim \ln \phi^{-1}$ for $n = 2$. In particular we also find $R^2 \dot{R} \sim 1$ and $R^2 \dot{R} \sim \frac{1}{\ln\left(\frac{d}{R}\right)}$ respectively, such that the mean radius should follow the growth law

$$\overline{R}(t) \sim \begin{cases} t^{1/3} & : \quad n \geq 3 \\ \frac{1}{(\ln \phi^{-1})^{1/3}} t^{1/3} & : \quad n = 2 \end{cases} . \tag{1.7}$$

This implies together with 1.5 that

$$E(t) \sim \begin{cases} \phi t^{-1/3} & : \quad n \geq 3 \\ \phi (\ln \phi^{-1})^{1/3} t^{-1/3} & : \quad n = 2. \end{cases} . \tag{1.8}$$

Our goal in this article will be to support these heuristics by rigorously establishing a corresponding weak lower bound in the spirit of [6]. This approach has also been successfully applied to other cases, such as phase separation in multicomponent systems or epitaxial growth [7], mean-field models for coarsening [2] or temperature dependent phase field models [3]. The approach is based on exploiting the relation of two suitably chosen global length scales. The first has already been introduced and is given by the inverse of the energy density E. In the following we use the notation from geometric measure theory and denote the perimeter of Ω with respect to Q by $\int_Q |\nabla\chi|$, where χ denotes here and in the following the characteristic function of the set Ω. It is well known (cf. e.g. [5]) that if $\partial\Omega$ is smooth, which we assume here for all but finitely many times, then the perimeter equals the surface area. Thus, the energy can also be written as $E = \fint_Q |\nabla\chi|$, where here and in the following $\fint_Q := \frac{1}{|Q|}\int_Q$.

The second length scale is, as in [6], a suitable negative norm of the characteristic function of Ω. We define

$$L := \Big(\fint_Q |\nabla^{-1}(\chi-\phi)|^2\,dx\Big)^{1/2}, \tag{1.9}$$

where $\|\nabla^{-1}u\|_{L^2} = \|u\|_{H^{-1}}$ denotes the H^{-1} norm for Q-periodic functions with mean value zero.

This choice of length scale is motivated by the interpretation of the Mullins-Sekerka evolution as a gradient flow on a Riemannian manifold. In fact, it is the gradient flow of the surface energy with respect to the scalar product given by the H^{-1}-norm in the bulk. In principle a natural choice for L would be the induced distance on the manifold between the phase configuration and the uniform reference state ϕ. However, we cannot easily compute this distance in the present setting due to the nonconvex constraint that χ takes only two integer values. Nevertheless one would expect that this distance is close to the distance within the Cahn-Hilliard model which is just given by 1.9. In fact, the main result formulated in the following theorem, can with similar arguments also be established within the Cahn-Hilliard framework.

Theorem 1.1. *Let E_0, L_0 be the initial energy density and initial length respectively and let $\phi \ll 1$. Then we have for $T \gg \phi^{-3/2}L_0^3$ that*

$$\fint_0^T E^2(t)\,dt \gtrsim \begin{cases} \phi^2\,T^{-2/3} & : \quad n \geq 3 \\ \phi^2\big(\ln\phi^{-1}\big)^{2/3}\,T^{-2/3} & : \quad n = 2 \end{cases}$$

Here and in the following we use the notation $A \lesssim B$ or $A \gtrsim B$ if there is a constant C such that $A \leq CB$ or $A \geq CB$ respectively. The constants will be independent of ϕ but may depend on the space dimension n. Such a dependence we sometimes indicate with $C = C(n)$.

The proof of Theorem 1.1 is based exactly as in [6] on three ingredients. The first is an interpolation (or isoperimetric) inequality which relates E and L and is independent of the dynamics of the evolution.

Lemma 1.2. *We have*

$$EL \gtrsim \begin{cases} \phi^{3/2} & : \quad n \geq 3 \\ \phi^{3/2}\big(\ln\phi^{-1}\big)^{1/2} & : \quad n = 2 \end{cases}$$

The proof of Lemma 1 – in particular for the case $n = 2$ – is the main contribution of this article. We provide it in the next section.

The second ingredient in the analysis of [6] is a diffusion inequality which relates the rate of change of the length L to the rate of change of the energy density.

LEMMA 1.3. *For almost all $t \in (0,T)$ we have*

$$|\dot L(t)|^2 \le -\dot E(t).$$

Proof. We denote by u the potential of the normal velocity v of $\partial\Omega$ and by w the potential for $\chi - \phi$, that is

$$-\Delta u = v_{|\partial\Omega} \qquad \text{and} \qquad -\Delta w = \chi - \phi \qquad \text{in } \mathcal{D}'(Q),$$

with periodic boundary conditions.

We assume that for all but a finite number of times the solution of the Mullins-Sekerka equation is smooth and for those times we have the relation $\partial_t w = u$. By definition $L^2(t) = ⨍_Q |\nabla w|^2\, dx$ and we find

$$\begin{aligned} L\dot L(t) &= ⨍_Q \nabla w \cdot \nabla u\, dx \\ &\le \Big(⨍_Q |\nabla u|^2\, dx\Big)^{1/2} \Big(⨍_Q |\nabla w|^2\, dx\Big)^{1/2} \\ &= \Big(⨍_Q |\nabla u|^2\, dx\Big)^{1/2} L. \end{aligned}$$

Hence, by an integration by parts and using 1.1-1.3, it follows

$$|\dot L(t)|^2 \le ⨍_Q |\nabla u|^2\, dx = \frac{1}{|Q|}\int_{\partial\Omega\cap Q} v\kappa\, d\mathcal{H}^{n-1} = -\dot E\,.$$

The last equality follows from the well-known fact that the mean curvature is the variation of the surface area with respect to kinematically admissible normal velocities of the surface (cf. e.g. [5], Ch. 10). □ The final ingredient to complete the proof of Theorem 1.1 is an ODE lemma which we can take directly from [6].

To that aim we consider the rescaled quantities

$$\begin{aligned} \hat E &= \begin{cases} \phi^{(1-n)/n} E & : \ n \ge 3 \\ \phi^{-1/2}\big(\ln\phi^{-1}\big)^{-1/2} E & : \ n = 2 \end{cases}, \\ \hat L &= \phi^{-(n+2)/2n} L, \\ \hat t &= \begin{cases} \phi^{-3/n} t & : \ n \ge 3 \\ \phi^{-3/2}\big(\ln\phi^{-1}\big)^{1/2} t & : \ n = 2 \end{cases}, \end{aligned} \tag{1.10}$$

and obtain

$$\hat E \hat L \gtrsim 1 \qquad \text{and} \qquad |\dot{\hat L}|^2 \le -\dot{\hat E}. \tag{1.11}$$

In order to arrive at 1.11 one has some freedom in the choice of factors in 1.10. The present choice is just the natural one, in the sense that for typical configurations the new quantities are of order one in ϕ.

The ODE Lemma (Lemma 3 in [6]) now yields for $\hat T \gg (\hat L_0)^3$

$$⨍_0^{\hat T} (\hat E(\hat t))^2\, d\hat t \gtrsim \hat T^{-2/3},$$

which in view of 1.10 implies the estimate in Theorem 1.1.

2. Proof of the interpolation inequality. Let χ denote the characteristic function of the set Ω. We claim that for small ϕ the following interpolation inequality is true.

$$\fint_Q (\chi-\phi)^2\,dx \lesssim \Big(\fint_Q |\nabla\chi|\,dx\Big)^{2/3}\Big(\fint_Q |\nabla^{-1}(\chi-\phi)|^2\,dx\Big)^{1/3} \tag{2.1}$$

To see that define $(\chi-\phi)_\varepsilon := (\chi-\phi)*\eta_\varepsilon$ and $\chi_\varepsilon := \chi*\eta_\varepsilon$, where η_ε is a standard sequence of mollifiers. We have

$$\fint_Q |\nabla\chi_\varepsilon|^2\,dx \lesssim \frac{1}{\varepsilon^2}\fint_Q |\chi-\phi|^2\,dx\,,$$

such that by duality

$$\fint_Q |(\chi-\phi)_\varepsilon|^2\,dx \lesssim \frac{1}{\varepsilon^2}\fint_Q |\nabla^{-1}(\chi-\phi)|^2\,dx \tag{2.2}$$

Hence, using $|\chi|, |\chi_\varepsilon| \le 1$, we find

$$\begin{aligned}\fint_Q (\chi-\phi)^2\,dx &\le \fint_Q |\chi-\chi_\varepsilon|^2 + \fint_Q |(\chi-\phi)_\varepsilon|^2\,dx\\ &\overset{2.2}{\lesssim} \fint_Q |\chi-\chi_\varepsilon| + \frac{1}{\varepsilon^2}\fint_Q |\nabla^{-1}(\chi-\phi)|^2\,dx\\ &\lesssim \varepsilon\fint_Q |\nabla\chi| + \frac{1}{\varepsilon^2}\fint_Q |\nabla^{-1}(\chi-\phi)|^2\,dx.\end{aligned}$$

The preceding inequality is optimal for

$$\varepsilon = \Big(\frac{\fint_Q |\nabla^{-1}(\chi-\phi)|^2\,dx}{\fint_Q |\nabla\chi|}\Big)^{1/3},$$

which proves 2.1. Since

$$\fint_Q (\chi-\phi)^2\,dx \ge \frac12\fint_Q \chi^2\,dx = \frac12\fint_Q \chi\,dx = \frac12\phi,$$

inequality 2.1 implies Lemma 1.2 for the case $n \ge 3$.

In case $n = 2$ we can even do better. We claim that

$$\fint_Q \chi^2\,dx \lesssim \frac{1}{\big(\ln\phi^{-1}\big)^{1/3}}\Big(\fint_Q |\nabla\chi|\,dx\Big)^{2/3}\Big(\fint_Q |\nabla^{-1}(\chi-\phi)|^2\,dx\Big)^{1/3} \quad \text{if } n=2. \tag{2.3}$$

We recall that by the definition of the dual norm we have for all $\zeta \in H^1_{per}(Q)$, that is the space of H^1 functions with periodic boundary conditions, that

$$\fint_Q |\nabla^{-1}(\chi-\phi)|^2\,dx \ge \frac{\Big(\fint_Q (\chi-\phi)\zeta\,dx\Big)^2}{\fint_Q |\nabla\zeta|^2\,dx}. \tag{2.4}$$

We are going to construct a periodic test function $\zeta \geq 0$ which satisfies the following properties:

$$\fint_Q \chi\zeta \, dx \geq \frac{1}{2} \fint_Q \chi \, dx, \tag{2.5}$$

$$\fint_Q \zeta \, dx \ll 1 \,, \tag{2.6}$$

$$\fint_Q |\nabla\zeta|^2 \, dx \lesssim \frac{\phi}{\mathcal{R}^2 \ln \phi^{-1}} \tag{2.7}$$

where

$$\mathcal{R} := \frac{\phi}{\fint_Q |\nabla\chi|}. \tag{2.8}$$

Then 2.5 and 2.6 imply

$$\fint_Q (\chi - \phi)\zeta \, dx \geq \frac{1}{4} \fint_Q \chi \, dx = \frac{1}{4} \fint_Q \chi^2 \, dx,$$

and using 2.4 and 2.7 we find

$$\Big(\fint_Q \chi^2 \, dx\Big)^2 \lesssim \Big(\fint_Q (\chi - \phi)\zeta \, dx\Big)^2 \leq \frac{\Big(\fint_Q |\nabla\chi|\Big)^2}{\phi \ln \phi^{-1}} \fint_Q |\nabla^{-1}(\chi - \phi)|^2 \, dx,$$

which, recalling $\fint_Q \chi^2 = \phi$, proves 2.3.

Hence it remains to construct a test function ζ with the properties 2.5-2.7. Fundamental for the construction is the following geometric lemma which says that for given Ω we can construct another set $\Omega_{\mathcal{R}}$ which covers a substantial part of Ω and behaves like a union of balls with radius larger than $\mathcal{R}$.

LEMMA 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a given Q-periodic set, χ its characteristic function and let $\mathcal{R}$ be such that*

$$\mathcal{R} \int_Q |\nabla\chi| \leq |\Omega \cap Q|.$$

Then there exists a Q-periodic set $\Omega_{\mathcal{R}} \subset \mathbb{R}^n$ such that

$$|\Omega \cap \Omega_{\mathcal{R}} \cap Q| \geq \frac{1}{2} |\Omega \cap Q|$$

and, for all $r > 0$,

$$|\Omega_{\mathcal{R}}^r \cap Q| \leq C |\Omega \cap Q| \Big(1 + \big(\frac{r}{\mathcal{R}}\big)^n\Big),$$

where $\Omega_{\mathcal{R}}^r = \{x \,|\, dist(x, \Omega_{\mathcal{R}}) \leq r\}$ and C is a constant which only depends on n. We postpone the proof of the lemma, choose $\mathcal{R}$ as in 2.8 and define

$$\zeta(x) := \psi(r(x)),$$

where $r(x) := \mathrm{dist}(x, \Omega_{\mathcal{R}})$ and

$$\psi(r) = \begin{cases} 0 & : \; r \geq d \\ \frac{\ln \frac{d}{r}}{\ln \frac{d}{\mathcal{R}}} & : \; r \in (\mathcal{R}, d) \\ 1 & : \; r \leq \mathcal{R} \end{cases} \,,$$

with $d = \frac{\mathcal{R}}{\phi^{1/2}} \gg \mathcal{R}$.

The fact that $|\nabla r(x)| = 1$ for almost all x and the coarea formula (Chs. 3.4.2 and 3.4.4 of [4]) imply

$$\begin{aligned}
\int_Q |\nabla \zeta|^2\, dx &= \int_Q |\psi'(r(x))|^2 |\nabla r(x)|^2\, dx \\
&= \int_0^\infty |\psi'(s)|^2 \mathcal{H}^{n-1}(\{x \in Q \,|\, r(x) = s\})\, ds \\
&= \int_{\mathcal{R}}^d |\psi'(s)|^2 \mathcal{H}^{n-1}(\{x \in Q \,|\, r(x) = s\})\, ds \\
&\le -\int_{\mathcal{R}}^d \frac{d}{ds} |\psi'(s)|^2 |\{x \in Q \,|\, r(x) < s\}|\, ds \\
&\qquad + |\psi'(d)|^2 |\{x \in Q \,|\, r(x) < d\}|\,.
\end{aligned}$$

Lemma 2.1 gives $|\{x \in Q \,|\, r(x) < s\}| \le C|\Omega \cap Q| \big(\frac{s}{\mathcal{R}}\big)^2$ and a straightforward computation yields

$$\begin{aligned}
\fint_Q |\nabla \zeta|^2\, dx &\lesssim \int_{\mathcal{R}}^d \frac{\phi}{s^3 |\ln \frac{d}{\mathcal{R}}|^2} \big(\frac{s}{\mathcal{R}}\big)^2\, ds \,+\, \frac{\phi}{|d \ln \frac{d}{\mathcal{R}}|^2} \big(\frac{d}{\mathcal{R}}\big)^2 \\
&\lesssim \frac{\phi}{\mathcal{R}^2 |\ln \frac{d}{\mathcal{R}}|},
\end{aligned}$$

which establishes 2.7.

Furthermore, an analogous computation yields

$$\begin{aligned}
\int_Q \zeta\, dx &= \int_{\mathcal{R}}^d \psi(s) \mathcal{H}^{n-1}(\{x \in Q \,|\, r(x) = s\})\, ds, \\
&= -\int_{\mathcal{R}}^d \psi'(s) |\{x \in Q \,|\, r(x) < s\}|\, ds + \psi(s) |\{x \in Q \,|\, r(x) < s\}| \Big|_{\mathcal{R}}^d . \qquad (2.9)
\end{aligned}$$

The second term on the right hand side is negative and Lemma 2.1 and the choice of d imply

$$\fint_Q \zeta\, dx \lesssim \frac{1}{\ln \frac{d}{\mathcal{R}}} \int_{\mathcal{R}}^d \frac{\phi s}{\mathcal{R}^2}\, ds \lesssim \frac{d^2}{\mathcal{R}^2 \ln \frac{d}{\mathcal{R}}} \phi = \frac{1}{\ln \phi^{-1/2}} \ll 1\,,$$

so that the desired property 2.6 follows.

Proof of Lemma 2.1: It is convenient to redefine $\mathcal{R}$ such that

$$\mathcal{R} \int_Q |\nabla \chi| \le \frac{1}{4} |\Omega \cap Q|. \qquad (2.10)$$

We notice that the isoperimetric inequality in particular implies that

$$\mathcal{R} \le \frac{1}{4} |\Omega \cap Q|^{\frac{1}{n}} \le \frac{1}{4} l. \qquad (2.11)$$

We set

$$\Omega_{\mathcal{R}} := \{x \,|\, |\Omega \cap B_{\mathcal{R}}(x)| > \frac{1}{2} |B_{\mathcal{R}}(x)|\} \qquad (2.12)$$

and claim that

$$|\Omega \cap \Omega_{\mathcal{R}} \cap Q| \geq \frac{1}{2}|\Omega \cap Q|. \tag{2.13}$$

We first notice that

$$\Omega_{\mathcal{R}} = \{x \,|\, \chi_{\mathcal{R}}(x) > \frac{1}{2}\}, \qquad \text{where} \qquad \chi_{\mathcal{R}}(x) := \frac{|\Omega \cap B_{\mathcal{R}}(x)|}{|B_{\mathcal{R}}(x)|},$$

and $\chi_{\mathcal{R}}$ can be considered as a convolution of the characteristic function χ of Ω. Furthermore we have

$$\chi - \chi_{\mathcal{R}} \geq 1 - \frac{1}{2} = \frac{1}{2} \qquad \text{on } \Omega\backslash\Omega_{\mathcal{R}},$$

so that

$$\begin{aligned}
|(\Omega\backslash\Omega_{\mathcal{R}}) \cap Q| &\leq 2\int_Q |\chi - \chi_{\mathcal{R}}|\, dx \\
&= 2 \sup_{|\psi|\leq 1} \int_Q \fint_{B_{\mathcal{R}}(0)} (\chi(x) - \chi(x+h))\psi(x) dx dh \\
&\leq 2 \sup_{|h|\leq \mathcal{R}} \int_Q |\chi(x) - \chi(x+h)|\, dx \\
&\leq 2\mathcal{R} \int_Q |\nabla \chi| \\
&\leq \frac{1}{2}|\Omega \cap Q|,
\end{aligned}$$

where the last inequality follows from 2.10. Thus

$$|\Omega \cap \Omega_{\mathcal{R}} \cap Q| = |\Omega \cap Q| - |(\Omega\backslash\Omega_{\mathcal{R}}) \cap Q| \geq \frac{1}{2}|\Omega \cap Q|,$$

which establishes 2.13.
Next we claim that there exists a set $A \subset \Omega_{\mathcal{R}} \cap Q$ such that

$$\#A \leq C\frac{|\Omega \cap Q|}{|B_{\mathcal{R}}|} \qquad \text{and} \qquad \Omega_{\mathcal{R}} \cap Q \subset \bigcup_{x\in A} B_{2\mathcal{R}}(x). \tag{2.14}$$

To that aim let $A \subset \Omega_{\mathcal{R}} \cap Q$ be a maximal family such that

$$\{B_{\mathcal{R}}(x)\}_{x\in A} \quad \text{are disjoint.} \tag{2.15}$$

We claim that

$$\Omega_{\mathcal{R}} \cap Q \subset \bigcup_{x\in A} B_{2\mathcal{R}}(x) \cap Q. \tag{2.16}$$

Indeed, assume that 2.16 were wrong. Then there exists $y \in \Omega_{\mathcal{R}} \cap Q$ such that for all $x \in A$ we have $y \notin B_{2\mathcal{R}}(x)$, that is for all $x \in A$ we find $B_{\mathcal{R}}(y) \cap B_{\mathcal{R}}(x) = \emptyset$. This contradicts the maximality of A.

We now have because of $A \subset \Omega_{\mathcal{R}} \cap Q$

$$\begin{aligned}
\#A|B_{\mathcal{R}}| &= \sum_{x\in A} |B_{\mathcal{R}}(x)| \\
&\overset{2.12}{<} 2\sum_{x\in A} |\Omega \cap B_{\mathcal{R}}(x)| \\
&\overset{2.15,2.11}{\leq} C(n)|\Omega \cap Q|.
\end{aligned}$$

Finally we claim

$$|\Omega^r_{\mathcal{R}} \cap Q| \leq C(n)|\Omega \cap Q|\Big(2 + \frac{r}{\mathcal{R}}\Big)^n. \tag{2.17}$$

In view of 2.16 we have the inclusion $\Omega_{\mathcal{R}} \cap Q \subset \bigcup_{x\in A} B_{2\mathcal{R}}(x) \cap Q$, which implies $\Omega^r_{\mathcal{R}} \subset \bigcup_{x\in A} B_{2\mathcal{R}+r}(x) \cap Q$. Thus

$$\begin{aligned}
|\Omega^r_{\mathcal{R}} \cap Q| &\leq \sum_{x\in A} |B_{2\mathcal{R}+r}(x) \cap Q| \\
&\overset{2.14}{\leq} C\frac{|\Omega \cap Q|}{|B_{\mathcal{R}}|}|B_{2\mathcal{R}+r}| \\
&= C|\Omega \cap Q|\Big(\frac{2\mathcal{R}+r}{\mathcal{R}}\Big)^n,
\end{aligned}$$

which proves 2.17 and thus completes the proof of Lemma 2.1.

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