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no. 306

Diese Arbeit ist mit Unterstützung des von der Deutschen Forschungsgemeinschaft getragenen Sonderforschungsbereiches 611 an der Universität Bonn entstanden und als Manuskript vervielfältigt worden.

Bonn, November 2006

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LORENZO GIACOMELLI, HANS KNÜPFER, AND FELIX OTTO

Abstract. In the simplest case of a linearly degenerate mobility, we view the thin–film equation as a classical free boundary problem. Our focus is on the regularity of solutions and of their free boundary in the "complete wetting" regime, which prescribes zero slope at the free boundary. In order to rule out of the analysis possible changes in the topology of the positivity set, we zoom into the free boundary by looking at perturbations of the stationary solution. Our strategy is based on a–priori energy–type estimates which provide "minimal" conditions on the initial datum under which a unique global solution exists. In fact, this solution turns out to be smooth for positive times and to converge to the stationary solution for large times. As a consequence, we obtain smoothness and large–time behaviour of the free boundary.

Contents

2000 *Mathematics Subject Classification.* 35A05, 35K65, 35Q35, 35R35, 76A20, 76D08, 76D27,
Key words and phrases. Existence and uniqueness, fourth order degenerate parabolic equations, thin-film equations, free boundary problems, thin fluid films, lubrication theory, Hele-Shaw flow.

Support is acknowledged by the SFB 611 "Singuläre Phänomene und Skalierung in mathematischen Modellen" and by the RTN-Programme "Fronts-Singularities" (HPRN-CT-2002-00274). L.G. and H.K. acknowledge the kind hospitality of, respectively, the Institut für Angewandte Mathematik in Bonn and the Dipartimento Me.Mo.Mat. in Rome.

1. Introduction and main results

The background. The thin–film equation

$$\partial_t h + \partial_y (h^n \ \partial_y^3 h) = 0 \tag{1}$$

arises in modeling the capillarity–driven evolution of a liquid film over a solid substrate in the regime of lubrication approximation. The unknown function $h(t, y)$ is the height of the (here assumed one–dimensional) film, and the parameter n describes the condition at the solid: $n = 3$ means no slip, whereas $n \in (0, 3)$ models various relaxed slip conditions. We refer to the review [20] for details. For $n = 1$, (1) may also be seen as the lubrication approximation of the two–dimensional Hele–Shaw flow in half–space (see [13]). Part of the interest of (1) lies in the fact that it preserves certain features of the full system (Navier–Stokes or Darcy equations with curvature–dependent free boundary conditions), such as the possibility that the support shrinks and that singularities form. A singularity in this context corresponds to a point where h attains the value zero, which may happen either for initially positive data (see [5]), or as a single droplet splits into two, even with the formation of dead–cores (see e.g. the numerical experiments in [16]). All this is intrinsic to (1) being of higher order, hence lacking a comparison principle.

In the case of "droplets", i.e. if the initial datum is compactly supported, the equation is already singular at the support's boundary, and (1) turns into a free boundary problem, the free boundary being given by $\partial\{h > 0\}$. Since the equation is of fourth order, three conditions are expected to be needed for well–posedness at least for $n < 3$, to which case we shall refer hereafter (for $n \geq 3$, the free boundary is expected to stand still due to the no–slip condition). In addition to the defining condition $h = 0$, the kinematics of the problem imply that for sufficiently regular solutions the speed V of the free boundary satisfies

$$V = \lim_{\{h>0\}\ni y\to\partial\{h>0\}} h^{n-1} \ \partial_y^3 h \qquad \text{on } \partial\{h > 0\}. \tag{2}$$

An alternative of (2), which is weaker but stable under (plausible, as explained above) topology changes, is to merely require conservation of mass:

$$\int_{\mathbb{R}} h(t, y) \ dy = \int_{\mathbb{R}} h(0, y) \ dy \qquad \text{for all } t > 0. \tag{3}$$

The third condition is modeling–wise not fully understood. The simplest choice is to assume that at triple junctions the droplet instantaneously attains its energetically most convenient configuration. By Young's law, this amounts to impose a fixed contact angle. In the so–called "complete wetting" regime (that is, whenever the surface is sufficiently hydrophilic), the angle is zero:

$$\partial_y h = 0 \qquad \text{on } \partial\{h > 0\}. \tag{4}$$

Weak solutions of (1)–(3)–(4) are well understood: after the pioneering work [6], an existence theory has been elaborated for the one–dimensional case in [5, 7]. We refer to [3, 14] for up–to–date reviews of the analytical theory (see also [21, 8] for the partial wetting case). On the other hand, only quite partial results are available concerning regularity: in particular, a weak solution is known to satisfy (4) only for almost every time, and seems not to be strong enough to infer (2) (though its support is known to stay bounded, see [15] and the references therein).

Motivated by the aim of understanding the attainment of (2) and, more generally, the regularity of solutions and of their free boundary, we shall view (1)–(2)–(4) as a classical free boundary problem. In the formulation we are going to introduce now, both (4) and (2) will henceforth be part of the problem.

The classical formulation. We consider the analytically simplest case, which corresponds to taking $n = 1$. Then (1)–(4) read as

$$\begin{cases} \partial_t h + \partial_y (h\, \partial_y^3 h) = 0 & \text{in } \{h > 0\}, \\ h = \partial_y h = 0 & \text{at } \partial\{h > 0\}, \\ h = h_0 & \text{at } \{t = 0\} \end{cases} \tag{5}$$

(the case $n = 1$ is peculiar in many aspects, see e.g. [21, 9]: as a naive example, it is the only case where explicit source type self–similar solutions are known). Assume that initially

$$\{h_0 > 0\} = (s_0, \infty), \tag{6}$$

$$\partial_y h_0(s_0) = 0, \tag{7}$$

and let

$$s(t) = \inf \{\ y \in \mathbb{R} : h(t, \cdot) > 0 \text{ in } (y, \infty)\ \}.$$

Then, as long as the topology of $\{h > 0\}$ does not change, we may rewrite (5) as

$$\begin{cases} \partial_t h + \partial_y (h\, \partial_y^3 h) = 0 & \text{in } (0, \infty) \times (s(t), \infty), \\ h = \partial_y h = 0 & \text{at } y = s(t), \\ h = h_0 & \text{at } \{t = 0\}. \end{cases} \tag{8}$$

In view of (2), for sufficiently regular solutions the evolution of the free boundary $s(t)$ is described by

$$\begin{cases} \dot{s}(t) = \partial_y^3 h(t, s(t)), \\ s(0) = s_0. \end{cases} \tag{9}$$

Hence, passing to a fixed spatial domain by letting $x = y - s(t)$, (8)–(9) read as follows:

$$\begin{cases} \partial_t h - \partial_x h\ \dot{s} + \partial_x(h\ \partial_x^3 h) = 0,\ \ h > 0 & \text{in } (0,\infty)^2, \\ h = \partial_x h = 0,\ \ \dot{s} = \partial_x^3 h & \text{at } x = 0, \\ h = h_0(\cdot + s_0),\ \ s = s_0 & \text{at } t = 0. \end{cases} \tag{10}$$

Even in this simplified context, the issues of global existence, uniqueness and regularity of generic solutions to (10) all have a non–local aspect: as we have seen above, solutions may rupture yielding changes in the topology of $\{h > 0\}$. In order to get rid of such non–local effects, we assume that the initial datum is a perturbation of a reference stationary solution, which (up to a translation in y and a rescaling of t and y) may be fixed to be $(y_+)^2/2$:

$$h_0(y) \approx \frac{1}{2}(y_+)^2. \tag{11}$$

The precise meaning of (11) will be given below. Here we just point out (compare (6)) that we do not require h_0 to have the same support as the reference steady state.

Global transformation onto fixed domain. The main results of this paper concern long–time existence, uniqueness, smoothness and long–time behavior of solutions to (10) provided h_0 satisfies (6), (7) and (11). The statements will be given and proved in terms of a transformed problem. We introduce the new variable F by

$$h(t,x) = \frac{1}{2}x^2 + xs(t) + F(t,x). \tag{12}$$

Then (10) turns into

$$\begin{cases} \partial_t F + \partial_x(\frac{1}{2}x^2\ \partial_x^3 F) = -\partial_x\left((F - x\ \partial_x F_{|x=0})(\partial_x^3 F - \partial_x^3 F_{|x=0})\right) & \text{in } (0,\infty)^2, \\ F = 0 & \text{on } x = 0, \\ F = F_0 & \text{on } t = 0. \end{cases} \tag{13}$$

The motivation for (12) is twofold. Firstly, it splits the operator into a linear and nonlinear part. Furthermore, the second term in the right–hand side of (12) keeps the linear part of the operator free of a transport term. As a consequence of (7) and (12), the position of the free boundary and the zero contact angle condition are encoded into F by

$$s_0 = -\partial_x F_0(0), \qquad\qquad s(t) = -\partial_x F(t,0),$$

whereas for the speed we still have

$$\dot{s}(t) = \partial_x^3 F(t,0). \tag{14}$$

These two equations can be easily seen to be compatible by formally differentiating the equation in (13) with respect to x and using (7). Therefore, provided (7) holds, (13) is equivalent to (10) through (12) .

Weighted Sobolev spaces. The structure of the linear operator motivates us to introduce the following semi–norms

$$[F]_{H_k} := \langle F, F\rangle_{H_k}^{1/2}, \tag{15}$$

where

$$\langle F, G\rangle_{H_k} := \int_0^\infty x^{k-1}\ \partial_x^k F\ \partial_x^k G\ dx.$$

For $m \geq 1$, the corresponding norm and space are given by:

$$\|F\|_{H_m}^2 := \sum_{k=1}^m [F]_{H_k}^2,$$

$$H_m := \begin{cases} \text{Completion of } \{F \in C_c^\infty([0,\infty)) : F(0) = 0\} \\ \text{with respect to } \|\cdot\|_{H_m}. \end{cases} \tag{16}$$

In fact the semi–norms $[F]_{H_m}$ are Lyapunov functionals for the linear operator, as we will see later on. Note that the first semi–norm, $k = 1$, is just the Dirichlet integral, which corresponds to the lubrication approximation of the capillary energy. Its boundedness ensures that the boundary condition is preserved under completion. Accordingly, we define the semi-norms

$$[F]_{L^2(H_m)} := \left(\int_0^\infty [F]_{H_m}^2\, dt\right)^{\frac{1}{2}}, \quad [F]_{C^0(H_m)} := \sup_{t\in(0,\infty)} [F]_{H_m}.$$

What is the relationship between these norms and the nonlinear part of the operator? Equation (13) is invariant under the transformation

$$x \mapsto \lambda x, \qquad t \mapsto \lambda^2 t, \qquad F \mapsto \lambda^2 F. \tag{17}$$

This transformation selects one single semi–norm out of the sequence of semi–norms $[F]_{H_m}$: the only invariant semi–norm under (17) is given by $[F]_{H_4}$. Can we get a minimal setting where we assume just to control the initial datum by the single semi–norm $[F_0]_{H_4}$? This seems not to be possible. The reason is the following: the setting is determined by the goal to control the nonlinear part of the operator by the linear one. In view of (13) it is then clear that one needs $\sup_t |F - x\,\partial_x F(t,0)| \ll x^2$. But unfortunately, the estimate one could hope for in terms of scaling barely fails:

$$\|x^{-2}(F - x\ \partial_x F_{|x=0})\|_{C^0} \not\lesssim C\ [F]_{H_4} \tag{18}$$

(a counterexample is given by $\partial_x^2 F = \ln(\ln 1/x)$ for $x \ll 1$).

Interpolation spaces. This motivates us to use interpolation semi-norms. They have the same scaling as the corresponding semi–norms (15) but are slightly stronger. For any $m \geq 2$, we define

$$[F]_{H_m^*} \ := \ \int_0^\infty \inf_{F=F_-+F_+} \left(s^{-1}[F_-]^2_{H_{m-1}} + s[F_+]^2_{H_{m+1}}\right)^{1/2} \frac{ds}{s}, \tag{19}$$

where $F_\pm \in H_{m\pm1}$. The corresponding norm and space are given by

$$\|F\|_{H_m^*} := \sum_{k=2}^m [F]_{H_k^*}, \tag{20}$$

$$H_m^* := \begin{cases} \text{Completion of } \{F \in C_c^\infty([0,\infty)) : F(0) = 0\} \\ \text{with respect to } \|\cdot\|_{H_m^*}. \end{cases} \tag{21}$$

It seems to be crucial to define H_m^* by completion rather than by finiteness of $\|\cdot\|_{H_m^*}$ (the two definitions disagree with each other, see Lemma A.5).
A comparison between the weighted Sobolev semi-norms and their interpolation counterparts is given in the next lemma:

Lemma 1.1. *For any even $m \geq 2$ and any function $F \in H_{m+1}$ we have*

$$\frac{1}{C}\ [F]_{H_m} \leq [F]_{H_m^*} \leq C\ [F]^{1/2}_{H_{m-1}}[F]^{1/2}_{H_{m+1}}. \tag{22}$$

The interpolation semi-norms are indeed strong enough to control supremum norms:

Lemma 1.2. *For any even $m \geq 2$ and any function $F \in H_m^*$ we have*

$$\|\partial_x^{m/2} F\|_{C^0} \ \leq \ C\ [F]_{H_m^*}. \tag{23}$$

In particular, we have

$$\|x^{-1}F\|_{C^0} \leq \|\partial_x F\|_{C^0} \leq [F]_{H_2^*}, \tag{24}$$

so that the boundary condition in (21) is preserved under completion. The proof of the Lemmata 1.1–1.2 can be found in the appendix. Note that by Lemma 1.2 and $F|_{x=0} = 0$, the statement corresponding to (18) holds true for the interpolation semi-norms:

$$\|x^{-2}(F - x\ \partial_x F|_{x=0})\|_{C^0} \ \leq \ C\ \|\partial_x^2 F\|_{C^0} \leq \ C\ [F]_{H_4^*}. \tag{25}$$

Finally, we introduce the corresponding parabolic semi-norms. For $m \geq 2$, let

$$\begin{aligned} [F]_{L^2(H_m)^*} &:= \int_0^\infty \inf_{F=F_-+F_+} \left(s^{-1}[F_-]^2_{L^2(H_{m-1})} + s[F_+]^2_{L^2(H_{m+1})}\right)^{1/2} \frac{ds}{s}, \\ [F]_{C^0(H_m)^*} &:= \int_0^\infty \inf_{F=F_-+F_+} \left(s^{-1}[F_-]^2_{C^0(H_{m-1})} + s[F_+]^2_{C^0(H_{m+1})}\right)^{1/2} \frac{ds}{s}. \end{aligned} \tag{26}$$

It seems to be crucial to interpolate the space-time semi-norms rather than taking the temporal norm of the spatial interpolation, as for

$$[F]_{L^2(H^*_m)} := \left(\int_0^\infty [F]^2_{H^*_m}\, dt\right)^{\frac{1}{2}}, \quad [F]_{C^0(H^*_m)} := \sup_{t\in(0,\infty)} [F]_{H^*_m}. \tag{27}$$

Indeed, these semi-norms do not coincide with (26) in general. However, there is a lower bound in general and a two-sided bound for tensor products:

Lemma 1.3.

i) *For any even $m \geq 2$ and any function $F \in C_c([0,\infty)^2)$ we have*

$$\begin{aligned} [F]_{L^2(H_m)^*} &\geq [F]_{L^2(H^*_m)}, \\ [F]_{C^0(H_m)^*} &\geq [F]_{C^0(H^*_m)}. \end{aligned} \tag{28}$$

ii) *For any even $m \geq 2$ and any pair of functions $\alpha, \zeta \in C_c([0,\infty))$ we have*

$$\begin{aligned} [\alpha\otimes\zeta]_{L^2(H_m)^*} &\leq \|\alpha\|_{L^2}\ [\zeta]_{H^*_m}, \\ [\alpha\otimes\zeta]_{C^0(H_m)^*} &\leq \|\alpha\|_{C^0}\ [\zeta]_{H^*_m}, \end{aligned} \tag{29}$$

where $(\alpha\otimes\zeta)(t,x) = \alpha(t)\ \zeta(x)$.

The proof of Lemma 1.3 can be found in the Appendix.

The semi-norms corresponding to (26) and (27) on a finite time interval $(0,T)$ are denoted by $[\cdot]_{L^2((0,T);H_m)^*}$, $[\cdot]_{C^0((0,T);H_m)^*}$, $[F]_{L^2((0,T);H^*_m)}$ and $[F]_{C^0((0,T);H^*_m)}$, respectively. The interpolation norms $\|\cdot\|_{L^2(H_m)^*}$ and spaces $L^2(H_m)^*$ are defined by completion exactly as in (20)-(21).

Statement of the results. The semi–norms defined in (26) provide an appropriate ambient space for existence and uniqueness of solutions to (13) under minimal (in the sense of (25) versus (18)) assumptions on the initial data. It is given, for $m \geq 4$, by:

$$\|F\|_{X^*_{m+2}} := \sum_{k=4}^{m} \left([\partial_t F]_{L^2(H_{k-2})^*} + [F]_{C^0(H_k)^*} + [F]_{L^2(H_{k+2})^*}\right)$$

and

$$X^*_m := \begin{cases} \text{Completion of } \{F \in C^\infty_c([0,\infty)^2) : F_{|x=0} = 0\} \\ \text{with respect to } \|\cdot\|_{X^*_m}. \end{cases}$$

It is straightforward to check that the trace of $F \in X^*_m$ at $t = 0$ is well defined (see Lemma A.6).

Here and after, universal constants are denoted by C, and C_k stands for a constant which is universal for fixed k. Our first main result is:

Theorem 1.1 (Existence and uniqueness)**.** *There exists an* $\epsilon > 0$ *such that if* $F_0 \in H_4^*$ *satisfies*

$$[F_0]_{H_4^*} < \epsilon, \tag{30}$$

then there exists a unique solution $F \in X_6^*$ *of* (13)*, and furthermore*

$$[\partial_t F]_{L^2(H_2)^*} + [F]_{C^0(H_4)^*} + [F]_{L^2(H_6)^*} \leq C\ [F_0]_{H_4^*}. \tag{31}$$

Equation (31) contains enough information to keep track of both the shape of the solution, as follows from

$$\|\partial_x^2 h - 1\|_{C^0((0,\infty)^2)} = \|\partial_x^2 F\|_{C^0((0,\infty)^2)} \overset{(25)}{\leq} C\ [F]_{C^0(H_4^*)} \overset{(28)}{\leq} C\ [F]_{C^0(H_4)^*}, \tag{32}$$

and the position of the free boundary, as follows from

$$\|\dot{s}\|_{L^2} \overset{(14)}{=} \|\partial_x^3 F_{|x=0}\|_{L^2} \overset{(23)}{\leq} C\ [F]_{L^2(H_6^*)} \overset{(28)}{\leq} C\ [F]_{L^2(H_6)^*}.$$

In particular, (32) implies that $h > 0$ for $\epsilon \ll 1$.

The minimal assumption (30) turns out to be sufficiently robust to keep all derivatives of the perturbation under control. This yields smoothness of the solution and of the free boundary for positive times:

Theorem 1.2 (Smoothness)**.** *There exists an* $\epsilon > 0$ *such that if* $F_0 \in H_4^*$ *satisfies* (30)*, then the solution* $F \in X_6^*$ *of* (13) *belongs to* $C^\infty((0,\infty)\times[0,\infty))$*; in particular, the free boundary* $s(t)$ *belongs to* $C^\infty((0,\infty))$*.*

It turns out that smoothness may be quantified in terms of $[F_0]_{H_4^*}$ via estimates of the decay of high derivatives of F. Not to overload the elaboration, we shall state and prove them with respect to the semi-norms (27):

Theorem 1.3 (Decay of high derivatives)**.** *There exists an* $\epsilon > 0$ *such that if* $F_0 \in H_4^*$ *satisfies* (30)*, then the solution* $F \in X_6^*$ *of* (13) *satisfies*

$$[t^{\frac{k-4}{4}} F]_{C^0(H_k^*)} + [t^{\frac{k-4}{4}} F]_{L^2(H_{k+2}^*)} \leq C_k\ [F_0]_{H_4^*} \qquad \text{for all } k \geq 4. \tag{33}$$

The temporal weights are such that estimate (33) is scale-invariant with respect to the transformation (17).

The bounds so far are oblivious of the value of $[F_0]_{H_1}$. In this sense, they are unable to single out the steady state $\tilde{F} = 0$ among the two-parameter family of stationary solutions to (13), $\tilde{F} = (Ax^2 - Bx)$. It is therefore not surprising that $[F_0]_{H_1}$ becomes quantitatively relevant when looking at the long–time behavior of the solution and of the free boundary.

Theorem 1.4 (Convergence to the steady state)**.** *There exists an $\epsilon > 0$ such that if $F_0 \in H_4^*$ satisfies (30), then the solution $F \in X_6^*$ satisfies*

$$[t^{\frac{k-1}{4}} F]_{C^0(H_k)} + [t^{\frac{k-1}{4}} F]_{L^2(H_{k+2})} \leq C_k \ ([F_0]_{H_1} + [F_0]_{H_4^*}) \qquad \textit{for all } k \geq 1.$$

In particular, the free boundary converges to zero:

$$|s(t)| \leq C \ (1+t)^{-\frac{1}{4}}([F_0]_{H_1} + [F_0]_{H_4^*}).$$

In terms of h, $[F_0]_{H_1} < \infty$ implies that the initial datum $h_0(y)$ is close to the reference steady state $(y_+)^2/2$ for $y \gg 1$. In other words, it is the behaviour at infinity (not the behaviour near $y = 0$) which selects the steady state to which the solution converges for large times. This explains why a general initial position of the free boundary, $s_0 \neq 0$, relaxes to zero in our setting.
In particular, Theorem 1.4 implies that solutions to the thin-film equation with $n = 1$ may shrink (take $s_0 < 0$). Such behavior was commonly believed to be possible in view of the existence of local travelling wave profiles of the form $h(t, y) = -\frac{1}{6}(y + t)^3 + A(y + t)_+^2$, $A > 0$, $(y + t)_+ \ll A$. However, to our knowledge it hadn't been rigorously observed so far.

The porous medium equation. There is a related second order diffusion equation, the porous medium equation:

$$\partial_t h - \partial_y(h^n \ \partial_y h) = 0.$$

As (1), the porous medium equation is degenerate parabolic, but it additionally obeys a comparison principle, which excludes both the formation of singularities and the contraction of the support. Long–time existence, uniqueness and regularity of solutions of the corresponding free boundary problem after the waiting time have been shown in [1, 2] in space dimension $d = 1$; short-time analogues of this result have later been obtained in [11] for $d = 2$ and in [18] for arbitrary d. The results in [1, 2, 11] hold for a generic initial datum (with suitable regularity and connected positivity set) rather than with perturbations of special solutions (stationary or self-similar). In fact, we expect that our results may also be extended (by using local maps) to the case of a single droplet with zero contact angle, after the waiting time (which may occur for the thin–film equation, too; see [12] and the references therein). However, unlike the one-dimensional porous medium equation, for the above mentioned reasons we can only expect short–time existence for generic initial data.

The aforementioned papers adopt quite different techniques. The analysis in [1, 2] relies on semi–group theory: the equation is written as an abstract parabolic boundary value problem to which the theory of Da Prato & Grisvard [10] can be seen to be applicable. The main tool in [11] are instead Schauder estimates in weighted Hölder spaces, obtained by an elaboration of the method of Safonov [22]. Finally, in [18] the analysis is based on the theory of singular integral operators and Gaussian estimates of the fundamental solution of the linearized parabolic equation.

Here we follow yet another strategy, whose advantage may be to identify the "minimal" assumptions on the initial datum under which the problem is well posed. It is based on a–priori energy–type estimates which we are going to describe now.

2. Outline of the proofs

In this section we describe the main ingredients of our method. This will lead us already to the proof of the uniqueness part of Theorem 1.1. The proof of the existence part of Theorem 1.1, as well as that of the other main results, will be outlined immediately afterwards, together with the plan of the paper. From here on, we write $f \lesssim g$, respectively $f \ll g$, whenever a universal $C > 1$ exists such that $f \leq C\, g$, resp. $f \leq C^{-1}\, g$.

We split the operator into its linear part,

$$AF := \frac{1}{2}\, \partial_x(x^2\, \partial_x^3 F),$$

and its nonlinear part $\mathcal{N}(F,F)$, where

$$\mathcal{N}(F,G) := -\partial_x \left((F - x\, \partial_x F_{|x=0})(\partial_x^3 G - \partial_x^3 G_{|x=0}) \right). \tag{34}$$

The bases of our argument are the symmetry and composition properties enjoyed by A, which induce the choice of the semi–norms $[\cdot]_{H_k}$, namely (see Lemmata 3.2 and 3.3)

$$2\langle AF, G\rangle_{H_k} = \langle F, G\rangle_{H_{k+2}}, \qquad\qquad 2[AF]_{H_k} = [F]_{H_{k+4}}. \tag{35}$$

As can be easily checked at a formal level, they imply the existence of a sequence of Lyapunov functionals for A (see Lemma 9.1):

$$\partial_t [F]_{H_k}^2 + [F]_{H_{k+2}}^2 = 4\langle AF, \partial_t F + AF\rangle_{H_{k-2}}, \qquad\qquad k \geq 3.$$

This yields the following existence and uniqueness result for the parabolic linear equation associated to (13):

Proposition 2.1. *Let $m \geq 4$. For any $F_0 \in H_m^*$ and any $G \in L^2(H_{m-2})^*$ there exists a unique $F \in X_{m+2}^*$ such that*

$$\begin{cases} \partial_t F + AF = G \\ F_{|t=0} = F_0. \end{cases} \tag{36}$$

It satisfies for all $4 \leq k \leq m$ and all $0 < T \leq \infty$ the estimate

$$\begin{aligned} &C^{-1}\, [\partial_t F]_{L^2((0,T);H_{k-2})^*} + [F]_{C^0((0,T);H_k)^*} + C^{-1}\, [F]_{L^2((0,T);H_{k+2})^*} \\ &\qquad \leq [F_0]_{H_k^*} + C\, [G]_{L^2((0,T);H_{k-2})^*}. \end{aligned} \tag{37}$$

The main ingredient in the proof of Theorem 1.1 is the following a–priori estimate for the nonlinear part of the operator:

Proposition 2.2. *For any given $F, G \in X_6^*$ and for any $0 < T \leq \infty$ we have $\mathcal{N}(F,G) \in L^2((0,T),H_2)^*$ and*

$$[\mathcal{N}(F,G)]_{L^2((0,T);H_2)^*} \lesssim [F]_{C^0((0,T);H_4)^*}\ [G]_{L^2((0,T);H_6)^*}. \tag{38}$$

The combination of (37) and (38) yields the a–priori estimate (31) which is at the core of Theorem 1.1:

Proposition 2.3. *There exists an $\epsilon > 0$ such that if $F \in X_6^*$ be a solution of* (13) *with $F_0 \in H_4^*$ such that*

$$[F_0]_{H_4^*} < \epsilon,$$

then

$$[F]_{C^0(H_4)^*} + [F]_{L^2(H_6)^*} \lesssim [F_0]_{H_4^*}. \tag{39}$$

Proof. It follows from (37) and (38) that

$$\begin{aligned}
\phi(T) &:= C^{-1}\ [\partial_t F]_{L^2((0,T);H_2)^*} + [F]_{C^0((0,T);H_4)^*} + C^{-1}\ [F]_{L^2((0,T);H_6)^*} \\
&\overset{(37)}{\leq} [F_0]_{H_4^*} + C\ [\partial_t F + AF]_{L^2((0,T);H_2)^*} \\
&\overset{(13)}{=} [F_0]_{H_4^*} + C\ [\mathcal{N}(F,F)]_{L^2((0,T);H_2)^*} \\
&\overset{(38)}{\leq} [F_0]_{H_4^*} + C\ [F]_{C^0((0,T);H_4)^*}[F]_{L^2((0,T);H_6)^*}.
\end{aligned} \tag{40}$$

It is easy to check (see Lemma A.6) that $\phi \in C([0,\infty))$ with $\phi(0) = [F_0]_{H_4^*}$. Hence (40) reads as

$$\phi(t) \overset{(30)}{\leq} \phi(0) + C\ \phi^2(t) \quad \text{for all } t > 0. \tag{41}$$

If $\phi(0) < 1/(4C) =: \epsilon$, then

$$C\ \phi^2 - \phi + \phi(0) \geq 0 \iff \begin{cases} \phi \leq \phi_1 = \frac{1-\sqrt{1-4C\phi(0)}}{2C} \\ \qquad\quad \text{or} \\ \phi \geq \phi_2 = \frac{1+\sqrt{1-4C\phi(0)}}{2C} \end{cases}.$$

Since $\phi(0) \leq \phi_1$ and ϕ is continuous, (41) implies that

$$\phi(t) \leq \phi_1 \lesssim \phi(0) \qquad\qquad \text{for all } t > 0,$$

and the proof is complete. □

We are now ready to prove the uniqueness of solutions.

Proof of Theorem 1.1 – uniqueness. Let $F_1, F_2 \in X_6^*$ be two solutions of (13) with initial datum $F_0 \in H_4^*$, and let $F = F_1 - F_2$. By (37), we have that

$$\begin{aligned}\phi &:= C^{-1}\ [\partial_t F]_{L^2(H_2)^*} + [F]_{C^0(H_4)^*} + C^{-1}\ [F]_{L^2(H_6)^*} \\ &\lesssim [\mathcal{N}(F_1, F_1) - \mathcal{N}(F_2, F_2)]_{L^2(H_2)^*}.\end{aligned} \tag{42}$$

It follows from the definition (34) of $\mathcal{N}$ that

$$\mathcal{N}(F_1, F_1) - \mathcal{N}(F_2, F_2) = \mathcal{N}(F_1, F) + \mathcal{N}(F, F_2).$$

Therefore, provided ϵ is sufficiently small for Proposition 2.3 to hold, we have that

$$\begin{aligned}[\mathcal{N}(F_1, F_1) - \mathcal{N}(F_2, F_2)]_{L^2(H_2)^*} &\lesssim [\mathcal{N}(F_1, F)]_{L^2(H_2)^*} + [\mathcal{N}(F, F_2)]_{L^2(H_2)^*} \\ &\overset{(38)}{\lesssim} [F_1]_{C^0(H_4)^*}[F]_{L^2(H_6)^*} + [F]_{C^0(H_4)^*}[F_2]_{L^2(H_6)^*} \\ &\overset{(39)}{\lesssim} \epsilon \left([F]_{C^0(H_4)^*} + [F]_{L^2(H_6)^*}\right).\end{aligned} \tag{43}$$

Inserting (43) into (42), we see that

$$\phi \lesssim \epsilon\ \phi,$$

and therefore $\phi = 0$ if ϵ is sufficiently small. In particular, it follows from Lemma 1.3 that $[\partial_t F(t)]_{H_2^*} = 0$ for a.e. t, which by (24) implies that $\partial_t F = 0$, that is $F(t, x) = F(0, x) = 0$. □

Of course, the a–priori estimate given by Proposition 2.3 is also at the core of the existence part of Theorem 1.1, whose proof will be given in section 7 using a fixed–point argument. In section 4 (see Proposition 4.1) we prove well–posedness for the resolvent equation associated to A,

$$F + AF = G,$$

which we use in section 5 to prove Proposition 2.1 via a time–discretization argument.

The results concerned with higher regularity rely on the fact that the symmetric structure of the linear part of the operator is preserved upon differentiation. This yields estimates similar to (38) for higher semi-norms of the nonlinear operator. We prove in Proposition 8.1 that

$$[\mathcal{N}(F, F)]_{H_k}\ \leq\ C\ [F]_{H_4^*}[F]_{H_{k+4}} + C_k\ [F]_{H_6^*}[F]_{H_{k+2}}.$$

Based on this a priori estimate, we prove Theorem 1.2 in Section 8, Theorem 1.3 in Section 9 and Theorem 1.4 in Section 10.

3. Basic properties of the linear operator

We first notice:

Lemma 3.1. *For all $F \in C^\infty((0,\infty))$ and all $k \geq 0$, there holds:*

$$\partial_x^k AF = \frac{1}{2x^{k-1}}\ \partial_x^2(x^{k+1}\ \partial_x^{k+2}F). \tag{44}$$

Proof. Easily obtained by induction. □

Identity (44) gives rise to symmetry with respect to each weighted semi–norm:

Lemma 3.2. *Let $k \geq 1$. For all $F \in H_{k+4}$ and all $G \in H_{k+2}$, there holds:*

$$\langle AF, G\rangle_{H_k} = \frac{1}{2}\langle F, G\rangle_{H_{k+2}}. \tag{45}$$

Proof. By (16), it suffices to consider $F, G \in C_c^\infty([0,\infty))$ (in fact, the boundary condition $F(0) = 0$ is not needed here). We have

$$\begin{aligned}\langle AF, G\rangle_{H_k} &= \int_0^\infty x^{k-1}\ \partial_x^k(AF)\ \partial_x^k G\ dx \\ &\overset{(44)}{=} \frac{1}{2}\int_0^\infty \partial_x^2(x^{k+1}\ \partial_x^{k+2}F)\ \partial_x^k G\ dx \\ &= \frac{1}{2}\int_0^\infty x^{k+1}\ \partial_x^{k+2}F\ \partial_x^{k+2}G\ dx.\end{aligned}$$

Note that all boundary terms at $x = 0$ vanish since $k \geq 1$. □

Applied twice, Lemma 3.2 yields:

Lemma 3.3. *Let $k \geq 1$. For all $F \in H_{k+4}$, there holds:*

$$[F]_{H_{k+4}} = 2\ [AF]_{H_k}. \tag{46}$$

The connection between the operator A and the spaces H_m^* can be seen even better in terms of the positive square root of A:

$$A^{1/2}F = -\frac{1}{\sqrt{2}}\ x\ \partial_x^2 F.$$

Indeed,

$$AF = \frac{1}{2}\ (x^2\ \partial_x^3 F) \overset{(44)}{=} \frac{1}{\sqrt{2}}\ x\ \partial_x^2(\frac{1}{\sqrt{2}}\ x\ \partial_x^2 F).$$

The operator $A^{1/2}$ is symmetric and positive definite with respect to all semi–norms $[\cdot]_{H_m}$. More precisely, there holds for all $k \geq 1$ and for all $F, G \in H_{k+1}$,

$$\langle A^{1/2}F, G\rangle_{H_k} = \frac{1}{\sqrt{2}}\ \langle F, G\rangle_{H_{k+1}}.$$

The last statement follows from an analogous identity to (44),

$$\partial_x^k(A^{1/2}F) = -\frac{1}{\sqrt{2}\ x^{k-1}}\ \partial_x(x^k\ \partial_x^{k+1}F).$$

This leads to the following representation for the semi–norms $[\cdot]_{H_m}$:

Lemma 3.4. *For all $k \geq 1$ and $F \in C_c^\infty([0,\infty))$ with $F(0) = 0$,*

$$[F]_{H_k} = 2^{\frac{k-1}{4}} \ \langle F, A^{(k-1)/2} F\rangle_{H_1}^{1/2}. \tag{47}$$

We conclude that the complete sequence of semi–norms $[\cdot]_{H_m}$, $m \geq 1$, is generated by $A^{1/2}$ and $\langle \cdot, \cdot \rangle_{H_1}$.

4. The linear elliptic equation

In this section we prove:

Proposition 4.1 (The resolvent equation)**.** *For all $G \in H_1$ there exists a unique solution $F \in H_3$ of*

$$\int_0^\infty \partial_x F \ \partial_x \phi \ dx + \frac{1}{2} \int_0^\infty x^2 \ \partial_x^3 F \ \partial_x^3 \phi \ dx = \int_0^\infty \partial_x G \ \partial_x \phi \ dx \quad \textit{for all } \phi \in H_3. \tag{48}$$

Furthermore $F \in H_5$,

$$F + AF = G, \tag{49}$$

and a positive constant C (independent of m) exists such that if $G \in H_m$, then $F \in H_{m+4}$ with

$$\sum_{j=0}^{4} [F]_{H_{m+j}} \lesssim \ [G]_{H_m}.$$

The proof of Proposition 4.1 proceeds as follows. We first prove existence and uniqueness of weak solutions by the Riesz representation theorem:

Lemma 4.1 (Weak solution)**.** *For all $G \in H_1$ there exists a unique solution $F \in H_3$ of (48), and*

$$\|F\|_{H_3} \lesssim [G]_{H_1}. \tag{50}$$

Then we prove that (49) holds:

Lemma 4.2 (Strong solution)**.** *For all $G \in H_1$, the solution F of* (48) *is such that $F \in H_5$,* (49) *holds, and*

$$\|F\|_{H_5} \lesssim [G]_{H_1}. \tag{51}$$

Finally, higher regularity follows by iterating the argument for Lemma 4.2. The rest of the section is concerned with the proofs of Lemma 4.1, Lemma 4.2 and Proposition 4.1.

Proof of Lemma 4.1. Clearly, H_3 is a Hilbert space equipped with the inner product

$$(F,G)=\sum_{k=1}^{3}\int_0^\infty x^{k-1}\ \partial_x^k F\ \partial_x^k G\ dx.$$

The form

$$b(F,G)=\int_0^\infty \partial_x F\ \partial_x G\ dx+\frac{1}{2}\int_0^\infty x^2\ \partial_x^3 F\ \partial_x^3 G\ dx$$

is bilinear and symmetric on H_3. By interpolation (see Lemma A.3)

$$\int_0^\infty x\ (\partial_x^2 F)^2\ dx \lesssim \int_0^\infty (\partial_x F)^2\ dx+\int_0^\infty x^2\ (\partial_x^3 F)^2\ dx, \tag{52}$$

hence b is also coercive. Existence and uniqueness now follow from the Riesz representation theorem upon the embedding i of H_1 into the dual space of H_3

$$i: H_1 \to \text{dual space of } H_3,$$

$$\langle i(G),\phi\rangle\ =\ \int_0^\infty \partial_x G\ \partial_x \phi\ dx$$

and (50) follows from (52) and from

$$\begin{aligned}\int_0^\infty (\partial_x F)^2\ dx+\frac{1}{2}\int_0^\infty x^2(\partial_x^3 F)^2\ dx &\overset{(48)}{=} \int_0^\infty \partial_x F\ \partial_x G\ dx\\ &\le \frac{1}{2}\int_0^\infty (\partial_x F)^2\ dx+\frac{1}{2}\int_0^\infty (\partial_x G)^2\ dx.\end{aligned}$$

□

Proof of Lemma 4.2. We set

$$H=x^2\ \partial_x^3 F.$$

Using only $\phi\in C_c^\infty((0,\infty))$ in (48), we gather

$$\partial_x(\partial_x F+\frac{1}{2}\ \partial_x^2 H-\partial_x G)=0$$

in a distributional sense. Since $[F]_{H_1}+[G]_{H_1}<\infty$ this yields $H\in H^2_{loc}([0,\infty))$ and

$$\frac{1}{2}\ \partial_x^2 H=\partial_x G-\partial_x F+C$$

almost everywhere. In particular, the traces $H(0)$ and $\partial_x H(0)$ exist. Since

$$\int_0^\infty \frac{H^2}{x^2}\ dx+\int_0^\infty (\partial_x G)^2\ dx+\int_0^\infty (\partial_x F)^2\ dx<\infty,$$

the constant vanishes:

$$\frac{1}{2}\ \partial_x^2 H=\partial_x G-\partial_x F. \tag{53}$$

We now use $\phi \in C_c^\infty([0,\infty))$ with $\phi(0) = 0$ in (48),

$$\begin{aligned} 0 &= \int_0^\infty \partial_x F \; \partial_x \phi \; dx + \frac{1}{2} \int_0^\infty H \; \partial_x^3 \phi \; dx - \int_0^\infty \partial_x G \; \partial_x \phi \; dx \\ &= \frac{1}{2}(H(0) \; \partial_x^2 \phi(0) - \partial_x H(0) \; \partial_x \phi(0)) + \int_0^\infty (\partial_x F + \partial_x^2 H - \partial_x G) \; \partial_x \phi \; dx \\ &\overset{(53)}{=} \frac{1}{2}(-H(0) \; \partial_x^2 \phi(0) + \partial_x H(0) \; \partial_x \phi(0)), \end{aligned}$$

to derive the Neumann boundary conditions

$$H(0) = \partial_x H(0) = 0. \tag{54}$$

In view of (54), Hardy's inequality in the form of Lemma A.1 yields

$$\begin{aligned} \int_0^\infty \frac{H^2}{x^4} \; dx + \int_0^\infty \frac{(\partial_x H)^2}{x^2} \; dx &\lesssim \int_0^\infty (\partial_x^2 H)^2 \; dx \\ &\overset{(53)}{\lesssim} [G - F]_{H_1}^2 \\ &\overset{(50)}{\lesssim} [G]_{H_1}^2. \end{aligned} \tag{55}$$

Next, observe that $\partial_x^5 F = \partial_x^2 (H/x^2)$, hence

$$\begin{aligned} \int_0^\infty x^4 (\partial_x^5 F)^2 \; dx &= \int_0^\infty x^4 (\partial_x^2 (\frac{H}{x^2}))^2 \; dx \\ &\lesssim \int_0^\infty \left((\partial_x^2 H)^2 + \frac{(\partial_x H)^2}{x^2} + \frac{H^2}{x^4} \right) \; dx \\ &\overset{(55)}{\lesssim} [G]_{H_1}^2. \end{aligned} \tag{56}$$

By interpolation (see Lemma A.3)

$$[F]_{H_4}^2 \lesssim [F]_{H_3}^2 + [F]_{H_5}^2 \overset{(50),(56)}{\lesssim} [G]_{H_1}^2.$$

Therefore $F \in H_5$ and (51) holds. Since $F(0) = G(0) = \partial_x H(0) = 0$ (cf. (54)), we obtain from (53)

$$\partial_x H = 2(G - F),$$

which in view of the definition of H turns into (49), i.e.

$$F + \frac{1}{2} \; \partial_x (x^2 \; \partial_x^3 F) = G.$$

□

Proof of Proposition 4.1. By induction on m we show that

$$G \in H_m \implies F \in H_{m+4} \text{ with } \sum_{j=0}^{4} [F]_{H_{m+j}} \lesssim [G]_{H_m}. \tag{57}$$

By Lemma 4.2, the claim is true for $m = 1$. If $G \in H_m$, $m \geq 2$, then by induction $F \in H_{m+3}$ and

$$\|F\|_{H_{m+3}} \lesssim \|G\|_{H_{m-1}} \leq \|G\|_{H_m}. \tag{58}$$

In a first step we argue qualitatively that $F \in H_{m+4}$. In a second step we will show the estimate in (57). Only in the second step we have to take care that constants do not depend on m.

Let $H = x^{m+1}\ \partial_x^{m+2} F$. We have

$$\begin{aligned} \partial_x^2 H &= \partial_x^2 (x^{m+1}\ \partial_x^{m+2} F) \\ &\overset{(44)}{=} 2\ x^{m-1}\ \partial_x^m AF \\ &\overset{(49)}{=} 2\ x^{m-1}\ \partial_x^m (G - F) \in L^2_{loc}([0,\infty)). \end{aligned} \tag{59}$$

We claim that

$$\exists\ x_n \to 0:\ \ H(x_n) \to 0, \tag{60}$$

$$\exists\ y_n \to 0:\ \ \partial_x H(y_n) \to 0. \tag{61}$$

Claim (60) follows immediately from

$$[F]_{H_{m+2}} < \infty \implies \exists\ x_n \to 0:\ x^{\frac{m+2}{2}}\ \partial_x^{m+2} F \to 0.$$

For (61), assume by contradiction that

$$\liminf_{x \to 0} |\partial_x H| > 0.$$

Since by (58) and (59) $H \in C^{\frac{3}{2}}([0,1])$, we may assume without loss of generality that $C > 0$ and $x_0 \in (0,1)$ exist such that $\partial_x H \geq C$ for $x \in (0, x_0)$. Then, using (60), $H \geq Cx$, that is $\partial_x^{m+2} F \geq Cx^{-m}$. But then, since $m \geq 2$,

$$\int_0^\infty x^{m+1} (\partial_x^{m+2} F)^2\ dx \geq \int_0^{x_0} x^{1-m}\ dx = \infty,$$

a contradiction. Hence (61) holds. In view of (60) and (61), we have by Lemma A.1:

$$\begin{aligned} \int_0^\infty \frac{H^2}{x^{m+3}}\ dx + \int_0^\infty \frac{(\partial_x H)^2}{x^{m+1}}\ dx &\leq C_m \int_0^\infty \frac{(\partial_x^2 H)^2}{x^{m-1}}\ dx \\ &\overset{(59)}{\leq} C_m\ [G - F]^2_{H_m} \overset{(58)}{\leq} C_m\ \|G\|^2_{H_m}. \end{aligned} \tag{62}$$

Therefore

$$\begin{aligned}\int_0^\infty x^{m+3}(\partial_x^{m+4}F)^2\,dx &= \int_0^\infty x^{m+3}(\partial_x^2(\frac{H}{x^{m+1}}))^2\,dx\\ &\le C_m\int_0^\infty \frac{H^2}{x^{m+3}}\,dx + C_m\int_0^\infty \frac{(\partial_x H)^2}{x^{m+1}}\,dx\\ &\quad+\int_0^\infty \frac{(\partial_x^2 H)^2}{x^{m-1}}\,dx\\ &\overset{(62)}{\le} C_m\,[G-F]_{H_m}^2.\end{aligned}$$

Hence $F\in H_{m+4}$ in view of Lemma A.4.

We now turn to the quantitative estimate in (57). In order to complete the proof, by the interpolation estimates in Lemma A.3 it suffices to show that

$$[F]_{H_m}+[F]_{H_{m+2}}\lesssim [G]_{H_m} \tag{63}$$

and

$$[F]_{H_{m+2}}+[F]_{H_{m+4}}\lesssim [G]_{H_m}. \tag{64}$$

For the first one, we differentiate (49) m times and test it with $x^{m-1}\,\partial_x^m F$:

$$[F]_{H_m}^2+\langle AF,F\rangle_{H_m}\overset{(45)}{=}[F]_{H_m}^2+\frac12[F]_{H_{m+2}}^2=\langle G,F\rangle_{H_m}\le\frac12\left([F]_{H_m}^2+[G]_{H_m}^2\right),$$

whence (63). For the second one, we differentiate (49) m times and test it with $x^{m-1}\,\partial_x^m AF$:

$$\langle AF,F\rangle_{H_m}+[AF]_{H_m}^2\overset{(45)}{=}\frac12[F]_{H_{m+2}}^2+[AF]_{H_m}^2=\langle AF,G\rangle_{H_m}\le\frac12\left([AF]_{H_m}^2+[G]_{H_m}^2\right).$$

Hence

$$\frac12\left([F]_{H_{m+2}}^2+[AF]_{H_m}^2\right)\le[G]_{H_m}^2,$$

and (64) follows from (46). □

5. The linear parabolic equation

In this section we prove Proposition 2.1. In fact, Proposition 2.1 follows from an equivalent statement in terms of the weighted spaces X_m, which are defined by

$$\|F\|_{X_{m+2}}:=\sum_{k=3}^m\left([\partial_t F]_{L^2(H_{k-2})}+[F]_{C^0(H_k)}+[F]_{L^2(H_{k+2})}\right), \tag{65}$$

$$X_m:=\begin{cases}\text{Completion of }\{F\in C_c^\infty([0,\infty)^2): F_{|x=0}=0\}\\ \text{with respect to }\|\cdot\|_{X_m}.\end{cases} \tag{66}$$

It reads as follows:

Proposition 5.1. *Let $m \geq 3$. For given $F_0 \in H_m$ and $G \in L^2(H_{m-2})$, there exists a unique solution $F \in X_{m+2}$ of*

$$\begin{cases} \partial_t F + AF = G \\ F_{|t=0} = F_0. \end{cases} \tag{67}$$

It satisfies for all $3 \leq k \leq m$ and all $0 < T \leq \infty$ the estimate

$$\begin{aligned} C^{-1}\ [\partial_t F]_{L^2((0,T);H_{k-2})} + [F]_{C^0((0,T);H_k)} + C^{-1}\ [F]_{L^2((0,T);H_{k+2})} \\ \leq [F_0]_{H_k} + C[G]_{L^2((0,T);H_{k-2})}. \end{aligned} \tag{68}$$

Before proving 5.1 by a time discretization argument, we first show how Proposition 2.1 follows from Proposition 5.1.

Proof of Proposition 2.1. We first prove (37) for $F_0 \in C^\infty([0,\infty))$ with $F(0) = 0$ and $G \in C_c^\infty([0,\infty)^2)$ with $G|_{x=0} = 0$. Let F be the corresponding solution of (67) as given by Proposition 5.1. For any smooth and compactly supported decomposition $F_0 = F_{0-} + F_{0+}$ and $G = G_- + G_+$, let $F_\pm$ be the corresponding solution of (67) with data $F_{0\pm}$, $G_\pm$. Due to the linearity of A we have

$$F = F_+ + F_-. \tag{69}$$

By Proposition 5.1, for every $s > 0$ and every $k \geq 3$ it holds:

$$\begin{aligned} &C^{-1}\left(s^{-1}[\partial_t F_-]_{L^2(H_{k-3})} + s[\partial_t F_+]_{L^2(H_{k-1})}\right) + s^{-1}[F_-]_{C^0(H_{k-1})} + s[F_+]_{C^0(H_{k+1})} \\ &+ C^{-1}\left(s^{-1}[F_-]_{L^2(H_{k+1})} + s[F_+]_{L^2(H_{k+3})}\right) \\ &\qquad \leq s^{-1}[F_{0-}]_{H_{k-1}} + s[F_{0+}]_{H_{k+1}} + C\left(s^{-1}[G_-]_{L^2(H_{k-3})} + s[G_+]_{L^2(H_{k-1})}\right). \end{aligned} \tag{70}$$

Equation (70) is preserved when taking the infimum over all decompositions $F_{0\pm}$ and $G_\pm$ of F_0 and G on both sides of the equation (cf. Lemma A.4). On the other hand, due to (69), the corresponding solutions $F_\pm$ are a subset of arbitrary decompositions of F. Therefore, we arrive at:

$$\begin{aligned} &\inf_{F=F_-+F_+}\left(s^{-1}[\partial_t F_-]_{L^2(H_{k-3})} + s[\partial_t F_+]_{L^2(H_{k-1})}\right) \\ &\quad + \inf_{F=F_-+F_+}\left(s^{-1}[F_-]_{C^0(H_{k-1})} + s[F_+]_{C^0(H_{k+1})}\right) \\ &\quad + \inf_{F=F_-+F_+}\left(s^{-1}[F_-]_{L^2(H_{k+1})} + s[F_+]_{L^2(H_{k+3})}\right) \\ &\lesssim \inf_{F_0=F_{0-}+F_{0+}}\left(s^{-1}[F_{0-}]_{H_{k-1}} + s[F_{0+}]_{H_{k+1}}\right) \\ &\quad + \inf_{G=G_-+G_+}\left(s^{-1}[G_-]_{L^2(H_{k-3})} + s[G_+]_{L^2(H_{k-1})}\right). \end{aligned}$$

Integrating in ds/s yields (37) for all $k \geq 4$. Since in particular $F_0 \in H_{m+1}$ and $G \in L^2(H_{m-1})$, we have due to (68) that $[F]_{X_{m+3}} < \infty$. With the help of (22), one then easily checks that $F \in X^*_{m+2}$.

Arbitrary data $F_0 \in H_m^*$ and $G \in L^2(H_m)^*$ can be approximated by smooth and compactly supported functions $F_{0\nu} \in H_m$ and $G_\nu \in L^2(H_m)$,

$$F_{0\nu} \to F_0 \text{ in } H_m^*, \qquad\qquad G_\nu \to G \text{ in } L^2(H_m)^*.$$

By (37), the corresponding solutions $F_\nu \in X_{m+2}^*$ converge to a solution $F \in X_{m+2}^*$ of (36) with data F_0 and G. Furthermore, F satisfies (37) for all $4 \le k \le m$. □

In order to prove Proposition 5.1, we begin with a discrete counterpart based on Proposition 4.1.

Lemma 5.1. *Let $m \ge 3$, $h > 0$. For all $F_0 \in H_m$ and all $G \in H_{m-2}$ there exists a solution $F \in H_{m+2}$ of*

$$F + h\ AF = F_0 + h\ G. \tag{71}$$

Furthermore, a positive constant C exists such that

$$C^{-1}h\ [\frac{F-F_0}{h}]^2_{H_{k-2}} + [F]^2_{H_k} + C^{-1}h\ [F]^2_{H_{k+2}} \le [F_0]^2_{H_k} + Ch\ [G]^2_{H_{k-2}} \tag{72}$$

for all $3 \le k \le m$.

Proof. The existence of a solution $F \in H_{m+2}$ which satisfies (71) follows by Proposition 4.1 and scaling in x: a solution of (49) with right–hand side $F_0(\hat{x}) + h\ G(\hat{x})$ turns into a solution of (71) by the change of variables $x = \sqrt{h}\ \hat{x}$.

By (71), we have for $3 \le k \le m$

$$\partial_x^{k-2}F + h\ \partial_x^{k-2}AF = \partial_x^{k-2}F_0 + h\ \partial_x^{k-2}G. \tag{73}$$

Testing (73) by $2x^{k-3}\ \partial_x^{k-2}AF$ and integrating, we see that

$$\begin{aligned}
[F]^2_{H_k} + 2h[AF]^2_{H_{k-2}} &\overset{(45)}{=} \langle F, AF\rangle_{H_{k-2}} + 2h\ [AF]^2_{H_{k-2}} \\
&\overset{(73)}{=} 2\ \langle AF, F_0 + h\ G\rangle_{H_{k-2}} \\
&\overset{(45)}{=} \langle F, F_0\rangle_{H_k} + 2h\ \langle AF, G\rangle_{H_{k-2}} \\
&\le \frac{1}{2}\ [F]^2_{H_k} + \frac{1}{2}\ [F_0]^2_{H_k} + h\ [AF]^2_{H_{k-2}} + h\ [G]^2_{H_{k-2}}.
\end{aligned}$$

Therefore, by (46),

$$[F]^2_{H_k} + C^{-1}h\ [F]^2_{H_{k+2}} \le [F_0]^2_{H_k} + h\ [G]^2_{H_{k-2}}. \tag{74}$$

It follows from (71) that

$$(F - F_0)/h = G - AF.$$

Hence

$$\begin{aligned}
h\ [\frac{F-F_0}{h}]^2_{H_{k-2}} &\le 2h\ [G]^2_{H_{k-2}} + 2h\ [AF]^2_{H_{k-2}} \\
&\overset{(46)}{\le} 2h\ [G]^2_{H_{k-2}} + Ch\ [F]^2_{H_{k+2}}.
\end{aligned} \tag{75}$$

Now (72) follows from (74) and (75). □

We turn to the proof of Proposition 5.1:

Proof of Proposition 5.1. We only consider the case $T=\infty$, since the proof directly transfers to arbitrary T. Uniqueness is straightforward: the difference F of two solutions solves (67) with $G=0$ and $F_{|x=0}=0$, and $\partial_x F \in H^1_{loc}([0,\infty)); L^2((0,\infty))$. Therefore, differentiating (67) once, testing it by $\partial_x F$ and integrating, we obtain

$$[F(t)]^2_{H_1} + \int_0^t [F]^2_{H_3}\, dt = 0,$$

whence $F=0$. The rest of the proof is thus concerned with existence.

For a fixed $h>0$ and $j \in \mathbb{N}$, we let $t^h_j := hj$ and

$$\begin{aligned} G^h_j &:= \frac{1}{h}\int_{t^h_j}^{t^h_{j+1}} G(\hat{t},\cdot)\, d\hat{t}, \\ F^h_0 &:= F_0, \\ F^h_{j+1} &:= \begin{cases} \text{solution of (71) with data} \\ F^h_j, G^h_j \text{ as given by Lemma 5.1.} \end{cases} \end{aligned}$$

By (72) we have for $3 \le k \le m$

$$C^{-1}h[\frac{F^h_{j+1}-F^h_j}{h}]^2_{H_{k-2}} + [F^h_{j+1}]^2_{H_k} + C^{-1}h[F^h_{j+1}]^2_{H_{k+2}} \le [F^h_j]^2_{H_k} + Ch[G^h_j]^2_{H_{k-2}}.$$

Summing over j, we obtain

$$C^{-1}h\sum_{j=0}^{\infty}\left([\frac{F^h_{j+1}-F^h_j}{h}]^2_{H_{k-2}} + [F^h_{j+1}]^2_{H_{k+2}}\right) + \sup_j [F^h_{j+1}]^2_{H_k} \le [F_0]^2_{H_k} + C[G]^2_{L^2(H_{k-2})}. \tag{76}$$

Define F_h by

$$F_h = \sum_{j=0}^{\infty}\left(\frac{t^h_{j+1}-t}{h}F^h_j + \frac{t-t^h_j}{h}F^h_{j+1}\right)\chi_{[t^h_j,t^h_{j+1})}$$

and G_h by

$$G_h = \sum_{j=0}^{\infty} G^h_j\ \chi_{[t^h_j,t^h_{j+1})}.$$

Clearly

$$G_h \to G \qquad \text{in } L^2(H_{m-2}).$$

We also have

$$F_{ht} = \frac{F^h_{j+1}-F^h_j}{h} \qquad \text{on } [t^h_j, t^h_{j+1}) \tag{77}$$

and

$$[F_h]_{H_k} \le \max\left\{[F_j^h]_{H_k}, [F_{j+1}^h]_{H_k}\right\} \qquad \text{on } [t_j^h, t_{j+1}^h).$$

Therefore, by (76), we have that

$$C^{-1}[F_{ht}]^2_{L^2(H_{k-2})} + [F_h]^2_{C^0(H_k)} + C^{-1}[F_h]^2_{L^2(H_{k+2})} \le [F_0]^2_{H_k} + C[G]^2_{L^2(H_{k-2})} \tag{78}$$

for all $3 \le k \le m$. With help of the time derivative of F and the initial datum, locally we can also control low semi–norms. For $T > 1$ and $M > 1$, we have:

$$\sup_{t\in(0,T)} [F_h]^2_{H_1} \lesssim [F_0]^2_{H_1} + T\int_0^T [\partial_t F_h]^2_{H_1}\, dt \overset{(78)}{\lesssim} T\left(\|F_0\|^2_{H_m} + \|G\|^2_{L^2(H_{m-2})}\right); \tag{79}$$

using Lemma A.3,

$$\sup_{t\in(0,T)} [F_h]^2_{H_2} \lesssim \sup_{t\in(0,T)} \left([F_h]^2_{H_1} + [F_h]^2_{H_3}\right) \overset{(78),(79)}{\lesssim} T\left(\|F_0\|^2_{H_m} + \|G\|^2_{L^2(H_{m-2})}\right); \tag{80}$$

using that $F(t,0) = 0$,

$$\sup_{t\in(0,T)} \int_0^M F_h^2\, dx \overset{(79)}{\le} M^2 \sup_{t\in(0,T)} [F_h]^2_{H_1} \overset{(79)}{\le} M^2 T\left(\|F_0\|^2_{H_m} + \|G\|^2_{L^2(H_{m-2})}\right) \tag{81}$$

and

$$\int_0^\infty\!\!\int_0^M \partial_t F_h^2\, dx\, dt \le M^2 \int_0^\infty [\partial_t F_h]^2_{H_1}\, dt \overset{(78)}{\le} M^2\left(\|F_0\|^2_{H_m} + \|G\|^2_{L^2(H_{m-2})}\right). \tag{82}$$

Collecting (78)-(82) we conclude that a subsequence exists such that

$$F_h \rightharpoonup F \text{ in } H^1_{loc}([0,\infty)); H^1_{loc}([0,\infty)) \cap L^2_{loc}([0,\infty); H^{m+2}_{loc}([0,\infty)))$$

and (68) holds. Furthermore, (78)-(81) imply that $F \in X_{m+2}$ (note that $F_h \in X_{m+2}$), and the compact embedding $H^1 \subset\subset C$ implies that $F_{|x=0} = F_0$. To prove (67), we note that F_h satisfies the approximate equation

$$\partial_t F_h + A\tilde{F}_h = G_h, \tag{83}$$

where

$$\tilde{F}_h := F_{j+1}^h \qquad \text{on } [t_j^h, t_{j+1}^h). \tag{84}$$

Also $\tilde{F}_h$ is uniformly bounded: by (76) and (84), we have for all $3 \le k \le m$

$$[\tilde{F}_h]^2_{C^0(H_k)} + [\tilde{F}_h]^2_{L^2(H_{k+2})} \lesssim [F_0]^2_{H_k} + [G]^2_{L^2(H_{k-2})}.$$

By (79)-(81), locally in space-time we also have uniform control on low semi–norms of $\tilde{F}_h$ since $\tilde{F}_h(t) = F_h(t_{j+1}^h)$ for $t \in [t_j^h, t_{j+1}^h)$. Therefore

$$\tilde{F}_h \rightharpoonup \tilde{F} \text{ in } L^2_{loc}([0,\infty)); H^1_{loc}([0,\infty)) \cap L^2_{loc}([0,\infty); H^{m+2}_{loc}(0,\infty)),$$

and passing to the limit in (83) we obtain that

$$\partial_t F + A\tilde{F} = G.$$

In order to see that $\tilde{F} = F$, it suffices to notice that for any $M > 0$

$$\begin{aligned}\int_0^\infty\int_0^M |\tilde{F}_h - F_h|^2 \, dx \, dt &= \sum_{j=0}^{\infty} \int_{t_j^h}^{t_{j+1}^h}\int_0^M (t_{j+1}^h - t)^2 \frac{(F_{j+1}^h - F_j^h)^2}{h^2} \, dx \, dt \\ &\overset{(82)}{\le} h^2 \int_0^\infty\int_0^M (\partial_t F_h)^2 \, dx \, dt \\ &\overset{(77)}{\to} 0 \qquad \text{as } h \to 0.\end{aligned}$$

□

6. The main estimate for the nonlinear operator

In this section we prove Proposition 2.2. We restrict ourselves to the case $T = \infty$; the proof directly extends to arbitrary T. The result will be achieved by splitting G into $\tilde{G}$ which additionally satisfies $\partial_x^3 \tilde{G}_{|x=0} = 0$ and a remainder of the form $\partial_x^3 G_{|x=0}(t)\ \eta(x)$, and by splitting F into $\tilde{F}$ which additionally satisfies $\partial_x^2 \tilde{F}_{|x=0} = 0$ and a remainder of the form $\partial_x^2 F_{|x=0}(t)\ \xi(x)$. This way, Proposition 2.2 will follow from the following two lemmata.

Lemma 6.1. *For any given $F, G \in C_c([0,\infty)^2)$ with $F_{|x=0} = 0$ and $\partial_x^3 G_{|x=0} = 0$ we have*

$$[\partial_x \left((F - x\ \partial_x F_{|x=0})\ \partial_x^3 G\right)]_{L^2(H_2)^*} \ \lesssim\ [F]_{C^0(H_4)^*}\ [G]_{L^2(H_6)^*}.$$

Lemma 6.2. *For any given $F, G \in C_c([0,\infty)^2)$ with $F_{|x=0} = 0$ and $\partial_x^2 F_{|x=0} = 0$ we have*

$$[\partial_x \left((F - x\ \partial_x F_{|x=0})\ (\partial_x^3 G - \partial_x^3 G_{|x=0})\right)]_{L^2(H_2)^*} \ \lesssim\ [F]_{C^0(H_4)^*}\ \|x^4\ \partial_x^8 G\|_{L^2(L^1)}.$$

We will need Lemma 1.2 in the following form:

Corollary 6.1. *Let $F \in [F]_{H_4^*}$. Then*

$$\|x^{-2}\ (F - x\ \partial_x F(0))\|_{C^0} + \|x^{-1}\ (\partial_x F - \partial_x F(0))\|_{C^0} + \|\partial_x^2 F\|_{C^0} + \|x\ \partial_x^3 F\|_{C^0} \lesssim [F]_{H_4^*}.$$

Proof. By Lemma 1.2, we have that $\|\partial_x^2 F\|_{C^0} \lesssim [F]_{H_4^*}$. Integrating yields

$$\sup_{\hat{x}\in(0,x)} |\partial_x F - \partial_x F(0)| \le \int_0^x |\partial_x^2 F|\ d\hat{x} \le x\ \|\partial_x^2 F\|_{C^0}, \tag{85}$$

$$\sup_{\hat{x}\in(0,x)} |F - x\ \partial_x F(0)| \le \int_0^x |\partial_x F - \partial_x F(0)|\ d\hat{x} \overset{(85)}{\lesssim} x^2\ \|\partial_x^2 F\|_{C^0}.$$

Finally, by application of Lemma A.2 and Lemma 1.1,

$$\|x\ \partial_x^3 F\|_{C^0} \lesssim [F]_{H_4} \lesssim [F]_{H_4^*}.$$

□

Proof of Lemma 6.1. We fix F and consider the linear map

$$G \quad \mapsto \quad \partial_x \left((F - x\ \partial_x F_{|x=0})\ \partial_x^3 G \right).$$

Since according to Lemma 1.3, $[F]_{C^0(H_4^*)} \le [F]_{C^0(H_4)^*}$, by the definition of the semi-norm $L^2(H_2)^*$ it is enough to show

$$[\partial_x \left((F - x\ \partial_x F_{|x=0})\ \partial_x^3 G \right)]_{L^2(H_1)} \lesssim [F]_{C^0(H_4^*)}\ [G]_{L^2(H_5)},$$
$$[\partial_x \left((F - x\ \partial_x F_{|x=0})\ \partial_x^3 G \right)]_{L^2(H_3)} \lesssim [F]_{C^0(H_4^*)}\ [G]_{L^2(H_7)}.$$

These two estimates can be "disintegrated" in time:

$$[\partial_x \left((F - x\ \partial_x F_{|x=0})\ \partial_x^3 G \right)]_{H_1} \lesssim [F]_{H_4^*}\ [G]_{H_5}, \tag{86}$$
$$[\partial_x \left((F - x\ \partial_x F_{|x=0})\ \partial_x^3 G \right)]_{H_3} \lesssim [F]_{H_4^*}\ [G]_{H_7}, \tag{87}$$

where now we think of F and G as functions of x only.

We start with (86):

$$\begin{aligned}
&[\partial_x \left((F - x\ \partial_x F(0))\ \partial_x^3 G \right)]_{H_1} \\
&\quad = \|\partial_x^2 \left((F - x\ \partial_x F(0))\ \partial_x^3 G \right)\|_{L^2} \\
&\quad \lesssim \|\partial_x^2 F\ \partial_x^3 G\|_{L^2} + \|(\partial_x F - \partial_x F(0))\ \partial_x^4 G\|_{L^2} + \|(F - x\ \partial_x F(0))\ \partial_x^5 G\|_{L^2} \\
&\quad \le \|\partial_x^2 F\|_{C^0}\ \|\partial_x^3 G\|_{L^2} + \|x^{-1}\ (\partial_x F - \partial_x F(0))\|_{C^0}\ \|x\ \partial_x^4 G\|_{L^2} \\
&\qquad + \|x^{-2}\ (F - x\ \partial_x F(0))\|_{C^0}\ \|x^2\ \partial_x^5 G\|_{L^2}.
\end{aligned}$$

It remains to appeal to Corollary 6.1 and to Hardy's inequality (cf. Lemma A.1)

$$\|\partial_x^3 G\|_{L^2} \ \lesssim\ \|x\ \partial_x^4 G\|_{L^2} \ \lesssim\ \|x^2\ \partial_x^5 G\|_{L^2} \ =\ [G]_{H_5}.$$

We now turn to (87):

$$\begin{aligned}
&[\partial_x \left((F - x\ \partial_x F(0))\ \partial_x^3 G \right)]_{H_3} \\
&\quad = \|x\ \partial_x^4 \left((F - x\ \partial_x F(0))\ \partial_x^3 G \right)\|_{L^2} \\
&\quad \lesssim \|x\ \partial_x^4 F\ \partial_x^3 G\|_{L^2} + \|x\ \partial_x^3 F\ \partial_x^4 G\|_{L^2} \\
&\qquad + \|x\ \partial_x^2 F\ \partial_x^5 G\|_{L^2} + \|x\ (\partial_x F - \partial_x F(0))\ \partial_x^6 G\|_{L^2} \\
&\qquad + \|x\ (F - x\ \partial_x F(0))\ \partial_x^7 G\|_{L^2} \\
&\quad \le \|x^{3/2}\ \partial_x^4 F\|_{L^2}\ \|x^{-1/2}\ \partial_x^3 G\|_{C^0} + \|x\ \partial_x^3 F\|_{C^0}\ \|\partial_x^4 G\|_{L^2} \\
&\qquad + \|\partial_x^2 F\|_{C^0}\ \|x\ \partial_x^5 G\|_{L^2} + \|x^{-1}\ (\partial_x F - \partial_x F(0))\|_{C^0}\ \|x^2\ \partial_x^6 G\|_{L^2} \\
&\qquad + \|x^{-2}\ (F - x\ \partial_x F(0))\|_{C^0}\ \|x^3\ \partial_x^7 G\|_{L^2}.
\end{aligned} \tag{88}$$

The last four terms in (88) can be treated as for (86), i. e. with Corollary 6.1 and Hardy's inequality

$$\|\partial_x^4 G\|_{L^2} \ \lesssim\ \|x\ \partial_x^5 G\|_{L^2} \ \lesssim\ \|x^2\ \partial_x^6 G\|_{L^2} \ \lesssim\ \|x^3\ \partial_x^7 G\|_{L^2} \ =\ [G]_{H_7}. \tag{89}$$

The first term in (88) requires a different argument: According to Lemma 1.1 we have

$$\|x^{3/2}\,\partial_x^4 F\|_{L^2} \;=\; [F]_{H_4} \;\lesssim\; [F]_{H_4^*}.$$

Finally, because of our assumption $\partial_x^3 G_{|x=0} = 0$ we have

$$\|x^{-1/2}\,\partial_x^3 G\|_{C^0} \;\leq\; \sup_x x^{-1/2}\int_0^x |\partial_x^4 G| \;\lesssim\; \|\partial_x^4 G\|_{L^2} \;\overset{(89)}{\lesssim}\; [G]_{H_7}.$$

□

Proof of Lemma 6.2. We fix G and consider the linear map

$$F \quad\mapsto\quad \partial_x\left((F - x\,\partial_x F_{|x=0})\,(\partial_x^3 G - \partial_x^3 G_{|x=0})\right).$$

By definition of the semi-norm $L^2(H_m)^*$, it is enough to show:

$$[\partial_x\left((F - x\,\partial_x F_{|x=0})\,(\partial_x^3 G - \partial_x^3 G_{|x=0})\right)]_{L^2(H_1)} \lesssim [F]_{C^0(H_3)}\,\|x^4\,\partial_x^8 G\|_{L^2(L^1)},$$
$$[\partial_x\left((F - x\,\partial_x F_{|x=0})\,(\partial_x^3 G - \partial_x^3 G_{|x=0})\right)]_{L^2(H_3)} \lesssim [F]_{C^0(H_5)}\,\|x^4\,\partial_x^8 G\|_{L^2(L^1)}.$$

These two estimates follow from the corresponding pointwise (in time) ones:

$$[\partial_x\left((F - x\,\partial_x F_{|x=0})\,(\partial_x^3 G - \partial_x^3 G_{|x=0})\right)]_{H_1} \lesssim [F]_{H_3}\,\|x^4\,\partial_x^8 G\|_{L^1}, \tag{90}$$
$$[\partial_x\left((F - x\,\partial_x F_{|x=0})\,(\partial_x^3 G - \partial_x^3 G_{|x=0})\right)]_{H_3} \lesssim [F]_{H_5}\,\|x^4\,\partial_x^8 G\|_{L^1}, \tag{91}$$

where now we think of F and G as functions of x only.

Before proving (90) and (91), we remark that we have

$$\|\partial_x^3 G\|_{C^0} + \|x\,\partial_x^4 G\|_{C^0} + \|x^2\,\partial_x^5 G\|_{C^0} + \|x^3\,\partial_x^6 G\|_{C^0} + \|x^4\,\partial_x^7 G\|_{C^0} \;\lesssim\; \|x^4\,\partial_x^8 G\|_{L^1}. \tag{92}$$

Indeed, if $f = \partial_x^3 G$ and $k \in \{0,1,2,3,4\}$, we have

$$x^k\,\partial_x^k f(x) \;=\; (-1)^{k+1}\,x^k \int_x^\infty \partial_x^5 f(x')\,\frac{1}{(4-k)!}(x-x')^{4-k}\,dx',$$

so that

$$\begin{aligned}|x^k\,\partial_x^k f(x)| &\lesssim \int_x^\infty |\partial_x^5 f(x')|\,x^k\,|x-x'|^{4-k}\,dx'\\ &\leq \int_x^\infty |\partial_x^5 f(x')|\,x'^4\,dx' \;\leq\; \|x^4\,\partial_x^5 f\|_{L^1}.\end{aligned}$$

We now turn to (90). We have:

$$\begin{aligned}
&[\partial_x \left((F - x\ \partial_x F(0))\ (\partial_x^3 G - \partial_x^3 G(0))\right)]_{H_1} \\
&\quad = \|\partial_x^2 \left((F - x\ \partial_x F(0))\ (\partial_x^3 G - \partial_x^3 G(0))\right)\|_{L^2} \\
&\quad \lesssim \|\partial_x^2 F\ (\partial_x^3 G - \partial_x^3 G(0))\|_{L^2} + \|(\partial_x F - \partial_x F(0))\ \partial_x^4 G\|_{L^2} \\
&\qquad + \|(F - x\ \partial_x F(0))\ \partial_x^5 G\|_{L^2} \\
&\quad \le 2\ \|\partial_x^2 F\|_{L^2}\ \|\partial_x^3 G\|_{C^0} + \|x^{-1}\ (\partial_x F - \partial_x F(0))\|_{L^2}\ \|x\ \partial_x^4 G\|_{C^0} \\
&\qquad + \|x^{-2}\ (F - x\ \partial_x F(0))\|_{L^2}\ \|x^2\ \partial_x^5 G\|_{C^0}.
\end{aligned}$$

This estimate implies (90) because of (92) and of Corollary 6.1.

We finally address (91).

$$\begin{aligned}
&[\partial_x \left((F - x\ \partial_x F(0))\ (\partial_x^3 G - \partial_x^3 G(0))\right)]_{H_3} \\
&\quad = \|x\ \partial_x^4 \left((F - x\ \partial_x F(0))\ (\partial_x^3 G - \partial_x^3 G(0))\right)\|_{L^2} \\
&\quad \lesssim \|x\ \partial_x^4 F\ (\partial_x^3 G - \partial_x^3 G(0))\|_{L^2} + \|x\ \partial_x^3 F\ \partial_x^4 G\|_{L^2} + \|x\ \partial_x^2 F\ \partial_x^5 G\|_{L^2} \\
&\qquad + \|x\ (\partial_x F - \partial_x F(0))\ \partial_x^6 G\|_{L^2} + \|x\ (F - x\ \partial_x F(0))\ \partial_x^7 G\|_{L^2} \\
&\quad \le 2\ \|x\ \partial_x^4 F\|_{L^2}\ \|\partial_x^3 G\|_{C^0} + \|\partial_x^3 F\|_{L^2}\ \|x\ \partial_x^4 G\|_{L^2} + \|x^{-1}\ \partial_x^2 F\|_{L^2}\ \|x^2\ \partial_x^5 G\|_{L^2} \\
&\qquad + \|x^{-2}\ (\partial_x F - \partial_x F(0))\|_{L^2}\ \|x^3\ \partial_x^6 G\|_{C^0} + \|x^{-3}\ (F - x\ \partial_x F(0))\|_{L^2}\ \|x^4\ \partial_x^7 G\|_{C^0} \\
&\quad \overset{(92)}{\lesssim} \|x^4 \partial_x^8 G\|_{L^1}\ \left(\|x\ \partial_x^4 F\|_{L^2} + \|\partial_x^3 F\|_{L^2} + \|x^{-1}\ \partial_x^2 F\|_{L^2}\right. \\
&\qquad \left. + \|x^{-2}\ (\partial_x F - \partial_x F(0))\|_{L^2} + \|x^{-3}\ (F - x\ \partial_x F(0))\|_{L^2}\right).
\end{aligned}$$

Because of $F(0) = \partial_x^2 F(0) = 0$, $F - x\ \partial_x F(0)$ vanishes to second order in $x = 0$; hence, by Hardy's inequality (cf. Corollary A.1) we have

$$\begin{aligned}
\|x^{-3}\ (F - x\ \partial_x F(0))\|_{L^2} \ &\lesssim\ \|x^{-2}\ (\partial_x F - \partial_x F(0))\|_{L^2} \ \lesssim\ \|x^{-1}\ \partial_x^2 F\|_{L^2} \\
&\lesssim\ \|\partial_x^3 F\|_{L^2} \ \lesssim\ \|x^2\ \partial_x^5 F\|_{L^2} \ =\ [F]_{H_5}.
\end{aligned}$$

and (91) follows. □

We are now ready to prove Proposition 2.2.

Proof of Proposition 2.2. We first assume that $F, G \in C_c^\infty([0,\infty))$ with $F|_{x=0} = G|_{x=0} = 0$. We fix two functions $\xi, \eta \in C_c^\infty([0,\infty))$ of the spatial variable only with

$$\begin{aligned}
\xi(0) \ &=\ \partial_x \xi(0) \ =\ 0, & \partial_x^2 \xi(0) \ &=\ 1, \\
\eta(0) \ &=\ \partial_x \eta(0) \ =\ \partial_x^2 \eta(0) \ =\ 0, & \partial_x^3 \eta(0) \ &=\ 1.
\end{aligned} \tag{93}$$

We use these function to split G and F into

$$F \ =\ \tilde{F} \ +\ \partial_x^2 F_{|x=0} \ \otimes\ \xi, \quad G \ =\ \tilde{G} \ +\ \partial_x^3 G_{|x=0} \ \otimes\ \eta.$$

Because of (93), we have $\partial_x^3 \tilde G_{|x=0} = 0$, so that we may apply Lemma 6.1 to the couple $(F, \tilde G)$. Likewise, we have $\partial_x^2 \tilde F_{|x=0} = 0$, so that we may apply Lemma 6.2 to the couple $(\tilde F, \partial_x^3 G_{|x=0} \otimes \eta)$. This yields

$$\begin{aligned}
&[\mathcal{N}(F,G)]_{L^2(H_2)^*} \\
&\quad\lesssim [\mathcal{N}(F,\tilde G)]_{L^2(H_2)^*} + [\mathcal{N}(\tilde F, \partial_x^3 G_{|x=0} \otimes \eta)]_{L^2(H_2)^*} \\
&\qquad + [\mathcal{N}(\partial_x^2 F_{|x=0} \otimes \xi, \partial_x^3 G_{|x=0} \otimes \eta)]_{L^2(H_2)^*} \\
&\quad\lesssim [F]_{C^0(H_4)^*} [\tilde G]_{L^2(H_6)^*} + [\tilde F]_{C^0(H_4)^*} \|x^4 \partial_x^8 (\partial_x^3 G_{|x=0} \otimes \eta)\|_{L^2(L^1)} \\
&\qquad + [(\partial_x^2 F_{|x=0} \partial_x^3 G_{|x=0}) \otimes \mathcal{N}(\xi,\eta)]_{L^2(H_2)^*} \\
&\quad\lesssim [F]_{C^0(H_4)^*} \big([G]_{L^2(H_6)^*} + [\partial_x^3 G_{|x=0} \otimes \eta]_{L^2(H_6)^*}\big) \\
&\qquad + \big([F]_{C^0(H_4)^*} + [\partial_x^2 F_{|x=0} \otimes \xi]_{C^0(H_4)^*}\big) \|\partial_x^3 G_{|x=0} \otimes (x^4 \partial_x^8 \eta)\|_{L^2(L^1)} \\
&\qquad + [(\partial_x^2 F_{|x=0} \partial_x^3 G_{|x=0}) \otimes \mathcal{N}(\xi,\eta)]_{L^2(H_2)^*}.
\end{aligned}$$

We now appeal to part ii) of Lemma 1.3:

$$\begin{aligned}
&[\mathcal{N}(F,G)]_{L^2(H_2)^*} \\
&\quad\lesssim [F]_{C^0(H_4)^*} \big([G]_{L^2(H_6)^*} + \|\partial_x^3 G_{|x=0}\|_{L^2} [\eta]_{H_6^*}\big) \\
&\qquad + \big([F]_{C^0(H_4)^*} + \|\partial_x^2 F_{|x=0}\|_{C^0} [\xi]_{H_4^*}\big) \|\partial_x^3 G_{|x=0}\|_{L^2} \|x^4 \partial_x^8 \eta\|_{L^1} \\
&\qquad + \|\partial_x^2 F_{|x=0}\|_{C^0} \|\partial_x^3 G_{|x=0}\|_{L^2} [\mathcal{N}(\xi,\eta)]_{H_2^*} \\
&\quad\lesssim [F]_{C^0(H_4)^*} \big([G]_{L^2(H_6)^*} + \|\partial_x^3 G_{|x=0}\|_{L^2}\big) \\
&\qquad + \big([F]_{C^0(H_4)^*} + \|\partial_x^2 F_{|x=0}\|_{C^0}\big) \|\partial_x^3 G_{|x=0}\|_{L^2} + \|\partial_x^2 F_{|x=0}\|_{C^0} \|\partial_x^3 G_{|x=0}\|_{L^2}.
\end{aligned}$$

We now evoke Lemma 1.2:

$$\begin{aligned}
&[\mathcal{N}(F,G)]_{L^2(H_2)^*} \\
&\quad\lesssim [F]_{C^0(H_4)^*} \big([G]_{L^2(H_6)^*} + [G]_{L^2(H_6^*)}\big) \\
&\qquad + \big([F]_{C^0(H_4)^*} + [F]_{C^0(H_4^*)}\big) [G]_{L^2(H_6^*)} + [F]_{C^0(H_4^*)} [G]_{L^2(H_6^*)}.
\end{aligned}$$

We conclude the proof of estimate (38) for smooth F and G using part i) of Lemma 1.3. By density, (38) holds for all $F, G \in X_6^*$. Finally, since $\mathcal{N}$ is an operator which maps $C_c^\infty([0,\infty))^2 \to C_c^\infty([0,\infty))$, we obtain that $\mathcal{N}$ maps $X_6^* \to L^2(H_2)^*$, and therefore $\mathcal{N}(F,G) \in L^2(H_2)^*$. □

7. Existence

In this section we complete the proof of Theorem 1.1. Uniqueness and (31) have already been shown in Section 2. Hence, we are left to prove:

Proposition 7.1. *There exists an $\epsilon > 0$ s.t. for all $F_0 \in H_6^*$ satisfying (30) there exists $F \in X_6^*$ which solves (13).*

Proof. For $\delta > 0$ to be chosen later, let

$$X = \{ \ F \in X_6^* : \|F\|_{X_6^*} \le \delta \ \} \tag{94}$$

and define

$$S(F) := L^* \mathcal{N}(F, F), \tag{95}$$

where L^*G is the unique solution $F \in X_6^*$ of (36) with initial datum F_0 as given by Proposition 2.1:

$$\partial_t S(F) + A(S(F)) = \mathcal{N}(F, F).$$

Hence, in order to prove Proposition 7.1 it suffices to show that S has a fixed point in X. In fact, we shall prove that S is a contraction in X. By Proposition 2.1, there holds:

$$\|S(F)\|_{X_6^*} \lesssim [F_0]_{H_4^*} + [\mathcal{N}(F, F)]_{L^2(H_2)^*}. \tag{96}$$

Furthermore, since the difference of two functions $F, \tilde{F} \in X$ satisfies

$$\partial_t(S(F) - S(\tilde{F})) + A(S(F) - S(\tilde{F})) = \mathcal{N}(F, F) - \mathcal{N}(\tilde{F}, \tilde{F}), \quad (F - \tilde{F})(0, x) = 0,$$

again by Proposition 2.1 and the definition (34) of $\mathcal{N}$ we see that

$$\begin{aligned} \|S(F) - S(\tilde{F})\|_{X_6^*} &\le [\mathcal{N}(F, F) - \mathcal{N}(\tilde{F}, \tilde{F})]_{L^2(H_2)^*} \\ &\lesssim [\mathcal{N}(F, F - \tilde{F})]_{L^2(H_2)^*} + [\mathcal{N}(F - \tilde{F}, \tilde{F})]_{L^2(H_2)^*}. \end{aligned} \tag{97}$$

We now argue as in the proof of Proposition 2.3: by Proposition 2.2, we have $\mathcal{N}(F, G) \in L^2(H_2)^*$ and

$$[\mathcal{N}(F, F)]_{L^2(H_2)^*} \lesssim \|F\|_{X_6^*}^2.$$

Therefore for $F \in X$

$$\|SF\|_{X_6^*} \overset{(96)}{\lesssim} \epsilon + \delta^2. \tag{98}$$

Note that, as a consequence of (98), $S(0) \in X$ for $\epsilon \ll \delta$, hence X is non–empty. In view of (97) and Proposition 10.1 we get

$$\begin{aligned} \|SF - S\tilde{F}\|_{X_6^*} &\lesssim C \ (\|F\|_{X_6^*} + \|\tilde{F}\|_{X_6^*}) \ \|F - \tilde{F}\|_{X_6^*}. \\ &\lesssim C \ \delta \ \|F - \tilde{F}\|_{X_6^*}. \end{aligned} \tag{99}$$

Choosing $\delta = \sqrt{\epsilon}$ and ϵ sufficiently small, (98) and (99) turn into

$$\|SF\|_{X_6^*} \le \delta,$$

$$\|SF - S\tilde{F}\|_{X_6^*} \le \frac{1}{2} \|F - \tilde{F}\|_{X_6^*},$$

and the proof is complete. □

8. Regularity

In this section we prove Theorem 1.2. We begin with higher–order estimates for the nonlinear operator $\mathcal{N}$, defined in (34). The key point of the next proposition is, that the constant in front of the highest order term $[F]_{H_{k+2}}$ does not depend on k:

Proposition 8.1. *Let $k \geq 5$ and $F \in H_{k+2}$. Then*

$$[\mathcal{N}(F,F)]_{H_{k-2}} \leq \; C\; [F]_{H_4^*}[F]_{H_{k+2}} + C_k' \; [F]_{H_6^*}[F]_{H_k}. \tag{100}$$

Proof. There holds

$$\begin{aligned}
[\mathcal{N}(F,F)]_{H_{k-2}} &= \|x^{\frac{k-3}{2}} \; \partial_x^{k-1} \big((F - x\; \partial_x F(0))(\partial_x^3 F - \partial_x^3 F(0)\big) \, \|_{L^2}. \\
&\leq \|x^{\frac{k-3}{2}} \; \partial_x^{k-1} F \; (\partial_x^3 F - \partial_x^3 F(0))\|_{L^2} \\
&\quad + \sum_{j=0}^{k-2} \tbinom{k-1}{j} \|x^{\frac{k-3}{2}} \; \big(\partial_x^j (F - x\; \partial_x F(0)) \; \partial_x^{k+2-j} F\big) \, \|_{L^2}.
\end{aligned}$$

We have by $\mathcal{N}$ by Lemma 1.2 and Corollary A.1:

$$\|x^{\frac{k-3}{2}} \; \partial_x^{k-1} F \; (\partial_x^3 F - \partial_x^3 F(0))\|_{L^2} \lesssim [F]_{H_6^*} \; \|x^{\frac{k-3}{2}} \; \partial_x^{k-1} F\|_{L^2} \lesssim [F]_{H_6^*}[F]_{H_k}.$$

Therefore it remains to estimate

$$\tbinom{k-1}{j} \|x^{\frac{k-3}{2}} \; \partial_x^j (F - x\; \partial_x F(0)) \; \partial_x^{k+2-j} F\|_{L^2}, \qquad j = 0, \ldots, k-2.$$

For $0 \leq j \leq 3$ we apply Corollary 6.1 and Corollary A.1 to get

$$\begin{aligned}
&\tbinom{k-1}{j} \|x^{\frac{k-3}{2}} \; \partial_x^j (F - x\; \partial_x F(0)) \; \partial_x^{k+2-j} F\|_{L^2} \\
&\lesssim k^j \; \|x^{j-2} \; \partial_x^j (F - x\; \partial_x F(0))\|_{C^0} \; \|x^{\frac{k+1-2j}{2}} \; \partial_x^{k+2-j} F\|_{L^2} \\
&\lesssim \; [F]_{H_4^*} \|x^{\frac{k+1}{2}} \; \partial_x^{k+2} F\|_{L^2} \\
&= [F]_{H_4^*}[F]_{H_{k+2}}
\end{aligned}$$

(note that the constant in Hardy's inequality ensures that the estimate is independent of k). This already proves (100) for $k = 5$. For $k \geq 6$ and $4 \leq j \leq k-2$, we estimate, using Lemma A.1, Lemma A.2 and Corollary A.2,

$$\begin{aligned}
&\tbinom{k-1}{j} \|x^{\frac{k-3}{2}} \; \partial_x^j (F - x\; \partial_x F(0)) \; \partial_x^{k+2-j} F\|_{L^2} \\
&\quad \leq C_k \; \|x^{\frac{j-2}{2}} \; \partial_x^j F\|_{C^0} \; \|x^{\frac{k-1-j}{2}} \; \partial_x^{k+2-j} F\|_{L^2} \\
&\quad \leq C_k \; \|x^{\frac{j-1}{2}} \; \partial_x^{j+1} F\|_{L^2} \|x^{\frac{k-1-j}{2}} \; \partial_x^{k+2-j} F\|_{L^2} \\
&\quad \leq C_k \; [F]_{H_{j+2}}[F]_{H_{k+4-j}} \\
&\quad \leq C_k \; [F]_{H_6}[F]_{H_k} \\
&\quad \overset{(22)}{\leq} \; C_k \; [F]_{H_6^*}[F]_{H_k}.
\end{aligned}$$

□

To the proof of Theorem 1.2 we premise the following intermediate result:

Proposition 8.2. *There exists an* $\epsilon > 0$ *such that if* $F_0 \in H_m$, $m \geq 5$, *satisfies* (30)*, then the solution* F *of* (13) *given by Theorem 1.1 is such that* $F \in X_{m+2}$ *(see* (66)*), and furthermore*

$$[F]_{C^0(H_k)} + [F]_{L^2(H_{k+2})} \leq C_k \, \|F_0\|_{H_k} \qquad \text{for all } 5 \leq k \leq m.$$

Proof. Since F has been obtained in Theorem 1.1 as the unique fixed point of the map S (see (95)) on X (see (94)), by Proposition 5.1 the sequence $F^{(n)}$ defined as the unique solution of

$$\begin{cases} \partial_t F^{(n+1)} + AF^{(n+1)} = \mathcal{N}(F^{(n)}, F^{(n)}) \\ F^{(n)}_{|x=0} = F_0 \end{cases}$$

(with, say, $F^{(0)} = 0$) satisfies $F \in X_{m+2}$ and converges to F in X. Hence, Proposition 8.2 follows immediately by dominated convergence once we have shown that

$$[\partial_t F^{(n+1)}]_{L^2(H_{k-2})} + [F^{(n+1)}]_{C^0(H_k)} + [F^{(n+1)}]_{L^2(H_{k+2})} \leq C_k \, \|F_0\|_{H_k} \tag{101}$$

for all $5 \leq k \leq m$ and all n sufficiently large (cf. Lemma A.4). To see (101), we write:

$$\begin{aligned} &[\partial_t F^{(n+1)}]_{L^2(H_{k-2})} + [F^{(n+1)}]_{C^0(H_k)} + [F^{(n+1)}]_{L^2(H_{k+2})} \\ &\overset{(68)}{\lesssim} [F_0]_{H_k} + [\mathcal{N}(F^{(n)}, F^{(n)})]_{L^2(H_{k-2})} \\ &\overset{(100)}{\lesssim} [F_0]_{H_k} + [F^{(n)}]_{C^0(H_4^*)}[F^{(n)}]_{L^2(H_{k+2})} + C_k' \, \|[F^{(n)}]_{H_6^*}[F^{(n)}]_{H_k}\|_{L^2}. \end{aligned} \tag{102}$$

We first assume that $k \leq 6$, hence in this case the constant C_k' is universal. We use Cauchy Schwarz in the form

$$\|[F^{(n)}]_{H_6^*}[F^{(n)}]_{H_k}\|_{L^2} \leq [F^{(n)}]_{L^2(H_6^*)}[F^{(n)}]_{C^0(H_k)}.$$

Since $\|F^{(n)} - F\|_X \to 0$, using (30) and (31) we may absorb the last two terms on the right hand side for n sufficiently large and ϵ sufficiently small, thus getting (101) for $5 \leq k \leq 6$:

$$[\partial_t F^{(n+1)}]_{L^2(H_{k-2})} + [F^{(n+1)}]_{C^0(H_k)} + [F^{(n+1)}]_{L^2(H_{k+2})} \lesssim [F_0]_{H_k}, \quad 5 \leq k \leq 6. \tag{103}$$

For $k \geq 6$ we only absorb the first term on the right hand side of (102), whereas for the second one we use Cauchy–Schwarz in the form

$$\|[F^{(n)}]_{H_6^*}[F^{(n)}]_{H_k}\|_{L^2} \leq [F^{(n)}]_{C^0(H_6^*)}[F^{(n)}]_{L^2(H_k)}$$

and arrive for n sufficiently large at

$$\begin{aligned}
&[\partial_t F^{(n+1)}]_{L^2(H_{k-2})} + [F^{(n+1)}]_{C^0(H_k)} + [F^{(n+1)}]_{L^2(H_{k+2})} \\
&\overset{(100)}{\lesssim} [F_0]_{H_k} + C_k' \, [F^{(n)}]_{C^0(H_6^*)}[F^{(n)}]_{L^2(H_k)}. \\
&\overset{(103)}{\lesssim} [F_0]_{H_k} + C_k' \, [F^{(n)}]_{L^2(H_k)}.
\end{aligned}$$

A straightforward induction on k starting from (103) concludes the proof. □

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Fix $\eta > 0$, and let $\tau_n \uparrow \eta$. In view of (31), (28) and (22), there exists $t_1 \in (0, \tau_1)$ such that $[F(t_1)]_{H_6} < \infty$. Choosing $F(t_1)$ as an initial datum in Proposition 8.2, we obtain in particular that

$$\int_{t_1}^{\infty} [F]_{H_8}^2 \, dt < \infty.$$

Hence, there exists $t_2 \in (\tau_1, \tau_2)$ such that $[F]_{H_8}^2 < \infty$. Iterating this argument, we conclude that $F(\cdot + \eta, \cdot) \in X_{m+2}$ for all m. A reiterated application of Corollary A.1 then implies that $\partial_x F(\cdot + \eta, \cdot) \in C^0([0, \infty), H^s([0, \infty))$ for all s. Regularity in time then follows by differentiating the equation, and the arbitrariness of η completes the proof. □

9. Decay of high derivatives

In this section we prove Theorem 1.3. We shall use the following:

Lemma 9.1. *Let $k \geq 3$. If $F \in X_{k+2}$, then $[F]_{H_k}^2 \in W_{loc}^{1,1}([0, \infty))$ and*

$$\partial_t [F]_{H_k}^2 + [F]_{H_{k+2}}^2 = 4\langle AF, \partial_t F + AF\rangle_{H_{k-2}}. \tag{104}$$

Proof. The lemma is not trivial since a priori there is no control on $\partial_x^k \partial_t F$. However its proof is standard, and we sketch it for completeness. For $T > 0$ fixed, let $\phi \in C_c^\infty([0, \infty))$ such that $\phi = 1$ on $(0, T)$ and let

$$\tilde{F} = \begin{cases} \phi(t)F(t) & t \geq 0 \\ \phi(-t)F(-t) & t < 0. \end{cases}$$

Let η be a mollifier, $\eta_\epsilon(t) = \epsilon^{-1}\eta(t\epsilon^{-1})$, and $\tilde{F}_\epsilon = \eta_\epsilon * \tilde{F}$. As is well known,

$$\begin{cases} [\tilde{F}_\epsilon]_{H_j} \;\to\; [\tilde{F}]_{H_j} \quad \text{in } L^2(\mathbb{R}) \quad \text{for } 1 \leq j \leq k+2, \\ [\partial_t \tilde{F}_\epsilon]_{H_j} \;\to\; [\partial_t \tilde{F}]_{H_j} \quad \text{in } L^2(\mathbb{R}) \quad \text{for } 1 \leq j \leq k-2. \end{cases} \tag{105}$$

Since each $\tilde{F}_\epsilon$ is smooth in time, we have:

$$[\tilde{F}_\epsilon(t_2)]_{H_k}^2 - [\tilde{F}_\epsilon(t_1)]_{H_k}^2 = 2\int_{t_1}^{t_2} \langle \tilde{F}_\epsilon, \partial_t \tilde{F}_\epsilon\rangle_{H_k} \, dt \overset{(45)}{=} 4\int_{t_1}^{t_2} \langle A\tilde{F}_\epsilon, \partial_t \tilde{F}_\epsilon\rangle_{H_{k-2}} \, dt. \tag{106}$$

Let $\epsilon, \delta > 0$. Choosing $t = t_2$ and $t_1 = -\infty$ in (106), we see that

$$[\tilde F_\epsilon(t)]_{H_k}^2 - [\tilde F_\delta(t)]_{H_k}^2 = 4\int_{-\infty}^t \Big(\langle A\tilde F_\epsilon, \partial_t \tilde F_\epsilon - \partial_t \tilde F_\delta\rangle_{H_{k-2}} + \langle A\tilde F_\epsilon - A\tilde F_\delta, \partial_t \tilde F_\delta\rangle_{H_{k-2}}\Big)\ dt$$
$$\overset{(46)}{\le} \left(\int_{-\infty}^\infty \Big([\tilde F_\epsilon - \tilde F_\delta]_{H_{k+2}}^2 + [\partial_t \tilde F_\epsilon - \partial_t \tilde F_\delta]_{H_{k-2}}^2\Big)\ dt\right)^{\frac12}$$
$$\left(\int_{-\infty}^\infty \Big([\tilde F_\epsilon]_{H_{k+2}}^2 + [\partial_t \tilde F_\delta]_{H_{k-2}}^2\Big)\ dt\right)^{\frac12}$$
$$\overset{(105)}{=} o_{\delta,\epsilon}(1).$$

Hence, $[\tilde F_\epsilon]_{H_k}^2$ is a Cauchy sequence in $C([0,T])$, and therefore $[\tilde F_\epsilon]_{H_k} \to [\tilde F]_{H_k}$ in $C([0,T])$ (the identification of the limit follows from (105)). Since $\tilde F = F$ in $[0,T]$ and T is arbitrary, this proves the continuity of $[F]_{H_k}^2$. Passing to the limit in (106) we see that for $0 \le t < t_2$

$$\frac{[F(t_2)]_{H_k}^2 - [F(t_1)]_{H_k}^2}{t_2 - t_1} = \frac{4}{t_2 - t_1}\int_{t_1}^{t_2} \langle AF, \partial_t F\rangle_{H_{k-2}}\ dt.$$

Passing to the limit as $t_2 \to t$ and using (46) we complete the proof. □

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. The starting point is:

$$\partial_t [F]_{H_k}^2 + [F]_{H_{k+2}}^2 \overset{(104)}{=} 4\langle AF, \mathcal N(F,F)\rangle_{H_{k-2}}$$
$$\overset{(46)}{\le} \frac12 [F]_{H_{k+2}}^2 + 2\ [\mathcal N(F,F)]_{H_{k-2}}^2$$
$$\overset{(100)}{\le} \frac12 [F]_{H_{k+2}}^2 + C\ [F]_{H_4^*}^2 [F]_{H_{k+2}}^2 + C_k'\ [F]_{H_6^*}^2 [F]_{H_k}^2,$$

which holds for all $t > 0$ and all $k \ge 5$. Therefore

$$\partial_t [F]_{H_k}^2 + [F]_{H_{k+2}}^2 \le C\ [F]_{H_4^*}^2 [F]_{H_{k+2}}^2 + C_k'\ [F]_{H_6^*}^2 [F]_{H_k}^2 \tag{107}$$

Note that the constant in the first term on the right-hand side of (107) does not depend on k, and that by Theorem 1.1 and (28), $\sup_t [F]_{H_4^*} \lesssim \epsilon$. Therefore, for ϵ sufficiently small we arrive at

$$\partial_t [F]_{H_k}^2 + [F]_{H_{k+2}}^2 \le C_k'\ [F]_{H_6^*}^2 [F]_{H_k}^2. \tag{108}$$

Bringing this together with the weights in time, as in the statement of the theorem, we get:

$$\partial_t (t^{\frac{k-4}{2}} [F]_{H_k}^2) + t^{\frac{k-4}{2}} [F]_{H_{k+2}}^2 = t^{\frac{k-4}{2}} (\partial_t [F]_{H_k}^2 + [F]_{H_{k+2}}^2) + C_k\ t^{\frac{k-6}{2}} [F]_{H_k}^2$$
$$\overset{(108)}{\lesssim} C_k'\ t^{\frac{k-4}{2}} [F]_{H_6^*}^2 [F]_{H_k}^2 + C_k\ t^{\frac{k-6}{2}} [F]_{H_k}^2. \tag{109}$$

In the case $k = 6$ this turns by interpolation (cf. Corollary A.2) into

$$\partial_t(t[F]^2_{H_6}) + t[F]^2_{H_8} \lesssim t[F]^2_{H_6^*}[F]^2_{H_6} + [F]^2_{H_6} \lesssim t[F]^2_{H_4}[F]^2_{H_8} + [F]^2_{H_6}.$$

For $\epsilon \ll 1$ we absorb on the left hand side and integrate in time. This yields

$$\sup_t \; t[F]^2_{H_6} + \int_0^\infty t[F]^2_{H_8}\; dt \lesssim \int_0^\infty [F]^2_{H_6}. \overset{(31)}{\lesssim} [F_0]^2_{H_4^*}. \tag{110}$$

The same argument, using also (110), yields the analogous statement for $k = 8$:

$$\sup_t \; t^2[F]^2_{H_8} + \int_0^\infty t^2[F]^2_{H_{10}}\; dt \lesssim [F_0]^2_{H_4^*}. \tag{111}$$

Interpolation between (31) and (111), using Lemma 1.1, yields

$$\sup_t \; t[F]^2_{H_6^*} + \int_0^\infty t[F]^2_{H_8^*}\; dt \lesssim [F_0]^2_{H_4^*}. \tag{112}$$

For arbitrary k, this argument would break down due to the k dependence of the constants. We instead integrate (109) directly to obtain

$$\begin{aligned}\sup_t \; (t^{\frac{k-4}{2}}[F]^2_{H_k}) + \int_0^\infty t^{\frac{k-4}{2}}[F]^2_{H_{k+2}}\; dt &\le C_k' \int_0^\infty t^{\frac{k-4}{2}}[F]^2_{H_6^*}[F]^2_{H_k}\; dt + C_k \int_0^\infty t^{\frac{k-6}{2}}[F]^2_{H_k} \\ &\overset{(112)}{\le} C_k \int_0^\infty t^{\frac{k-6}{2}}[F]^2_{H_k}\; dt.\end{aligned}$$

The last equation is the basis for an induction argument, starting from $k = 6$, which yields for all even $k \ge 4$

$$\sup_t \; (t^{\frac{k-4}{2}}[F]^2_{H_k}) + \int_0^\infty t^{\frac{k-4}{2}}[F]^2_{H_{k+2}}\; dt \le C_k\; [F_0]^2_{H_4^*}. \tag{113}$$

Interpolation between the case $k = 4$ (cf. (31)) and (113) yields (33) for all interpolation norms in between, and completes the proof of Theorem 1.3. □

10. Convergence to the steady state

In this section we prove Theorem 1.4. We premise another short lemma about the nonlinear part of the operator:

Proposition 10.1. *Let $F \in H_6^*$. Then*

$$[\mathcal{N}(F, F)]_{H_1} \lesssim [F]_{H_3}[F]_{H_6^*}. \tag{114}$$

Proof. We write using Lemma 1.2, Corollary 6.1 and Corollary A.1:

$$
\begin{aligned}
[\mathcal{N}(F,F)]_{H_1} &= \|\partial_x^2\left((F - x\,\partial_x F(0))(\partial_x^3 F - \partial_x^3 F(0))\right)\|_{L^2} \\
&\lesssim \|\partial_x^2 F\ (\partial_x^3 F - \partial_x^3 F(0))\|_{L^2} + \|(\partial_x F - \partial_x F(0))\ \partial_x^4 F\|_{L^2} \\
&\quad + \|(F - x\,\partial_x F(0))\ \partial_x^5 F\|_{L^2} \\
&\le \|\partial_x^2 F\|_{L^2}\|\partial_x^3 F - \partial_x^3 F(0)\|_{C^0} \\
&\quad + \|x^{-1}\ (\partial_x F - \partial_x F(0))\|_{C^0}\ \|x\ \partial_x^4 F\|_{L^2} \\
&\quad + \|x^{-2}\ (F - x\,\partial_x F(0))\|_{C^0}\|x^2\ \partial_x^5 F\|_{L^2} \\
&\le [F]_{H_3}[F]_{H_6^*} + [F]_{H_4^*}[F]_{H_5} \\
&\le [F]_{H_3}[F]_{H_6^*}.
\end{aligned}
$$

□

Proof of Theorem 1.4. We first show that

$$
\sup_t\ [F]_{H_1}^2 + \int_0^\infty [F]_{H_3}^2\ dt \lesssim [F_0]_{H_1}^2. \tag{115}
$$

We write using Cauchy–Schwarz

$$
\begin{aligned}
\partial_t [F]_{H_1}^2 &\overset{(104)}{=} 2\langle F, \partial_t F\rangle_{H_1} \\
&= 2\langle F, -AF + \mathcal{N}(F,F)\rangle_{H_1} \\
&\overset{(45)}{\le} -[F]_{H_3}^2 + 2[F]_{H_1}[\mathcal{N}(F,F)]_{H_1} \\
&\overset{(114)}{\le} -[F]_{H_3}^2 + C\ [F]_{H_1}[F]_{H_3}[F]_{H_6^*} \\
&\le -\frac{1}{2}[F]_{H_3}^2 + C\ [F]_{H_1}^2[F]_{H_6^*}^2.
\end{aligned} \tag{116}
$$

Therefore

$$
\begin{aligned}
\log\left(\frac{[F(t)]_{H_1}^2}{[F_0]_{H_1}^2}\right) &\lesssim \int_0^t [F]_{H_6^*}^2\ dt \\
&\overset{(31)}{\lesssim} [F_0]_{H_4^*}^2,
\end{aligned}
$$

which implies that $[F]_{H_1} \lesssim [F_0]_{H_1}$. Inserting this information into (116) we obtain (115). By an analogous argument as in the proof of Theorem 1.3 this leads to

$$
\begin{aligned}
&\sup_t \left(t^{\frac{k-1}{2}}[F]_{H_k}^2\right) + \int_0^\infty t^{\frac{k-1}{2}}[F]_{H_{k+2}}^2\ dt \\
&\lesssim\ C_k' \int_0^\infty t^{\frac{k-1}{2}}[F]_{H_6^*}^2[F]_{H_k}^2\ dt + C_k \int_0^\infty t^{\frac{k-3}{2}}\ [F]_{H_k}^2\ dt \\
&\overset{(112)}{\lesssim}\ C_k \int_0^\infty t^{\frac{k-3}{2}}[F]_{H_k}^2\ dt
\end{aligned}
$$

for every $k \geq 1$. An induction argument as in the proof of Theorem 1.3 completes the proof. □

Appendix: Weighted Sobolev spaces and their interpolation spaces

Hardy inequality. The basic tool for the weighted Sobolev spaces is the Hardy inequality, introduced in [17] (see also the detailed survey for inequalities in weighted spaces in [19]). Let us mention that similar weighted spaces and tools are also used in [23]. For the convenience of the reader we derive the form of Hardy inequality as we need it:

Lemma A.1 (Hardy inequality). *Let $k \neq -1$. Assume that $F \in H^1_{loc}((0,\infty))$ is such that*

$$\|x^{(k+2)/2}\ \partial_x F\|_{L^2} < \infty$$

and

$$\exists \alpha_n \downarrow 0 : \ F(\alpha_n) \to 0 \qquad \text{if } k < -1,$$
$$\exists \beta_n \uparrow \infty : \ F(\beta_n) \to 0 \qquad \text{if } k > -1.$$

Then

$$\|x^{k/2}\ F\|_{L^2} \leq \frac{2}{k+1}\ \|x^{(k+2)/2}\ \partial_x F\|_{L^2}. \tag{A.1}$$

Proof. First we observe that if $k < -1$, then

$$\begin{aligned} F(x) &= F(\alpha_n) + \int_{\alpha_n}^{x} \partial_x F\ dx \\ &\leq o_n(1) + \left(\int_{\alpha_n}^{x} x^{k+2}(\partial_x F)^2\ dx\right)^{\frac{1}{2}} \left(\int_{\alpha_n}^{x} x^{-k-2}\ dx\right)^{\frac{1}{2}}. \end{aligned}$$

Hence, passing to the limit as $n \uparrow \infty$,

$$F(x) = x^{-(k+1)/2}\ o(1) \qquad \text{as } x \to 0.$$

If $k > -1$, the same argument applied to $F(1/x)$ yields that

$$F(x) = x^{-(k+1)/2}\ o(1) \qquad \text{as } x \to \infty.$$

Let now $0 < \alpha < \beta$. Taking also into account the sign of one of the boundary terms, we have:

$$\begin{aligned} \int_{\alpha}^{\beta} x^k F^2\ dx &= \left[\frac{1}{k+1} x^{k+1} F^2\right]_{\alpha}^{\beta} - \frac{2}{k+1} \int_{\alpha}^{\beta} x^{k+1} F\ \partial_x F\ dx \\ &\leq o(1) + \frac{1}{2} \int_{\alpha}^{\beta} x^k F^2\ dx + \frac{2}{(k+1)^2} \int_0^{\infty} x^{k+2} (\partial_x F)^2\ dx \quad \text{as } \alpha, \frac{1}{\beta} \to 0, \end{aligned}$$

and Lemma A.1 follows by monotone convergence. □

It follows immediately from Lemma A.1 that:

Corollary A.1. *Let $k \neq -1$. Assume that $F \in H^1_{loc}((0,\infty))$ is such that*

$$\|x^{(k+2)/2}\, \partial_x F\|_{L^2} < \infty$$

and

$$\begin{aligned} \|x^{-1/2} F\|_{L^2((0,1))} &< \infty && \text{if } k < -1, \\ \|x^{-1/2} F\|_{L^2((1,\infty))} &< \infty && \text{if } k > -1. \end{aligned}$$

Then (A.1) *holds.*

Hardy inequality implies the following supremum estimates:

Lemma A.2. *Let $k \geq 1$. Assume $F \in H^1_{loc}((0,\infty))$ is such that*

$$\|x^{(k-1)/2}\, F\|_{L^2} + \|x^{(k+1)/2}\, \partial_x F\|_{L^2} < \infty.$$

Then

$$\|x^{k/2} F\|_{C^0} \lesssim \|x^{(k+1)/2}\, \partial_x F\|_{L^2}.$$

Proof. The integrability of F at $x = 0$ implies that a sequence $\alpha_n \to 0$ exists such that $\alpha_n^k F^2(\alpha_n) \to 0$. Therefore

$$\begin{aligned} \sup_{x \in (\alpha_n, \infty)} x^k F^2 &\leq o_n(1) + k \int_0^\infty x^{k-1} F^2 \, dx + 2 \int_0^\infty x^k F \, \partial_x F \, dx \\ &\leq o_n(1) + (k+1) \int_0^\infty x^{k-1} F^2 \, dx + \int_0^\infty x^{k+1} (\partial_x F)^2 \, dx \\ &\overset{(A.1)}{\lesssim} o_n(1) + \int_0^\infty x^{k+1} (\partial_x F)^2 \, dx. \end{aligned}$$

Passing to the limit as $n \to \infty$ completes the proof. □

Interpolation inequalities for H_m and H_m^*. We derive the interpolation inequalities in two different forms. In the first lemma we use less assumptions on the integrability of the estimated function. We also use general weights. Secondly, we give the proof of Lemma 1.1.

Lemma A.3 (Interpolation inequality)**.** *Let $k \geq 0$. A universal constant C exists such that for all $F \in H^2_{loc}((0,\infty))$ such that*

$$\int_0^\infty x^k F^2 \, dx + \int_0^\infty x^{k+2} (\partial_x^2 F)^2 \, dx < \infty,$$

there holds:

$$\int_0^\infty x^{k+1} (\partial_x F)^2 \, dx \leq C \left(\int_0^\infty x^k F^2 \, dx \right)^{\frac{1}{2}} \left(\int_0^\infty x^{k+2} (\partial_x^2 F)^2 \, dx \right)^{\frac{1}{2}}.$$

Proof. We claim that

$$\exists \beta_n \to \infty : \partial_x F(\beta_n) \to 0. \tag{A.2}$$

If not, then we would have $|\partial_x F(x)| \geq C$ for $x \in (x_0, \infty)$ for some $C > 0$ and $x_0 > 0$. By the continuity of $\partial_x F$, without loss of generality we would have $\partial_x F \geq C$, and therefore $F(x) \geq F(x_0) + Cx$, in (x_0, ∞), in contradiction with the integrability assumption of F. Hence (A.2) holds, and by Lemma A.1 we obtain that

$$\int_0^\infty x^k (\partial_x F)^2 \, dx \lesssim (k+1)^{-2} \int_0^\infty x^{k+2} (\partial_x^2 F)^2 \, dx. \tag{A.3}$$

We have for $0 < \alpha < \beta$:

$$\begin{aligned}
\int_\alpha^\beta x^{k+1} (\partial_x F)^2 \, dx &= - \int_\alpha^\beta x^{k+1} \; F \; \partial_x^2 F \, dx \\
&\quad - (k+1) \int_\alpha^\beta x^k F \; \partial_x F \, dx + [x^{k+1} F \; \partial_x F]_\alpha^\beta \\
&\overset{(A.3)}{\lesssim} \left(\int_\alpha^\beta x^k F^2 \, dx \right)^{\frac{1}{2}} \left(\int_\alpha^\beta x^{k+2} (\partial_x^2 F)^2 \, dx \right)^{\frac{1}{2}} + [x^{k+1} F \; \partial_x F]_\alpha^\beta.
\end{aligned}$$

It remains to show that sequences $\alpha_n \to 0, \beta_n \to \infty$ exist such that

$$[x^{k+1} F \; \partial_x F]_{\alpha_n}^{\beta_n} = o_n(1).$$

For β_n, assume on the contrary that C and x_0 exist such that

$$C \leq x^{k+1} |F \; \partial_x F| \leq x^{k+1} F^2 + x^{k+1} (\partial_x F)^2 \qquad \text{for all } x > x_0.$$

Then

$$\infty = \int_{x_0}^\infty \frac{C}{x} \, dx \leq \int_{x_0}^\infty x^k F^2 + x^k (\partial_x F)^2 \, dx,$$

in contradiction with (A.3). The argument for α_n is identical. □

We have also been using the following corollary of Lemma A.3:

Corollary A.2. *Let $1 \leq l < k < m$. There exists a constant C_{lm} such that for all $F \in H_m$, there holds:*

$$[F]_{H_k} \leq C_{lm} \; [F]_{H_l}^{\frac{m-k}{m-l}} [F]_{H_m}^{\frac{k-l}{m-l}} \tag{A.4}$$

and

$$[F]_{H_k^*} \leq C_{lm} \; [F]_{H_l}^{\frac{m-k}{m-l}} [F]_{H_m}^{\frac{k-l}{m-l}}. \tag{A.5}$$

Proof. (A.4) follows by repeated application of Lemma A.3, and (A.5) follows from (A.4) and Lemma 1.1. □

We now prove Lemma 1.1.

Proof of Lemma 1.1. The proof of

$$[F]_{H_m^*} \lesssim [F]_{H_{m-1}}^{1/2}\, [F]_{H_{m+1}}^{1/2}. \tag{A.6}$$

is straightforward. Decompose

$$F = F\,\chi_{(0,s^*)} + F\,\chi_{(s^*,\infty)} =: F_- + F_+.$$

Using this decomposition in (19) and optimizing in s^* yields (A.6).

We turn to the proof of

$$[F]_{H_m} \lesssim [F]_{H_m^*}. \tag{A.7}$$

The first step is to argue that

$$[F]_{H_m^*} \gtrsim \left(\int_0^\infty \inf_{F=F_-+F_+} \left(s^{-1}\,[F_-]_{H_{m-1}}^2 + s\,[F_+]_{H_{m+1}}^2\right) \frac{ds}{s}\right)^{1/2}. \tag{A.8}$$

Hence in terms of

$$K^2(s) := \inf_{F=F_-+F_+} \left(s^{-1}\,[F_-]_{H_{m-1}}^2 + s\,[F_+]_{H_{m+1}}^2\right),$$

we have to show

$$\left(\int_0^\infty K^2(s)\,\frac{ds}{s}\right)^{1/2} \lesssim \int_0^\infty K(s)\,\frac{ds}{s}.$$

This follows from the stronger statement that

$$\sup_{s\in(0,\infty)} K(s) \lesssim \int_0^\infty K(s)\,\frac{ds}{s},$$

which in turn follows from

$$K(s) \lesssim K(s') \qquad \text{for } \frac{s}{2} \le s' \le s, \tag{A.9}$$

since

$$K(s) = \frac{2}{s}\int_{s/2}^s K(s)\,ds' \overset{(A.9)}{\lesssim} \frac{1}{s}\int_{s/2}^s K(s')\,ds' \lesssim \int_{s/2}^s K(s')\,\frac{ds'}{s'}.$$

Inequality (A.9) can be seen as follows:

$$\begin{aligned} K(s) &= \inf_{F=F_-+F_+} \left(s^{-1}\,[F_-]_{H_{m-1}}^2 + s\,[F_+]_{H_{m+1}}^2\right) \\ &\le \inf_{F=F_-+F_+} \left(s'^{-1}\,[F_-]_{H_{m-1}}^2 + 2s'\,[F_+]_{H_{m+1}}^2\right) \\ &\le 2\,K(s') \quad \text{for } s' \le s \le 2s'. \end{aligned}$$

The second step is to argue that

$$\frac{\pi}{2}\,\langle F, A^{\frac{m-1}{2}}\,F\rangle_{H_1} = \left(\int_0^\infty \inf_{F=F_-+F_+} \left(s^{-1}\,[F_-]_{H_{m-1}}^2 + s\,[F_+]_{H_{m+1}}^2\right) \frac{ds}{s}\right)^{1/2}. \tag{A.10}$$

By density, we may assume that $F \in C_c^\infty([0,\infty))$ with $F(0) = 0$. In view of (47), we have

$$\inf_{F=F_-+F_+} \left(s^{-1}\ [F_-]^2_{H_{m-1}} + s\ [F_+]^2_{H_{m+1}}\right)$$
$$= \inf_{F=F_-+F_+} \left(s^{-1}\ \langle F_-, A^{\frac{m-2}{2}}\ F_-\rangle_{H_1} + s\ \langle F_+, A^{\frac{m}{2}}\ F_+\rangle_{H_1}\right).$$

The minimization can now be carried out explicitly. The minimizers are given by

$$F_- = s^2 A\ (\mathrm{id} + s^2 A)^{-1}\ F, \qquad\qquad F_+ = (\mathrm{id} + s^2 A)^{-1}\ F$$

(the invertibility of $I + s^2 A$ follows from Proposition 4.1), so that

$$\inf_{F=F_-+F_+} \left(s^{-1}\ \langle F_-, A^{\frac{m-2}{2}}\ F_-\rangle_{H_1} + s\ \langle F_+, A^{\frac{m}{2}}\ F_+\rangle_{H_1}\right)$$
$$= s\ \langle F, A^{\frac{m}{2}}\ (\mathrm{id} + s^2\ A)^{-1}\ F\rangle_{H_1}.$$

Hence,

$$\int_0^\infty \inf_{F=F_-+F_+} \left(s^{-1}\ [F_-]^2_{H_{m-1}} + s\ [F_+]^2_{H_{m+1}}\right)\ \frac{ds}{s}$$
$$= \langle F, A^{\frac{m-2}{2}} \int_0^\infty (\mathrm{id} + s^2\ A)^{-1}\ AF\ ds\rangle_{H_1}$$
$$= \frac{\pi}{2}\langle F, A^{\frac{m-2}{2}}\ A^{\frac{1}{2}}\ F\rangle_{H_1}.$$

We have used the following representation formula for $A^{\frac{1}{2}}$:

$$A^{\frac{1}{2}} F = \int_0^\infty (\mathrm{id} + s^2\ A)^{-1}\ AF\ ds, \tag{A.11}$$

which holds in view of Proposition 4.1 (see [4]). Now (A.7) follows from (47):

$$[F]^2_{H_m} \overset{(47)}{=} C\ \langle F, A^{\frac{m-1}{2}} F\rangle_{H_1} \overset{(A.8),(A.10)}{\lesssim} [F]^2_{H_m^*}.$$

□

Proof of Lemma 1.2. We first appeal to Hardy's inequality

$$[F]_{H_{m-1}} = \|x^{\frac{m-2}{2}}\ \partial_x^{m-1} F\|_{L^2} \gtrsim \|\partial_x^{\frac{m}{2}} F\|_{L^2},$$
$$[F]_{H_{m+1}} = \|x^{\frac{m}{2}}\ \partial_x^{m+1} F\|_{L^2} \gtrsim \|\partial_x^{\frac{m}{2}+1} F\|_{L^2},$$

to obtain

$$[F]_{H_m^*} \gtrsim \int_0^\infty \left(\inf_{F=F_-+F_+} \left(s^{-1}\ \|\partial_x^{\frac{m}{2}} F_-\|^2_{L^2} + s\ \|\partial_x^{\frac{m}{2}+1} F_+\|^2_{L^2}\right)\right)^{1/2}\ \frac{ds}{s}.$$

It is convenient to introduce $f = \partial_x^{\frac{m}{2}} F$. Because of the above, it is enough to show

$$\|f\|_{C^0} \lesssim \int_0^\infty \left(\inf_{f=f_-+f_+} \left(s^{-1}\ \|f_-\|^2_{L^2} + s\ \|\partial_x f_+\|^2_{L^2}\right)\right)^{1/2}\ \frac{ds}{s}. \tag{A.12}$$

By even reflection, we may prove (A.12) for functions f on the real line instead of the half–line. This allows as to use the Fourier transform $\hat f$. Because of $\sup_x |f| \lesssim \int_{-\infty}^{\infty} |\hat f|\ dk$, it is enough to show

$$\int_{-\infty}^{\infty} |\hat f|\ dk \ \lesssim\ \int_0^{\infty} \left(\inf_{\hat f = \hat f_- + \hat f_+} \left(s^{-1} \int_{-\infty}^{\infty} |\hat f_-|^2\ dk + s \int_{-\infty}^{\infty} k^2\ |\hat f_+|^2\ dk \right) \right)^{1/2} \frac{ds}{s}. \tag{A.13}$$

The minimizer on the right-hand side can be explicitly computed to be $\hat f_+ = (1 + s^2 k^2)^{-1} \hat f$, so that

$$\begin{aligned} &\inf_{\hat f = \hat f_- + \hat f_+} \left(s^{-1} \int_{-\infty}^{\infty} |\hat f_-|^2\ dk + s \int_{-\infty}^{\infty} k^2\ |\hat f_+|^2\ dk \right) \\ &\quad = \int_{-\infty}^{\infty} \inf_{\hat f = \hat f_- + \hat f_+} \left(s^{-1}\ |\hat f_-|^2\ + s\ k^2\ |\hat f_+|^2 \right)\ dk \\ &\quad = \int_{-\infty}^{\infty} \frac{s\ k^2}{1 + s^2\ k^2}\ |\hat f|^2\ dk. \end{aligned}$$

Hence we obtain in particular

$$\begin{aligned} &\int_0^{\infty} \left(\inf_{\hat f = \hat f_- + \hat f_+} \left(s^{-1} \int_{-\infty}^{\infty} |\hat f_-|^2\ dk + s \int_{-\infty}^{\infty} k^2\ |\hat f_+|^2\ dk \right) \right)^{1/2} \frac{ds}{s} \\ &\quad \gtrsim \int_0^{\infty} \left(s^{-1} \int_{s^{-1}}^{2\ s^{-1}} |\hat f|^2\ dk \right)^{1/2} \frac{ds}{s}. \end{aligned} \tag{A.14}$$

On the other hand, we have by Cauchy–Schwarz

$$\begin{aligned} \int_{-\infty}^{\infty} |\hat f|\ dk \ &\sim\ \int_0^{\infty} \int_{s^{-1}}^{2\ s^{-1}} |\hat f|\ dk\ \frac{ds}{s} \\ &\lesssim \int_0^{\infty} \left(s^{-1} \int_{s^{-1}}^{2\ s^{-1}} |\hat f|^2\ dk \right)^{1/2} \frac{ds}{s}. \end{aligned} \tag{A.15}$$

Hence (A.13) follows from (A.14) and (A.15). □

We turn to the proof of Lemma 1.3:

Proof of Lemma 1.3. We start with part i). It is convenient to introduce the abbreviations

$$\begin{aligned}
K(s,t) &:= \inf_{F(t)=F_-+F_+} \left(s^{-1}\,[F_-]^2_{H_{m-1}} + s\,[F_+]^2_{H_{m+1}}\right)^{1/2},\\
K_2(s) &:= \inf_{F=F_-+F_+} \left(s^{-1}\,[F_-]^2_{L^2(H_{m-1})} + s\,[F_+]^2_{L^2(H_{m+1})}\right)^{1/2}\\
&= \inf_{F=F_-+F_+} \|\left(s^{-1}\,[F_-(\cdot)]^2_{H_{m-1}} + s\,[F_+(\cdot)]^2_{H_{m+1}}\right)^{1/2}\|_{L^2}\\
&= \|\inf_{F(\cdot)=F_-+F_+} \left(s^{-1}\,[F_-]^2_{H_{m-1}} + s\,[F_+]^2_{H_{m+1}}\right)^{1/2}\|_{L^2}\\
&= \|K(s,\cdot)\|_{L^2}, \qquad\qquad (A.16)\\
K_\infty(s) &:= \inf_{F=F_-+F_+} \left(s^{-1}\,[F_-]^2_{C^0(H_{m-1})} + s\,[F_+]^2_{C^0(H_{m+1})}\right)^{1/2}\\
&\geq \inf_{F=F_-+F_+} \|\left(s^{-1}\,[F_-(\cdot)]^2_{H_{m-1}} + s\,[F_+(\cdot)]^2_{H_{m+1}}\right)^{1/2}\|_{C^0}\\
&= \|\inf_{F(\cdot)=F_-+F_+} \left(s^{-1}\,[F_-]^2_{H_{m-1}} + s\,[F_+]^2_{H_{m+1}}\right)^{1/2}\|_{C^0}\\
&= \|K(s,\cdot)\|_{C^0}. \qquad\qquad (A.17)
\end{aligned}$$

We now obtain by the triangle inequality in L^2 and L^∞ respectively:

$$\begin{aligned}
[F]_{L^2(H_m)^*} &= \int_0^\infty K_2(s)\,\frac{ds}{s}\\
&\overset{(A.16)}{=} \int_0^\infty \|K(s,\cdot)\|_{L^2}\,\frac{ds}{s}\\
&\geq \|\int_0^\infty K(s,\cdot)\,\frac{ds}{s}\|_{L^2}\\
&= [F]_{L^2(H_m^*)},\\
[F]_{C^0(H_m)^*} &= \int_0^\infty K_\infty(s)\,\frac{ds}{s}\\
&\overset{(A.17)}{\geq} \int_0^\infty \|K(s,\cdot)\|_{C^0}\,\frac{ds}{s}\\
&\geq \|\int_0^\infty K(s,\cdot)\,\frac{ds}{s}\|_{C^0}\\
&= [F]_{C^0(H_m^*)}.
\end{aligned}$$

We turn to part ii) and fix $\alpha(t)$. We consider the linear operator $\zeta \mapsto \alpha \otimes \zeta$. Then the inequalities follow from interpolating the standard estimates

$$\begin{aligned}
[\alpha\otimes\zeta]_{L^2(H_{m-1})} &\leq \|\alpha\|_{L^2}\,[\zeta]_{H_{m-1}},\\
[\alpha\otimes\zeta]_{L^2(H_{m+1})} &\leq \|\alpha\|_{L^2}\,[\zeta]_{H_{m+1}}
\end{aligned}$$

and

$$[\alpha \otimes \zeta]_{C^0(H_{m-1})} \leq \|\alpha\|_{C^0} \, [\zeta]_{H_{m-1}},$$
$$[\alpha \otimes \zeta]_{C^0(H_{m+1})} \leq \|\alpha\|_{C^0} \, [\zeta]_{H_{m+1}}$$

respectively. □

Density & non–density. In (16), we have defined H_m as completion of

$$\mathcal{D} = \{C_c^\infty([0,\infty)) : F(0) = 0\}$$

with respect to $\|\cdot\|_{H_m}$. Similarly, in (21), we have defined H_m^* as completion of $\mathcal{D}$ with respect to $\|\cdot\|_{H_m^*}$. For the convenience of the reader we show in the next two lemmata that

$$H_m = \{F \in H_{loc}^m : F(0) = 0, \|F\|_{H_m} < \infty\}, \tag{A.18}$$

but

$$H_m^* \subsetneqq \{F \in H_{loc}^m : F(0) = 0, \|F\|_{H_m^*} < \infty\}. \tag{A.19}$$

For (A.18), we have to prove:

Lemma A.4. *For all $m \geq 1$,*

$\mathcal{D}$ is dense in $W_m := \{F \in H_{loc}^m : F(0) = 0, \|F\|_{H_m} < \infty\}$ with respect to $\|\cdot\|_{H_m}$.

Proof. Locally, density for standard Sobolev spaces translates directly to weighted norms, i.e.

$$\mathcal{D}_0 := C^\infty((0,\infty)) \cap W_m \qquad \text{is dense in } W_m \text{ with respect to } \|\cdot\|_{H_m}.$$

Therefore it suffices to consider $F \in \mathcal{D}_0$. We first show that

$$\mathcal{D}_1 := C^\infty([0,\infty)) \cap W_m \qquad \text{is dense in } \mathcal{D}_0 \text{ with respect to } \|\cdot\|_{H_m}. \tag{A.20}$$

Define for all $\delta > 0$

$$F_\delta(x) := \int_0^x \partial_{\hat{x}} F(\hat{x} + \delta) \, d\hat{x}.$$

Of course $F_\delta \in \mathcal{D}_1$ and for all $1 \leq k \leq m$

$$\lim_{\delta \to 0} \int_a^{\frac{1}{a}} x^{k-1} \, (\partial_x^k F_\delta - \partial_x^k F)^2 \, dx = 0. \tag{A.21}$$

for any $a > 0$. On the other hand

$$\limsup_{\delta \to 0} \int_0^a x^{k-1} (\partial_x^k F_\delta - \partial_x^k F)^2 \, dx \leq 2 \limsup_{\delta \to 0} \int_0^{a+\delta} x^{k-1} (\partial_x^k F)^2 \, dx = o_a(1) \tag{A.22}$$

and

$$\limsup_{\delta \to 0} \int_{\frac{1}{a}}^\infty x^{k-1} (\partial_x^k F_\delta - \partial_x^k F)^2 \, dx \leq 2 \limsup_{\delta \to 0} \int_{\frac{1}{a}}^\infty x^{k-1} (\partial_x^k F)^2 \, dx = o_a(1). \tag{A.23}$$

Convergence of $F_\delta \to F$ in W_m follows from (A.21), (A.22), (A.23). Hence (A.20) holds, and it remains to show that

$$\mathcal{D} \text{ is dense in } \mathcal{D}_1 \text{ with respect to } \|\cdot\|_{H_m}.$$

Note that, since $F(0) = 0$,

$$\int_0^\infty \frac{F^2}{x^2}\, dx \lesssim \int_0^\infty (\partial_x F)^2\, dx \qquad \text{(by Lemma A.1).} \tag{A.24}$$

Let η be a cut–off function s.t. $\eta = 0$ on $(0, 1)$ and $\eta = 1$ on $(2, \infty)$, and $\eta_R(x) = \eta(x/R)$. Letting $F_R := (1 - \eta_R)F$, for every $R > 1$ and every $1 \le k \le m$ we have:

$$\begin{aligned}
[F - F_R]^2_{H_k} &= \int_0^\infty x^{k-1}(\partial_x^{k-1}(\eta_R \partial_x F))^2\, dx \\
&\le \int_0^\infty x^{k-1}\eta_R^2\, (\partial_x^k F)^2\, dx + C_k \sum_{j=1}^{k-1} \int_0^\infty x^{k-1}(\partial_x^{k-j}\eta_R)^2(\partial_x^j F)^2\, dx \\
&\quad + \int_0^\infty x^{k-1}(\partial_x^k \eta_R)^2 F^2\, dx
\end{aligned}$$

Since $\partial_x^{k-j}\eta_R \lesssim R^{j-k}\text{supp}(\partial_x^{k-j}\eta_R)$, it follows that

$$\begin{aligned}
[F - F_R]^2_{H_k} &\le \int_R^\infty x^{k-1}(\partial_x^k F)^2\, dx + C_k \sum_{j=1}^{k-1} \int_R^{2R} x^{k-1}R^{2j-2k}(\partial_x^j F)^2\, dx \\
&\quad + \int_R^{2R} x^{k-1}R^{-2k}F^2\, dx \\
&\overset{R>1}{\le} C_k \sum_{j=1}^{k} \int_R^\infty x^{j-1}(\partial_x^j F)^2\, dx + \int_R^{2R} x^{-2}F^2\, dx \\
&\overset{\text{(A.24)}}{\le} C_k \sum_{j=1}^{k} \int_R^\infty x^{j-1}(\partial_x^j F)^2\, dx \\
&\to 0 \qquad\qquad \text{for } R \to \infty.
\end{aligned}$$

This concludes the proof. □

The statement (A.19) is a consequence of

Lemma A.5. *For all even $m \ge 2$ there holds*

$$\|x\|_{H_m^*} < \infty, \qquad \textit{but} \qquad x \notin H_m^*.$$

Proof. It follows directly from Lemma 1.2 that $x \notin H_m^*$, since

$$1 = \|\partial_x x\|_{C^0} \overset{(23)}{\lesssim} [F]_{H_2^*},$$

and therefore x cannot be approximated in H_2^* by functions with compact support.

In order to prove the first claim we note that

$$[x]_{H_1} = \infty, \qquad\qquad [x]_{H_k} = 0 \text{ for all } k \geq 2.$$

Therefore $[x]_{H_k^*} = 0$ for all $k \geq 3$ and it remains to prove that $[x]_{H_2^*} < \infty$. By (21), it is enough to find a decomposition $x = F_- + F_+$ which ensures finiteness of $[x]_{H_2^*}$. Let η be a cut–off function s.t. $\eta = 0$ on $(0,1)$ and $\eta = 1$ on $(2,\infty)$. We decompose

$$x = \frac{x\ \eta(x/s)}{1+\ln^2 s} + x\ (1 - \frac{\eta(x/s)}{1+\ln^2 s}) =: F_- + F_+,$$

A straightforward calculation and using (21) yields $[x]_{H_2^*} < \infty$. □

Continuity in X_6^*. We conclude the Appendix by pointing out simple properties of X_6^*.

Lemma A.6. $X_6^* \subset C([0,\infty); H_4^*)$. *In particular, for any* $F \in X_6^*$ *the trace* $F|_{t=0}$ *is well defined in* H_4^*. *In addition, for any* $F \in X_6^*$ *the function*

$$\phi(T) = [\partial_t F]_{L^2((0,T);H_2)^*} + [F]_{C^0((0,T);H_4)^*} + [F]_{L^2((0,T);H_6)^*}$$

is continuous in $[0,\infty)$ *with* $\phi(0) = [F|_{t=0}]_{H_4^*}$.

Proof. By translation invariance, it is enough to show continuity of $t \mapsto F(t)$ in H_4^* at $t = 0$. Let $F_\nu \in C_c^\infty([0,\infty)^2)$ such that $\|F - F_\nu\|_{X_6^*} \to 0$. Then, for a given $\epsilon > 0$ there exists $\nu^* \in \mathbb{N}$ such that $\sup_t [F(t) - F_{\nu^*}(t)]_{H_4^*} < \epsilon/4$, and since F_{ν^*} is smooth, there exists a $\delta > 0$ s.t. $[F_{\nu^*}(t) - F_{\nu^*}(s)]_{H_4^*} < \epsilon/2$ for all $0 < s < t < \delta$. Hence,

$$[F(t) - F(s)]_{H_4^*} < \frac{\epsilon}{2} + [F_{\nu^*}(t) - F_{\nu^*}(s)]_{H_4^*} < \epsilon \qquad \text{for all } 0 < s < t < \delta.$$

The completeness of H_4^* now implies that F is continuous in this space at $t = 0$. The second statement follows by the same argument, noting that

$$[F]_{L^2((0,T);H_k)^*} \lesssim [F]^{1/2}_{L^2((0,T);H_{k-1})} [F]^{1/2}_{L^2((0,T);H_{k+1})} \tag{A.25}$$

for all $k \geq 2$ ((A.25) is an easy generalization of (A.6) which we leave to the reader). □

References

1. Sigurd Angenent, *Analyticity of the interface of the porous media equation after the waiting time*, Proc. Amer. Math. Soc. **102** (1988), no. 2, 329–336. MR MR920995 (89f:35103)
2. ______, *Local existence and regularity for a class of degenerate parabolic equations*, Math. Ann. **280** (1988), no. 3, 465–482. MR MR936323 (89e:35072)
3. Lidia Ansini and Lorenzo Giacomelli, *Doubly nonlinear thin-film equations in one space dimension*, Arch. Ration. Mech. Anal. **173** (2004), no. 1, 89–131. MR MR2073506 (2005e:35106)
4. A. V. Balakrishnan, *Fractional powers of closed operators and the semigroups generated by them*, Pacific J. Math. **10** (1960), 419–437. MR MR0115096 (22 #5899)

5. Elena Beretta, Michiel Bertsch, and Roberta Dal Passo, *Nonnegative solutions of a fourth-order nonlinear degenerate parabolic equation*, Arch. Rational Mech. Anal. **129** (1995), no. 2, 175–200. MR MR1328475 (96b:35116)
6. Francisco Bernis and Avner Friedman, *Higher order nonlinear degenerate parabolic equations*, J. Differential Equations **83** (1990), no. 1, 179–206. MR MR1031383 (91c:35078)
7. A. L. Bertozzi and M. Pugh, *The lubrication approximation for thin viscous films: regularity and long-time behavior of weak solutions*, Comm. Pure Appl. Math. **49** (1996), no. 2, 85–123. MR MR1371925 (97b:35114)
8. Michiel Bertsch, Lorenzo Giacomelli, and Georgia Karali, *Thin-film equations with "partial wetting" energy: existence of weak solutions*, Phys. D **209** (2005), no. 1-4, 17–27. MR MR2167440
9. J. A. Carrillo and G. Toscani, *Long-time asymptotics for strong solutions of the thin film equation*, Comm. Math. Phys. **225** (2002), 551–571. MR MR1888873 (2002m:35115)
10. Giuseppe Da Prato and Pierre Grisvard, *Equations d'évolution abstraites non linéaires de type parabolique*, Ann. Mat. Pura Appl. (4) **120** (1979), 329–396. MR MR551075 (81d:34052)
11. P. Daskalopoulos and R. Hamilton, *Regularity of the free boundary for the porous medium equation*, J. Amer. Math. Soc. **11** (1998), no. 4, 899–965. MR MR1623198 (99d:35182)
12. Lorenzo Giacomelli and Günther Grün, *Lower bounds on waiting times for degenerate parabolic equations and systems*, Interfaces Free Bound. **8** (2006), 111–129.
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15. ———, *Droplet spreading under weak slippage: the waiting time phenomenon*, Ann. Inst. H. Poincaré Anal. Non Linéaire **21** (2004), no. 2, 255–269. MR MR2047357 (2005c:35146)
16. Günther Grün and Martin Rumpf, *Simulation of singularities and instabilities arising in thin film flow*, European J. Appl. Math. **12** (2001), no. 3, 293–320, The dynamics of thin fluid films. MR MR1936040 (2003m:76050)
17. G. H. Hardy, *Note on a theorem of Hilbert*, Math. Z. **6** (1920), no. 3-4, 314–317. MR MR1544414
18. Herbert Koch, *Non–euclidean singular integrals and the porous medium equation*, Habilitation Thesis, 1999.
19. Alois Kufner and Lars-Erik Persson, *Weighted inequalities of Hardy type*, World Scientific Publishing Co. Inc., River Edge, NJ, 2003. MR MR1982932 (2004c:42034)
20. Alexander Oron, Stephen H. Davis, and S. George Bankoff, *Long-scale evolution of thin liquid films*, Rev. Mod. Phys. **69** (1997), no. 3, 931–980.
21. Felix Otto, *Lubrication approximation with prescribed nonzero contact angle*, Comm. Partial Differential Equations **23** (1998), no. 11-12, 2077–2164. MR MR1662172 (99m:35276)
22. M. V. Safonov, *The classical solution of the elliptic Bellman equation*, Dokl. Akad. Nauk SSSR **278** (1984), no. 4, 810–813. MR MR765302 (86f:35081)
23. Dejan Slepčev, *Linear stability of selfsimilar solutions of unstable thin–film equations*, submitted.

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