COLLATZ-WOCHE

Pattern formation & Partial Differential Equations

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Pattern formation for three specific examples

A) crystal growth under deposition
   — roughness of crystal surface

B) demixing of polymers
   — labyrinth-like pattern of concentration field

C) Temperature gradient triggers flow
   — mushroom-like pattern of temperature field

Few elementary mechanisms (diffusion, viscosity, ...)
   — complex Pattern
Partial differential equations

**Ordinary differential equation** for a quantity $c$

\[
\frac{dc}{dt}(t) = F(c(t))
\]

**Partial differential equation** for a field $c(x)$

\[
\frac{\partial c}{\partial t}(t, x) = F(c(t, x), \frac{\partial c}{\partial x}(t, x), \frac{\partial^2 c}{\partial x^2}(t, x), \cdots)
\]

Predictability

$c(t = 0, x)$ known $\leadsto c$ determined
Prediction

\[ \text{Solution of partial differential equations} \]

Explicit solutions
\[ c(t, x) = \frac{1}{td/2} e^{-\frac{|x|^2}{4t}} \]

Asymptotic approximate solutions
\[ \frac{dc}{dt} = F(\epsilon, c) \]
\[ c(t) = c_0(t) + \epsilon c_1(t) + \epsilon^2 c_2(t) + \cdots \]

Numerical approximate solutions
\[ \frac{1}{h}(c(t) - c(t - h)) = F(c(t)) \]

Qualitative properties of ensemble of solutions
\[ \int c \, dx, \cdots \]

Here: Existence & uniqueness & regularity ok
A. Crystal growth and Kuramoto-Sivashinsky equation
A. Relevant mechanisms

Crystal lattice favors certain slopes of the surface

Exposed positions are disfavored

Vertical growth rate depends on slope

\[
\frac{\partial h}{\partial t} = -\frac{\partial}{\partial x} \left( (1 - (\frac{\partial h}{\partial x})^2) \frac{\partial h}{\partial x} \right) - \frac{\partial^4 h}{\partial x^4} + f \left( 1 + \frac{1}{2} (\frac{\partial h}{\partial x})^2 \right)
\]
A. Qualitatively different behavior for small/large deposition rate $f$

Initial data $h(t = 0) =$ white noise of small amplitude

Deposition rate $f \ll 1$
- slow growth
- facets with preferred slope $\pm 1$
- number of facets decreases

Deposition rate $f \gg 1$
- fast growth
- slope $\ll 1$
- number of maxima/minima $\approx$ constant
A. Regime of strong deposition: Kuramoto-Sivashinsky equation

For $f \gg 1$, expressed in $u = -\frac{\partial h}{\partial x}$:

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x} \left( \frac{1}{2} u^2 \right) - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^4 u}{\partial x^4}$$
A. Three terms — three simple mechanisms

\[ \frac{\partial^2 u}{\partial x^2} \quad \text{Growth} \]

\[ \frac{\partial^4 u}{\partial x^4} \quad \text{Decay} \]

\[ \frac{\partial}{\partial x} \left( \frac{1}{2} u^2 \right) \quad \text{Shear} \]

Periodic configurations \( u(t, x + L) = u(t, x) \); large system \( L \gg 1 \)
A. Dynamic equilibrium

Initial data: \( u(t = 0) = \) white noise of small amplitude

Observations:

Initial phase

- 1. Smoothing \( \frac{\partial^4 u}{\partial x^4} \)
- 2. Growth \( \frac{\partial^2 u}{\partial x^2} \)
- 3. Shear \( \frac{\partial}{\partial x} \left( \frac{1}{2} u^2 \right) \)

Dynamic equilibrium

- average amplitude \( \sim 1 \)
- average wave length \( \sim 1 \)
- chaotic behavior in space & time

Shear contains exponential growth
A. Butterfly effect

\[ \tilde{u}(t = 0) = u(t = 0) + \text{localized perturbation of small amplitude} \]

**Observations:**

- 1. Perturbation stays “invisible”
- 2. Perturbation becomes visible at other place
- 3. \( \tilde{u} \) and \( u \) differ significantly everywhere
- \( \tilde{u} \) and \( u \) behave “statistically” similar

**Effective unpredictability of details**

— robust qualitative behavior
A. Energy spectrum

Decomposition of spatial signal into waves of length $L, \frac{L}{2}, \frac{L}{3}, \cdots$:

$$(\mathcal{F}u(t, \cdot))(k) := L^{-1} \int_0^L e^{ikx} u(t, x) \, dx \quad \text{(Fourier series)}$$

Contribution of wave number $(k, k + dk)$ to total energy:

$$L |(\mathcal{F}u(t, \cdot))(k)|^2 \, dk$$

Time average:

$$\lim_{t_0 \uparrow \infty} t_0^{-1} \int_0^{t_0} L |(\mathcal{F}u(t, \cdot))(k)|^2 \, dt$$
A. Equipartition of energy

Observations:

- Equipartition of energy over wave numbers $|k| \ll 1$
- Energy spectrum independent of $L \gg 1$

“Universal” behavior
A. Challenge for mathematics

Observation:
After initial phase, there is a dynamic equilibrium, with statistics independent of the initial data $u(t = 0)$ and of the system size $L$.

Challenge for theory of partial differential equations:
Why?

In mathematics: “Why ?” = “How can it be proved?”

A good proof gives insight into “why”
A. Modest state of mathematical insight

Only statements of the following form have been proved:

space-time averages of $|u|$, $|\frac{\partial u}{\partial x}|$, $|\frac{\partial^2 u}{\partial x^2}| \lesssim 1$,
for all initial data $u(t = 0)$, system sizes $L$

These statements have been proved step-by-step:

space-time averages of $|u|$, $|\frac{\partial u}{\partial x}|$, $|\frac{\partial^2 u}{\partial x^2}| \lesssim L^p$,
for all initial data $u(t = 0)$

Nicolaenko & Scheurer & Temam ’85, Goodman ’94: $\lesssim L^2$
Collet & Eckmann & Epstein & Stubbe ’93: $\lesssim L^{11/10}$
Bronski & Gambill ’06: $\lesssim L$, Giacomelli & O. ’05: $\ll L$
O. ’09 $\lesssim \ln^{5/3} L$

$\sim$ bounds on $\text{dim}(\text{Attractor})$, $\text{dim}(\text{Inertial Manifold})$, Fojas et. al.
A. Insight from proof

Three methods have been developed. Insight of last method:

Shear term $\frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right)$ behaves like a coercive term, i.e.

$$\int_0^L \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) u \, dx \quad \text{as} \quad \int_0^L \left|\frac{\partial}{\partial x}\right|^{1/3} u^3 \, dx$$

despite actually being conservative, i.e.

$$\int_0^L \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) u \, dx = 0.$$
A. Insight: Conservative acts as coercive in forced inviscid Burgers

Consider $f(t, x), g(t, x)$ with \[ \frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left( \frac{1}{2} u^2 \right) = \frac{\partial g}{\partial x}, \]
smooth, periodic in $x$, compactly supported in $t$. Then

\[ \int \int \left| \frac{\partial}{\partial x} \right|^{\frac{1}{3}} u \right|^{3} \, dx \, dt \lesssim \log \int \int \left| \frac{\partial}{\partial x} \right|^{\frac{2}{3}} g \right|^{\frac{3}{2}} \, dx \, dt, \]

more precisely expressed in interpolation spaces

\[ \| u; [\dot{H}_{\infty}^1, L_2]_{\frac{1}{3}, \infty} \|^{3} \lesssim \| g; [\dot{H}_2^1, L_1]_{\frac{2}{3}, 1} \|^{\frac{3}{2}}. \]
A. Insight: Connection with Onsager’s conjecture on level of forced viscous Burgers

On the one hand, for \(\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left( \frac{1}{2} u^2 \right) - \nu \frac{\partial^2 u}{\partial x^2} = \frac{\partial g}{\partial x}\) have uniform estimate in \(\nu \downarrow 0\)

\[
\|u; [\dot{H}^1_\infty, L^2_2]_{\frac{1}{3}, \infty}\|^3 + \nu \|u; \dot{H}^1_2\|^2 \lesssim \|g; [\dot{H}^1_2, L^1_1]_{\frac{3}{3}, 1}\|^{\frac{3}{2}}.
\]

On the other hand, at \(\nu = 0\) if \(u \in [\dot{H}^1_\infty, L^2_2]_{\frac{1}{3}, p}\) with \(p < \infty\), would have conservation of energy

\[
\frac{\partial}{\partial t} \int \frac{1}{2} u^2 \, dx = \int u \frac{\partial g}{\partial x} \, dx.
\]
A. Different physics — same mathematics

Magnetization in thin ferromagnetic films

Local minima of energy functional ($u$ gray scale)

\[
\epsilon \int \int \left( \frac{\partial u}{\partial x_1} \right)^2 dx_1 dx_2 + \int \int \left( | \frac{\partial}{\partial x_1} |^{-\frac{1}{2}} \left( \frac{\partial u}{\partial x_2} - \frac{\partial}{\partial x_1} \left( \frac{1}{2} u^2 \right) \right) \right)^2 dx_1 dx_2
- H \int \int u^2 dx_1 dx_2
\]
A. Similar math: Interpolation inequalities

Kuramoto-Sivashinsky:

\[
\epsilon \iint (\frac{\partial}{\partial x_1} u)^2 \, dx_1 dx_2 + \frac{1}{\epsilon} \iint \left( |\frac{\partial}{\partial x_1}|^{-1} \left( \frac{\partial u}{\partial x_2} - \frac{\partial}{\partial x_1} \left( \frac{1}{2} u^2 \right) \right) \right)^2 \, dx_1 dx_2
\]

Magnetism:

\[
\epsilon \iint (\frac{\partial}{\partial x_1} u)^2 \, dx_1 dx_2 + \left( \ln \frac{1}{\epsilon} \right) \iint \left( |\frac{\partial}{\partial x_1}|^{-1} \left( \frac{\partial u}{\partial x_2} - \frac{\partial}{\partial x_1} \left( \frac{1}{2} u^2 \right) \right) \right)^2 \, dx_1 dx_2
\]
B. Demixing and Cahn-Hilliard equation
B. Relevant mechanisms

\[ \frac{\partial u}{\partial t} = -\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)(u - u^3) - \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^2 u \]
B. Dynamics in initial phase

Periodic: \( u(t, x + L) = u(t, x) \). Large system: \( L \gg 1 \).

Initial data: \( u(t = 0) = \) white noise of small amplitude

Observations:

- 1. average wave length \( \sim 1 \),
- 2. average amplitude \( \sim 1 \)
B. Dynamics in later phase

Observations:

width of transition layer $\sim 1$

$\ll$ size of domains $\sim R$

$\ll$ size of system $\sim L$
B. A geometric evolution equation

Curvature Flow

\[ v = -\kappa \]

second order

Mullins-Sekerka

\[ -\Delta \varphi = 0 \]

third order

Pego,
Alikakos&Bates&Chen,
Röger & Schätzle

Surface Diffusion

\[ v = \frac{d \kappa}{ds^2} \]

fourth order
B. Statistical self-similarity

earlier  later  later, rescaled, periodically extended
B. Rate of energy decay

After initial phase: Energy $E(u) \approx$ length of transition layer

Energy $E$ vs. time $t$, double logarithmic plot:

$L^{-(d=2)} E(u) \sim t^{-1/3}$
B. Modest state of mathematical insight

Only the following statement has been proved:

\[ L^{-d} E(u) \gtrsim t^{-1/3} \quad \text{for } t \gg 1 \]

for all initial data \( u(t = 0) \) close to \( u = 0 \).

Kohn & O. '02
B. Insight from proof

Dynamics is steepest descent in energy landscape

- energy ↔ heights,
- dissipation mechanism ↔ distances

“Flat” energy landscape \(\implies\) slow energy decay
B. Dissipation mechanism influences dynamics

Energy functional \[ E \approx \text{length of interfacial layer} \]

- Flow, viscosity: \[ L^{-d}E(u) \lesssim t^{-1} \]
  Brenier & O. & Seis

- Bulk diffusion, friction: \[ L^{-d}E(u) \lesssim t^{-1/3} \]

- Surface diffusion, friction: \[ L^{-d}E(u) \lesssim t^{-1/4} \]
B. Dissipation mechanism determines geometry

Distance on configuration space

Monge-Kantorowicz distance with cost function

\[ c(x_1, x_2) = \ln(1 + |x_1 - x_2|) \]

\[ H^{-1} \text{ norm} \]

\[ \left( \int |\nabla|^{-1}(u_1 - u_2)^2 \right)^{\frac{1}{2}} \]

Monge-Kantorowicz distance with cost function

\[ c(x_1, x_2) = |x_1 - x_2| \]
B. Energy landscape via interpolation estimates

Flatness of energy landscape

\[
\frac{L^{-d} H^{d-1}}{(\partial \{ u \approx 1 \})} \left( L^{-d} \int \| \nabla |^{-1} u \|^2 dx \right)^{\frac{1}{2}} \gtrsim 1 \approx \left( L^{-d} \int |u|^\frac{4}{3} dx \right)^{\frac{3}{2}}
\]

\[
\nabla u \|_{L^1}^{\frac{1}{2}} \| \nabla |^{-1} u \|_{L^2}^{\frac{1}{2}} \gtrsim \| u \|_{L^\frac{4}{3}}
\]

Cohen & Dahmen & Daubechies & DeVore, Ledoux
B. Different physics — same mathematics

Domain branching in ferromagnets

Same interpolation estimate:

\[\|u\|_{L^4_3} \lesssim \|\nabla u\|_{L^1_2} \|\nabla^{-1} u\|_{L^2_2}\]
B. A universal pattern

Domain branching in ferromagnets

Domain branching in superconductors

Twin-splitting in shape memory alloys

Hubert, Choksi & Kohn

Landau, Choksi & Kohn & O.

Kohn & Müller, Conti
C. Rayleigh-Bénard convection
### C. Relevant mechanisms

<table>
<thead>
<tr>
<th>Diffusion</th>
<th>Buoyancy</th>
<th>Convection</th>
</tr>
</thead>
<tbody>
<tr>
<td>temperature $T$</td>
<td>flow velocity $u$</td>
<td></td>
</tr>
<tr>
<td>driving boundary cond.</td>
<td>limiting boundary cond.</td>
<td></td>
</tr>
</tbody>
</table>

- **Diffusion**
  - $T = 0$
  - $\frac{\partial T}{\partial z} - \Delta T = 0$
  - $z = H$
  - $T = 1$

- **Buoyancy**
  - $u = 0$
  - $-\Delta u + \nabla p = T(\varphi)$
  - $\nabla \cdot u = 0$

- **Convection**
  - $\partial_t T + \nabla \cdot (Tu) = 0$
  - $u = 0$
C. Dynamic equilibrium

Periodic in horizontal direction: \( T(t, x + L, z) = T(t, x, z) \)
large system \( H \gg 1, \ L \gg 1 \).

Initial data: \( T(t = 0) = \) linear profile + small amplitude white noise

Observation:

Initial stage  
- 1. large rolls  
- 2. boundary layer, plumes over half height

Dynamic equilibrium  
- plumes over entire height  
- chaotic behavior in space & time
C. Nusselt number

Diffusion and convection \(\leadsto\) vertical heat flux

heat flux \( q = T u - \nabla T \), vertical heat flux \( = q \cdot (0)_1 \)

Nusselt number = space-time average of \( q \cdot (0)_1 \)

\[
\begin{align*}
\text{Nu} &= \lim_{t_0 \to \infty} \frac{1}{t_0} \frac{1}{L^{d-1}} \frac{1}{H} \int_0^{t_0} \int_{(0, L)^{d-1} \times (0, H)} q \cdot (0)_1 \, dx \, dt 
\end{align*}
\]
C. Nusselt number $\textit{Nu}$ independent of height $H \gg 1$

Experiments: $\textit{Nu}$ independent of container height $H \gg 1$

Simulation:

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
 & $H$ & $\textit{Nu}$ & Varianz \\
\hline
BLAU: & 250 & 0.0648 & 3.61e−04 \\
ROT: & 500 & 0.0584 & 2.15e−04 \\
SCHWARZ: & 1000 & 0.0545 & 1.19e−04 \\
\hline
\end{tabular}
\end{table}
C. Modest state of mathematical insight

Constantin & Doering ('99): \( Nu \lesssim \ln^{2/3} H \), Stokes maximal regularity in \( L^\infty \)

Doering & O. & Reznikoff ('06): \( Nu \lesssim \ln^{1/3} H \), logarithmic background temperature profile

O. & Seis (in preparation): \( Nu \lesssim \ln^{1/15} H \), logarithmic background profile (optimal)

O. & Seis (in preparation): \( Nu \lesssim \ln^{2/3} \ln H \), Stokes maximal regularity in \( L^\infty \) and logarithmic background profile
C. Insight from proof

Two different methods of proof. Insights:

1) **Non monotonicity** of background temperature profile improves stability

   \[ \text{stable for } H_0 \lesssim \log^{-1/15} H \]

   yields \[ \text{Nu} \lesssim \log^{1/15} H \]

2) Optimal background profile *has no physical meaning*
Summary:

Scaling laws in spatially extended systems

B) Gradient flows

C) Driven gradient flows

A) Non-gradient dynamics
Summary

Simple mechanisms — complex patterns

Statistical properties of pattern are universal

Only partial mathematical understanding

Different physics — similar mathematics

Opposite bounds for generic initial data?